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COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCE
DEPARTMENT OF MATHEMATICS

GRADUATE THESIS REPORT ON ANALYTIC SOLUTIONS OF ELLIPTIC PDEs

Submitted in partial fulfilment of the requirement for the Degree of master of science in Mathematics

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Contents

Abstract

This thesis is concerned with some Fundamental solution and Green's function for a System of second order and more, for Linear Elliptic partial differential equations in two or more independent variables.
Fundamental Solutions and a number of Green's Functions are given for cases when the coefficient in the equations are constant.

0.1 Notations

Ω - non-empty open subsets of $\mathbb{R}^n$

$\partial\Omega$ - the boundary of Ω

$\overline{\Omega}$ - closure of Ω

C(Ω) - the space of continuous functions on Ω

$C^1(\Omega)$ - the space of once continuously differentiable functions on Ω

$C^2(\Omega)$ - the space of twice continuously differentiable functions on Ω

$C^\infty(\Omega)$ - the set of all infinitely differentiable function on Ω with compact support
D' - Distribution

$D^{|\alpha|} = (\frac{\partial}{\partial x_1})^{\alpha_1} + ... + (\frac{\partial}{\partial x_n})^{\alpha_n}$ - n times derivatives
$|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$ - multi index
$B(x_0, R)$ - ball of radius R about $x_0 \quad in \quad \Omega$

$\partial B(x_0, R)$ - ball of radius R about x_0 on the boundary of Ω

$V(R) \quad = \alpha(n)$ - volume of unit ball in $\quad \Omega$

$A(R) = n\alpha(n)R^{n-1}$ - area of unit ball in Ω
$\frac{\partial u}{\partial n}$ $=n.\nabla U$ - normal derivative of U

0.2 Introduction

This thesis is concerned with assessing analytic solution of PDE and fundamental solution for linear elliptic second order PDE using Green's Function.
The primary attention of this thesis is devoted to analytic solution of elliptic PDE with in a region and on the region. Because of their physical formulation elliptic equation typically arise as associated with boundary condition.
Consider the second order elliptic equation

$$LU(x) = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu = F \tag{1}$$

It is required to find a solution U in some open region Ω of a space with condition imposed on $\partial\Omega$ (the boundary of Ω) or at infinity.
By fundamental solution of elliptic equation we mean a function $U(x, y)$ such that

$$LU(x, y) = -\delta(x, y) \tag{2}$$

$x, y \in \Omega \subseteq \mathbb{R}^n$ and $\delta(x, y)$ is the Dirac delta distribution.

Elliptic partial differential equation can be written in more concise form

$$LU(x) = f(x) \tag{3}$$

Where L denotes the second order elliptic differential operator $x \in \Omega \subseteq \mathbb{R}^n$
Elliptic equation in particular (Laplace equation) is arguably one of the most important differential equation in all of applied mathematics,It arises in an astonishing variety of physical and mathematical systems ranging through electromagnetism, fluid mechanics,potential theory, solid mechanics,heat conduction and so on
Green's function is a tool to solve nonhomogeneous linear equation.
Hence fundamental solution and green's function of elliptic equation can actually be regarded as a solution of homogeneous and inhomogeneous Laplace equation.

Chapter 1

Preliminary idea

In this section we will consider basic definitions,terminologies and ideas that are important in this thesis.

1.1 The Dirac delta function

Unit impulse Mechanical systems are often acted on by external force of large magnitude that acts only for a given period of time.
A function that approximate $\delta(x-\xi) = \lim_{a\to 0} \delta_a(x-\xi)$
In physical problem one often encounter idealized concepts such as force concentrated at a point ξ or an impulsive force that acts instantaneously. These forces are described by the Dirac delta function $\delta(x-\xi)$ which has several significant properties.

$$\delta_a(x-\xi) = 0 \qquad Singular\ at\ x = \xi$$

$$\int_a^b \delta_a(x-\xi)dx = \begin{cases} 0, & \text{for} \quad a,b < \xi, or, \xi < a,b \\ 1, & \text{for} \quad a \le \xi \le b \end{cases} \qquad and$$

$$\int_{-\infty}^{\infty} \delta(x-\xi)dx = 1 \tag{1.1}$$

Equation(1.1) is a special case of the general formula

$$\int_{-\infty}^{\infty} \delta(x-\xi)f(x)dx = f(\xi) \tag{1.2}$$

1.2 Properties of n×n matrix.

Definition 1. *A symmetric matrix A is said to be positive definite if* $V^T AV > 0$ *for any non zero* $V \in \Omega \subseteq \mathbb{R}^n$

Let A be an $n \times n$ matrix on $\Omega \subseteq \mathbb{R}^n$. If A is symmetric then one consequence of this assumption is that A has only real eigenvalues.

1.3 Green's Function and Divergence Theorem

Green's functions helps as to solve inhomogeneous BVPs.
Green's function is a special fundamental solution satisfying

$$\begin{aligned} \triangle G(x,\xi) &= \delta(x) \; for \; x \in \Omega \\ G(x,\xi) &= 0 \; for \; x \; \in \partial\Omega \end{aligned}$$

The Divergence theorem ,Let Ω be a C^1 domain and w $\in C^1(\Omega)$ be a vector field then $\int_{\partial\Omega} w.nds = \int_\Omega div.w(x)dx$ where ds is the $n-1$ dimensional Lebegue measure of $\partial\Omega$.
dx $= dx_1...dx_n$
If

$$w = V\Delta U$$

assuming that $U, V \in C^2(\Omega) \cap C^1(\overline{\Omega})$, then

$$\begin{aligned} w.n &= (V\triangle U).n = V(\nabla U.n) = V\frac{\partial U}{\partial n} \\ &= div(V\nabla.U) = \nabla.(V\nabla U) \\ &= \nabla V.\nabla U + V\triangle U \end{aligned}$$

$$\begin{aligned} \int_{\partial\Omega} w.nds &= \int_{\partial\Omega} V\frac{\partial U}{\partial n}ds \\ &= \int_\Omega (\nabla V.\nabla U + V\triangle U)dx \quad (G_1) \; This \; is \; Green's \; first \; identity \end{aligned}$$

$let \quad U, V \in C^2(\overline{\Omega}) \; from \; (G_1)$ and interchanging U and V

$$\begin{aligned} \int_{\partial\Omega} U\frac{\partial V}{\partial n}ds &= \int_\Omega (\nabla U.\nabla V + U\triangle V)dx \; (G_2); \; subtracting \; (G_1) \; and \; (G_2) \\ \int_{\partial\Omega} (U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n})ds &= \int_\Omega (U\triangle V - V\triangle U)dx \quad This \; is \; Green's \; 2^{nd} \; identity \; (G_2) \end{aligned}$$

$If\ we\ set\ U = V\ in\ G_1\ \int_{\partial\Omega} U\frac{\partial U}{\partial n}ds = \int_{\Omega}((\nabla U)^2{+}U\triangle U)dx \quad This\ is\ Green's\ 3^{rd}\ identity$

Green's second identity in open space Ω smooth domain with positively oriented boundary $\partial\Omega$ and U,V twice continuously differentiable function.

$\int\int_{\Omega}(U\triangle V - V\triangle U)dA = \int_{\Omega}(U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n})ds$ where dA = dxdy and ds = surface element(Area)

1.4 Adjoint and Green's Function

Consider a linear PDE of the form $LU(x){=}F(x)$ in Ω where L is a linear (self-adjoint) differential operator $U(x)$ is unknown and F(x) is known inhomogeneous term.

Definition 2. *Given a linear map* $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *there is a unique linear map* $L^* : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ *is called adjoint of L.*

Definition 3. *An inner product of two arbitrary functions U and V which are function of x is given by* $\langle U, V\rangle = \int VUdx$ *where* $\langle , \rangle$ *denote inner product.*

Now consider a linear operator L, the following relationship holds true if L is self adjoint. $\langle U, LV\rangle = \langle LU, V\rangle$

Definition 4. *A linear map* $L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *is called Self adjoint if* $L : L^*$.

1.5 Self-adjoint Operator

Consider an m^{th} -order differential operator $LU = \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha U$

assume U,V vanish near the boundary $\partial\Omega$

The integration by parts formula gives

$$\int_{\Omega} U_{xk}Vdx = \underbrace{\int_{\partial\Omega} UVn_k ds}_{0} - \int_{\Omega} UV_{xk}dx \quad \vec{n} = (n_1...n_n) \in \Omega \subseteq \mathbb{R}^n \tag{1.3}$$

$$\int_{\Omega} U_{xk}Vdx = -\int_{\Omega} UV_{xk}dx \tag{1.4}$$

Generally,we can repeat integration by part with any combination of derivatives

$$\int_{\Omega} D^\alpha UVdx = (-1)^{|\alpha|}\int_{\Omega} UD^\alpha Vdx \qquad |\alpha| \leq m$$

We have,

$$
\begin{aligned}
\int_{\Omega}(LU)V\,dx &= \int_{\Omega}\Big(\sum_{|\alpha|\leq m} a_{\alpha}(x)D^{\alpha}U\Big)V\,dx \\
&= \sum_{|\alpha|\leq m}\int_{\Omega} a_{\alpha}(x)VD^{\alpha}U\,dx \\
&= \sum_{|\alpha|\leq m}(-1)^{|\alpha|}\int_{\Omega} D^{\alpha}(a_{\alpha}(x)V)U\,dx \\
&= \int_{\Omega}\underbrace{\sum_{|\alpha|\leq m} D^{\alpha}(a_{\alpha}(x)V)}_{L^*(V)}U\,dx \\
&= \int_{\Omega}(L^*V)U\,dx \ for\ all\ U\in C^{|\alpha|}(\Omega)\ and\ V\in C_0^{\infty}
\end{aligned}
$$

$$
The\ operator\ L^*(V)=\sum_{|\alpha|\leq m} D^{\alpha}(a_{\alpha}(x)V) \tag{1.5}
$$

is called the adjoint of L.

$$
The\ operator\ is\ self\ adjoint\ if\ L^*=L
$$

$$
and\ also\ L\ is\ self-adjoint\ if\ \int_{\Omega}VL(U)dx=\int_{\Omega}UL(V)dx \tag{1.6}
$$

i.e $L{=}L^* \Leftrightarrow \langle LU,V\rangle = \langle U,L^*V\rangle = \langle U,LV\rangle$

1.6 Formulation of Area and Volume of Sphere/ Ball

It will be convenient to continue in polar coordinates thus

$$\int_{\mathbb{R}^n} f(x)dx = \int_0^\infty \int_{\partial B(x_0,r)} f(x)dsdr \tag{1.7}$$

Here ds represents surface measure on the $n-1$ dimensional sphere $\partial B(x_0, r)$,the total surface measure of the sphere is proportional to r^{n-1} and the constant will be taken so that it is by definition $n\alpha(n)r^{n-1}$.
Euler's integral definition of the Gamma function is

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}du \tag{1.8}$$

convergence of the integral requires that $x-1 > -1$ or $x > 0$.
As an example , we can take $f(x) = e^{-x^2}$ and $x_0 = 0$. Then

$$\int_{\mathbb{R}^n} e^{-x^2}dx = n\alpha(n)\int_0^\infty e^{(-r^2)}r^{n-1}dr$$

Here the total surface measure of the ball is defined to be $n\alpha(n)r^{n-1}$
We can also write this as

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-x^2}dx &= n\alpha(n)\frac{1}{2}\int_0^\infty u^{\frac{n}{2}-1}e^{-u}du \\ &= n\alpha(n)\frac{1}{2}\int_0^\infty u^{\frac{n}{2}}u^{-1}e^{-u}du \\ &= n\alpha(n)\frac{1}{2}\Gamma(\frac{n}{2})\end{aligned}$$

When n= 2 this says that the value of the integral is π.
The differential of Volume of a hyper sphere of radius $r = R$ is

$$V_n = \upsilon_n r^n \;\; \Rightarrow \;\; dV_n = \upsilon_n.nr^{n-1}dr \; where \; \upsilon_n \; is \; equal \; to \; \alpha(n).$$

from Gaussian integral we have $\int_{-\infty}^{\infty} e^{-x^2}dx = \pi^{\frac{1}{2}}$ multiplying this integral by it self n times subscripting each dummy variable x by a different index i

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty} e^{(-\sum_{i=1}^n x_i^2)\Pi_{i=1}^n}dx_i = \pi^{\frac{n}{2}}$$

however the summation is simply equal to r^2 in n-dimension and the product of differentials is just the n-dimensional volume element.

It follows from above equation for arbitrary dimension the value of the integral is $\pi^{\frac{n}{2}}$.

$$Thus \quad \pi^{\frac{n}{2}} = n\alpha(n)\frac{1}{2}\Gamma(\frac{n}{2})$$

This proves the basic fact that area of the unit $n-1$ sphere is

$$n\alpha(n) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} = A(R) \tag{1.9}$$

$$The\ volume\ of\ the\ unit\ ball\ is\ thus\ \alpha(n) = \frac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})},\ \alpha(n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} \tag{1.10}$$

The main properties of Gamma function are :

$$\Gamma(x+1) = x\Gamma(x) \qquad \Gamma(n+1) = n!$$

To find $\Gamma(\frac{1}{2})$ if we set $x = \frac{1}{2}$

$$\Gamma(\frac{1}{2}) = \int_0^\infty u^{\frac{-1}{2}} e^{-u} du$$

$$let\ u = t^2 \quad \Gamma(\frac{1}{2}) = \int_0^\infty 2e^{-t^2} dt$$

But $\int_0^\infty 2e^{-t^2} dt = \int_0^\infty 2e^{-v^2} dv$

$$(\Gamma(\frac{1}{2}))^2 = (2\int_0^\infty e^{-t^2} dt)(2\int_0^\infty e^{-v^2} dv)$$
$$= 4\int_0^\infty \int_0^\infty e^{-(t^2+v^2)} dt dv$$

Bivariate transformation
$t = r\cos\theta$
$v = r\sin\theta$ will transform the integral problem from cartesian to polar coordinates.
The region R which defines the first quadrant, is the region of integration for the integral to polar coordinates(r, θ) and these new variable will range from $0 \le r \le \infty$ and $0 \le \theta \le \frac{\pi}{2}$ for the first quadrant the Jacobian transformation $rcos^2\theta + rsin^2\theta = r$ coordinates to polar coordinates and let $r^2 = u$

$$(\Gamma(\frac{1}{2}))^2 = 4\int_0^\infty\int_0^\infty e^{-(t^2+v^2)}dtdv = 4\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}rdrd\theta$$

$$(\Gamma(\frac{1}{2}))^2 = \pi$$

$$\Gamma(\frac{1}{2}) = \sqrt{\pi} \tag{1.11}$$

$\Gamma(\frac{3}{2}) = \Gamma(1+\frac{1}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$

Note. In general $\Gamma(1+\frac{1}{2}+n) = \frac{(2n+1)!}{2^{2n+1}n!}\sqrt{\pi}$ for all n= 0 , 1, 2,...

1.7 Some Important Definitions

Definition 5. *A linear functional on* Ω *is a linear map if* $L:\Omega \to \mathbb{R}$ *, we denote the value of L acting on a test function* ϕ *by* $\langle L, \phi\rangle$ *thus L is linear if*

$$\langle L, \lambda\phi + \mu\psi\rangle = \lambda\langle L, \phi\rangle + \mu\langle L, \psi\rangle$$

for all $\lambda, \mu \in \mathbb{R}$ *and* $\phi, \psi \in \Omega$

A function L is continuous if $\phi_n \to \phi$ *in the sense of test function implies that* $\langle L, \phi_n\rangle \longrightarrow \langle L, \phi\rangle$ *in* Ω

Definition 6. *A function f is locally integrable in* $\mathbb{R}^n$ *if* $\int_\Omega |f(x)|dx < \infty$ *for every bounded region in* Ω

The set of distribution that are more useful are those generated by locally integrable function. Indeed every locally integrable function f(x) generates a distribution through the formula

$\langle f, \phi\rangle = \int_{R^n} f(x)\phi(x)dx$ *a function defined by called regular.*

A distribution which are generated by locally integrable function are regular distribution.

Definition 7. *The support of a function f is the closure of the set all points x such that* $f(x) \neq 0$

$$Supp(f) = (\overline{x \in \mathbb{R}^n / f(x) \neq 0})$$

Chapter 2

Elliptic Equation

Elliptic equation are typically associated with steady-state behavior.
The prototypes of elliptic equation are the laplace equation and poisson's equation.
$\triangle U = 0$ *homogeneous*
$-\triangle U = f$ *Poisson's equation (inhomogeneous) and describes*
- Steady-in rotational flows
- Electro static,potential in the absence of charge
- Equilibrium temperature distribution in a medium.

Because of their physical origin elliptic equation typically arise as boundary value problems(BVPs)
Solving a BVP for the second order elliptic equation

$$LU(x) = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu = F$$

All eigenvalues of the matrix A $=(a_{ij})$ i,j = 1, 2,...,n are non-zero and have the same sign
Our aim is to find a solution U in some open region Ω with conditions imposed on $\partial\Omega$ -the boundary of Ω

2.1 Hadamard Concept of Well-posed problem

The notation of well-posedness in the sense of Hadamard is related to the requirement that can be expected in solving a partial differential equations.

Definition 8. *A given problem for a partial differential equation is said to be well-posed if*

• *A solution exists*

• *The solution is unique*

• *The solution depends continuously on the given data. The BVP of elliptic equation consists of the above three things.*

2.2 Types of Boundary Condition

We will consider three types of boundary conditions for well-posed BVP.

2.2.1 Dirichlet Condition (First boundary value problem)

Let f,g be continuous functions on $\partial\Omega$ boundary the problem of finding a function $U \in C^2(\Omega) \bigcap C(\overline{\Omega})$ *such that*

$$\begin{cases} \triangle U = f, & \text{in } \Omega \\ U = g, & \text{on } \partial\Omega \end{cases}$$

is called the Dirichlets(first boundary Value)

i.e. U takes prescribed values on the boundary $\partial\Omega$

2.2.2 Neumann Condition

The normal derivative $\frac{\partial U}{\partial n}$=$n.\nabla U$ is prescribed on the boundary Ω in this case we have to check computability condition.

Example 1.

$$\triangle U = F \text{ on } \Omega$$

and

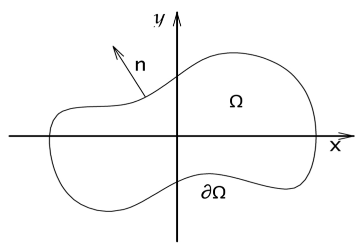
$$n.\nabla U = \partial_n U = f \text{ on } \partial\Omega$$

Then by compatibility condition

$$
\begin{aligned}
\int_{\Omega} \Delta U dv &= \int_{\Omega} \nabla . \nabla U ds \\
&= \int_{\partial\Omega} \nabla U . n ds \\
&= \int_{\partial\Omega} \frac{\partial U}{\partial n} ds \text{ Divergence theorem} \\
\int_{\Omega} F dv &= \int_{\partial\Omega} f ds \quad \text{for the problem to be wellposed}
\end{aligned}
$$

2.2.3 Robin Condition

A combination of U and its normal derivative such as $\frac{\partial U}{\partial n} + \alpha U$ is prescribed on the boundary Ω .



$$
\begin{cases}
\triangle U = 0 & \text{in } \Omega \\
\frac{\partial U}{\partial n} + \alpha U = g & \text{on } \partial\Omega
\end{cases}
$$

where α is continuous function on the boundary of Ω is called the Robin problem or (the third BVP for the Laplace equation) If Ω encloses a finite region we have an interior problem if,however,Ω is unbounded we have an exterior problem and we must impose condition at infinity

2.3 Harmonic Function

Definition 9. *A function satisfying Laplace's equation in an open region Ω, with continuous first and second order derivative is called an harmonic function.*
A function U in $C^2(\Omega)$ with $\triangle U \geq 0$ is called Subharmonic
A function U in $C^2(\Omega)$ with $\triangle U \leq 0$ is called Super harmonic

2.4 Fundamental Solution

2.4.1 Derivation of Fundamental Solution

Consider Laplace's equation in $\Omega \subseteq \mathbb{R}^n$,that is
$\triangle U = 0 \quad x \in \Omega$
There are lots of functions U which satisfy this equation in particular any constant function can be a solution to the above equation and any function of the form
$U(x) = b_1x_1 + b_2x_2 + ... + b_nx_n$ for a constant b_i is a solution for Laplace's equation
It is noted that in looking for explicit solution and it is often wise to restrict attention to classes of function with certain symmetry properties since laplace's equation is invariant under rotations, it consequently seems a devisable to search for radial solution that is function of the form $r = |x| = \sqrt{(x_1^2 + x_2^2 + ... + x_n^2)}$.
We say that Laplace's equation is invariant under rigid motion which are the translation (transformation)and rotations
That is $x \to x'$ $x' = x + a$ for some vector a
$y' = y + b$ for some vector b

$$U_{xx} + U_{yy} = U_{x'x'} + U_{y'y'} = 0$$

If a function is harmonic in variable (x, y) it must also be harmonic in the variable (x', y') this is invariance under translations.
The radial solution of Laplace's equation we makes change to polar coordinate . so radial solution means $U(r, \theta) = V(r)$ that is the function depends only one variables and as a consequence the PDE will be reduced to an ODE.

$$\begin{aligned} U(x) &= V(r) = V(|x|) \; |x| \neq 0 \\ U_{x_i} &= \frac{x_i}{|x|}V'(|x|) \qquad r = |x| = \sqrt{(x_1^2 + x_2^2 + ... + x_n^2)} \end{aligned}$$

which implies

$$U_{x_ix_i} = (\frac{x_i'|x|}{|x|^2} - \frac{x_i^2}{|x|^3})V'(|x|) + \frac{x_i^2}{|x|^2}V''(|x|)$$

$$\triangle U = V''(r) + \frac{n-1}{r}V'(r)$$

hence $\triangle U = 0$ if and only if

$$V''(r) + \frac{n-1}{r}V'(r) = 0, \; V' > 0$$

$$\frac{V''(r)}{V'(r)} = \frac{1-n}{r}$$

integrating both sides

$$\ln(V') = (1-n)\ln(r) + c \qquad V' > 0 \qquad \Rightarrow V' = \frac{c_1}{r^{n-1}}$$

Which implies

$$V(x) = \begin{cases} c_1 \ln|x| + c_2, & \text{for n=2} \\ \frac{c_1}{(2-n)|x|^{n-2}}, & \text{for } n \geq 3 \end{cases} \tag{2.1}$$

where c_1 and c_2 are constants.

The function

$$U(x) = \begin{cases} c_1 \ln|x| + c_2, & \text{for n=2} \\ \frac{c_1}{(2-n)|x|^{n-2}}, & \text{for } n \geq 3 \end{cases} \tag{2.2}$$

For $x \in \mathbb{R}^n$ $|x| \neq 0$ is a solution of Laplace's equation in $\mathbb{R}^n \setminus [0]$

We notice that the function U defined in (2.2) satisfies

$$\begin{aligned} \triangle U(x) &= 0 \; for \; x \neq 0 \\ \langle \delta_0, \phi \rangle &= \phi(0) \end{aligned}$$

Claim 1 . Choose c_1 and c_2 appropriately so that

$$\triangle U = \delta_0 \tag{2.3}$$

in the sense of distribution and where δ_0 the Dirac delta distribution

Let $\phi \in D'$ where D' is distribution function from Dirac delta distribution in $\mathbb{R}^n$ we have

$$\begin{aligned} \langle \delta(x-\xi), \phi(x) \rangle &= \phi(\xi) \\ So \; \langle \delta(x-0), \phi(x) \rangle &= \phi(0) \\ \Rightarrow \langle \delta_0, \phi \rangle &= \phi(0) \end{aligned}$$

Assume that we can find c_1 ,c_2 such that U defined (2.2) satisfies (2.3)

By construction the function $x \longmapsto \phi(x)$ is harmonic for $x \neq 0$. if we shift the origin to a new point y the PDE is unchanged and so $x \longmapsto \phi(x-y)$ is also harmonic as a function of x, $x \neq y$.

Let us now take $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and note that the mapping $x \longmapsto \phi(x-y)$ f(y) $(x \neq y)$ is harmonic for each point $y \in \mathbb{R}^n$ and thus so is the sum of finitely many such expressions built for different point y.
This reasoning suggest that

$$V(x) \;=\; \int_{R^n} \phi(x-y)f(y)dy \quad \textit{be a solution of poissons equation}$$

Let ϕ denote the solution of (**2.3**) then define

$$V(x) = \int_{R^n} \phi(x-y)f(y)dy \tag{2.4}$$

We can compute the Laplace's of V as follows

$$\begin{aligned} -\triangle_x V &= -\int_{R^n} \triangle_x \phi(x-y)f(y)dy \\ &= -\int_{R^n} \triangle_y \phi(x-y)f(y)dy \\ &= -\int_{R^n} \delta_x f(y)dy \\ &= f(x) \end{aligned}$$

That is V is a solution of poisson's equation
Now,let us use the radial solution (**2.2**) and define the function $\phi(x)$ as follows for $|x| \neq 0$

$$\phi(x) = \begin{cases} \frac{-1}{2\pi} \ln |x|, & \text{for } n = 2 \\ \frac{1}{n(2-n)\alpha(n)} \cdot \frac{1}{|x|^{n-2}}, & \text{for } n \geq 3 \end{cases} \tag{2.5}$$

Where $\alpha(n)$ is the volume of the unit ball \ sphere in $\mathbb{R}^n$ and ϕ satisfies the Laplace's equation on $\mathbb{R}^n \setminus (0)$
Claim 2 For ϕ defined by **(2.5)** ϕ satisfies

$$\triangle_x \phi = \delta_0$$

Let $g \in D'(distribution)$

$$-\int_{R^n} \phi(x) \triangle_x g(x)dx \;=\; g(0)$$

Proof : Let F_ϕ be the distribution associated with the fundamental solution ϕ
That is let $F_{\triangle_x\phi} : D' \longrightarrow \mathbb{R}$ be defined such that

$$\langle F_{\triangle_x\phi}, g\rangle = \int_{R^n} \triangle_x\phi(x)g(x)dx \quad for\ all\ g \in D'$$

define $\langle G, g\rangle = -\langle F, g'\rangle$ that the derivative of distribution F is defined as the distribution G such that

$$\int_{R^n} \triangle_x\phi(x)g(x)dx = -\int_{R^n} \phi(x)\triangle_x g(x)dx$$

$$\langle F_{\triangle_x\phi}, g\rangle = -\langle F_\phi, \triangle_x g\rangle$$

$$\langle \delta_0, g\rangle = g(0)$$

Therefore $\langle \delta_0, g\rangle$ =$g(0)$ which means $\triangle_x\phi = \delta_0$ in the sense of distribution by definition

$$\langle F_\phi, \triangle g\rangle = \int_{R^n} \phi(x)\triangle g(x)dx$$

Now,we would like to apply the divergence theorem but ϕ has a singularity at $x = 0$ by breaking up the integral in to two pieces , one piece consisting of the ball of radius R a bout the origin, $B(0, R)$ and the other piece consisting of the complement of this ball in $\mathbb{R}^n$.
Therefore we have

$$\begin{aligned}
\langle F_\phi, \triangle g\rangle &= \int_{R^n} \phi(x)\triangle g(x)dx \\
&= \underbrace{\int_{B(0,R)} \phi(x)\triangle g(x)dx}_{I} + \underbrace{\int_{R^n/B(0,R)} \phi(x)\triangle g(x)dx}_{J} \\
&= I + J
\end{aligned}$$

We look first at term I for n=2 we went to show term I is bounded as follows.‘

$$I = |\int_{B(0,R)} \phi(x)\triangle g(x)dx| \leq |\triangle g|_{L^\infty}|\int_{B(0,R)} \phi(x)\triangle dx|$$

for n=2 $\phi(x) = \frac{1}{2\pi}\ln|x|$

$$\begin{aligned}
I &= |\int_{B(0,R)} \frac{1}{2\pi}\ln|x|\triangle g(x)dx| \leq C|\triangle g|_{L^\infty}|\int_{B(0,R)} \ln|x|dx| \\
&\leq C|\int_0^{2\pi}\int_0^R \ln|r|rdrd\theta| \\
&\leq C\ln|R|R^2
\end{aligned}$$

Therefore,the term I is bounded for $n = 2$

For $n \geq 3$ term I is bounded as follows

$$\begin{aligned}
I = |\int_{B(0,R)} \frac{1}{n(n-2)\alpha(n)}.\frac{1}{|x|^{n-2}} \triangle g(x)dx| &\leq C|\triangle g|_{L^\infty} \int_{B(0,R)} \frac{1}{|x|^{n-2}}dx \\
&\leq C|\triangle g|_{L^\infty} \int_0^R (\int_{\partial B(0,r)} \frac{1}{|y|^{n-2}} ds(y))dr \\
&= \int_0^R \frac{1}{r^{n-2}} (\int_{\partial B(0,r)} ds(y))dr \\
&= \int_0^R \frac{1}{r^{n-2}} \underbrace{n\alpha(n)r^{n-1}}_{A(R)} dr \\
&= n\alpha(n) \int_0^R rdr \\
&= \frac{n\alpha(n)}{2} R^2
\end{aligned}$$

$R \longrightarrow 0^+$ $I \longrightarrow 0$

Therefore,for n =2 and $n \geq 3$ I is bounded .

Let us look at term J .

Applying the divergence theorem we have

$\int_{R^n} \phi(x) \triangle_x g(x)dx = \int_{R^n/B(0,R)} \triangle_x \phi(x)g(x)dx$ - $\int_{\partial(R^n/B(0,R))} \frac{\partial \phi}{\partial \nu} g(x)ds(x) + \int_{\partial(R^n/B(0,R))} \phi(x) \frac{\partial g}{\partial \nu} ds(x)$

$$= -\underbrace{\int_{\partial(R^n/B(0,R))} \frac{\partial \phi}{\partial \nu} g(x)ds(x)}_{J_1} + \underbrace{\int_{\partial(R^n/B(0,R))} \phi(x) \frac{\partial g}{\partial \nu} ds(x)}_{J_2}$$

$$= J_1 + J_2$$

Using the fact that $\triangle_x \phi(x) = 0$ for $x \in R^n/B(0,R)$.

We first look term J_1 now by assumption $g \in D'$ and therefore g vanish at ∞ .

Consequently,we only need to calculate the integral over $\partial B(0,\epsilon)$ where the normal derivative ν is the outer normal to $R^n/B(0,R)$.

$\nabla_x \phi(x) = \frac{-x}{n\alpha(n)|x|^n}$

The outer unit normal to $R^n/B(0,R)$ on $B(0,R)$ is given by

$$\nu = \frac{-x}{|x|}$$

Therefore, the normal derivative of ϕ on $B(0,R)$ is given by

$$\frac{\partial\phi}{\partial\nu} = |\frac{-x}{n\alpha(n)|x|^n}|.|\frac{|x|}{x}| \quad = \quad \frac{1}{n\alpha(n)|x|^{n-1}}$$

Therefore J_1 can be written as follows
$J_1 =$ -$\int_{\partial B(0,R)} \frac{1}{n\alpha(n)|x|^n} g(x)ds(x)$=$\frac{-1}{n\alpha(n)R^{n-1}} \int_{\partial B(0,R)} g(x)ds(x)$
now if g is continuous function then
$J_1 = \frac{-1}{n\alpha(n)R^{n-1}} \int_{\partial B(0,R)} g(x)ds(x)$ the mean of g on $\partial B(0,R)$
$⨍ g(x)ds(x) \longrightarrow -g(0)$ as $R \longrightarrow 0$.
Let look at term J_2. Using the fact g vanishes as $|x| \longrightarrow +\infty$ we only need to integrate over $\partial B(0,R)$ since $g \in D'$ and therefore infinitely differentiable
We have

$$|\int_{\partial B(0,R)} \phi(x)\frac{\partial g}{\partial\nu} ds(x)| \leq |\frac{\partial g}{\partial\nu}_{L^\infty(\partial B(0,R))} \int_{\partial B(0,R)} |\phi(x)|ds(x)$$

$$\leq \int_{\partial B(0,R)} |\phi(x)|ds(x)$$

For n=2

$$\begin{aligned} J_2 = \int_{\partial B(0,R)} |\phi(x)|ds(x) &= C\int_{\partial B(0,R)} |\ln|x||ds(x) \\ &\leq \frac{C}{R^{n-2}} \int_{\partial B(0,R)} ds(x) \\ &= \frac{C}{R^{n-2}} n\alpha(n)R^{n-1} \leq CR \end{aligned}$$

Therefore we conclude that J_2 *is bounded*

$$\begin{cases} CR|\ln R|, & \text{for } n = 2 \\ CR, & \text{for} n \geq 3 \end{cases}$$

Hence $J_2 \to 0$ as $R \to 0^+$ By combining these estimates we see that

$$\int_{R^n} \phi(x)\triangle_x g(x)dx = \lim_{R\to 0^+} I + J_1 + J_2 = -g(0)$$

Therefore our claim is proved.

2.4.2 Solving Poisson's Equations

We now return to solve poisson's equation $x \in \Omega \subseteq \mathbb{R}^n$

$$-\triangle U = f \tag{2.6}$$

From our discussion above(**claim1**) We expect the function

$$V(x) = \int_{R^n} \phi(x-y)f(y)dy$$

to be a solution of poisson's equation.we now prove that this is in fact true.

Note : By definition function has compact support if it is zero out side of compact set.

If we hope that the function above solves the poisson's equation we must first verify that this integral actually Converges.

If we assume f has compact support on some bounded set $\Omega \; in \; \mathbb{R}^n$, then we see that

$$\int_{R^n} \phi(x-y)f(y)dy \leq |f|_{L^\infty} \int_{\Omega} |\phi(x-y)|dy$$

If we additionally assume that f is bounded then $|f|_{L^\infty} \leq C$

Since $\int_\Omega |\phi(x-y)|dy < +\infty$ on any compact setΩ ,

$$\int_{R^n} \phi(x-y)f(y)dy < \infty$$

Theorem 1. *Assume $f \in C^2(R^n)$ and has compact support .*
let

$$U(x) = \int_{R^n} \phi(x-y)f(y)dy$$

Where ϕ is the fundamental solution of Laplace's equation (2.4) then
i) $U \in C^2(R^n)$
ii)$-\triangle U = f \; in \; R^n$

proof

i) By change of variables we write $\int_{R^n} \phi(x-y)f(y)dy = \int_{R^n} \phi(y)f(x-y)dy$
Let $e_i = (...,0,1,0,...)$ e_i is a unit vectors in R^n then

$$\frac{U(x+he_i)-U(x)}{h} = \int_{R^n} \phi(y)[\frac{(x+he_i-y)-f(x-y)}{h}]dy \quad Now f \in C^2$$

implies

$$\frac{(x+he_i-y)-f(x-y)}{h} \longrightarrow \frac{\partial f}{\partial x_i}(x-y)$$

as $h \longrightarrow 0$ *Uniformly on* R^n *therefore* $\frac{\partial U(x)}{\partial x_i} = \int_{R^n} \phi(y)\frac{\partial f}{\partial x_i}(x-y)dy$ *Differentiating with respect to* x

$$\frac{\partial^2 U(x)}{\partial x_i \partial x_i} = \int_{R^n} \phi(y)\frac{\partial^2 f}{\partial x_i \partial x_i}(x-y)dy$$

This function is continuous because the right-hand side is continuous.
ii) By the above calculation and claim(1) *we see that*

$$\triangle_x U(x) = \int_{R^n} \phi(y)\triangle_x f(x-y)dy$$

$$= \int_{R^n} \phi(y)\triangle_y f(x-y)dy$$

$$= -f(x)$$

Therefore,we conclude that

$$-\triangle_x U(x) = f(x)$$

2.5 Properties of Laplace's and Poisson's Equation

2.5.1 Mean Value Properties

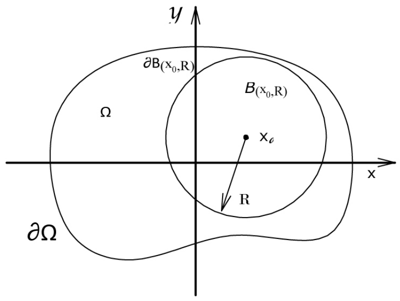
The function U has mean value property at a point $x_0 \in \Omega$ if

$$U(x_0) = \frac{1}{A(R)}\int_{\partial B(x_0,R)} U(x)ds$$

For every $R > 0$ such that $B(x_0, R)$ is contained in Ω

$$U(x_0) = \frac{1}{V(R)}\int_{B(x_0,R)} U(x)dx$$

where V(R)is the volume of open ball $B(x_0, R)$ we say that $U(x_0)$ has the second mean value property at a point $x_0 \in \Omega$ then two mean value properties are equivalent.



For a function U defined on $B(x_0, R)$ the mean value of U on $B(x_0, R)$ is given by

$$\fint_{B(x_0,R)} U(y)dy = \frac{1}{\alpha(n)r^n} \int_{B(x_0,R)} U(y)dy$$

For a function U defined by $\partial B(x_0, R)$ the mean value U $\partial B(x_0, R)$ is given by

$$\fint_{\partial B(x_0,R)} U(y)dy = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x_0,R)} U(y)ds(y)$$

Theorem 2. *Mean Value Formula*
Let $\Omega \subseteq \mathbb{R}^n$. If $U \in C^2(\Omega)$ is harmonic then

$$\fint_{\partial B(x_0,R)} U(y)ds(y) = \fint_{B(x_0,R)} U(y)dy$$

for every ball $B(x_0, R) \subset \Omega$
Proof *Assume $U \in C^2(\Omega)$ is harmonic for $R > 0$*
Define

$$\phi(R) = \fint_{\partial B(x_0,R)} U(y)ds(y)$$

For $R = 0$ define $\phi(R) = U(x_0)$. note that if U is a smooth function the $\lim_{R\to 0} \phi(R) = U(x_0)$ and therefore ϕ is a continuous function.Now if we can show that $\phi'(R) = 0$ then we conclude that ϕ is a constant function and therefore

$$\begin{aligned} U(x_0) &= \fint_{\partial B(x_0,R)} U(y)ds(y) \\ &= \fint_{\partial B(0,1)} U(x + Rz)ds(z) \end{aligned}$$

$$\begin{aligned}
Therefore \qquad \phi'(R) &= ⨍_{\partial B(0,1)} \nabla U(x_0+Rz).z ds(z) \\
&= ⨍_{\partial B(x_0,R)} \nabla U(y).\frac{y-x_0}{R} ds(y) \\
&= ⨍_{\partial B(x_0,R)} \frac{\partial U}{\partial \upsilon}(y) ds(y) \\
&= \frac{1}{n\alpha(n)R^{n-1}} \int_{\partial B(x_0,R)} \frac{\partial U}{\partial \upsilon}(y) ds(y) \\
&= \frac{1}{n\alpha(n)R^{n-1}} \int_{\partial B(x_0,R)} \nabla.(\nabla U) d(y) \quad (by\ the\ divergence\ theorem) \\
\phi'(R) &= \frac{1}{n\alpha(n)R^{n-1}} ⨍_{\partial B(x_0,R)} \triangle U(y) d(y) \\
&= \frac{1}{n\alpha(n)R^{n-1}} ⨍_{\partial B(x_0,R)} \triangle U(y) d(y) = 0
\end{aligned}$$

since U is harmonic we conclude that $\phi(R)$ is constant .

Now we prove that

$$U(x_0) = ⨍_{\partial B(x_0,R)} U(y) d(y)$$

From the first result

$$\begin{aligned}
\int_{B(x_0,R)} U(y) dy &= \int_0^R \Big(\int_{\partial B(x_0,R)} U(y) ds(y)\Big) ds \\
&= \int_0^R \Big(n\alpha(n) s^{n-1} \underbrace{⨍_{\partial B(x_0,s)} U(y) ds(y)}_{U(x_0)}\Big) ds \\
&= \int_0^R n\alpha(n) s^{n-1} U(x_0) ds \\
&= n\alpha(n) U(x_0) \int_0^R s^{n-1} ds \\
&= \alpha(n) U(x_0) R^n \\
Therefore,\ \int_{B(x_0,R)} U(y) dy &= \alpha(n) R^n U(x_0)
\end{aligned}$$

$$\begin{aligned}
Which \quad implies\ U(x_0) &= \frac{1}{\alpha(n)R^n} \int_{B(x_0,R)} U(y) dy \\
&= ⨍_{B(x_0,R)} U(y) dy
\end{aligned}$$

Next we proof that $\lim_{R\to 0^+}\phi(R) = U(x_0)$ *since* ϕ *is constant*
Let R *=t*

$$\phi(R) = \lim_{t\to 0^+}\phi(t) = lim_{t\to 0}\fint_{\partial B(x_0,t)} U(y)ds(y) = U(x_0)$$

$$\phi(R) = \lim_{t\to 0}\phi(t)$$

$$= \lim_{t\to 0}\fint_{\partial B(x_0,t)} U(y)ds(y) = U(x_0)$$

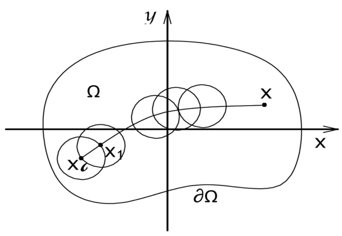
$$= \fint_{\partial B(x_0,t)} U(y)ds(y) = 1$$

$$\begin{aligned} Therefore, \qquad |U(x_0 - \fint_{\partial B(x_0,t)} U(y)ds(y)| &= |\fint_{\partial B(x_0,t)} (U(x_0) - U(y))ds(y)| \\ &\leq \int_{\partial B(x_0,t)} |U(x_0) - U(y)|ds(y) \\ &\leq max_{y\in\partial B(x_0,t)}|U(x_0) - U(y)| \end{aligned}$$

This converges to zero for $t \to 0$ *because* U *is continuous at* x_0 R *=t from our assumption* $R \to 0$
Therefore $\phi(R) = U(x_0) = 1$

Theorem 3. *Suppose that* U *has the mean value property in a bounded region* Ω *and that* U *is continuous in* $\overline{\Omega} = \Omega \cup \partial\Omega$ *if* U *is not constant in* Ω *then* U *attains its maximum value on the boundary of* Ω *, not in the interior of* Ω*.*

Proof : *U is continuous in the closed bounded domain* $\overline{\Omega}$ *then it contains its maximum* M *some where in* $\overline{\Omega}$ *our aim is to show that if U attains its max at an interior point of* Ω *, then U is constant in* $\overline{\Omega}$*.*
suppose $U(x_0) = M$ *and* x^* *be some other point of* Ω *join these points with a path covered by a sequence of overlapping balls* B_R



Consider the ball with x_0 at its center ,since U has the mean value property

$$M = U(x_0) = \frac{1}{AR}\int_{B(x_0,R)} U ds \leqslant M$$

This equality must hold through out this statement and $U = M$ through out sphere surrounding x_0 since the balls overlaps there is x_1 center of the next ball such that $U(x_1) = M$
The mean value property implies that $U = M$ through out Ω and by continuity like this gives $U(x^*) = M$ since x^* is arbitrary we conclude that $U = M$ through out Ω and by continuity throughout $\overline{\Omega}$.
Thus if U is not a constant in Ω it can attain its maximum value only on the boundary $\partial\Omega$.

2.5.2 The Weak Maximum Principle

Theorem 4. *Let Ω be bounded domain and $U \in C^2(\Omega) \cap C(\overline{\Omega})$ be a harmonic in Ω. then the maximum of U in Ω is achieved on the boundary $\partial\Omega$.*
Suppose that the function U satisfies $\triangle U = F$ in Ω with $F > 0$ in Ω recall from diferential calculus of two variables that at a point of interior maximum,$\frac{\partial^2 U}{\partial x^2} \leq 0$ and $\frac{\partial^2 U}{\partial y^2} \leq 0$. As a consequence,$\Delta U \leq 0$ at an interior maximum point.Thus if V is a function such that $\Delta V > 0$ in Ω The idea to prove Weak maximum principle is to find such a function V starting from the given harmonic function U.

$$\triangle U = F \; in \; \Omega \; with \; F > 0 \; in \; \Omega$$

proof *Consider a function $V \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying $\triangle V > 0$ in Ω we argue that V can not have a local maximum point in Ω.*

$$Let \qquad (x_0, y_0) \in \Omega \; is \; a \; local \; maximum \; point \; of \; U \; then$$

$$\frac{\partial U}{\partial x} = 0, \; \frac{\partial U}{\partial y} = 0$$

$$\frac{\partial^2 U}{\partial x^2} \leq 0 \; and \; \frac{\partial^2 U}{\partial y^2} \leq 0$$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \leq 0$$

Which contradicts $F > 0$ since U is harmonic the function

$$V(x, y) = U(x, y) + \varepsilon(x^2 + y^2)$$

$$\triangle V = \triangle U + 4\varepsilon$$

$$\triangle V = 4\varepsilon$$

$$\triangle V > 0 \qquad for\ any\ \varepsilon > 0$$

Set $M = \max_{\partial\Omega}(x^2 + y^2)$ from our argument about V it follows that $V \leq M + \varepsilon L$ in Ω. Since $U = V - \varepsilon(x^2 + y^2)$ it follows that $U \leq M + \varepsilon L$ in Ω because ε can be made arbitrary small we obtain $U \leq M$ in Ω.

2.6 Application of the Maximum Principle

We shall illustrate the importance of the maximum principle by using it to prove uniqueness and stability of the solution to Dirichlet problem.

Theorem 5. *Consider the Dirichlet problem in a bounded domain Ω.*

$$\begin{aligned} \triangle U &= f(x, y)\ (x, y) \in \Omega \\ U(x, y) &= g(x, y)\ (x, y) \in \partial\Omega \end{aligned}$$

The problem has at most one solution in $C^2(\Omega) \cap C(\overline{\Omega})$

Proof. *Assume the contrary that there exist two solutions U_1 and U_2*

$$\begin{aligned} U_1(x, y) &= g_1(x, y) \\ U_2(x, y) &= g_2(x, y) \\ V = U_1 - U_2 &= g_1(x, y) - g_2(x, y) \end{aligned}$$

Since V is harmonic in Ω then it vanishes on $\partial\Omega$

$$\begin{aligned} \triangle V &= \triangle(U_1 - U_2) \\ &= \triangle U_1 - \triangle U_2 \end{aligned}$$

The weak maximum principle implies then $0 \leq V \leq 0$
Thus $V \equiv 0$

Chapter 3

Green's Function

More generally consider an equation with a linear PDE operator L.
$LU(x) = F(x)\ in\ \Omega$
where L a linear invertible (self adjoint)differential operator. U(x) is the unknown function F(x) is the known inhomogeneous term since L is self-adjoint $L = L^*$ defined by

$$\begin{aligned} \langle V, LU \rangle &= \langle L^* V, U \rangle \\ Where\ \langle V, U \rangle &= \int V(x)W(x)U(x)dx\ where\ W(x) = weightfunction \end{aligned}$$

The solution to the PDE can be written formally $U(x) = L^{-1}F(x)$ where L^{-1} the inverse of L .
A Green's function $G(x, \xi)$ of a linear operator L is a solution of the equation.

$$\begin{aligned} LU(x) &= F(x) \\ &= \int_\Omega LG(x,\xi)F(\xi)d\xi \\ &= L\int_\Omega G(x,\xi)F(\xi)d\xi \end{aligned}$$

$$\begin{aligned} U(x) &= L^{-1}L\int_\Omega G(x,\xi)F(\xi)d\xi \\ &= -\int_\Omega G(x,\xi)F(\xi)d\xi \end{aligned}$$

Hence, the Green's function $G(x, \xi)$ satisfies
$U(x) = -\int_\Omega G(x,\xi)F(\xi)d\xi$ with $LG(x,\xi) = -\delta(x,\xi) \quad where\ (x,\xi) \in \Omega$

3.1 Green's Function for Laplace Operator

Green's theorem States that a line integral a round the boundary of a plane boundary region Ω can be computed as a double integral over Ω. Consider poisson's equation in the open bounded region Ω with boundary S.

$$\Delta U = F \; in \; \Omega$$

The Green's theorem (n is normal to S outward from Ω)

$$\int_\Omega (U\Delta V - V\Delta U)dv = \int_S (U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n})ds$$

for any function U and V with $\frac{\partial U}{\partial n} = n.\nabla U$ becomes

$$\begin{aligned} \int_\Omega U\Delta V dv &= \int_\Omega V\Delta U dv + \int_s (U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n})ds \\ &= \int_\Omega VF dv + \int_s (U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n})ds \end{aligned}$$

so if we choose $V = V(x, \xi)$ singular at $(x = \xi)$ such that $\Delta V = -\delta(x - \xi)$ then U is solution of the equation

$$U(\xi) = -\int_\Omega VF dv - \int_s (U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n})ds \tag{3.1}$$

which is an integral equation since U appears in the integrand.
To address this we consider another function $W = W(x, \xi)$ regular at$(x = \xi)$ such that $\Delta W = 0 \; in \; \Omega$ hence apply Green's theorem to the function U and W .

$$\begin{aligned} \int_s (U\frac{\partial W}{\partial n} - W\frac{\partial U}{\partial n})ds &= \int_\Omega (U\Delta W - W\Delta U)dv = -\int_\Omega WF dv \\ &= -\int_s (U\frac{\partial W}{\partial n} - W\frac{\partial U}{\partial n})ds - \int_\Omega WF dv = 0 \end{aligned}$$

Combining this equation with (3.1) we find

$$U(\xi) = -\int_\Omega (V + W)F dv - \int_s (U\frac{\partial (V + W)}{\partial n} - (V + W)\frac{\partial U}{\partial n})ds$$

So suppose we consider the Fundamental Solution of laplace equation $G = V + W$ such that

$$\begin{aligned} \Delta G &= -\delta(x - \xi) \; in \; \Omega \\ U(\xi) &= -\int_\Omega GF dv - \int_s (U\frac{\partial G}{\partial n} - G\frac{\partial U}{\partial n})ds \end{aligned}$$

The way to remove U or $\frac{\partial U}{\partial n}$ from the right hand side of the above equation depends on the choice of boundary condition.

3.2 Dirichlet Boundary Value Problems

Here the solution to equation $\Delta U = F$ in Ω satisfying the condition $U = f$ on S. we choose W such that $W + V = 0$ which is W = -V on S. That is G = 0 on S in order to eliminate $\frac{\partial U}{\partial n}$ from the equation.

$$U(\xi) = -\int_{\Omega} GFdv - \int_{s} (U\frac{\partial G}{\partial n} - G\frac{\partial U}{\partial n})ds$$

Then the solution of the Dirichlet BVP for poisson's equation.

$$\begin{aligned} \Delta U &= F \ in \ \Omega \\ U &= f \ on \ S. \end{aligned}$$

$$U(\xi) = \int_{s} f\frac{\partial G}{\partial n}ds \tag{3.2}$$

where G= V + W, W regular at $x = \xi$ with $\Delta V = -\delta(x - \xi)$ and
$\Delta W = 0 \ in \ \Omega$
So the Green's function G is a solution of the Dirichlet BVP with

$$\Delta G = -\delta(x - \xi) \ in \ \Omega$$

$$V + W \ = 0 \ on \ S$$

3.3 Neumann Boundary Value Problem

To solve $\triangle U = F$ in Ω satisfying the condition $\frac{\partial U}{\partial n} = f$ on S we choose W such that $\frac{\partial W}{\partial n} =-\frac{\partial V}{\partial n}$ on S.That is $\frac{\partial G}{\partial n} = 0$ on S. In order to eliminate U from

$$U(\xi) \ = \ -\int_{\Omega} GFdv - \int_{s} (U\frac{\partial G}{\partial n} - G\frac{\partial U}{\partial n})ds$$

However the Neumann BVP

$$\triangle G \ = \ -\delta(x - \xi) \ in \ \Omega$$

This does not satisfy a compatibility equations. We recall that Neumann BVP

$$\begin{aligned}\triangle U &= F\, in\, \Omega \\ \frac{\partial U}{\partial n} &= f\, on\, S\, is\, illposed \\ if \int_\Omega F dv &\neq \int_s f ds\end{aligned}$$

So we need to alter the Green's function a little to satisfy the compatibility equation. Put $\triangle G = -\delta + c$ where c is constant.
Then the compatibility equation for the Neumann BVP for G becomes

$$\int_\Omega (-\delta + c) dv = \int_s 0 ds = 0\; Put\; c = \frac{1}{v} where\; v\; is\; the\; volume\; of\; V$$

Now applying Green's theorem to G and U

$$\begin{aligned}\int_\Omega (G\triangle U - U\triangle G) dv &= \int_S (G\frac{\partial U}{\partial n} - U\frac{\partial G}{\partial n}) ds\; we\; get \\ U(\xi) &= -\int_\Omega GF dv + \int_s G f ds + \underbrace{\frac{1}{v}\int_\Omega U dv}_{\overline{U}}\end{aligned}$$

This shows the solution of poisson's equation with Dirichlet boundary condition is unique ,and the solution of the Neumann problem is unique up to additive constant $\overline{U}$ which is the mean value of U over Ω. Thus the solution of the Neumann BVP for poisson's equation

$$\begin{aligned}\triangle U &= F\, in\, \Omega \\ \frac{\partial U}{\partial n} &= f\, on\, S \\ U(\xi) &= \overline{U} - \int_\Omega GF dv + \int_{\partial\Omega} G f ds\; Where\; G = V + W, \quad W\; regular\; at\; (x = \xi) \\ \triangle V &= -\delta(x-\xi), and\; \triangle W = \frac{1}{v}\; in\; \Omega \\ \frac{\partial W}{\partial n} &= -\frac{\partial V}{\partial n}\; on\; S\end{aligned}$$

So Green's function G is solution of the Neumann BVP

$$\begin{aligned}\triangle G &= -\delta(x-\xi) + \frac{1}{v}\; in\; \Omega \\ \frac{\partial G}{\partial n} &= 0\; on\; S\end{aligned}$$

Example 2. *Consider the Neumann BVP for laplace's equation in the upper half plane*

$$\begin{aligned}
\triangle U &= \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \; in \; y > 0 \\
with \; \frac{\partial U}{\partial n} &= -\frac{\partial U}{\partial y} = f(x) \; on \; y = 0 \\
G(x, y, \xi, \eta) &= -\frac{1}{4\pi} \ln((x-\xi)^2 + (y-\eta)^2) - \frac{1}{4\pi} \ln((x-\xi)^2 + (y+\eta)^2)
\end{aligned}$$

Note that the Green function $G(x, y, \xi, \eta)$ acts like a weighting function for (x ,y)and neighboring points in the plane. the solution U(x,y) involves integrals of the weighting $G(x, y, \xi, \eta)$ times the boundary condition $f(\xi, \eta)$ and forcing function $F(\xi, \eta)$ on the boundary S, y =0.

$$\frac{\partial G}{\partial y} = -\frac{1}{4\pi}\left(\frac{2(y-\eta)}{(x-\xi)^2 + (y-\eta)^2} + \frac{2(y+\eta)}{(x-\xi)^2 + (y+\eta)^2}\right)$$

and as required for Neumann BVP

$$\begin{aligned}
\frac{\partial G}{\partial n} \,|_s = -\frac{\partial G}{\partial y} \,|_{y=0} &= \frac{1}{4\pi}\left(\frac{-2\eta}{(x-\xi)^2 + (\eta)^2} + \frac{2\eta}{(x-\xi)^2 + (\eta)^2}\right) = 0 \\
since \quad G(x, 0, \xi, \eta) &= \frac{-1}{2\pi} ln((x-\xi)^2 + \eta^2) \\
U(\xi, \eta) &= \frac{-1}{2\pi} \int_{-\infty}^{\infty} f(x) \ln((x-\xi)^2 + \eta^2) dx \\
That \; is \; U(x, y) &= \frac{-1}{2\pi} \int_{-\infty}^{\infty} f(\xi) \ln((x-\xi)^2 + y^2) dx
\end{aligned}$$

3.4 Robin Boundary Value Problems

The solution to equation $\triangle U = F$ Satisfies the condition

$$\frac{\partial U}{\partial n} + \alpha U = f$$

on S we choose W such that

$$
\begin{aligned}
\frac{\partial W}{\partial n} + \alpha W &= -\frac{\partial V}{\partial n} - \alpha V \ on\ S. \\
i.e \qquad \frac{\partial G}{\partial n} + \alpha G &= 0 \ on\ S \\
\int_S (U\frac{\partial G}{\partial n} - G\frac{\partial U}{\partial n})ds &= \int_s (U\frac{\partial G}{\partial n} + G(\alpha U - f))ds \\
&= \int_S (\underbrace{U\frac{\partial G}{\partial n} + G\alpha U}_{=0} - Gf))ds \\
&= -\int_S Gfds
\end{aligned}
$$

Hence the solution of Robin BVP for Poisson's equation

$$
\begin{aligned}
\triangle U &= F \ in\ \Omega \\
\frac{\partial U}{\partial n} + \alpha U &= f \ on\ S \\
U(\xi) &= -\int_\Omega GFdv + \int_S Gfds \\
Where\ G &= V + W \ (W\ regular\ at\ x = \xi)\ with \\
\triangle V &= -\delta(x - \xi)\ and \\
\triangle W &= 0 \ in\ \Omega
\end{aligned}
$$

3.5 Free Space Green's Function

Green's Function G is a linear hence we decompose the solution in the form of

$$G(x, \xi) = V(x, \xi) + W(x, \xi)$$

$$\triangle V = -\delta(x - \xi) \quad in\ \Omega$$

where V, W satisfy

$$LV = f(x)$$

$$LW = 0$$

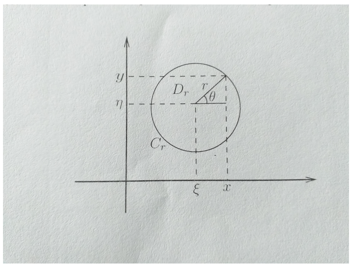
How do we find free space Green's function V ? Define V such that

$$\triangle V = -\delta(x - \xi) \quad in\ \Omega$$

Note that V does not depend on the form of the boundary .(The function V is a source term and for laplace's equation it is the potential due to a point source at the point $x = \xi$)
So we can drive that in two dimension

$$V = \frac{-1}{2\pi} \ln r$$

$$= \frac{-1}{4\pi} \ln((x-\xi)^2 + (y-\eta)^2)$$



We move to polar coordinate around (ξ, η)

$$x - \xi = r\cos\theta \quad and$$

$$y - \eta = r\sin\theta$$

and look for a solution of laplace's equation which is independent of θ and which is singular as $r \longrightarrow 0$ Laplace's equation in polar coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial V}{\partial r}) = \frac{1}{r}(\frac{\partial}{\partial r}(r)\frac{\partial V}{\partial r} + r\frac{\partial^2 V}{\partial r^2})$$

$$= \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial r^2} = 0$$

Which has solution

$$V = B\ln r + A$$

with A and B constant put$A = 0$ and to determine the constant B,apply Green's theorem to V and 1 in small disc D_r.
D_r = Small disc with radius r.
C_r = The boundary around the circle of radius r around the origin (ξ, η)
$\int_{C_r} \frac{\partial V}{\partial n} ds = \int_{D_r} \triangle V dv = - \int_{D_r} \delta(x-\xi) dv = -1$

So we choose B to make

$$\int_{C_r} \frac{\partial V}{\partial n} ds = -1$$

Now in polar coordinate $\frac{\partial V}{\partial n} = \frac{\partial V}{\partial r} = \frac{B}{r}$
and $\frac{ds}{d\theta} = r$, $ds = rd\theta$ $(going \quad around \quad circle \quad C_r)$
So

$$\int_0^{2\pi} \frac{B}{r} r d\theta = B \int_0^{2\pi} d\theta = -1 \qquad B = \frac{-1}{2\pi}$$

Hence

$$\begin{aligned} V &= \frac{-1}{2\pi} \ln r \\ &= \frac{-1}{4\pi} \ln r^2 \\ &= \frac{-1}{4\pi} \ln((x-\xi)^2 + (y-\eta)^2) \end{aligned}$$

So the free space Green's function V for the laplace's equation in n dimension is

$$V(x,\xi) = \begin{cases} \frac{-1}{2\pi} \ln r, & \text{for } , n = 2 \\ \frac{-1}{(2-n)A_n(1)} r^{2-n}, & \text{for}, n \geq 3 \end{cases}$$

where x and ξ are distinct point, $A_n(1) =$ The area of the unit disc.

3.6 Method of Images

In order to solve BVPs for poisson's equation ,such as $\triangle U = F$ in an open region Ω with some conditions on the boundary S , we seek a Green's function G such that V is a functions in Ω .
Having found the Green's function V

$$G(x,\xi) = V(x,\xi) + W(x,\xi) \; where \; \triangle V = -\delta(x-\xi)$$

and

$$\triangle W = 0 \; or \; \frac{1}{V(v)} \; volume$$

Green's function V which does not depend on the boundary conditions and so is the same for all problems.
We still need to find the function W, solution of Laplace's equation and regular in $(x = \xi)$ which fixes the boundary conditions (V does not satisfy the boundary conditions required for G by it self).
So we look for the function which satisfies

$$\triangle W = 0 \; or \; \frac{1}{V(v)} \; in \; \Omega$$

ensuring W is regular at (ξ, η) with W = -V (i.e G = 0) on S for Dirichlet Boundary conditions.

$$\frac{\partial W}{\partial n} = -\frac{\partial V}{\partial n}$$

$(\frac{\partial G}{\partial n} = -\frac{\partial V}{\partial n}) = 0$ on S for Neumann Boundary conditions.
To obtain such a function we superpose function with singularities at the image points of (ξ, η).
This may be regarded as adding appropriate point sources and seeks to satisfy the boundary conditions.
Note also that since G and V are symmetric then W must be symmetric too.($W(x, \xi) = W(\xi, x)$)
Example 3. *Suppose we wish to solve the Drichlet BVP for Laplace equation*

$$\triangle U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

in $y > 0$ *with* $U = f(x)$ *on* $y = 0$
We know that in 2-D the free space function is

$$V = \frac{-1}{4\pi} \ln((x-\xi)^2 + (y-\eta)^2)$$

If we super pose to V the function

$$W = \frac{+1}{4\pi} \ln((x-\xi)^2 + (y+\eta)^2)$$

Since solution of $\triangle W = 0$ *in* Ω *and regular* $(x = \xi, y = \eta)$ *then*

$$G(x, y, \xi, \eta) = V + W = \frac{-1}{4\pi} \ln(\frac{(x-\xi)^2 + (y-\eta)^2}{(x-\xi)^2 + (y+\eta)^2})$$

Note that setting $y = 0$ in this equation

$$\frac{-1}{4\pi}\ln\left(\frac{(x-\xi)^2+\eta^2}{(x-\xi)^2+\eta^2}\right) = 0$$

as required . The solution is given by

$$U(\xi,\eta) = -\int_s f\frac{\partial G}{\partial n}ds$$

Now, we want $\frac{\partial G}{\partial n}$ for the boundary $y = 0$ which is

$$\frac{\partial G}{\partial n}\mid_S = -\frac{\partial G}{\partial n}\mid y_= 0$$

$$G(x,y,\xi,\eta) = \frac{-1}{4\pi}\ln\left(\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta)^2}\right)$$

$$\frac{\partial G}{\partial n}\mid_{y=0} = \frac{-1}{4\pi}\frac{((x-\xi)^2+(y+\eta)^2 2(y-\eta)-((x-\xi)^2+(y-\eta)^2 2(y-\eta)))}{(x-\xi)^2+(y+\eta)^2}\cdot\frac{1}{(x-\xi)^2+(y-\eta)^2}$$

$$= \frac{-1}{4\pi}\frac{((x-\xi)^2+\eta^2)2\eta-((x-\xi)^2+\eta^2)2\eta)}{(x-\xi)^2+\eta^2}\cdot\frac{1}{(x-\xi)^2+\eta^2}$$

$$= \frac{-1}{4\pi}\cdot\frac{((x-\xi)^2+\eta^2)(2\eta+2\eta)}{(x-\xi)^2+\eta^2}\cdot\frac{1}{(x-\xi)^2+\eta^2}$$

$$= \frac{-4}{4\pi}\cdot\frac{\eta}{(x-\xi)^2+\eta^2}$$

$$= \frac{-1}{\pi}\cdot\frac{\eta}{(x-\xi)^2+\eta^2}$$

Thus

$$U(\xi,\eta) = \frac{\eta}{\pi}\int_{-\infty}^{\infty}\frac{f(x)}{(x-\xi)^2+\eta^2}dx$$

and the solution to Dirichlet BVP for $y > 0$ is given by

$$U(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(x)}{(x-\xi)^2+\eta^2}dx$$

3.7 Summary

• The primary attention of this thesis was to show the analytic solution of elliptic PDE and its Fundamental Solution in a region and on the region of a given sphere /Balls.

• The general equation of second order elliptic equation can be represented by

$$LU(x) = \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu = F$$

$$\phi(x) = \begin{cases} \frac{-1}{2\pi}\ln|x|, & \text{for } n = 2 \\ \frac{1}{n\alpha(n)(n-2)} \cdot \frac{1}{|x|^{n-2}}, & \text{for} n \geq 3 \end{cases}$$

Called fundamental solution of laplace equation has stable (solution)

We considered three types BVPs.

• Drichlet boundary value problems $\triangle U = F \ in \ \Omega$

$U = g \quad on \ \Omega$

For instance the solution for Dirichlet boundary in Green's can be given by

$$U(\xi) = \int_s f\frac{\partial G}{\partial n}ds$$

• Neumann boundary value problems

$$\triangle U = F \ in \ \Omega$$

$$n.\nabla U = g \ on \ \partial\Omega$$

For instance the solution for Neumann boundary in Green's can be given by

$$U(\xi) \ = \ -\int_\Omega GF dv + \int_s Gf ds + \underbrace{\frac{1}{v}\int_\Omega U dv}_{\overline{U}}$$

• Robin boundary value problems

$$\frac{\partial U}{\partial n} + \alpha U = f$$

The solution of Robin boundary value problem in Green's can be given by

$$U(\xi) \ = \ -\int_\Omega GF dv + \int_s Gf ds$$

• Some properties of Laplace's and Poisson's equation such as

i)The mean value property for a function U defined on $B(x_0, R)$ the mean value of U is given by

$$\fint_{B(x_0,R)} U(y)dy = \frac{1}{\alpha(n)r^n} \int_{B(x_0,R)} U(y)dy$$

ii)Maximum principle of the solution occurs on the boundary and its consequence of this is that the problem is stable.

• We discussed a Green's function is a tool to solve nonhomogeneous linear equation

$$\begin{aligned} LU(x) &= F(x) \quad x \in \Omega \\ LG(x,\xi) &= -\delta(x-\xi) \quad (x,\xi) \in \Omega \end{aligned}$$

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