A GRADUATE SEMINAR REPORT

ON

GENERALIZED FUNCTIONS AND APPLICATION ON AERODYNAMICS

COMPILED BY

ANDUALEM TAMIRU

ADVISOR

DR. TSGAYE GEDIF

SCHOOL OF GRADUATE STUDIES
ADDIS ABABA UNIVERSITY

July 8, 2003
Addis Ababa

Preface

This Seminar Report contains six parts:

The first part deals with the concept of generalized functions.

The second part contains differentiation and integration of generalized functions.

The third part deals with direct product and convolution of generalized functions.

The fourth part contains generalized functions of slow growth.

The fifth part deals with multidimensional delta functions

The sixth part deals with the application of generalized functions in Reynolds Transport theorem.

Acknowledgment

I would like to thank my advisor Tsegaye Gedif not only for his willingness to be my advisor but also for providing me with the necessary materials and ideas that made my wishes a reality.

My special gratitude goes to Dr. Edessa who provided me with the necessary materials and ideas.

I would like to thank W/o Tobiaw Tefera who was very much helpful. Finally I would like to thank all who were with me in my endeavor for success, especially my parents.

Contents

I. Generalized Function and Their Properties

1. Test Functions and Generalized Functions

1.1 Introduction: At the end of the 1920's Paul Dirac introduced for the first time in his quantum mechanical studies the so called delta function (δ-function) which has the following properties.

$$\delta(x) = 0, \; x \neq 0$$

$$\int \varphi(x)\, \delta(x) dx = \varphi(0)$$

where φ is continuous function.

But this δ-function is not a function in the classical meaning. Therefore it required the efforts of many mathematicians to find a mathematically proper definition of the delta function, of its derivatives and, generally to introduce a generalized function.

At present, the theory of generalized functions has advanced substantially and is becoming a very important tool of the mathematician, engineer and physicist.

1.2 The Space of Test Functions

Definition A functional on a space of functions Ω is a mapping (a rule) of Ω into a scalars (real or complex)

Examples: Take Ω as space of differentiable functions. The following are functionals on Ω, $\phi \in \Omega$.

1. $F[\phi]\ \phi'(0) + 2\phi(1)$
2. $F[\phi] = \int_0^1 \phi^2(x)dx$
3. $F[\phi] = \sin[\phi(0)]$
4. $F[\phi] = 2\phi(1) + \int_{-1}^1 \phi(x)dx$

Definition Support of a function ϕ is the closure of the set on which $\phi \neq 0$. In theory, the functionals act on various test function spaces depending on the problem. We define generalized functions on the following test function space.

Definition **Space D of Test Function:** Infinitely differentiable functions with bounded support.

Example of Functions in D.

1. $$\phi(x; a) = \begin{cases} \exp\left[\dfrac{a^2}{x^2 - a^2}\right] & |x| < a \\ 0 & |x| \geq a \end{cases}$$

$\Rightarrow \phi(x; a) \in D$.

2. Let g(x) be any continuous function, then

$\psi(x) = \int_b^c g(y)\, \phi(x\text{-}y,a)dy$, where [b,c] is a finite interval, belongs to D. We can show that supp $\psi(x) = [b - a, c + a]$.

Definition: A functional on D is linear if $F[\alpha\phi, + \beta\phi_2]$

$$= \alpha F[\phi_1] + \beta F[\phi_2]$$

for all ϕ_1 and ϕ_2 in D. and $\alpha, \beta \in K$

Examples: $\phi \in D$

1. $F[\phi] = \phi(0)$ is linear
2. $F[\phi] = 2\phi'(1) - \int f\phi dx$, f an ordinary function, is linear
3. $F[\phi] = \phi^2(0)$ is nonlinear

Definition: A sequence of functions $\{\phi_n\}$ in D converges to zero in D, written as $\phi_n \xrightarrow{D} 0$, if ϕ_n and all its derivatives converge uniformly to zero and supp ϕ_n C I for all n where I is fixed bounded interval.

Definition: A functional F on D is continuous if $F[\phi_n] \to 0$ when $\phi_n \xrightarrow{D} 0$

Examples: 1. Let $\phi_n = \frac{1}{n}\phi(x;a) \Rightarrow \phi_n \xrightarrow{D} 0$

2. $\delta[\phi] = \phi(0)$, $\phi \in D$ is continuous (it is linear)

3. $\delta[\phi] = \phi(0)$, $\phi \in D$ is linear an continuous,

(a) $\delta[\lambda\phi_1 + \mu\phi_2] = (\lambda\phi_1 + \mu\phi_2)(0)$

$= \lambda\phi_1(0) = \mu\phi_2(0)$

$= \lambda\delta[\phi_1] + \mu\delta[\phi_2]$

Thus $\delta[\phi]$ is linear

(b) To show continuity.

Let $\{\phi_n\}$ be a sequence in D such that $\phi_n \xrightarrow{D} 0$ (that is, ϕ_n and all its derivatives converge uniformly to zero and supp ϕ_n C I for all n were I is fixed bounded integral)

Then $\delta[\phi_n] = \phi_n(0)$

But $\phi_n \to 0$.

$\Rightarrow \phi_n(0) \to 0$

Thus $\delta[\phi_n] \to 0$.

hence $\delta[\phi]$ is continuous

1.3 The space of Generalized Functions D′(Ω)

An ordinary function f defines a continuous linear functional on D by the relation $(f, \phi) := \int f\phi dx$, $\phi \in D(\Omega)$. But ordinary functions do not exhaust all continuous linear functionals on D.

Definition: A continuous linear functional on space D defines a generalized function. A generalized function f is denoted by (f, ϕ), $\phi \in D$

Examples: $\phi \in D$

1. $(\delta, \phi) = \phi(0)$ defines a generalized function. We have already shown that (δ, ϕ) is a linear continuous functional. There is no ordinary function f(x) such that $\int f(x)\, \phi(x) dx = \phi(0)$. This means means that D′ is larger than the space of ordinary functions.
2. $(f, \phi) = a\phi'(1) + \int f\, \phi dx$, where f is an ordinary function, defines a generalized function

To show this

(i) Trivially it is a functional

(ii) linearity

$$(f, \lambda\phi_1 + \mu\phi_2)$$

$$= 2[\lambda\phi_1 + \mu\phi_2]'(1) + \int f(\lambda\phi_1 + \mu\phi_2)dx$$

$$= 2\lambda\phi_1(10 + 2\mu\phi_2^1(1) + \int \lambda f\phi_1 dx + \int \mu f\phi_2 dx$$

$$= \lambda[2\phi_2^1(1) + \int f\phi_1 ds] + \mu[1\phi_2^1(1) + \int f\phi_2 dx]$$

$$= \lambda(f_1\phi_1) + \mu(f_1\phi_2)$$

Thus it is linear

(iii) Continuity

Let $\{\phi_n\}$ be a sequence in $D(\Omega)$ such that $\phi_n \xrightarrow{D} 0$

then $(f, \phi_n) = 2\phi_2^1(1) + \int f\phi_n dx$

But $\phi_2^1(1) \to 0$

and $\int f\phi_n\, dx \to 0$.

Thus $(f, \phi_n 0 \to 0$

Hence f is continuous.

Therefore f is a generalized function.

Ordinary function are called regular generalized functions. Other generalized functions are called singular generalized functions.

Generalized functions f and g are said to be equal in Ω if they are equal as functions on the set of basic functions; that is,

$$\forall \phi \in D(\Omega)$$

$$(f, \phi) = (g, \phi) \text{ and we write}$$

$$f = g \text{ in } \Omega.$$

$D'(\Omega)$ is the set of generalize functions specified in Ω.

Convergence in $D'(\Omega)$: A sequence of generalized function $f_1, f_2, \ldots$ in $D'(\Omega)$ converges to a generalized function $f \in D'(\Omega)$ if for any $\phi \in D(\Omega)$ $(f_k, \phi) \to (f, \phi)$ as

$k \to \infty$. And we write $f_k \to f$ in $D'(\Omega)$

This convergence is called weak convergence.

The set of generalized functions is linear

i.e for $f, g \in D'(\Omega)$ and $\lambda = \mu \in k$

$$(\lambda f + \mu g, \phi) = \lambda(f, \phi) + \mu(s, \phi),\ \phi \in D(\Omega)$$

(f, φ) is bilinear form (linear in f and linear in φ separately).

Definition: the linear set $D'(\Omega)$ together with convergence which it is equipped is called the space $D'(\Omega)$ of generalized functions

1.4 Support of Generalized Functions.

Generalized functions don't have values at separate points. We can speak about vanishing of a generalized functions in an open set.

$f \in D'(\Omega)$ vanish in $\Omega' \subset \Omega$ if its restriction to Ω' is a zero functional; that is, $(f, \phi) = 0\ \forall \phi \in D(\Omega')$.

If a generalized function in $D'(\Omega)$ vanishes in some neighborhood of every point of the open set Ω, then it vanishes in the whole set Ω. Suppose $f \in D'(\Omega)$, the union of the neighborhoods where $f = 0$ forms an open set Of which is called the zero set of the generalized function f. Furthermore Of is the largest open set in which f vanishes.

Definition: The support of a generalized function f is the complement O_f to Ω.

$$\text{supp } f = \Omega \backslash O_f.$$

Remark: If the support Of f from the set of generalized functions and ϕ from $D(\Omega)$ don't have any points in common then $(f, \phi) = 0$

1.5 Some Operation On Generalized Functions

1.5.1 Multiplication of generalized functions $F[\phi]$ with a c^{∞} function a(x):

$a(f,\phi) = (f,a\phi)$ (left side is defined by right side).

Example $a(\delta, \phi) = (\delta,a\phi) = a(0)\,\phi(0)$ or symbolically $a(x)\,\delta(x) = a(0)\,\delta(x)$, an important result.

<u>**Note**</u>: Multiplication of two singular generalized functions or a regular and a singular generalized functions may not be defined.

1.5.2 Addition of generalized functions:

$$(f + g,\phi) = (f,\phi) + (g,\phi).$$

1.5.3 Shifting Operation: $E_h(f,\phi) = (f,E_h\phi)$ where $E_{-h}\phi = \phi(x\text{-}h)$

Example: $E_h(\delta,\phi) = (\delta,E_h\phi) = \phi(-h)$ or symbolically

$$\int E_h\delta(x)\,\phi(x)dx = \int\delta(x+h)\,\phi(x)dx = \phi(-h)$$

2. Differentiation and Integration

2.1 Derivative of Generalized functions

Let $f \in c^k(\Omega)$. Then for all α $|\alpha| \le k$ and for any $\varphi \in D(\Omega)$ we have the following integration by parts formula.

$$(D^{\alpha}f, \varphi) = \int D^{\alpha}f(x)\,\varphi(x)dx$$
$$= (-1)^{|\alpha|}\int f(x)\,D^{\alpha}\varphi(x)dx$$
$$= (-1)^{|\alpha|}(f,D^{\alpha}\varphi).$$

Thus we define a generalized derivative $D^{\alpha}f$ of the generalized function f in $D'(\Omega)$ by the formula

$$(D^{\alpha}f, \varphi) = (-1)^{|\alpha|} (f, D^{\alpha}\varphi), \ \varphi \in D(\Omega)$$

since the operation $\varphi \to (-1)^{|\alpha|} D^{\alpha}\varphi$ is linear and continuous from $D(\Omega)$, the functional $D^{\alpha}f$ defined in equation above is a generalized function in $D'(\Omega)$. That is

i) $(D^{\alpha}f, \varphi)$ is real or complex number

ii) $D^{\alpha}f$ is a liner functional on $D(\Omega)$;

i.e $(D^{\alpha}f, \lambda\varphi + \mu\psi) = \lambda(D^{\alpha}f, \varphi) + \mu(D^{\alpha}f, \psi)$

$\varphi, \psi \in D$ & $\lambda, \mu \in C$

iii) $D^{\alpha}f$ is continuous functional on $D(\Omega)$

i.e if $\varphi_k \to \varphi$ as $k \to \infty$ in $D(\Omega)$, then

$(D^{\alpha}f, \varphi^{k}) \to (D^{\alpha}f, \varphi)$ as $k \to \infty$, $\varphi \in D(\Omega)$, in $D'(\Omega)$

Proof

ii) $(D^{\alpha}f, \lambda\varphi + \mu\psi)$

$= (-1)^{|\alpha|} (f, D^{\alpha}(\lambda\varphi + \mu\psi), \ \varphi, \psi \in D(\Omega)$ and $\lambda, \mu \in C$

$= (-1)^{|\alpha|} (f, \lambda D^{\alpha}\varphi + \mu D^{\alpha}\psi)$

$= (-1)^{|\alpha|} (f, \lambda D^{\alpha}\varphi) + (-1)^{|\alpha|} (f, \mu D^{\alpha}\psi)$

$= \lambda(-1)^{|\alpha|} (f, D^{\alpha}\varphi) + \mu(-1)^{|\alpha|} (f, D^{\alpha}\psi)$

$= \lambda(D^{\alpha}f, \varphi) + \mu(D^{\alpha}f, \psi)$

iii) $(D^{\alpha}f \ \varphi_k) = (-1)^{|\alpha|}(f, D^{\alpha}\varphi_k) \to (-1)^{|\alpha|} (f, D^{\alpha}\psi)$

$= (D^{\alpha}f, \varphi), \ \varphi \in D(\Omega)$ as $k \to \infty$

Thus we conclude that the functional in (1) is a Generalized function in $D'(\Omega)$.

in particular if $f = \delta$, then

$$(D^{\alpha}\delta, \varphi) = (-1)^{|\alpha|} D^{\alpha}\varphi(0), \ \varphi \in D(\Omega)$$

Example: Let $\theta(x)$ be Heaviside function such that

$$\theta(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

The generalized derivative of $\theta(x)$ is given by (δ, φ). To see this we write:

$$(\theta'(x), \varphi) = (-1)\,(\theta(x), \varphi'(x))$$

$$= -\int_{-\infty}^{\infty} \theta(x)\varphi'(x)dx$$

$$= -\int_{0}^{\infty} \theta(x)\varphi'(x)dx$$

$$= \varphi(x)\Big|_{0}^{\infty}$$

$$= \varphi(0)$$

$$= (\delta, \varphi)$$

Hence $(\theta'(x), \varphi) = (\delta, \varphi)$

$\Rightarrow \theta'(x) = \delta(x)$ in $D'(\Omega)$ in the generalized sense.

Properties

The following properties of the operation D^{α} of generalized function hold true.

a) The operation D^{α} is linear and continuous from $D'(\Omega)$ into $D'(\Omega)$

(i) Linearity

for all f and g in D′ and $\lambda, \mu \in k$

$$(D^{\alpha}(\lambda f + \mu g), \varphi) = (-1)^{|\alpha|} (\lambda f + \mu g, D^{\alpha}\varphi)$$
$$= (-1)^{|\alpha|} [(\lambda f, D^{\alpha}\varphi) + (\mu g, D^{\alpha}\varphi)]$$
$$= (-1)^{|\alpha|} (\lambda f, D^{\alpha}\varphi) + (-1)^{|\alpha|} (\mu g, D^{\alpha}\varphi)$$
$$= (-1)^{|\alpha|} \lambda(f, D^{\alpha}\varphi) + (-1)^{|\alpha|} \mu(g, D^{\alpha}\varphi)$$

(ii) Continuity

Suppose $f_k \to 0$ as $k \to \infty$ in $D'(\Omega)$.

Then for all $\varphi \in D(\Omega)$, we have

$D^{\alpha}(f_k, \varphi) = (-1)^{|\alpha|} (f_k, D^{\alpha}\varphi) \to 0$ as $k \to \infty$

This shows that

$$D^{\alpha} f_k \to 0 \text{ as } k \to \infty \text{ in } D'(\Omega)$$

b) If $f \in D'(\Omega)$ and $a \in c^{\infty}(\Omega)$, then the Leibniz formula for differentiation of a product $af \in D'(\Omega)$ holds. That is

$$D^{\alpha}(af) = \sum_{0 \le \beta \le \alpha} \binom{\alpha}{\beta} D^{\beta} a D^{\alpha - p} f \text{ in } D'(\Omega)$$

For example if $\alpha = (1, 0, 0, \ldots, 0)$, then

$$\frac{\partial(af)}{\partial x_1} = \frac{\partial af}{\partial x_1} + \frac{\partial af}{\partial x_i}$$

Indeed, for any φ in $D(\Omega)$ we have

$$\left(\frac{\partial(af)}{\partial x_1}, \varphi\right) = \mathbf{(-1)}^{|\alpha|} \left(af, \frac{\partial \varphi}{\partial x_1}\right) = \mathbf{-1} \left(f, \frac{a\partial \varphi}{\partial x_1}\right)$$

$$= \left(f, \frac{\partial(a\varphi)}{\partial x_1} - \frac{\partial a}{\partial x_1}\varphi \right)$$

$$= \left(f, \frac{\partial(a\varphi)}{\partial x_1} \right) + \left(f, \frac{\partial a}{\partial x_1}\varphi \right)$$

$$= \left(\frac{\partial f}{\partial x_1}, a\varphi \right) + \left(\frac{\partial a}{\partial x_1} f, \varphi \right)$$

$$= \left(a\frac{\partial f}{\partial x_1}, \varphi \right) + \left(\frac{\partial a}{\partial x_1} f, \varphi \right)$$

$$= \left(a\frac{\partial f}{\partial x_1} + \frac{\partial a}{\partial x_1} f, \varphi \right)$$

c) Any generalized function $f \in D'(\Omega)$ is infinitely differentiable in the generalized sense indeed, if $f \in D'(\Omega)$, it follows that

$$\frac{\partial f}{\partial x_i} \in D'(\Omega). \text{ That is}$$

$$(D^\alpha f, \varphi) = (-1)^{|\alpha|}(f, D^\alpha \varphi) \in D'(\Omega)$$

This is to mean that the derivative of a generalized function is again a generalized function

Now let $\frac{\partial f}{\partial x_i} = g$, the $\frac{\partial f}{\partial x_i} \in D'(\Omega)$

Thus $f \in D'(\Omega)$ is infinitely differentiable

d) If a generalized function $f = 0$, for all x in Ω_1, the $D^\alpha f = 0\ \forall_x \in \Omega$, so that

supp $D^\alpha f \subset$ supp f

indeed, if $f \in D'(\Omega)$, then for all $\varphi \in D(\Omega_f)$, we have $D^\alpha \varphi \in D(\Omega_f)$

and $(D^{\alpha}f, \varphi) = (-1)^{|\alpha|}(f, D^{\alpha}\varphi) = \Omega$, it follows that $D^{\alpha}\varphi \in D(\Omega_{D^{\alpha}f})$

Then combining the above two equations we have

$$D(\Omega_f) \subset D(\Omega_{D^{\alpha}f}) \text{ and this}$$

implies $\Omega_f \subset \Omega_{D^{\alpha}f}$

Hence supp $D^{\alpha}f \subset$ supp f.

e) Generalized derivative and ordinary derivative coincide on continuous regular generalized functions.

2.2 Integration of Generalized Functions

Definition: We say (g, φ) is an integral of (f, φ) if

$$(g', \varphi) = (f, \varphi)$$

Symbolically $\int f(x)dx = g(x)$

Example

$$\begin{aligned}(\theta'(x), \varphi(x)) &= -(\theta(x), \varphi'(x)) \\ &= \varphi(0) \\ &= (\delta, \varphi)\end{aligned}$$

And written:

$$\int \delta(x)\, dx = \theta(x)$$

Let (k, ϕ) be a generalized function such that

$$(k', \phi) = 0 \text{ for all } \phi \in D$$

Then if (g, ϕ) is integral of (f, ϕ), it follows that $((g + k), \phi) = (g, \phi) + (k, \phi)$ is also an integral of (f, ϕ).

Some important Results of Generalized Function Theory.

- Sequences of Generalized functions: A sequence (f_n,ϕ) of GF_s is convergent if $\forall\phi\in D$, the sequence of numbers (f, ϕ) is convergent.
- Theorem: The spaces D′ is complete. This theorem implies that a convergent sequence of GF_s gives a GF.
- Exchange of Limit process: We can exchange limit processes when we are dealing with GF_s. This result is very important in applications.

Examples:

$$\frac{\bar{\partial}^2 f}{\partial x_i \partial x_j} = \frac{\bar{\partial}^2 f}{\partial x_i \partial x_j}$$

$$\sum_i \int_\Omega \cdots = \int_\Omega \sum_i \cdots$$

$D(\Omega_f) \subset D(\Omega_{D^\alpha f})$ and this implies

$\Omega_f \subset \Omega_{D^\alpha f}$

Hence $\text{supp}\, D^\alpha f \subset \text{supp}\, f$.

3. The Direct Product and Convolution of Generalized Functions

3.1 Definition of the direct product

Let $f(x)$ and $g(y)$ be locally summable functions in the open sets $\Omega_1 \subset \Re^n$ and $\Omega_2 \subset \Re^m$ respectively. Then the function $f(x)\, g(y)$ will also be locally summable function in $\Omega_1 \times \Omega_2$.

Definition: The direct product of the generalized functions $f(x) \in D'(\Omega_1)$ and $g(y) \in D'(\Omega_2)$ is a generalized function $f(x).\ g(y)$ in $D'(\Omega_1 \times \Omega_2)$ which is given by the formula. $\forall \varphi \in D(\Omega_1 \times \Omega_2)$

$$(f(x) \times g(y), \varphi(x,y)) = (f(x), (g(y), \varphi(x,y))) \quad (1)$$

and $$(g(y) \times f(x), \varphi(x,y)) = (g(y), (f(x), \varphi(x,y))) \quad (2)$$

Let us see whether this definition is properly posed, i.e whether the right hand sides of equation (1) defines a continuous linear functional on $D(\Omega_1 \times \Omega_2)$ or not.

We first consider the following Lemma.

Lemma: For any $g \in D'(\Omega_2)$ and $\varphi \in D(\Omega_1 \times \Omega_2)$, the function $\psi(x) = (g(y), \varphi(x,y))$ belongs to $D(\Omega_1)$ and

$$D^{\alpha}\psi(x) = (g(y), D_x^{\alpha} \varphi(x,y)) \quad \forall \alpha. \quad (3)$$

Further, if $\varphi_k \to 0$ as $k \to \infty$ in $D(\Omega_1 \times \Omega_2)$, then

$$\psi_k(x) = (g(y), \varphi_k(x,y)) \to 0,\ k \to \infty \quad \text{in} \quad D(\Omega_1).$$

Let us return to the definition of the direct product. According to the Lemma $\psi(x) = (g(y), \varphi(x,y)) \in D(\Omega_1)$ for all $\varphi \in D(\Omega_1 \times \Omega_2)$. Therefore the right hand side of equation (1), which is (f, ψ), has meaning for every generalized functions f and g, and hence, defines a functional on $D(\Omega_1 \times \Omega_2)$. More over, the linearity of this functional follows from the linearity of the functionals f and g.

We will now show that the functional we have just constructed is continuous on $D(\Omega_1 \times \Omega_2)$.

Suppose that $\varphi_k \to 0$ as $k \to \infty$ in $D(\Omega_1 \times \Omega_2)$. Then according to the Lemma, $(g(y), \varphi_k(x,y)) \to 0,\ k \to \infty$ in $D(\Omega_1)$, and therefore by virtue of the fact that the functional f is continuous on $D(\Omega_1)$, we have $(f(x), (g(y), \varphi_k(x,y))) \to 0$ as $k \to \infty$. Thus the right hand side of equation (1) is continuous. Therefore we have shown that the functional $f(x) \times g(y) \in D'(\Omega_1 \times \Omega_2)$, i.e it is a generalized function.

3.2 Properties of Direct Product

i) **Commutativity:** For any two generalized functions $f(x) \in D'(\Omega_1)$ and $g(y) \in D'(\Omega_2)$ we have

$$f(x) \times g(y) = g(y) \times f(x) \qquad (3)$$

In deed; on the basic functions $\varphi \in D(\Omega_1 \times \Omega_2)$ of the form

$$\varphi(x,y) = \sum_{i=1}^{N} ui(x)\, vi(y) \quad , \; ui \in D(\Omega_1),\; vi \in D(\Omega_2) \qquad (4)$$

We have, $(f(x) \times g(y), \varphi) = (f(x), (g(y), \varphi(x,y)))$

$$= \int f(x)\,.\, g(y) \sum_{i=1}^{N} ui(x)\, vi(y)\, dydx$$

$$= \int f(x)\, ui(x) \sum_{i=1}^{N} g(y)\, vi(y) dydx$$

$$= \sum_{i=1}^{N} \int f(x)\, ui(x)\, g(y)\, vi(y)\, dydx$$

$$= \sum_{i=1}^{N} \int_{\Omega_1} f(x)\, ui(x) dx \int_{\Omega_2} g(y)\, vi(y) dy$$

$$= \sum_{i=1}^{N} (f, ui)\, (g, vi).$$

Similarly

$$(g(y) \times f(x), \varphi(x,y)) = \sum_{i=1}^{N} (f, ui)\, (g, vi)$$

i.e $(f(x) \times g(y), \varphi(x,y)) = \sum_{i=1}^{N} (f, ui)\, (g, vi) = (g(y) \times f(x), \varphi(x,y)).$

Hence the direct product Operation is commutative.

ii) **Associativity** : If $f \in D'(\Omega_1)$, $g \in D'(\Omega_2)$ and $h \in D'(\Omega_3)$ then

$$[\, f(x) \times g(y)\,] \times h(z) = f(x) \times [\, g(y) \times h(z)\,] \qquad (5)$$

Proof:

Suppose $\varphi \in D(\Omega_1 \times \Omega_2 \times \Omega_3)$, then

$$([f(x) \times g(y)] \times h(z), \varphi) = (f(x) \times g(y), (h(z), \varphi(x,y,z)))$$
$$= (f(x), (g(y), (h(z), \varphi(x,y,z))))$$
$$= (f(x), (g(y) \times h(z), \varphi(x,y,z)))$$
$$= (f(x) \times [g(y) \times h(z)], \varphi).$$

Then by taking into account the commutativity and associativity of the operation of the direct product we will write the following

$$(f \times g) \times h = f \times g \times h = f \times (g \times h).$$

iii) **Differentiation:** If $f(x) \in D'(\Omega_1)$ and $g(y) \in D'(\Omega_2)$, then

$$D_x^{\alpha}[f(x) \times g(y)] = D^{\alpha} f(x) \times g(y)$$

and $$D_y^{\alpha}[f(x) \times g(y)] = f(x) \times D^{\alpha} g(y) \qquad (6)$$

Proof:

If $\varphi \in D(\Omega_1 \times \Omega_2)$, then

$$D_x^{\alpha}[f(x) \times g(y), \varphi] = (-1)^{\alpha}(f(x) \times g(y), D_x^{\alpha}\varphi)$$
$$= (-1)^{|\alpha|}(g(y), (f(x), D_x^{\alpha}\varphi(x,y)))$$
$$= (g(y), (D^{\alpha} f(x), \varphi))$$
$$= (D^{\alpha} f(x) \times g(y), \varphi).$$

iv, **Multiplication:** Let $f \in D'(\Omega_1)$, $g \in D'(\Omega_2)$ and $\varphi \in (\Omega_1 \times \Omega_2)$

If $a \in C^{\infty}(\Omega_1)$ then

$$a(x)[f(x) \times g(y)] = a(x) f(x) \times g(y) \qquad (7)$$

Proof:

$$(a(x)\,[\,f(x) \times g(y)\,],\ \varphi\,) = (f(x) \times g(y),\ a(x)\,\varphi(x,y))$$
$$= (f(x),\ (g(y),\ a(x)\,\varphi(x,y)))$$
$$= (f(x),\ a(x)\,(g(y),\ \varphi(x,y)))$$
$$= (a(x)\,f(x),\ (g(y),\ \varphi(x,y)))$$
$$= (a(x)\,f(x) \times g(y),\ \varphi(x,y)).$$

Similarly

$$a(x)\,b(y)\,[\,f(x) \times g(y)\,] = [\,a(x)\,f(x)\,] \times [\,b(y)\,g(y)\,].$$

v. **Translation:** Let $f(x) \in D'(\Omega_1)$, $g(y) \in D'(\Omega_2)$ and $\varphi \in D(\Omega_1 \times \Omega_2)$ then

$$(f \times g)\,(x + h,\ y) = f(x + h) \times g(y) \qquad (8)$$

Proof:

$$((\,f \times g\,)\,(x + h,\ y),\ \varphi) = (f(x) \times g(y),\ \varphi\,(x - h,\ y\,))$$
$$= (g(y),\ (f(x),\ \varphi\,(\,(x\text{-}h),\ y\,)))$$
$$= (g(y),\ (f\,(x + h),\ \varphi(x,y)))$$
$$= f(x + h) \times g(y)$$

Similarly

$$(f \times g)\,(x + h,\ y + k\,) = f(x + h) \times g(y + k).$$

vi. **Sup** $(f \times g)$ = Supp $f \times$ supp g (9)

Proof:

Suppose $(x_o\,,\ y_o) \in$ supp $f \times$ supp g and $U(x_o,\ y_o)$ is the neighbourhood of the point $(x_o\,,\ y_o)$ lying in $\Omega_1 \times \Omega_2$. Then there exist neighbourhood U_1 of x_o and U_2 of y_o such that $U_1 \times U_2 \subset U(x_o\,,\ y_o)$. From the definition of support, there are functions $\varphi_1 \in D(U_1)$ and $\varphi_2 \in D(U_2)$ such that $(f,\ \varphi_1) \neq 0$ and $(g,\ \varphi_2) \neq 0$. Then we have $(f \times g,\ \varphi_1\varphi_2) = (f,\ \varphi_1)\ (g,\ \varphi_2) \neq 0$. Therefore $(x_o,\ y_o) \in$ supp $(f \times g)$, and this implies supp $f \times$ supp $g \subset$ supp $(f \times g)$.

To prove the other inclusion, consider $\varphi \in D(\Omega_1 \times \Omega_2)$ such that supp $\varphi \subset (\Omega_1 \times \Omega_2) \setminus (\text{supp } f \times \text{supp } g)$. Then there is a neighborhood U of the set supp f such that for every $x \in U$, supp $\varphi(x,y) \subset \Omega_2 \setminus \text{supp } g$.

Therefore $\psi(x) = (g(y), \varphi(x,y)) = 0$, $x \in U$ and hence supp $\psi \cap \text{supp } f = \phi$ and so $(f \times g, \varphi) = (f, \psi) = 0$.

Thus the zero set $\Omega_{f \times g}$ contains $(\Omega_1 \times \Omega_2) \setminus (\text{supp } f \times \text{supp } g)$, from this we have supp $(f \times g) \subset \text{supp } f \times \text{supp } g$.

Therefore

$$\text{Supp } (f \times g) = \text{Supp } f \times \text{supp } g.$$

3.3 Convolution of Generalized Functions.

Definition: Let f(x) and g(x) be functions locally summable in $\Re^n$, where the function

$$h(x) = \int |g(y) f(x-y)| dy$$

is locally summable in $\Re^n$. The function

$$(f * g)(x) = \int f(y) g(x-y) dy$$

$$= \int g(y) f(x-y) dy = (g * f)(x) \qquad (10)$$

is known as the convolution f * g of these functions.

Function given by (10) is locally summable in $\Re^n$ and therefore defines a (regular) generalized functions acting on the basic function $\varphi \in D(\Re^n)$ according to the rule

$$(f * g, \varphi) = \int (f * g)(\xi) \varphi(\xi) d\xi$$

$$= \int [\int g(y) f(\xi - y) dy] \varphi(\xi) d\xi$$

$$= \int g(y) [\int f(\xi - y) \varphi(\xi) d\xi] dy$$

$$= \int g(y) [\int f(x) \varphi(x + y) dx] dy$$

(by virtue of Fubini's theorem), that is,

$$(f * g, \varphi) = \int f(x) g(y) \varphi(x + y)\, dx\, dy, \quad \varphi \in D(\Re^n) \qquad (11)$$

We shall note three cases in which the condition of local summability of the function h(x) is satisfied and so the convolution f * g defined by (10) exists.

1. One of the functions f or g has compact support, for instance supp $g \subset U_{R1}$:

$$\int_{U_R} h(x)dx = \int_{UR_1} | g(y) | \int_{U_R} | f(x-y) | dxdy$$

$$\leq \int_{UR_1} |g(y)| dy \int_{UR+R_1} |f(\xi)| d\xi < \infty$$

2. The function f and g become zero when $x < 0$, $(n = 1)$:

$$\int_{-R}^{R} h(x)dx = \int_0^R\int_0^R | g(y) | | f(x-y) | dydx$$

$$= \int_0^R | g(y) | \int_0^R | f(x-y) | dxdy$$

$$\leq \int_0^R |g(y)| dy \int_0^R |f(\xi)| d\xi < \infty$$

3. The function f and g are integrable over $\Re^n$:

$$\int h(x)dx = \int | g(y) | \int | f(x-y)| dxdy$$

$$= \int | g(y) | dy \int | f(\xi) | d\xi < \infty$$

In this case the convolution f * g is integrable over $\Re^n$.

We shall say that the sequence $\{\eta_k\}$ of basic functions belonging to $D(\Re^n)$ converges to 1 in $\Re^n$ if for any compact K there is a number N such that $\eta_k(x) = 1$ where $x \in K$, $k \geq N$, and the η_k are uniformly bounded in $\Re^n$ together with all their derivatives $| D^\alpha \eta_k(x) | < C_\alpha$ for $x \in \Re^n$, $k = 1,2, \ldots$. And such sequences always exist: for example, $\eta_k(x) = \eta(x/k)$, where $\eta \in D$, $\eta(x) = 1$ in U_1.

Remark:

1. We note that equation (11) may be written in the form

$$(f * g, \varphi) = \lim_{k \to \infty} (f(x) \times g(x), \eta_k(x;y)\, \varphi(x+y)), \varphi \in D(\Re^n) \qquad (12)$$

where $\eta_k(x;y)$ for k = 1,2, …, is any sequence converging to 1 in $\Re^{2n}$.

2. The functional

$$(f * g, \varphi) = (f(x) \times g(y), \varphi(x+y))$$
$$= \lim_{k \to \infty} (f(x) \times g(y), \eta_k (x;y)\, \varphi(x+y)), \varphi \in D(\Re^n)$$

is known as the convolution f * g if it is continuous in $D(\Re^n)$.

3.4 <u>Properties of Convolutions</u>

The convolution operation possesses a number of properties. Some of them are given here:

a) Commutativity:

If f * g exists then g * f will also exist and

$$f * g = g * f \qquad (13)$$

The statement follows from the definition of a convolution and from the commutativity of the direct product:

$$(f * g, \varphi) = \lim_{k \to \infty} (f(x) \times g(y), \eta_k (x; y)\, \varphi(x+y))$$
$$= \lim_{k \to \infty} (g(y) \times f(x), \eta_k (k;y)\, \varphi(x+y))$$
$$= (g * f, \varphi) , \varphi \in D.$$

b) Convolution with the δ function:

The convolution of any generalized function f ∈ D′ with the δ function exists and is equal to f

$$\text{i.e} \quad f * \delta = \delta * f = f \qquad (14)$$

In other words any generalized function f may be expanded interms of the δ function, which formally expressed as

$$f(x) = \int f(\xi)\, \delta(x-\xi)\, d\xi.$$

Let us show that $f * \delta = f$.

Suppose $\varphi \in D(\Re^n)$ and $\{\eta_k\}$ be a sequence of functions in $D(\Re^{2n})$ that converges to 1 in $\Re^{2n}$. then

$$\eta_k(x; 0)\, \varphi(x) \to \varphi,\ k \to \infty \text{ in } D(\Re^n),$$

and so
$$(f * \delta, \varphi) = \lim_{k \to \infty} (f(x) \times \delta(y), \eta_k(x; y)\, \varphi(x + y))$$

$$= \lim_{k \to \infty} (f(x), \eta_k(x; 0)\, \varphi(x))$$

$$= (f, \varphi).$$

Thus $f * \delta = f$.

c) The shift of a convolution.

If the convolution $f * g$ exists then so also does the convolution $f(x + h) * g(x)$. and for all $h \in \Re^n$ and $f, g \in D'$ we've

$$f(x + h) * g(x) = (f * g)\ (x + h) \qquad (15)$$

i.e the convolution is invariant under the translation.

Indeed, let $\{\eta_k\}$ be a sequence of functions in $D(\Re^{2n})$ such that $\eta_k \to 1$ in $\Re^n$ as $k \to \infty$; then for any $h \in \Re^n$, $\eta_k(x - h, y) \to 1$, $k \to \infty$ in $\Re^{2n}$.

Now using the definition of shift and convolution we obtain for all $\varphi \in D(\Re^n)$, the following ;

$$((f * g)(x + h), \varphi) = (f * g, \varphi(x - h))$$

$$= \lim_{k \to \infty} (f(x) \times g(y), \eta_k(x - h, y)\, \varphi(x - h + y))$$

$$= \lim_{k \to \infty} (f(x + h) \times g(y), \eta_k(x; y)\, \varphi(x + y))$$

$$= (f(x + h) * g, \varphi)$$

d) Differentiating a convolution

For any two generalized functions f and g in D' the derivative of $f * g$ if it exists is given by the following equality;

$$(D^\alpha f * g) = D^\alpha (f * g) = f * D^\alpha g \qquad (16)$$

we prove this assertion for the 1st derivative, i.e D_j^1, j = 1, 2, 3, …, n.

Let $\varphi \in D(\Re^n)$ and let $\{\eta_k\} \in D(\Re^{2n})$ such that $\eta_k \to 1$. $k \to \infty$ in $\Re^{2n}$;

then the sequence $\{\eta_k + D_j \eta_k\} \to 1$ in $\Re^{2n}$, $k \to \infty$. then

$$(D_j^1 (f * g), \varphi) = (-1)^{|\alpha|} (f * g, D_j^1 \varphi), \quad |\alpha| = 1$$

$$= -\lim_{k\to\infty} (f(x) \times g(y), \eta_k \frac{\partial(x+y)}{\partial x_j})$$

$$= -\lim_{k\to\infty} \left(f(x) x g(y), \frac{\partial}{\partial x_j}[\eta_k \varphi(x+y)] \frac{\partial_{\eta_k}}{\partial x_j} \right)$$

$$= \lim_{k\to\infty} \left(\frac{\partial}{\partial x_j} [f(x) x g(y)], \eta_k \varphi(x+y) \right)$$

$$+ \lim_{k\to\infty} (f(x) \times g(y), \left[\eta_k + \frac{\partial \eta_k}{\partial x_j} \right] \varphi(x+y))$$

$$- \lim_{k\to\infty} (f(x) \times g(y), \eta_k \varphi (x+y))$$

$$= \lim_{k\to\infty} (D_j f(x) \times g(y), \eta_k \varphi(x+y)) + (f * g, \varphi) - (f * g, \varphi).$$

$$= \lim_{k\to\infty} (D_j f(x) \times g(y), \eta_k \varphi (x+y))$$

$$= (D_j f * g, \varphi).$$

Thus from the commutativity property and the last result,

$$(D_j (f * g), \varphi) = (D_j f * g, \varphi) \text{ we get}$$

$$(D_j (f * g), \varphi) = (D_j (g * f), \varphi)$$

$$= (D_j g * f, \varphi)$$

More over from the property of convolution with the δ- function and the property of differentiating convolution we get

$$D^\alpha f = D^\alpha \delta * f = \delta * D^\alpha f, \quad f \in D'.$$

e) Associativity of a convolution

In general, the operation of convolution is not associative. For example

$$(1 * \delta') * \theta = 1' * \theta = 0 * \theta = 0, \text{ but}$$

$$1 * (\delta' * \theta) = 1 * \delta = 1$$

However if $f * g$, $g * h$, $f * h$ and $f * g * h$ exists so does $(f * g) * h$, $f * (g * h)$ and we have

$$(f * g) * h = f * g * h = f * (g * h) \qquad (17)$$

which in this case means the convolution is associative.

4. Generalized Functions Of Slow Growth

One of the most effective ways of solving problems in mathematical physics is to employ the Fourier transform method. We will set out the Fourier transform theory for the So-called generalized functions of slow growth (tempered distribution) in the next section. For this reason here we will deal with the space of generalized functions of slow growth.

4.1 The space S of basic (rapidly diminishing) functions

4.1.1 Definition of the space S, Norm and convergence in S

Definition : All function infinitely differentiable in $\Re^n$ that decreases together with all their derivatives as $|x| \to \infty$ faster than any power of $|x|^{-1}$ is known as basic functions of slow growth and denoted by $S = S(\Re^n)$

For example the function $f(x) = e^{-|x|^2}$ belongs to the set S;

Let $m \in N_0$, $\alpha \in N_0^n$ where $\alpha_j \in N$ and $|x| = \sqrt{x_1^2 + ... + x_n^2}$, then we introduce Norm in the space S by

$$P_{m,\alpha}(\varphi) = \sup_{x \in \Re^n} [\, (1 + |x|)^m \, | D^{\alpha}\varphi(x)| \,], \varphi \in S \qquad (1)$$

Definition: Let φ_k, $\varphi \in S$, then $\varphi_k \to \varphi$ in S if $\forall \alpha \in N_o^n$ and $\forall m \in N_o$ $P_{m,\alpha}(\varphi_k - \varphi) \to 0$ as $k \to \infty$.

Thus the set S together with such convergence is called the space of basic functions of slow growth and has the same symbolism as the set of slow growth $k \to \infty$.

Remark: $D(\Re^n) \subset S(\Re^n) \subset W_p^l(\Re^n)$.

Where $W_p^l(\Re^n)$ is known as the sobelov space and it is defined

$$W_p^l(\Re^n) = \{ f: \| f \|_{W_p^l(\Re^n)} = \sum_{0 \le |\alpha| \le l} \| D^\alpha f \|_{L_p} < \infty \}$$

If $\ell = 0$ in the Sobelov space we have the relation:

$W_p^0(\Re^n) = L_p(\Re^n)$ and from this we get $S(\Re^n) \subset L_p(\Re^n)$.

In particular if $p = 1$, we've $S \subset L_1$.

4.1.2 Operators in the space S

Let $A: S \to S$ be an operator. We say that the operator A is linear if

i, $\forall \varphi \in S \Rightarrow A\varphi \in S$

ii, $A(\alpha_1 \varphi_1 + \alpha_2 \varphi_2) = \alpha_1 A\varphi_1 + \alpha_2 A\varphi_2$, $\varphi_1, \varphi_2 \in S$ & $\alpha_1, \alpha_2 \in \mathbb{C}$ and it is continuous if $\varphi_k \to \varphi$, $k \to \infty$ in S, then $A\varphi_k \to A\varphi$, $k \to \infty$ in S.

Lemma: A linear Operator $A: S \to S$ is continuous if $\forall$ (m, α), $m \in N_0, \alpha \in N_0^n$, there exists a finite set of pairs (m', α'), $m' \in N_0$, $\alpha' \in N_0^n$ and a constants $C_{m,\alpha} > 0$ such that

$$P_{m,\alpha}(A\varphi) \le C_{m,\alpha} \sum_{(m',\alpha') \in N_{m,\alpha}} P_{m',\alpha}(\varphi) \qquad \forall \varphi \in S \qquad (2)$$

Proof: Let $\varphi_k \to \varphi$ in S, $(k \to \infty)$. We want to show that A is continuous (i.e $A\varphi_k \to A\varphi$ as $k \to \infty$). For all pairs (m, α), $P_{m,\alpha}(A\varphi_k - A\varphi) \to 0$ as $k \to \infty$.

$P_{m,\alpha}(A\varphi_k - A\varphi) = P_{m,\alpha}(A(\varphi_k - \varphi))$, since A is linear. But $\varphi_k \to \varphi$, $(k \to \infty)$, then

$P_{m,\alpha}(A\varphi_k - A\varphi) = P_{m,\alpha}(A(\varphi_k - \varphi)) \to 0$

So there exists $C_{m,\alpha} > 0$, constant such that

$$P_{m,\alpha}(A\varphi_k - A\varphi) \le C_{m,\alpha} \sum_{(m',\alpha') \in N_{m,\alpha}} P_{m',\alpha}(\varphi_k - \varphi) \to 0$$ Therefore $A\varphi_k \to A\varphi$ in S as $k \to \infty$. i.e A is continuous.

Example: Let $D^{\beta}: S \rightarrow S$ $\beta \in N_0^n$ be an operation of differentiation Then D^{β} is linear and continuous from S to S.

Indeed, for all $\varphi \in S$ we have $D^{\beta}\varphi \in S$ and $D^{\beta}(\alpha_1\varphi_1 + \alpha_2\varphi_2) = \alpha_1 D^{\beta}\varphi_1 + \alpha_2 D^{\beta}\varphi_2$, $\varphi_1, \varphi_2 \in S$. Thus $D^{\beta}: S \rightarrow S$ is linear.

and $P_{m,\alpha}(D^{\beta}\varphi) \leq C_{m,\alpha} \sum_{(m',\alpha') \in Nm,\alpha} P_{m',\alpha'}(\varphi)$ $\varphi, \in S$.

Since $D^{\alpha}(D^{\beta}\varphi) = D^{\alpha+\beta}\varphi$, we have,

$$P_{m,\alpha}(D^{\beta}\varphi) = \sup_{x \in \Re^n} [(1 + |x|)^m |D^{\alpha+\beta}\varphi|] \leq P_{m,\alpha+\beta}(\varphi), \ \forall \varphi \in S.$$

Hence $D^{\beta}: S \rightarrow S$ is continuous.

Definition: We say that λ is from the set of multipliers $\theta_M(\Re^n)$ if the following conditions are satisfied

i, $\lambda \in C^{\infty}(\Re^n)$

ii, $\forall \beta \in N_0^n$, there exists a constant $C_{\beta} > 0$ and $m_{\beta} \in N_0^n$ such that

$$|D^{\beta}\lambda(x)| \leq C_{\beta}(1 + |x|)^{m_{\beta}}, \ x \in \Re^n \quad (3)$$

Example: Let $M_{\lambda}: S \rightarrow S$ such that $M_{\lambda}\varphi = \lambda\varphi$, $\forall \varphi \in S$, $\lambda \in \theta_M(\Re^n)$. Assume that $|D^{\beta}\lambda(x)| \leq C_{\beta}(1 + |x|)^{m_{\beta}}$ then M_{λ} is continuous.

Indeed, since $M_{\lambda}\varphi = \lambda\varphi$, we have

$$P_{m,\alpha}(\lambda\varphi) = \sup_{x \in \Re^n} [(1 + |x|)^m |D^{\alpha}(\lambda\varphi)|]$$

$$\leq \sup_{x \in \Re^n} [(1 + |x|)^m \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |D^{\beta}\varphi D^{\alpha-\beta}\lambda|$$ Leibniz rule of differentiation.

But $|D^{\alpha-\beta}\lambda| \leq C_N(1 + |x|)^N$, $x \in \Re^n$ and N is the smallest integer not less than the maximum of m_{β}. From this it follows that

$$P_{m,\alpha}(\lambda\varphi) \leq C_{m,\alpha} \sup_{x \in \Re^n} [(1 + |x|)^{m+N} |D^{\alpha}\varphi|],$$

$$= C_{m,\alpha} \sum_{m,\alpha \in N_{m'\alpha'}} P_{m,\alpha}(\varphi)$$

Hence, $P_{m.\alpha}(M_\lambda\varphi) = P_{m.\alpha}(\lambda\varphi) \le C_{m.\alpha} \sum_{m,\alpha\in Nm',\alpha'} P_{m',\alpha'}(\varphi)$ which shows that M_λ is continuous

Remark: Multiplication by an infinitely differentiable function may take one out side the domain of S. For example

If $U(x) = e^{|x|^2} \in C^\infty$ but not in S and $\varphi(x) = e^{-|x|^2} \in S \subset C^\infty$, then

$U(x) . \varphi(x) = e^{|x|^2} . e^{-|x|^2} = 1$ which is not in S.

But for a function $U(x) \in \theta_M(\Re^n)$, the product $U(x). \varphi(x)$ is again in S.

4.1.3 <u>Embedding of $S(\Re^n)$ in $W_p^l(\Re^n)$, $1 \le p \le \infty$</u>

The space $S(\Re^n)$ is said to be embedded in $W_p^l(\Re^n)$ if

i) $\forall\varphi\in S(\Re^n)$, $\varphi\in W_p^l(\Re^n)$ i.e $S(\Re^n) \subset W_p^l(\Re^n)$

ii) $\|\varphi\|_{W_p^l(\Re^n)} \le C \|\varphi\|_{S(\Re^n)}$, where C is a constant. We verify this as follows.

Let $\varphi \in S(\Re^n)$,

Fix $m > n/p$, where m is taken from the definition of norm of φ i.e

$$\|\varphi\| = P_{m.\alpha}(\varphi) = \sup_{x\in\Re^n} (1 + |x|)^m |D^\alpha\varphi|, \varphi \in S(\Re^n).$$

$$\forall\alpha, |\alpha| \le \ell \ \& \ m \in N_0.$$

Now, $\|\varphi\|_{S(\Re^n)} = P_{m.\alpha}(\varphi) = \sup_{x\in\Re^n} (1 + |x|)^m |D^\alpha\varphi)|$.

This implies $P_{m.\alpha}(\varphi) \ge (1 + |x|)^m |D^\alpha\varphi|$

$$|D^\alpha\varphi| \le P_{m.\alpha}(\varphi)(1 + |x|)^{-m}$$

$$\left(\int_{\Re^n} |D^\alpha\varphi|^p dx\right)^{\frac{1}{p}} \le P_{m.\alpha}(\varphi)\left(\int_{\Re^n} (1 + |x|)^{-mp} dx\right)^{\frac{1}{p}}$$

$$\|D^\alpha\varphi\|_p \le C_n P_{m.\alpha}(\varphi) \left(\int_0^\infty \frac{r^{n-1}}{(1+r)^{mp}} dr\right)^{\frac{1}{p}} < \infty, \ \forall mp > n.$$

Put $C'_n = C_n \left(\int_0^{\infty} \frac{r^{n-1}}{(1+r)^{mp}} dr \right)^{\frac{1}{p}}$; it follow that $\| D^{\alpha}\varphi \|_p \leq C'_n P_{m,\alpha}(\varphi) = C'_n \| \varphi \|_s$

But if $\varphi \in W_p^l(\Re^n)$, ... (a)

then $\| \varphi \|_{W_p^l(\Re^n)} \leq \| D^{\alpha}\varphi \|_p \leq \sum_{0 \leq |\alpha| \leq l} C'_n P_{m,\alpha}(\varphi)$

this implies $\| \varphi \|_{W_p^l(\Re^n)} \leq C \| \varphi \|_{S(\Re^n)}$ .... (b)

where $C = \sum_{0 \leq |\alpha| \leq l} C'_n$ and $\| \varphi \|_{S(\Re^n)} = P_{m,\alpha}$

Thus from (a) and (b) we get that $S(\Re^n)$ is embedded in $W_p^l(\Re^n)$.

4.2 The Space S′ of Generalized Functions of Slow Growth

4.2.1 The definition the space S′

Definition: A continuous linear functional on the space S of basic functions is called generalized functions of slow growth on $\Re^n$ (Temperd distributions).

We denote the set of generalized functions of slow growth by $S' = S'(\Re^n)$. The value of the functional f on the basic function $\varphi \in S$ is given by (f, φ). That is

i) there corresponds a complex valued number to (f, φ)

ii) (f, φ) is linear functional

that is $\forall \alpha_1, \alpha_2 \in C$ and $\varphi_1, \varphi_2 \in S$,

$$(f, \alpha_1 \varphi_1 + \alpha_2 \varphi_2) = \alpha_1 (f, \varphi_1) + \alpha_2 (f, \varphi_2)$$

iii) (f, φ) is continuous functional that is if $\varphi_k \to \varphi$, $k \to \infty$ in S, then $(f, \varphi_k) \to (f, \varphi)$, $k \to \infty$ in S′, $\varphi \in S$.

We are now in position to define convergence in S′.

Definition: A sequence $\{f_k\}$ of generalized functions from S′ converges to a generalized function f in S′ if for any basic function $\varphi \in S$,

$(f_k, \varphi) \to (f, \varphi)$, $k \to \infty$ in S′ and we write

$f_k \to f$, $k \to \infty$ in S′.

A convergence of this type is called weak convergence and a linear set S′ equipped with this convergence is called the space of generalized functions of slow growth, $S' = S'(\Re^n)$.

Remark: $S' \subset D'$ and from the definition of convergence it follows that convergence in S′ implies convergence in D′.

Indeed, if $f \in S'$, then $f \in D'$, since $D \subset S$ and convergence in D implies convergence in S. Further more, if $f_k \to 0$ as $k \to \infty$ in S′, then $(f_k, \varphi) \to 0$ as $k \to \infty$ for all $\varphi \in D \subset S$ and hence $f_k \to 0$ as $k \to \infty$ in D′.

Lemma: A linear functional f: $S' \to S'$ is continuous if and only if there exists C_f, constant and $N_f = \{ (m,\alpha) : m \in N., \alpha \in N_0^n \}$ finite such that

$$| (f, \varphi) | \le C_f \sum_{(m,\alpha)\in N_f} P_{m,\alpha}(\varphi) \tag{4}$$

where $\sum_{(m,\alpha)\in N_f} P_{m,\alpha}(\varphi) = \sup_{x\in\Re^n} [(1 + |x|)^m | D^\alpha \varphi |]$

Proof: ($\Rightarrow$) Suppose f is continuous in S′, we need to show $| (f, \varphi) | \le C_f P_{m,\alpha}(\varphi_k)$

Assume the contrary; then there is a sequence of functions $\{\varphi_k\}$, k = 1, 2, ... in S

Such that $| (f, \varphi_k) | \ge k \sum_{(m,\alpha)\in N_f} P_{m,k}(\varphi_k)$

But the sequence $\psi_k = \dfrac{\varphi_k(x)}{\sqrt{k} \sum_{(m,k)\in N_f} P_{m,k}(\varphi_k)} \to 0, k \to \infty$ in S

Since $\forall_k \ge |\beta|$, we have

$$|D^\beta \psi_k(x)| = \left| D^\beta \frac{\varphi_k(x)}{\sqrt{k} \sum_{(m,k)\in N_f} P_{m,k}(\varphi_k)} \right| \le \frac{1}{\sqrt{k}} \to 0, k \to \infty$$

Therefore $(f, \varphi_k) \to 0, k \to \infty$. On the other hand we get

$$|(f, \psi_k)| = \frac{|(f, \varphi_k)|}{\sqrt{k} \sum_{(m,k)\in N_f} P_{m,k}(\varphi_k)} \geq \sqrt{k} \rightarrow \infty, k \rightarrow \infty$$

Which implies $0 > \infty$, (contradication).

So the assumption is wrong

Hence $|(f, \varphi)| \leq p_{m,\alpha}(\varphi)$

($\Leftarrow$) Suppose now $|(f, \varphi)| \leq C_f \sum_{(m,\alpha)\in N_f} P_{m,\alpha}(\varphi)$

To show that f is continuous in S′

Let $\varphi_k \rightarrow \varphi$ in S. i.e $P_{m,\alpha}(\varphi_k - \varphi) \rightarrow 0$ as $k \rightarrow \infty$ then since

$|(f,\varphi)| \leq \sum_{(m,\alpha)\in N_f} P_{m,\alpha}(\varphi)$ we have that

$$|(f, \varphi_k - \varphi)| \leq C_f \sum_{(m,\alpha)\in N_f} P_{m,\alpha}(\varphi_k - \varphi) \rightarrow 0, (k \rightarrow \infty)$$

$(f, \varphi_k - \varphi) \rightarrow 0, \quad (k \rightarrow \infty)$

i.e $(f, \varphi_k - \varphi) \rightarrow 0$ as $k \rightarrow \infty$ whenever $(\varphi_k - \varphi) \rightarrow 0, (k \rightarrow \infty)$.

Hence f is continuous.

Remark: The Lemma just proved imply that each generalized function of slow growth is a continuous functional with respect to a certain norm $P_{m,\alpha}$ (in other word it has a finite order).

4.2.2 Regular generalized functions in S′

We say that locally summable functions f has a slow growth if the following conditions hold:

i. f is locally summable in, $\Re^n$, $f \in L^1_{loc}(\Re^n)$.

ii. There exists $\ell \in N_o$ Such that

$$\int_{\Re^n} |f(x)| (1 + |x|)^{-\ell} dx = c(f) < \infty \qquad (5)$$

where $f \in \theta_m$. In particular, if $\ell = 0$, then $f \in L_1(\Re^n)$.

i.e for $\ell = 0$, $\int_{\Re^n} |f(x)|\,(1+|x|)^0 dx = \int_{\Re^n} |f(x)|dx = L_1(\Re^n)$

Therefore $f \in L_1$

If we define a functional (f, φ) for $f \in S'$ by the formula $\int_{\Re^n} f\varphi dx$, $\varphi \in S$, then f is generalized functions from S'. this means that

i, the value of the functional, (f, φ), corresponds to a (complex) number.

ii, $(f, \varphi) = \int_{\Re^n} f\varphi dx$ is linear in S'

Indeed, if $\varphi_1, \varphi_2, \in S$ and $\alpha_1, \alpha_2 \in \mathbb{C}$, then

$$(f, \alpha_1\varphi_1 + \alpha_2\varphi_2) = \int_{\Re^n} f(\alpha_1\varphi_1 + \alpha_2\varphi_2)dx.$$

$$= \int_{\Re^n} f(\alpha_1\varphi_1)dx + \int_{\Re^n} f(\alpha_2\varphi_2)dx$$

$$= \alpha_1 \int_{\Re^n} f\varphi_1 dx + \alpha_2 \int_{\Re^n} f\varphi_2 dx$$

$$= \alpha_1(f, \varphi) + \alpha_2(f, \varphi_2)$$

Hence (f, φ) is linear.

iii, (f, φ) is continuous in S'

Indeed, $\forall \varphi \in S$, $|(f, \varphi)| = |\int_{\Re^n} f\varphi dx|$,

$$\leq \int_{\Re^n} |f|\,|\varphi| dx$$

$$= \int_{\Re^n} |f(x)|\,(1+|x|)^{-\ell}\,(1+|x|)^{\ell}\,|\varphi(x)|dx$$

$$\leq P_{\ell,0}(\varphi) \int_{\Re^n} |f(x)|\,(1+|x|)^{-\ell}dx, \ \alpha = 0,\ m = -\ell$$

$= C(f)\,P_{\ell,0}(\varphi) < \infty$. Which means f is continuous.

Definition: Let $f \in L_p(\Re^n)$ $1 \leq p \leq \infty$, then the functional (f, φ), $\varphi \in S$ that is generated by $f \in L_p$ is called regular generalized functions from S'.

Corollary: If $f \in L_p(\Re^n)$: $1 \leq p \leq \infty$, then a functional which is generated by $f \in L_p(\Re^n)$ is a regular generalized function.

Proof:

$$\text{Since } C(f) = \int_{\Re^n} |f(x)|\,(1+|x|)^{-\ell}dx,\quad \ell \in N,\ f \in L_p(\Re^n)$$

$$\leq \Big(\int_{\Re^n} |f(x)|^p dx\Big)^{\frac{1}{p}} \Big(\int_{\Re^n} (1+|x|)^{-\ell q}\Big)^{\frac{1}{q}}$$

$$\leq \|f\|_{L_p(\Re^n)}\, C_f$$

$$= C_f \|f\|_{L_p(\Re^n)} < \infty$$

Hence f is a regular generalized function.

Example: The δ-function (Dirac function) is a generalized function. (i.e $\delta(x) \in S'$); since by definition $(\delta,\varphi) = \varphi(0)$ and by equation (4) this can be written as:

$$|(\delta,\varphi)| = |\varphi(0)| \leq P_{0,0}(\varphi) \text{ i.e } m = 0, \propto = 0 \text{ and } C_f = 1.$$

But δ is not a regular generalized function.

In other words generalized functions that is not regular is said to be singular generalized function.

Example: The δ- function is best example of singular generalized function.

4.2.3 Linear and Continuous Operators in S′

An operator $A: S' \to S'$ is said to be linear and continuous if the following conditions are satisfied:

i, $A(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 Af_1 + \alpha_2 Af_2$ $\forall_{\alpha_1, \alpha_2} \in C$, $f_1, f_2 \in S'$

ii, from the fact that $f_k \to f$ in S' as $k \to \infty$, it follows that $Af_k \to Af$ as $k \to \infty$ $\forall f \in S'$. i.e

$((f_k, \varphi) \xrightarrow{k\to\infty} (f, \varphi)) \Rightarrow ((Af_k,)(\varphi) \xrightarrow{k\to\infty} (Af,\varphi))$ in S′.

Linear continuous operators in S′ generate linear continuous operators in S′ as conjugate operations, and this will be verified by the following examples:

Example_ 1. the operator of differentiation D^β, i.e D^β: S′ → S′ defined by $(D^\beta f, \varphi) = (-1)^\beta (f, D^\beta\varphi)$, $\forall\varphi\in S$. is linear and continuous operator in S′.

Indeed, $\forall\alpha_1, \alpha_2\in C$ and $f_1, f_2 \in S'$,

$$(D^\beta(\alpha_1 f_1 + \alpha_2 f\alpha_2, \varphi)) = (-1)^\beta (\alpha_1 f_1 + \alpha_2 f_2, D^\beta\varphi)$$

But $\varphi \in S \Rightarrow D^\beta\varphi\in S$., if follows that,

$$\begin{aligned}(D^\beta(\alpha_1 f_1 + \alpha_2 f_2), \varphi) &= (-1)^\beta [(\alpha_1 f_1, D^\beta\varphi) + (\alpha_1 f_2, D^\beta\varphi)] \\ &= (-1)^\beta (\alpha_1 f_1, D^\beta\varphi) + (-1)^\beta (\alpha_2 f_2, D^\beta\varphi) \\ &= \alpha_1(-1)^\beta(f_1, D^\beta\varphi) + \alpha_2(-1)^\beta(f_2, D^\beta\varphi) \\ &= \alpha_1(D^\beta f_1, \varphi) + \alpha_2(D^\beta f_2, \varphi).\end{aligned}$$

Thus $D^\beta(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 D^\beta f_1 + \alpha_2 D^\beta f_2$.

Hence D^β is linear.

For Continuity of D^β we use equation. (4), i.e

$$|(D^\beta f, \varphi)| = |(-1)^\beta (f, D^\beta\varphi)| = C_f \sum_{(m.\alpha)\in N_f} P_{m.\alpha} (D^\beta\varphi) = C_f \sum_{(m.\alpha+\beta)\in N_f} P_{m.\alpha+\beta} (\varphi)$$

thus $f \in S' \Rightarrow D^\beta f \in S'$ and so D^β is continuous in S′.

Example 2: $\mu\lambda$: S′ → S′ where $\mu\lambda(f) = \lambda f$ is linear continuous in S′

$(\mu\lambda (f), \varphi) = (\lambda f, \varphi) = (f, \lambda\varphi)$; $\forall\varphi\in S$ since $\varphi\in S \Rightarrow \lambda\varphi\in S$ and $\lambda \in\theta_m$, the definition $(\lambda f, \varphi) = (f, \lambda\varphi)$ is correct.

5. Multidimensional Delta Functions

In multidimensional, $\delta(x)$ has the simple interpretation given by

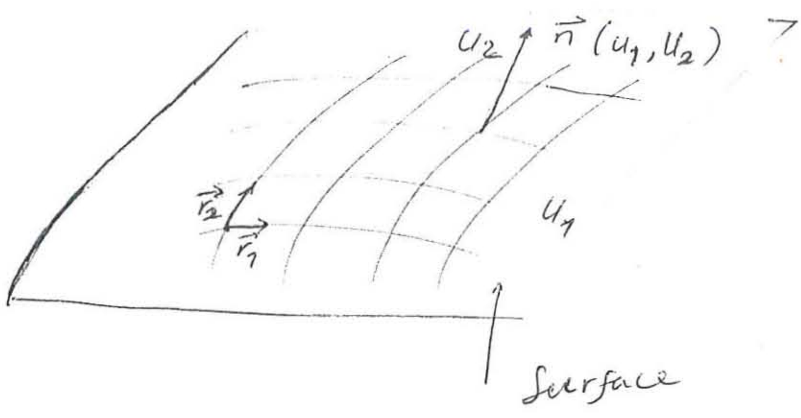
$$\int \phi(x)\, \delta(x\backslash)dx = \phi(0)$$

$$\delta(x) = \delta(x_1) \ldots \delta(x_n)$$

where $x = (x_1, x_2, \ldots, x_n)$

5.1 Three Dimensional Space

Here we confine ourselves to three-dimensional space. Of interest in application is $\delta(f)$ where $s = \{x \in \Re^3 / f(x) = 0\}$ is a surface in three dimensional space. We can always assume that f is defined so that $|\nabla f|=1$ at every point on s. If f does not have this property, then $f_1 = f / |\nabla f|$ does.



5.2 Interpretation of $\delta(f)$

Consider the integral

$$I = \int \phi(x)\ \delta(f)\ dx \qquad (1)$$

Assume that we define a curvilinear coordinate system (u_1, u_2) on the surface s and extend these variables locally to the space nean this surface along local normal. Let $s_3 = \{x \in \Re^3 / f(x) = u_3\}$ which, because

$|\nabla f| = \frac{df}{du_3} = 1$, u_3 is the local distance from the surface. Thus $f = u_3 =$ constant $\neq 0$ is a surface "parallel" to $f = 0$. Of course, we assume u_3 is mall.

From Differential Geometry

$$dx = \sqrt{g_{(2)}(u_1, u_2, u_3)}\ du_1\ du_2\ du_3 \qquad (2)$$

where $\sqrt{g_{(2)}(u_1, u_2, u_3)} = |J_T(u_1, u_2, u_3)|$

using equation (2) in equation (1) and integrating with respect to u_3 gives

$$I = \int \phi[x(u_1, u_2, u_3)\ \delta(u_3)\ \sqrt{g_{(2)}(u_1, u_2, u_3)}\ du_1\ du_2\ du_3$$

$$= \int \phi[x(u_1, u_2, 0)]\ \sqrt{g_{(2)}(u_1, u_2,)}\ du_1\ du_2$$

$$= \int_{f=0} \phi(x) ds \qquad (3)$$

That is, I is the surface integral of ϕ over the surface s.

6. Application

6.1 **Aerodynamics**: As part of mechanics, is a science which studies laws of motion of air and laws of the interaction between air and a solid body moving in it.

- The practical problems confronting mankind in connection with flights helped the development of aerodynamics as a science.
- Theoretically it is founded on Hydrodynamics whose corner stones were laid by the two Russian scientists L.Euler and D.Bernoulli.

In Solid mechanics we follow the same system forever

For example: - we follow a piston as it oscillates back and forth

- We follow a rocket system all the way to Mars.

But fluid systems don't need such concentrated attention. It is rare that we wish to follow the ultimate path of a specific particle of fluid. Instead we want to know the effect of the fluid on our product.

6.2 **Reynolds Transport Theorem.**

For the two examples cited above, we wish to know

- the fluid pressure on the piston and
- the drag and lift loads on the rocket.

This requires that the basic laws (of conservation of mass, linear momentum and angular momentum and energy) be rewritten to apply to a specific region called control volume in the neighborhood of our product.

Flow conditions away from this local region are then irrelevant.

In order to convert a system analysis into a control volume analysis, we must convent our mathematics to apply to a specific region rather than to individual masses. This conversion, called the Reynolds transport theorem, can be applied to all the basic laws:

- Conservation of mass: m_{syst} = const or

$$\frac{dm}{dt} = 0$$

- Conservation of Linear Momentum

$$F = mn = \frac{mdv}{dt} = \frac{d}{dt}(mv)$$

(This is a vector law)

- Conservation of angular Momentum

$M = \dfrac{dH}{dt}$ where H is angular momentum of system.

- Conservation of Energy

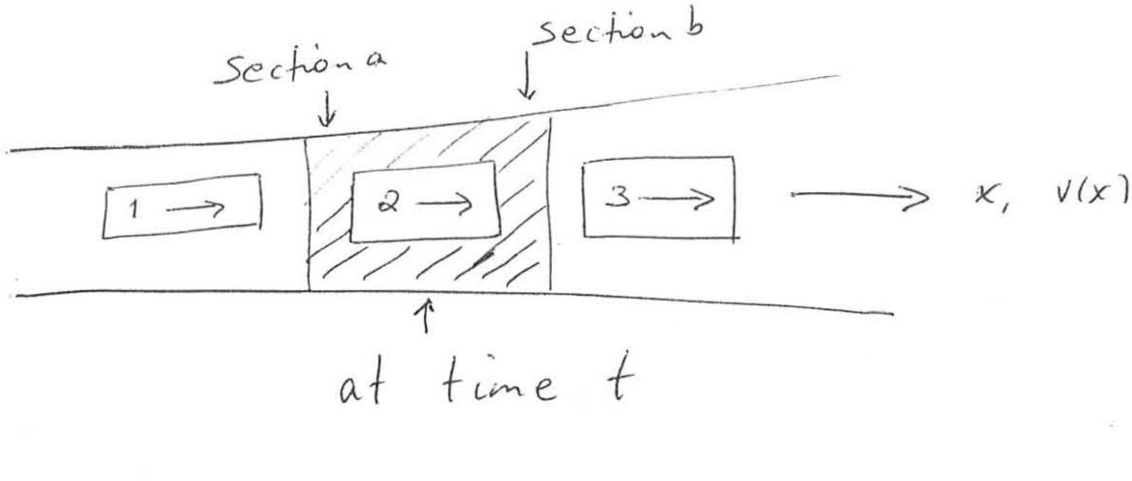
$$\frac{dQ}{dt} - \frac{dw}{dt} = \frac{dE}{dt}$$

Examining the basic laws above we see that they are all concerned with time derivative of fluid properties: m, v, H and E. Therefore what we need is to

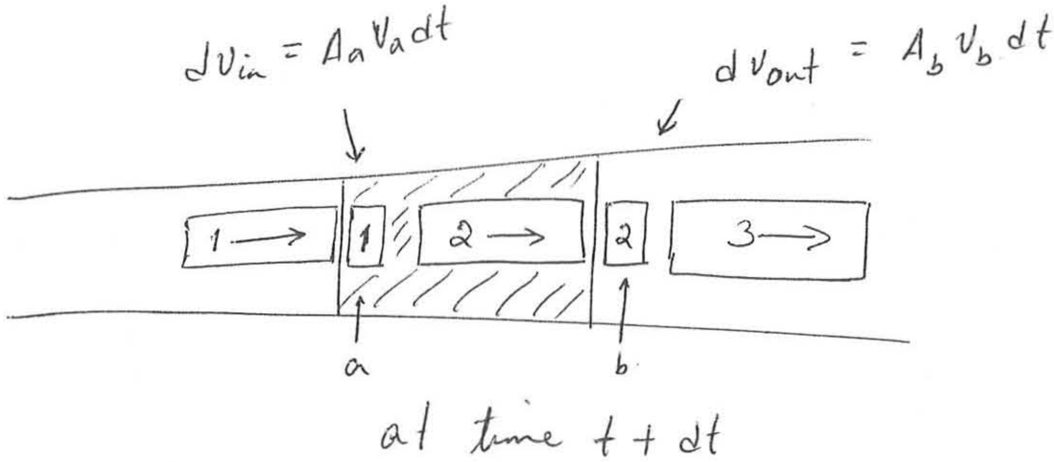
relate the time derivative of a system property to the rate of change of that property within a certain region. The desired conversion formula differs slightly according as the control volume is fixed moving or deformable.

6.2.1 One-Dimensional Fixed Control Volume

As a simple first example, consider a duct or stream tube with a nearly one – dimensional flow v = v(x). The selected control volume is a portion of the duct which happens to be filled exactly by system 2 at a particular instant t. At time t + dt, system 2 has begin to move out and sliver of system 1 has entered from the left. The shaded areas show an outflow sliver of volume $A_b v_b dt$ and an inflow volume $A_a v_a dt$.



(a) Control volume fixed in space.



(b)

fig 1

Example of inflow and outflow as three systems pass through a control volume:

(a) System 2 fills the control volume at time t,

(b) at t + dt system 2 begins to leave and system 1 enters

Now let B be any property of the fluid (energy, momentum, etc) and let $\beta = {dB}/{dm}$ be the intensive value or the amount of B per unit mass in any small portion of the fluid. The total amount of B is the control volume is thus

$$B_{cv} = \int_{cv} \beta\rho dV.$$

Where ρdV is a differential mass of the fluid. We want to relate the rate of change of B_{cv} to the rate of change of the amount of B in system 2 which happens to coincide with the control volume at time t. The time derivative of B_{cv} is defined by the calculus limit

$$\frac{d}{dt}(B_{cv}) = \frac{1}{dt}(B_{cv})(t + dt) - \frac{1}{dt}(B_{cv})(t)$$

$$= \frac{1}{dt}[\ \underbrace{B_2(t+dt)}_{Bsyst} - (\beta\rho dv)_{out} + (\beta\rho dv)_{in}\ ] - \frac{1}{dt}[B_2(t)]$$

$$= \frac{1}{dt}[\ \underbrace{B_2(t+dt) - B_2(t)}_{Bsyst}\ ]\ (\beta\rho Av)_{out} + (\beta\rho Av)_{in} \quad (1)$$

The first term on the right is the rate of change of B within system 2 at the instant it occupies the control volume. By rearranging (1) we have the desired conversion formula relating changes in any property B of a local system to one-dimensional computations concerning a fixed control volume, which instantaneously encloses the system.

$$\frac{d}{dt}(B_{syst}) = \frac{d}{dt}\left(\int B\rho dV\right) + (B\rho Av)_{out} - (B\rho Av)_{in} \quad (2)$$

rate of change of B within the control volume	flux of B passing out of the control volume	flux of B passing into the control volume.

The above equation can readily be generalized to an arbitrary flow pattern, as follows.

6.2.2 Arbitrary Fixed Control Volume

The figure shows generalized fixed control volume with an arbitrary flow pattern passing through. The only additional complication is that there are variable slivers of inflow and outflow of fluid all about the control surface. In general, each differential area dA of surface will have a different velocity v making different angle θ with the local normal to dA. Some elemental areas will have inflow volume $(vA\cos\theta)_{in}$ dt and others will have outflow volume $(vA\cos\theta)_{out}$ dt, as seen in the figure. Some surfaces might correspond to streamline ($\theta = 90^0$) or solid walls ($v = 0$) with neither inflow nor outflow. Equation (2) generalizes to

$$\frac{d}{dt}(B_{syst})$$

$$= \frac{d}{dt}\left(\iiint_v \beta\rho dv\right) + \iint_{cs} \beta\rho v \cos\theta dA_{out} - \iint_{cs} \beta\rho v \cos\theta dA_{in} \qquad (3)$$

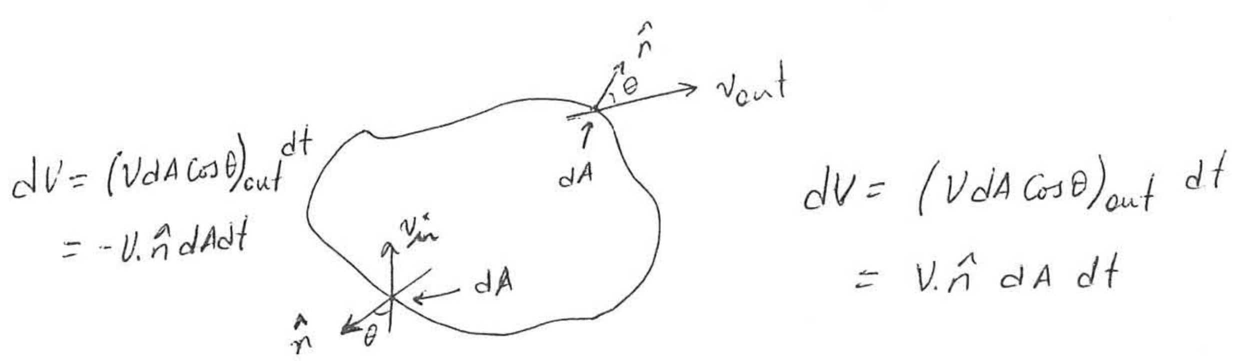


Fig 2. Generalization of Fig 1 to arbitrary control volume with an arbitrary flow pattern.

This is Reynolds transport theorem for an arbitrary fixed control volume. Equation (3) expresses the basic formula that a system derivative equals the rate of change of B within the control volume plus the flux of B out of the control surface minus the flux of B into the control surface. The quantity B (or β) may be any vector or scalar property of the fluid.

$$\text{Flux terms} = \iint_{cs} \beta\rho(v.n)dA \qquad (4)$$

The compact form of the Reynolds transport theorem is thus

$$\frac{d}{dt}(\text{Bsyst}) = \frac{d}{dt}\left(\iiint_{cv} \beta\rho dv\right) + \iint_{cs} \beta\rho(v.n)dA \qquad (5)$$

6.2.3 Control Volume Moving at constant Velocity

if the control volume is moving uniformly at velocity v_s, an observer fixed at the control volume will see a relative velocity v_r, of fluid crossing the control surface, defined by

$$\vec{v}_r = \vec{v} - \vec{v}_s \qquad (6)$$

where v is the fluid velocity relative to the same coordinate system in which the control volume motion v_s is observed. The flux terms will be proportional to v_r, but the volume integral is unchanged because the control volume moves as a fixed shape without deforming. The RTT for this case of uniformly moving control volume is

$$\frac{d}{dt}(B_{syst}) = \frac{d}{dt}\left(\iiint_{cs} \beta\rho dv\right) + \iint_{cs} \beta\rho(v_r n)dA \qquad (7)$$

which reduces to equation 6 if $v_s \equiv 0$.

*For control volume of constant shape but variable velocity $v_s(t)$, the volume elements don't change with time and the boundary relative velocity

$v_r = v(r,t)$

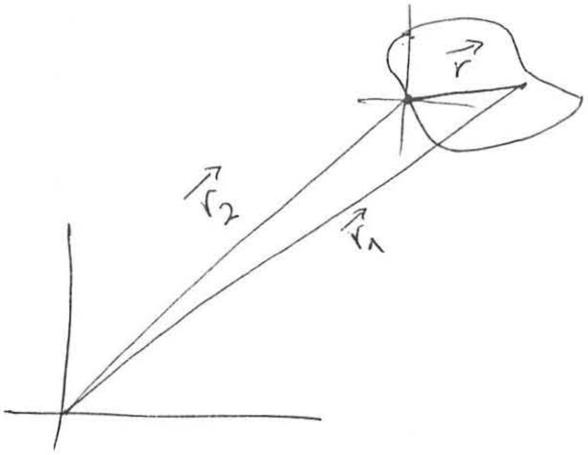
Then equation (8) doesn't change.

$$\vec{r} = \bar{r}_2 + \bar{r}_1$$

$$\dot{\vec{r}} = \dot{\bar{r}}_2 + \dot{\bar{r}}_1$$

$$v = v_s + r_r$$

$$\Rightarrow \quad v_r = v - v_s$$



6.2.4 Arbitrary moving and Deformable Control Volume

The most general situation is when the control volume is both moving and deforming arbitrarily. The flux of volume across the control surface is again proportional to the relative normal velocity component $v_r.n$, as in equation (7). However, since the control surface has a deformation, its velocity $v_s = v_s(r, t)$, so that the relative velocity $v_r = v(r, t) - v_s(r, t)$. The flux integral is the same as in equation (7). While, the volume integral in equation (7) must allow the volume elements to distort with time. Thus the time derivative must be applied after integration. For the deforming control volume, then, the transport theorem takes the form

$$\frac{d}{dt}(B_{syst}) = \frac{d}{dt}\left(\iiint_{cv} \beta\rho dv\right) + \iint_{cs} \beta\rho(v_r.n)dA \qquad (8)$$

This is the most general case. This equation contains two complications:

1. The time derivative of the triple integral must be taken outside.
2. The double integral involves the relative velocity v_r between the third system and the central surface.

   The first one can be resolved using generalized function theory.

6.3 **Two Transport Theorems**

We give one of the two results here that are used in the derivation of conservation laws. We want to take the time derivative inside the integral

$$I = \frac{d}{dt} \int_{\Omega(t)} Q(x,t)\, dx. \qquad (1)$$

where $\Omega(t)$ is a time-dependent region of space and $Q(x, t)$ is c' function. Let us assume the boundary $\partial\Omega(t)$ of Ω is piecewise smooth and is given by the surface $f = 0$ such that $f > 0$ in Ω. Assume also that $\nabla f = n'$ where n' is the unit in ward normal to the surface. Suppose we can ascertain that the integral in equation (1) is continuous in time. Then, we can replace d/dt with $\bar{d}/dt$ and bring the derivative inside the integral. We write

$$\begin{aligned} I &= \frac{\bar{d}}{dt} \int h(f)\, Q(x, t) dx \\ &= \int \left[\frac{\partial f}{\partial t} \delta(f) Q(x,f) + h(f) \frac{\partial Q}{\partial t} \right] dx \\ &= \int_{\partial\Omega(t)} \frac{\partial f}{\partial t} Q(x,f) ds + \int_{\Omega(t)} \frac{\partial Q}{\partial t} dx \end{aligned} \qquad (2)$$

where $h(f)$ is the Heaviside function. Here we have used equation (3) of section 3.2 to integrate $\delta(f)$ in the second step above.

$$\frac{\partial f}{\partial t} = -v_{n'} = v_n \qquad (3)$$

where $v_{n'}$ and v_n are the local normal velocities in the direction of inward and outward normals, respectively. Thus,

$$\mathrm{I} = \int_{\partial\Omega(t)} v_n Q(x, f)\,ds + \int_{\Omega(t)} \frac{\partial Q}{\partial t}\,dx \qquad (4)$$

Reference

1. V.s Vladmirov.
   Generalized Functions in Mathematical Physics Mir Publishers Moscow 1979

2. V.S. Vladmirov
   Equations of Mathematical Physics
   Marcel Dekkey. Inc, New York 1971\

3. Frank M.White
   Fluid Mechanics McGraw – Hill Book compay New York. London. Madrid. Mexico. New Delhi Tokyo

3. A graduate Seminar Report on Generalized functions & Fundamental solution of D ifferential Operators by Yesuf Obsie June, 2001 Addis Ababa.

4. Introduction to generalized functions with applications in Aerodynamics and Aeroacoustics NASA Administration. Lagley Reasearch Center. Hampton verginia 23 681 - 0001