UNIFIED THEORIES AND FAMILY PROBLEMS IN PARTICLE PHYSICS

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PREFACE

We have broadly divided this project along the following lines.

Our principal aim is to achieve proficiency in dealing with the basic ingredients that make up a gauge theory. We start by retracing the origin of the gauge idea leading upto the revival of gauge theories in the last few decades. We then discuss the weak interaction phenomenology (IVB theory) and this combined with electromagnetism (QED) which is well known to be a gauge theory leads one to the electroweak theory of Weinberg, Salam and Glashow - the so called GSW model. This development constitutes the largest part of our studies.

However, gauge theories have now advanced far beyond the electroweak era. We, next, study these developments leading us to the gauge theory of strong interactions - quantum chromodynamics (QCD) and subsequently to the grand unified theories (GUTs). We could not afford to study these developments in the same detail as the GSW theory given the limited scope of this project.

GUTs have implications on cosmology. Using some relevant information from cosmology an analysis of neutrino masses is presented in the last chapter.

The concluding remarks are devoted to recording the future directions suggested by our present effort.

A few appendices are included to standardize our notations and conventions, to record the barest essentials of the group theory we need and to discuss the rather unfamiliar concept of Dirac and Majorana masses.

We have tried to present a comprehensive list of references. It includes those we have exploited directly and those which contain additional material. Inspite of the vastness of the list it was impossible to do justiceto all.

CONTENTS

ABSTRACT

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The work discusses the basic techniques of gauge theories. The problem of neutrino masses is examined and the limitations it imposes on the allowed numbers of lepton and quark families is analyzed. It is argued that several existing models of unified theories may not be realistic at all.

CHAPTER I

A HISTORICAL PERSPECTIVE

1. General

Gauge theories represent perhaps the most significant development in physics of this century. Man's age old dream of unifying the fundamental forces of nature now seems a tangible reality. We believe that electromagnetism and weak nuclear forces have now been successfully unified. This remarkable success of the Glashow-Salam-Weinberg (GSW) theory[1] and the leasons it has taught us encourages us to think that all the fundamental forces of nature may be unified into a single gauge theory. It is worth noting that the potential areas of application of gauge theories encompass several other branches of physics[2] and even pure mathematics[3]. This, then, is a very general area of study not confined merely to elementary particle physics.

We begin with a historical survey of the subject. This would be very illuminating and exciting and would give us a lot of insight into the working of gauge theories which are often masked in the complexity of mathematical equations[4].

Gauge invariance is an old idea first proposed by Hermann Weyl in 1919[5]. It is a classic example of a good idea presented prematurely. Weyl's interpretation of it was easy to discredit in the absence of quantum mechanics, but fortunately being a known symmetry of Maxwell's equations it continued to be accepted as an accidental symmetry of electromagnetism.

The early history can be divided into the pre 1950 era and the post 1950 era. We shall first explore the question as to what motivated Weyl to propose the idea whose mathematical form survives as such till today and how Weyl himself was responsible for the rebirth of his idea with the help of quantum mechanics[6]. We shall then come to the new era beginning with the work of Yang and Mills[7] to extend gauge theory to strong interactions. We will find that many ideas of Weyl were rediscovered and incorporated in the modern theory.

2. The Einstein Connection

By 1919 the theory of relativity, both special and general, were on a sound footing. They told us that there are no absolute frames of reference in the universe.

In special relativity we introduce the equivalence of inertial reference frames (say S and S') which move uniformly with respect to each other. In general relativity we have to worry about the gravitational field while considering relative motion. We learn that in a freely falling elevator we can forget about the gravitational field if it is uniform. But gravitational fields vary with distance. The Lorentz transformation or rather the Lorentz symmetry group of special relativity is an example of a global symmetry. The Lorentz transformations between S and S' are the same as long as their relative velocity is the same. But the falling elevator in the general theory defines a reference frame within an infinitesimal region over which the gravitational field can be assumed to be constant. Thus the reference frame can be defined locally – at a single point in a gravitational field. Observers located at different points in the field are not inertial and hence not related by just a Lorentz transformation.

How do we relate them? Einstein told us that to relate these non-inertial observers we need a connection. Consider, as an example, a four vector A_μ. (See Appendix A for notations and conventions). Let A_μ change by an amount dA_μ as seen by an observer at x (in S) while another observer at x' (in S') notes a change dA'_μ. Special relativity tells us that

$$dA'_\nu = \frac{\partial x^\mu}{\partial x'^\nu} dA_\mu \tag{1.1}$$

expressing the linearity of the Lorentz transformation between x and x'. In general relativity we have,

$$\begin{aligned} dA'_\nu &= \frac{\partial x^\mu}{\partial x'^\nu} dA_\mu + A_\mu d\left(\frac{\partial x^\mu}{\partial x'^\nu}\right) \\ &= \frac{\partial x^\mu}{\partial x'^\nu} dA_\mu + A_\mu \frac{\partial^2 x^\mu}{\partial x'^\nu \partial x'^\lambda} dx'^\lambda \end{aligned} \tag{1.2}$$

Non-linear terms like the second term in the last equation are familiar to us while dealing with curvilinear coordinate systems. A little consideration would show that such terms can be interpreted as a kind of co-efficient which arises from curvilinear rather than linear transformation between x and x'. These curvilinear co-efficients are denoted by the special symbol,

$$\Gamma^\mu_{\nu\lambda} \equiv \frac{\partial^2 x^\mu}{\partial x'^\nu \partial x'^\lambda} \tag{1.3}$$

and are called the components of a "connection" or affine connections or Christoffel symbols[8]. The value of the connections at each spacetime point is dependent on the properties of the gravitational field. The analogy with curvilinear coordinate systems finally led Einstein to the idea of replacing gravity by the curvature of space-time in general relativity[9].

Let us briefly recapitulate what we have learnt:

Only local coordinates can be defined in a gravitational field due to its physical behaviour. This leads naturally to the idea of a connection between local coordinate systems. Special relativity is a global theory while general relativity is a local one. The local property is the key to Weyl's theory.

3. Weyl's Gauge Theory[5]

Weyl raised the following question: if gravitational force can be accounted for as a connection giving the relative orientation between local frames in space-time, may be the same can be done with electromagnetism (the only other then known force). Generalizing the concept that all physical measurements are relative he proposed that the norm of a physical vector should also depend on its location in space-time. A new connection would be needed to relate the scales of measurements at different positions. This idea became known as scale or gauge invariance. Weyl was emphasizing the local property of a gauge theory. The idea of locality was to achieve supreme importance as we shall see, determining the general structure as well as many detailed properties of gauge theories.

Consider a vector at x with norm f(x). If we shift the vector to x+dx, its norm, correct to first order is

$$f(x+dx) = f(x)+\partial_\mu f\, dx^\mu \qquad (1.4)$$

with $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$. We introduce a gauge change through a multiplicative scale factor S(x). We can visualize this as the change in the size of a meter stick. Set S(x) = 1 at x. Then,

$$S(x+dx) = 1 + \partial_\mu S\, dx^\mu . \qquad (1.5)$$

Thus the norm of the vector at x+dx is

$$Sf = f+(\partial_\mu S)f dx^\mu + (\partial_\mu f)dx^\mu \qquad (1.6)$$

and for the special case of a constant vector the norm changes by an amount

$$\Delta|f| = (\partial_\mu + \partial_\mu S)f dx^\mu . \qquad (1.7)$$

The derivative $\partial_\mu S$ is the new mathematical "connection" associated with the gauge change. Weyl set $A_\mu \equiv \partial_\mu S$. This looks reasonable, for a second gauge change connected with a scale factor Λ implies,

$$\partial_\mu S \longrightarrow \partial_\mu S + \partial_\mu \Lambda \qquad (1.8)$$

or

$$A_\mu \longrightarrow A_\mu + \partial_\mu \Lambda \qquad (1.9)$$

Thus Weyl's assertion was compatible with electromagnetism.

Unfortunately his idea of scale invariance ran into trouble with quantum mechanics. Quantum mechanics tells us that the compton wavelength λ of a particle is related to its mass m by $\lambda \sim \frac{1}{m}$ and associates a natural scale with it. But m does not depend on position and hence the same with λ. Weyl's proposition failed. But the idea of local gauge invariance luckily remained because Maxwell's equations had this invariance.

4. Canonical Momentum and Electromagnetic Potential

A very important clue to the meaning of gauge invariance came from classical mechanics. We can get the Maxwell's equations and the equations of motion for charged particles from a single physical principle - the principle of least action[10]. All we need is to replace the four momentum p_μ by $p_\mu - eA_\mu$. We then construct the Lagrangian density,

$$\mathcal{L} = \frac{1}{2m}(p_\mu - eA_\mu)^2 - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \qquad (1.10)$$

where,

$$F_{\mu\nu} \equiv \partial_\nu A_\mu - \partial_\mu A_\nu \qquad (1.11).$$

Hamilton's principle tells us that we get the desired result by minimizing the action,

$$S \equiv \int L(q,t)dt \qquad (1.12)$$

i.e. to say that the system follows the path for which S is a minimum. The use of the above path integral is now familiar in quantum mechanics[11]. The minimum of the action is found by varying the generalized coordinates q and velocities dq/dt and setting the variation $\delta S = 0$. We get the equations of motions as

$$\partial_\lambda \left(\frac{\partial \mathcal{L}}{\partial(\partial_\lambda q_\mu)}\right) - \frac{\partial \mathcal{L}}{\partial q_\mu} = 0 \qquad (1.13)$$

where the coordinate q_μ is now defined to be a four vector. If q_μ is identified with spatial coordinates we get the Lorentz force equation and if we set $q_\mu = A_\mu$ we recover Maxwell's equations. Thus, in the Hamiltonian formalism the potential acquires an added significance in that it is now an integral part of the canonical momentum and a generalized coordinate in the Euler-Lagrange equations. These facts were crucial to the rebirth of gauge invariance.

5. Quantum Mechanics and Gauge Theory

Though initially quantum mechanics seemed to invalidate Weyl's idea, he and others soon realized that gauge theory could be given a new meaning. The Schroedinger equation of a charged particle in an electromagnetic field is easily seen to be invariant under the familiar gauge transformation

$$A_\mu \longrightarrow A_\mu - \partial_\mu \lambda(x) \qquad (1.14)$$

provided the wave function undergoes the perfectly permissible change,

$$\psi \longrightarrow \psi e^{-ie\lambda} \qquad (1.15)$$

The phase of the wave function satisfies the requirement of a new local variable. The phase is not involved in the measurement of a space-time quantity like the length of a vector and thus the previous objection to the Weyl theory no longer applies. A different choice of phase at each point in space can then be accommodated easily by interpreting A_μ as a connection relating phases at different points. Gauge invariance now becomes an invariance under change of phase. The arbitrariness of the potential now translates into the freedom of choosing the phase of a wave function locally without affecting the equations of motion. Gauge invariance was thus rediscovered.

6. Aharanov-Bohm Effect[12]

Our brief excursion into the history of gauge theories will be incomplete without a mention of the Aharanov-Bohm Effect. Thirty years after Weyl's rediscovery of gauge invariance Aharanov and Bohm showed that A_μ can produce observable effects and in fact in quantum mechanics A_μ is more fundamental than the electric and magnetic fields[13]. The following comment by Feynman[14] on this is illuminating.

> It is interesting that something like this should be around for thirty years, but because of certain prejudices of what is and is not significant, continues to be ignored.

The Aharanov-Bohm effect contradicted the accepted notion that only electric and magnetic fields can produce observable effects. Potential now acquires the status of a physical field that is directly observable and its arbitrariness does not matter.

We have thus come to realize that electromagnetic theory could be interpreted as a local gauge theory in quantum mechanics. In analogy with Weyl's original theory, the phase of a particle wave function can be identified as a new physical degree of freedom which is dependent on the space-time position. The phase value can be arbitrarily altered. This implies a connection between phase values at nearby points. The potential A_μ provides this connection. The Aharanov-Bohm effect demonstrates the intimate relationship between A_μ and the phase change. Interpreting the phase as a local variable instead of the norm of a vector electromagnetism can be interpreted as a local gauge theory as Weyl had envisioned.

Let us also note that the set of all gauge transformations form a U(1) group. This does not arise from any form of coordinate transformation like the more familiar spin-rotation group SU(2) or the Lorentz group. Thus one had lost the original interpretation proposed by Weyl of a new space-time symmetry. Even with the Aharanov-Bohm effect becoming known gauge transformations seemed more like a property of quantum mechanics rather than of electromagnetism. For these reasons local gauge invariance could not still be recognized as a fundamental principle of physics even after its rediscovery. Further, Maxwell's equations, came first and gauge invariance later and it thus played no role in defining the dynamical content of electromagnetism, unlike the case of general relativity.

Attitudes changed after the 1950's as we shall see in the coming sections.

7. Towards a New Gauge Theory

Two developments were crucial. The first was Yukawa's suggestion[15] (with an eye on the photon being a mediator of the long range electromagnetic forces) that exchange of "massive photons" - the π-mesons were responsible for nuclear forces. Three suitable π-mesons were subsequently discovered.

The second was Heisenberg's suggestion[15] that the charge independence of nuclear forces implied the existence of an abstract symmetry group SU(2). Strong interactions were SU(2) symmetric. We know now that the nucleons form a doublet of SU(2) and the pions a triplet. Thus pions were both carriers of the strong force and eigenstates of the isospin symmetry group SU(2).

But, still, the exchange of π-mesons does not lead to a gauge theory of strong interactions for the SU(2) group we mentioned was introduced as a global symmetry. Pions were still not the same vis-a-vis strong interactions as photons were vis-a-vis electromagnetism. Isospin was an internal quantum number independent of space-time position. So there was no need for a connection or an isospin potential field whose quantum could be identified with the pion. Gauge theories of pion exchange were constructed[16] but were unsuccessful. Inspite of these difficulties the advent of isospin heralded a new era which was to lead one to modern gauge theories.

8. Yang-Mills Gauge Theory[7]

In 1954 Yang and Mills set out to describe strong interactions as a gauge invariant field theory. Their gauge group was the isospin group SU(2) - the only candidate they could think of. This idea puts our definition of a particle in jeopardy. Assume for a moment that electromagnetism is switched off and we forget about the small mass difference between the neutron and the proton. The proton and neutron become totally indistinguishable. Let us label the proton as the "up" state of the nucleon and the neutron as the "down" state. SU(2) being an exact gauge symmetry the up state at one location may be the down state at another. Local symmetry leaves us free to choose the up and the down states at each point independently!

Having made the choice at one location we need a rule to compare it with the choice at another. Weyl told us that we need a connection for this. Accordingly Yang and Mills proposed a new isospin potential field. SU(2) is a non-Abelian group unlike the simple U(1) of electromagnetism where the potential A_μ provides a connection between phase values of ψ at different locations. Here the phase has to be replaced by a more complicated local variable specifying the direction of the isospin (details in subsequent chapters). Qualitatively, the connection arises as follows. SU(2) is also the group of rotations in a 3-dimensional space. Visualize the "up" component of isospin as a vector in an abstract "isospin space". To relate the "up" states at x and y we ask how much the "up" state at x must be rotated so as to be in the same direction as the "up" state at y. The connection must act like a rotation in isospin space. The connection thus performs the same job as the SU(2) group transformations themselves. Thus the isospin connection and hence the potential acts like the symmetry group itself. This concept is at the heart of a gauge theory. This is how the symmetry group is built into the dynamics of the interaction between particles and fields. Remember also that the amount of phase change must be proportional to the potential to ensure that the particle wave equation is gauge invariant. Thus the most general form of the potential can be seen to be,

$$A_\mu = \Sigma A_\mu^i (x)\, T_i \qquad (1.16)$$

where $A_\mu^i(x)$ depend on space-time position and T_i are the generators of SU(2). (For electromagnetism T_i is just a multiple of the unit matrix). A_μ is thus a field in space-time and an operator in the isospin space. It carries internal quantum numbers. The potential must have three charge components corresponding to the three angular momentum-like operators T_+, T_- and T_3. The component $T_+(T_-)$ can change a neutron(proton) into a proton (neutron). The Yang-Mills field must thus carry charge unlike the photon and the Yang-Mills field can interact with itself unlike the photon. But like the photon the Yang-Mills field must remain massless, for a mass term in the Lagrangian has the form $m^2 A_\mu A^\mu$ and breaks gauge invariance. Our troubles continue. Such a field exhibits long range behaviour and cannot reproduce the short range nature of the strong force. The story of the revival of the Yang-Mills theories came through developments in seemingly unrelated areas and would be taken up in detail in the rest of the work. All the same the seeds of the modern gauge theories were already sown. The SU(2) gauge transformations could not be considered merely as a phase change. We have learnt that local gauge symmetry was a powerful principle that could provide insight into the newly discovered internal quantum numbers like isospin. Isospin was not merely a label for the charge states of particles but it was crucially involved in determining the fundamental form of the interaction.

Yang-Mills theory revived the idea that elementary particles might have new degrees of freedom in some "internal" space. By showing how these internal degrees of freedom could be unified in a nontrivial way with the dynamical motion in space-time a new type of geometry was discovered in physics. Unfortunately, we shall not discuss this point any further here.[4]

In summary we have gone through roughly four decades of history of the subject during which gauge invariance evolved from an accidental symmetry of Maxwell's equations to a basic principle governing the fundamental forces of nature.

We shall persue the gauge idea in the coming chapters. To this end we now turn to weak interactions.

CHAPTER 2

WEAK INTERACTIONS

I. Intermediate Vector Boson (IVB) Theory

We know that weak interactions are well described by the V-A theory[17,18,19] which represents a contact interaction. However, we now believe that all interactions are mediated by suitable bosonic quanta. Hence we shall consider here the modern version of the theory known as the IVB theory[20].

Consistency with experimental results suggests that the lepton fields enter the weak interaction Lagrangian in the following combination*.

$$J_\alpha = \sum_\ell \bar{\ell}\,\gamma_\alpha(1-\gamma_5)\nu_\ell \qquad (2.1)$$

$$J^\dagger_\alpha = \sum_\ell \bar{\nu}_\ell\,\gamma_\alpha(1-\gamma_5)\ell \qquad (2.2)$$

where J_α and $J^\dagger_\alpha$ are the leptonic currents and ℓ and ν_ℓ are the quantized fields of leptons and neutrinos respectively.

In analogy with the electromagnetic interaction which is transmitted by photons we assume that the weak interactions are transmitted by quanta called W bosons. The QED interaction Lagrangian density

$$\mathcal{L}_{QED} = e\bar{\ell}\gamma^\alpha \ell A_\alpha \qquad (2.3)$$

then suggests that the leptonic interactions of the IVB theory be taken as

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*Refer to Appendix [illegible] for notations.

$$\mathcal{L}_I = -g_W J^{\dagger}_{\alpha} W_{\alpha} - g_W J^{\alpha} W^{\dagger}_{\alpha} \tag{2.4}$$

where g_W is a dimensionless coupling constant analogous to e in QED and W_α are the fields that describe the W particles as A_α describes the photons in QED.

Assuming that the neutrinos have zero mass the operator

$$\nu^L_\ell = \tfrac{1}{2}(1-\gamma_5)\nu_\ell \tag{2.5}$$

can annihilate only negative helicity neutrinos and creates positive helicity antineutrinos since ν_ℓ is linear in neutrino absorption and antineutrino creation operators.

We can also write an equation similar to equation (2.5) for high energy leptons ($m_\ell = 0$ effectively)

$$\ell^L = \tfrac{1}{2}(1-\gamma_5)\ell \tag{2.6}$$

Equation (2.1) can now be recast to read,

$$J_\alpha = 2\sum_\ell \bar{\ell}^L \gamma_\alpha \nu^L_\ell \tag{2.7}$$

The Feynman rules for the IVB theory can be worked out in analogy with QED[20]. The W propagator turns out to be

$$iD_F^{\alpha\beta}(k,m_W) = \frac{i\left(-g^{\alpha\beta} + \frac{k^\alpha k^\beta}{m_W^2}\right)}{k^2 - m_W^2 + i\epsilon} \tag{2.8}$$

For $k^2 \ll m^2_W$, $iD_F^{\alpha\beta}(k,m_W) \longrightarrow \frac{i}{m_W^2} = \text{constant}$ (2.9)

This enables one to recover the contact V-A interaction in the above limit. Examples of processes describable by lowest order IVB theory are indicated in (Fig. 2.1) and (Fig 2.2).

2. Difficulties of the IVB Theory

The IVB theory has been applied to various weak processes and successfully accounts for many but fails to describe some perfectly allowed processes like

$$\nu_\mu + e^- \longrightarrow \nu_\mu + e^- \qquad (2.10)$$

With the interaction (2.4) the Feynman diagram for process (2.10) is shown in (Fig. 2.3) at the end of Chapter 2. Thus we see that process (2.10) necessarily involves the exchange of two W bosons.

On the other hand processes like

$$\nu_e + e^- \longrightarrow \nu_e + e^- \qquad (2.11)$$

take place through an exchange of only one W boson. The Feynman diagram for process (2.11) is shown in (Fig. 2.4).

Since the coupling constant g_W is small we expect the cross-section for (2.10) to be smaller than that of (2.11). But the loop integrals to which the Feynman graphs in (Fig. 2.3) give rise are divergent and the IVB theory is not renormalizable. The IVB theory thus allows one to calculate only those processes which do not involve loop integrals in the lowest order of perturbation theory.

The experimentally measured cross-sections for process (2.10) are comparable to that of process (2.11). We notice that process (2.10) cannot take place through one W boson exchange without violation of the law of conservation of lepton number. This suggests the existence of an uncharged boson, the Z^0 vector boson, which is absent in the IVB theory, under the exchange of one of which process (2.10) can take place and which saves, ofcourse, lepton number conservation law. This requires a modification of the IVB interaction(2.4) by adding to it terms which allow the $\nu_\mu - e$ scattering to occur as a one boson exchange process. These extra terms lead to Feynman diagrams like that shown in (Fig. 2.5).

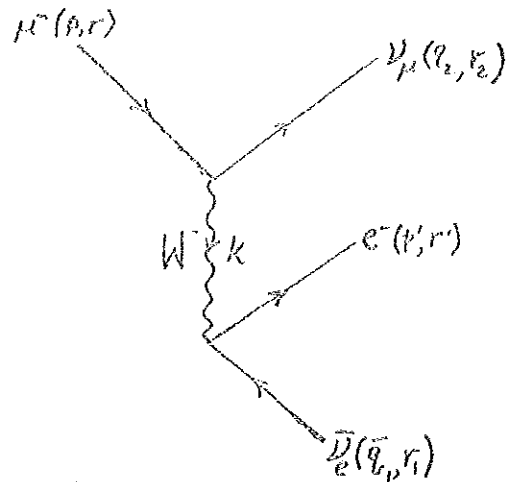
The currents involving the exchange of a neutral vector boson Z^o are called neutral currents unlike the charged currents J_α and $J_\alpha^\dagger$ which involve the exchange of a charged vector boson W.

The attempt to incorporate such a neutral current term eventually brings us to the electroweak theory as we shall see in the later chapters.

Another difficulty connected with the non-renormalizability of the IVB theory is worth noting. Even the lowest order calculations seem to run into trouble with the unitarity of the S-matrix. The $\nu\bar{\nu}$ scattering cross-section grows as E^2 where E is the total center of mass energy. This cross-section can grow arbitrarily large when E is several hundred GEV and would ultimately violate the maximum value allowed by ordinary quantum mechanics which is knowe as the unitarity limit.

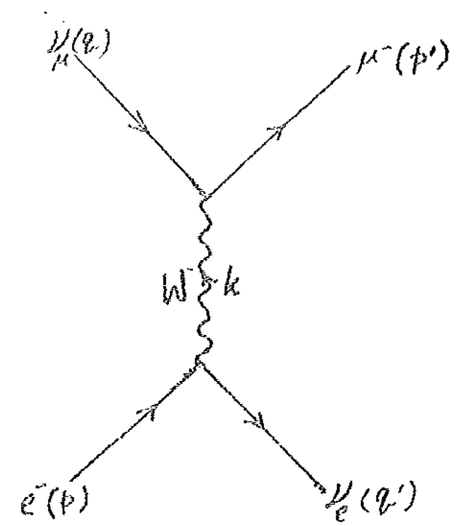
A further difficulty arises in describing simple processes like $\pi \dashrightarrow \mu + \nu$ which are easily described by the contact V-A theory. This [illegible] too few particles to permit a W boson exchange mechanism. [illegible] we can see a pointer here towards the composite structure of the pion. This difficulty is resolved when we consider hadrons as composites of quarks.

In the next chapter we turn to the simplest gauge theory - electromagnetism. We shall then see that difficulties of the IVB theory are pointing towards a gauge theory of description of the weak interactions.

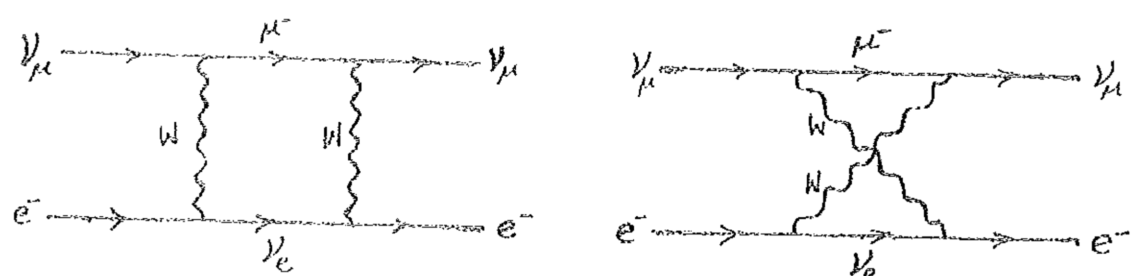


(Fig. 2.1) $\mu^- \longrightarrow e^- + \bar{\nu}_e + \nu_\mu$
(Muon decay)

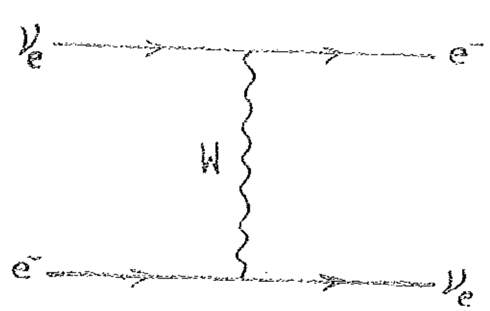
(r = spin index)



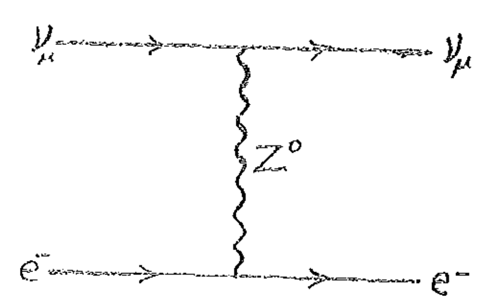
(Fig. 2.2) $\nu_\mu + e^- \longrightarrow \nu_e + \mu^-$
(Inverse muon decay)



(Fig 2.3) $\nu_\mu + e^- \longrightarrow \nu_\mu + e^-$
(leading contributions as given by the IVB theory)



(Fig. 2.4) $\nu_e + e^- \longrightarrow \nu_e + e^-$
(leading contribution given by the IVB theory)



(Fig. 2.5) $\nu_\mu + e^- \longrightarrow \nu_\mu + e^-$
(not contained within the IVB Theory)

CHAPTER 3

ELECTROMAGNETISM - A SIMPLE GAUGE THEORY

We consider this trivial example to appreciate some typical characteristics of a gauge theory. Consider the free field Lagrangian density of a massive fermion field:

$$\mathcal{L}_0 = \bar{\psi}(i\not{\partial} - m)\psi \qquad (3.1)$$

Let U be a one parameter unitary transformation defined as

$$U = e^{+i\lambda}, \lambda \text{ real} \qquad (3.2)$$

These transformations form an Abelian group which is appropriately called a U(1) group. If λ is space-time independent then U is said to be a global transformation and if $\lambda \equiv \lambda(x)$, U is said to be a local trnsformation. Correspondingly, we call U(1) a global or a local group respectively. In the sequel the local transformations will be referred to as gauge transformations in accordance with Weyl's usage and the corresponding group as a gauge group. The Lagrangian $\mathcal{L}_1$ is invariant under a global U(1) transformation. We say that U(1) is a global symmetry of $\mathcal{L}_0$. This means that when

$$\psi \longrightarrow \psi' = U\psi = e^{+i\lambda}\psi \qquad (3.3)$$

$\mathcal{L}_0$ remains invariant provided λ is independent of x.

How does $\mathcal{L}_0$ behave under a local transformation? Clearly, it is not invariant, for, now $\partial_\mu\psi$ does not transform as ψ itself and an additional term appears. We now want to see how one can restore the invariance. Well, we must replace $\partial_\mu\psi$ by $D_\mu\psi$ such that

$$(D_\mu\psi)' = U(D_\mu\psi) \qquad (3.4)$$

We shall henceforth call such a $D_\mu\psi$ a covariant derivative of ψ since it transforms exactly like ψ. Let us set

$$D_\mu \equiv \partial_\mu + iqA_\mu \qquad (3.5)$$

and investigate the behaviour of A_μ. Combining equations (3.4) and (3.5) we find,

$$(\partial_\mu + iqA'_\mu)U\psi = U(\partial_\mu + iqA_\mu)\psi \qquad (3.6)$$

or $$(\partial_\mu U)\psi + iqA'_\mu U\psi = iqUA_\mu\psi$$

or $$iqU^{-1}A'_\mu U = iqA_\mu - U^{-1}\partial_\mu U$$

or $$A'_\mu = UA_\mu U^{-1} + \frac{i}{q}(\partial_\mu U)U^{-1} \qquad (3.7)$$

or $$A'_\mu = A_\mu - \frac{1}{q}\partial_\mu\lambda \qquad (3.8)$$

Thus a D_μ defined by equation (3.5) with A_μ constrained to transform under the U(1) gauge transformation according to equation (3.7) gives us a Lagrangian density

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi \qquad (3.9)$$

$$= \bar{\psi}(i\not{\partial} - m)\psi - q\bar{\psi}\gamma^\mu\psi A_\mu \qquad (3.10)$$

Identifying A_μ with the electromagnetic field and q with the charge of the field annihilated by ψ we have recovered the Lagrangian of the fermion field interacting with the electromagnetic field A_μ in accord with the standard minimal prescription. Now, one might wonder what really is the purpose of the whole exercise when the result was so easily obtainable through the minimal substitution. It is like using a sledge hammer to cruck a nut.

The point is this: By demanding the gauge invariance of a free Lagrangian, we have learnt that a four-vector field A_μ must exist. This field must couple in a specific manner to the matter fields and transform in a precisely difined way along with the matter fields under the group transformation. The constant q introduced infront of the field A_μ in the definition of the covariant derivative appears as a coupling constant determining the strength of the interaction between the gauge field A_μ and the matter field. The gauge field must be massless for a mass term $m^2 A_\mu A^\mu$ is not invariant under the trnsformation law imposed upon the gauge field.

The fact that the gauge prescription generates the correct electromagnetic interaction is not so important (though very satisfying) for we already know the interaction from the classical limit. But when the classical limit is not known such a prescription is immensely useful. This turns out to be so as we shall realize in the chapters to follow.

The local symmetry group of electromagnetism is an Abelian group. We shall be shortly dealing with non-Abelian gauge groups. Some new features would emerge. Equation (3.7) already suggests this. We note that the photon has no electric charge and does not couple to itself since it couples only to charged objects. The gauge fields of non-Abelian theories will have self-interactions.

The existence of a symmetry group implies conserved currents in accordance with Noether's theorem. In the present case we have the conservation of the electromagnetic current. The same conserved current emerges whether we use a global or a local U(1). However the local group generates interactions and new Feynman vertices.

Gauge invariance yields a photon propagator behaving as $\frac{1}{k^2}$ for large k. On the other hand the W propagator behaves as a constant for large k and so the loop integrals diverge far more severely than in the photon case. This fact renders the IVB theory nonrenormalizable. Thus if the IVB theory could be transformed into a gauge theory, renormalizability could be expected. We are thus motivated to build a gauge theory to describe weak interactions.

We can anticipate one trouble if we do succeed. The W-bosons would turn out to be massless and describe a long range interaction.

If a mass term is included at the Lagrangian level then gauge invariance and hence renormalizability would be lost. We shall see that this paradox can be resolved and both the desired objectives namely, the invariance of the Lagrangian at the bare level and the inclusion of the mass term can be achieved.

Our present exercise has revealed some important building blocks of a gauge theory. These are the matter fields, the gauge group and the gauge bosons. The comments of the last paragraph indicate that we have to incorporate a suitable mass generating mechanism about which our bare Lagrangian should not know anything.

In the next chapter we follow the clues and look for an underlying symmetry group for weak interactions so that we can gauge it and watch the fun.

CHAPTER 4

TOWARDS A GAUGE THEORY OF WEAK INTERACTIONS

1. The Global Symmetry Group of Weak Interactions

To keep matters simple we consider leptons only. We treat them as massless. The free Lagrangian density is

$$\mathcal{L}_o = i\{\bar{\ell}\not\partial\ell + \bar{\nu}_\ell\not\partial\nu_\ell\} \qquad (4.1)$$

Summation over $\ell(=e,\mu,\tau,\ldots)$ is assumed. Since weak interactions seem to use only left-handed fields we rewrite equation (4.1) in terms of left and right handed fermion fields using

$$\psi^{L,R} = \tfrac{1}{2}(1 \mp \gamma_5)\psi, \psi^L + \psi^R = \psi \qquad (4.2)$$

A little algebra tells us that

$$\mathcal{L}_o = i\{\bar{\ell}^L\not\partial\ell^L + \bar{\nu}^L_\ell\not\partial\nu^L_\ell + \bar{\ell}^R\not\partial\ell^R + \bar{\nu}^R_\ell\not\partial\nu^R_\ell\} \qquad (4.3)$$

We now combine the left handed fields into a two-component field

$$L \equiv \binom{\nu_\ell^L}{\ell^L} \equiv \binom{\nu_\ell}{\ell}_L \qquad (4.4)$$

and correspondingly

$$\bar{L} = (\bar{\nu}_\ell, \bar{\ell})_L \qquad (4.5)$$

Thus, $$\mathcal{L}_o = i\{\bar{L}\not\partial L + \bar{\ell}^R\not\partial\ell^R + \bar{\nu}^R_\ell\not\partial\nu^R_\ell\} \qquad (4.6)$$

Clearly, equation (4.6) looks very unsymmetrical between the left and right fields, unlike equation (4.3). We shall see that the left-right asymmetry of weak interactions can be described in terms of different transformation properties of the left and right-handed fields. Now for the two component fields arises the possibility of two-dimensional transformations which leave bilinear forms $\bar{L}(\cdots)L$ invariant. To this end we introduce the Pauli matrices τ_1, τ_2, τ_3 (See Appendix B) satisfying the SU(2) algebra

$$[\tau_i, \tau_j] = 2i\varepsilon_{ijk}\tau_k \tag{4.7}$$

A typical SU(2) transformation is

$$U(\vec{\omega}) = e^{i\omega_j T_j} \tag{4.8}$$

where, for the two-dimensional case, the generator $T_k \equiv \frac{1}{2}\tau_k$, $\vec{\omega} \equiv (\omega_1, \omega_2, \omega_3)$. Under this transformation we have

$$L \to L' = U(\vec{\omega})L \text{ and } \bar{L} \to \bar{L}' = \bar{L}U^{\dagger}(\vec{\omega}) \tag{4.9}$$

The term $\bar{L}\not{\partial}L$ in $\mathcal{L}_o$ is invariant under a global SU(2) transformation given in equation (4.8) i.e. when the ω's are not functions of x. We call L a weak isodoublet. The origin of this terminology should be self-evident. We take the right-handed fields to be isoscalars so that they are invariant under the global transformations considered here. Thus the entire $\mathcal{L}_o$ is invariant under global SU(2) ($\equiv$ weak isospin group) transformations.

From the invariance of $\mathcal{L}_o$, we have, by the Noether's theorem, three conserved currents

$$j_i^{\alpha} = \tfrac{1}{2}\bar{L}\gamma^{\alpha}\tau_i L \, , \; i = 1,2,3 \tag{4.10}$$

Equations (4.10) follow immediately by considering infinitesimal SU(2) transformations under which

$$L \to L' = (1 + i\omega_j \tau_j/2) L \text{ and}$$

$$\ell^R \to \ell'^R = \ell^R \qquad (4.11)$$

$$\nu_\ell^R \to \nu_\ell'^R = \nu_\ell^R$$

These (eqn. 4.10) are called weak isospin currents. The currents of the IVB theory are simply

$$J^\alpha = 2(J_1^\alpha - iJ_2^\alpha), \quad J^{\alpha +} = 2(J_1^\alpha + iJ_2^\alpha) \qquad (4.12)$$

and are ofcourse conserved.

Corresponding to J_i^α we have three conserved charges

$$I_i^W = \int d^3\vec{x} J_i^o, \quad i = 1,2,3 \qquad (4.13)$$

called weak isospin charges.

In addition to the IVB currents we have an extra conserved current, namely,

$$J_3^\alpha = \tfrac{1}{2}\bar{L}\gamma^\alpha \tau_3 L$$

$$= \tfrac{1}{2}\{\bar{\nu}_\ell^L \gamma^\alpha \nu_\ell^L - \bar{\ell}^L \gamma^\alpha \ell^L\} \qquad (4.14)$$

This is a neutral current since it couples either electrically neutral leptons or electrically charged leptons like the electromagnetic current.

$$S^\alpha = -e\bar{\ell}\gamma^\alpha \ell \qquad (4.15)$$

This contrasts with the charge changing currents of the IVB theory. It is remarkable that the second term in equation (4.14) is proportional to the electromagnetic current S^α. Here, then, is the first signal that in our theory weak and electromagnetic interactions would appear to be interconnected.

Borrowing from our experience with the isospin symmetry of strong interactions we define the weak hypercharge current

$$\begin{aligned} J_y^\alpha &= \frac{S^\alpha}{e} - J_3^\alpha \\ &= -\tfrac{1}{2}\bar{L}\gamma^\alpha L - \bar{\ell}^R\gamma^\alpha \ell^R \end{aligned} \tag{4.16}$$

The corresponding charge - the weak hypercharge is

$$Y = \int d^3\vec{x}\, J_y^o \tag{4.17}$$

From equations (4.16 and (4.17) we get

$$Y = \frac{Q}{e} - I_3^W \tag{4.18}$$

where Q is the electric charge.

Thus $Y = -\frac{1}{2}$ for the left-handed leptons, $Y = -1$ for the right handed ℓ^- leptons and $Y = 0$ for ν_ℓ^R

Defining a global U(1) transformation

$$U_y(1) = e^{i\theta y} \tag{4.19}$$

the invariance of $\mathcal{L}_o$ under the above U, implies directly the conservation of Y.

We have thus verified that the Lagrangian $\mathcal{L}_o$ has a global symmetry SU(2) and a global symmetry $U_y(1)$. In the next section we shall consider these symmetries as local symmetries.

2. The Gauge Invariant Electroweak Interaction

We now consider local SU(2) and U(1) transformations. First consider SU(2). A general local transformation is

$$U(\vec{\omega}) = e^{ig\omega_j T_j} \tag{4.20}$$

where ω_j are now functions of x and T_j are three generators of SU(2). Acting on isodoublets $T_j = \frac{1}{2}\tau_j$ and acting on isosinglets T_j are unit matrices.

Thus
$$U(\vec{\omega})L = e^{ig\omega_j \frac{1}{2}\tau_j}L \tag{4.21}$$
$$U(\vec{\omega})R = R$$

where R stands for any right handed fields which are all isosinglets in our picture. We now demand the invariance of $\mathcal{L}_o$ under (4.21). Our experience of the previous chapter tells us that we must replace $\partial_\mu L$ by a suitable covariant derivative $D_\mu L$. Analogously we define

$$D_\mu L \equiv (\partial_\mu + ig\tau_j W^\mu_j/2)L \tag{4.22}$$

so that
$$\mathcal{L}_o + \mathcal{L}_g' = i\{\bar{L}\not{D}L + \bar{\ell}^R\not{\partial}\ell^R + \bar{\nu}^R_\ell\not{\partial}\nu^R_\ell\} \tag{4.23}$$

We have introduced three real gauge fields W^μ_j compared to one gauge field A_μ of QED, as there are now three conserved charges I^W_j and the gauge transformation contains three arbitrary functions ω_j.

Defining
$$W_\mu \equiv \tfrac{1}{2}\tau_j W_{j\mu} \tag{4.24}$$

and recalling equation (3.7) we immediately deduce that gauge invariance will be ensured if under the gauge transformation

$$W_\mu \rightarrow W'_\mu = UW_\mu U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1} \tag{4.25}$$

Taking ω_j to be infinitesimal we can easily deduce that

$$W'_{i\mu} = W_{i\mu} - \partial_\mu \omega_i - g\varepsilon_{ijk}\omega_j W_{k\mu} \tag{4.26}$$

We stop here to first consider the effect of local $U_y(1)$ transformation. We write it as

$$U_y(\theta) = \exp\{ig'y\theta\} \tag{4.27}$$

where θ now is a function of x. This is exactly analogous to the case of QED so that the invariance of the Lagrangian requires a covariant derivative

$$D_\mu = \partial_\mu + ig'YB_\mu \tag{4.28}$$

where the associated gauge field B_μ must transform as

$$B_\mu \rightarrow B'_\mu = B_\mu - \partial_\mu \theta \tag{4.29}$$

Making the replacements given by equations (4.22) and (4.28) simultaneously in the Lagrangian density (4.1) we obtain

$$\mathcal{L} = i\{\bar{L}\not{D}L + \bar{\ell}^R\not{D}\ell^R + \bar{\nu}_\ell^R\not{D}\nu_\ell^R\} \tag{4.30}$$

where

$$D_\mu L = (\partial_\mu + ig\tau_j W_{j\mu}/2 - \frac{ig'B_\mu}{2})L \tag{4.31}$$

$$D_\mu \ell^R = (\partial_\mu - ig'B_\mu)\ell^R \tag{4.32}$$

$$D_\mu \nu_\ell^R = \partial_\mu \nu_\ell^R \tag{4.33}$$

We choose the fields $W_{i\mu}$ to be invariant under $U_y(1)$ i.e. they have $y = 0$ and B_μ to be invariant under SU(2) i.e. B_μ is an isoscalar. Then $\mathcal{L}$ is both SU(2) and $U_y(1)$ gauge invariant. It is said to be SU(2)xU(1) gauge invariant.

We now write

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 \tag{4.34}$$

$$\mathcal{L}_1 = - gJ_i^\mu W_{i\mu} - g' J_y^\mu B_\mu \tag{4.35}$$

$\mathcal{L}_1$ represents the interaction of the weak isospin currents and weak hypercharge current defined previously with the gauge fields $W_{i\mu}$ and B_μ.

We now define

$$W_\mu \equiv \frac{1}{\sqrt{2}} \{W_{1\mu} - iW_{2\mu}\}$$

$$W_\mu^\dagger \equiv \frac{1}{\sqrt{2}} \{W_{1\mu} + iW_{2\mu}\}$$

The first two terms of $\mathcal{L}_1$ become

$$-g \sum_{i=1}^{2} J_i^\mu W_{i\mu} = - \frac{g}{2\sqrt{2}} \{J^{\mu\dagger} W_\mu + J^\mu W_\mu^\dagger\} \tag{4.37}$$

We rewrite the remaining two terms of $\mathcal{L}_1$ by introducing two Hermitian fields A_μ and Z_μ as

$$W_3 = \text{Cos}\theta_W Z_\mu + \text{Sin}\theta_W A_\mu \tag{4.38}$$

$$B = -\text{Sin}\theta_W Z_\mu + \text{Cos}\theta_W A_\mu$$

θ_W is called the Weinberg angle. Combining (4.38) with $J_y = S^\mu/e - J_3^\mu$ we obtain

$$- g J_3^\mu W_{3\mu} - g' J_Y^\mu B_\mu$$

$$= -\frac{g'}{e} S^\mu [-\, \mathrm{Sin}\,\theta_W Z_\mu + \mathrm{Cos}\,\theta_W A_\mu]$$

$$- J_3^\mu \{ g[\mathrm{Cos}\,\theta_W Z_\mu + \mathrm{Sin}\,\theta_W A_\mu] \tag{4.39}$$

$$- g'[-\,\mathrm{Sin}\,\theta_W Z_\mu + \mathrm{Cos}\,\theta_W A_\mu]\}$$

We now identify A_μ with the electromagnetic field. Hence the coefficient of $S^\mu A_\mu$ must be -1 and that of $J_3^\mu A_\mu$ must vanish. We obtain

$$g\,\mathrm{Sin}\,\theta_W = g'\,\mathrm{Cos}\,\theta_W = e \tag{4.40}$$

Thus finally,

$$\mathcal{L}_I = -S^\mu A_\mu - \frac{g}{2\sqrt{2}} \{ J^{\mu\dagger} W_\mu + J^\mu W_\mu^\dagger \}$$

$$- \frac{g}{\mathrm{Cos}\,\theta_W} \{ J_3^\mu - \mathrm{Sin}^2\,\theta_W S^\mu/e \} Z_\mu \tag{4.41}$$

In addition to the standard QED and IVB theory this SU(2)xU(1) gauge invariant Lagrangian density first introduced by Glashow in 1961[1c] contains an additional interaction represented by the third term. From the second term we have

$$g_W = \frac{g}{2\sqrt{2}} \tag{4.42}$$

Thus the quanta of the gauge field W are just $W^\pm$ vector bosons. The third term represents a neutral current coupled to a neutral vector boson Z^o. The neutral boson suspected in the IVB theory appears naturally in this formalism and comparing the predictions of neutral current processes based on this term consistency with experiment obtains for

$$\sin^2\theta \approx 0.22 \qquad (4.43)$$

$\theta_W = 0$ would lead to a pure SU(2) theory ruled out by experiment. The observed agreement between theory and experiment is a strong support for the unified theory of weak and electromagnetic interactions. We shall discuss this in detail later.

3. Properties of the Gauge Bosons

Our Lagrangian density (4.34) describes free massless leptons and their interactions with massless gauge fields. The complete Lagrangian must contain the kinetic energy terms for these gauge fields. We ignore the mass terms for the present. We must require these additional kinetic energy terms to be SU(2)xU(1) gauge invariant. The B-field corresponds to a U(1) group. Thus the kinetic energy term is simple $-\frac{1}{4} B_{\mu\nu}B^{\mu\nu}$(with $B_{\mu\nu} \equiv \partial_\nu B_\mu - \partial_\mu B_\nu$) in exaxt analogy to the photon term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and is gauge invariant.

A similar term constructed for the W-fields is not gauge invariant as should be clear from the more complicated transformation properties of the W's. Howe[illegible] we can easily show that the term

$$\mathcal{L}_G = -\frac{1}{4} G_{i\mu\nu}G_i^{\mu\nu} \qquad (4.44)$$

is gauge invariant where

$$G_i^{\mu\nu} = \partial^\nu W_i^\mu - \partial^\mu W_i^\nu + g\epsilon_{ijk} W_j^\mu W_k^\nu \qquad (4.45)$$

$\mathcal{L}_G$ is ofcourse U(1) gauge invariant.

Defining $$G^{\mu\nu} = \tfrac{1}{2}\tau_i G_i^{\mu\nu} \qquad (4.46)$$

and using $$T_r(\tau_i \tau_j) = 2\delta_{ij} \qquad (4.47)$$

we immediately have

$$\mathcal{L}_G = -\tfrac{1}{2} T_r(G_{\mu\nu} G^{\mu\nu}) \qquad (4.48)$$

The $\mathcal{L}_G$ defined above serves as the required kinetic energy term. From the structure of $\mathcal{L}_G$ it is clear that it involves products of three and four W fields. These will lead to Feynman vertices with three and four external gauge field lines. The W-fields are self-interacting since they couple to anything carrying weak isospin and hence to themselves. The origin of these terms can be directly traced to the non-Abelian nature of the SU(2) group and is a feature that was totally absent in the Abelian U(1) gauge theory.

4. Lepton and Gauge Boson Masses

The gauge field mass terms are clearly not gauge invariant. The same is true of fermion mass terms. A typical fermion field mass term is of the type $\bar{\psi}\psi$.

But $\bar{\psi}\psi = \bar{\psi}(P_L + P_R)\psi = \bar{\psi}^L\psi^R + \bar{\psi}^R\psi^L$. In our picture the left-handed fields are SU(2) doublets whereas the right-handed fields are singlets and hence the mass term is not i[illegible]iant.

In the next chapter we explore the possibility of including masses without destroying the gauge invariance of the Lagrangian.

CHAPTER 5

SYMMETRY BREAKING

1. Spontaneous Symmetry Breaking (SSB)[21]

A symmetry is said to be spontaneously broken if it is present in the basic equations of the theory but not in the solution. A ferromagnet offers a typical example. In quantum field theory this possibility can be realized if the vacuum state i.e. the ground state of the theory is not invariant under the symmetry transformation. (Recall that all physically relevant amplitudes are vacuum expectation values of appropriate operators).

When the equations of motion of a theory have a symmetry an asymmetric solution may or may not exist depending upon the dynamics. When both symmetric and asymmetric solutions exist the one corresponding to a lower minimum of the potential is physically relevant. Usually it is the broken symmetry or the asymmetric solution.

Let G be a continuous Lie group. Spontaneous symmetry breaking occurs when some of the generators T_a do not annihilate the vacuum. i.e.

$$e^{i\theta_a T_a}|0\rangle \neq |0\rangle \tag{5.1}$$

so that

$$T_a|0\rangle \neq 0 \tag{5.2}$$

The generators for which $T_a|0\rangle = 0$ correspond to unbroken symmetry and generate a subalgebra of G.

As a consequence of equation (5.2) some operators ϕ which transform nontrivially under G acquire non-zero vacuum expectation values (VEV), $<0|\phi|0> \equiv <\phi>_o \neq 0$. If the vacuum is invariant under G then $<\phi>_o = 0$ trivially. SSB can be implemented in field theory by giving non-zero VEV to appropriately chosen fields. For example, if $\phi_k (k=1,2,3)$ are an isotriplet of fields and $<\phi_3>_o \neq 0$ then remembering that $[T_j, \phi_k] = i\epsilon_{jk\ell}\phi_\ell$, T_1, T_2 are broken.

Two important properties of VEV's of ϕ must be noted

1. For a translationally invariant vacuum $<\phi>_o$ are space-time independent. i.e.

$$P_\mu|0> = 0 \Rightarrow <0|\phi(x)|0> = <0|\phi(0)|0>$$

2. Let $\phi_\alpha (\alpha = 1,\ldots N)$ constitute an N-dimensional representation R of G and let F_a be the $N \times N$ matrix representation of T_a in R so that

$$[T_a, \phi_\alpha] = (F_a)_{\alpha\beta}\phi_\beta \qquad (5.3)$$

Let ϕ be the column vectors of the fields ϕ_α and set $<\phi>_o = \eta$. Taking VEV of equation (5.3) we have the following result

$$T_a|0> = 0 \Rightarrow F_a \eta = 0 \qquad (5.4)$$

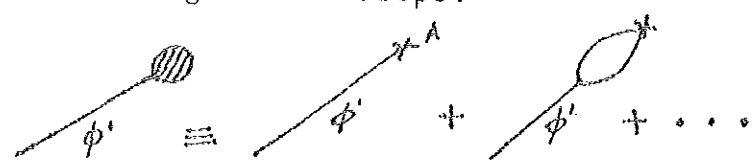
How does one calculate VEV's in a given theory? First of all fields with $<\phi>_o = 0$ must be Lorentz scalars. The ϕ's may be fields in the Lagrangian or, for example, bilinears in fermi fields. We take them to be scalar fields. We take them to be scalar fields in the Lagrangian. The G invariant Lagrangian is usually of the form

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi_\alpha)(\partial^\mu\phi_\alpha) - V(\phi) + \text{terms involving other fields} \qquad (5.5)$$

Let $\langle\phi_\alpha\rangle_o \equiv \eta_\alpha$ and $\phi'_\alpha \equiv \phi_\alpha - \eta_\alpha$

Then $$\langle\phi'_\alpha\rangle_o = 0 \tag{5.6}$$

The quantities $\langle\phi'\rangle_o$ are one point functions. The perturbation expansion of this quantity consists of a term with a single vertex in which a ϕ' quantum vanishes into the vacuum and other terms involving closed loops:



(5.7)

Now if we expand $V(\phi)$ around η we get

$$V(\phi) = V(\eta)+\phi' \frac{\partial V}{\partial \phi}\Big|_{\phi=\eta} + \text{higher terms} \tag{5.8}$$

Then the vertex A in the above diagram is proportional to $\frac{\partial V}{\partial \phi}\Big|_{\eta}$. In the tree approximation only this term contributes so that equation (5.6) gives

$$\frac{\partial V}{\partial \phi_\alpha}\Big|_{\phi=\eta} = 0 \tag{5.9}$$

These equations determine the VEV's in the tree approximation. We shall not consider the exact determination of VEV's.

We note that if η_α constitute a solution of equation (5.9) then so do

$$\eta'_\alpha = (e^{i\theta_a T_a})_{\alpha\beta}\eta_\beta \tag{5.10}$$

for the potential is G-invariant. Thus, the G invariance of the original Lagrangian manifests itself through this arbitrariness of the broken symmetry solution.

2. The Goldstone Theorem[22]

The SSB mechanism leads to a very interesting consequence which is best summed up by stating the Goldstone theorem.

> If in a manifestly Lorentz covariant theory, the equations of motion are covariant with respect to a continuous symmetry transformation which does not leave the vacuum state invariant, then the Hilbert space of the theory must contain states with light like four momentum[23].

We illustrate the theorem with a simple example. Take G =SO(3) (the group of rotations in three dimensions)

Let $\phi_\alpha (\alpha = 1,2,3)$ be a triplet of scalar fields. Consider the G-invariant Lagrangian

$$= \tfrac{1}{2}(\partial_\mu \phi_\alpha)(\partial^\mu \phi_\alpha) - \tfrac{1}{2}m^2 \phi_\alpha \phi_\alpha - \lambda\,(\phi_\alpha \phi_\alpha)^2 \qquad (5.11)$$

λ is positive so that the energy spectrum is bounded from below. Using equation (5.9) the VEV's are determined by

$$0 = \frac{\partial V}{\partial \phi_\alpha}\Big|_{\phi=\eta} = -m^2 \eta_\alpha - 4\lambda \eta_\alpha \eta_\beta \eta_\beta, \qquad (5.12)$$

$$\alpha = 1,2,3$$

One solution is $\eta_\alpha = 0$ which corresponds to unbroken symmetry. There is another solution

$$\eta_\beta \eta_\beta = -\frac{m^2}{4\lambda}$$

Such a solution exists only for $m^2 < 0$. This is allowed for m^2 is only the bare mass squared. The m^2 is just a real parameter in the theory.

Now $$V(\phi = \eta) = -\frac{m^4}{16\lambda} < 0 \qquad (5.14)$$

is lower than $V(\phi=0) = 0$. The broken symmetry solution has thus a lower energy than the symmetric one. In fact $\phi = 0$ corresponds to a local maximum of V. Thus the broken symmetry solution represents the true ground state. Only $\eta_\alpha \eta_\alpha$ is determined and not η_α itself. We have thus a whole class of solutions related to each other by G-transformations. By a suitable G transformation any η can be brought into the form in which only the third component is nonvanishing. We are just saying that any vector can be made to point in the Z-direction through a suitable rotation. Thus we write

$$\eta = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}, \quad v = \left(\frac{-m^2}{4\lambda}\right)^{\frac{1}{2}} \qquad (5.15)$$

Now in the regular representation we have

$$(F_\alpha)_{\beta\gamma} = -i\epsilon_{\alpha\beta\gamma} \qquad (5.16)$$

Let $b_1T_1+b_2T_2+b_3T_3$ be the most general combination of generators which annihilates η, i.e.,

$$(b_1T_1+b_2T_2+b_3T_3)\eta = 0 \qquad (5.17)$$

We find that $b_1=b_2 = 0$. Thus the unbroken subgroup is U(1) generated by T_3. Putting $\phi = \eta + \chi$ in equation (5.11) the mass term for the χ fields is seen to be

$$\mathcal{L}_{mass} = -4\lambda v^2 \chi_3^2 \qquad (5.18)$$

There are no mass terms for χ_1 and χ_2. These are the Goldstone fields in this example. Actually, we have only shown that χ_1

and χ_2 are massless in the tree approximation. Infact their masses vanish exactly. We cannot discuss this point further.

The fields χ_1 and χ_2 have the same quantum numbers as the broken generators T_1 and T_2. There is always a one-to-one correspondence between the broken generators and the Goldstone particles.

It is often useful to consider a polar decomposition of fields. We state it here.

Let $$\phi = U\rho \ , \ \rho = \begin{pmatrix} 0 \\ 0 \\ \rho_3 \end{pmatrix} \ , \ U = \exp i(\theta_1 F_1 + \theta_2 F_2) \qquad (5.19)$$

It can be seen that for non-vanishing ρ_3 there is a one-to-one and invertible correspondence between $\phi_{1,2,3}$ and ρ_3, θ_1, θ_2. The Lagrangian can now be seen to be

$$\mathcal{L} = \tfrac{1}{2}\{(\partial_\mu \rho^\dagger)(\partial^\mu \rho) + \rho^\dagger(\partial_\mu U^\dagger)(\partial^\mu U) + (\partial_\mu \rho^\dagger)U^\dagger(\partial^\mu U)\rho$$
$$+ \rho^\dagger(\partial_\mu U^\dagger)U(\partial^\mu \rho)\} - \tfrac{1}{2}m^2\rho^\dagger\rho - \lambda(\rho^\dagger\rho)^2 \qquad (5.20)$$

It is the ρ-field which acquires non-zero VEV. Proceeding exactly as above, we find, for the broken symmetry case

$$\eta = \langle\rho\rangle_0 = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} \ ; \ v = \left(-\frac{m^2}{4\lambda}\right)^{\frac{1}{2}} \qquad (5.21)$$

Setting $\rho = \eta + \rho'$ in equation (5.20) one finds that the θ-fields have no mass terms and these serve as the Goldstone fields.

3. Spontaneously Broken Symmetries and the Higgs-Kibble Mechanism

A remarkable thing happens when a gauge symmetry is spontaneously broken. The Goldstone fields become the longitudinal components of the gauge fields corresponding to the broken symmetry, which, thereby, become massive. The gauge fields corresponding to unbroken symmetry remain massless. The θ-fields, ofcourse, disappear from the Lagrangian. This is referred to as the Higgs - Kibble mechanism or simply Higgs phenomenon.[24,25,26]

We continue with the example of SO(3) and make it a gauge symmetry. Our G-invariant Lagrangian now reads

$$\mathcal{L}' = \tfrac{1}{2}(D_\mu\phi)_\alpha (D^\mu\phi)_\alpha - \tfrac{1}{4}F_{\alpha\mu\nu}F_\alpha^{\mu\nu} - \tfrac{1}{2}m^2\phi_\alpha\phi_\alpha - \lambda(\phi_\alpha\phi_\alpha)^2 \qquad (5.22)$$

where $D_\mu\phi \equiv (\partial_\mu + igA_\mu)\phi$

and $F_{\alpha\mu\nu} = \partial_\nu A_{\alpha\mu} - \partial_\mu A_{\alpha\nu} + g\varepsilon_{\alpha\beta\gamma} A_{\beta\mu}A_{\gamma\nu}$

Now equation (5.19) can be considered as an SO(3) gauge transformation of the field ρ into the field ϕ. Under the same guauge transformation the field A_μ will be related to the field B_μ as

$$A_\mu = UB_\mu U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1} \qquad (5.23)$$

Thus A_μ and ϕ are the gauge transforms of the fields B_μ and ρ respectively, the transformation being given by $U = e^{i(\theta_1 T_1 + \theta_2 T_2)}$. Since our Lagrangian (5.22) is gauge invariant it is also given by

$$\mathcal{L}' = \tfrac{1}{2}(D_\mu\rho^\dagger)(D^\mu\rho) - \tfrac{1}{4}G_{\alpha\mu\nu}G_\alpha^{\mu\nu} - \tfrac{1}{2}m^2\rho^\dagger\rho - \lambda(\rho^\dagger\rho)^2 \qquad (5.24)$$

where $D_\mu \rho \equiv (\partial_\mu + igB_\mu)\rho$ and

$$G_{\alpha\mu\nu} = \partial_\nu B_{\alpha\mu} - \partial_\nu B_{\alpha\nu} + g\varepsilon_{\alpha\beta\gamma} B_{\beta\mu} B_{\gamma\nu}$$

Notice that the θ-fields have been eliminated from the Lagrangian as promised. Now, as before, the ρ-field acquires VEV as in equation (5.21). Writing $\rho = \eta + \rho'$ we find a mass term for the B-fields:

$$\mathcal{L}^B_{mass} = \tfrac{1}{2} g^2 \eta^\dagger B_\mu B^\mu \eta = \tfrac{1}{2} g^2 \eta^\dagger T_\alpha T_\beta B_{\alpha\mu} B^\mu_\beta \eta$$

$$= \tfrac{1}{2} g^2 v^2 (B_{1\mu} B^\mu_1 + B_{2\mu} B^\mu_2) = g^2 v^2 W^\dagger_\mu W^\mu \qquad (5.25)$$

where in the last line we have gone to the spherical basis with $W_\mu = \frac{1}{\sqrt{2}}(B_{1\mu} - iB_{2\mu})$. No mass term appears for the B_3 field which corresponds to the unbroken generator.

The value of the W-mass namely gv depends on v besides depending on g. But v depends on the parameters m and λ in the Higgs potential. We can now see the prospects of obtaining a unified description of long range interactions mediated by massless vector particles (e.g. electromagnetic interaction) and short range interactions mediated by massive vector particles (e.g. weak interactions) in the framework of spontaneously broken gauge symmetry group.

The most general renormalizable Lagrangian invariant under G is of the form [27]

$$\mathcal{L} = -\tfrac{1}{4} F^{\mu\nu}_\alpha F_{\alpha\mu\nu} + \tfrac{1}{2} (D_\mu \phi)_\alpha (D^\mu \phi)_\alpha - V(\phi)$$
$$+ \bar{\psi}(i\not{\partial} - M)\psi + \bar{\psi}\Gamma_\alpha \psi \phi_\alpha \qquad (5.26)$$

where ψ is a column of Dirac fields and M is a mass matrix. The structure of the terms is self explanatory. In line with our discussion in this chapter ultimately we shall set $\phi = \langle\phi\rangle_o + \chi$ and one can expect a contribution to fermion masses from the Yukawa coupling terms. Sometimes this may be the only source of fermion masses for G-invairance may not permit a $\bar{\psi}M\psi$ term at all. This is exactly what happens in the GSW model to which we turn in the next chapter.

CHAPTER 6

THE STANDARD ELECTRO-WEAK THEORY

In Chapter 4 we wrote down the $SU(2) \times U(1)$ gauge invariant Lagrangian. It contains both electromagnetism and the IVB theory. However the weak vector bosons and the fermi fields turned out to be massless.

In Chapter 5 we saw how the mass problem can be resolved in a spontaneously broken gauge symmetry. All we need is a suitable set of Higgs fields coupled to matter and gauge fields. By giving non-zero expectation values to appropriately chosen Higgs fields masses can be generated for gauge particles. By working in the unitary gauge the Goldstones can be eliminated from the Lagrangian. At the end of Chapter 5 we indicated that the Yukawa couplings of the Higgs fields to matter fields can generate masses for the matter fields as well. We have seen also how part of the symmetry can be maintained by ensuring conditions imposed by equation (5.4).

Thus to get to the standard GSW model we only need to choose the Higgs fields and their vacuum expectation values. For illustration, we consider the minimal version of the model in which only the charged leptons acquire mass and the neutrinos remain massless.

We choose the scalar Higgs field to be a complex weak isodoublet

$$\phi = \begin{pmatrix} \phi_a \\ \phi_b \end{pmatrix}, \quad Y = \tfrac{1}{2} \qquad (6.1)$$

with the gauge invariant Lagrangian density

$$\mathcal{L}_H = (D^\mu\phi)^\dagger(D_\mu\phi) - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \tag{6.2}$$

and
$$D_\mu\phi = \{\partial_\mu + ig\frac{\tau_j}{2}W_{j\mu} + ig'YB_\mu\}\phi \tag{6.3}$$

The gauge invariant Higgs lepton coupling is given by

$$\mathcal{L}_{LH} \equiv - g_\ell\{\bar{L}\ell^R\phi + \phi^\dagger\bar{\ell}^R L\}$$

Using the above equations we get the Lagrangian of the GSW model as

$$\begin{aligned}\mathcal{L}_{GSW} = & \; i\{\bar{L}\not{D}L + \bar{\ell}^R\not{D}\ell^R + \bar{\nu}_\ell^R\not{D}\nu_\ell^R\} \\ & + \{-\tfrac{1}{4}B_{\mu\nu}B^{\mu\nu} - \tfrac{1}{4}G_{i\mu\nu}G^{i\mu\nu}\} + \mathcal{L}_H + \mathcal{L}_{LH}\end{aligned} \tag{6.5}$$

where B_μ, $G_{i\mu\nu}$ and D_μ have already been defined.

The Lagrangian $\mathcal{L}_{GSW}$ is clearly $SU(2) \times U(1)$ invariant. For $\mu^2 < 0$ the $SU(2) \times U(1)$ gauge symmetry is spontaneously broken as explained in Chapter 5.

We set,
$$\langle 0|\phi|0\rangle \equiv \phi = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \tag{6.6}$$

where
$$v = (-\frac{\mu^2}{\lambda})^{\frac{1}{2}} \quad (>0) \tag{6.7}$$

In the unitary gauge the Higgs field simply becomes

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + \sigma \end{pmatrix} \tag{6.8}$$

All the Goldstone particles have been gauged away. Three gauge particles now gain mass but the photon remains massless since the charge operator is $Q = I_3^W + Y$ and we can check that

$$Q\phi_0 = 0 \tag{6.9}$$

Notice that the electrically neutral component of the Higgs fields has been given a VEV so that $U_{em}(1)$ has been preserved.

Finally, then, the total Lagrangian can be written as:

$$\mathcal{L}_{GSW} = \mathcal{L}_{I} + \mathcal{L}_{e.m.} + \mathcal{L}_{H} + \mathcal{L}_{LH} + \text{Kinetic energy terms of matter and gauge fields.} \quad (6.10)$$

By writing out the Lagrangian in full and looking at the coefficients of mass terms we immediately deduce that

$$m_W = \tfrac{1}{2} vg, \quad m_Z = \frac{m_W}{\cos\theta_W} \quad (6.11)$$

$$m_H = \sqrt{-2\mu^2}, \quad m_\ell = \frac{vg_\ell}{\sqrt{2}}$$

Further, we already know that

$$g \sin\theta_W = g' \cos\theta_W = e \quad (6.12)$$

and $\sin^2\theta_W = 0.227 \pm 0.014$ (experimentally) (6.13)

Clearly the masses of all the bosons and leptons are related to the parameters of the theory - $g, g', -\mu^2, \lambda$ and g_ℓ.

Using $\frac{G}{\sqrt{2}} = \left(\frac{g_W}{m_W}\right)^2$, $g_W = \frac{g}{2\sqrt{2}}$ and $m_W = \frac{1}{2} vg$ we immediately find that

$$v = (G\sqrt{2})^{-\frac{1}{2}} \quad (6.14)$$

Combining equations (6.11), (6.12) and (6.14) one has

$$m_W = \left(\frac{\alpha\pi}{G\sqrt{2}}\right)^{\frac{1}{2}} \frac{1}{\sin\theta_W}, \quad m_Z = \left(\frac{\alpha\pi}{G\sqrt{2}}\right)^{\frac{1}{2}} \frac{2}{\sin 2\theta_W} \quad (6.15)$$

where the fine structure constant

$$\alpha = \frac{e^2}{4\pi} = \frac{1}{137.04} \quad (6.16)$$

and the Fermi coupling constant

$$G = 1.166 \times 10^{-5}\ \text{GeV}^{-2} \qquad (6.17)$$

Using these known values of α and G we get

$$m_W = (78.3 \pm {}^{2.5}_{2.3})\ \text{GeV and } m_Z = (89\ {}^{+2.1}_{-1.8})\text{GeV} \qquad (6.18)$$

which are in very good agreement with the experimentally determined values.

The radiative corrections[28] modify our predictions slightly but the results remain consistent with experimental values. The experimental detection of the gauge bosons at the expected masses has been a major triumph of the electroweak theory[29].

The parameters g and g' are also determined in terms of α and θ_W. The parameters g_ℓ are calculable in terms of the known lepton masses. It is only the parameter λ which remains undetermined. Since $-\mu^2 = \lambda v^2$, λ occurs as a coupling constant in the Higgs self-coupling terms but there is no chance of measuring these at present. Thus the Higgs boson mass cannot be predicted from known data, for,

$$m_H = \sqrt{-2\mu^2} = \sqrt{2\lambda v^2} \qquad (6.19)$$

This is unfortunate, for, knowing its mass would be of great help in searching for it. The existence of the Higgs is essential. Without them the theory is not renormalizable. For m_H too small or too large the higher order corrections become large. The success of the theory in lowest order calculations restricts m_H to the extremely wide range[28].

$$7\ \text{GeV} \lesssim m_H \lesssim 110^3\ \text{GeV} \qquad (6.20)$$

The detection of Higgs becomes difficult because readily available particle beams and targets consist of photons, electrons, muons, pions, kaons and nucleons). The Higgs particle does not couple to the photons. The remaining couplings are very weak (they generate the small masses of these particles).

A major success story of the GSW model has been the prediction of weak neutral currents. These effects have been well confirmed quantitatively in the elastic electron-neutrino scattering and neutrino-nucleon scattering[28].

This concludes our proposed study of the GSW model. However, we would like to record some additional features which we did not study in the present programme. Some of these are:

1. We have assumed the neutrinos to be massless. A neutrino mass term can easily be included by using an additional Higgs doublet

$$\tilde{\phi} = -i[\phi\tau_2]^T = \begin{pmatrix} \phi_b^* \\ -\phi_a^* \end{pmatrix} \tag{6.21}$$

with an additional gauge invariant piece in the Lagrangian given by

$$\mathcal{L}' = -g_{\nu\ell}\{\bar{L}\nu_\ell^R\tilde{\phi} + \tilde{\phi}^\dagger\bar{\nu}_\ell^R L\} \tag{6.22}$$

The SSB mechanism now leads to a neutrino mass

$$m_{\nu\ell} = v g_{\nu_\ell}/\sqrt{2} \tag{6.23}$$

2. A generalization of considerable interest is to replace $\mathcal{L}'$ by

$$-G_{\ell'\ell}\bar{L}_{\ell'}\nu_\ell^R\tilde{\phi} - G^*_{\ell'\ell}\tilde{\phi}^\dagger\bar{\nu}_\ell^R L_{\ell'} \tag{6.24}$$

where G is a Hermitian coupling matrix. We can write equation (6.24) in the unitary gauge and diagonalize it to obtain eigenstate neutrinos $\nu_i (i = 1,2,...)$ with masses m_i. The leptonic neutrinos $\nu_\ell (\ell = e,\mu,\tau)$ are linear combinations of these eigenstate neutrinos ν_i each of which would have its characteristic time dependence. As a result neutrino mixing will occur. (This is just the well known time dependence of a superposition of energy eigenstates of a system). Thus, a pure initial ν_μ beam can become a mixture of ν_μ and ν_e particles and then purely a ν_e beam. Neutrino mixing will occur. Such effects are called neutrino oscillations and occur if neutrinos have non-zero masses and observing such effects would provide information on neutrino masses. Although non-zero masses do not lead to oscillations necessarily observing these will be in the spirit of lepton-quark symmetry that we shall discuss shortly. Experiments to detect neutrino oscillations are being planned.[30]

3. We have mentioned that the GSW model represents a renormalizable gauge theory. The proof is due to 't Hooft, Veltman and others[31]. It is beyond our scope to discuss this. We only note that such procedures do not use the unitary gauge. The Goldstone particles are reintroduced into the theory. There exist no real particles corresponding to their quantized fields although the Feynmann propagator for the Goldstone field can be interpreted in terms of the exchange of virtual scalar bosons. The properties of these "ghost particles" are analogous to those of the longitudinal and scalar photons in QED which also do not exist as real free particles but contribute as virtual intermediate

quanta to the photon propagator. The detailed properties of these fields are complicated and gauge dependent. However, all observable quantities are gauge invariant like in QED[21].

4. We have considered electroweak interactions of leptons only. The theory is easily extended to include quarks and hence hadrons so that semileptonic processes can be included. This extension is necessary to prove renormalizability.

5. The successes of the GSW model are compelling enough to believe that it must at least be a part of a larger truth. However there is no compelling reason to believe that it represents the only possible description of electroweak interactions. Several other models exist in the literature[28].

In the next chapter we shall consider a gauge theory of strong interactions. The GSW model provides enough reason to consider such a possibility.

CHAPTER 7

STRONG INTERACTIONS AS A GAUGE THEORY

1. Introduction

In view of the proliferation of hadron states and the absence of a comprehensive theory of strong interactions the first rational step was to simplify and classify this profusion of hadron states. Such an approach has been familiar to us from the time of Mendeleef's formulation of the periodic table of the elements. The following facts provide some justification for the attempts to "build" hadrons from simpler subunits.

i) Hadrons have a measurable size ($\sim 10^{-15}$m)

ii) Hadrons are complex. The nucleous, for example appear to be granular.

iii) Hadrons exist in many varieties both as fermions and bosons. The contrast between hadrons and leptons (which appear to be elementary or at least did so till some time ago) is obvious.

2. Unitary Symmetries

The most successful such attempt at classification came through the works of Gell-mann and Nee'man[32]. They proposed SU(3) as the global symmetry group of strong interactions. Hadrons fall into multiplets of this group. The rudiments of this group are summarized in Appendix C. The particle multiplets can be charted on a two-dimensional weight diagram with the third component of isospin I_3 and hypercharge Y as axes.

Thus SU(3) was an enlargement of SU(2) with the quantum number Y added to I_3. The small mass differences between members of SU(2) multiplets arise from the breaking of SU(2) by electromagnetic interaction. Similarly the much larger mass differences within members of an SU(3) multiplet arise due to symmetry breaking by strong interactions themselves. The SU(3) group could thus be only an approximate symmetry of strong interactions.

An outstanding success of the SU(3) scheme concerns the $J^P = \frac{3}{2}^+$ baryon decuplet. In 1964 the $\Delta(1232)$, $\Sigma(1385)$ and $\Xi(1530)$ were established. All had $J^P = \frac{3}{2}^+$. They could not be members of an octet since the I-spin of $\Delta(1332)$ is too large. Accomodating them in a decuplet left one location vacant. The missing member was predicted to have B = 1, S = -3, $J^P = \frac{3}{2}^+$ and a mass of about 1670 MeV. The Ω^- was found[33].

Within SU(3) framework the proton, Λ, Σ, Ξ are manifestations of the same particle just as proton and neutron are of the nucleon within SU(2). The presence of Ω^- "particle" in a decuplet which is otherwise composed of "resonances" is an illustration of the marginal difference between "particles" and "resonances". In higher symmetry schemes which incorporate spin and parity, the Ω^- can appear in the family of nucleons and does.

SU(3) symmetry provided a good description of experimental data and its success implies that it must contain some measure of truth.

3. Quarks[34]

Of all the multiplets which are allowed by the SU(3) scheme only a few - the singlets, octets, decuplets appear to correspond to physical particles. This may be understood in terms of a quark model[34].

The basic multiplet in SU(3) is a triplet which appears in $Y\text{-}I_3$ plot as a doublet and singlet of isospin and corresponds to the SU(3) quantum numbers of the proton, neutron and lambda respectively. Sakata[35] tried to use the proton, neutron and lambda as members of this triplet but the attempt to build the known hadrons from this basic triplet met with no success. Gell-mann and Zweig[32] postulated that the members of the triplets are three quarks whose SU(3) quantum numbers have very unusual properties. The three quarks are called u(up), d(down) and s(sideways or strange) quarks. The three quark varieties are called quark flavours.

The quark model says that a meson consists of a quark-antiquark ($q\bar{q}$) pair and a baryon is made up of three quarks (qqq). A quark baryon number $\frac{1}{3}$ is required in order to satisfy the requirement of $B = 0$ for mesons and $B = 1$ for baryons. Since $Q = I_3 + Y/2$, it follows that quarks must have fractional Y and Q, The $q\bar{q}$ and qqq hypothesis is consistent with the fact that only a few SU(3) representations have been observed in nature.

Mesons can be built from the u,d,s triplet (3) and $\bar{u},\bar{d},\bar{s}$ anti-triplet ($\bar{3}$) to form singlets and octets.

$$3 \times \bar{3} = 1 + 8 \qquad (7.1)$$

Similarly three quarks (qqq) baryons form singlets, octets and decuplets.

$$3 \times 3 \times 3 = 1 + 8 + 8 + 10 \qquad (7.2)$$

Combinations like

$$3 \times 3 = \bar{3} + 6 \qquad (7.3)$$

produce representations that do not materialize in nature.

The experimental data does not require $qq\bar{q}\bar{q}$ mesons or $qqqq\bar{q}$ baryons though the model does not forbid these.

The quark model gives a good account of the quantum numbers of the lowest excited hadron states. With the addition of orbital excitations the model can be extended to include higher masses. The available data does not provide any crucial test of the model. The model has also been successful in predicting relations between cross-sections and magnetic moments[34].

4. Colour and Charm[36,37]

Inspite of the successes of the three flavour quark model, it suffers from a serious defect. It seems to violate the Pauli principle. The three quarks inside a baryon seem to form states which are symmetric under quark exchanges. The Ω^- seems to be made of three identical quarks in the same quantum state! The same is true of Δ^- and Δ^{++} resonances. Quarks are fermions and yet behave like bosons.

In 1964 Greenberg proposed that quarks should have another attribute-conventionally called colour. The colour quantum number should take at least three distinct values. The three quarks inside Ω^-, Δ^- or Δ^{++} are identical in all respects except colour. The baryons and mesons are colourless. Thus the problem of statistics can be evaded without assigning observed hadrons any extra degree of freedom. The symmetry of the three quark colours can be dealt within a colour SU(3) scheme analogous to the SU(3) scheme discussed above. The generators of this colour SU(3) will act on the three-valued colour index and would take a quark of one colour into another through appropriate SU(3) colour transformations. This scheme can explain the rules concerning quark colours if all hadrons are to be colour singlets. Clearly the group being SU(3) the notion of the "colour isotopic spin" and the "colour hypercharge" emerge. These then lead to the notion of a "colour charge" analogous to the electric charge. The importance of this scheme can be easily visualized. The three different coloured quarks inside a baryon have unlike values of the colour charge and could then be bound together by attractive forces analogous to the electromagnetic interactions between electrically charged particles. But the electromagnetic interactions are transmitted by photons. In a similar manner quark-quark forces require intermediate particles. These intermediate particles would be the glue that bind together the quarks inside hadrons. They are thus called gluons. We can now build a gauge theory based on the group SU(3) colour. This will lead to eight gauge bosons (gluons) with nontrivial SU(3) colour transformation properties capable of changing a quark of one colour into that of another without changing the quark flavour. We have now a possibility of describing quark-quark forces as a gauge theory with the spin one gluons as the gauge bosons. They carry no flavour but only colour.*

*A possible bonus of the colour hypothesis is contained in a scheme due to Han and Nambu[38] whereby hadrons can be built out of integrally charged quarks.

The introduction of colour trebles the number of quarks without changing the number of hadrons which are colourless or $SU_c(3)$ singlets. Now there is a marked similarity between the properties of quarks and leptons - both behave like point particles. We have long known two pairs of leptons (e,ν_e) and (μ,ν_μ). In 1964 the quark model required only three quarks u,d,s. The u,d seemed to fall naturally into an isodoublet and the s quark seemed alone. The appealing idea of lepton quark symmetry led to the postulate of a fourth quark flavour called charm by Glashow and Bjorken[37]. To distinguish it from other quark flavours one needed a new distinctive quantum number which they called charm C with a value +1 for quarks c and -1 for antiquarks $\bar{c}$. The charm quantum number behaves very much like strangeness, being conserved in strong and electromagnetic interactions and not conserved in weak interactions.

Further support for the existence of charmed quarks came from weak interactions where one could salvage the $\Delta S = \Delta Q$ rule by introducing a charmed quark.

With the introduction of the charmed quark, a new triplet of coloured quarks is added to the quark model. The fourth quark flavour can be accomodated by adding a third dimension, charm to the SU(3) symmetry scheme weight diagram leading to the group SU(4) for particle classification. In addition to existing mesons and baryons SU(4) would predict new families of charmed particles.

We noted that leptons and quarks appear to have a point like distribution of electric charge. Now the rates of production of particles with point like charge in e^+e^- annihilations can be calculated using QED. The rates of productions are found to be proportional to the squares of the point charges.

Consider the ratio

$$R = \frac{\sigma(e^+e^- \longrightarrow \text{hadrons})}{\sigma(e^+e^- \longrightarrow \mu^+\mu^-)} \qquad (7.4)$$

Since hadrons are ultimately derived from $q\bar{q}$ pairs in e^+e^- collisions, then, if the quark hypothesis is correct, R should be given by

$$R = \text{sum of quark charges} \qquad (7.5)$$
$$= \Sigma Q_i^2$$

almost independent of energy.

R provides a very direct test of the quark models. At centre-of-mass energies above 4 GeV we know that experimentally $R \sim 5$.[34] The values given by the quark models are

$$u,d,s \text{ model } R = (\tfrac{2}{3})^2 + (-\tfrac{1}{3})^2 + (-\tfrac{1}{3})^2 = \tfrac{2}{3}$$

$$u,d,s + \text{colour } R = \tfrac{2}{3} \times 3 = 2$$

$$u,d,s,c + \text{colour } R = 3(\tfrac{2}{3} + (\tfrac{2}{3})^2) = \tfrac{10}{3}$$

$$\text{Han-Nambu Model } R = 1^2 + 1^2 + (-1)^2 + (-1)^2 = 4$$

The experimentally determined value of R is higher than any of the above and suggests that there is room for even more quarks - an indication that time has since shown to be correct.

The charmed particles were ultimately discovered by two separate groups of experimenters in 1974[39]. In addition two more quarks known as t and b have been established since[40]. Their charges are $\frac{2}{3}$ and $-\frac{1}{3}$ respectively. The six-quark model with three colours would give 5 as required. Our notion of lepton-quark similarity has also been strengthened by the discovery of the third lepton τ[41]. It's neutrino ν_τ is almost certainly believed to be there. We thus have three quarks and lepton families. We can now have enough faith in the basic quark model and the associated colour scheme. (The three are incidentally usually denoted as red, blue and green). Thus in effect we believe now in 6x3 = 18 basic quark fields and six lepton fields.

5. QCD

The field theory of strong interactions based on the above discussions is a gauge theory based on the group $SU_c(3)$ and is appropriately called quantum chromodynamics (QCD)[42].

The eight generators of SU(3) colour are represented in the fundamental representation by the Gell-Mann matrices $\frac{\lambda_a}{2}$ (a = 1,...8) with the normalization

$$T_r(\lambda_a \lambda_b) = 2\delta_{ab} \tag{7.6}$$

The gauge fields (gluon fields) are denoted by G^μ_α (α = 1,...8).

$$G^\mu \equiv \tfrac{1}{2}\lambda_a G^\mu_a \tag{7.7}$$

and

$$F_{\mu\nu} = \partial^\nu G^\mu - \partial^\mu G^\nu + i g_o [G^\mu, G^\nu]$$

where g_o is a dimensionless number - the unrenormalized gauge oupling constant.

The matter fields consist of spinor quark fields denoted collectively by q with components q_α^{fi} with

$i = 1,2,3$ (colour index: red, blue, green)

$f = 1,\ldots 6$ (flavour index: u,d,s,c,t,b)

$\alpha = 1,\ldots,4$ (spinor index)

In the by now standard manner the complete Lagrangian density of QCD is

$$= -\tfrac{1}{4}F_a^{\mu\nu} F_{a\mu\nu} + \bar{q}\,(i\not{D} - M)q \qquad (7.8)$$

where $D_\mu = \partial_\mu + ig_o G_\mu$

and M is a colour independent mass matrix in the flavour indices. The theory is renormalizable.

6. Asymptotic Freedom, Infrared Slavery and the Running Coupling Constant[4]

Asymptotic freedom is a new feature of strong interactions which was initially indicated by a series of beautiful experiments[43]. These experiments are the high energy analogues of the Rutherford experiments with α-particles. High energy electrons were scatterred on proton targets in a bid to study the structure of the proton. The momentum Q of the photon is related inversely to the size of the region probed as per the uncertainty principle. For Q greater than a few GeV the region probed is smaller than the size of the proton. The results showed that the nucleons seem to consist of point like particles later identified with quarks. As Q increases the quark-quark coupling grows weaker and weaker. This behaviour was predicted earlier by Bjorken[44] and is called Bjorken scaling. The traditional belief that strong forces become stronger at shorter distances was thus proved wrong. Bjorken scaling seems to imply that at shorter and shorter distances quarks behave more and more like free particles. This idea is known as asymmptotic freedom. Thus the quark-quark coupling is a function of Q^2 which decreases as Q^2 increases and vice versa.

To understand this phenomena let us recall our experience with QED[21]. In QED the electron is surrounded by virtual e^+e^- pairs created by the electric field from vacuum. We know that Lamb shift originates this way. The virtual positrons are attracted towards the electron and the electrons repelled. The space around the electron is polarized like a dielectric medium. The positrons partially shield the electrons charge so that at larger distances the effective charge of the electron and hence the QED coupling constant appears smaller.

In QCD a similar vacuum polarization from $q\bar{q}$ pairs is to be expected. Consider the colour force on two quarks in a proton. The magnitude of this force is determined by the effective colour charge seen by each quark. If we decrease the distance between the quarks, each quark will penetrate further inside the cloud of virtual quarks pairs surrounding the other quark and will see a larger effective colour charge. Since the net quark-quark force depends on the net colour charge seen by the quarks, the force between quarks should become stronger. Thus with increasing Q the force should increase in contradiction with the experimental finding. But, we forgot the gluons.

Unlike photons which carry no electric charge the gluons carry colour charge and can interact with anything that carries colour. Unlike the $q\bar{q}$ pairs which carry no net colour, the virtual gluons carry part of the total colour charge. Thus what seems to be happening is that virtual gluons spread the colour charge throughout some finite volume like the old fashioned Thompson model of the atom. When we decrease the distance between quarks each quark will be in the virtual gluon cloud of the other quark thus seeing effectively a smaller fraction of the colour charge on the other quark. Thus the vacuum polarization effect from $q\bar{q}$ pairs and gluons produce opposite effects. The experiments indicate that the antiscreening from gluons dominates. The effective coupling constant of strong interactions should thus be a function of Q^2 decreasing with increasing Q^2.

The calculation of the effective colour charge from vacuum polarization in QCD was originally done with perturbative quantum field theory techniques[45]. It turns out however that the QCD vacuum can also be treated semiclassically like a dielectric medium[46]. In this picture the potential between two charges is given by

$$V(r) = \frac{q_1 q_2}{4\pi\varepsilon(r)\, r} \tag{7.9}$$

where all vacuum polarization effects are absorbed into an effective $\varepsilon(r)$ of the vacuum. The semiclassical approach leads to the same result as the field theoretic calculation. The approach is the same as that used for a dielectric medium in electrodynamics[47]. The classical energy density of the medium in an external magnetic field, for example, depends on the susceptibility χ which is related to the permeability μ and the product $\epsilon\mu$ is related to the wave velocity in the medium. Thus, knowing the energy of the vacuum determines $\epsilon(\gamma)$. The results lead to a remarkable conclusion. If the gluon antiscreening has to dominate then the number of quark flavours should be at most 16. This, ofcourse, is well above the presently known flavours which are 6. Another interesting fact emerges. The QED coupling constant α also varies with distance (Q^2) due to vacuum polarization effects of e^+e^- pairs. The low energy cut-off for QED calculations of the Lamb shift is determined by the Bohr radius. This depends on the electron mass which thus provides a natural low energy cut-off. But the zero mass of the gluons means that in QCD there is no natural low energy cut-off and hence there is no intrinsic scale. This has led to a powerful mathematical approach in QCD called 'renormalization group techniques'. Unfortunately we cannot study this here.[48]

7. Discussions

The observed asymptotic freedom in deep inelastic scattering provides one of the most compelling arguments that the strong interaction is a colour gauge theory. Asymptotic freedom is in fact a unique property of non-Abelian gauge theories[45]. Unfortunately the effective coupling constant depends only logarithmically on Q so that extremely high energies are needed even to detect the change in the coupling[49].

We can now gain some insight into the properties of hadrons as bound states of quarks. One knows that the spectrum of excited states of charmonium($c\bar{c}$) resembles that of positronium. c and $\bar{c}$ being heavy move slowly in low angular momentum states. The separation between the quarks is small and hence the coupling is weak. Under such conditions the colour field is like a coulomb field producing positronium like spectrum. For heavier quarks like t and b this description should be even more accurate. Surely this picture must not apply to lighter mass hadrons. Size goes inversely as the mass and so lighter hadrons are relatively larger compared to charmonium. The quarks are well separated and the coupling strong. The motion of the quarks will be relativistic and there is no reason to expect a positronium like spectrum.

We can now also appreciate why free quarks and gluons have not been seen. Imagine pulling a quark out of a proton. As it moves away from other quarks the coupling increases - the binding force increases. If effective coupling were to increase indefinitely it would need infinite energy to pull it out. In such a case free quarks or gluons would not appear in decays or collisions. In deep inelastic scattering the photon can transfer a large momentum Q to a quark. This quark will try to move away but will be restrained by the ever increasing coupling constant. If Q is sufficiently large and as the coupling increases the energy density of the colour field will become large enough to produce $q\bar{q}$ pairs from the vacuum. These quarks have low energies and would recombine to form mesons and baryons.

Similiarly if a high energy gluon is produced in the collision by "bremsstrahlung" the coupling of the gluon will increase in the same manner. Thus copious mesons and baryons can be produced in deep inelastic scattering but no free quarks or gluons. However, it must be noted that it is very hard to prove that QCD coupling constant increases indefinitely since perturbation theory cannot be used beyond a certain limit. The challenge of QCD which is consistent with all known strong interaction phenomenology will be to find non-perturbative techniques.

CHAPTER 8

GRAND UNIFIED THEORIES[28] (GUTs)

I. Shortcomings of the Standard Model

The standard model ($SU(2)_L \times U_y(1) \times SU_c(3)$ successfully describes or is at least consistent with all known facts of elementary particle phenomenology. It is a mathematically consistent field theory in which all known interactions are basically gauge interactions. Inspite of its observed correctness it surely is not the ultimate theory of elementary particles. The model has no inconsistencies but it has a lot of arbitrariness.

Some of the unexplained features are:

1. The pattern of groups and representations is complicated and arbitrary. Why should the gauge group be a product of three different factors. Whereas $SU_c(3)$ is a parity conserving theory the $SU_L(2) \times U_y(1)$ violates parity.

2. What is the purpose of the second and the third lepton families which are merely massive repetitions of the first family. This is the new version of the old question "Why the muon?"

3. The three coupling constants have three very different values. Why do they have these specific values and why are they so different?

4. Why does the Weinberg angle take the particular value it does?

5. Electric charge is not quantized.

6. What are the Higgs? The Higgs sector involves too many

uncontrolled parameters. There are in all 19 free parameters in the theory. If neutrinos are massive there should be 26.

7. There are several quantities in the model which are arbitrary and appear to be unnaturally small. For example the ratio of the neutrino masses (if non-zero) to other fermion masses, the ratio of fermion masses to other W and Z masses.

8. Gravity is not included.

2. GUTs

These were partly motivated by the desire to constrain some of the quantities that are arbitrary at the level of the standard model.

The basic idea in a grand unified theory is that if $G_s = SU_c(3) \times SU_L(2) \times U_y(1)$ is embedded in a larger underlying group G then the additional symmetries may reduce some of the arbitrariness of the standard model. A typical consequence of this embedding is that the new symmetry generators and their associated gauge bosons involve both flavour and colour. The new interactions violate baryon number (B) and lepton number (L) conservation and can change a quark into a lepton and vice versa. The proton can decay. Since the proton life time $\tau_p > 10^{31}$ years [50] one requires the baryon number violating interactions to be very weak. For models in which the proton can decay via the exchange of a single gauge boson X one needs $m_X > 10^{14}$ GeV$\sim 10^{12}$ times larger than m_W or m_Z! An unfortunate consequence of such a value of m_X is that other experimentally observable consequences are practically non-existent.

If G is simple which essentially means that it is not a direct product of factors like SU(3) or SU(2) or if it is a direct product of identical simple groups related by discrete symmetries, G has only one gauge coupling constant. For $Q^2 > m_X^2$ where all symmetry breaking effects can be ignored then strong, weak, electromagnetic and baryon number violating interactions all look basically similar and there is a single coupling constant. We know that for large Q^2 the SU(3) and SU(2) coupling constants decrease, more rapidly for SU(3) and the U(1) coupling constant increases, in typically logarithmic manner. It can transpire that the three approach a common value at $Q^2 = m_X^2$ and all types of interactions merge into a gauge theory with one overall coupling constant. Quarks, leptons, antiquarks and antileptons will all be fundamentally similar. They can be placed in the same representation of G. Only for $Q^2 < m_X^2$ the SSB effects become important and the running coupling constants become different making the various observed interactions appear different at low energies.

The values of the coupling constants measured at $Q^2 \lesssim m_W^2$ can be used in simpler models to predict m_X. Since the coupling constants differ widely at low energies and their Q^2 variation is only logarithmic, m_X turns out to be very large. In the georgi-Glashow model, for example, $m_X \sim 10^{14}$ GeV - the same scale needed to explain the approximate stability of the proton.

3. The SU(5) MOdel [28,51]

Since we want SU(3)xSU(2)xU(1) to be a subgroup of G, the group G must have a minimum rank of 4. The simplest choice that meets this requirement is SU(5). The unified model based on SU(5) was proposed by Georgi and Glashow[52]. We study this model here in some detail. This group has $5^2-1 = 24$ generators and gauging the group would mean introducing 24 gauge fields. Eight of these will be identified with gluons and four with $W^{\pm}$, Z and γ. Thus there would be 12 additional gauge fields leading to new interactions. There would be one overall coupling constant g_o.

If ψ is a multiplet belonging to a SU(5) representation then the covariant derivative will be

$$D_\mu \psi = (\partial_\mu + i g_o G_\mu)\psi \tag{8.1}$$

where $G_\mu = G_{\mu k} T_k,\ k = 1,\ldots,24$

We will assume at least three basic fermion families to exist, namely

$$u_1 d_1,\ u_2 d_2,\ u_3 d_3,\ \nu_e\ e,$$

$$c_1 s_1,\ c_2 s_2,\ c_3 s_3,\ \nu_\mu \mu\ ,$$

$$t_1 b_1,\ t_2 b_2,\ t_3 b_3,\ \nu_\tau \tau\ ,$$

As before, we consider only one family.

The SU(5) matrices will act on a basic quintuplet which we take to consist of the three right-handed colour components of a given quark flavour of the first family, call them q_1, q_2, q_3 plus the right-handed positron and electronic-antineutrino:

$$\psi^R = \text{column}(q_1, q_2, q_3, e^c, -\nu_e^c)_R \equiv R \tag{8.2}$$

This is because in the Weinberg-Salam model we postulated the left handed doublet $\binom{\nu_e}{e}_L$ since the neutrinos are left polarized. So the antineutrinos are right handed and its lepton companion must be e_R^c, c designates charge conjugate. The origin of the minus sign is explained in Appendix C.

The generators T_k in the five-dimensional representation are

$$T_k = \tfrac{1}{2} t_k \qquad (8.3)$$

where t_k can be written down as 5×5 matrices using the general tensor representation discussed in Appendix C. We number them such that $k = 1,\ldots,8$ correspond to SU(3) matrices and $k = 9,\ldots,14$ correspond to SU(4) matrices. t_{15} and t_{24} are chosen to be the remaining two diagonal matrices[51]. Thus t_3, t_8, t_{15} and t_{24} are the diagonal matrices and, of course, all 24 are traceless. Let us note down t_{15} which is of immediate interest to us. (See Appendix C).

$$t_{15} = \mathrm{diag}(1,1,1,-3,0) \qquad (8.4)$$

We can now identify the quark flavours in $q_1 q_2 q_3$. As the electromagnetic field will be one of the $G_{\mu k}$, the generator T_k associated with it will be the charge Q which then will also be traceless. Applied to ψ^R it will give $3Q_q + Q_e^c = 0$ or $Q_q = -\frac{1}{3}$ i.e. q=d

Hence $$R = \text{column}\ (d_1, d_2, d_3, e^c, -\nu^c)_R \qquad (8.5)$$

As we want the Higgs fields to generate masses for d_1, d_2, d_3 and for the electron, we shall also need the left-handed components for these fields. We must also introduce the u field which too acquires a mass. Thus we have to consider 15 operators,

$$d_1^R \quad d_2^R \quad d_3^R \quad d_1^L \quad d_2^L \quad d_3^L$$

$$u_1^R \quad u_2^R \quad u_3^R \quad u_1^L \quad u_2^L \quad u_3^L$$

$$e^R \quad e^L \quad \nu^L \quad (\text{or } e^{Rc} \quad e^{Lc} \quad \nu^{Rc})$$

Since five of these operators are in the right-handed quintuplet we need a left-handed decuplet for the remaining fields.

With the quintuplet we can make an irreducible ten-dimensional representation of SU(5) obtained by making the antsymmetrized product of two quintuplets. Calling

$$\psi = \text{column } (\psi_1, \ldots, \psi_5) \tag{8.6}$$

the antisymmetrized product is

$$\psi_{ab} = \frac{1}{\sqrt{2}} \{\psi_a(1)\psi_b(2) - \psi_a(2)\psi_b(1)\} \text{ , } a,b=1,\ldots,5 \tag{8.7}$$

It has ten independent components. The first three ψ_a, a=1,2,3 are a colour triplet. Hence ψ_{ab} for a,b=1,2,3 represents a product of two SU(3) triplets which decompose as $3\times3=6+\bar{3}$. The sextuplet is symmetric and the antitriplet is antisymmetric. Thus ψ_{ab} for a,b=1,2,3 form an anticolour triplet. They have charges $-\frac{2}{3}$ and therefore we may identify them with anti-u-quarks. We shall put

$$\psi_{ab} \equiv \varepsilon_{abk}(U_k^c)_L \text{ , } a,b = 1,2,3 \tag{8.8}$$

since the charge conjugate of a right-handed spinor is left-handed:

$$(\psi^R)^c = (\psi^c)_L$$

Now ψ_{a4} represents a colour triplet with charge $\frac{2}{3}$ while ψ_{a5} is a colour triplet with charge $-\frac{1}{3}$. Hence we make the following identification.

$$\psi_{a4} = (u_a)_L, \psi_{a5} = (d_a)_L, \quad a = 1,2,3 \tag{8.9}$$

ψ_{45} is a colour singlet and isospin singlet with charge +1. We identify it with e^{Lc}, i.e. $\psi_{45} = e^{Lc}$.

We thus postulate the following decuplet

$$\psi^L = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & u_3^c & -u_2^c & u_1 & d_1 \\ -u_3^c & 0 & u_1^c & u_2 & d_2 \\ u_2^c & -u_1^c & 0 & u_3 & d_3 \\ -u_1 & -u_2 & -u_3 & 0 & e^c \\ -d_1 & -d_2 & -d_3 & -e^c & 0 \end{pmatrix}_L \equiv L \tag{8.10}$$

We now have the gauge invariant Lagrangian

$$\begin{aligned} \mathcal{L}_o &\equiv i\bar{R}\not{D}R + i\bar{L}\not{D}L \\ &= i\bar{R}_a\gamma^\mu\{\partial_\mu\delta_{aa'} + ig_oG_{\mu k}(T_k)_{aa'}\}R_{a'} \\ &+ i\bar{L}_{ab}\gamma^\mu\{\partial_\mu\delta_{ab,a'b'} + ig_oG_{\mu k}(T_k)_{ab,a'b'}\}L_{a'b'} \end{aligned} \tag{8.11}$$

The first term contains generator matrices $(T_k)_{aa'}$ in the five-dimensional representation while the second contains the generator matrices $(T_k)_{ab,a'b'}$ in the ten-dimensional representation. Now an infinitesimal gauge transformation on the quintuplet is given by

$$\psi'_a = \{\delta_{ab} + ig_o\Lambda_k(T_k)_{ab}\}\psi_b, \quad T_k = \frac{t_k}{2} \tag{8.12}$$

Hence the transformation induced on the decuplet is

$$\begin{aligned} \psi_{ab} &= \tfrac{1}{2}\{[\delta_{aa'} + ig_o\Lambda_k(T_k)_{aa'}][\delta_{bb'} + ig_o\Lambda_k(T_k)_{bb'}] \\ &- [\delta_{ba'} + ig_o\Lambda_k(T_k)_{ba'}][\delta_{ab'} + ig_o\Lambda_k(T_k)_{ab'}]\}\psi_{a'b'} \end{aligned} \tag{8.13}$$

If we compare this with the transformation

$$\psi'_{ab} = [\delta_{ab'}\delta_{bb'}+ig_o A_k(T_k)_{ab,a'b'}]\psi_{a'b'}$$

we get

$$(T_k)_{ab,a'b'}=\tfrac{1}{2}\{\delta_{aa'}(T_k)_{bb'}+\delta_{bb'}(T_k)_{aa'}-\delta_{ab'}(T_k)_{ba'}-\delta_{ba'}(T_k)_{ab'}\} \tag{8.14}$$

From this we get

$$\bar{\psi}_{ab}(T_k)_{ab,a'b'}\psi_{a'b'} = 2\bar{\psi}_{ab}(T_k)_{aa'}\psi_{a'b}$$

$$= \bar{\psi}_{ab}(t_k)_{aa'}\psi_{ab'} \tag{8.15}$$

This tells us how to operate with the ten-dimensional representation of the generators.

The interaction of the fermion fields with the gauge fields is contained in the covariant derivative terms: it is

$$\mathcal{L}_I = -g_o\{\bar{R}_a\gamma^\mu(\tfrac{t_k}{2})_{aa'}R_{a'}+\bar{L}_{ab}\gamma^\mu(\tfrac{t_k}{2})_{ab,a'b'}L_{a'b'}\}G_{\mu k} \tag{8.16}$$

The fields k=1,...,8 are expected to describe the gluon fields.

Given the expression of T_{15} we find in the five-dimensional representation

$$-g_o(\bar{R}\gamma^\mu T_{15}R)G_{\mu,15}=\frac{g_o}{2\sqrt{6}}\{\bar{R}\gamma^\mu \mathrm{diag}(1,1,1,-3,0)R\}G_{\mu,15} \tag{8.17}$$

Setting $G_{\mu,15} = A_\mu$ we get

$$\frac{g_o}{2\sqrt{6}} = \frac{e}{3} \tag{8.18}$$

which fixes g_o.

The decuplet part will contribute the left-handed part of the electromagnetic current so as to give the correct electromagnetic interactions of u,d and e. We can identify $G_{\mu,24}$ with Z_μ.

For the electron-neutrino sector this must reproduce the required term in the GSW model, i.e. it must give

$$- \frac{e}{\sin\theta_W \cos\theta_W} \{ \bar{L}\gamma^\mu \frac{\tau_3}{2} L - \sin^2\theta_W J^\mu_{e.m.} \} Z_\mu$$

If we call $T_3^L \equiv \int \bar{L}\gamma^o \frac{\tau_3}{2} L d^3 \vec{x}$, we have for the neutral charge T^Z

$$T^Z = T_3^L - \sin^2\theta_W Q \qquad (8.19)$$

Since $\quad T_r(T_k T_{k'}) = \frac{1}{2}\delta_{kk'}$, we have

$$T_r(T^Z Q) = T_r\{T_3^L Q - \sin^2\theta_W Q^2 = 0$$

or

$$\sin^2\theta_W = T_r(T_3^L)^2 / T_r Q^2 \qquad (8.20)$$

where $\quad Q = T_3^L + Y/2$

Taking the trace over the quintuplet

$$\sin^2\theta_W = \frac{\frac{1}{2}}{1+(3\times\frac{1}{9})} = \frac{3}{8} \qquad (8.21)$$

The value of the Weinberg angle is thus fixed in the limit of exact symmetry. This is different from the GSW prediction where SSB was introduced. In our case all coupling constants are equal. Elaborate calculations where the Q^2 variation of the coupling constants are taken into account[28] lead to values consistent with the GSW model. These calculations also indicate that the three coupling constants approach a common value at energies of $\sim 10^{15}$ GeV.

The coupling of the gauge fields to fermion fields leads to the usual as well as some unusual vertices involving quark-lepton transitions. Some processes are indicated in (Fig.8.1) at the end of this chapter.

These interactions clearly imply the instability of the proton since the following decays will be possible (B-L is conserved in this model):

$$p \to e^+\pi^0,\ \nu_e\pi^+,\ \mu^+K^0$$
$$n \to \nu_e\pi^0,\ \text{etc.}$$

The masses of X and Y must be extremely large to keep $\tau_p \geq 10^{31}$ years. Proton decay processes are shown in (Fig.8.2). The Y propagator enters this diagram. The decay rate goes as m_y^{-4} and the life time as m_y^4. The Higgs mechanism must give very large masses to X and Y in order to suppress these decays since these are estimated to be $\sim 10^{15}$ GeV. The Higgs mechanism should be carried out in two stages. First SU(5) is broken down to SU(3)xSU(2)xU(1) and conserves the rank of the group. A set of 24 Higgs fields are required with non-zero VEV. The X and Y acquire mass but the gluons, $W^\pm$, Z, A_μ remain massless. A second set of Higgs fields is then introduced to give masses to $W^\pm$, Z^0. $m_{X,Y}$ sets the scale at which SU(5) is broken down to $SU_c(3)\times SU_L(2)\times U_y(1)$.

At the scale $m_{Z,W}$ (~100 GeV) the symmetry is broken down further to $SU_c(3) \times U_{e.m.}(1)$. We shall not discuss this matter further.

We note that in the SU(5) model discussed the neutrino remains massless because there is no room for a right-handed neutrino. One way to introduce a neutrino mass would be to incorporate an SU(5) singlet identified as ν^R. The electric charge is quantized since the charge appears as a generator of SU(5).

4. Other Models[28]

Besides SU(5) (with several variations on Higgs structure) other groups have also been considered for grand unification. The most interesting among them is SO(10). This admits a 16-dimensional representation which can accomodate all fermions in a single family and the right-handed neutrino unlike the SU(5) case where they are put in two different irreducible representations. It allows more than one symmetry breaking scheme[53], the simplest of which is

$$SO(10) \rightarrow SU(5) \rightarrow SU_c(3) \times SU_L(2) \times U_Y(1) \rightarrow SU_c(3) \times U_{e.m.}(1)$$

Pati and Salam[53] have considered a series of groups of the form $G = G^S \times G^W$ where G^S and G^W are identical strong and weak groups related by a discrete symmetry. An interesting special case is $G = SU(4) \times SU(4)$ where twelve integrally charged quarks, four flavours with three colours each and four leptons are placed in a $(4, \bar{4})$ representation.

In all these models SSB is carried out as if there existed a heirarchy of broken symmetries. The "grand" group is broken down to another one by Higgs fields which acquire an extremely large VEV and give rise to super-heavy vector mesons. Then the latter group is broken down to a succeeding one with another set of Higgs fields which give rise to less heavy vector mesons.

All these models predict proton decay. No GUT model considered so far is compelling. Gell-Mann and Ramond and Slansky[53] have carried out an extensive study of the various possible candidates for grand unification.

The baryon non-conservation predicted by GUTs finds a very important application in cosmology, namely, in the explanation of the baryon asymmetry of the universe[54]. GUTs can account for the small baryon to photon ratio (10^{-9}) of the universe. They can provide a mechanism by which baryon asymmetry can be dynamically induced starting out with a symmetric universe.

5. Problems with GUTs and Superunification

There are, however, still many unsatisfactory features of GUTs:

i) They do not provide an explanation of the origin of families.

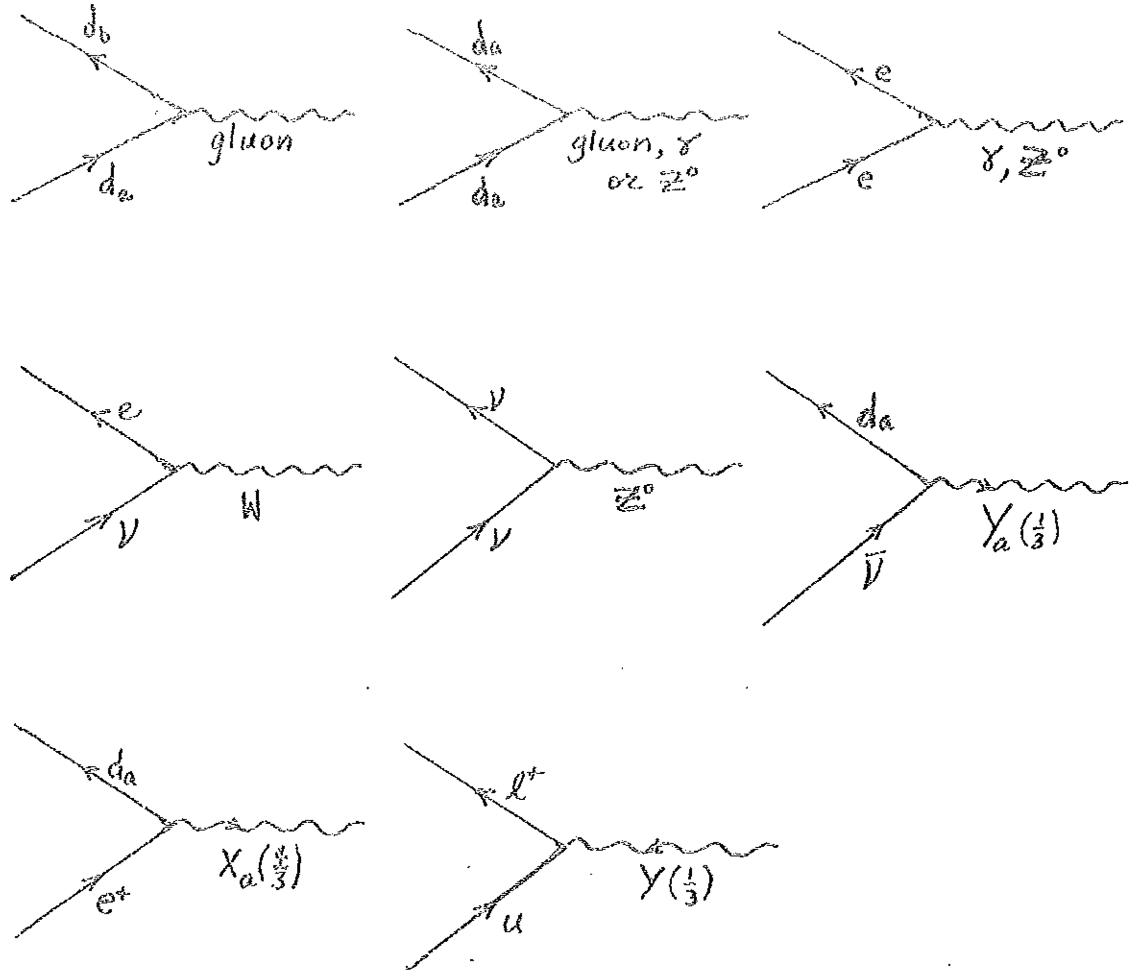
ii) Gauge boson masses jump from 10^2 to 10^{14} GeV with nothing in-between.

iii) The theories contain several free parameters.

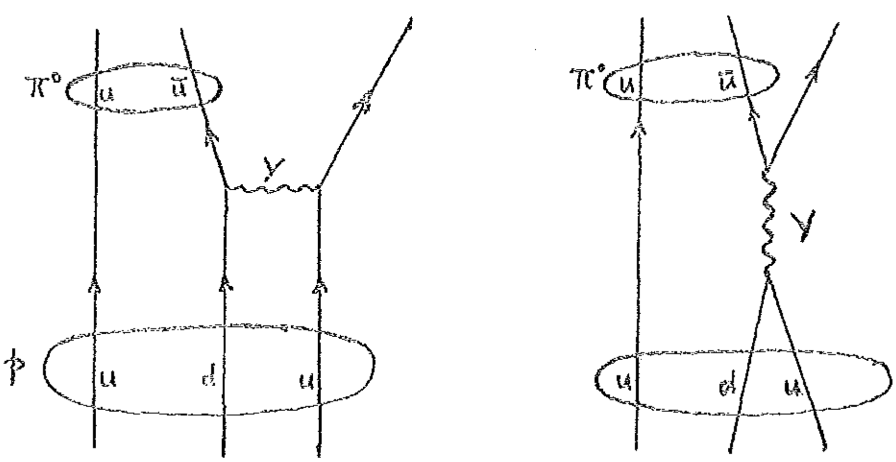
iv) Gravity is not included.

Already we are talking about at least 24 basic fermion fields and the question of their structure naturally arises. The composite nature of quarks and fermions is now under extensive study[55].

The question of unification of strong, weak and electromagnetic interaction with gravity hinges on supersymmetry and supergravity theories[56] which started from the attempt at deeply correlating and unifying fermion and boson fields, are being developed and point out to a possible superunification of the forces of nature bringing into the picture gravitons and perhaps masseless spin - $\frac{3}{2}$ particles.



(Fig 8.1) Some SU(5) Processes



(Fig 8.2) Proton Decay

CHAPTER 9

NEUTRINO MASSES AND THE GENERATION PROBLEM

In the last chapter we discussed GUT models. We noted that none of them is compelling. Further, the models neither explain the repitition of families nor seem to restrict their number. The only theoretical clue comes from QCD where the experimentally supported asymptotic freedom can be ensured by restricting the number of quark generations to eight. Combined with the aesthetically appealing idea of lepton-quark symmetry we are restricted to eight lepton families. This lack of handle on the number of lepton or equivalently neutrino types has been freely exploited in the literature in a bid to build various GUT models[57]. We wish to argue here that although such a priviledge cannot be ruled out the validity of such models is highly dubious. Our arguements draw upon plausibility and naturalness.

The experimental limit on the number of neutrino types is around 20[58]. These come from the width of Z^o decay. Cosmological considerations limit this number to between three and six[59]. These limits are derived in the big bang theory - the so-called standard model of cosmology[54]. According to the big bang theory the Helium and Deuterium contents of stellar objects are essentially primordial in origin. These contents are reasonably well known and each neutrino type contributes a certain amount to the formation of these elements in the early history of the universe.

Here, we have an example of the interface between cosmology and particle physics. We have a very important case of cosmological constraints providing a limitation on model building in particle physics. The cosmological considerations provide a further hard bound on the sum of the neutrino masses of the light type[60]. It says that $\Sigma m_\nu \lesssim 50eV$. The summation counts light neutrinos of masses less than one MeV. If the neutrino of a higher generation has a mass greater than one MeV it could possibly be argued to be consistent with observation through its decays[61]*.

We wish to argue that minimal left-right symmetric GUTs must provide some neutrinos in the forbidden range 50eV - 1 MeV unless the number of generations is limited. Any attempt to escape this conclusion would entail rediculous fine tuning of quark masses.

In GUT models neutrino masses can be introduced through the Dirac mass term and through Majorana mass terms. Upon diagonalization of the mass matrix the Dirac mass term induces a further Majorana mass term. In the class of models we shall consider, the neutrino mass term, in the final analysis is the difference of two terms - the Majorana mass and the induced Majorana mass. We shall see that both contributions can be estimated. Appendix D is devoted to the Dirac and Majorana masses and the reader is referred to it for relevant details.

A few words about the choice of models to be considered are in order here. We do not consider SU(5). Minimal SU(5) is already not consistent with proton decay[50]. Non-minimal SU(5) has no aesthetic appeal and offers no appealing reason at all to

*These bounds are derived by considering the matter content of the universe. Higher mass neutrinos decouple much earlier from the initial cosmic soup and are not expected to affect the considerations leading to the above bound.

be considered. Further, SU(5) does not offer a natural place for a massive neutrino though they can be accomodated. In addition SU(5) has other features which can be understood only if it were a subgroup of a larger group like SO(10), for example.

For definiteness, we shall consider a two step breaking of minimal SO(10):[28,53]

$$SO(10) \xrightarrow{m_X} SU_c(4)\times SU_L(2)\times SU_R(2) \xrightarrow{m_R} SU_c(3)\times SU_L(2)\times U_Y(1) \xrightarrow{m_W} SU_c(3)\times U_{e.m.}(1) \qquad (9.1)$$

This scheme is attractive for the massive neutrinos have a natural place here in the 16-dimensional representation along with the fifteen basic quarks and leptons of one family which were forced into a 5+10 of SU(5) in the last chapter. Further left-right symmetry is maintained upto a certain level - a possibility not allowed in SU(5). SO(10) could have been broken via SU(5) but this path leads to very low neutrino masses and would not give us any useful information. And, more importantly, there is no real motivation for such a route to be followed. We thus restrict ourselves to the aesthetically appealing left-right symmetric models. We would get the same results if we had used a similar breaking of SU(16). It is really the second symmetry breaking scale m_R which is the crucial scale for the present considerations.

We stated before that the neutrino mass has two sources[62]. The Majorana mass $m_a(\nu)$ depends on various parameters - the Higgs fields and their Yukawa couplings and has been argued to have a "natural" value of $10^{\pm 1}$eV for all families. We shall argue shortly that this term cannot be much greater in any case.

The induced mass of the light neutrino $m_b(\nu)$ is related to the $+\frac{2}{3}$ charge quark mass $m(q)$ in the same family by the relation:

$$m_b(\nu) \simeq \frac{m^2(q)}{m_R} \qquad (9.2)$$

where m_R is the second symmetry breaking scale.* This relation suffers corrections if higher dimensional Higgs fields are used, e.g. a 126 of SO(10). These corrections are large for the lower families but are insignificant for the higher generations. Thus from the third family onwards equation (9.2) can be assumed to be a reasonable result.

$m_b(\nu)$ is negligible for the lower generations but can become substantial, even dominant, for the higher ones as the quark masses rise rapidly, unless the quark mass spectrum becomes inexplicabily degenerate.

Now, first of all, we note that equation (9.2) gives an upper limit to quark masses. For this take $m_b(\nu) \lesssim m(\nu) \lesssim 50$eV (cosmological bound). We know that $m_R \lesssim 10^{14}$ GeV at most. We then find m (highest quark) $\lesssim$2TeV. Thus if a quark heavier than say a few TeV were ever to be found that would be the end of either GUT models we are examining or big bang cosmology. One such

*In Appendix D it will be argued that the denominator of equation (9.2) is actually the mass of the right handed neutrino. Since this mass can be generated at the second symmetry breaking stage it is expected to be $\sim m_R$

quark, would, by itself violate the cosmological bound on neutrino masses.

Before coming to the Russian β-decay experiment[63] which predicts a lower bound of 10eV for the electron neutrino mass with 90% confidence level and puts $m_{\nu e}$ between 25eV and 40eV with reasonable confidence let us consider the following, somewhat hypothetical situation.

Let $m_{\nu e} \approx 0$. This implies that $m_a(\nu) \approx 0$ for all generations. This in turn implies that $m_{\nu\mu} \approx 0$. Now the top quark is expected to be ~ 50 GeV. Hence $m_b(\nu_\tau) \approx \frac{25\times 10^2 \times 10^9}{10^{14}} = 25 \times 10^{-3}$ eV. This leaves us about two orders below the cosmological bound. Now, one knows that there is no way to lower this bound by that much. So we are strongly tempted to believe that a fourth quark in the TeV range should appear so that the corresponding neutrino would be in the eV range. Such a scenario will be welcome in certain models which work with four families[57]. In such a scheme more quarks could be introduced between the 100 GeV - 1 TeV range. Such a quark spectrum would appear quite unnatural if we induct more than one or two quarks in such a range. Thus even such a scheme would be unattractive if say more than six quarks are to be fitted. It is clear that such a scenario would most naturally accomodate four quarks. But, then, the burden shifts to finding loopholes in the Russian experiment and they are hard to find. Let us now concentrate on the realistic situation created by this fairly reliable experiment.

The Russian experiment claims $25eV < m_{\nu e} < 40eV$. This experiment is measuring m_a and not m_b. It would imply an m_a around the above value for all families. But then $m_{\nu\mu}$ would also be in the above range. The cosmological bound clearly rules out the upper bound. It rules out the lower bound as well. This would give $m_{\nu e} \stackrel{\sim}{\sim} m_{\nu\mu} \stackrel{\sim}{\sim} 25eV$. These two neutrinos saturate the bound. This means that ν_τ must be massless. Thus m_a and m_b must cancel out for ν_τ. But this would need a mass $\sim$ 1 TeV for the top quark. This is well above the expected mass of 40 GeV for the top quark. By assuming that the m_a's are slightly different i.e. by playing with the various Yukawa couplings we can bring the requirement of $m_a(\nu_\tau)$ to be of the order of an electron volt which can then be made to cancel $m_b(\nu_\tau)$. Thus with some difficulty the ν_τ can be accomodated within the bound. In other words the lower bound of $m_{\nu e} \stackrel{\sim}{\sim} 25eV$ can be made consistent with the existence of three families. We conclude that the Russian experiment and left right symmetric models and the cosmological bounds can all be acceptable provided there are three generations only. Any further quark would come in by a very artificial tuning of its mass so that the corresponding neutrino will not contribute to the bound.

The experiment tells us that the 90% confidence level bound is only 10eV. If we accept this value then our argument suggests that upto six families can be sneaked in with the sixth neutrino massless. The top most quark would have a mass of about 1 TeV according to equation (9.2).

We can now ask the following question. How can we construct left-right symmetric GUTs with more than three quarks. One way, as we have seen above is to appeal to the near degeneracy of quark masses of the higher generation plus some adjustments of various Yukawa couplings. A simpler and more acceptable way in our opinion is to appeal to the slight uncertainty in the experimental result. Suppose we want to have four families. Then we can set $m_a(\nu) \overset{\sim}{\sim} 17eV = m_\nu$ for the first three generations. No unnatural adjustments of the previous paragraph are required for this. For the fourth quark m_a must cancel m_b so that the quark mass must be ~ 1.3 TeV. Further quarks get in only if the quark mass spectrum becomes degenerate.

The message from our analysis is clear. If the current estimate of $m_{\nu e}$ is reasonable then only three or four families could be fitted in a left right symmetric GUT broken in two steps. If the experimental bound is relaxed then may be six generations can be squeezed in. The larger the number of generations the less naturally they can be brought in.

Thus if we do not stretch the GUT models too far, they bring us to the same conclusion as the cosmological ones, namely, we should expect three to four generations only.- far below the theoretical bound of eight and the experimental bound of 20. We thus see no justification in proposing GUT models employing eight families, for example.

Whereas the experimental search for quarks in the TeV range is still a far cry, it would be very worthwhile to refine the

the experimental data on the electron neutrino mass. This information alone may determine the fate of GUTs as well as big bang cosmology.

CONCLUDING REMARKS

We have come a long way discussing gauge theories. The successes of the theory are impressive, indeed. However, the very promise this theory holds towards an unravelling of Nature makes it meaningful to ask questions which one would not have dared to ask only a few years ago. Some of these are:

1. Why does Nature employ both quarks and leptons with leptons having no role to play in strong interactions? Why does each family of leptons come with at least three families of quarks? Why are there so many basic lepton and quark fields when we believe that Nature works economically and intelligently? Are they really elementary?

2. Why does Nature employ two statistics? Is there a deeper boson-fermion symmetry?

3. What are the Higgs? Recall that the Higgs sector is the most troublesome part of any gauge theory and yet we need these Higgs. Are the Higgs composite objects?

4. Where does gravity fit into the general scenario?

The above questions lead us into the areas of studies currently under intense investigation. These are the composite models of quarks and leptons - the so called preonic theories, supersymmetry and supergravity[55,56]. Our present studies naturally motivate us to explore some of these fields in the future.

Coming back to what we have done we can conclude that two parameters would play a very crucial role in deciding the fate of the gauge theory models. These are the proton decay life-time and the mass of the electron neutrino. If the experiments can push the proton life-time further by a few orders of magnitude then perhaps all existing GUT models would be in severe difficulty. If ν_e mass or in fact any other light neutrino mass falls in the range 50eV to 1 MeV then a major conflict would arise between GUTs and big bang cosmology. A conflict between two theories we believe to be correct, would be a certain signal that we are missing something deep somewhere.

Finally, we wish to humbly acknowledge that even within the programme we have carried out we have been forced to admit major gaps due to the fact that we had only half a year to complete these studies. These areas include, for example, a critical study of group theory, renormalization group techniques and a sounder understanding of low energy phenomenology in particle physics. These three areas would form the immediate fields of our future studies.

APPENDIX A

NOTATIONS AND CONVENTIONS

We have used natural units throughout, i.e. we have set $\hbar = c = 1$.

The electronic charge = $-e, e>0$

The metric tensor $g_{\mu\nu}$ has components

$$g_{oo} = - g_{11} = - g_{22} = - g_{33} = 1$$

$$g_{\mu\nu} = 0, \quad \mu \neq \nu \qquad \text{(A.1)}$$

Contravariant and covariant four-vectors are defined respectively as

$$A^{\mu} = (A^{o}, \vec{A}\ ), \quad A_{\mu} = (A^{o}, -\vec{A}) \qquad \text{(A.2)}$$

$$A_{\mu} = g_{\mu\nu} A^{\nu}, \quad \nu \text{ summed over} \qquad \text{(A.3)}$$

The scalar product of two four-vectors is

$$a^{\mu} b_{\mu} = a_{\mu} b^{\mu} = a^{o} b^{o} - \vec{a}.\vec{b} \equiv ab \qquad \text{(A.4)}$$

For simplicity of notation the following symbols are used in the text wherever no confusion arises.

A lepton field $\psi_{\ell}(x)$ is written as ℓ

A neutrino field $\psi_{\nu_{\ell}}(x)$ is written as ν_{ℓ}

A quark field $\psi_{q}(x)$ is written as q

A field ψ destroys the particle and creates the antiparticle.

A suitable collection of left (right) handed fermion fields will be denoted by $L_{\ell}(\ell_R)$. For example

$\binom{\nu}{\ell}\ell)_L = \frac{1}{2}(1-\gamma_5)\binom{\nu}{\ell}\ell) \equiv L_\ell$ = left-handed lepton doublet. (Subscript in L_ℓ may be deleted whenever no confusion arises) and $\ell^R = \frac{1}{2}(1+\gamma_5)\ell$ is a right-handed lepton singlet.

A diagonal matrix with diagonal entries $b_1, b_2, b_3, \ldots, b_n$ is written as $\mathrm{diag}(b_1, b_2, b_3, \ldots, b_n)$

The trace and determinant of a matrix A are denoted as TrA and det A respectively.

APPENDIX B

THE DIRAC EQUATION

The Dirac equation is

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \qquad (B.1)$$

The γ-matrices satisfy the conditions

$$[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu} \qquad (B.2)$$

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0 \qquad (B.3)$$

The γ-matrices may be put in terms of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (B.4)$$

as

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad k = 1,2,3 \qquad (B.5)$$

Defining $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (B.6)

Then γ^5 has the properties

$$[\gamma^\mu, \gamma^5]_+ = 0 \, , (\gamma^5)^2 = 1 \, , \gamma^{5\dagger} = \gamma^5 \qquad (B.7)$$

The projection operators, trace identities, etc, are precisely as listed in Appendix A of reference 21.

Note that for zero mass particles (m=0) the helicity projection operators are

$$\Pi^{\pm} = \tfrac{1}{2}(1 \pm \gamma_5) \qquad (B.8)$$

APPENDIX C

ESSENTIALS OF SU(N) GROUPS

I. General Internal Symmetries

Particles may fall into multiplets forming representations of SU(N) - the group isomorphic to that of NxN unitary unimodular matrices ($U^\dagger U = 1$ and $\det U = 1$). The condition $\det U = 1$ singles out a connected subgroup of the group of matrices. The requirement $U^\dagger U = 1$ insures that the norms of particle states are preserved under the group transformations.

A general NxN matrix has $2N^2$ arbitrary real parameters. The requirement $U^\dagger U = 1$ imposes N^2 conditions and $\det U = 1$ imposes one condition. Hence SU(N) has $N^2 - 1$ generators T_α obeying

$$[T_\alpha , T_\beta] = i f_{\alpha\beta\gamma} T_\gamma \, , \; f_{\alpha\beta\gamma} = \text{str. constants} \qquad (C.1)$$

An arbitrary infinitesimal element of the group is

$$U = 1 - i \omega_\alpha T_\alpha \qquad (C.2)$$

The smallest non-trivial irreducible representation (the fundamental representation) is of dimensionality N by definition. There always exists the adjoint representation whose dimensionality equals the number of generators with

$$(T_\alpha)_{\beta\gamma} = -i f_{\alpha\beta\gamma} \tag{C.3}$$

The possible dimensionalities of other representations depend on N.

2. SU(2) - Isospin

Experimental evidence suggests that the nucleons and π-mesons may be grouped into the following multiplets.

$$N = \begin{pmatrix} p \\ n \end{pmatrix}, \quad \Pi = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} \tag{C.4}$$

where the observed pions $\pi^{\pm} = \frac{1}{\sqrt{2}}(\pi_1 \pm i\pi_2)$ and $\pi^0 = \pi_3$. Members of each multiplet have nearly equal masses and the small mass differences can be thought of as due to electromagnetic corrections. Apart from these corrections, systems of nucleons and pions are invariant under matrix transformations representing the SU(2) group

$$\begin{aligned} N &\to N + \delta N, \quad \delta N = -i\omega_\alpha T_\alpha N \\ \Pi &\to \pi + \delta\pi, \quad \delta\pi = -i\omega_\alpha T_\alpha \pi \end{aligned} \tag{C.5}$$

where ω_α are arbitrary infinitesimal real numbers and the components of isospin T_α ($\alpha = 1,2,3$) are the generators of SU(2) obeying the commutation relations $[T_\alpha, T_\beta] = i\varepsilon_{\alpha\beta\gamma} T_\gamma$ (C.6)

The nucleon doublet forms a basis for a two-dimensional representation in which $2T_\alpha$ is represented by 2×2 Pauli matrices τ_α

$$T_\alpha = \tfrac{1}{2}\tau_\alpha \Rightarrow T_\alpha N = \tfrac{1}{2}\tau_\alpha N$$

and (C.7)

$$\delta N_i = -\tfrac{i}{2}\{\omega_1(\tau_1)_{ij} + \omega_2(\tau_2)_{ij} + \omega_3(\tau_3)_{ij}\}N_j$$

The π-meson triplet forms a basis for a three-dimensional irreducible representation

$$(T_\alpha)_{\beta\gamma} = -i\varepsilon_{\alpha\beta\gamma}$$

$$(T_\alpha\pi)_\beta = (T_\alpha)_{\beta\gamma}\pi_\gamma = +i\varepsilon_{\alpha\beta\gamma}\pi_\gamma$$

i.e. $$\delta\pi_\beta = -i\varepsilon_{\alpha\beta\gamma}\omega_\alpha\pi_\gamma$$

or $$\delta\vec{\pi} = i\vec{\omega}\times\vec{\pi}$$ (C.8)

The representation 3 is special, in that its dimensionality equals the number of generators and that the matrix representation of T_α is obtainable directly from the structure constants $\varepsilon_{\alpha\beta\gamma}$. It is called the adjoint representation. Other irreducible representations of SU(2) are familiar from the theory of angular momentum. Their possible dimensionalities are $2T+1$ $(T = 0, 1, 2)$.

The adjoint representation can also be represented in an alternative form as follows:

If x_i $(i=1,2)$ transforms as 2, i.e.

$$\delta x_i = -\tfrac{i}{2}\omega_\alpha(\tau_\alpha)_{ij}x_j$$

Then $y_\alpha (\alpha = 1,2,3)$ which transforms as 3 may be represented as

$$y_\alpha \equiv x^\dagger T_\alpha x = \tfrac{1}{2} x_i^* (\tau_\alpha)_{ij} x_j$$

or

$$\vec{y} = \tfrac{1}{2}(x^\dagger \vec{\tau} x) \qquad (C.9)$$

To check this we compute δy_α

$$\delta y_\alpha = (\delta x^\dagger T_\alpha x) + (x^\dagger T_\alpha \delta x)$$

$$= i\omega_\beta \{x^\dagger [T_\beta, T_\alpha] x\}$$

$$= - \varepsilon_{\beta\alpha\delta} \omega_\beta y_\delta \quad \text{as required}$$

Next consider the somewhat unfamiliar idea of the complex conjugate representation. Since N transforms as

$$\begin{aligned} N \rightarrow N' &= UN \\ N^* \rightarrow N^{*\prime} &= U^* N^* \text{ with } N^* = \binom{p^*}{n^*} \end{aligned} \qquad (C.10)$$

Thus N^* also generates a representation of SU(2) in terms of the matrices U^* (In field theory p designates a field that destroys a proton and creates an antiproton and p^* is an adjoint field creating a proton and destroying an antiproton). For SU(2) the complex conjugate representation is equivalent (not identical) to the representation having the spinor N as the basis. Equivalence implies that a matrix S exists such that

$$U = SU^* S^{-1} \qquad (C.11)$$

Using the infinitesimal version of U we have

$$N \rightarrow (1 + \frac{i}{2} \vec{\omega} \cdot \vec{\tau}) N$$

$$N^* \rightarrow (1 - \frac{i}{2} \vec{\omega} \cdot \vec{\tau}^*) N^*$$

For equivalence all we need is

$$\tau \to -S\tau^* S^{-1} \tag{C.12}$$

With $\tau_1 = \tau_1^*$, $\tau_3 = \tau_3^*$ and $\tau_2 = -\tau_2^*$ we see that $S \propto \tau_2$ is the required matrix. Then the spinor N_c with

$$N_c = SN^* = i\tau_2 N^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \bar{p} \\ \bar{n} \end{pmatrix} = \begin{pmatrix} \bar{n} \\ -\bar{p} \end{pmatrix} \tag{C.13}$$

transforms identically to N:

$$N_c \to N_c' = (1 + \tfrac{i}{2}\,\vec{\omega}.\vec{\tau})N_c \tag{C.14}$$

3. SU(3)

The group SU(3) has eight generators $T_\alpha (\alpha = 1,\ldots,8)$ with

$$[T_\alpha, T_\beta] = i f_{\alpha\beta\gamma} T_\gamma \tag{C.15}$$

where $f_{\alpha\beta\gamma}$ are real and totally antisymmetric in α,β,γ. The fundamental representation 3 and the generators in this are $T_\alpha = \frac{1}{2}\lambda_\alpha$; λ_α are 3x3 traceless hermitian matrices. They act on basis vectors of the form

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{C.16}$$

An infinitesimal element of the group is represented by the transformation $x' = Sx$ with $S = 1 - \frac{i}{2}\omega_\alpha\lambda_\alpha$ where ω_α are real infinitesimal parameters. A set of matrices λ_α satisfying the above commutation relations are

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad (C.17)$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

By construction the first three are isospin generators. The last one which is diagonal and commutes with isospin is identified with $\sqrt{3}Y$. These are called Gell-Mann matrices and are generalizations of the Pauli matrices. The structure constants $f_{\alpha\beta\gamma}$ can be calculated by explicit commutation of λ_α. They are of course, the same for any representation.

λ_4, λ_5 form two members of another set of Pauli-like matrices (called U spin) and so do λ_6 and λ_7 (V-spin). The missing member in each case can be obtained from the commutations and expresed as appropriate linear combinations of λ_3 and λ_8 giving two other useful equivalent sets.

The dimensionality of SU(3) irreducible representations depend on two positive integers m and n and are given by

$$D(m,n) = \tfrac{1}{2}(n+1)(m+1)(n+m+2) \qquad (C.18)$$

The representations (m,n) and (n,m) have the same dimensionality. Some typical representations

$$D(1,0)=3, \; D(1,1)=8, \; D(2,0)=6, \; D(3,0)=10$$

D(1,1) corresponds to the adjoint representation with

$$(T_\alpha)_{\beta\gamma} = -i f_{\alpha\beta\gamma} \quad (C.19)$$

The complex conjugate representation is distinct, e.g. 3* is distinct from 3. This can be seen from the simple fact that there is no S such that $S\lambda S^{-1} = -\lambda^*$ for all λ and the $-\frac{1}{2}\lambda^*$ generates an independent set of representation matrices. If an S were to exist as required above then $S\lambda_8 S^{-1} = -\lambda_8^* = -\lambda_8$. This means $\pm\lambda_8$ would have the same eigenvalues.

For the sake of completeness, let us mention one useful set of generators used extensively by Okubo. One defines a set of nine real traceless matrices A^i_j with matrix elements $(A^i_j)_{\alpha\beta}$ $(i,j,\alpha,\beta = 1,2,3)$.

$$(A^i_j)_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta} - \frac{1}{3}\delta_{ij}\delta_{\alpha\beta} \quad (C.20)$$

Only eight are independent since $A^1_1 + A^2_2 + A^3_3 = 0$. The commutation rules are easily computed to be

$$[A^i_j, A^k_\ell] = \delta^i_\ell A^k_j - \delta^k_j A^i_\ell \quad (C.21)$$

The correspondence with T_α can be worked out. The above representation can be taken over to SU(N) groups by replacing $-\frac{1}{3}$ by $-\frac{1}{N}$ in equation (C.20) and letting the indices run from 1 to N.

In the Gell-Mann scheme the basic triplet is identified with the quarks -

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \equiv \begin{pmatrix} u \\ d \\ s \end{pmatrix} \quad (C.22)$$

Isospin and hypercharge are defined as

$$T_\alpha = \tfrac{1}{2}\lambda_\alpha, \ (\alpha = 1,2,3) \quad Y = \frac{1}{\sqrt{3}}\lambda_8 \quad (C.23)$$

Thus $T_3 = \mathrm{diag}(\frac{1}{2}, -\frac{1}{2}, 0)$, $Y = \mathrm{diag}(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$

$T^2 = T(T+1) = \mathrm{diag}(\frac{3}{4}, \frac{3}{4}, 0)$

$S = Y - B = \mathrm{diag}(0, 0, -1)$, ($B = \frac{1}{3}$ for u, d, s)

$Q = T_3 + \frac{Y}{2} = \mathrm{diag}(\frac{2}{3}, \frac{1}{3}, -\frac{1}{3})$

T_3 and Y for antiquarks are the negatives of the above values for quarks.

APPENDIX D

MAJORANA AND DIRAC MASSES

Since fermion number violations are permitted in GUTs, two kinds of neutrino mass terms can be introduced. We illustrate this with the example of one family: $(\nu_e\ e)_L$, ν_R, e_R we have the usual Dirac mass term $m_D \bar{\nu}^L \nu^R$ + Hermitian Conjugate term generated by the Higgs doublets. We can add a Majorana mass term:

$$m_M \bar{\nu}_L^{\mathcal{C}} \nu_R + m_M^* \bar{\nu}_R \nu_L^{\mathcal{C}} = m_M \nu_R^T C \nu_R' + m_M^* \nu_L^{CT} C \nu_L^{\mathcal{C}} \qquad (D.1)$$

where C is the charge conjugation matrix obeying

$$C^{-1} \gamma_\mu \mathcal{C} = -\gamma_\mu^T \qquad (D.2)$$

The Majorana term can be introduced into the Lagrangian as a bare term since ν_R is a singlet under $SU(2) \times U(1)$ or else from the Yukawa couplings to a singlet Higgs field. Note that

$$\nu_{L,R} = P_{L,R} \nu \qquad (D.3)$$

where ν is a four component field, annihilate L and R neutrinos and that

$$\nu^C_{L,R} = P_{L,R}\nu^C = P_{L,R}C\bar{\nu}^T = C\bar{\nu}^T_{R,L} \qquad (D.4)$$

annihilate L and R antineutrinos; ν_L and ν^C_R are members of doublets while ν_R and ν^C_L are singlets. Majorana mass terms violate lepton and fermion numbers by two units.

By redefining phases of the fields we can take m_D and m_M to be real so that the overall mass term is

$$m_D(\bar{\nu}_L\nu_R + \bar{\nu}_R\nu_L) + m_M(\bar{\nu}^C_L\nu_R + \bar{\nu}_R\nu^C_L) = m_D\bar{\nu}\nu + m_M\bar{n}n \qquad (D.5)$$

with $\nu = \nu_L + \nu_R$, $n = \nu^C_L + \nu_R$

In the usual case $m_M = 0$, we identify ν as the mass eigenstate. If $m_D = 0$ and $m_M \neq 0$ one must regard (ν_L, ν^C_L) and (ν^C_R, ν_R) as pairs of independent left and right-handed fields. Linear combinations with definite mass but not definite lepton numbers must be chosen to diagonalize the mass matrix. For one family the most general mass matrix is

$$(\bar{\nu}_L, \bar{\nu}^C_L)\begin{pmatrix} a & d \\ d & s \end{pmatrix}\begin{pmatrix} \nu^C_R \\ \nu_R \end{pmatrix} \qquad \text{with } d = \tfrac{1}{2} m_D \text{ and } s = m_M$$

The off-diagonal elements are equal because $\bar{\nu}_L\nu_R = \bar{\nu}^C_L\nu^C_R$. The term in a is not gauge invariant and is thus forbidden. However it can be introduced through a Higgs triplet (ρ_1, ρ_2, ρ_3). Then the Yukawa coupling $(\bar{\nu}_L \bar{e}_L)(\vec{\tau}\cdot\vec{\rho})\begin{pmatrix} e^C_R \\ -\nu^C_R \end{pmatrix}$ will contain the Majorana term a provided $\langle\rho_1\rangle_0 \neq 0$. It will violate lepton number by two

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