The Efficiency of A Simple Quantum Heat Engine

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
AT
ADDIS ABABA UNIVERSITY
ADDIS ABABA,ETHIOPIA

By
WARGA CHEGENO
NOVEMBER 2011

© Copyright by WARGA CHEGENO, 2011

Addis Ababa University
College of Natural Sciences
School of Graduate Studies
Faculty of Chemical and Physical Science
Department of Physics

The undersigned hereby certify that they have read and recommend to the School of Graduate Studies for acceptance a thesis entitled "**The Efficiency of A Simple Quantum Heat Engine**" by **WARGA CHEGENO** in partial fulfillment of the requirements for the degree of Master of Science (MSc) in physics.

Dated: November 2011

Approved by the Examining Committee

Dr. Mulugeta Bekele, Advisor ———————————.

Dr. Tatek Yergou, Examiner ———————————.

ADDIS ABABA UNIVERSITY

Date: **November 2011**

Author: **WARGA CHEGENO**

Title:

Department: **Physics**

Degree: **M.Sc.** Convocation: **November 2011** Year:

Permission is herewith granted to Addis Ababa University to circulate and to have copied for non-commercial purposes, at its discretion, the above title upon the request of individuals or institutions.

Signature of Author

THE AUTHOR RESERVES OTHER PUBLICATION RIGHTS, AND NEITHER THE THESIS NOR EXTENSIVE EXTRACTS FROM IT MAY BE PRINTED OR OTHERWISE REPRODUCED WITHOUT THE AUTHOR'S WRITTEN PERMISSION.

THE AUTHOR ATTESTS THAT PERMISSION HAS BEEN OBTAINED FOR THE USE OF ANY COPYRIGHTED MATERIAL APPEARING IN THIS THESIS (OTHER THAN BRIEF EXCERPTS REQUIRING ONLY PROPER ACKNOWLEDGEMENT IN SCHOLARLY WRITING) AND THAT ALL SUCH USE IS CLEARLY ACKNOWLEDGED.

Table of Contents

List of Figures

List of Tables

Abstract

We study a quantum heat engine with a working substance as a two level single spin-1/2 system immersed in a magnetic field region. Work is produced when the spin-1/2 system is being carried from high magnetic field region to a low magnetic field region. Heat is transfered when the occupation probabilities of the spin-1/2 system is changed while the energy level remains the same due to constant magnetic field. We have found that the efficiency of the engine at quasi-static limit is equal to the Carnot efficiency. We have also found that the efficiency at maximum power of our model heat engine is equal to that of the Curzon-Ahlborn efficiency.

Acknowledgements

First of all I would like to thank my instructor and advisor Dr. Mulugeta Bekele for his unreserved support and generous guidance throughout my work and sharing his experience of research and social life.

Secondly, I would like to express my sincere thanks to Statistical and Computational Physics group in providing me all the necessary material for this work.

Chapter 1

Introduction

A heat engine typically uses energy provided in the form of heat to do work and then exhausts the heat which cannot be used to do work.

To convert heat into work, you need at least two places with different temperatures. If you take in Q_{In} at temperature T_H you must dump at least Q_{out} at temperature T_C. The amount of work you get out of a heat engine is $W = Q_{In} - Q_{out}$. Heat engines such as automobile engines operate in a cyclic manner, adding energy in the form of heat in one part of the cycle and using that energy to do useful work in another part of the cycle. Heat engines have played a major role in the development of thermodynamics, and most modern general physics and thermodynamics texts use heat engines to introduce the second law of thermodynamics[1]. In physics, the efficiency of heat engines has been treated as a basic subject of thermodynamics[2]. A major consequence of the second law is Carnot's theorem for for heat engines. A reversible Carnot cycle, operating between fixed reservoir temperatures T_H and T_C in which T_C is less than T_H, has the highest thermal efficiency possible for a heat engine operating between these temperatures, namely

$$\eta_c = 1 - \frac{T_C}{T_H} \tag{1.1}$$

where T_H and T_C are the hot and the cold reservoir temperatures. Although the performance limits of reversible processes like the Carnot efficiency provide upper bounds for real irreversible processes they may not be good enough to be a useful guide in the improvement of real processes[3]. In spite of the high efficiency, η_C is usually realized only in the quasistatic limit. This means that the Carnot heat engine is useless as a real engine because the power defined as output work per unit time is 0. Real engines should work for a finite time and produce a finite power.

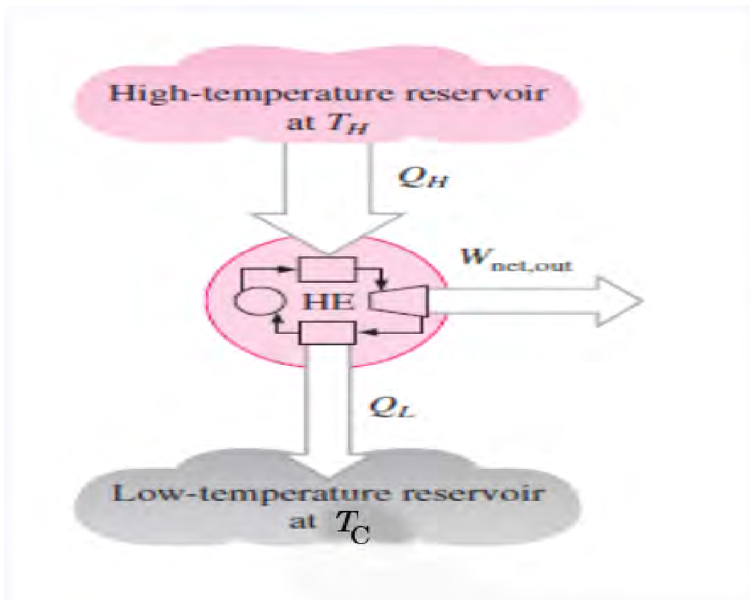

Figure 1.1: Schematic representation of a heat engine. Heat is taken at the high temperature T_H . Some is converted to work and the remainder is released at the low temperature T_C .

Therefore, the finite-time extension of the quasistatic heat engines is an important subject of thermodynamics. Equilibrium thermodynamics compares real processes to reversible processes proceeding without losses at an infinite slow speed. Curzon and Ahlborn considered such an extension of the Carnot cycle and derived a simple and beautiful result: the efficiency at the maximal power output is given by[4],[5],[6],[7].

$$\eta_{CA} = 1 - \sqrt{\frac{T_C}{T_H}} \tag{1.2}$$

This can also be written as

$$\eta_{CA} = \frac{\eta_C}{2} + \frac{\eta_c^2}{8} + ... \tag{1.3}$$

which is considerably lower than the corresponding Carnot efficiency.

Chapter 2

Quantum Thermodynamic Cycle and Quantum Heat Engine

2.1 Basic Quantum Thermodynamic Process

A. Quantum first law of thermodynamics

To define quantum isothermal and quantum isochoric processes, we need to first consider the working substance. An arbitrary quantum system with a finite number of energy levels is used here as the working substance [8](see Fig. 2.1). The Hamiltonian of the working substance can be written as

$$H = \sum_n E_n |n\rangle\langle n|. \tag{2.1}$$

where $|n\rangle$ is the n^{th} eigen state of the system and E_n is its corresponding eigen energy. The internal energy U of the working substance can be expressed as

$$U = \sum_n E_n p_n \tag{2.2}$$

for a given occupation distribution with probabilities p_n in the n^{th} eigen state. To clearly define quantum isothermal and isochoric processes, we need to identify the quantum analogues of the heat exchange dQ and the work performed dW. From Eq. (2.2) we have

$$dU = \sum_n [E_n dp_n + p_n dE_n] \tag{2.3}$$

In classical thermodynamics, the first law of thermodynamics is expressed as

$$dU = dQ + dW \tag{2.4}$$

where $dQ = TdS$, and $dW = \sum_i Y_i dy_i$ [5]; T is the temperature and S is the entropy; y_i is the generalized coordinates and Y_i is the generalized force conjugated to y_i. Due to the relationship $S = k_B \sum_i p_i ln p_i$ between the entropy S and the probabilities P_i, we can make the following identification

$$dQ = \sum_n E_n dp_n, \tag{2.5}$$

$$dW = \sum_n p_n dE_n. \tag{2.6}$$

Equation (2.6) implies that the work performed corresponds to the change in the eigen energies E_n, and this is in accordance with the fact that work can only be performed through a change in the generalized coordinates of the system, which in turn gives rise to a change in the eigen energies [9]. Thus the quantum version of the first law of thermodynamics $dU = dQ + dW$ just follows from Eq.(2.3) with the quantum identifications of heat exchange and work performed in Eqs.(2.5) and (2.6). Different from dQ = T dS, which is applicable only to the thermal equilibrium case, below we will see that Eqs. (2.5) and (2.6) are applicable to both the thermal equilibrium case.

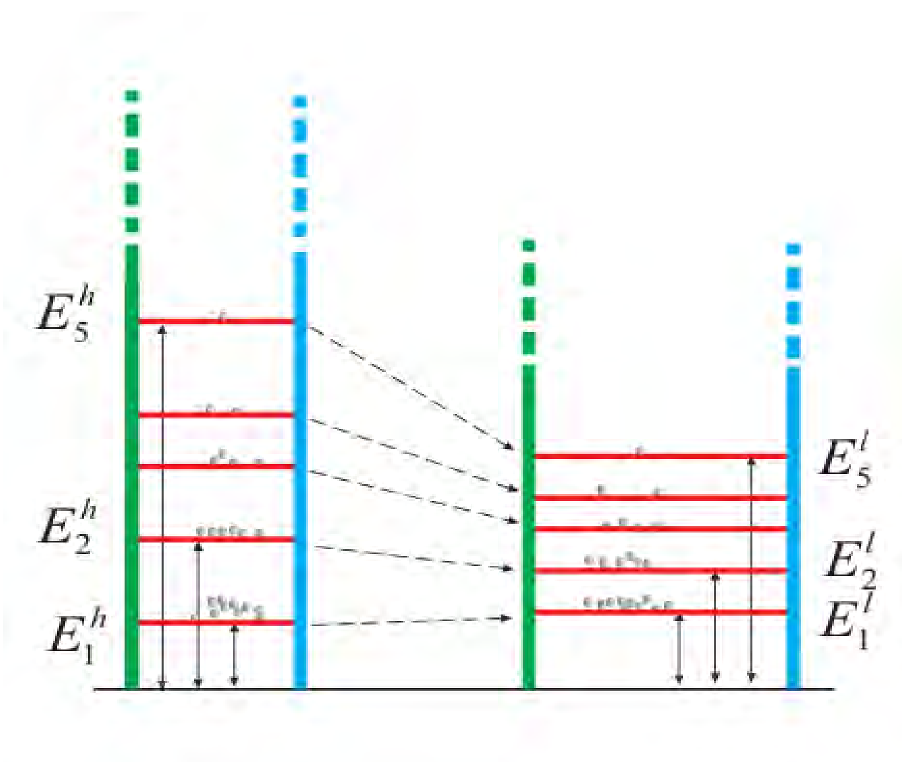

Figure 2.1: (Color online) Schematic diagram of multi-level quantum system as the working substance for a QHE. E_n^h and E_n^l are the n^{th} eigen energy of the working substance in the two isochoric processes.

B. Quantum isothermal process

Consider the quantum versions of some thermodynamic processes [10]. In quantum isothermal processes, the working substance, such as a particle confined in a potential energy well, is kept in contact with a heat bath at a constant temperature [11]. The particle can perform positive work to the outside, and meanwhile absorb heat from the bath. Specifically, they consider a two level system with the excited state $|e\rangle$, the ground state $|g\rangle$, and a single energy spacing Δ with instantaneous eigen energy $(E_e(t)$ $(E_g(t))$ depending on time t. They modeled this two-level system as a spin 1/2 particle in an external magnetic field. It interacts with a heat bath of inverse temperature β, which can be universally modeled as a collection of many bosons. Initially, the two-level system be thermalized to equilibrium. Then they alter the magnetic field slowly so that the energy level spacing $\Delta(t)$ slowly changes from Δ_A to Δ_B. During the controlling process illustrated by the smooth curve AB in Fig.2.2, work is done on the system. In the infinitely slow process, which can be alternatively regarded as a quantum isothermal process the two-level system is in the thermal equilibrium at every instant.

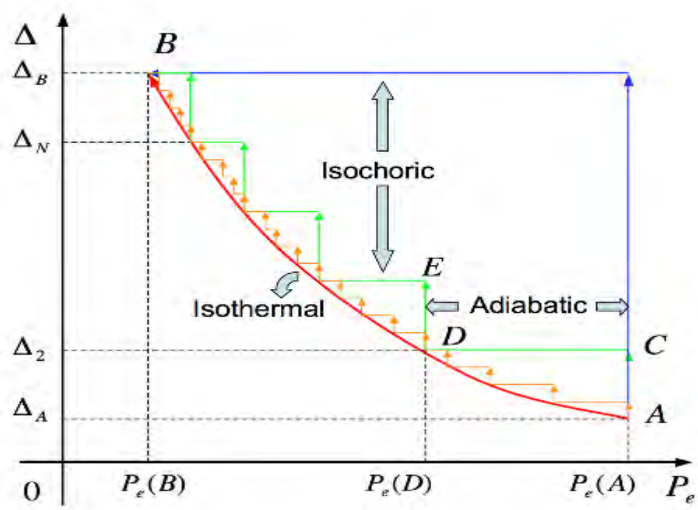

Figure 2.2: Schematic illustration of a quantum isothermal process AB. Here the horizontal axis P_e is the occupation probability in the excited state of the two-level system, and the vertical axis indicates the level spacing of the two-level system. The smooth curve AB represents the isothermal process, whose equation of state can be expressed as $\Delta(t) = -\beta^{-1} ln(p_e^{-1} - 1)$. The horizontal and vertical lines represent QIC and QAP.

It should be pointed out that, during the isothermal process, there is a heat exchange between the two-level system and the heat bath. For such an isothermal process, it is difficult to calculate the microscopic work distribution directly. However, this process can be simulated by a series of QAP and QIP. In QAP (QIC) processes, there is only work done (heat exchange). Hence, using the changes of eigen-energies of microscopic state at instant t = A, C, D, they indirectly calculate the microscopic work done (heat exchange) $dW = E_\alpha(C) - E_\alpha(A)$ and $dQ = E_\alpha(D) - E_\beta(C))$ for $\alpha,\beta = e, g$. In the parameter space, these QAP and QIP series processes are represented by the stair path $(A \to C \to D \to ... \to B)$ in Fig 2.2 . When every step of the stair path becomes infinitesimal, the stair path becomes equivalent to the isothermal process AB. In this way they simulate the quantum isothermal process with N equal-height steps with the small height $\Delta = \frac{(\Delta_B - \Delta_A)}{N}$ where Δ_A and Δ_B are the level spacings at point A and point B respectively. The level spacing of the two-level system after the $(j-1)^{th}$ QIC is

$$\Delta_j = \Delta_A + (j-1)\Delta. \tag{2.7}$$

for $j = 1, 2, , N+1$. The initial and final point A and B corresponds to $j = 1$ and $j = N+1$ respectively. When they fix the initial point A, and the final point B, the jump Δ in every step decrease with the increase of the step number N, and Δ approaches zero when N becomes infinity. Obviously, when N $\rightarrow \infty$, the stair path approaches its asymptotic behavior of the isothermal path (see Fig. 2.2). When the system reaches thermal equilibriums, the occupation probabilities obeys the Gibbs distribution defined by

$$p_e^j = \frac{e^{-\beta\Delta_j}}{1+e^{-\beta\Delta_j}}; p_g^j = p_e^j e^{\beta\Delta_j} \tag{2.8}$$

Having defined the path in the parameter space (Δ, p_e) space, they further introduce the microscopic work and its corresponding probabilities for a given path. In the above path divided into many steps, the first step $A \rightarrow C \rightarrow D$ consists of a QAP $A \rightarrow C$, and a QIP $C \rightarrow D$. At the beginning (the point A of Fig. 2.2), the system is initially in a thermal equilibrium state , which implies that the system is either in its microscopic state $|g\rangle$ or$|e\rangle$ with probabilities p_g and p_e respectively. We choose the ground state in the energy reference point so that the microscopic energy $E(A)$ of the system at initial pint A can take $E_e(A) = \Delta_A$ or $E_g(A) = 0$, with probability p_e and $1 - p_e$ respectively. In the first QAP $A \rightarrow C$, the system remains in its microscopic state $|g\rangle(|e\rangle)$ if the system is initially in its microscopic state $|g\rangle(|e\rangle)$. As there is no heat exchange in the quantum adiabatic process, the work done by external controller is just the change of the microscopic energy $W = E(C) - E(A)$ for $\ = e, g$. Correspondingly the work done during $A \rightarrow C$ can be either $\Delta_C - \Delta_A$ or 0 with probabilities P_e or $1 - P_e$ respectively. This also agrees with the definition of work in quantum mechanical system: work is associated with the change of the level spacing.

C. Quantum isochoric process

A quantum isochoric process has similar properties to that of a classical isochoric processes. In a quantum isochoric process, the working substance is placed in contact with a heat bath. No work is done in this process while heat is exchanged between the working substance and the heat bath [8]. This is the same as that in a classical isothermal process. In a quantum isochoric process the occupation probabilities p_n and thus the entropy S changes, until the working substance finally reaches thermal equilibrium with the heat bath. In classical isochoric process the pressure P and the temperature T changes, and the working substance reaches thermal equilibrium with the heat bath only at the end of this process. For example, if the working substance is chosen to be a particle confined in a infinite square well potential, no work is done during a quantum isochoric process when heat is absorbed or released, and the occupation probabilities in every eigen state satisfy Boltzmann distribution at the end of the isochoric process.

D. Quantum adiabatic process

A classical adiabatic thermodynamic process can be formulated in terms of a microscopic quantum adiabatic thermodynamic process. Because quantum adiabatic processes proceed slow enough such that the generic quantum adiabatic condition is satisfied, then the population distributions remain unchanged, $dp_n = 0$.

According to Eq. (2.5), dQ = 0, there is no heat exchange in a quantum adiabatic process, but work can still be nonzero according to Eq. (2.6). A classical adiabatic process, however, does not necessarily require the occupation probabilities to be kept invariant.

2.2 Quantum Carnot Engine Cycle(QCE)

Here we will see the work of H.T. Quan and Franco Nori about the QCE and its properties. The QCE cycle (see Fig. 2.3) for an example of a QCE based on a two-level system. just like its classical counterpart, consists of two quantum isothermal processes ($A \to B$ and $C \to D$) and two quantum adiabatic processes ($B \to C$ and $D \to A$). During the isothermal expansion process from A to B, the particle confined in the potential well is kept in contact with a heat bath at temperature T_h, while the energy levels of the system change much slower than the relaxation of the system, so that the particle is always kept in thermal equilibrium with the heat bath [11].

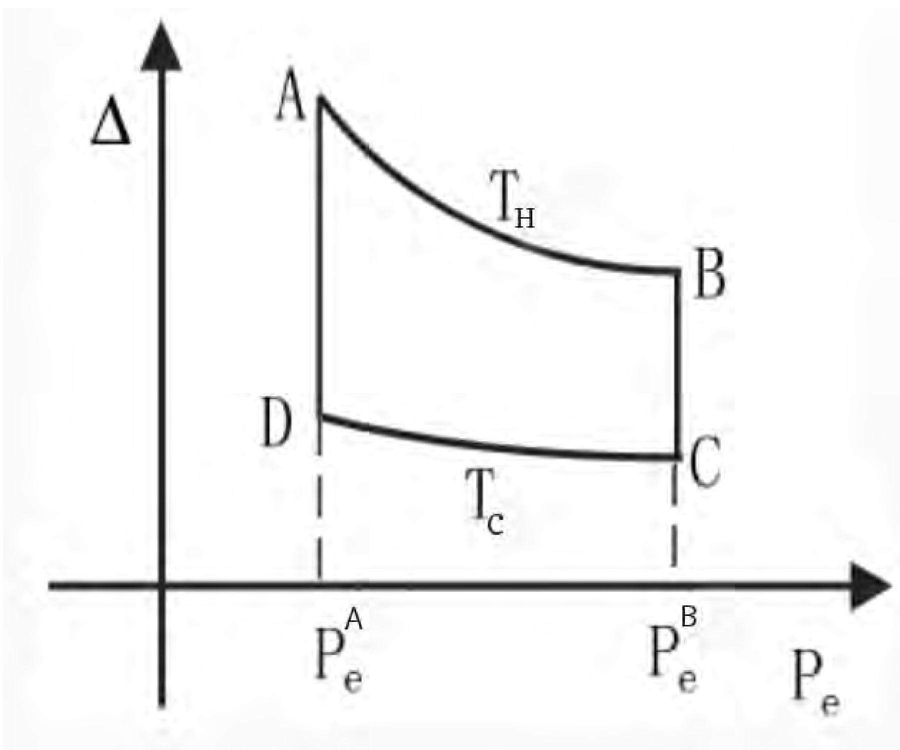

Figure 2.3: A schematic diagram of a quantum Carnot engine based on a two-level quantum system. Δ is the level spacing between the two energy levels. P_e is the occupation probability in the excited state. The process from A to B (C to D) is the isothermal expansion (compression) process, in which the working substance is put in contact with the high (low) temperature heat bath. The processes from B to C and from D to A are two adiabatic processes.

Now we will see how they analyze the operation efficiency η_C of the QCE. For simplicity they use
$dQ = TdS$ to calculate the heat exchange dQ in any Quantum IsoThermal (QIT) process. Because the temperature of the heat bath is kept invariant in the quantum isothermal process, the heat absorbed Q_{In} and released Q_{Out} in the quantum isothermal expansion and compression processes can be calculated as follows.

$$Q_{In} = T_H[S(B) - S(A)] > 0 \tag{2.9}$$

$$Q_{Out} = T_C[S(D) - S(C)] > 0 \tag{2.10}$$

where T_H and T_C are the temperatures of the two different heat baths.

$$S(i) = -k_B \sum_n p_i ln p_i \tag{2.11}$$

are the entropies of the working substance at different instants $i = A, B, C, D$ and calculating the work W_{cyc} done during a QCE cycle and its operation efficiency η_C. From Eqs.(2.9) and (2.10) and the first law of thermodynamics we obtain the net work done during a QCE cycle.

$$W_{Cyc} = Q_{In} - Q_{Out} \tag{2.12}$$

$$W_{Cyc} = (T_H - T_C)[S(B) - S(A)] \tag{2.13}$$

where we have used the relations $S(B) = S(C)$ and $S(A) = S(D)$. This equivalence is due to the fact that the occupation probabilities and thus the entropy remain invariant in any quantum adiabatic process. The efficiency η_C of the QCE is

$$\eta_c = \frac{W_C}{Q_{In}} \tag{2.14}$$

$$\eta_c = 1 - \frac{T}{T_H} \tag{2.15}$$

Chapter 3

The Efficiency Of A Two Level Quantum Heat Engine

3.1 Quantum Thermodynamic Processes

The study of heat engines has always been an important part of thermodynamics. In order to act as a heat engine, a physical system usually has to go through a cyclic process enforced by its environment. Here we want to discuss how this concept of thermodynamic machines can be carried over to small quantum(spin 1/2) system. The system can perform positive work to the outside and meanwhile absorb heat from the bath. Both the energy gap and the occupation probabilities need to change simultaneously, along the continuous path B to C or D to A in fig 3.2 so that the system remains in an equilibrium state with the bath at every instant. This two level system can be modeled as a spin 1/2 system in an external magnetic field. It interacts with a heat bath of inverse temperature β, which can be universally modeled as a collection of many Bosons.

The basic problems that we address here are the infinite and finite-time performance and also efficiency at maximum power of this engine as it runs through the following four standard stages of a Carnot cycle (see figure 3.2) based on [10]

I Adiabatic expansion process

The spin which was initially at equilibrium with the cold heat reservoir is disconnected and the energy level is changed from Δ_A to new level Δ_B with in a short period of time by increasing the magnetic field. Since the process of increasing the magnetic field is done a short enough time, the population of the level does not change during this process. Hence, there is no heat exchange during this process. However, the change of the energy level releases a corresponding amount of work. We assume that the operation time of this step is very short, in particular negligibly small compared to that of the time taken for the isothermal process.

II) Isothermal expansion process

The spin 1/2 system is connected with a hot reservoir at temperature T_H in addition to external magnetic field B. The magnetic field is increased from B_i to B_f during a time interval of duration N in appropriate time unit. Work is done on the system and it absorbs heat from the hot reservoir during this process (B to C).

III) Adiabatic compression process

The system is disconnected from the hot reservoir and the magnetic field is decreased which leads to the decrease in the energy level. Here no heat is transfered but the system do work on the environment. Again, we assume that the operation time of this process is negligibly small.

IV) Isothermal process

The spin is then connected to a cold heat reservoir at temperature T_C. The energy level is restored from Δ_D to the initial energy level Δ_A, by releasing a heat and doing a corresponding amount of work during a time interval of length N'.

The above procedure defines one cycle of the thermal engine, requiring a total time taken $N + N'$. The change in magnetic field in step II and III must be designed in such a way that the thermodynamic state of the system, in our case the occupation probability p_e of the quantum level, returns to the same initial value after every cycle. Since there is no change in occupation probability during the adiabatic stages I and II, the change in occupation probability from, say p_e^B to p_e^C, during process II, must necessarily be compensated by a change back from p_e^D to p_e^A during process IV.

The mean energy, $\langle U \rangle$, of the system is given by

$$\langle U \rangle = \sum_n E_n p_n \tag{3.1}$$

where E_n is the energy of the n^{th} energy level of the spin 1/2 system and p_n is the probability to be in the state n. The infinitesimal change $d\langle U \rangle$ in the internal energy of the system is

$$d\langle U \rangle = d\sum_n E_n p_n. \tag{3.2}$$

$$d\langle U \rangle = \sum_n p_n dE_n + \sum_n E_n dp_n \tag{3.3}$$

The equivalent equation of first law of thermodynamics is

$$dU = dW + dQ. \tag{3.4}$$

where

$$dW = \sum_n p_n dE_n \tag{3.5}$$

and

$$dQ = \sum_n E_n dp_n. \tag{3.6}$$

So, by using equations (3.5) and (3.6) we can determine the heat absorbed(released) and the work done(input). Since our system is a single two level spin 1/2 system with eigen-energies E_e and E_g, the infinitesimal heat absorbed(released) is

$$dQ = E_e dp_e + E_g dp_g. \tag{3.7}$$

since the sum of occupation probabilities is normalized

$$1 - p_e = p_g \tag{3.8}$$

Substituting this into equation (3.7) we will get

$$dQ = (E_e - E_g)dp_e. \tag{3.9}$$

Similarly the infinitesimal work done is

$$dW = p_e(dE_e - dE_g) + dE_g. \tag{3.10}$$

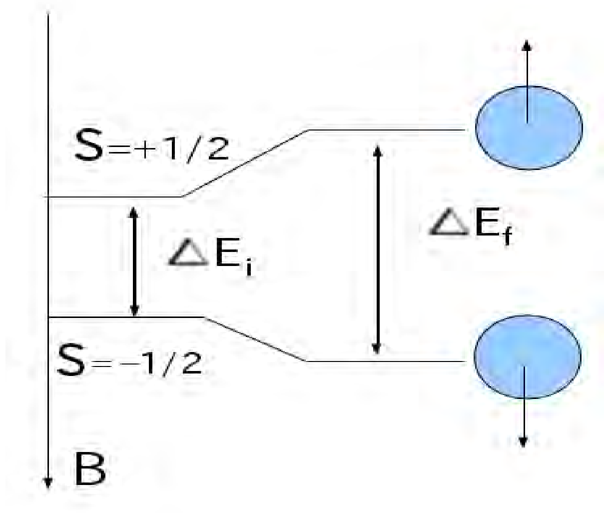

Figure 3.1: Allowed Energy States for a Spin 1/2 System

3.2 Quasi-static Limit

Consider a one cycle process described in the above section. It is difficult for such a cyclic process containing an isothermal process to calculate the microscopic work and heat transfer directly. So, we can divide the isothermal expansion and compression into a series of N steps of quantum adiabatic processes (QAP) and quantum isochoric processes(QIC) in which the magnetic field is changed slowly that our system is quasi static [10]. The system does work (work is done on the system) as the energy level is changed analogues to the change of coordinate in mechanical system and heat is transfered as the occupation probability is changed. In Fig 3.2 bellow the path is divided into many steps.

The first step $B \rightarrow 1 \rightarrow 2$ of the isothermal expansion consists of a QIC $B \rightarrow 1$, and a QAP $1 \rightarrow 2$. and the next step is from $2 \rightarrow 3 \rightarrow 4$ consists of a QIC $2 \rightarrow 3$, and a QAP $3 \rightarrow 4$ and so on until the last step $N \rightarrow N+1 \rightarrow B$. Similarly we can also subdivide the isothermal compression process in the same way up to N' which is equal to N.

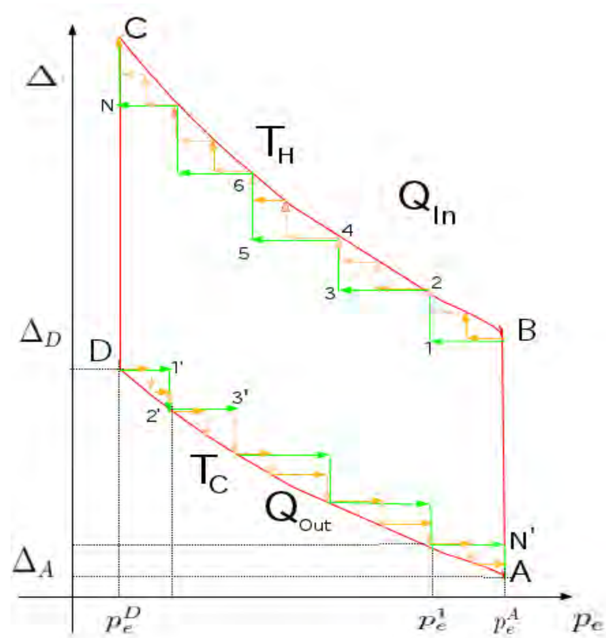

Figure 3.2: Schematic illustration of a Quantum Carnot Cycle. Here the horizontal axis P_e indicates the occupation probability in the excited state of the two-level system, and the vertical axis indicates the level spacing of the two-level system

We can calculate the heat absorbed in the consecutive isochoric process in the following way. In the first step of isochoric process($1 \rightarrow 2$) it will absorb an amount of heat dQ^1 where

$$dQ^1 = (E_e^1 - E_g^1)(p_e^2 - p_e^1) \tag{3.11}$$

similarly

$$dQ^2 = (E_e^3 - E_g^3)(p_e^4 - p_e^3) \tag{3.12}$$

$$dQ^3 = (E_e^5 - E_g^5)(p_e^6 - p_e^5) \tag{3.13}$$

The heat absorbed in the last isochoric step $N \rightarrow N+1$ is

$$dQ^N = (E_e^N - E_g^N)(p_e^{N+1} - p_e^N). \tag{3.14}$$

The total heat absorbed as our system expands isothermally from B to C is the sum of the heat absorbed in each step of the isochoric process.

$$Q^{B\rightarrow C} = dQ^1 + dQ^2 + dQ^3 + ... + dQ^N \tag{3.15}$$

$$\begin{aligned} Q^{B\rightarrow C} \quad &= (E_e^1 - E_g^1)(p_e^2 - p_e^1) + (E_e^3 - E_g^3)(p_e^4 - p_e^3) + (E_e^5 - E_g^5)(p_e^6 - p_e^5) + \\ &\quad .... + (E_e^N - E_g^N)(p_e^{N+1} - p_e^N) \end{aligned} \tag{3.16}$$

$$\begin{aligned} Q^{B\rightarrow C} \quad &= \Delta_B(p_e^1 - p_e^B) + (\Delta_B + \Delta)(p_e^4 - p_e^3) + (\Delta_B + 2\Delta)(p_e^6 - p_e^5) + \\ &\quad .... + (\Delta_B + N\Delta)(p_e^{N+1} - p_e^N) \end{aligned} \tag{3.17}$$

where Δ_B is equal to $E_e^1 - E_g^1$ and Δ is the energy spacing between any two consecutive energy levels and is given by
$\Delta = \frac{\Delta_C - \Delta_B}{N}$.

Since we have subdivided the total energy spacing between B and C into N intervals Eq.(3.17)can be written as

$$
\begin{aligned}
Q^{B\to C} \quad &= \Delta_B p_e^1 - \Delta_B p_e^B + (\Delta_B+\Delta)p_e^3 - (\Delta_B+\Delta)p_e^2 + (\Delta_B+2\Delta)p_e^5 - (\Delta_B+2\Delta)p_e^4 \\
&+ ..... + (\Delta_B+N\Delta)p_e^{N+1} - (\Delta_B+N\Delta)p_e^N
\end{aligned}
\tag{3.18}
$$

$$
\begin{aligned}
Q^{B\to C} \quad &= \Delta_B p_e^1 - \Delta_B p_e^B + \Delta_B p_e^3 + \Delta p_e^3 - \Delta_B p_e^2 - \Delta p_e^2 + \Delta_B p_e^5 + 2\Delta p_e^5 \\
&-\Delta_B p_e^4 - 2\Delta p_e^4 + ... + (\Delta_B+N\Delta)p_e^{N+1} - \Delta_B p_e^N - N\Delta p_e^N
\end{aligned}
\tag{3.19}
$$

Note that the occupation probabilities $p_e^1 = p_e^2, p_e^3 = p_e^4, p_e^5 = p_e^6, ..., p_e^{N+1} = p_e^C$. Then simplifying equation (3.19)

$$Q^{B\to C} = (\Delta_B + N\Delta)p_e^{N+1} - \Delta_B p_e^B - \Delta(p_e^1 + p_e^3 + p_e^5 + ... + p_e^N) \tag{3.20}$$

$$Q^{B\to C} = [\Delta_B + (N+1)\Delta]p_e^{N+1} - \Delta_B p_e^B - \Delta[p_e^1 + p_e^3 + p_e^5 + ... + p_e^{N+1}] \tag{3.21}$$

As N$\to\infty, \Delta\to 0$

the term in the second bracket above becomes zero while the term in the first bracket will be Δ_C. Then Eq.(3.21) is simplified to

$$Q^{B\to C} = \Delta_C p_e^C - \Delta_B p_e^B. \tag{3.22}$$

So that the heat absorbed during isothermal expansion can be written as

$$Q_{In} = (E_e^C - E_g^B)p_e^C - (E_e^B - E_g^B)p_e^B. \tag{3.23}$$

Note that Δ_C and Δ_B are the energy spacing at point C and B respectively.

Similarly the heat released during isothermal compression from D to A is the sum of the infinitesimal heat released at fixed external parameter (the magnetic field) while the occupation probabilities are changed. So,

$$Q^{D\to A} = dQ^{1'} + dQ^{2'} + dQ^{3'} + ... + dQ^{N'+1} \tag{3.24}$$

Using equation (3.6)

$$dQ^{1'} = (E_e^{1'} - E_g^{1'})(p_e^{2'} - p_e^{1'}) \tag{3.25}$$

$$dQ^{2'} = (E_e^{3'} - E_g^{3'})(p_e^{4'} - p_e^{3'}) \tag{3.26}$$

$$dQ^{3'} = (E_e^{5'} - E_g^{5'})(p_e^{6'} - p_e^{5'}) \tag{3.27}$$

$$dQ^{4'} = (E_e^{7'} - E_g^{7'})(p_e^{8'} - p_e^{7'}) \tag{3.28}$$

The heat released in the last isochoric compression step

$$dQ^{N'+1} = (E_e^{N'+1} - E_g^{N'+1})(p_e^{D} - p_e^{N'+1}) \tag{3.29}$$

Using from equation (3.24) to equation (3.28) equation (3.23) becomes

$$\begin{aligned} Q^{D\to A} \quad &= \Delta_D(p_e^{1'} - p_e^{D}) + (\Delta_D - \Delta)(p_e^{3'} - p_e^{2'}) + (\Delta_D - 2\Delta)(p_e^{5'} - p_e^{4'}) + \\ &....... + (\Delta_D + N'\Delta)(p_e^{N'} - (p_e^{N'-1}) \end{aligned} \tag{3.30}$$

$$\begin{aligned} Q^{D\to A} \quad &= \Delta_D p_e^{1'} - \Delta_D p_e^{D} + \Delta_D p_e^{3'} - \Delta p_e^{3'} - \Delta_D p_e^{2'} + \Delta p_e^{2'} + \Delta_D p_e^{5'} - 2\Delta p_e^{5'} - \\ &\Delta_D p_e^{4'} + 2\Delta p_e^{4'} + ... + (\Delta_D - N'\Delta)p_e^{N'} - \Delta_D p_e^{N'-1} + N'\Delta p_e^{N'-1} \end{aligned} \tag{3.31}$$

With a certain manipulation as before we will get

$$Q^{D\to A} = (\Delta_D - N'\Delta)p_e^{N'} - \Delta_D p_e^{D'} + \Delta(p_e^{1'} + p_e^{3'} + p_e^{5'} + ... + p_e^{N'}) \tag{3.32}$$

This equation can be more simplified to

$$Q^{D\to A} = \Delta_A p_e^A - \Delta_D p_e^D + \Delta(p_e^{3'} + p_e^{5'} + ... + p_e^{N'}) \tag{3.33}$$

With $p_e^A = p_e^B$ and $p_e^C = p_e^D$,

$\Delta_A = E_e^A - E_g^A$

$\Delta_D = E_e^D - E_g^D$ and as

$N \to \infty$ $\Delta \to 0$, the heat released during isothermal compression process is

$$Q_{Out} = (E_e^A - E_g^A)p_e^A - (E_e^D - E_g^D)p_e^D. \tag{3.34}$$

We can also determine the work done in the process for one complete cycle by calculating the infinitesimal work done by changing the magnetic field slowly while the occupation probability remains the same or when the system expands (compressed) adiabatically. From equation (3.5) and since our system is two level system the infinitesimal work done is given by

$$dW = p_e dE_e + p_g dE_g. \tag{3.35}$$

Using the normalization condition this equation can be written as

$$dW = p_e(dE_e - dE_g) + dE_g. \tag{3.36}$$

The work done during adiabatic expansion that is from A to B is

$$dW = p_e^B(dE_e - dE_g) + dE_g. \tag{3.37}$$

Integrating this equation

$$W^{A\to B} = p_e^B((E_e^B - E_g^B) - (E_e^A - E_g^A)) + (E_g^B - E_g^A) \tag{3.38}$$

The work done during isothermal expansion that is from B to C will be

$$W^{B\to A} = dW^1 + dW^2 + dW^3 + ... + dW^N. \tag{3.39}$$

From Eq.(3.5) the infinitesimal work done in each step can be determined as follow

$$dW^1 = p_e^2((E_e^2 - E_g^2) - (E_e^1 - E_g^1)) + (E_g^2 - E_g^1) \tag{3.40}$$

$$dW^2 = p_e^4((E_e^4 - E_g^4) - (E_e^3 - E_g^3)) + (E_g^4 - E_g^3) \tag{3.41}$$

$$dW^3 = p_e^6((E_e^6 - E_g^6) - (E_e^5 - E_g^5)) + (E_g^6 - E_g^5) \tag{3.42}$$

The work done in the last adiabatic expansion process is

$$dW^N = p_e^C((E_e^{N+1} - E_g^{N+1}) - (E_e^N - E_g^N)) + (E_g^{N+1} - E_g^N). \tag{3.43}$$

Then

$$\begin{aligned} W^{B\to C} &= p_e^2((\Delta_A + \Delta) - \Delta_A)) + p_e^4((\Delta_A + 2\Delta) - (\Delta_A - \Delta)) + p_e^6((\Delta_A + 3\Delta) - (\Delta_A + 2\Delta)) \\ &+ .... + p_e^{N+1}((\Delta_A + (N+1)\Delta) - (\Delta_A + N\Delta)) + (E_g^{N+1} - E_g^A) \end{aligned} \tag{3.44}$$

Simplifying this equation with $(E_g^{N+1} = E_g^C)$ and $(E_g^1 = E_g^B)$

$$W^{B\to C} = (E_g^C - E_g^B) + \Delta(p_e^2 + p_e^4 + p_e^6 + ... + p_e^N) \tag{3.45}$$

The work done during adiabatic compression that is from C to D will be

$$W^{C\to D} = p_e^C((E_e^D - E_g^D) - (E_e^C - E_g^C)) + (E_e^D - E_g^C) \tag{3.46}$$

In a similar way dividing the isothermal compression process in to a series of adiabatic compression and isochoric process we can determine the infinitesimal work and then the total work done during isothermal compression as follow.

$$dW^{1'} = p_e^{1'}((E_e^{2'} - E_g^{2'}) - (E_e^{1'} - E_g^{1'})) + (E_g^{2'} - E_g^{1'}) \tag{3.47}$$

$$dW^{2'} = p_e^{3'}((E_e^{4'} - E_g^{4'}) - (E_e^{3'} - E_g^{3'})) + (E_g^{4'} - E_g^{3'}) \tag{3.48}$$

$$dW^{3'} = p_e^{5'}((E_e^{6'} - E_g^{6'}) - (E_e^{5'} - E_g^{5'})) + (E_g^{6'} - E_g^{5'}) \tag{3.49}$$

$$dW^{N'} = p_e^{N'}((E_e^A - E_g^A) - (E_e^{N'} - E_g^{N'})) + (E_g^A - E_g^{N'}) \tag{3.50}$$

Then the total work done in this process will be

$$W^{D\to A} = dW^{1'} + dW^{2'} + dW^{3'} + ... + dW^{N'} \tag{3.51}$$

$$\begin{aligned} W^{D\to A} \quad &= p_e^{1'}((\Delta_D - \Delta) - \Delta_D)) + p_e^{3'}((\Delta_D - 2\Delta) - (\Delta_D - \Delta)) + \\ &(p_e^{5'}((\Delta_D - 3\Delta) - (\Delta_D - 2\Delta)) + ... + p_e^{N'}\Delta_A \\ &-(\Delta_D + N'\Delta)) + (E_g^{1'} - E_g^C) + \\ &(E_g^{2'} - E_g^{1'}) + (E_g^{4'} - E_g^{3'}) + ... + (E_g^A - E_g^{N'}) \end{aligned} \tag{3.52}$$

Simplifying this equation

$$W^{D\to A} = (E_g^A - E_g^D) - \Delta(p_e^{1'} + p_e^{3'} + p_e^{5'} + ... + p_e^{N'}) \tag{3.53}$$

The negative work done in this step shows that the system provides external work. Then the total work done in a cycle is the sum of work in the four process.

$$W_{cyc} = W^{A\to B} + W^{B\to C} + W^{C\to D} + W^{D\to A} \tag{3.54}$$

$$\begin{aligned} W_{cyc} \quad &= \Delta(p_e^1 + p_e^3 + p_e^5 + ... + p_e^N) - \Delta(p_e^1 + p_e^{3'} + p_e^{5'} + ... + p_e^{N'}) + \\ &p_e^D[(E_e^D - E_g^D) - (E_e^C - E_g^C)] + p_e^A[(E_e^B - E_g^B) - (E_e^A - E_g^A)] \end{aligned} \tag{3.55}$$

For $N \to \infty$ $\Delta \to 0$ Eq.(3.55) is simplified to

$$W_{cyc} = p_e^C[(E_e^D - E_g^D) - (E_e^C - E_g^C)] + p_e^B[(E_e^B - E_g^B) - (E_e^A - E_g^A)] \tag{3.56}$$

This equation can be rewritten as

$$W_{cyc} = p_e^B(E_e^B - E_g^B)\left[1 - \frac{E_e^A - E_g^A}{E_e^B - E_g^B}\right] - p_e^C(E_e^C - E_g^C)\left[1 - \frac{E_e^D - E_g^D}{E_e^C - E_g^C}\right] \tag{3.57}$$

The temperature of the system remains the same in the isothermal expansion(compression) process at T_H(T_C). The system satisfies the the Boltzmann distribution.

$$p_n = \frac{e^{-\beta^\alpha E_n}}{Z} \tag{3.58}$$

where α represents either the hot or cold temperature value and Z is the partition function given by

$$Z = \sum_n e^{-\beta^\alpha E_n}. \tag{3.59}$$

The occupation probabilities $p_{e(g)}$ of the working substance are kept unchanged during the adiabatic process ($A \to B$) and ($C \to D$).

Thus, for the eigenstate $|e\rangle$ and $|g\rangle$, the occupation probabilities $P_{e(g)}$ satisfy

$$\frac{p_e^B}{p_g^B} = \frac{p_e^A}{p_g^A} \tag{3.60}$$

This equation can be written as

$$e^{-\beta^H(E_e^B - E_g^B)} = e^{-\beta^L(E_e^A - E_g^A)} \tag{3.61}$$

which relates the temperature ratios with the energy at B and A as

$$\frac{T_C}{T_H} = \frac{E_e{}^A - E_g^A}{E_e{}^B - E_g^B} \tag{3.62}$$

where T_C and T_H are the cold and hot temperature value and Eq(3.57) can be written as

$$W_{cyc} = \left[1 - \frac{E_e^A - E_g^A}{E_e^B - E_g^B}\right]\left[p_e^B(E_e^B - E_g^B) - p_e^B(E_e^C - E_g^C)\right]. \tag{3.63}$$

The term in the second bracket is simply the negative of Eq(3.22) that is the heat absorbed during isothermal expansion step of our system. Then Eq(3.63) can be written as

$$W_{cyc} = \left[1 - \frac{E_e^A - E_g^A}{E_e^B - E_g^B}\right][-Q_{In}]. \tag{3.64}$$

This equation can be rewritten as

$$\frac{W_{cyc}}{[-Q_{In}]} = \left[1 - \frac{E_e^A - E_g^A}{E_e^B - E_g^B}\right]. \tag{3.65}$$

We know that the ratio of the work done to the heat absorbed is the efficiency of the engine.

$$\eta = \frac{W_{cyc}}{[-Q_{In}]} \tag{3.66}$$

From the above equation the efficiency of our engine will be

$$\eta^* = 1 - \frac{E_e^A - E_g^A}{E_e^B - E_g^B}. \tag{3.67}$$

And we have shown before that the ratio of the energy difference at the low to high temperature level equals to the ratio of cold to hot temperature value. So we have arrived to the Carnot efficiency when we divide the isothermal expansion and compression in to infinite number of infinitesimal adiabatic and isochoric expansion and compression.

$$\eta^* = 1 - \frac{T_C}{T_H} \tag{3.68}$$

i.e $\eta^* = \eta_c$

3.3 Processes Beyond Quasi Static Limit

The quasi-static limit allows for reversibility but at the cost of making the power out put of heat engines tend to zero. To obtain a finite power output the cycle must take a finite time. And finite-time thermodynamics extends thermodynamic analysis to include finite-time constraints, in our case changing the magnetic field with finite value[14]. We use the same concept in the previous section where $N = 3$

We also assume the system is to be equilibrium at point A initially and also gets to equilibrium after consecutive isochoric expansions(compression). Using extended concept of the quasi static limit we can determine the heat absorbed(released) and work done in every step. In the first step of subdivided isothermal expansion process that is when the system expands isochorically to point 1 and then isothermally to point 2 which becomes equilibrium will absorb ΔQ^1 amount of heat given by

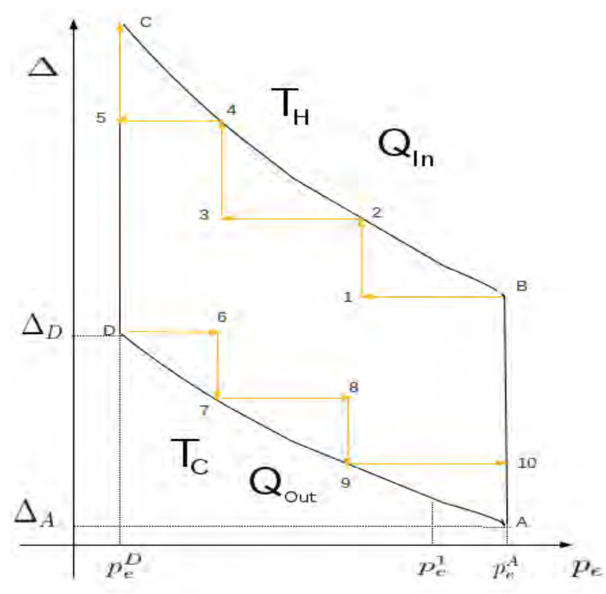

Figure 3.3: Schematic illustration of a quantum Carnot cycle for finite time division of the isothermal expansion and compression

$$\Delta Q^1 = \sum_n E_n^1 \Delta p_n^1 \tag{3.69}$$

$$\Delta Q^1 = (E_e^B - E_g^B)\Delta p_e^1 \tag{3.70}$$

In other words, in the first step it will absorb an amount of heat

$$\Delta Q^1 = (E_e^B - E_g^B)(p_e^1 - p_e^B) \tag{3.71}$$

Similarly in the second step

$$\Delta Q^2 = (E_e^2 - E_g^2)(p_e^3 - p_e^2), \tag{3.72}$$

And at the last step

$$\Delta Q^3 = (E_e^4 - E_g^4)(p_e^5 - p_e^4). \tag{3.73}$$

Then the total heat absorbed will be

$$Q_{In} = \Delta Q^1 + \Delta Q^2 + \Delta Q^3 \tag{3.74}$$

$$Q_{In} = \Delta_B(p_e^1 - p_e^B) + (\Delta_B + \Delta)(p_e^3 - p_e^2) + (\Delta_B + 2\Delta)(p_e^5 - p_e^4) \tag{3.75}$$

Taking the occupation probabilities $p_e^1 = p_e^2$, $p_e^3 = p_e^4$ and $p_e^5 = p_e^C$

Q_{In} can be written as

$$Q_{In} = (E_e^C - E_e^C)p_e^C - (E_e^B - E_e^B)p_e^B - \Delta(p_e^2 + p_e^4 + p_e^C). \tag{3.76}$$

In a similar fashion we can calculate the heat released during isothermal compression by subdividing it into three adiabatic compressions and three isochoric processes. The heat released in the first step as it move from point D to 6 is given by

$$\Delta Q^{1'} = (E_e^D - E_g^D)(p_e^6 - p_e^D). \tag{3.77}$$

In the second step (as it moves from point 7 to 8) the amount of heat release will be

$$\Delta Q^{2'} = (E_e^7 - E_g^7)(p_e^8 - p_e^7). \tag{3.78}$$

In the last step(as it moves from 9 to 10) it will release

$$\Delta Q^{3'} = (E_e^9 - E_g^9)(p_e^{10} - p_e^9). \tag{3.79}$$

The total heat released during the isothermal compression is

$$Q_{Out} = \Delta Q^{1'} + \Delta Q^{2'} + \Delta Q^{3'} \tag{3.80}$$

$$Q_{Out} = \Delta_D(p_e^6 - p_e^D) + (\Delta_D - \Delta)(p_e^8 - p_e^7) + (\Delta_D - 2\Delta)(p_e^{10} - p_e^9) \tag{3.81}$$

Noting that $p_e^6 = p_e^7$, $p_e^8 = p_e^9$ and $p_e^A = p_e^{10}$ equation 3.78 can be rewritten as

$$Q_{Out} = (E_e^A - E_e^A)p_e^A - (E_e^D - E_e^D)p_e^D + \Delta(p_e^7 + p_e^9 + p_e^A) \tag{3.82}$$

Then the work done in one cycle will be

$$W_{cyc} = Q_{In} + Q_{Out} \tag{3.83}$$

Following a similar approach as before

$$W_{cyc} = \left[1 - \frac{E_e^A - E_g^A}{E_e^B - E_g^B}\right][p_e^C(E_e^C - E_g^C) - p_e^B(E_e^B - E_g^B)] + \Delta[(p_e^B + p_e^9 + p_e^7) - (p_e^C + p_e^4 + p_e^2)] \tag{3.84}$$

And the efficiency will be

$$\eta_{ft} = \frac{W_{cyc}}{Q_{In}} \tag{3.85}$$

Substituting the value of work done and heat absorbed we can determine the efficiency of our modeled engine with in a finite time taken.

$$\eta_{ft} = \frac{\left[1 - \frac{E_e^A - E_g^A}{E_e^B - E_g^B}\right][p_e^C(E_e^C - E_g^C) - p_e^B(E_e^B - E_g^B)] + \Delta[(p_e^B + p_e^9 + p_e^7) - (p_e^C + p_e^4 + p_e^2)]}{p_e^C(E_e^C - E_g^C) - p_e^B(E_e^B - E_g^B) - \Delta(p_e^C + p_e^4 + p_e^2)} \tag{3.86}$$

This equation can be rewritten as

$$\eta_{ft} = \frac{\eta_c - x + y}{1 + y} \tag{3.87}$$

where

$$x = \frac{\Delta[p_e^B + p_e^9 + p_e^7]}{[p_e^B(E_e^B - E_g^B) - p_e^C(E_e^C - E_g^C)]} \tag{3.88}$$

$$y = \frac{\Delta[p_e^C + p_e^4 + p_e^2]}{[p_e^B(E_e^B - E_g^B) - p_e^C(E_e^C - E_g^C)]} \tag{3.89}$$

Equation 3.86 shows that efficiency at finite time η_{ft}, with N = 3 is smaller than the value of Carnot efficiency.

3.4 Efficiency at maximum power out put

Here our goal is to maximize the power output and to evaluate the corresponding efficiency. We consider the work done and the respective heat transfer for N number of subdivisions of the isothermal process. We assume the adiabatic process is completely reversible(no heat exchange with the surrounding) where as in the isothermal process heat is transfered and work is done as we have seen before. From section 3.2 Eq(3.55) for N subdivisions of the isothermal expansion(compression)the work done can be written as

$$\begin{aligned} W_{cyc} = \quad & \eta_c\Big[p_e^B(E_e^B - E_g^B) - p_e^C(E_e^C - E_g^C)\Big] + \Delta\Big[(p_e^1 + p_e^3 + p_e^5 + ... \\ & +p_e^N) - (p_e^{1'} + p_e^{3'} + p_e^{5'} + ...p_e^{N'})\Big]. \end{aligned} \tag{3.90}$$

We can write this equation as

$$W_{cyc} = \eta_c[\Delta_B p_e^B - \Delta_C p_e^C] + \Delta[p_e^C - p_e^B]. \tag{3.91}$$

And the heat absorbed as it expands isothermally will be

$$Q_{In} = \Delta_C p_e^C - \Delta_B p_e^B - \Delta(p_e^1 + p_e^3 + p_e^5 + ... + p_e^C). \tag{3.92}$$

The power out put becomes

$$P = \frac{W}{t} \tag{3.93}$$

Because the time taken for adiabatic expansion and compression process is much smaller as compared to isothermal process we can approximate the time taken for one complete cycle by

$$t = N + N'. \tag{3.94}$$

where N and N' are the number of subdivisions for the isothermal expansion and compression and equal to the time taken. Since the number of subdivisions are equal the time taken for isothermal expansion and compression are equal. So that $t = 2N$.

And the energy spacing Δ can be written in terms of the time taken for one complete cycle and the energies at point B and C as

$$\Delta = \frac{2(\Delta_C - \Delta_B)}{t} \tag{3.95}$$

Using equation 3.91 the power out put will be

$$P = \frac{\eta_c[\Delta_B p_e^B - \Delta_C p_e^C] + \Delta[p_e^C - p_e^B]}{t} \tag{3.96}$$

Substituting Eq 3.95 into Eq 3.96

$$p = \frac{\eta_c[p_e^B \Delta_B - p_e^C \Delta_C]}{t} + \frac{2[\Delta_C - \Delta_B][p_e^C - p_e^B]}{t^2} \tag{3.97}$$

Maximizing the power with respect to the time t i.e

$\frac{\partial p}{\partial t} = 0$

$$\frac{\partial p}{\partial t} = -\frac{\eta_c[p_e^B \Delta_B - p_e^C \Delta_C]}{t^2} - \frac{4[\Delta_C - \Delta_B][p_e^C - p_e^B]}{t^3} \tag{3.98}$$

$$\eta_c t[p_e^B \Delta_B - p_e^C \Delta_C] = 4[\Delta_B - \Delta_C][p_e^C - p_e^B] \tag{3.99}$$

Simplifying this equation, the time taken for one complete cycle where the power out put becomes maximum will be

$$t^* = \frac{4[\Delta_B - \Delta_C][p_e^C - p_e^B]}{\eta_c[p_e^B \Delta_B - p_e^C \Delta_C]} \tag{3.100}$$

And the the maximum power output

$$P_{max} = \frac{W}{t^*} \tag{3.101}$$

Substituting the work done per cycle and the time at which the power is maximum

$$P_{max} = \frac{\eta_c[p_e^B \Delta_B - p_e^C \Delta_C]}{4\frac{[\Delta_B - \Delta_C][p_e^C - p_e^B]}{\eta_c[p_e^B \Delta_B - p_e^C \Delta_C]}} + \frac{2[\Delta_C - \Delta_B][p_e^C - p_e^B]}{\left[\frac{4[\Delta_B - \Delta_C][p_e^C - p_e^B]}{\eta_c[p_e^B \Delta_B - p_e^C \Delta_C]}\right]^2} \tag{3.102}$$

$$P_{max} = \frac{\eta_c^2[p_e^B\Delta_B - p_e^C\Delta_C]^2}{4[\Delta_B - \Delta_C][p_e^C - p_e^B]} - \frac{\eta_c^2[p_e^B\Delta_B - p_e^C\Delta_C]^2}{8[\Delta_B - \Delta_C][p_e^C - p_e^B]} \quad (3.103)$$

Simplifying this equation the maximum power out put can be written as

$$P_{max} = \frac{\eta_c^2[p_e^B\Delta_B - p_e^C\Delta_C]^2}{8[\Delta_B - \Delta_C][p_e^C - p_e^B]} \quad (3.104)$$

The corresponding efficiency at maximum power is

$$\eta_{mp} = \frac{\dot{W}(t^*)}{\dot{Q}(t^*)} \quad (3.105)$$

Note that

$$\frac{W_{cyc}}{t^*} = P_{max}(t^*) \quad (3.106)$$

And the heat absorbed per time t^*, $\dot{Q}(t^*)$ is

$$[-\dot{Q}(t^*)] = \frac{[p_e^B\Delta_B - p_e^C\Delta_C]}{\frac{4[\Delta_B-\Delta_C][p_e^C-p_e^B]}{\eta_c[p_e^B\Delta_B-p_e^C\Delta_C]}} + \frac{2[\Delta_C - \Delta_B][p_e^1 + p_e^3 + ... + p_e^C]}{\left[\frac{4[\Delta_B-\Delta_C][p_e^C-p_e^B]}{\eta_c[p_e^B\Delta_B-p_e^C\Delta_C]}\right]^2} \quad (3.107)$$

$$[-\dot{Q}(t^*)] = \frac{\eta_c[p_e^B\Delta_B - p_e^C\Delta_C]^2}{4[\Delta_B - \Delta_C][p_e^C - p_e^B]} - \frac{\eta_c^2[p_e^B\Delta_B - p_e^C\Delta_C]^2[p_e^1 + p_e^3 + ... + p_e^C]}{8[\Delta_B - \Delta_C][p_e^C - p_e^B]^2} \quad (3.108)$$

Dividing equation 3.104 by equation 3.108 we will get

$$\eta_{mp} = \frac{\frac{\eta_c}{2}}{\left[1 - \frac{\eta_c(p_e^1+p_e^3+...+p_e^C)}{2(p_e^C-p_e^B)}\right]} \quad (3.109)$$

For a very small Carnot efficiency that means when T_H closes to T_c, $\eta_c = 1 - \frac{T_C}{T_H}$ approaches to zero.

Then the above equation can be approximated by using Taylor series as follows

$$\eta_{mp} = \frac{\eta_c}{2}[1 + \eta_c x + \frac{\eta_c^2 x^2}{2} + \frac{\eta_c^3 x^3}{6} + ... + O] \tag{3.110}$$

Where $x = \frac{[p_e^1 + p_e^3 + ... + p_e^C]}{2[p_e^C - p_e^B]}$.

We have arrived at the Curzon and Ahlborn efficiency at maximum power when the efficiency of our modeled heat engine is approximated up to the first order of Carnot efficiency. i.e $\eta_{mp} = \frac{\eta_c}{2}$

Chapter 4

Summary and Conclusion

In this work, we consider a single spin-1/2 system with a two level energy as a working substance subjected to a magnetic field which can be increased or decreased. The system interact with a heat bath universally modeled as many Bosons.

By defining the quantum version of basic thermodynamic processes, we study the simple quantum heat engine which consists of two quantum adiabatic and two quantum isothermal processes. We divide the two isothermal processes in to a number of steps by altering the magnetic field in such a way that in the quantum adiabatic process all energy gaps of the working substance is changed by the same ratio and the system gets equilibrium after every consecutive quantum adiabatic processes.

Considering the mean energy of the working substance we determine the work done and the corresponding heat transfered during one complete cycle. In the assumption when the system reaches thermal equilibriums, the occupation probabilities obeys the Gibbs distribution and the occupation probabilities p_n of the working substance are kept unchanged during the adiabatic process we gets the change of the ratio of the energy gaps is equal the ratio of the temperatures of the two heat baths.

We determine the efficiency of engine in the quasi static limit and found that is equal to the Carnot efficiency. This is the case where the time taken is infinite or the power out put is zero in which no real engine operates.

We also calculate the finite time performance of the engine both for the finite power out put and maximum power. Our calculation is in agreement with the known universality properties. In particular, the efficiency at maximum power out put is equal to half of the Carnot efficiency ($\eta_{mp} = \frac{\eta_c}{2}$).

Bibliography

[1] Harvay S.Leff *Thermal efficiency at maximum work output new results for old heat engines*, American Journal of Physics 55,602(1987)

[2] Y. Izumida and K. Okuda, *Molecular kinetic analysis of a finite-time Carnot cycle*, arXiv:0802.3759v3[cond-mat.stat-mech](2008)

[3] H. Hoffmann *An Introduction To Endoreversible Thermodynamics*, Institute of Physics, Chemnitz, Germany (2008)

[4] Massimiliano Esposito,Ryoichi Kawai,Katja Lindenberg,Christian Van den Broeck,*Quantum-dot Carnot engine at maximum power*, arXiv:1001.2192v2[cond-mat.stat-mech](2010)

[5] F. Angulo-Brown, N. Snchez-Salas and G. Ares de Parga, *The Curzon-Ahlborn efficiency for three different energy converters*, Lat. Am. J. Phys. Educ. 4, 1(2010)

[6] Mahmoud Huleihil,Bjarne Andresen, *Convective heat transfer law for an endoreversible engine*, Journal of applied physics 100, 014911 (2006)

[7] Yang,Wang and Z. C. Tu, *Efficiency at maximum power output of Carnot-like heat engines*, arXiv:1108.2873v2[cond-mat.stat-mech](2011)

[8] H. T. Quan, Yu-xi Liu, C. P. Sun, and Franco Nori1 *Quantum Thermodynamic Cycles and Quantum Heat Engines*, arXiv:quant-ph/0611275v2(2007)

[9] F.Reif, *Fundamental of statistical and Thermal Physics*, Education development center, united state of America (1965)

[10] H. T. Quan, S.Yang and C.P Sun, *Microscopic Work distribution of small system in Quantum Isothermal Process* arXiv:0804.1312v2[cond-math.stat-mech](2008)

[11] Sumiyoshi Abe and Shinji Okuyama, *Similarity between quantum mechanics and thermodynamics: Entropy, temperature and Carnot cycle*, Phys.Rev.E.83, 021121(2011)

[12] J. Arnaud, L. Chusseau, and F. Philippe, *A simple quantum heat engine*, arXiv:quant-ph/0211072v2(2003)

[13] J. Gemmer M. Michel G. Mahler, *Quantum Thermodynamics*, Springer(2009)

[14] Herbert B. Callen, *Thermodynamics and an introduction to Thermostatics*, John Wiley(1985)

[15] Carl M. Bender, Dorje C. Brody, and Bernhard K. Meister,arXiv;quant-phy/0007002v1(2000)

[16] B. Andresen, R. Stephen Berry A.Nitzan,and P. Salamon, *Thermodynamics in finite time. I. the step Carnot cycle*, Phys.Rev. A. 15, 5(1977)

[17] Jacques Arnaud, Laurent Chusseau, Fabrice Philippe,*A simple model for Carnote heat engines*, arXiv: 0812.2757v2[cond-mat.stat-mech] (2009)

[18] Jan Birjukov, Thomas Jahnke , and G nter Mahler, *Quantum thermodynamic processes: A control theory formachine cycles*, arXiv:0712.0534v2[cond-mat.stat-mech](2008)

[19] M. Esposito, R. Kawai, K. Lindenberg,Van den Broeck, *Efficiency at maximum power of low dissipation Carnot engines*, arXiv:1008.2464v2[cond-math. stat-mech](2010)

DECLARATION

I hereby declare that this thesis is my original work and has not been presented for a degree in any other university. All sources of material used for the thesis have been duly acknowledged.

Name: **Warga Chegeno**

Signature:− − − − − − − − − − − −

Place and time of submission: Addis Ababa University, November 2011

This thesis has been submitted for examination with my approval as University advisor.

Name: **Dr. Mulugeta Bekele**

Signature:− − − − − − − − − − −