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Multiplicity of Positive Solutions of a Class of Ordinary Differential Equations with Nonlinear Boundary Conditions

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Abstract

In this paper we study the multiplicity solution of the boundary value problem

$$u^{''} + a(t)u^{'} + f(u) = 0;\ B_1(u(0)) - u^{'}(0) = 0,\ B_2(u(b)) + u^{'}(b) = 0,$$

where $0 < t < b < \infty$, B_1 and $B_2 \in C^1[0,\infty), a \in C[0,\infty)$ with $a \leq 0$ on $[0,\infty)$ and $f \in C[0,\infty) \cap C^1(0,\infty)$ satisfy suitable conditions.

Notations

$\mathbb{R}$	The set of all real numbers.
$\mathbb{N}$	The set of all natural numbers.
$\mathbb{R}^n$	n-dimensional Euclidean spaces.
$\|t\|$	Euclidean norm of $t = (t_1, t_2, ..., t_n) \in \mathbb{R}^n$, i.e. $\|t\| = (\sum_{i=1}^{n} t_i^2)^{1/2}$.
$\bar{B}(y_0, a)$	Closed ball of radius a about the point y_0.
$C[0, \infty)$	The set of continuous function on $[0, \infty)$.
$C^1[0, \infty)$	The set of 1-times continuously differentiable function on $[0, \infty)$.

Chapter 1

Introduction

We cannot say that all problems in mathematics have a solution. And for those which have solutions it is not such an easy task to know weather they have another solution or not unless some specific criterion is in place. Indeed, the two dominating problems in mathematics are existence and uniqueness, in general, of a solution to a problem subject to some specific conditions.

A differential equation is an equation consisting of independent variable(s), dependent variable(s) (or an unknown function(s)) and its derivatives. And depending on the number of independent variables that occur in the equation or equivalently depending on the type of derivatives that is (are) involved in the equation, differential equations can broadly be classified in two groups as ordinary differential equation [ODE] and partial differential equation [PDE] whereby the former ones are those consisting of only one independent variable or those involving ordinary derivatives and PDEs are those consisting of more than one independent variables or involving partial derivatives. In ordinary differential equations (ODE) the two typical classes of problems are boundary value problems (BVP) and initial value problems (IVP).[2]

Initial value problem of second order explicit equation is a problem of finding a solution $y(x)$ of

$$y^{''} = f(x, y, y^{'})$$

subject to the conditions

$$\begin{aligned} y(a) &= \alpha, \\ y^{'}(a) &= \beta. \end{aligned}$$

A boundary value problem of linear second order equation is a problem of finding a solution to a differential equation, for instance,

$$y^{''} + a_1(x)y^{'} + a_0(x)y = f(x), \qquad x \in (a, b)$$

subject to the boundary condition

$$\begin{aligned} B_1: \quad & \alpha_1 y(a) + \alpha_2 y^{'}(a) = \eta_1, \\ B_2: \quad & \beta_1 y(b) + \beta_2 y^{'}(b) = \eta_2. \end{aligned}$$

The subject of this paper is not to discuss ordinary differential equations as a whole, rather to prove the uniqueness(multiplicity) of positive solutions of a class of ordinary differential equations with nonlinear boundary conditions.

More precisely, we consider the uniqueness(multiplicity) of positive solutions of boundary value problem

$$u^{''} + a(t)u^{'} + f(u) = 0 \tag{1.1}$$

$$B_1(u(0)) - u^{'}(0) = 0 \tag{1.2}$$

$$B_2(u(b)) + u^{'}(b) = 0$$

where $0 < t < b < \infty$. To this end, the following three conditions are supposed to be true:

(C1) $f \in C[0, \infty) \cap C^1(0, \infty)$ with $f(0) = 0$,

$$f(u) > 0, \quad uf^{'}(u) < f(u), \text{ for } u > 0; \tag{1.3}$$

(C2) $a \in C[0, \infty)$ with $a(t) \leq 0$ for $t \geq 0$;

(C3) $B_i \in C^1[0, \infty)$ satisfies
$B_i(0) = 0, B_i(x) > 0$ for $x > 0$,
$B_i^{'}(x)$ is nondecreasing on $(0, \infty)(i = 1, 2)$.

Indeed, it is shown that if (C1)-(C3) hold, then problem (1.1), (1.2) has at most one positive solution.

Here, we say $u(t)$ is a ***positive solution*** of (1.1), (1.2), if $u(t) > 0$ on $[0, b]$ and satisfies the differential equation (1.1) as well as the boundary conditions (1.2).[7]

The proof of uniqueness(multiplicity) is based on the shooting method. The rest of the paper is organized as follows. In chapter 2, we will define basic definitions, state and prove some important theorems. In chapter 3, we will try to say very little about the shooting method and properties of solutions of IVPs. The proof of uniqueness of positive solution of BVPs will be given in chapter 4.

Chapter 2

Basic Concepts And Auxiliary Notions

In this part we recall some concepts from analysis and state pertinent theorems, so as to furnish the ground for the main work.

2.1 Definitions

Definition 2.1.1. *[6] The set $S \subseteq \mathbb{R}^m$, for $m \in \mathbb{N}$ is compact if for each bounded sequence $\{a_n\}$ there is a convergent subsequence $\{a_{n_m}\}$ of $\{a_n\}$ which converges in S.*

Definition 2.1.2. *[3] Suppose $T \subseteq \mathbb{R}^n$. A function $f : T \to \mathbb{R}^n$ satisfies a Lipschitz condition in T if there is a constant L such that for any $x, y \in T, \|f(x) - f(y)\| \leq L\|x - y\|$.*

Definition 2.1.3. *[3] Suppose $S \subseteq \mathbb{R} \times \mathbb{R}^n$. A function $f : S \to \mathbb{R}^n$ which satisfies a Lipschitz condition in x for each fixed y, with a single Lipschitz constant L independent of y, is said to satisfy a Lipschitz condition with respect to x in S.*

Definition 2.1.4. *[3] Suppose $M \subseteq \mathbb{R} \times \mathbb{R}^n$ is open. A function $f : M \to \mathbb{R}^n$ which satisfies a Lipschitz condition in x on each compact subset K of M is said to satisfy the local Lipschitz condition with respect to x in M.*

2.2 Existence and Uniqueness Theorems for IVPs

The theorem which we state and prove in this section are very important in understanding the whole work of this paper.

Theorem 2.2.1. *[4](uniqueness Theorem for First order IVP)*
Let $I \subset \mathbb{R}$ be an open interval, $U \subset \mathbb{R}^n$ be an open set and $(x_0, y_0) \in I \times U$ be given and $a_1, a_2 > 0$ such that

$$[x_0 - a_1, x_0 + a_1] \subset I \quad \textit{and} \quad \bar{B}(y_0, a_2) \subset U.$$

Define $K = [x_0 - a_1, x_0 + a_1] \times \bar{B}(y_0, a_2)$.
If $f : I \times U \to \mathbb{R}^n$ satisfies locally Lipschitz condition and

$$M := \max_{(x,y)\in K} \|f(x, y)\|, \qquad \alpha := \min\{a_1, \frac{a_2}{M}\},$$

then the initial value problem

$$y'(x) = f(x, y) \tag{2.1}$$
$$y(x_0) = y_0$$

has a unique solution on $[x_0 - \alpha, x_0 + \alpha]$.

Proof. ***i) existence***
Define $K = [x_0 - a_1, x_0 + a_1] \times \bar{B}(y_0, a_2) \leftarrow$ is compact.
Since $f : I \times U \to \mathbb{R}^n$ is locally Lipschitzian, it follows that it is continuous on K.
Let

$$M := \max_{(x,y)\in K} \|f(x, y)\|.$$

Since f is continuous on the compact K, existence of M makes sense.

Now, we will show that $\alpha := \min\{a_1, \frac{a_2}{M}\}$ satisfies the IVP. To this end, we employ Picard Method of Successive Approximation.
Evidently (2.1) is equivalent to the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))ds. \tag{2.2}$$

If we set up the successive approximation, as follows

$$\begin{aligned} y_0(x) &= y_0 \\ y_{n+1}(x) &= y_0 + \int_{x_0}^{x} f(s, y_n(s))ds, \quad n = 0, 1, 2, ... \end{aligned} \tag{2.3}$$

We can then verify (by induction) that the sequence $\{y_n(x)\}$ are well-defined for $|x - x_0| \le \alpha$, and that $(x, y_n(x)) \in K,\ |x - x_0| \le \alpha$. In fact,

$$\|y_1(x) - y_0\| \le \int_{x_0}^{x} \|f(s, y_0)\|ds \le M|x - x_0| \le a_2,$$

$$\|y_n(x) - y_0\| \le \int_{x_0}^{x} \|f(s, y_{n-1}(s))\|ds \le M|x - x_0| \le a_2, \quad n = 2, 3, ...,$$

if $|x - x_0| \le \frac{a_2}{M}$.
One can also show that $\{y_n(x)\}$ is a Cauchy sequence for $|x - x_0| \le \alpha$.

$$\begin{aligned} \|y_{n+1}(x) - y_n(x)\| &\le \int_{x_0}^{x} \|f(s, y_n(s)) - f(s, y_{n-1}(s))\|ds \\ &\le L \int_{x_0}^{x} \|y_n(s) - y_{n-1}(s)\|ds, \quad n = 1, 2, ... \end{aligned}$$

and

$$\|y_1(x) - y_0(x)\| \le \int_{x_0}^{x} \|f(s, y_0)\|ds \le M|x - x_0|, \quad |x - x_0| \le \alpha.$$

Thus

$$\|y_2(x) - y_1(x)\| \le ML|x - x_0|^2/2!, \quad |x - x_0| \le \alpha,$$

and

$$\|y_3(x) - y_2(x)\| \le ML^2|x - x_0|^3/3!, \quad |x - x_0| \le \alpha,$$

and, for n=1,2,...

$$\|y_{n+1}(x) - y_n(x)\| \leq \frac{ML^n|x-x_0|^{n+1}}{(n+1)!} \leq \frac{M}{L}\frac{(L\alpha)^{n+1}}{(n+1)!}.$$

Thus, for $0 \leq m \leq n$, we have

$$\|y_n(x) - y_m(x)\| \leq \frac{M}{L}\sum_{k=m}^{n-1}\frac{(L\alpha)^{k+1}}{(k+1)!}, \quad |x-x_0| \leq \alpha. \tag{2.4}$$

Consequently the solution to the IVP is obtained as the uniform limit of $\{y_n(x)\}$. Cauchy's criterion and (2.4) imply that

$$\lim_{n\to\infty} y_n(x) = y(x)$$

uniformly for $x \in [x_0-\alpha, x_0+\alpha]$ and $y(x)$ is continuous. In the equation (2.3), let $n \to \infty$, we verify that $y(x)$ is a solution to the integral equation (2.2). Thus $y(x)$ is differentiable and satisfies the IVP.

ii) uniqueness
Recall that y is a solution of the initial value problem (2.1) implies that

$$y(x) = y(x_0) + \int_{x_0}^{x} f(s,y(s))ds, \qquad \forall x \in J := [x_0-a_1, x_0+a_1]$$

Let y_1, y_2 be two solutions to the IVP(2.1).
We want to show that $y_1 = y_2$ on J.

$$\begin{aligned} y_1(x) - y_2(x) &= y_1(x_0) - y_2(x_0) + \int_{x_0}^{x} f(s,y_1(s))ds - \int_{x_0}^{x} f(s,y_2(s))ds \\ &= \int_{x_0}^{x}[f(s,y_1(s)) - f(s,y_2(s))]ds, \quad \forall x \in J \end{aligned}$$

this implies

$$\begin{aligned} \|y_1(x) - y_2(x)\| &\leq \int_{x_0}^{x} \|f(s,y_1(s)) - f(s,y_2(s))\|ds \\ &\leq \int_{x_0}^{x} L\|y_1(s) - y_2(s)\|ds, \end{aligned}$$

since f satisfies locally Lipschitz condition.
This implies

$$\underbrace{\|y_1(x) - y_2(x)\|}_{:=g(x)} \leq L\int_{x_0}^{x} \underbrace{\|y_1(s) - y_2(s)\|}_{:=g(s)} ds$$

$$\Rightarrow g(x) \leq \underbrace{L\int_{x_0}^{x} g(s)ds}_{:=h(x)}$$

Set $\gamma(x) := \exp\big(-L(x-x_0)\big), x > x_0$

$$h(x) = L\int_{x_0}^{x} g(s)ds \Rightarrow h'(x) = Lg(x) \leq Lh(x)$$

$$\begin{aligned}
&\Rightarrow \gamma(x)h'(x) \leq \underbrace{L\gamma(x)}_{-\gamma'(x)} h(x) \\
&\Rightarrow \gamma(x)h'(x) \leq -\gamma^{'}(x)h(x) \\
&\Rightarrow \gamma(x)h'(x) + \gamma^{'}(x)h(x) \leq 0 \\
&\Rightarrow (\gamma(x)h(x))^{'} \leq 0 \\
&\Rightarrow (\gamma h)(x) \text{ is decreasing} \\
&\Rightarrow \gamma(x)h(x) \leq \underbrace{\gamma(x_0)}_{=1}\underbrace{h(x_0)}_{=0} = 0 \\
&\Rightarrow \gamma(x)h(x) \leq 0 \\
&\Rightarrow h(x) \leq 0, \text{ since } \gamma(x) > 0 \ \forall x.
\end{aligned}$$

It follows that

$$g(x) \leq 0$$

$$\begin{aligned}
&\Rightarrow \|y_1(x) - y_2(x)\| \leq 0 \\
&\Rightarrow \|y_1(x) - y_2(x)\| = 0 \text{ as } g(x) \geq 0, \ \forall x \\
&\Rightarrow y_1 = y_2.
\end{aligned}$$

□

Theorem 2.2.2. *[4](Uniqueness Theorem for Higher order IVP)*
Let $I \subset \mathbb{R}$ be an open interval, $U \subset \mathbb{R}^n$ be open set. $(x_0, y_0) \in I \times U$ be given, where $y_0 = (\omega_0, \omega_1, \omega_2, ..., \omega_{n-1}) \in \mathbb{R}^n$, and $a_1, a_2 > 0$ be constants such that $[x_0 - a_1, x_0 + a_1] \subset I$ and $\bar{B}(y_0, a_2) \subset U$.
If $f : I \times U \to \mathbb{R}^n$ satisfies the local Lipschitz condition, then the initial value problem

$$y^{(n)} = f(x, y, y^{'}, y^{''}, ..., y^{(n-1)}) \tag{2.5}$$

$$y(x_0) = \omega_0 \tag{2.6}$$

$$y^{'}(x_0) = \omega_1$$

$$y^{''}(x_0) = \omega_2$$

$$\vdots$$

$$y^{(n-1)}(x_0) = \omega_{n-1}$$

has a unique solution on $[x_0 - \alpha, x_0 + \alpha]$.

Proof. First we transform equation (2.5) in to a system of n first order differential equations for n functions $y_1(x), y_1(x), y_2(x), ..., y_n(x)$; such that

$$\begin{aligned} y_1^{'} &= y_2 \\ y_2^{'} &= y_3 \\ &\vdots \\ y_{n-1}^{'} &= y_n \\ y_n^{'} &= f(x, y_1, y_2, ..., y_n). \end{aligned} \tag{2.7}$$

Equation (2.5) and the system of equation (2.7) are equivalent in the following sense: if $y(x)$ is a solution of (2.5) then the vector function

$$y = (y_1, y_2, ..., y_n) := (y, y^{'}, y^{''}, ..., y^{(n-1)})$$

is a solution of (2.7). Conversely, if y is a (differentiable) solution of (2.7) and one sets $y_1(x) := y(x)$ then $y(x)$ is an n times differentiable,

$$y_2(x) = y^{'}(x),\ y_3(x) = y^{''}(x),\ \dots\ ,\ y_n(x) = y^{(n-1)}(x),$$

and equation (2.5) holds. And once after we have changed the n^{th} order differential equation to a first order differential equation, we can apply **thm (2.2.1)** to our case. And with this we complete the proof. □

2.3 The Implicit Function Theorem

Theorem 2.3.1. *[5](Implicit Function Theorem)*
let $\mathcal{O}$ be an open subset of the plane $\mathbb{R}^2$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable. Let (x_0, y_0) be a point in $\mathcal{O}$ at which $f(x_0, y_0) = 0$ and consider the expression $f(x, y) = 0$.
If $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$, then exists a C^1 function $y = g(x)$ defined on an open interval I about the point x_0 such that

i) $f(x, g(x)) = 0$ for all x in I

ii) $g(x_0) = y_0$

iii) $g^{'}(x) = -\frac{\frac{\partial f}{\partial x}(x,y(x))}{\frac{\partial f}{\partial y}(x,y(x))}$ for all x in I

Proof. We assume that $\frac{\partial f}{\partial y}(x_0, y_0) > 0$. Since $\mathcal{O}$ is open and the function $\frac{\partial f}{\partial y} : \mathcal{O} \to \mathbb{R}$ is continuous and positive at the point (x_0, y_0), we can choose positive numbers a and c such that the closed square $R = [x_0 - a, x_0 + a] \times [y_0 - a, y_0 + a]$ is contained in $\mathcal{O}$ and

$$\frac{\partial f}{\partial y}(x, y) \geq c \qquad \text{for all points } (x, y) \text{ in } R. \tag{2.8}$$

It follows from the Mean Value Theorem for scalar functions of a single real variable that

$$f(x, y_1) < f(x, y_2) \quad \text{if } |x - x_0| \leq a \ \text{ and } \ y_0 - a \leq y_1 < y_2 \leq y_0 + a. \tag{2.9}$$

In particular, since $f(x_0, y_0) = 0$, it follows that $f(x_0, y_0 - a) < 0 < f(x_0, y_0 + a)$. Moreover, the function $f : \mathcal{O} \to \mathbb{R}$ is continuous since it is continuously differentiable. Thus, we can choose a positive number r less than a such that, if we let $I = (x_0 - r, x_0 + r)$,

$$f(x, y_0 - a) < 0 < f(x, y_0 + a) \quad \text{for all } x \text{ in } I.$$

Let x be a point in I. Since $f(x, y_0 - a) < 0$ and $f(x, y_0 + a) > 0$, according to the Intermediate Value Theorem, there is some point y between $y_0 - a$ and $y_0 + a$ at which $f(x, y) = 0$, and (2.9) implies that there is only one such point. Define $g(x)$ to be this point. This clearly defines a function $g : I \to \mathbb{R}$ having properties (i) and (ii).

We now show that $g : I \to \mathbb{R}$ is continuously differentiable and that the differentiation formula (iii) holds at the point x_0 . Indeed, let $x_0 + h$ be a point in I. Then, by definition, $f(x_0 + h, g(x_0 + h)) = 0$ and $f(x_0, g(x_0)) = 0$. In particular,

$$0 = f(x_0 + h, g(x_0 + h)) - f(x_0, g(x_0))$$

According to the Mean Value Theorem for scalar functions of two real variables, there is some point on the segment between the points $(x_0, g(x_0))$ and $(x_0 + h, g(x_0 + h))$, which we label $\mathbf{p}(h)$, at which

$$f(x_0 + h, g(x_0 + h)) - f(x_0, g(x_0)) = \frac{\partial f}{\partial x}(\mathbf{p}(h))h + \frac{\partial f}{\partial y}(\mathbf{p}(h))[g(x_0 + h) - g(x_0)].$$

But the left-hand side is 0, and hence,

$$g(x_0 + h) - g(x_0) = -[\frac{\frac{\partial f}{\partial x}\mathbf{p}(h)}{\frac{\partial f}{\partial y}\mathbf{p}(h)}]h. \tag{2.10}$$

Since the function $\frac{\partial f}{\partial x} : \mathcal{O} \to \mathbb{R}$ is continuous and the closed square R is a sequentially compact subset of the plane, by the Extreme Value Theorem we can choose a positive number M such that

$$\left|\frac{\partial f}{\partial x}(x, y)\right| \leq M \quad \text{for all points } (x, y) \in R.$$

Using this inequality, together with inequality (2.8), it follows from formula (2.10) that

$$|g(x_0 + h) - g(x_0)| \leq \frac{M}{c}|h| \quad \text{if } x_0 + h \text{ is in } I.$$

Hence the function $g : I \to \mathbb{R}$ is continuous at the point x_0 . Since the point $\mathbf{p}(h)$ lies on the segment between the points $(x_0, g(x_0))$ and $(x_0 + h, g(x_0 + h))$, we conclude that

$$\lim_{h \to 0} \mathbf{p}(h) = (x_0, y_0).$$

If we now divide (2.10) by h and use the continuity of the first-order partial derivatives of $f : \mathcal{O} \to \mathbb{R}$ at the point (x_0, y_0), it follows from (2.10) that

$$\lim_{h \to 0} \frac{g(x_0 + h) - g(x_0)}{h} = -\frac{\frac{\partial f}{\partial x}(x_0, y_0)}{\frac{\partial f}{\partial y}(x_0, y_0)},$$

which means that g is differentiable at x_0 and formula (iii) holds at x_0. But any other point x in the interval I satisfies the same assumptions as does the point x_0 , and hence (iii) holds at all points in I. $\square$

Note:- The assumption in *Implicit Function Theorem* that $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$ can be replaced by the assumption that $\frac{\partial f}{\partial x}(x_0, y_0) \neq 0$ and the conclusion remains the same except that the roles of x and y are interchanged.

Example 2.2.2. Consider the equation

$$\exp\left(x - 2 + (y-1)^2\right) - 1 = 0, \quad (x, y) \text{ in } \mathbb{R}^2. \tag{2.11}$$

Define

$$f(x, y) = \exp\left(x - 2 + (y-1)^2\right) - 1 \quad \text{for } (x, y) \text{ in } \mathbb{R}^2.$$

Then the point (x, y) is a solution of equation (2.11) if and only if $f(x, y) = 0$. Observe that the point $(2, 1)$ is a solution of equation (2.11) and that

$$\frac{\partial f}{\partial x}(2, 1) = 1 \text{ and } \frac{\partial f}{\partial y}(2, 1) = 0.$$

Implicit Function Theorem, with the roles of the variables x and y interchanged, implies that there is a positive number r and a continuously differentiable function $g : J \to \mathbb{R}$, where J is the open interval $(1 - r, 1 + r)$, such that

$$\exp\left(g(y) - 2 + (y-1)^2\right) - 1 = 0 \quad \text{for all } y \text{ in } J.$$

Moreover, if (x, y) is a solution of equation (2.11) with $|x - 2| < r$ and $|y - 1| < r$, then $x = g(y)$. Finally, $g^{'}(1)$ is determined by the formula

$$\frac{\partial f}{\partial x}(2, 1)g^{'}(1) + \frac{\partial f}{\partial y}(2, 1) = 0,$$

so $g^{'}(1) = 0$.

Chapter 3

The Shooting Method And Property Of IVPs

Roughly speaking, shooting method is applying a numerical solution of initial value problems to solution of a boundary value problem, assuming that there is a solution for the given boundary value problem.

3.1 The Shooting Method

Consider a 2^{nd} order ordinary differential equation with boundary conditions

$$y^{''} = f(x, y, y^{'}), \quad a < x < b$$
$$y(a) = \alpha$$
$$y(b) = \beta$$

where a, b, α, β are given constants, y is the unknown function of x, f is a given function that specifies the differential equation.

The basic idea of **shooting method** is to replace the above BVP by an IVP. But of course, we do not know the derivative of y at $x = a$. But we can guess and then further improve the guess iteratively.
More precisely, we treat $y^{'}(a)$ as the unknown, and use secant method or Newtons method (or other methods for solving nonlinear equations) to determine $y^{'}(a)$.

We introduce a function u, which is a function of x, but it also depends on a parameter t. Namely, $u := u(x; t)$. We use $u^{'}$ and $u^{''}$ to denote the partial derivative of u, with respect to x. We want u to be exactly y, if t is properly chosen. But u is defined for any t, by

$$u^{''} = f(x, u, u^{'})$$
$$u(a, t) = \alpha$$
$$u^{'}(a, t) = t.$$

If you choose some t, you can then solve the above IVP of u. In general u is not the same as y, since $u^{'}(a) = t \neq y^{'}(a)$. But if t is $y^{'}(a)$, then u is y. Since we do not know $y^{'}(a)$, we determine it from the boundary condition at $x = b$. Namely, we solve t from:

$$\phi(t) = u(b, t) - \beta = 0.$$

If a solution t is found such that $\phi(t) = 0$, that means $u(b,t) = \beta$. Therefore, u satisfies the same boundary conditions at $x = a$ and $x = b$, as y. In other words, $u = y$. Thus, the solution t of $\phi(t) = 0$ must be $t = y^{'}(a)$.

If we can solve the IVP of u (for arbitrary t) analytically, we can write down a formula for $\phi(t) = u(b,t) - \beta$. Of course, this is not possible in general. However, without an analytic formula, we can still solve $\phi(t) = 0$ numerically. For any t, a numerical method for IVP of u can be used to find an approximate value of $u(b,t)$ (thus $\phi(t)$). The simplest method is to use the secant method.

$$t_{j+1} = t_j - \frac{t_j - t_{j-1}}{\phi(t_j) - \phi(t_{j-1})}\phi(t_j), \quad j = 1, 2, 3, ...$$

For that purpose, we need two initial guesses: t_0 and t_1. We can also use Newtons method:

$$t_{j+1} = t_j - \frac{\phi(t_j)}{\phi^{'}(t_j)}, \quad j = 0, 1, 2, ...$$

We need a method to calculate the derivative $\phi(t)$. Since $\phi(t) = u(b,t) - \beta$, we have

$$\phi^{'}(t) = \frac{\partial u}{\partial t}(b,t) - 0 = \frac{\partial u}{\partial t}(b,t)$$

If we define $v(x,t) := \frac{\partial u}{\partial t}$, we have the following IVP for v:

$$v^{''} = f_u(x,u,u^{'})v + f_{u^{'}}(x,u,u^{'})v^{'}$$
$$v(a,t) = 0$$
$$v^{'}(a,t) = 1$$

Here $v^{'}$ and $v^{''}$ are the first and second order partial derivatives of v, with respect to x. The above set of equations are obtained from taking partial derivative with respect to x for the system for u. The chain rule is used to obtain the differential equation of v. Now, we have $\phi^{'}(t) = v(b,t)$.[8]

3.2 Some Properties of Solutions of IVPs

In this section we state and prove the preliminary lemmas and some remarks which together helps us to prove the main result.[7]

Remark 3.2.1. Condition(C3) implies that $B_i^{'}(x) \geq 0$ for $x \geq 0 (i = 1, 2)$.
In fact, we have from $B_i(0) = 0$ and $B_i(x) > 0$ for $x > 0$ that

$$B_i^{'}(0) = \lim_{x\to 0} \frac{B_i(x) - B_i(0)}{x - 0}$$
$$= \lim_{x\to 0} \frac{B_i(x)}{x}.$$

Applying the L'Hopitals rule, we have

$$B_i^{'}(0) = \lim_{x\to 0} B_i^{'}(x).$$

This together with the assumption $B_i^{'}(x)$ is nondecreasing on$(0,\infty)$ implies that $B_i^{'}(x) \geq 0$ for $x \geq 0$.

To apply the shooting method, we need some properties of the solutions of the initial value problem

$$u^{''} + \bar{a}(t)u^{'} + \bar{f}(u) = 0 \tag{3.1}$$

$$u(0) = \eta \tag{3.2}$$

$$u^{'}(0) = \lambda.$$

Lemma 3.2.1. *Let $\bar{a} \in C[0,\infty), \bar{f} \in C[0,\infty) \cap C^1(0,\infty)$ with $\bar{f}(0) = 0$ and $\bar{f}(s) > 0$ for $s > 0$. Let $\eta \in (0,\infty)$ and $\lambda \in \mathbb{R}$ be two given constants. Then (3.1),(3.2) has a unique solution u satisfying either*

(I) $u(t) > 0$ for $t \in [0,\infty)$; or

(II) there exists $\rho \in (0,\infty)$ such that

$$u(t) > 0 \ \text{ on } t \in [0,\rho), \ \ u(\rho) = 0, \ \ u^{'}(\rho) < 0. \tag{3.3}$$

Proof. For any $r \in (0,\infty)$, let

$$\Omega_r := \{(t,u,p) \ | t \in [0,r], u > 0\} \tag{3.4}$$

and consider the function F on Ω_r defined by

$$F(t,u,p) := \bar{a}(t)p + \bar{f}(u). \tag{3.5}$$

First we need to show that F satisfies the locally Lipschitz condition in Ω_r

a) consider $(x,u_1,p),(x,u_2,p) \in \Omega_r$, then

$$\begin{aligned}|F(t,u_1,p) - F(t,u_2,p)| &= |\bar{a}(t)p + \bar{f}(u_1) - \bar{a}(t)p - \bar{f}(u_2)| \\ &= |\bar{f}(u_1) - \bar{f}(u_2)| \\ &= \frac{|\bar{f}(u_1) - \bar{f}(u_2)|}{|u_1 - u_2|}|u_1 - u_2|, \qquad u_1 \neq u_2.\end{aligned}$$

Now applying the mean value theorem, we have $c \in (u_1, u_2)$ such that

$$|F(t,u_1,p) - F(t,u_2,p)| = \bar{f}'(c)|u_1 - u_2|.$$

This implies that F satisfies locally Lipschitz condition in Ω_r with respect to u.

b) consider $(x,u,p_1),(x,u,p_2) \in \Omega_r$, then

$$\begin{aligned}|F(t,u,p_1) - F(t,u,p_1)| &= |\bar{a}(t)p_1 + \bar{f}(u) - \bar{a}(t)p_1 - \bar{f}(u)| \\ &= |\bar{a}(t)p_1 - \bar{a}(t)p_2| \\ &= |\bar{a}(t)||p_1 - p_2|.\end{aligned}$$

Since $\bar{a}$ is continuous in $[0,\infty)$ it is bounded in each compact subset of $[0,\infty)$. This implies there exists a positive real number k such that $|\bar{a}(t)| \leq k$.

Thus

$$|F(t,u,p_1) - F(t,u,p_1)| \leq k|p_1 - p_2|.$$

Therefore F satisfies locally Lipschitz condition in Ω_r with respect to p.

Hence F satisfies locally Lipschitz condition in Ω_r. And by theorem 2.2.2 (3.1),(3.2) has a unique solution $u(t)$ such that one of the following cases must occur

(i) $u > 0$ on $[0, \infty)$;

(ii) there exists $\rho \in (0, \infty)$ such that $u > 0$ on $[0, \rho)$, and $\lim_{t\to\rho^-} u(t) = 0$;

(iii) there exists $T \in (0, \infty)$ such that $u > 0$ on $[0, T)$, and $\limsup_{t\to T^-} u(t) = \infty$.

We claim that (iii) can not occur.
Assume on the contrary that (iii) occurs, then

$$\limsup_{t\to T^-} u'(t) = \infty. \tag{3.6}$$

On the other hand multiplying (3.1) by $\exp\big(\int_0^t \bar{a}(s)ds\big)$, we have that

$$\Big(u'(t)\exp\Big(\int_0^t \bar{a}(s)ds\Big)\Big)' + \exp\Big(\int_0^t \bar{a}(s)ds\Big)\bar{f}(u) = 0, \ \ t \in [0, T) \tag{3.7}$$

this implies that

$$\Big(u'(t)\exp\Big(\int_0^t \bar{a}(s)ds\Big)\Big)' = -\exp\Big(\int_0^t \bar{a}(s)ds\Big)\bar{f}(u)$$

which together with the condition $\bar{f}(s) > 0$ for $s > 0$ implies that

$$\Big(u'(t)\exp\Big(\int_0^t \bar{a}(s)ds\Big)\Big)' < 0.$$

It follows that

$$u'(t)\exp\Big(\int_0^t \bar{a}(s)ds\Big) \text{ is srtictly decreasing on } [0, T). \tag{3.8}$$

However this contradicts the fact (3.6).
Therefore either (i) or (ii) must occur.

Suppose on the contrary that (ii) occurs and $u'(\rho) = 0$. Using the similar argument of proving (3.8), we conclude that $u'(t)\exp(\int_0^t \bar{a}(s)ds)$ is strictly decreasing on $[0, \rho)$. This implies

$$u'(t)\exp\Big(\int_0^t \bar{a}(s)ds\Big) > 0 \text{ on } [0, \rho) \text{ (since by assumption } u'(\rho) = 0)$$

It follows that

$$u'(t) > 0, \quad \text{on } [0, \rho)$$

This implies $u(t)$ is increasing and

$$u(0) = \delta < \lim_{t\to\rho^-} u(t) = 0.$$

However this is a contradiction. Therefore $u'(\rho) < 0$ if (ii) occurs. This completes the proof. □

In order to prove uniqueness of positive solution of BVPs, we introduce an initial value problem

$$u'' + a(t)u' + f(u) = 0 \tag{3.9}$$
$$u(0) = \alpha > 0 \tag{3.10}$$
$$u'(0) = B_1(\alpha).$$

For any $\alpha > 0$, we know from Lemma 3.2.1 that (3.9), (3.10) has a unique solution u such that one of the cases occurs:

(i) $u > 0$ in $[0, \infty)$;

(ii) there exists a unique $\rho = \rho(\alpha) \in (0, \infty)$ such that $u(t) > 0$ on $[0, \rho), u(\rho) = 0$ and $u'(\rho) < 0$.

Now let us define T_α by

$$T_\alpha = \begin{cases} \infty, & \text{if (i) occurs} \\ \rho(\alpha), & \text{if (ii) occurs} \end{cases} \tag{3.11}$$

and

$$u(t, \alpha) := u(t).$$

From $\alpha > 0$, we have that

$$u(0, \alpha) = u(0) = \alpha > 0$$
$$u'(0, \alpha) = u'(\alpha) = B_1(\alpha) > 0,$$

and consequently

$$B_2(u(0, \alpha)) + u'(0, \alpha) = B_2(\alpha) + B_1(\alpha) > 0. \tag{3.12}$$

Now (3.12) together with continuity of B_2 and u implies there exists $\epsilon \in (0, T_\alpha)$ such that

$$B_2(u(t, \alpha)) + u'(t, \alpha) > 0, \quad t \in [0, \epsilon). \tag{3.13}$$

Denote

$$B(t, \alpha) := B_2(u(t, \alpha)) + u'(t, \alpha). \tag{3.14}$$

When $B(t, \alpha)$ vanishes at some $t_0 \in (0, T_\alpha)$, we define $b(\alpha)$ to be the first zero of $B(t, \alpha)$ in $(0, T_\alpha)$. More precisely, $b(\alpha)$ is a function of α which has the properties

$$B(b(\alpha), \alpha) = 0, \qquad B(t, \alpha) > 0, \qquad t \in [0, b(\alpha)). \tag{3.15}$$

If $B(t, \alpha)$ is positive in $[0, T_\alpha)$, then we define $b(\alpha) = T_\alpha$.
Let

$$N := \{\alpha : \alpha > 0,\ b(\alpha) < T_\alpha\}. \tag{3.16}$$

We recall that $u(t, \alpha)$ is a positive solution means $u(t, \alpha) > 0$ in $[0, b]$. So in the case $B(T_\alpha, \alpha) = 0,\ u(t, \alpha)$ is not a positive solution of (1.1),(1.2) since $u(T_\alpha, \alpha) = 0$. Hence we suppose $N \neq \varnothing$.

Remark 3.2.2. It is worth remarking here that if (ii) occurs, and accordingly $u(\rho(\alpha), \alpha) = 0$, then $b(\alpha) \in (0, \rho(\alpha))$,

$$B(b(\alpha), \alpha) = 0, \qquad B(t, \alpha) > 0 \quad \text{ on } [0, b(\alpha)). \tag{3.17}$$

In fact, we have from lemma 3.2.1 and (C3)

$$\begin{aligned} B(\rho(\alpha),\alpha) &= B_2(u(\rho(\alpha),\alpha)) + u^{'}(\rho(\alpha),\alpha) \\ &= B_2(0) + u^{'}(\rho(\alpha),\alpha)) \\ &= u^{'}(\rho(\alpha),\alpha)) < 0, \end{aligned} \tag{3.18}$$

which together with

$$B(0,\alpha) = B_2(u(0,\alpha)) + u^{'}(0,\alpha) = B_2(\alpha) + B_1(\alpha) > 0$$

and the Intermediate Value theorem assures the zero of $B(t,\alpha)$ in $(0,\rho(\alpha))$.

Lemma 3.2.2. *Let (C1)-(C3) hold and let* $\alpha \in N$*. Let* $u(t,\alpha)$ *be the unique solution of (3.9),(3.10) on* $[0,T_\alpha)$*. Then*

$$u(t,\alpha) > 0, \qquad t \in [0,b(\alpha)], \tag{3.19}$$

$$u^{'}(b(\alpha),\alpha) < 0.$$

***Proof*.** By remark 3.2.2, $b(\alpha) \in (0,\rho(\alpha))$. Applying lemma 3.2.1, we get

$$u(t,\alpha) > 0, \qquad t \in [0,b(\alpha)]. \tag{3.20}$$

Now, from the definition of $b(\alpha)$, we have

$$B(b(\alpha),\alpha) = B_2(u(b(\alpha),\alpha)) + u^{'}(b(\alpha),\alpha) = 0.$$

This implies

$$u^{'}(b(\alpha),\alpha) = -B_2(u(b(\alpha),\alpha)).$$

But from equation (3.20) and (C3)

$$B_2(u(b(\alpha),\alpha)) > 0.$$

It follows that

$$u^{'}(b(\alpha),\alpha) < 0. \tag{3.21}$$

□

Lemma 3.2.3. *Let (C1)-(C3) hold. Let* $u(t,\alpha)$ *be the unique solution of (3.9),(3.10) on* $[0,T_\alpha)$*. If* $\eta \in (0,T_\alpha)$ *is such that*

$$B(\eta,\alpha) = 0, \tag{3.22}$$

then

$$B(t,\alpha) > 0, \qquad t \in [0,\eta). \tag{3.23}$$

***Proof*.** Multiplying (3.9) by $\exp\big(\int_0^t a(s)ds\big)$ gives

$$\Big(u^{'}\exp\Big(\int_0^t a(s)ds\Big)\Big)^{'} + \exp\Big(\int_0^t a(s)ds\Big)f(u) = 0. \tag{3.24}$$

Since $u(t,\alpha) > 0$ for all $t \in [0,\eta]$ and equation (3.24) together with (C1), we have

$$\Big(u^{'}(t,\alpha)\exp\Big(\int_0^t a(s)ds\Big)\Big)^{'} = -\exp\Big(\int_0^t a(s)ds\Big)f(u(t,\alpha)) < 0, \quad \forall t \in [0,\eta]. \tag{3.25}$$

To show

$$B(t,\alpha) > 0, \quad t \in [0,\eta).$$

Suppose on the contrary that there exists $\tau_2 \in [0,\eta)$ such that

$$B(\tau_2,\alpha) = B_2(u(\tau_2,\alpha)) + u^{'}(\tau_2,\alpha) = 0. \tag{3.26}$$

Then we have from condition (C3) and the fact $u(\tau_2,\alpha) > 0$ that

$$u^{'}(\tau_2,\alpha) = -B_2(u(\tau_2,\alpha)) < 0 \tag{3.27}$$

and accordingly

$$u^{'}(\tau_2,\alpha)\exp\Big(\int_0^{\tau_2} a(s)ds\Big) < 0. \tag{3.28}$$

This together with (3.25) implies that

$$u^{'}(t,\alpha)\exp\Big(\int_0^{t} a(s)ds\Big) < 0, \quad t \in [\tau_2,\eta] \tag{3.29}$$

and consequently

$$u^{'}(t,\alpha) < 0, \quad t \in [\tau_2,\eta]. \tag{3.30}$$

This implies

$$u(\tau_2,\alpha) > u(\eta,\alpha). \tag{3.31}$$

By remark 3.2.1 and (3.31), we get

$$B_2(u(\tau_2,\alpha)) \geq B_2(u(\eta,\alpha)). \tag{3.32}$$

From (3.30) and (C1)-(C2) and the fact $u^{''}(t,\alpha)+a(t)u^{'}(t,\alpha)+f(u(t,\alpha)) = 0$, we have

$$u^{''}(t,\alpha) = -a(t)u^{'}(t,\alpha) - f(u(t,\alpha)) < 0, \quad t \in [\tau_2,\eta] \tag{3.33}$$

and consequently

$$u^{'}(\tau_2,\alpha) > u^{'}(\eta,\alpha), \tag{3.34}$$

which together with (3.32) implies that

$$\begin{aligned} B(\tau_2,\alpha) &= B_2(u(\tau_2,\alpha)) + u^{'}(\tau_2,\alpha) \\ &> B_2(u(\eta,\alpha)) + u^{'}(\eta,\alpha) \\ &= B(\eta,\alpha) = 0. \end{aligned} \tag{3.35}$$

However this contradicts (3.26).
Thus

$$B(t,\alpha) > 0, \quad t \in [0,\eta).$$

□

Remark 3.2.3. From Lemmas 3.2.2 and 3.2.3, we have that if $\eta \in (0,T_\alpha)$ satisfies

$$B(\eta,\alpha) = 0. \tag{3.36}$$

Then

$$\eta = b(\alpha). \tag{3.37}$$

In other words, if $\alpha \in N$, then $b(\alpha)$ is the unique zero of $B(t,\alpha) = 0$ in $[0,\rho(\alpha))$. Therefore to prove that (1.1), (1.2) has at most one positive solution, it is sufficient to show that for any $l > 0$, there exists at most one $\alpha \in N$ such that $b(\alpha) = l$.

Now we denote the *variation* of $u(t,\alpha)$ by

$$\phi(t,\alpha) = \partial u(t,\alpha)/\partial\alpha.$$

Then, $\phi(t,\alpha)$ satisfies

$$\phi^{''} + a(t)\phi^{'} + f^{'}(u)\phi = 0, \tag{3.38}$$

$$\phi(0,\alpha) = 1, \tag{3.39}$$

$$\phi^{'}(0,\alpha) = B_1^{'}(\alpha).$$

To show this, we consider equations (3.9),(3.10)
i.e.

$$u^{''} + a(t)u^{'} + f(u) = 0$$

$$u(0) = \alpha > 0$$

$$u^{'}(0) = B_1(\alpha)$$

and from the nature of $u(t,\alpha)$ the partial derivatives with respect to t can be treated as the derivatives of u as $u(t,\alpha)$ is dependent only on t, in other words

$$\frac{\partial}{\partial t}u(t,\alpha) = u^{'}(t,\alpha)$$

and for the same reason

$$\frac{\partial}{\partial t}\phi(t,\alpha) = \phi^{'}(t,\alpha),$$

then

$$\begin{aligned}\phi^{'}(t,\alpha) &= \frac{\partial}{\partial t}\phi(t,\alpha)\\ &= \frac{\partial}{\partial t}[\frac{\partial}{\partial\alpha}u(t,\alpha)]\\ &= \frac{\partial}{\partial\alpha}[\frac{\partial}{\partial t}u(t,\alpha)], \text{(equality of mixed partial)}\\ &= \frac{\partial}{\partial\alpha}u^{'}(t,\alpha).\end{aligned}$$

And also

$$\begin{aligned}\phi^{''}(t,\alpha) &= \frac{\partial}{\partial t}[\frac{\partial}{\partial\alpha}u^{'}(t,\alpha)]\\ &= \frac{\partial}{\partial\alpha}[\frac{\partial}{\partial t}u^{'}(t,\alpha)]\\ &= \frac{\partial}{\partial\alpha}u^{''}(t,\alpha).\end{aligned}$$

Thus
$\phi^{''}(t,\alpha) + a(t)\phi^{'}(t,\alpha) + f^{'}(u(t,\alpha))\phi(t,\alpha)$

$$\begin{aligned}&= \frac{\partial}{\partial\alpha}u^{''}(t,\alpha) + a(t)\frac{\partial}{\partial\alpha}u^{'}(t,\alpha) + f^{'}(u(t,\alpha))\frac{\partial}{\partial\alpha}u(t,\alpha)\\ &= \frac{\partial}{\partial\alpha}u^{''}(t,\alpha) + a(t)\frac{\partial}{\partial\alpha}u^{'}(t,\alpha) + \frac{\partial}{\partial\alpha}(f(u(t,\alpha))).\end{aligned}$$

Now from the linearity of derivatives and (1.1) follows that

$$\phi'' + a(t)\phi' + f'(u)\phi = \frac{\partial}{\partial \alpha}(u'' + \alpha(t)u' + f(u)) = \frac{\partial}{\partial \alpha}(0) = 0.$$

And

$$\phi(0, \alpha) = \frac{\partial}{\partial \alpha} u(0, \alpha) = \frac{\partial}{\partial \alpha}\alpha = 1,$$

$$\phi'(0, \alpha) = \frac{\partial}{\partial \alpha} u'(0, \alpha) = \frac{\partial}{\partial \alpha} B_1(\alpha) = B_1'(\alpha).$$

Hence (3.38), (3.39) holds.

Lemma 3.2.4. *Suppose that*

$$B_2'(u(b(\alpha), \alpha))\phi(b(\alpha), \alpha) + \phi'(b(\alpha), \alpha) \neq 0, \qquad \alpha \in N. \tag{3.40}$$

Then one of the following cases must occur

(i) N is an open interval;

(ii) $N = (0, j_1) \cup (j_2, \infty)$ with $0 < j_1 < j_2 < +\infty$. Moreover, $b'(\alpha) > 0$ for all $(0, j_1)$; $b'(\alpha) < 0$ for all (j_2, ∞).

***Proof**.* We firstly show that $b(\alpha) \in C^1(N)$ and $b'(\alpha) \neq 0$.

From lemma 3.2.2 we have

$$u(t, \alpha) > 0, \quad \text{for} \quad t \in [0, b(\alpha)] \quad \text{and} \quad u'(b(\alpha), \alpha) < 0$$

and from (C1)-(C2) we have

$$a(t) < 0 \quad \text{for} \quad t > 0 \quad \text{and} \quad f(t) \geq 0 \quad \text{for} \quad t \in [0, \infty)$$

such that

$$u''(b(\alpha), \alpha) + a(b(\alpha))u'(b(\alpha), \alpha) + f(u(b(\alpha), \alpha)) = 0.$$

This implies that

$$u''(b(\alpha), \alpha) = -a(b(\alpha))u'(b(\alpha), \alpha) - f(u(b(\alpha), \alpha)) < 0. \tag{3.41}$$

This together with

$$B(b(\alpha), \alpha) = 0 \tag{3.42}$$

and (C3) and (3.19) implies that

$$\begin{aligned} \frac{\partial}{\partial t} B(t, \alpha)\big|_{t=b(\alpha)} &= \frac{\partial}{\partial t}[B_2(u(t, \alpha)) + u'(t, \alpha)]\big|_{t=b(\alpha)} \\ &= B_2'(u(t, \alpha))u'(t, \alpha) + u''(t, \alpha)\big|_{t=b(\alpha)} \\ &= B_2'(u(b(\alpha), \alpha))u'(b(\alpha), \alpha) + u''(b(\alpha), \alpha) < 0. \end{aligned} \tag{3.43}$$

So by Implicit Function theorem, $b(\alpha)$ is well-defined as a function of α on N and $b(\alpha) \in C^1(N)$. Furthermore, it follows from (3.43) that N is an open set.

Now differentiating both sides of (3.42) with respect to α we get zero.
i.e.

$$\frac{\partial}{\partial\alpha}B(b(\alpha),\alpha)=0.$$

This together with chain rule implies that

$$\begin{aligned}
\frac{\partial}{\partial\alpha}B(b(\alpha),\alpha) &= \frac{\partial}{\partial\alpha}(B_2(u(b(\alpha),\alpha))+u^{'}(b(\alpha),\alpha)) \\
&= B_2^{'}(u(b(\alpha),\alpha))[u^{'}(b(\alpha),\alpha)b^{'}(\alpha)+\frac{\partial}{\partial\alpha}u(b(\alpha),\alpha)] \\
&\quad + u^{''}(b(\alpha),\alpha)b^{'}(\alpha)+\frac{\partial}{\partial\alpha}u^{'}(b(\alpha),\alpha) \\
&= B_2^{'}(u(b(\alpha),\alpha))[u^{'}(b(\alpha),\alpha)b^{'}(\alpha)+\phi(b(\alpha),\alpha)] \\
&\quad + u^{''}(b(\alpha),\alpha)b^{'}(\alpha)+\phi^{'}(b(\alpha),\alpha)=0 \qquad (3.44)
\end{aligned}$$

that is

$$\begin{aligned}
&[B_2^{'}(u(b(\alpha),\alpha))u^{'}(b(\alpha),\alpha)+u^{''}(b(\alpha),\alpha)]b^{'}(\alpha) \\
&\quad + B_2^{'}(u(b(\alpha),\alpha))\phi(b(\alpha),\alpha)+\phi^{'}(b(\alpha),\alpha)=0. \qquad (3.45)
\end{aligned}$$

This together with (3.40) implies that

$$[B_2^{'}(u(b(\alpha),\alpha))u^{'}(b(\alpha),\alpha)+u^{''}(b(\alpha),\alpha)]b^{'}(\alpha)\neq 0$$

implies

$$b^{'}(\alpha)\neq 0. \qquad (3.46)$$

Next we show that if $\bar{\alpha}\in(0,\infty)\backslash N$ is such that there is a sequence $\{\alpha_n\}\subset N$ and $\alpha_n\to\bar{\alpha}$ as $n\to\infty$, then $b(\alpha_n)\to+\infty$.

Suppose on the contrary that $b(\alpha_n)\nrightarrow+\infty$, then there exists a subsequence of $\{b(\alpha_n)\}$ which converges to a limit number t^*. Without loss of generality, we may suppose that

$$b(\alpha_n)\to t^* \quad \text{as} \quad n\to\infty$$

and consequently

$$B(t^*,\bar{\alpha})=\lim_{n\to\infty}B(b(\alpha_n),\alpha_n)=\lim_{n\to\infty}0=0$$

since $\alpha_n\in N$ and this by definition of N implies $B(b(\alpha_n),\alpha_n)=0$. Thus

$$B(t^*,\bar{\alpha})=0. \qquad (3.47)$$

However this contradicts $\bar{\alpha}\notin N$.

Finally we show that if N is not an open interval, then (ii) occur.

Suppose $J_1=(j_0,j_1)$ and $J_2=(j_2,j_3)$ are two distinct components of N with $0<j_1<j_2<\infty$. Then

$$\lim_{\alpha\to j_1^-}b(\alpha)=\lim_{\alpha\to j_2^+}b(\alpha)=+\infty. \qquad (3.48)$$

Since $b(\alpha)$ is strictly monotonic in each component of N, we have that $b^{'}(\alpha)>0$ in J_1, and $b^{'}(\alpha)<0$ in J_2. Meanwhile

$$\lim_{\alpha\to j_0^+}b(\alpha)<+\infty, \qquad \lim_{\alpha\to j_3^-}b(\alpha)<+\infty. \qquad (3.49)$$

It follows that $j_0 = 0$ and $j_3 = +\infty$, and accordingly

$$N = (0, j_1) \cup (j_2, \infty)$$

with $b'(\alpha) > 0$ in $(0, j_1)$ and $b'(\alpha) < 0$ in (j_2, ∞). $\square$

Chapter 4

Uniqueness Of Positive Solution Of BVPs

In this chapter the proof of uniqueness of positive solution of BVPs will be given, as stated earlier. The proof totally depends on preliminary lemmas and remarks given in previous chapter.

4.1 Proof of Uniqueness Of Positive Solution Of BVPs

By remark 3.2.3, we only need to show that for any $l > 0$, there exists at most one $\alpha \in N$ such that $b(\alpha) = l$.
Recall that for any given $\alpha \in N,\ B(b(\alpha), \alpha) = 0$.
And from (3.43)

$$\frac{\partial}{\partial t} B(t, \alpha)\big|_{t=b(\alpha)} = B_2^{'}(u(b(\alpha), \alpha))u^{'}(b(\alpha), \alpha) + u^{''}(b(\alpha), \alpha) < 0$$

and from (3.45)

$$[B_2^{'}(u(b(\alpha), \alpha))u^{'}(b(\alpha), \alpha) + u^{''}(b(\alpha), \alpha)]b^{'}(\alpha)$$
$$+B_2^{'}(u(b(\alpha), \alpha))\phi(b(\alpha), \alpha) + \phi^{'}(b(\alpha), \alpha) = 0$$

holds true.
If we can show that

$$B_2^{'}(u(b(\alpha), \alpha))\phi(b(\alpha), \alpha) + \phi^{'}(b(\alpha), \alpha) > 0, \quad \alpha \in N \tag{4.1}$$

then it follows from (3.43) and (3.45) that

$$b^{'}(\alpha) > 0 \quad \alpha \in N. \tag{4.2}$$

Thus by Lemma 3.2.4, N must be an open interval. Moreover we know from(4.2) that $b(\alpha)$ is a strictly increasing function on N. Thus, for any given $l > 0$, there is at most one $\alpha \in N$ such that $b(\alpha) = l$, and consequently, (1.1), (1.2) has at most one positive solution.
Now we proceed to the proof

Proof. Now we prove (4.1)

First we claim that

$$\phi(t,\alpha) > 0, \qquad t \in [0, b(\alpha)]. \tag{4.3}$$

Suppose on the contrary that $\phi(t,\alpha)$ has a zero in $(0, b(\alpha)]$. We denote the first zero of $\phi(t,\alpha)$ in $(0, b(\alpha)]$ by t_3, then $0 < t_3 \le b(\alpha)$ and

$$\begin{aligned} &[u^{'}(t,\alpha)\phi(t,\alpha) - u(t,\alpha)\phi^{'}(t,\alpha)]\big|_{t=t_3} \\ &= u^{'}(t_3,\alpha)\phi(t_3,\alpha) - u(t_3,\alpha)\phi^{'}(t_3,\alpha) \\ &= -u(t_3,\alpha)\phi^{'}(t_3,\alpha) \end{aligned}$$

since $\phi(t_3,\alpha) = 0$ and $\phi(t,\alpha) > 0$ on $(0, t_3)$ implies that

$$\phi^{'}(t_3,\alpha) \le 0$$

consequently, it follows that

$$u^{'}\phi - u\phi^{'}\big|_{t=t_3} = -u(t_3,\alpha)\phi^{'}(t_3,\alpha) \ge 0. \tag{4.4}$$

Notice that

$$\phi^{''} + a(t)\phi^{'} + f^{'}(u)\phi = 0 \tag{4.5}$$

so that using (C1) and (1.1) we can compute

$$\begin{aligned} \Big[\exp\Big(\int_0^t a(s)ds\Big)(u^{'}\phi - u\phi^{'})\Big]^{'} &= \Big[\exp\Big(\int_0^t a(s)ds\Big)\Big]^{'}(u^{'}\phi - u\phi^{'}) \\ &\quad + \Big[\exp\Big(\int_0^t a(s)ds\Big)\Big](u^{'}\phi - u\phi^{'})^{'} \\ &= a(t)\exp\Big(\int_0^t a(s)ds\Big)(u^{'}\phi - u\phi^{'}) \\ &\quad + \Big[\exp\Big(\int_0^t a(s)ds\Big)\Big](u^{''}\phi - u\phi^{''}) \\ &= \exp\Big(\int_0^t a(s)ds\Big)[a(t)(u^{'}\phi - u\phi^{'}) + (u^{''}\phi - u\phi^{''})] \\ &= \exp\Big(\int_0^t a(s)ds\Big)[\phi(u^{''} + a(t)u^{'}) - u(\phi^{''} + a(t)\phi^{'})] \end{aligned}$$

but from (1.1) we have

$$-f(u) = u^{''} + a(t)u^{'}$$

and from (4.5) we have

$$-f^{'}(u)\phi = \phi^{''} + a(t)\phi^{'}$$

thus

$$\Big[\exp\Big(\int_0^t a(s)ds\Big)(u^{'}\phi - u\phi^{'})\Big]^{'} = \exp\Big(\int_0^t a(s)ds\Big)[uf^{'}(u) - f(u)]\phi < 0 \tag{4.6}$$

for $t \in (0, t_3)$. Since from (C1), we have $uf^{'}(u) - f(u) < 0$ and $\phi > 0$ for $t \in (0, t_3)$. Next we compute

$$\exp\Big(\int_0^t a(s)ds\Big)(u^{'}\phi - u\phi^{'})|_{t=0} = u^{'}(0,\alpha)\phi(0,\alpha) - u(0,\alpha)\phi^{'}(0,\alpha)$$

but from (3.10) and (3.39) we have

$$u(0,\alpha)=\alpha>0, \qquad u^{'}(0,\alpha)=B_1(\alpha)$$

and

$$\phi(0,\alpha)=1, \qquad \phi^{'}(0,\alpha)=B_1^{'}(\alpha).$$

Then it follows that

$$\begin{aligned}\exp\Big(\int_0^t a(s)ds\Big)(u^{'}\phi-u\phi^{'})|_{t=0} &= B_1(\alpha)-\alpha B_1^{'}(\alpha)\\ &=\alpha[\frac{B_1(\alpha)}{\alpha}-B_1^{'}(\alpha)]\\ &=\alpha[\frac{B_1(\alpha)-B_1(0)}{\alpha-0}-B_1^{'}(\alpha)]\end{aligned}$$

since $B_1(0)=0$.
Now by the mean value theorem, there is $\xi_1(\alpha)\in(0,\alpha)$ such that

$$B_1^{'}(\xi_1(\alpha))=\frac{B_1(\alpha)-B_1(0)}{\alpha-0}.$$

This implies that

$$\exp\Big(\int_0^t a(s)ds\Big)(u^{'}\phi-u\phi^{'})|_{t=0}=[B_1^{'}(\xi_1(\alpha))-B_1^{'}(\alpha)]\alpha\leq 0. \tag{4.7}$$

This means that

$$\exp\Big(\int_0^t a(s)ds\Big)(u^{'}\phi-u\phi^{'})|_{t=t_3}<0 \tag{4.8}$$

since by (4.6) $\exp\Big(\int_0^t a(s)ds\Big)(u^{'}\phi-u\phi^{'})$ is strictly decreasing.
And accordingly

$$(u^{'}\phi-u\phi^{'})|_{t=t_3}<0. \tag{4.9}$$

However this contradicts (4.4). Therefore

$$\phi(t,\alpha)>0, \quad t\in[0,b(\alpha)].$$

Using (4.5),(1.1),(C1) and (4.3), we can conclude

$$\Big[\exp\Big(\int_0^t a(s)ds\Big)(u^{'}\phi-u\phi^{'})\Big]^{'}=\exp\Big(\int_0^t a(s)ds\Big)[f^{'}(u)u-f(u)]\phi<0, \quad t\in(0,b(\alpha)] \tag{4.10}$$

which together with (4.7) implies that

$$(u^{'}\phi-u\phi^{'})|_{t=b(\alpha)}<0. \tag{4.11}$$

Since

$$\begin{aligned}0&=B(b(\alpha),\alpha)\\ &=B_2(u(b(\alpha),\alpha))+u^{'}(b(\alpha),\alpha)\\ &=\frac{B_2(u(b(\alpha),\alpha))-B_2(0)}{u(b(\alpha),\alpha)-0}(u(b(\alpha),\alpha))+u^{'}(b(\alpha),\alpha)\end{aligned}$$

applying the mean value theorem, we have $\xi_2(\alpha) \in (0, u(b(\alpha), \alpha))$ such that

$$B_2^{'}(\xi_2(\alpha)) = \frac{B_2(u(b(\alpha), \alpha)) - B_2(0)}{u(b(\alpha), \alpha) - 0}$$

thus

$$B(b(\alpha), \alpha) = B_2^{'}(\xi_2(\alpha))u(b(\alpha), \alpha) + u^{'}(b(\alpha), \alpha) = 0 \tag{4.12}$$

from this it follows that

$$u^{'}(b(\alpha), \alpha) = -B_2^{'}(\xi_2(\alpha))u(b(\alpha), \alpha). \tag{4.13}$$

This together with (4.11) implies

$$\begin{aligned} &-u(b(\alpha), \alpha)[B_2^{'}(\xi_2(\alpha))\phi(b(\alpha), \alpha) + \phi^{'}(b(\alpha), \alpha)] \\ &\quad = -B_2^{'}(\xi_2(\alpha))u(b(\alpha), \alpha)\phi(b(\alpha), \alpha) - u(b(\alpha), \alpha)\phi^{'}(b(\alpha), \alpha) \\ &\quad = u^{'}(b(\alpha), \alpha)\phi(b(\alpha), \alpha) - u(b(\alpha), \alpha)\phi^{'}(b(\alpha), \alpha) \qquad (4.14) \\ &\quad = u^{'}\phi - u\phi^{'}|_{t=b(\alpha)} < 0 \end{aligned}$$

and consequently

$$B_2^{'}(\xi_2(\alpha))\phi(b(\alpha), \alpha) + \phi^{'}(b(\alpha), \alpha) > 0. \tag{4.15}$$

Now we have from (C3) and the fact $\xi_2(\alpha) < u(b(\alpha), \alpha)$ and $\phi(b(\alpha), \alpha) > 0$ that

$$B_2^{'}(u(b(\alpha), \alpha))\phi(b(\alpha), \alpha) + \phi^{'}(b(\alpha), \alpha) \geq B_2^{'}(\xi_2(\alpha))\phi(b(\alpha), \alpha) + \phi^{'}(b(\alpha), \alpha) > 0. \tag{4.16}$$

Therefore (4.1) holds.[7] □

4.2 Illustration

We conclude this section with example of nonlinear boundary problem satisfying our assumptions. Let see example for the clear of this paper.

Example:-[7] Consider the following nonlinear boundary value problem

$$u^{''} + a(t)u^{'} + u^p = 0, \quad 0 < t < b < \infty \tag{4.17}$$

$$\begin{aligned} (u(0))^k - u^{'}(0) = 0 \qquad (4.18) \\ (u(b))^l + u^{'}(b) = 0, \end{aligned}$$

where $p \in (0, 1)$, $k, l \in (1, \infty)$ are given, $a \in C[0, \infty)$ with $a \leq 0$ on $[0, \infty)$.

Now we need to check that weather the three conditions are satisfied or not.
Let

$$\begin{aligned} f(u) &= u^p \\ B_1(u) &= u^k \qquad (4.19) \\ B_2(u) &= u^l \end{aligned}$$

where p, k, l are as stated in (4.18).

Check for

C1-

Since f is a root function for $p \in (0,1)$, it follows that $f \in C[0,\infty)$ and $f \in C^1(0,\infty)$, and therefore $f \in C[0,\infty) \cap C^1(0,\infty)$.

And from elementary calculus we have that

$$f^{'}(u) = pu^{p-1}. \tag{4.20}$$

Now multiplying (4.20) by u we get

$$uf^{'}(u) = upu^{p-1} = pu^p$$

since $p \in (0,1)$, it follows that

$$pu^p < u^p$$

consequently

$$uf^{'}(u) < f(u), \quad for \;\; u > 0.$$

Thus the first condition is satisfied.

C2-

Since $a \in C[0,\infty)$ and $a(t) \leq 0$ on $[0,\infty)$ is already given, it follows that the second condition is satisfied.

C3-

From equation (4.19) we have

$$B_1(x) = x^k, B_2(x) = x^l$$

this implies that

$$B_1^{'}(x) = kx^{k-1} \geq 0 \;\; x \in [0,\infty) \tag{4.21}$$
$$B_2^{'}(x) = lx^{l-1} \geq 0 \;\; x \in [0,\infty)$$

and it follows that $B_1^{'} \in C[0,\infty)$ and $B_2^{'} \in C[0,\infty)$. Consequently $B_i \in C^1[0,\infty)$ for all $i = 1,2$. Moreover

$$B_1(0) = 0 = B_2(0)$$

and

$$B_1(x) = x^k > 0, \quad x \in (0,\infty) \tag{4.22}$$
$$B_2(x) = x^l > 0, \quad x \in (0,\infty)$$

from (4.21) and (4.22) follows the third condition.

Therefore the boundary value problem (4.17) and (4.18) has at most a positive solution for any $b \in (0,\infty)$.

Summary

Multiplicity and existence results for explicit first order initial value problem(IVP) can be established by Picard Lindelof Theorem, provided that Lipshitz condition is in place. These results can be extended to higher order ODEs with initial conditions by introducing new variables(as many as the order of the ODE) thereby generating a system of first order ODEs with initial data in vector form.

Uniqueness of solution of a boundary value problem(BVP) can also be seen within the confines of initial value problems(IVP). To this end, one needs to employ Shooting Method so as to approximate (transform) a given BVP by (into) the pertinent IVP.

Bibliography

[1] **E.A.Coddington and N.Levinson**, *Theory of ordinary differential equations*, TATA McGRAW-HILL publishing Co.LTD, New Delhi, (1972).

[2] **George F. Simmons**, *Differential Equationas With Applications and Historical Notes*, Second Edition, McGraw-Hill, Inc.(1991).

[3] **Lawrence.P**, *Differential Equations and Dynamical Systems*, Third Edition, springer (2000).

[4] **M.Brun**, *Differential Equations and Their Applications*, Third Edition, springer-Verlag, (1983).

[5] **P.M.Fitzpartick**, *Advanced Calculus*, Second Edition, Thomson Books/cool, (2006)

[6] **R.Deumlich**, *Functional Analysis I*, (1998) unpublished.

[7] **Ruyun Ma and Yulian An**, *Uniqueness of positive solutions of a class of ODE with nonlinear Boundary Conditions*, 2006 Hindawi Publishing Corporation, boundary value problems 2005:3 (2005) 289-298, DOI:10.1155/BVP.2005.289.

[8] **YA YAN LU**, *Numerical Methods for Differential Equations*, Third Edition, Springer-verlang,(1998).