ADDIS ABABA UNIVERSITY
SCHOOL OF GRADUATE STUDIES

ACTION OF A RELATIVISTIC CHARGED POINT PARTICLE ON ITSELF AND OTHER CHARGES

BY
NEGUSSIE TIRFESSA

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ABSTRACT

The different expressions for the Green function of electromagnetic field of a point charge have been considered with the special attention to the advanced contribution that is shown to be of importance for the self-action of particle responsible for the radiation reaction. Simultaneous expansions for electromagnetic fields of a point charge are obtained in two different ways, by means of Lagrange expansions of the fields and the Taylor expansion of the Green function. The Lagrangian description for a system of charged particles and their electromagnetic field is given in terms of particle variables only, with the field expressed through the higher derivatives of the particles velocities. It is shown that the self-actions of particles may be included in their equations of motion with all divergent terms cancelled. An intrinisic relation between the retarded and advanced fields of charged particles has been also obtained. The results are important for an exact description of a system of relativistic charged particles.

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INTRODUCTION

A charged particle in motion produces an electromagnetic field. At all points in space this field is governed by the Maxwell's equations and it is described by their solutions known as retarded Lienard-Wiechert potentials. These potentials can be obtained by means of a Green function (integrated over the particle's position and the retarded time) provided this Green function by itself satisfies a certain inhomogeneous equation with a singular source called "the Green function equation."

However, the fields obtained from the retarded Lienard-Wiechert potentials are expressed in terms of the particle's position, velocity and acceleration at the retarded time but not at the time of observation (The difference between these moments of time is the time necessary for the fields to arrive at the point of observation and is proportional to the distance between the particle's position at the retarded time and the point of observation). As a result, a complete description of this field at the point of interest requires a knowledge of the whole past history of the particle. But this is not an easy task to find it even for the simple trajectories (e.g. circular and elliptical). We can imagine how complex it would be to deal with a system of particles (e.g. in a statistical description of this system) in a completely arbitrary motion, where we are interested in the evolution of the state of the whole system at some moment of laboratory time.

Whenever a charged particle is accelerated it emits radiation that carries away energy with it. The process has to obey the principle of energy conservation. This requires a force that does negative work on this particle. It is assumed that the force that the particle exerts on itself to be the sole source of this force which is the radiation reaction force. But since this force appears to be divergent at the site of a point particle the calculation of this force has never been satisfactory. Lorentz calculated this force for an electron (a small charged sphere in his model) having a finite radius by considering the retarded force exerted by one part of the electron on other parts. This force was expressed as a series in powers of the radius (The first term of which was the radiation reaction force). However, his calculation was approximate and limited to non-relativistic motion and divergent as radius of the particle tends to zero. Dirac assumed an expression for this force to be half the difference between retarded and advanced fields of a particle which is non-singular at any point in space. He evaluated this force at the site of the particle and obtained an intersting result for the radiation reaction force. Eventhough the basis of his assumption is not clear his results are still valid and are being used in classical electrodynamics. In an attempt to give a physical explanation to Dirac's assumption while accepting his result Wheeler and Feynman in their absorber theory of radiation, assumed this force to arise from other particles of the surrounding absorber as a response to the field produced by the radiating particle. In this theory a particle does not act on itself. The source of the radiation reaction force is the *absorber. Gordeyev has also obtained the same result as that of Dirac but from the retarded Lorentz force exerted by a particle on itself. However, in this work some assumptions have been made with respect to some terms that are justified in our present consideration.*

The Maxwell equations are symmetric in time. Although the retarded and advanced solutions are mathematically equivalent and satisfy these equations, the advanced solution is in most cases rejected because it appears to contradict with our experience and causality. However, in the four dimensional space-time of relativity an interaction between particles involves advanced interaction. Wheeler and Feynman have also shown the importance of advanced fields in their calculation of the radiation reaction force. It can also be seen from one of our calculations of this same force in this work. But in the case of Wheeler and Feynman the appearance of the advanced field is due to rather artificial idea of absorber.

In the first chapter, we shall first make a review of field of a charged particle in arbitrary motion and consider the retarded and advanced Green functions. Here, it may be incredible but the commonly used Green function is shown to be not a solution of the "Green function equation" because it does not satisfy this basic equation. The correct retarded Green function is particularly important in handling the problem of the action of a particle on itself, in which the importance of advanced field (that is usually rejected) can be seen. Hence it is this Green function that should be used to obtain the field of a particle at any point in space, including the center of the particle itself. Next the potentials and fields of a relativistic *charged particle are expanded in terms of position, velocity, acceleration, first order derivative of acceleration, second order derivatives of acceleration, ... of this particle at a time simultaneous with the observation time. The simultaneous expansion of the electromagnetic field of a relativistic charged particle in arbitrary motion was first given by Gordeyev (using Lagrange expansion) and later on commented by Weert (using expansion of Green function). Both approaches are presented here and are used in the remaining chapters and a comparison is made.*

The second chapter is devoted to the calculation of the self-action force. With the simultaneous expansions for potentials and fields the problem is treated in two different ways. In the first method we use a purely retarded field of the particle in the Lorentz force that the particle exerts on itself and the symmetry of delta function, which makes the divergent terms vanish (This justifies Gordeyev's assumptions). In the second method by using the correct retarded Green function an advanced field of the particle is also included and that results in the cancelation of the divergent terms (This one justifies Dirac's assumption). Both of our calculations are exact and leads to the same result. Moreover, we do not use any approximations at all.

The last chapter deals with the Lagrangian description of a system of relativistic charged particles and relation between retarded and advanced fields. The usual Lagrangian of a system of relativistic charged particles contains field variables and generalized particle variables, which are functions of the retarded time. Variation of the *action integral with respect to field variables in the conventional approach gives the Maxwell equations and variation with respect to the particle's variables gives its equations of motion. But in view of the simultaneous expansion the field variables are expressed in terms of position, velocity, acceleration and all order accelerations of the particle producing the field. In this case the Lagrangian of a system of relativistic charged particles would be a function of these variables only. The Lagrange equations of motion for such system were derived by Gordeyev. These equations contains higher derivatives of velocity with respect to* $x_0 = ct$ *and are derived here.*

As a result of the simultaneous expansion for potentials a relation between the retarded and advanced fields of interacting particles was obtained by Gordeyev (by using the Lagrange expansion for potentials). This relation is of such a nature that it can be included into the Lagrangian of interacting particles without affecting their equations of motion. Here we demonstrate that the Green function approach also gives the same relation.

A lot of attempts have been made to make a relativistic generalization of Statistical Mechanics. The need arises practically due to the existence of high temperature plasma and its possible use for controlled thermonuclear reactions. Also theoretically according to the principle of relativity Statistical Mechanics should be formulated in such a way that it is Lorentz invariant. One of the problems that arise in relativistic Statistical Mechanics, which can be seen from the series of works of Hakim and others, is the absence of a universal laboratory time for all particls. Although with the *simultaneous expansion this problem appears to be removed, the relativistic generalization of Statistical Mechanics requires the Hamiltonian description of the system of charged particles. However, such a generalization is not straight forward for a Lagrangian depending on higher derivatives of velocities of particles as is the case with simultaneous expansion, and therefore this problem should be a subject of future research.*

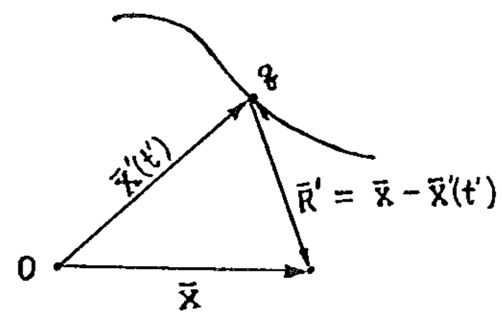
CHAPTER 1

SIMULTANEOUS EXPANSION OF THE ELECTROMAGNETIC FIELD OF A RELATIVISTIC CHARGED PARTICLE

1.1. Field of A Charged Particle in Arbitrary Motion

Consider a point charge particle in an arbitrary motion.

fig. 1.



The field produced by the moving charge satisfies the following non-homogeneous Maxwell equations.

$$\bar{\nabla}\cdot\bar{E} = 4\pi\rho \qquad\qquad \bar{\nabla}\times\bar{E} + \frac{1}{c}\frac{\partial\bar{H}}{\partial t} = 0$$

$$\bar{\nabla}\times\bar{H} - \frac{1}{c}\frac{\partial\bar{E}}{\partial t} = \frac{4\pi}{c}\bar{J} \qquad\qquad \bar{\nabla}\cdot\bar{H} = 0 \qquad (1.1)$$

Where all the sources and fields are taken at the point of observation $\bar{x}$ and time of observation t.

For a point charge the sources are given interms of delta function

$$\rho = e\,\delta[\bar{x} - \bar{x}(t)] \quad \text{and } \bar{J} = e\,\bar{\vartheta}\,\delta[\bar{x} - \bar{x}(t)]$$

If we introduce the scalar and vector potentails $\phi(\bar{x},t)$ and $\bar{A}(\bar{x},t)$, respectively, then the fields can be described interms of these potentials as

$$\bar{E} = -\bar{\nabla}\phi - \frac{1}{c}\frac{\partial\bar{A}}{\partial t} \quad ; \quad \bar{H} = \bar{\nabla}\times\bar{A} \qquad (1.2)$$

If the potentials satisfy the Lorentz Condition,

$$\bar{\nabla}\cdot\bar{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0 \tag{1.3}$$

then they are solutions of the wave equation

$$\Box \left\{ \begin{array}{l} \phi(\bar{x},t) \\ \bar{A}(\bar{x},t) \end{array} \right\} = -4\pi \left\{ \begin{array}{l} \rho(\bar{x},t) \\ 1/c\ \bar{J}(\bar{x},t) \end{array} \right\} \tag{1.4}$$

where $\Box = \nabla^2 - \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}$ is the d' Alembert's Operator.

To solve these wave equations, we look for an appropriate Green function $G(\bar{x},t;\bar{x}',t')$ that should satisfy the equation

$$\Box G = -\,4\pi\delta[\bar{x} - \bar{x}']\delta(t-t') \tag{1.5}$$

With the help of this Green function the solutions of the wave equations, eq.(1.4), can be given by

$$\phi(\bar{x},t) = \int G(\bar{x},t;\bar{x}',t')\rho(\bar{x}',t')d^3x'dt' \tag{1.6a}$$

and

$$\bar{A}(\bar{x},t) = \frac{1}{c}\int G(\bar{x},t;\bar{x}',t')\bar{J}(\bar{x}',t')d^3\bar{x}'dt' \tag{1.6b}$$

It can at once be seen that (i.e. Applying the d'Alembert's Operator to equations (1.6a) and (1.6b)) these potentials are the solutions of the wave equations provided that the Green function is a solution of (1.5). Of course, we can not start evaluating the integrals in equation (1.6) before having the solutions of equation (1.5) at hand.

1.2. The Retarded and Advanced Green Functions

The solutions of the Green function equation, eq.(1.5), may be found by using the Fourier transform method in the following form.

Using the Fourier representations of the (four dimensional) delta function

$$\delta(\bar{x} - \bar{x}')\delta(t - t') = \left[\frac{1}{2\pi}\right]^4 \int e^{i[\bar{k}\cdot(\bar{x}-\bar{x}') - \omega(t-t')]} \, d^3\bar{k} \, d\omega$$

and the Green function

$$G(\bar{x}-\bar{x}';t-t') = \left[\frac{1}{2\pi}\right]^4 \int \tilde{G}(\bar{k},\omega) \, e^{i[\bar{k}\cdot(\bar{x}-\bar{x}') - \omega(t-t')]} \, d^3\bar{k} \, d\omega$$

in equation (1.5) and solving for the Fourier transform $\tilde{G}(\bar{k},\omega)$ of $G(\bar{x}-\bar{x}';t-t')$, the required Green function becomes

$$G(R,\tau) = \frac{c^2}{4\pi^3} \int \frac{e^{i[\bar{k}\cdot\bar{R} - \omega\tau]}}{(ck)^2 - \omega^2} \, d^3\bar{k} \, d\omega$$

where $\bar{R} = \bar{x} - \bar{x}'$ and $\tau = t-t'$.

Depending on the choice made in avoiding the singularities in this last integral, we arrive at the following two different solutions, which are the retarded and advanced Green functions

$$G_r(R,\tau) = \frac{\theta(\tau)}{R}\left[\delta(\tau - \frac{R}{c}) - \delta(\tau + \frac{R}{c})\right] \qquad (1.7a)$$

and

$$G_a(R,\tau) = \frac{\theta(-\tau)}{R}\left[\delta(\tau + \frac{R}{c}) - \delta(\tau - \frac{R}{c})\right] , \qquad (1.7b)$$

respectively.

where $\theta(\tau)$ is a unit step-function given by

$$\theta(\tau) = \begin{cases} 1, & \tau > 0 \\ 0, & \tau < 0 \end{cases} \qquad (1.8)$$

For $R \neq 0$, the Green functions in equation (1.7) are usually written in the form

$$G_r(R,\tau) = \frac{\theta(\tau)}{R}\,\delta(\tau - \frac{R}{c}) = 2c\theta(\tau)\;\delta\left[\tau^2 - \frac{R^2}{c^2}\right] \qquad (1.9a)$$

$$G_a(R,\tau) = \frac{\theta(-\tau)}{R}\delta(\tau + \frac{R}{c}) = 2c\theta(-\tau)\;\delta\left[\tau^2 - \frac{R^2}{c^2}\right] \qquad (1.9b)$$

Our interest, at first, is to check whether the usual Green functions, equations(1.9a) and (1.9b), satisfy the Green function equation, eq.(1.5), for R = 0 too. when R = 0 the Green functions in this form do not satisfy equation (1.5). And the result obtained after applying the d'Alembert's Operator on these functions is different from the delta function on the right hand side of (1.5). Thus, for R = 0 these Green functions are not solutions of (1.5). In fact,had it not been for the homogenity of the Green function equation for $R \neq 0$, they would have not satisfied this equation at all. But if we take the Green functions in their original form (eqs.(1.7a) and (1.7b)) then (1.5) is found to be satisfied. This implies that for R = 0 the advanced Green function appearing in the retarded one, eq.(1.7a), or the retarded in the advanced Green function (1.7b), should not be rejected (see Appendix A).

Taking expression (1.7a) as a correct retarded Green function that should be used as a solution for R = 0, we next look for the value of the theta function when R = 0 equivalently when $\tau = 0$. As it is given in (1.8), at $\tau = 0$, $\theta(\tau)$ is not defined. But $\theta(\tau)$ can be given by

$$\theta(\tau) = \frac{1 + \mathcal{E}(\tau)}{2}\;; \qquad \text{where}\quad \mathcal{E}(\tau) = \begin{cases} 1, & \tau > 0 \\ -1, & \tau < 0 \end{cases}$$

Since $\mathcal{E}(\tau) = - \mathcal{E}(-\tau)$, at $\tau = 0$ $\mathcal{E}(\tau)$ should be zero.

Hence, $\theta(0) = 1/2$.

Moreover, from Fourier integral theorem, a discontinuous function like $\theta(\tau)$ can be represented in the form

$$\theta(\tau) = \frac{1}{2}\left[\theta(\tau + 0) + \theta(\tau - 0)\right]$$

where $\theta(\tau+0)$ and $\theta(\tau - 0)$ denote the limits of $\theta(\tau+\epsilon)$ and $\theta(\tau - \epsilon)$, respectively, as ϵ (>0) tends to zero. At $\tau = 0$ this representation also gives the same value

$$\theta(0) = 1/2$$

Thus, for $R \to 0$, the Green function in (2.7a) becomes

$$G^{s}\Big|_{R\to 0} = \frac{1}{2}\left[\frac{\delta(\tau - R/c)}{R} - \frac{\delta(\tau + R/c)}{R}\right]_{R\to 0}$$

which can be written as

$$G^{s}\Big|_{R\to 0} = \frac{1}{2}\left(G_r - G_a\right)\Big|_{R\to 0} \tag{1.10}$$

where G_r and G_a are the retarded and advanced Green functions, respectively, for $R \neq 0$ and the field obtained by using (1.10) should be evaluated for $R = 0$.

Equation(1.10) is a very important but simple mathematical result. It tells us how we may describe the field of a charged particle when it interacts with itself. Correspondingly, the superscript "S" stands for this "self-action" of the particle. Thus, the field produced by a charged particle at any point in space including the particle's position should be described by using the correct Green functions given in (1.7).

1.3. Simultaneous Expansions for Potentials and Fields By Using The Lagrange Expansion

The solutions of the Maxwell equations (1.1) are given by the retarded and advanced Lienard-Wiechert potentials, which can be obtained by using (1.9) in (1.6). The retarded Lienard-Wiechert potentials are

$$\phi(\bar{x},t) = \frac{e}{R'[1 - \bar{n}'\cdot\bar{\beta}']} \tag{1.11a}$$

$$\bar{A}(\bar{x},t) = \frac{e\,\bar{\beta}'}{R'[1 - \bar{n}'\cdot\bar{\beta}']} \tag{1.11b}$$

where $\bar{R}' = \bar{x} - \bar{x}'(t')$ is the vector directed from the position of the charge at the earlier time t' to the required point of observation; $\bar{n}' = \bar{R}'/R$ is a unit vector in this same vector direction and $c\bar{\beta}' = \bar{\vartheta}'(t')$ is the velocity of the charge at time t'.

Due to the finite speed of propagation of interaction in relativity the time t' and t are related to each other as

$$c(t - t') = R(t')$$

$$t' = t - \frac{R(t')}{c} \tag{1.12}$$

In principle knowing the trajectory of the particle $\bar{x}'(t')$ one can solve (1.12). But leaving alone a particle in arbitrary motion we face a great difficulty when we try to solve (1.12) even for a uniform motion of a particle in a circular path. The problem will be more difficult when a system of relativistic charged particles is considered. So it appears advantegeous to have expressions for potentials and hence fields interms of the simultaneous time t of observation.

The scalar potential (1.11a) being a function of t′ given by (1.12) can be expanded by using the Lagrange expansion (see Appendix B)

$$\phi(\bar{x},t) = \frac{e}{R'(t')-\bar{R}'(t')\cdot\bar{\beta}(t')} = \frac{e}{R-\bar{R}\cdot\bar{\beta}} + \sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\left(\frac{1}{c}\right)^k \frac{d^{k-1}}{dt^{k-1}}\left[R^k\frac{d}{dt}\left[\frac{e}{R-\bar{R}\cdot\bar{\beta}}\right]\right]$$

where $\bar{R} = \bar{x} - \bar{x}'(t)$ and $\bar{\beta} = \bar{\beta}(t) = \frac{\bar{\vartheta}(t)}{c}$

$$= \frac{e}{R-\bar{R}\cdot\bar{\beta}} + e\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\left(\frac{1}{c}\right)^k \frac{d^k}{dt^k}\left[\frac{R^k}{R-\bar{R}\cdot\bar{\beta}}\right] + \frac{e}{c}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(k-1)!}\left(\frac{1}{c}\right)^{k-1}\frac{d^{k-1}}{dt^{k-1}}\left[\frac{R^{k-1}}{R-\bar{R}\cdot\bar{\beta}}\frac{dR}{dt}\right]$$

Combining the first two terms under a single summation and changing the index of summation in the last term, we get

$$\phi(\bar{x},t) = e\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{1}{c}\right)^k\frac{d^k}{dt^k}\left[\frac{R^k}{R-\bar{R}\cdot\bar{\beta}}\left[1+\frac{1}{c}\frac{dR}{dt}\right]\right]$$

$$\frac{dR}{dt} = \frac{d}{dt}\left(\left|\bar{x}-\bar{x}'(t)\right|\right) = -\,\bar{n}\cdot\bar{\vartheta}(t) \tag{1.13}$$

where $\bar{n} = \bar{R}/R$ is a unit vector directed from the charge's positon at time t to the observation point at the same time and $\bar{\vartheta}(t)$ is now the velocity of the charge at time t but not at the earlier time t′.

Using (1.13) the above last expansion for the scalar potential $\phi(\bar{x},t)$ becomes

$$\phi(\bar{x},t) = e\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{1}{c^k}\frac{d^k}{dt^k}\left(R^{k-1}\right)$$

In a similar way we can find an expansion for the vector potential $\bar{A}(\bar{x},t)$ to be

$$\bar{A}(\bar{x},t) = e\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{1}{c^k}\frac{d^k}{dt^k}\left(R^{k-1}\,\bar{\vartheta}\right)$$

By choosing a suitable system of units in which the speed of light is taken to be unity, we can avoid the different powers of c which appear in the above expansions. According to this convention, our expressions for $\phi(\bar{x},t)$ and $\bar{A}(\bar{x},t)$ can be written as

$$\phi(\bar{x},t) = e\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{d^k}{dt^k}R^{k-1} \tag{1.14}$$

$$\bar{A}(\bar{x},t) = e\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{d^k}{dt^k}\left[R^{k-1}\,\bar{\vartheta}\right] \tag{1.15}$$

These are the required potentials interms of a single time t, which is also used in determining the equations of motion of the charged particle.

Having found expressions for potentials, (1.14) and (1.15), the fields can be determined by making use of (1.2)

$$\bar{E} = -e\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{d^k}{dt^k}\left[(k-1)R^{k-2}\,\bar{\nabla}R + \frac{\partial}{\partial t}\left[R^{k-1}\,\bar{\vartheta}\right]\right]$$

Since $\frac{\partial x}{\partial t} = 0$, we have $\frac{\partial}{\partial t}\left[R^{k-1}\,\bar{\vartheta}\right] = \frac{d}{dt}\left[R^{k-1}\,\bar{\vartheta}\right]$

$$\bar{E} = -e\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{d^k}{dt^k}\left[(k-1)R^{k-2}\,\bar{n}\right] - e\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{d^{k+1}}{dt^{k+1}}\left[R^{k-1}\,\bar{\vartheta}\right]$$

$$\bar{E} = e\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{d^k}{dt^k}\left[R^{k-2}\,\{\bar{n} - k(\bar{n}-\bar{\vartheta})\}\right] \tag{1.16}$$

And

$$\bar{H} = e\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{d^k}{dt^k}\left[(k-1)R^{k-2}\,\bar{n}\times\bar{\vartheta}\right] \tag{1.17}$$

Applying Leibnitz formula for the k-th derivative of a product of functions the vector potential (1.15) becomes

$$\bar{A}(\bar{x},t) = e \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{(-1)^k}{(k-m)!\,m!} \frac{d^{k-m}}{dt^{k-m}} \bar{\vartheta} \frac{d^m}{dt^m} R^{k-1}$$

$$\bar{A}(\bar{x},t) = e \sum_{k,m=0}^{\infty} \frac{(-1)^{k+m}}{k!\,m!} \overset{k}{\bar{\vartheta}} \frac{d^m}{dt^m} R^{k+m-1}$$

where $\overset{k}{\bar{\vartheta}} = \dfrac{d^k \bar{\vartheta}}{dt^k}$

If we define the scalar potential of the k-th order ϕ_k as

$$\phi_k = e \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{d^m}{dt^m} R^{k+m-1} \tag{1.18}$$

then interms of this potentials equations (1.14) and (1.15) can be written as

$$\phi = \phi_0 \tag{1.19}$$

$$\bar{A} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \overset{k}{\bar{\vartheta}} \phi_k \tag{1.20}$$

using a formula for the m-th derivative of a compound function we have the following expression for the k-th order scalar potential (1.18)

$$\phi_k = e \sum \frac{(-1)^{m+p}}{i!\,j!\ldots h!} \frac{\partial^p R^{m+k-1}}{\partial \bar{R}^p} (\bar{\vartheta})^i \left[\frac{\dot{\bar{\vartheta}}}{2!} \right]^j \ldots \left[\frac{\overset{q-1}{\bar{\vartheta}}}{q!} \right]^h \tag{1.21}$$

where the indices are to satisfy the equations

$$\begin{aligned} i + 2j + \ldots + qh &= m \\ i + j + \ldots + h &= p \end{aligned} \qquad (1.22)$$

Equation (1.21) can be expressed interms of accelerations of all orders which does not contain expansions in ϑ (i.e. in powers of ϑ/c). The required expression is [5]

$$\phi_k = e \sum_{m=0}^{\infty} (-1)^m R^{m+k-1} B_k^m \qquad (1.23)$$

where $$B_k^m = \sum \frac{1}{i!j!\ldots h!} \frac{\partial^s \Phi_{k+m-1}}{\partial \vartheta^s} \left[\frac{\dot{\vartheta}}{2!}\right]^i \left[\frac{\ddot{\vartheta}}{3!}\right]^j \cdots \left[\frac{\overset{q}{\vartheta}}{(q+1)!}\right]^h ,$$

The equation for the indices is the same as that of (1.22), but here p is replaced by s.

$$\Phi_{k+m-1} = \frac{\gamma^{k+m+1}[z + (1+z^2)^{1/2}]^{k+m}}{(1 + z^2)^{1/2}} , \qquad (1.24)$$

$\gamma = (1 - \vartheta^2)^{1/2}$ and $z = \bar{n}\cdot\bar{u} = \gamma(\bar{n}\cdot\bar{\vartheta})$, $\bar{u}$ being the spatial component of the four velocity of the particle. By using the same Leibnitz differentation formula for the field expressions (1.16) and (1.17) and arranging the indices of summation, we get

$$\bar{E} = e \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (k-1)\, C_{k-1} \overset{k}{\bar{R}} \qquad (1.25)$$

and

$$\bar{H} = e \sum_{k,m=0}^{\infty} \frac{(-1)^{k+m}}{k!m!}\, m\, C_{k+m-2}\, \overset{k}{\bar{R}} \times \overset{m}{\bar{R}} \qquad (1.26)$$

where
$$C_k = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}(k+i) \frac{d^i}{dt^i} R^{k+i-2} \tag{1.27}$$

$$\overset{k}{\bar{R}} = \frac{d^k}{dt^k} \bar{R} \qquad \text{and} \qquad \overset{k}{\bar{R}} = -\overset{k-1}{\bar{\vartheta}} \quad , \ k \geq 1$$

Let us consider expression (1.25)

$$\bar{E} = e \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^{k+i}}{k!\,i!} (k-1)(k+i-1) \frac{d^i R^{k+i-3}}{dt^i} \overset{k}{\bar{R}}$$

$$= e \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^k}{(k-i)!\,i!} (k-i-1)(k-1) \frac{d^i R^{k-3}}{dt^i} \overset{k-i}{\bar{R}}$$

$$= e \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (-1)(k-1) \frac{d^k R^{k-3}}{dt^k} \bar{R} + \sum_{k=0}^{\infty} \sum_{i=0}^{k-1} \frac{(-1)^k}{(k-i)!\,i!} (k-i-1)(k-1) \frac{d^i R^{k-3}}{dt^i} \overset{k-i}{\bar{R}}$$

$$= e \sum_{k=0}^{\infty} (-1)^k \left\{ -\frac{(k-1)}{k!} \frac{d^k R^{k-3}}{dt^k} \bar{R} + \sum_{i=0}^{k} \frac{(k-i)}{(k-i+1)!} \, k \, \frac{d^i R^{k-2}}{dt^i} \overset{k-i}{\bar{\vartheta}} \right\}$$

$$\bar{E} = e \sum_{k=0}^{\infty} (-1)^k \bar{E}_k \tag{1.28}$$

where $\bar{E}_k = -C_k^{-1} \bar{R} + \sum_{i=0}^{k} \frac{(k-i)}{(k-i+1)!} C_i^{k-i} \overset{k-i}{\bar{\vartheta}}$

Similarly

$$\bar{H} = e \sum_{k=0}^{\infty} (-1)^k \bar{H}_k \qquad ; \qquad \bar{H}_k = \sum_{j=0}^{\infty} H_j^k \tag{1.29}$$

Where $\bar{H}_j^k = C_j^{k-j-1}\,\bar{R}\times\overset{k-j}{\bar{\vartheta}} + C_j^{k-j}\sum_{i=0}^{E[(k-j)/2]}\frac{(k-j-2i)\,\overset{i}{\bar{\vartheta}}\times\overset{k-j-i}{\bar{\vartheta}}}{(i+1)!\,(k-j-i+1)!}$

Here $E[(k-j)/2]$ refers to the integral part of $(k-j)/2$. From (1.27) we have the following relation for the functions C_j^k

$$C_k = \sum_{j=0}^{\infty}(-1)^j\,C_j^k$$

Similar to the expressions given for the scalar potential of the k-th order, these scalar functions can also be expressed interms of the accelarations of all orders as follows

$$C_0^k = \Omega_k\ ,\quad C_\ell^k = \sum\frac{1}{i!\,j!\dots h!}\,\frac{\partial^s\Omega_{k+\ell}}{\partial\vartheta^s}\left[\frac{\dot{\bar{\vartheta}}}{2!}\right]^i\left[\frac{\ddot{\bar{\vartheta}}}{3!}\right]^j\cdots\left[\frac{\overset{q}{\bar{\vartheta}}}{(q+1)!}\right]^h \tag{1.30}$$

Here again the indices satisfy (1.22)

where

$$\Omega_k(\bar{R},\bar{\vartheta}) = R^{k-2}\,\gamma^{k+2}\,\frac{[z+(1+z^2)^{1/2}]^k\,[k(1+z^2)^{1/2}-z]}{(1+z^2)^{3/2}} \tag{1.31}$$

1.4. Simultaneous Expansion For Potentials And Fields In Terms Of Green Function

Even though the results obtained by using the Lagrange expansion are exact and quite interesting, their derivations were not direct and very easy to handle. Here we shall see that these results can be found by making use of δ-function and Taylor expansion in a very simple way.

The retarded Lienard-Wiechert potentials, which can be seen by substituting (1.9a) into (1.6), are given by

$$\phi(\bar{x},t) = 2e\int_{-\infty}^{\infty}\theta(t-t')\delta[(\bar{x} - \bar{x}'(t'))^2 - (t-t')^2]dt'$$

and

$$\bar{A}(\bar{x},t) = 2e\int_{-\infty}^{\infty}\theta(t-t')\bar{\vartheta}(t')\delta[(\bar{x} - \bar{x}'(t'))^2 - (t-t')^2]dt'$$

Let us make a change of variable as

$$t - t' = \tau \quad , \quad dt' = -d\tau$$

with the new variable of integration, we have

$$\phi(\bar{x},t) = 2e\int_{-\infty}^{\infty}\theta(\tau)\delta[(\bar{x} - \bar{x}'(t-\tau))^2 - \tau^2]d\tau \tag{1.32}$$

and

$$\bar{A}(\bar{x},t) = 2e\int_{-\infty}^{\infty}\theta(\tau)\bar{\vartheta}(t - \tau)\delta[(\bar{x} - \bar{x}'(t-\tau))^2 - \tau^2]d\tau \tag{1.33}$$

If we make Taylor expansion of $\delta[(\bar{x} - \bar{x}'(t-\tau))^2 - \tau^2]$ about $\tau = 0$

$$\delta[(\bar{x} - \bar{x}'(t-\tau))^2 - \tau^2] = \sum_{k=0}^{\infty}\frac{(-\tau)^k}{k!}\left[\frac{\partial^k}{\partial t'^k}\delta[(\bar{x} - \bar{x}'(t'))^2 - \tau^2]\right]_{t'=t}$$

then

$$\phi(\bar{x},t) = 2e\int_{-\infty}^{\infty}\theta(\tau)\sum_{k=0}^{\infty}\frac{(-\tau)^k}{k!}\frac{\partial^k}{\partial t'^k}\delta[(\bar{x} - \bar{x}'(t'))^2 - \tau^2]d\tau$$

$$\phi(\bar{x},t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\partial^k}{\partial t^k} \left\{ 2e \int_{-\infty}^{\infty} \theta(\tau)\tau^k \delta[(\bar{x} - \bar{x}'(t))^2 - \tau^2] d\tau \right\}$$

$$\phi(\bar{x},t) = e \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{d^k}{dt^k} \left\{ \theta(R)\ R^{k-1} - \theta(-R)(-R)^{k-1} \right\}$$

where $\bar{R} = |\bar{x} - \bar{x}'(t)|$

For $R > 0$, we have $\theta(R) = 1$, $\theta(-R) = 0$ and $\phi(\bar{x},t)$ at once reduces to

$$\phi(\bar{x},t) = e \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{d^k}{dt^k} \left\{ R^{k-1} \right\}$$

A similar expansion for the velocity and delta function in (1.33) gives

$$\bar{A}(\bar{x},t) = e \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{k+m}}{k!\,m!} \frac{d^k}{dt^k} \bar{\vartheta} \frac{d^m}{dt^m} R^{k+m-1}$$

This is the same as (1.20), the double sum can be reduced to a single sum and the result will be identical.

Now let us make Taylor expansion of $\bar{x}'(t-\tau)$ about $\tau = 0$

$$\bar{x}'(t-\tau) = \sum_{k=0}^{\infty} \frac{(-\tau)^k}{k!} \left[\frac{\partial^k}{\partial t^k} \bar{x}'(t-\tau) \right]_{\tau=0}$$

$$\bar{x}'(t-\tau) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \tau^k \frac{d^k}{dt^k} \bar{x}'(t)$$

with this we have

$$\bar{x} - \bar{x}'(t-\tau) = \bar{x} - \bar{x}'(t) - \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \tau^k \frac{d^k \bar{x}'(t)}{dt^k}$$

which is the same as

$$\bar{x} - \bar{x}'(t-\tau) = \bar{R}(t) + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \tau^{k+1} \overset{k}{\bar{\vartheta}}(t)$$

where $\bar{R}(t) = \bar{x} - \bar{x}'(t)$ is a vector directed from the postion of the charge at time t to the observation point at the same time.

$$\bar{x} - \bar{x}'(t-\tau) = \bar{R} + \tau\bar{\vartheta} + \tau \sum_{k=1}^{\infty} \frac{(-1)^k}{(k+1)!} \tau^k \overset{k}{\bar{\vartheta}}$$

If the summation in this equation is represented by

$$\bar{g}\,(t,\tau) = \sum_{k=1}^{\infty} \frac{(-1)^k}{(k+1)!} \tau^k \overset{k}{\bar{\vartheta}} \tag{1.34}$$

then

$$\bar{x} - \bar{x}'(t-\tau) = \bar{R} + \tau\bar{\vartheta} + \tau\bar{g}$$

substituting this into $\delta[(\bar{x} - \bar{x}'(t-\tau))^2 - \tau^2]$ and making Taylor expansion of the resulting $\delta[(\bar{R} + \tau\bar{\vartheta} + \tau\bar{g})^2 - \tau^2]$ about $\bar{g} = 0$, we get

$$\delta[(\bar{R} + \tau\bar{\vartheta} + \tau\bar{g})^2 - \tau^2] = \sum_{m=0}^{\infty} \frac{\bar{g}^{\,m}}{m!} \frac{\partial^m}{\partial\bar{g}^m} \delta[(\bar{R} + \tau\bar{\vartheta} + \tau\bar{g})^2 - \tau^2]\Big|_{\bar{g}=0}$$

But

$$\left[\frac{\partial}{\partial\bar{g}} \delta[(\bar{R} + \tau\bar{\vartheta} + \tau\bar{g})^2 - \tau^2]\right]_{\bar{g}=0} = \frac{\partial}{\partial\bar{\vartheta}} \left[\delta[(\bar{R} + \tau\bar{\vartheta} + \tau\bar{g})^2 - \tau^2]\Big|_{\bar{g}=0}\right]$$

$$= \frac{\partial}{\partial\bar{\vartheta}} \delta[(\bar{R} + \tau\bar{\vartheta})^2 - \tau^2]$$

Therefore we can write the δ-function expansion as

$$\delta[(\bar{R} + \tau\bar{\vartheta} + \tau\bar{g})^2 - \tau^2] = \sum_{m=0}^{\infty} \frac{\bar{g}^{\,m}}{m!} \frac{\partial^m}{\partial\bar{\vartheta}^{\,m}} \delta[(\bar{R} + \tau\bar{\vartheta})^2 - \tau^2]$$

or

$$\delta[(\bar{R} + \tau\bar{\vartheta} + \tau\bar{g})^2 - \tau^2] = e^{\bar{g}\partial/\partial\bar{\vartheta}} \delta[(\bar{R} + \tau\bar{\vartheta})^2 - \tau^2] \tag{1.35}$$

with this the integral in (1.32) becomes

$$\phi(\bar{x}, t) = 2e \int_{-\infty}^{\infty} \theta(\tau)\, e^{\bar{g}\partial/\partial\bar{\vartheta}} \delta[(\bar{R} + \tau\bar{\vartheta})^2 - \tau^2]\, d\tau \tag{1.36}$$

Let $\quad f = (\bar{R} + \tau\bar{\vartheta})^2 - \tau^2$

The zeros of f are at

$$\tau^{+} = \gamma R\left[z + \sqrt{1+z^2}\,\right] \quad \text{and} \quad \tau^{-} = \gamma R\left[z - \sqrt{1+z^2}\,\right] < 0 \tag{1.37}$$

Where γ and z are the same as those used in (1.24).

The result of the integral in (1.36) is then given by

$$\phi(\bar{x}, t) = 2e \frac{\left[\theta(\tau)\, e^{\bar{g}\partial/\partial\bar{\vartheta}}\right]_{\tau=\tau^{+}}}{\left|\left(\frac{\partial f}{\partial\tau}\right)_{\tau=\tau^{+}}\right|} + 2e \frac{\left[\theta(\tau)\, e^{\bar{g}\partial/\partial\bar{\vartheta}}\right]_{\tau=\tau^{-}}}{\left|\left(\frac{\partial f}{\partial\tau}\right)_{\tau=\tau^{-}}\right|}$$

Since $\tau^{-} < 0$, the value of the theta function in the second term is zero. Thus $\phi(\bar{x}, t)$ becomes

$$\phi(\bar{x},t) = \frac{e\,\gamma\, e^{\bar{g}(t,\tau^{+})\partial/\partial\bar{\vartheta}}}{R\sqrt{1+z^2}}$$

From (1.34)

$$\bar{g}(t,\tau_{+}) = \sum_{k=1}^{\infty} \frac{(-1)^k}{(k+1)!}\tau_{+}^{k}\overset{k}{\bar{\vartheta}} = -\tau_{+}\frac{\dot{\bar{\vartheta}}}{2!} + \tau_{+}^{2}\frac{\ddot{\bar{\vartheta}}}{3!} - \tau_{+}^{3}\frac{\overset{3}{\bar{\vartheta}}}{4!} + \ldots$$

(Note: τ_{+} is the same as τ^{+})

Hence $$e^{\bar{g}(t,\tau_{+})\partial/\partial\bar{\vartheta}} = \exp\left[-\tau_{+}\frac{\dot{\bar{\vartheta}}}{2!}\frac{\partial}{\partial\bar{\vartheta}}\right]\exp\left[\tau_{+}^{2}\frac{\ddot{\bar{\vartheta}}}{3!}\frac{\partial}{\partial\bar{\vartheta}}\right]\exp\left[-\tau_{+}^{3}\frac{\overset{3}{\bar{\vartheta}}}{4!}\frac{\partial}{\partial\bar{\vartheta}}\right]\ldots$$

$$= \sum_{k_1=0}^{\infty}\frac{\left[-\tau_{+}\frac{\dot{\bar{\vartheta}}}{2!}\frac{\partial}{\partial\bar{\vartheta}}\right]^{k_1}}{k_1!}\sum_{k_2=0}^{\infty}\frac{\left[\tau_{+}^{2}\frac{\ddot{\bar{\vartheta}}}{3!}\frac{\partial}{\partial\bar{\vartheta}}\right]^{k_2}}{k_2!}\sum_{k_3=0}^{\infty}\frac{\left[-\tau_{+}^{3}\frac{\overset{3}{\bar{\vartheta}}}{4!}\frac{\partial}{\partial\bar{\vartheta}}\right]^{k_3}}{k_3!}\ldots$$

thus $$e^{\bar{g}(t,\tau_{+})\partial/\partial\bar{\vartheta}} = \sum\frac{(-1)^s}{k_1!k_2!\ldots}\frac{\partial^{\ell}}{\partial\bar{\vartheta}^{\ell}}\tau_{+}^{s}\left[\frac{\dot{\bar{\vartheta}}}{2!}\right]^{k_1}\left[\frac{\ddot{\bar{\vartheta}}}{3!}\right]^{k_2}\left[\frac{\overset{3}{\bar{\vartheta}}}{4!}\right]^{k_3}\ldots$$

where the indices satisfy the equations

$$S = k_1 + 2k_2 + 3k_3 + \ldots = \sum ik_i \text{ and } \ell = k_1 + k_2 + k_3 + \ldots = \sum k_i \tag{1.39}$$

Substituting this last expansion into (1.38) and using (1.37) for τ_{+}, we have

$$\phi(\bar{x},t) = e\sum\frac{(-1)^s}{k_1!k_2!\ldots}\frac{\partial^{\ell}}{\partial\bar{\vartheta}^{\ell}}\left[\frac{\gamma^{s+1}[z+(1+z^2)^{1/2}]^s}{\sqrt{1+z^2}}R^{s-1}\right]\left[\frac{\dot{\bar{\vartheta}}}{2!}\right]^{k_1}\left[\frac{\ddot{\bar{\vartheta}}}{3!}\right]^{k_2}\ldots$$

If we define

$$\phi_s(\bar{n},\bar{\vartheta}) = \frac{\gamma^{s+2}[z + (1+z^2)^{1/2}]^{s+1}}{\sqrt{1 + z^2}} \tag{1.40}$$

then the retarded scalar potential $\phi(\bar{x},t)$ can be written as

$$\phi(\bar{x},t) = e \sum \frac{(-1)^s}{k_1!k_2!\ldots} \frac{R^{s-1}\,\partial^\ell}{\partial\bar{\vartheta}^\ell} \phi_{s-1} \left(\frac{\dot{\bar{\vartheta}}}{2!}\right)^{k_1} \left(\frac{\ddot{\bar{\vartheta}}}{3!}\right)^{k_2} \left(\frac{\overset{3}{\bar{\vartheta}}}{4!}\right)^{k_3} \ldots$$

We can also obtain an expansion for the vector potential given in (1.33) in the following way. A Taylor expansion of $\bar{\vartheta}(t-\tau)$ about $\tau = 0$ gives

$$\bar{\vartheta}(t-\tau) = \sum_{m=0}^{\infty} \frac{(-\tau)^m}{m!} \frac{d^m}{dt^m} \bar{\vartheta}(t) \tag{1.42}$$

with this and the expansion (1.35) for δ-function. Eq. (1.33) becomes

$$\bar{A}(\bar{x},t) = e \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \overset{m}{\bar{\vartheta}} \left\{ 2\int \theta(\tau)\ \tau^m\ e^{\bar{g}(t,\tau_+)\partial/\partial\bar{\vartheta}} \delta[(\bar{R}+\tau\bar{\vartheta})^2 - \tau^2] d\tau \right\}$$

The integral in this equation can be evaluated by applying a similar method employed in (1.36) which leads to the following expression for the vector potential

$$\bar{A}(\bar{x},t) = e \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \overset{m}{\bar{\vartheta}}\, \Phi_m \tag{1.43}$$

where

$$\Phi_m = \sum \frac{(-1)^s \; R^{s+m-1}}{k_1!k_2!\ldots} \frac{\partial^\ell}{\partial\bar{\vartheta}^\ell} \phi_{s+m-1} \left(\frac{\dot{\vartheta}}{2!}\right)^{k_1} \left(\frac{\ddot{\vartheta}}{3!}\right)^{k_2} \left(\frac{\overset{3}{\vartheta}}{4!}\right)^{k_3} \ldots$$

and $\phi_{s+m} = \phi_{s+m} (\bar{n}, \bar{\vartheta})$ is the same as the function defined in (1.40)

Expansions for the advanced scalar and vector potential can be obtained in a quite similar way. They are

$$\phi^-(\bar{x}, t) = e \sum \frac{(-1)^s \; R^{s-1}}{k_1!k_2!\ldots} \frac{\partial^\ell}{\partial\bar{\vartheta}^\ell} \phi_{s-1} \left(\frac{\dot{\vartheta}}{2!}\right)^{k_1} \left(\frac{\ddot{\vartheta}}{3!}\right)^{k_2} \left(\frac{\overset{3}{\vartheta}}{4!}\right)^{k_3} \ldots \tag{1.44}$$

and

$$\bar{A}^-(\bar{x}, t) = e \sum_{m=0}^{\infty} \frac{1}{m!} \bar{\vartheta}^{m} \Phi^-_m \tag{1.45}$$

where

$$\Phi^-_m = \sum \frac{(-1)^s \; R^{s+m-1}}{k_1!k_2!\ldots} \frac{\partial^\ell}{\partial\bar{\vartheta}^\ell} \phi^-_{s+m-1} \left(\frac{\dot{\vartheta}}{2!}\right)^{k_1} \left(\frac{\ddot{\vartheta}}{3!}\right)^{k_2} \left(\frac{\overset{3}{\vartheta}}{4!}\right)^{k_4} \ldots$$

with
$$\phi^-_s = \frac{\gamma^{s+2}[z - (1+z^2)^{1/2}]^{s+1}}{\sqrt{1 + z^2}} \tag{1.46}$$

CHAPTER 2

THE SELF - ACTION FORCE

An accelerated charge emits radiation and hence looses energy. As a result the equation of motion of this particle can't be determined by external forces without the radiation reaction. This radiation reaction force is assumed to be accounted by the force that the particle exerts on itself.

As it can be seen from the works of Wheeler and Feynman and other books on electrodynamics, the equation of motion of a relativistic charged particle in a system of such particles is determined by the action of the field due to other particles in the system (i.e. the field external to the particle under consideration) and a radiative reaction force, in the case of radiative motion. According to our speculation the equation of motion of this particle and any other particle in this system can be found from a single expression, that includes the action of a particle on itself, for the force acting on this particle. This radiation reaction force is then found when we consider the force exerted by the particle on itself.

According to Dirac [1938], for a given particle in a radiative motion, the radiation reaction force can be found when half retarded minus half advanced field of this particle is evaluated at the position of this particle. His result was used by Wheeler and Feynman [1945] but they have discarded his idea of the radiation field. According to their absorber theory of radiation, an absorber is an essential element in the mechanism of radiation. Any accelerated charged particle of the absorber produces a field which is half advanced and half retarded. This field makes all other particles of the absorber generate a field (absorber response) which is equal to half the retarded minus half the advanced field of the source, which they referred to as the absorber reaction. The absorber reaction is then considered as the source of the radiation reaction force. Using their assumption and Dirac's result of the radiation reaction force, they have also arrived at the following equation of motion, for a^{th} relativistic charged particle in a completely absorbing medium,

$$m_a \ddot{a}_n = e_a \sum_{k \neq a} \bar{F}^{(k)}_{n\alpha} \dot{a}^{\alpha} + \frac{2}{3} e^2 (\dot{a}_n \ddot{a}_{\alpha} - \ddot{a}_n \dot{a}_{\alpha}) \dot{a}^{\alpha}$$

where a^{α} is the space time coordinates of the particle and $\bar{F}^{(k)}_{n\alpha}$ is the retarded field of the k-th particle.

In the present approach no assumption of an absorber will be made. Instead Dirac's description, which seems to have its root in (1.10), is well accepted and his expression for the radiation reaction force will be discussed and calculated. In any case (i.e. either by using the retarded field alone or Dirac's method) our calculation in arriving at the required result has involved no approximation.

For low-velocity motion, $\vartheta \ll c$, the expression for the radiation reaction force reduces to that derived from Larmor radiation formula.

2.1. Removal of Singular Terms Due to Symmetry of Delta Function

The equation of motion of a relativistic charged particle can be written as

$$\frac{d\bar{p}}{dt} = \int \bar{f}(\bar{x},t)\, d^3\bar{x} \qquad \text{and} \qquad \frac{d\mathcal{E}}{dt} = e(\bar{\vartheta}\cdot\bar{E}) \tag{2.1}$$

where $\bar{p} = \gamma m\bar{\vartheta}$ is the spatial component of the particle's four momentum, m is the rest mass of this same particle and $\bar{f}(\bar{x},t)$ is the Lorentz force density.

$$\bar{f}(\bar{x},t) = \rho(\bar{x},t) \left[\bar{E}(\bar{x},t) + \bar{\vartheta}(t) \times \bar{H}(\bar{x},t) \right]$$

From the expressions of the simultaneous expansion for fields given in (1.28) and (1.29) , we have

$$\bar{E} = e \left[R^{-2}\, \bar{d}_{-2} + R^{-1}\, \bar{d}_{-1} + R^{0}\, \bar{d}_{0} + \left(\begin{matrix} \text{terms with +ve} \\ \text{powers of R} \end{matrix} \right) \right]$$

and (2.2)

$$\bar{H} = e \left[R^{-2}\, \bar{h}_{-2} + R^{-1}\, \bar{h}_{-1} + R^{0}\, \bar{h}_{0} + \left(\begin{matrix} \text{terms with +ve} \\ \text{powers of R} \end{matrix} \right) \right]$$

where

$$\bar{d}_{-2} = -\grave{\Omega}_{-1}\bar{n} \qquad , \qquad \bar{d}_{-1} = \grave{C}_1^{-1}\,\bar{n} - \frac{\grave{\Omega}_1}{2}\,\dot{\bar{\vartheta}}$$

$$\bar{d}_{0} = -\grave{C}_2^{-1}\bar{n} + \frac{1}{2}\grave{C}_1^{1}\,\dot{\bar{\vartheta}} + \frac{2}{3!}\grave{\Omega}_2\ddot{\bar{\vartheta}}$$

and $\bar{h}_{-2} = {}^{\backprime}\Omega_1 \bar{n} \times \dot{\bar{\vartheta}}$, $\bar{h}_{-1} = -\left[{}^{\backprime}\Omega_0\, \bar{n} \times \dot{\bar{\vartheta}} + \frac{1}{2}{}^{\backprime}\Omega_1\, \bar{\vartheta} \times \dot{\bar{\vartheta}} + {}^{\backprime}C_1^{-1}\, \bar{n} \times \bar{\vartheta}\right]$

$$\bar{h}_0 = {}^{\backprime}C_2^{-1}\, \bar{n} \times \bar{\vartheta} + {}^{\backprime}C_1^0\, \bar{n} \times \dot{\bar{\vartheta}} + \frac{1}{2}{}^{\backprime}\Omega_1\, \bar{n} \times \ddot{\bar{\vartheta}} + \frac{1}{2}{}^{\backprime}C_1^1\, \bar{\vartheta} \times \dot{\bar{\vartheta}} + \frac{2}{3!}{}^{\backprime}\Omega_2 \bar{\vartheta} \times \ddot{\bar{\vartheta}}$$

The primed scalar functions ${}^{\backprime}C_\ell^k$ and ${}^{\backprime}\Omega_k$ used here are related to their corresponding unprimed scalar functions C_ℓ^k and Ω_k , respectively, given in (1,30) and (1.31) by

$${}^{\backprime}C_\ell^k = R^{-(k+\ell-2)}\, C_\ell^k \qquad \text{and} \quad {}^{\backprime}\Omega_k = R^{-(k-2)}\Omega_k$$

with this relation we have

$${}^{\backprime}\Omega_{\pm1} = \pm \frac{\gamma^3}{(1+z^2)^{3/2}} \qquad , \quad {}^{\backprime}\Omega_0 = -\frac{\gamma^2 z}{(1+z^2)^{3/2}}$$

$${}^{\backprime}\Omega_2 = \frac{\gamma^4 z}{(1+z^2)^{3/2}} + \frac{2\gamma^4 z}{(1+z^2)^{1/2}} + 2\gamma^4$$

$${}^{\backprime}C_1^{-1} = \frac{\partial\, {}^{\backprime}\Omega_0}{\partial \bar{\vartheta}} \cdot \frac{\dot{\bar{\vartheta}}}{2!} \qquad {}^{\backprime}C_2^{-1} = \frac{\partial\, {}^{\backprime}\Omega_1}{\partial \bar{\vartheta}} \cdot \frac{\ddot{\bar{\vartheta}}}{3!} \qquad (2.3)$$

$${}^{\backprime}C_1^0 = \frac{\partial\, {}^{\backprime}\Omega_1}{\partial \bar{\vartheta}} \cdot \frac{\dot{\bar{\vartheta}}}{2!} \qquad {}^{\backprime}C_1^1 = \frac{\partial\, {}^{\backprime}\Omega_2}{\partial \bar{\vartheta}} \cdot \frac{\dot{\bar{\vartheta}}}{2!}$$

The Lorentz force exerted on a charge located at the postion $\bar{x}$ by the field produced by another charge at $\bar{x}'$, after substituting for the fields from (2.2) into the Lorentz force density and representing the point charge and charge density interms of δ-function, can be written as

$$\bar{F} = e^2 \iint \delta[\bar{x} - \bar{r}(t)]\delta[\bar{x}' - \bar{r}'(t)]\Big\{(\bar{d}_{-2} + \bar{\vartheta} \times \bar{h}_{-2})R^{-2} + (\bar{d}_{-1} + \bar{\vartheta} \times \bar{h}_{-1})R^{-1}$$

$$+ (\bar{d}_0 + \bar{\vartheta} \times \bar{h}_0)R^0 + \begin{bmatrix}\text{terms with +ve}\\ \text{powers of R}\end{bmatrix}\Big\}\, d^3\bar{x}\, d^3\bar{x}' \qquad (2.4)$$

where $R = |\bar{x} - \bar{x}'|$ and $\bar{r}(t)$ and $\bar{r}'(t)$ are the trajectories of the charge on which the field acts and the charge producing the field, respectively.

We now calculate the force that the particle producing the field exerts on itself. This requires putting $\bar{r}(t) = \bar{r}'(t)$ and then $R = 0$. The latter condition will make all terms containing positive powers of R zero. And for $\bar{r}(t) = \bar{r}'(t)$, we have

$$\delta[\bar{x} - \bar{r}(t)]\delta[\bar{x}' - \bar{r}'(t)] = \delta[\bar{x} - \bar{x}']\, \delta[\bar{x}' - \bar{r}'(t)]$$

with these (2.4) becomes

$$\bar{F} = e^2\int \delta[\bar{x}' - \bar{r}'(t)]d^3\bar{x}' \int\delta[\bar{x} - \bar{x}'][(\bar{d}_{-2} + \bar{\vartheta} \times \bar{h}_{-2})R^{-2}$$

$$+ (\bar{d}_{-1} + \bar{\vartheta} \times \bar{h}_{-1})R^{-1} + (\bar{d}_0 + \bar{\vartheta} \times \bar{h}_0)]d^3\bar{x} \qquad (2.5)$$

Let us consider the integration over $\bar{x}$

since $\bar{x} - \bar{x}' = \bar{R}$

$$d^3\bar{x} = d^3\bar{R}$$

then

$$\int \delta[\bar{x} - \bar{x}']\Big[(\bar{d}_{-2} + \bar{\vartheta} \times \bar{h}_{-2})R^{-2} + (\bar{d}_{-1} + \bar{\vartheta} \times \bar{h}_{-1})R^{-1} + (\bar{d}_0 + \bar{\vartheta} \times \bar{h}_0)\Big]d^3\bar{x}$$

$$= \int(\bar{d}_{-2} + \bar{\vartheta} \times \bar{h}_{-2})\,\frac{\delta(\bar{R})d^3\bar{R}}{R^2} + \int(\bar{d}_{-1} + \bar{\vartheta} \times \bar{h}_{-1})\,\frac{\delta(\bar{R})d^3\bar{R}}{R}$$

$$+ \int(\bar{d}_0 + \bar{\vartheta} \times \bar{h}_0)\delta(\bar{R})d^3\bar{R} \qquad (2.6)$$

Consider the first integral on the right hand side of this last inequality. Since $\bar{d}_{-2} = -\grave{\Omega}_{-1}\bar{n}$ and $\bar{h}_{-2} = \grave{\Omega}_{-1}\bar{n} \times \bar{\vartheta}$, it is enough to consider $\int \grave{\Omega}_{-1}\bar{n}\,\frac{\delta(\bar{R})}{R^2}\,d^3\bar{R}$

Substituting for $\grave{\Omega}_{-1}$ from (2.3)

$$\int \grave{\Omega}_{-1}\bar{n}\,\frac{\delta(\bar{R})}{R^2}\,d^3\bar{R} = \int_0^\pi \frac{\sin\theta d\theta}{\left[\gamma^{-2} + (\bar{n}\cdot\bar{\vartheta})^2\right]^{3/2}} \int_0^{2\pi} \bar{n}\, d\varphi \int_0^\infty (R)d\bar{R}$$

Choosing $\bar{\vartheta} \mathbin{//} \hat{k}$, we have $\gamma^{-2} + (\bar{n}\cdot\bar{\vartheta})^2 = 1 - \vartheta^2 \sin^2\theta$

$$\int \grave{\Omega}_{-1}\bar{n}\,\frac{\delta(\bar{R})}{R^2}\,d^3\bar{R} = \int_0^\pi \frac{\sin\theta d\theta}{\left[1 - \vartheta^2\sin^2\theta\right]^{3/2}} \int_0^{2\pi} \begin{Bmatrix} \sin\theta\ \sin\varphi \\ \sin\theta\ \cos\varphi \\ \cos\theta \end{Bmatrix} d\varphi \int_0^\infty \delta(\bar{R})dR$$

$$\int \grave{\Omega}_{-1}\bar{n}\,\frac{\delta(\bar{R})}{R^2}\,d^3\bar{R} = 2\pi\left[\frac{1}{\vartheta[1-\vartheta^2\sin^2\theta]^{1/2}}\right]_0^\pi \int_0^\infty \delta(\bar{R})dR \;=\; 0$$

The integral containing integrand terms proportional to R^{-1} (i.e. the second integral on the right hand side of (2.6)) also vanishes when a similar integration over θ and φ is performed. Thus, in the field expressions, terms proportional to R^{-2} and R^{-1} do not contribute to the Lorentz force of the self-action. The whole contribution comes from the terms proportional to R^{0}. Let us evaluate the integral containing these terms (i.e. the third integral on the right hand side of (2.6)).

Substituting the expressions for the vectors $\bar{d}_0$ and $\bar{h}_0$, we have

$$\int(\bar{d}_0 + \bar{\vartheta} \times \bar{h}_0)\delta(\bar{R})d^3\bar{R} = \int \delta(\bar{R}) \Big\{-\grave{C}_2^{-1} \left[\bar{n} - \bar{\vartheta} \times (\bar{n} \times \bar{\vartheta})\right] + \grave{C}_1^{0}\, \bar{\vartheta} \times (\bar{n} \times \dot{\bar{\vartheta}})$$

$$+ \frac{1}{2}\grave{\Omega}_1 \bar{n} \times \ddot{\bar{\vartheta}} + \frac{1}{2}\grave{C}_1^{1} \left[\dot{\bar{\vartheta}} + \bar{\vartheta} \times \dot{\bar{\vartheta}}\right] + \frac{2}{3!}\grave{\Omega}_1 \left[\ddot{\bar{\vartheta}} + \bar{\vartheta} \times (\bar{\vartheta} \times \ddot{\bar{\vartheta}})\right]\Big\} d^3\bar{R}$$

The function $\grave{\Omega}_1$ given in (2.3) is

$$\grave{\Omega}_1 = \frac{\gamma^3}{(1+z^2)^{3/2}} = \frac{1}{(\gamma^{-2} + (\bar{n}\cdot\bar{\vartheta})^2)^{3/2}}$$

Hence

$$\int \grave{\Omega}_1 \bar{n}\delta(\bar{R})d^3\bar{R} = \frac{1}{2\pi}\int_0^{\pi} \frac{\sin\theta d\theta}{[\gamma^{-2} + (\bar{n}\cdot\bar{\vartheta})^2]^{3/2}} \int_0^{2\pi} \bar{n}d\varphi \int_0^{\infty} \delta(R)dR = 0$$

since $\grave{C}_2^{-1} = \frac{\partial}{\partial\bar{\vartheta}}\grave{\Omega}_1\cdot\frac{\ddot{\bar{\vartheta}}}{3!}$ and $\grave{C}_1^{0} = \frac{\partial}{\partial\bar{\vartheta}}\grave{\Omega}_1\cdot\frac{\dot{\bar{\vartheta}}}{2!}$, the values of the integrals containing $\grave{C}_2^{-1}$ and $\grave{C}_1^{0}\,\bar{n}$ are also zero.

Thus,

$$\int(\bar{d}_0 + \bar{\vartheta} \times \bar{h}_0)\delta(\bar{R})d^3\bar{R} = \frac{1}{2}\left[\dot{\bar{\vartheta}} + \bar{\vartheta} \times (\bar{\vartheta} \times \dot{\bar{\vartheta}})\right] \int \grave{C}_1^1\delta(\bar{R})d^3\bar{R}$$

$$+ \frac{2}{3!}\left[\ddot{\bar{\vartheta}} + \bar{\vartheta} \times (\bar{\vartheta} \times \ddot{\bar{\vartheta}})\right] \int \grave{\Omega}_2\delta(\bar{R})d^3\bar{R}$$

From (2.3) $\quad \grave{\Omega}_2 = \dfrac{\gamma^4 z}{(1+z^2)^{3/2}} + \dfrac{2\gamma^4 z}{(1+z^2)^{1/2}} + 2\gamma^4$

Integration of the first two terms of this expression gives still zero.

Hence

$$\int\Omega_1\bar{n}\delta(\bar{R})d^3\bar{R} = 2\gamma^4\int\delta(\bar{R})d^3\bar{R} = 2\gamma^4$$

With the help of the expression for $\grave{C}_1^1$, given in (23), the result just obtained can be used to evaluate the integral of $\grave{C}_1^1$.

$$\int \grave{C}_1^1\ \delta(\bar{R})d^3\bar{R} = \left(\frac{\dot{\bar{\vartheta}}}{2!}\cdot\frac{\partial}{\partial\bar{\vartheta}}\right)\int \grave{\Omega}_2\delta(\bar{R})d^3\bar{R} = 4\gamma^6(\bar{\vartheta}\cdot\dot{\bar{\vartheta}})$$

With these results of integration, we have

$$\int(\bar{d}_0 + \bar{\vartheta} \times \bar{h}_0)\delta(\bar{R})d^3\bar{R} = \frac{2}{3}\gamma^4\left[\ddot{\bar{\vartheta}} + \bar{\vartheta} \times (\bar{\vartheta} \times \ddot{\bar{\vartheta}})\right] + 2\gamma^6(\bar{\vartheta}\cdot\dot{\bar{\vartheta}})\left[\dot{\bar{\vartheta}} + \bar{\vartheta} \times (\bar{\vartheta} \times \dot{\bar{\vartheta}})\right]$$

Collecting all our results of integrations and substituting them into (2.6) (Note that for the action of a particle on itself all the points $\bar{r}(t)$, $\bar{r}'(t)$, $\bar{x}$, and $\bar{x}'$ should be the same), we get the Lorentz force that the relativistic charged particle exerts on itself to be

$$\bar{F} = \frac{2}{3}e^2\gamma^4\left[\ddot{\bar{\vartheta}} + \bar{\vartheta} \times (\bar{\vartheta} \times \ddot{\bar{\vartheta}})\right] + 2e^2\gamma^6(\bar{\vartheta}\cdot\dot{\bar{\vartheta}})\left[\dot{\bar{\vartheta}} + \bar{\vartheta} \times (\bar{\vartheta} \times \dot{\bar{\vartheta}})\right] \qquad (2.7)$$

This expression is identical to that obtained by Gordeyev (1975). But in this paper (2.7) was obtained by assuming that as $\bar{R} \to 0$, $\bar{n} = \bar{R}/R$ and $Z = \bar{n}\cdot\gamma\bar{\vartheta}$ also become zero. In addition the singular terms (i.e. terms ~ R^{-2} and R^{-1}) were to be avoided by means of mass renormalization. In the present calculation all these assumptions are not made, the symmetry of delta function was just enough to carry out the exact calculations.

The fourth component of the equation of motion can be found by taking the scalar product of $\bar{F}$ with $\bar{\vartheta}$.
Thus

$$\frac{d\mathcal{E}}{dt} = (\bar{\vartheta}\cdot\bar{F}) = \frac{2}{3} e^2\gamma^4(\bar{\vartheta}\cdot\ddot{\bar{\vartheta}}) + 2e^2\gamma^6(\bar{\vartheta}\cdot\dot{\bar{\vartheta}})^2 \tag{2.8}$$

For Low-Velocity motion , $\bar{\vartheta} \ll c$ and $\gamma \to 1$, the Lorentz force of the self action reduces to

$$\bar{F} = \frac{2}{3} e^2 \ddot{\bar{\vartheta}},$$

which is exactly the same as that derived from Larmor radiation formula.

The Lorentz-force of the self action (2.7) can be written as

$$\bar{F} = \frac{d\bar{P}}{d\tau} = \frac{2}{3} e^2 \left[\frac{d^2\bar{U}}{d\tau^2} - \bar{U}\left[\frac{d}{d\tau}U_\rho\right]^2\right]$$

where $U_\rho = (\bar{U}=\gamma\bar{\vartheta}, i\gamma)$ is the four velocity of the particle and $d\tau = dt/\gamma$ is an element of the particles proper time. Combining this with the fourth component (2.8) gives a four vector self-action force F^s_σ in a covariant form.

$$F^s_\sigma = \frac{2}{3} e^2 \left[\ddot{U}_\sigma - U_\sigma (\dot{U}_\rho)^2 \right] \tag{2.9}$$

This is the same as the expression given by Dirac (1938) for a radiation reaction force on a classical relativistic point charged particle.

2.2. Cancelation of Singular Terms By Using The Correct Retarded Green Function.

The self-action force in (2.7) was obtained from a purely retarded field of the particle only. But it can also be calculated from an expression for a field that contains not only the retarded field but also an advanced field of the particle. According to Dirac's assumption this field is given by

$$\overline{F}^s = \frac{1}{2} \left(\overline{F}_{ret} - F_{adv} \right) \tag{2.10}$$

Where $\overline{F}_{ret}$ and $\overline{F}_{adv}$ denote the retarded and advanced fields, respectively, of the particle.

The field obtained in this way is supposed to be free from singularities everywhere in space and hence at the postion of the particle producing the field.

The field given above is actually not an assumption in the present calculation but, as we have shown in (1.10), it happens to be a consequence of mere mathematical result that comes out of using the correct retarded Green function. We shall find the self-action force by using this Green function in the delta-function approach to the simultaneous expansion for potentials and fields.

Using (1.7a) in (1.6a) and (1.6b) and with the system of units we have adopted in the simultaneous expansion for potentils and fields (i.e. c = 1), we can write the scalar and vector potentials as

$$\phi = e \int \frac{\theta(\tau)}{|\bar{R}(t-\tau)|} \Big[\delta(\tau - |\bar{R}(t-\tau)|) - \delta(\tau + |\bar{R}(t-\tau)|)\Big] d\tau$$

$$\bar{A} = e \int \frac{\theta(\tau)}{|\bar{R}(t-\tau)|} \bar{\vartheta}(t-\tau)\Big[\delta(\tau - |\bar{R}(t-\tau)|) - \delta(\tau + |\bar{R}(t-\tau)|)\Big] d\tau \tag{2.11}$$

When the interaction of the particle with other particles is considered (i.e. $\tau > 0$) the potentials in (2.11) at once reduce to the usual retarded Lienard-Wiechert potentials. But for the interaction of the particle with itself (i.e. $\tau = 0$) with $\theta(0) = 1/2$ these potentials (2.11) lead to the field given in (2.10).

Let us shift the argument of the unit step function in (2.11) by small ϵ (> 0) so that

$$\phi = \lim_{\epsilon\to 0} e \int \frac{\theta(\tau+\epsilon)}{|\bar{R}(t-\tau)|} \Big[\delta(\tau - |\bar{R}(t-\tau)|) - \delta(\tau + |\bar{R}(t-\tau)|)\Big] d\tau$$

$$\bar{A} = \lim_{\epsilon\to 0} e \int \frac{\theta(\tau+\epsilon)}{|\bar{R}(t-\tau)|} \bar{\vartheta}(t-\tau)\Big[\delta(\tau - |\bar{R}(t-\tau)|) - \delta(\tau + |\bar{R}(t-\tau)|)\Big] d\tau$$

Starting from these expressions and following a similar path to that used to arrive at (1.41) and (1.43), we get

$$\phi = e \sum \frac{(-1)^{s}\ R^{s-1}}{k_1!k_2!\ldots} \left[\frac{\dot{\vartheta}}{2!}\right]^{k_1} \left[\frac{\ddot{\vartheta}}{3!}\right]^{k_2} \ldots \frac{\partial^{\ell}}{\partial\bar{\vartheta}^{\ell}} \left[\theta(\tau^{+}+\epsilon)\phi^{+}_{s-1} - \theta(\tau^{-}+\epsilon)\phi^{-}_{s-1}\right] \tag{2.12}$$

where $\tau^{\pm} = \gamma R[z \pm (1+z^2)^{1/2}]$ given by (1.37) and their corres - ponding, scalar functions

$$\phi^{\pm}_{s} = \gamma^{s+2} \frac{[z \pm (1+z^2)^{1/2}]^{s+1}}{(1+z^2)^{1/2}}$$

are the same as those given in (1.40) and (1.46) for ϕ^{+}_{s} and ϕ^{-}_{s}, respectively,

and

$$\bar{A}\ (\bar{x},t) = e \sum_{m=0}^{\infty} \frac{(-1)^{m}}{m!} \overset{m}{\bar{\vartheta}}\ \Phi_{m} \tag{2.13}$$

where

$$\Phi_{m} = \sum \frac{(-1)^{s+m}}{k_1!k_2!\ldots} R^{s+m-1} \left[\frac{\dot{\vartheta}}{2!}\right]^{k_1} \left[\frac{\ddot{\vartheta}}{3!}\right]^{k_2} \ldots \frac{\partial^{\ell}}{\partial\bar{\vartheta}^{\ell}} \left[\theta(\tau^{+}+\epsilon)\phi^{+}_{s+m-1} - \theta(\tau^{-}+\epsilon)\phi^{-}_{s+m-1}\right] \tag{2.14}$$

with

$$\phi^{\pm}_{s+m} = \gamma^{s+m+2} \frac{[z \pm (1+z^2)^{1/2}]^{s+m+1}}{(1+z^2)^{1/2}}$$

The indices in (2.12) and (2.14) are still restricted by the same equation (1.39). Note that it should be understood that the potentials in (2.12) and (2.13) are going to be evaluated in the limit as $\epsilon \to 0$.

For $R \to 0$ the terms that we need to consider in these above expansions (2.12) and (2.13) for potentials are those proportional to R^{-1}, R^{0} and R (of course these terms are equivalent to those considered for fields in the preceeding calculation). For the scalar potential they correspond to $s = 0$, $s = 1$, and $s = 2$.

For $s = 0$, $k_1 = k_2 = \ldots = 0$, we have

$$\phi^{0} = \lim_{\substack{R\to 0\\ \epsilon\to 0}} e\left[R^{-1}\theta(\tau^{+}+\epsilon)\phi^{+}_{-1} - R^{-1}\,\theta(\tau^{-}+\epsilon)\phi^{-}_{-1}\right]$$

$$\phi^{0} = e\,\theta(0)\lim_{R\to 0}\left[R^{-1}(\phi^{+}_{-1} - \phi^{-}_{-1})\right] = 0$$

For $s = 1$, $k_1 = 1$, $k_2 = k_3 = \ldots = 0$

$$\phi^{1} = \lim_{\substack{R\to 0\\ \epsilon\to 0}} e\left\{-\left(\frac{\dot{\bar{\vartheta}}}{2!}\right)\cdot\frac{\partial}{\partial\bar{\vartheta}}\left[\theta(\tau^{+}+\epsilon)\phi^{+}_{0} - \theta(\tau^{-}+\epsilon)\,\phi^{-}_{0}\right]\right\}$$

$$\phi^{1} = -\,e\theta(0)\left(\frac{\dot{\bar{\vartheta}}}{2!}\right)\cdot\frac{\partial}{\partial\bar{\vartheta}}\left[\,2\gamma^{2}\,\right] = -\,e\gamma^{4}(\bar{\vartheta}\cdot\dot{\bar{\vartheta}})$$

For $s = 2$, the possible values of the indices satisfying (1.39) are $k_1 = 2$, $k_2 = k_3 = \ldots = 0$ and $k_1 = 0$, $k_2 = 1$, $k_3 = k_4 = \ldots = 0$. with this

$$\phi^{2} = \lim_{\substack{R\to 0\\ \epsilon\to 0}} e\left[\frac{1}{2}\left(\frac{\dot{\bar{\vartheta}}}{2!}\right)^{2}\cdot\left(\frac{\partial}{\partial\bar{\vartheta}}\right)^{2} + \frac{\ddot{\bar{\vartheta}}}{3!}\cdot\frac{\partial}{\partial\bar{\vartheta}}\right]\left[\theta(\tau^{+}+\epsilon)\phi^{+}_{1} - \theta(\tau^{-}+\epsilon)\phi^{-}_{1}\right]$$

$$\phi^{2} = e\theta(0)\lim_{R\to 0}\left[\frac{1}{2}\left(\frac{\dot{\bar{\vartheta}}}{2!}\right)^{2}\cdot\left(\frac{\partial}{\partial\bar{\vartheta}}\right)^{2} + \frac{\ddot{\bar{\vartheta}}}{3!}\cdot\frac{\partial}{\partial\bar{\vartheta}}\right]\left[2\gamma^{4}(\bar{R}\cdot\bar{\vartheta})\right]$$

$$\phi^2 = e \left\{\gamma^6(\bar{R}\cdot\bar{\vartheta})\left[\frac{4}{3}\,(\bar{\vartheta}\cdot\ddot{\bar{\vartheta}}) + 6\gamma^2(\bar{\vartheta}\cdot\dot{\bar{\vartheta}})^2 + \dot{\bar{\vartheta}}^2\right] + 2\gamma^6(\bar{R}\cdot\bar{\vartheta})(\bar{\vartheta}\cdot\dot{\bar{\vartheta}}) + \frac{1}{3}\,\gamma^4(\bar{R}\cdot\ddot{\bar{\vartheta}})\right\}$$

Thus for the scalar potential, we have as $R \to 0$,

$$\phi = \phi^0 + \phi^1 + \phi^2$$

$$\phi = e\gamma^4\left\{-\,(\bar{\vartheta}\cdot\dot{\bar{\vartheta}}) + \gamma^2(\bar{R}\cdot\bar{\vartheta})\left[\frac{4}{3}\,(\bar{\vartheta}\cdot\ddot{\bar{\vartheta}}) + 6\gamma^2(\bar{\vartheta}\cdot\dot{\bar{\vartheta}})^2 + \dot{\bar{\vartheta}}^2\right] + 2\gamma^2(\bar{R}\cdot\dot{\bar{\vartheta}})(\bar{\vartheta}\cdot\dot{\bar{\vartheta}}) + \frac{1}{3}(\bar{R}\cdot\ddot{\bar{\vartheta}})\right\} \qquad (2.15)$$

Similarly

$$\bar{A} = \bar{\vartheta}\phi - e\gamma^2\,\dot{\bar{\vartheta}}\left[1 - 4\gamma^4(\bar{R}\cdot\bar{\vartheta})(\bar{\vartheta}\cdot\dot{\bar{\vartheta}}) - (\bar{R}\cdot\dot{\bar{\vartheta}})\gamma^2\right] + e\gamma^4(\bar{R}\cdot\bar{\vartheta})\ddot{\bar{\vartheta}} \qquad (2.16)$$

We are now in a position to find the fields $\bar{E}$ and $\bar{H}$. This can be done by differentiating the scalar and vector potential expression (2.15) and (2.16), respectively, as in (1.2). Carrying out the differentation and tending R to zero in the resulting field expressions, we get

$$\bar{E} = \frac{2}{3}\,e\,\gamma^4\left[\,\ddot{\bar{\vartheta}} + 3\gamma^2\,(\bar{\vartheta}\cdot\dot{\bar{\vartheta}})\,\dot{\bar{\vartheta}}\,\right]$$

and

$$\bar{H} = \bar{\vartheta}\times\bar{E}$$

with these expressions for fields we once again arrive at an expression for the Lorentz force of the self-action identical to (2.7) but here in a very interesting way(without divergencies in integrals over R).

CHAPTER 3

LAGRANGIAN DESCRIPTION OF A SYSTEM OF RELATIVISTIC CHARGED PARTICLES.

3.1. Lagrange Equations of Motion

The Lagrangian description of a system of relativistic charged particles requires a complete specification of position, velocity and the field variables describing the field produced by each of the individual particles of this system. All these variables are functions of the particle's proper time, implicitly. But since the field variable, four potential, is already found interms of the simultaneous position, velocity and all orders of accelerations, the Lagrangian of such system also contains these simultaneous position, velocity and all orders of accelerations instead of the field variables. As such we may write

$$L = L(\bar{x}_a , \dot{\bar{x}}_a , \ddot{\bar{x}}_a , \ldots , \overset{k}{\bar{x}}_a , \ldots)$$

where a = 1,2,3, ... , N is the number of particles in the system.
The principle of least action for the motion of this system is

$$\delta S = \delta \int_{t_1}^{t_2} L dt = 0$$

$$\delta S = \delta \int_{t_1}^{t_2} \left[\sum^{\infty} \frac{\partial L}{\partial \overset{k}{x}} \delta \overset{k}{\bar{x}}_a \right] dt = 0 \qquad (3.1)$$

By using a mathematical induction it can be shown that if φ and ψ are arbitrary functions of x_0, then

$$\overset{k}{\varphi}\,\psi = (-1)^k \varphi\,\overset{k}{\psi} + \frac{d}{dx_0}\sum_{j=0}^{k-1}(-1)^j\,\overset{j}{\varphi}\,\overset{k-j-1}{\psi}\,, \quad k \ge 1 \tag{3.2}$$

Using (3.2) , the integrand in (3.1) can be written as

$$\frac{\partial L}{\partial \overset{k}{x}}\,\delta\overset{k}{x}_a = (-1)^k \frac{d^k}{dx_0^k}\left(\frac{\partial L}{\partial \overset{k}{x}_a}\right)\delta x_a + \frac{d}{dx_0}\sum_{j=0}^{k-1}(-1)^j \frac{d^j}{dx_0^j}\left(\frac{\partial L}{\partial \overset{k}{x}_a}\right)\delta\overset{k-j-1}{x}_a$$

with this

$$\int_{t_1}^{t_2}\left(\sum_{k=0}^{\infty}\frac{\partial L}{\partial \overset{k}{x}}\,\delta\overset{k}{x}_a\right)dt = \int_{t_1}^{t_2}\sum_{j=0}^{k-1}\left[(-1)^k \frac{d^k}{dx_0^k}\left(\frac{\partial L}{\partial \overset{k}{x}_a}\right)\delta x_a\right]dt$$

$$+\frac{1}{c}\sum_{k=1}^{\infty}\sum_{j=0}^{k-1}(-1)^j\frac{d^j}{dx_0^j}\left(\frac{\partial L}{\partial \overset{k}{x}_a}\right)\delta\overset{k-j-1}{x}_a\Bigg|_{t_1}^{t_2} = 0$$

Arranging the indices of summation in the second expression, we have

$$\int_{t_1}^{t_2}\left(\sum_{k=0}^{\infty}\frac{\partial L}{\partial \overset{k}{x}}\,\delta\overset{k}{x}_a\right)dt = \int_{t_1}^{t_2}\sum_{j=0}^{k-1}\left[(-1)^k \frac{d^k}{dx_0^k}\left(\frac{\partial L}{\partial \overset{k}{x}_a}\right)\delta x_a\right]dt$$

$$+\frac{1}{c}\sum_{j,k=0}^{\infty}(-1)^j\frac{d^j}{dx_0^j}\left(\frac{\partial L}{\partial \overset{k+j+1}{x}_a}\right)\delta\overset{k}{x}_a\Bigg|_{t_1}^{t_2} = 0$$

If $\delta\overset{k}{x}_a(t_1) = \delta\overset{k}{x}_a(t_2) = 0$ for $k = 0, 1, 2, \ldots$, then

$$\int_{t_1}^{t_2}\sum_{k=0}^{\infty}\left[(-1)^k \frac{d^k}{dx_0^k}\left(\frac{\partial L}{\partial \overset{k}{x}_a}\right)\delta x_a\right]dt = 0$$

This equation holds true only if

$$\sum_{k=0}^{\infty}\left[(-1)^k \frac{d^k}{dx_0^k}\left(\frac{\partial L}{\partial \overset{k}{x}_a}\right)\right] = 0 \tag{3.3}$$

which are the required Lagrange equations of motion for the system of relativistic charged particles. This equation consists of a system of 3N Lagrange equations each with infinite order of derivatives. The solutions of these equations then need infinite number of intial conditions. These initial conditions are the position, velocity and all orders of accelerations of each particle in the system at the intial state of this system. That is equivalent to knowledge of the fields at any point in space at initial moment.

We next consider the energy and momentum conservation laws for the motion of this system. We may follow the usual procedure of deriving these conservation laws for a conservative system.

The Lagrangian of a conservative system can't depend explicitly on time. Hence,

$$\frac{dL}{dx_0} = \sum_{a=1}^{\infty}\sum_{k=0}^{\infty}\frac{\partial L}{\partial \overset{k}{x}_a}\overset{k+1}{x}_a = \sum_{a=1}^{\infty}\sum_{k=0}^{\infty}\frac{\partial L}{\partial \overset{k}{x}_a}\overset{k}{\overline{V}}_a \tag{3.4}$$

By making use of the identity given in (3.2) and with the help of the Largange equations (3.3), eq. (3.4) can be written as

$$\frac{dL}{dx_0} = \frac{d}{dx_0}\sum_{a=1}^{N}\sum_{k=1}^{\infty}\sum_{j=0}^{k-1}(-1)^j \frac{d^j}{dx_0^j}\left(\frac{\partial L}{\partial \overset{k}{x}_a}\right)\overset{k-j-1}{\overline{V}}_a$$

After arranging the indices of summation, we have

$$\frac{d}{dx_0}\left[\sum_{a=1}^{N}\sum_{k,j=0}^{\infty}(-1)^j \overset{k}{\bar{V}}_a \frac{d^j}{dx_0^j}\left(\frac{\partial L}{\partial \overset{k+j}{\bar{v}}_a}\right) - L\right] = 0$$

From this we get the energy conservation as

$$\sum_{a=1}^{N}\sum_{k,j=0}^{\infty}(-1)^j \overset{k}{\bar{V}}_a \frac{d^j}{dx_0^j}\left(\frac{\partial L}{\partial \overset{k+j}{\bar{v}}_a}\right) - L = \text{Constant} \qquad (3.5)$$

The Lagrange equations of motion (3.3) can be written as

$$\frac{\partial L}{\partial x_0} = \frac{d}{dx_0}\sum_{k=0}^{\infty}(-1)^k \frac{d^k}{dx_0^k}\left(\frac{\partial L}{\partial \overset{k}{\bar{v}}_a}\right)$$

If we perform summation over all particles of the system, then

$$\sum_{a=1}^{N}\frac{\partial L}{\partial x_a} = \frac{d}{dx_0}\sum_{a=1}^{N}\sum_{k=1}^{\infty}(-1)^k \frac{d^k}{dx_0^k}\left(\frac{\partial L}{\partial \overset{k}{\bar{v}}_a}\right) \qquad (3.6)$$

That gives the conservation of linear momentum in the system including the field momentum, expressed in terms of the particles' accelerations. If there is no external field acting on the system, then the Larangian depends on the coordinates due to the potentials, which are functions of the coordinates in the form $\bar{R}_{ab} = \bar{x}_a - \bar{x}_b$. Thus, for any interacting particles a and b of the system, we have

$$\frac{\partial L}{\partial x_a} = -\frac{\partial L}{\partial x_b}$$

Hence, $$\sum_{a=1}^{N} \frac{\partial L}{\partial x_a} = 0$$

with this eq. (3.6) gives an equation for the conservation of momentum of this system to be

$$\sum_{a=1}^{N} \sum_{k=0}^{\infty} (-1)^k \frac{d^k}{dx_0^k} \left[\frac{\partial L}{\partial \overset{k}{\bar{v}}_a} \right] = \text{constant} \tag{3.7}$$

3.2. The Retarded And Advanced Fields

For a relativistic charged particle in an arbitrary motion, the Maxwell equations as we have already said are satisfied by both solutions, the retarded and advanced Lienard-Wiechert potentials. These potentials (expressed interms of the particles postion, velocity and all orders of accelerations, which are simultaneous with the observation time) are given by

$$\phi^{+}(\bar{x},t) = e\Phi_0^{+} \qquad ; \qquad \bar{A}(\bar{x},t) = e\sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{\ell!} \overset{\ell}{\bar{\vartheta}}\, \Phi_{\ell}^{+} \,,$$

and

$$\phi^{-}(\bar{x},t) = e\Phi_0^{-} \qquad ; \qquad \bar{A}^{-}(\bar{x},t) = e\sum_{\ell=0}^{\infty} \frac{1}{\ell!} \overset{\ell}{\bar{\vartheta}}\, \Phi_{\ell}^{-}$$

where

$$\Phi_{\ell}^{+}(\bar{x}-\bar{x}', \dot{\bar{x}}', \ddot{\bar{x}}', \ldots, \overset{k}{\bar{x}}', \ldots) = \sum \ldots \frac{(-1)^m}{i_k! \ldots} \frac{\partial^p R^{m+\ell-1}}{\partial \bar{x}'^{p}} \ldots \left[\frac{\overset{k}{\bar{x}}'}{k!} \right]^{i_k} \ldots \tag{3.8}$$

and

$$\Phi_{\ell}^{-}(\bar{x}-\bar{x}', \dot{\bar{x}}', \ddot{\bar{x}}', \ \ldots, \ \overset{k}{\bar{x}}', \ \ldots) = \sum \frac{}{\ldots i_k! \ldots} \frac{\partial^{P} R^{m+\ell-1}}{\partial \bar{x}'^{P}} \cdots \left[\frac{\overset{k}{\bar{x}}'}{k!} \right]^{i_k} \cdots$$

(3.9)

(Here $m = \sum ki_k$, $P = \sum i_k$) are the retarded and advanced ℓ-th order potentials, respectively. As the above expressions show, the retarded and advanced potentials have the same expressions except for an aleternating sign difference.

When an interaction between such particles is considered, a relation between a retarded potential of one particle and an advanced potential of another second particle was established in the following way:

Consider an interaction between two relativistic charged particles (hereafter named as particle 1 and particle 2).

The ℓ-th order retarded potential at the position of particle 2 due to particle 1 is given by (3.8)

$$\Phi_{\ell}^{+} (\bar{x}_2 - \bar{x}_1, \dot{\bar{x}}_1, \ddot{\bar{x}}_1, \ \ldots) \ = \ \Phi_{\ell}^{1+}$$

Let us consider the m-th total derivative of this potential with respect to $x_0 = ct$.

$$\frac{d^m}{dx_0^m} \Phi_{\ell}^{1+} = \sum \frac{m!}{i! \ldots j_k!} \frac{\partial^{p+i}}{\partial \bar{X}^{p} \partial x_0^{i}} \Phi_{\ell}^{1+} \cdots \left[\frac{\overset{k}{\bar{x}}_2}{k!} \right]^{j_k}$$

where the indices satisfy the equations

$$m = i + \sum kj_k \quad \text{and} \quad p = \sum j_k$$

$$\sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m}{dx_0^m} \Phi^{1+}_{\ell+m} = \sum \frac{1}{\ldots j_k! \ldots} \ldots \left(\frac{\overset{k}{\bar{x}}_2}{k!} \right)^{j_k} \ldots \frac{\partial^p}{\partial \bar{x}_2^p} \sum_{i=0}^{\infty} \frac{1}{i!} \frac{\partial^i}{\partial \bar{x}_0^i} \Phi^{1+}_{\ell+m+i} \tag{3.10}$$

where $\quad m = \sum kj_k \qquad P = \sum j_k$

By using the relation

$$\sum_{i=0}^{\infty} \frac{(-1)^i s!}{(s-i)!\,i!} = \delta_{os} = (1-1)^s$$

we can write (3.10) as

$$\sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m}{dx_0^m} \Phi^{1+}_{\ell+m} = \sum \frac{1}{\ldots j_k! \ldots} \frac{\partial^p}{\partial \bar{x}_2^p} R^{\ell+m-1} \ldots \left(\frac{\overset{k}{\bar{x}}_2}{k!} \right)^{j_k} \ldots \tag{3.11}$$

The right hand side of this last equation is the ℓ-th order advanced potential Φ^{2-}_{ℓ} at the positon of the first particle produced by the second particle.

Thus

$$\sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m}{dx_0^m} \Phi^{1+}_{\ell+m} = \Phi^{2-}_{\ell}$$

$$\Phi^{1+}_{\ell} + \sum_{m=1}^{\infty} \frac{1}{m!} \frac{d^m}{dx_0^m} \Phi^{1+}_{\ell+m} = \Phi^{2-}_{\ell}$$

Hence

$$\Phi_{\ell}^{2-} - \Phi_{\ell}^{1+} = \frac{d}{dx_0}\left[\sum_{m=0}^{\infty} \frac{1}{(m+1)!}\frac{d^m}{dx_0^m}\Phi_{\ell+m+1}^{1+}\right] \tag{3.12}$$

Thus the difference between advanced potential of the second particle at the position of the first and the retarded potential at the positoon of the second is a total time derivative of some function of $\{\bar{x}_2-\bar{x}_1, \dot{\bar{x}}_1, \ddot{\bar{x}}_1, \dddot{\bar{x}}_1, \ldots\}$. And its being a total time derivative suggests that we can include this difference in the Lagrangian function of this system without affecting the equations of motion of the system. Moreover, due to this relation , in a purely retarded consideration of the interaction between such particles, advanced interactions can directly enter into the description without any postulate made to include them.

As an example illustrating an important consequence of (3.12) we shall obtain the equations of motion of a two particle system (of course they are relativistic charged particles).

The Lagrangian of this system, constructed from a purely retarded potential, is given by

$$L = -m_1c^2\sqrt{1-\vartheta_1^2} - m_2c^2\sqrt{1-\vartheta_2^2} - \frac{1}{2}\left\{ e_1(\varphi^{2+} - \bar{\vartheta}_1\cdot\bar{A}^{2+}) + e_2(\varphi^{1+} - \bar{\vartheta}_2\cdot\bar{A}^{1+})\right\} \tag{3.13}$$

Substituting this into the Largrange equations of motion,

$$\frac{\partial L}{\partial x_a} - \frac{d}{dx_0}\left[\frac{\partial L}{\partial v_a}\right] + \sum_{k=2}^{\infty}\frac{(-1)^k}{} \frac{d^k}{dx_0^k}\left[\frac{\partial L}{\partial \overset{k}{x}_a}\right] = 0,$$

and carrying out the differentations with respect to postion, velocity and (k-1)-th order of acceleration of one of the particles, say a = 1, we get

$$\frac{d}{dx_0}\left[\frac{m_1 c^2 \bar{\vartheta}_1}{\sqrt{1-\vartheta_1^2}}\right] = \frac{e_1}{2}\left[\bar{E}_2^{+} + \bar{\vartheta}_1 \times \bar{H}_2^{+}\right] - \frac{e_1 e_2}{2}\sum_{k=0}^{\infty}\frac{1}{k!}\frac{d^k}{dx_0^k}\left[\frac{\partial}{\partial \bar{x}_1}\overset{1+}{\varphi}_k - \sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}\,\ell}{\ell!}(\bar{\vartheta}_1\cdot\bar{\vartheta}_2)\frac{\partial}{\partial \bar{x}_1}\overset{1+}{\varphi}_{\ell+k} + k\bar{\vartheta}_2\overset{1+}{\varphi}_{k-1}\right] \qquad (3.14)$$

The first term on the right hand side is half-the usual retarded Lorentz force exerted on particle 1 by 2. The second term, after making use of the relation between retarded and advanced potential (eq. (3.12)), gives the Lorentz force on this same particle 1 due to half-advanced field of the second particle. Equation (3.14) then becomes

$$\frac{d}{dx_0}\left[\frac{m_1 c^2 \bar{\vartheta}_1}{\sqrt{1-\vartheta_1^2}}\right] = \frac{e_1}{2}\left[\bar{E}_2^{+} + \bar{\vartheta}_1 \times \bar{H}_2^{+}\right] + \frac{e_1}{2}\left[\bar{E}_2^{-} + \bar{\vartheta}_1 \times \bar{H}_2^{-}\right]$$

Thus the forece exerted on particle 1 by the other particle 2 in the system can be given by

$$\left.\begin{aligned} &\frac{d\bar{P}_1}{dx_0} = \frac{1}{2}\left(\bar{F}^{2+} + \bar{F}^{2-}\right) \\ &\text{Similarly} \\ &\frac{d\bar{P}_2}{dx_0} = \frac{1}{2}\left(\bar{F}^{1+} + \bar{F}^{1-}\right) \end{aligned}\right\} \qquad (3.15)$$

This result is in agreement with the theory of action at a distance, which is the basis of Wheeler and Feynman absorber theory of radiation, which describes such interactions by a time symmetric solution in the form

$$\bar{F}_{ext} = \frac{1}{2} (\bar{F}_{ret} + \bar{F}_{adv}) \tag{3.16}$$

Although the description of the external force (3.15) is the same, the methods used in the present by Gordeyev (1978) and Wheeler and Feynman (1945) to arive at this description are completely different.

In the Wheeler and Feynman approach the advanced interactions are accepted as a postulate just because the retarded fields alone are not enough to obey the law of action and reaction. While (3.16) was derieved from the least action principle, which is given interms of each particle's space-time coordinates. Here (3.16) is a result of the relation between advanced and retarded interactions between the particles (3.12).

3.3. Retarded and Advanced Fields in the Delta Function Approach

One of the important consequences of the simultaneous expansion for potentials and fields, in addition to providing the possibility of common time (i.e. observaion time) description of the interaction between relativistic charged point particles which may avoid the need for the knowledge of the past history of the individual particle producing the field, is the relation established between a retarded and advanced fields of two interacting particles. This relation has led to a half retarded plus half advanced field description of the interaction of these particles.

We have used in parallel both the Lagrange expansion method or its results and delta function approach in the simultaneous expansion for potentials and fields and calculation of the self-action force. It is clearly seen that the latter approach at least for its being simple can be taken to be advantageous than the former approach.

We now obtain the relation between retarded and advanced fields found in (3.12) and derive the equations of motion for the system of two relativistic charged particles in the delta function approach.

Let us consider the interaction of the same two particles considered to arrive at (3.12). The retarded scalar and vector potentials at the position of particle 2 due to particle 1 are given by (1.32) and (1.33), respectively, as

$$\phi^{1+}(\bar{x}_2,t) = 2e_1 \int \theta(\tau)\ \delta\left[(\bar{x}_2(t) - \bar{x}_1(t-\tau))^2 - \tau^2\right] d\tau$$

$$\bar{A}^{1+}(\bar{x}_2,t) = 2e_1 \int \theta(\tau)\ \bar{\vartheta}_1(t-\tau)\ \delta\left[(\bar{x}_2(t) - \bar{x}_1(t-\tau))^2 - \tau^2\right] d\tau$$

with the Taylor expansion of $\bar{\vartheta}_1(t-\tau)$ given in (1.42), we can write these potentials as

$$\phi^{1+} = \Phi_0^{1+} \qquad\qquad \bar{A}^{1+} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \overset{m}{\bar{\vartheta}}_1\ \Phi_m^{1+}$$

where

$$\Phi^{1+}_{m} = 2e_1 \int \theta(\tau)\, \tau^m\, \delta[(\bar{x}_2(t) - \bar{x}_1(t-\tau))^2 - \tau^2] d\tau$$

is the m-th order retarded scalar potential at the position of particle 2 due to particle 1.

In a similar way the $(m+\ell)$-th order retarded scalar potential may be written as

$$\Phi^{1+}_{m+\ell} = 2e_1 \int \theta(\tau)\, \tau^{m+\ell}\, \delta[(\bar{x}_2(t) - \bar{x}_1(t-\tau))^2 - \tau^2] d\tau$$

Expanding $\bar{x}_2(t-\tau)$ in powers of τ, we have

$$\bar{R}(t-\tau) = \bar{x}_2(t) - \bar{x}_1(t-\tau) = \bar{R}(t) + \tau\bar{\vartheta}_1(t) + \tau\, \bar{g}(t,\tau)$$

where $\bar{R}(t) = \bar{x}_2(t) - \bar{x}_1(t)$ and $\bar{g}(t,\tau)$ is the same as (1.34)

$$g(t,\tau) = \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{(\ell+1)!} \tau^\ell \overset{\ell}{\vartheta}_1$$

With this the argument of the delta function becomes

$$(\bar{x}_2(t) - \bar{x}_1(t-\tau))^2 - \tau^2 = (\bar{R} + \tau\, \bar{\vartheta}_1 + \tau\, \bar{g})^2 - \tau^2 = f(t,\tau)$$

so that

$$\Phi^{1+}_{m+\ell} = 2e_1 \int \theta(\tau)\, \tau^{m+\ell}\, \delta[f(t,\tau)]\, d\tau$$

Now let us take the m-th total derivative of $\Phi^{1+}_{m+\ell}$ with respect to $x_0 = ct = t$ (with $c = 1$)

$$\frac{d^m}{dx_0^m} \Phi^{1+}_{m+\ell} = 2e \int \theta(\tau)\, \tau^{m+\ell} \frac{d^m}{dt^m} \delta[f(t,\tau)] d\tau$$

then

$$\sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m}{dx_0^m} \Phi^{1+}_{m+\ell} = 2e \int \theta(\tau)\tau^{\ell} \left\{ \sum_{m=0}^{\infty} \frac{\tau^m}{m!} \frac{d^m}{dt^m} \delta[f(t,\tau)] \right\} d\tau$$

The term in the curled brackets is just an expression for the Taylor expansion of $\delta[f(t+\tau,\tau)]$ in powers of τ. Thus we have

$$\sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m}{dx_0^m} \Phi^{1+}_{m+\ell} = 2e \int \theta(\tau)\tau^{\ell}\delta[f(t+\tau,\tau)]d\tau$$

But $f(t+\tau,\tau) = [\bar{R}(t+\tau) + \tau\, \bar{\vartheta}_1(t+\tau) + \tau\, \bar{g}(t+\tau,\tau)]^2 - \tau^2$

$$= (\bar{x}_2(t+\tau) - \bar{x}_1(t))^2 - \tau^2$$

Hence,

$$\sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m}{dx_0^m} \Phi^{1+}_{m+\ell} = 2e \int \theta(\tau)\tau^{\ell}\delta[(\bar{x}_1(t) - \bar{x}_2(t+\tau))^2 - \tau^2]d\tau$$

The right hand side of this last equation is the ℓ-th order advanced scalar potential Φ^{2-}_{ℓ} at the position of particle 1 due to particle 2. Or

$$\sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m}{dx_0^m} \Phi^{1+}_{m+\ell} = \Phi^{2-}_{\ell}$$

This is the required relation which can be written as in (3.12) but here it was obtained in a much less obscured way than before.

To get the Lagrange equations of motion for the system of two particles in this approach we shall once again use the Lagrangian in (3.13) and the Lagrange equations of motion,

$$\frac{\partial L}{\partial x_1} - \frac{d}{dx_0}\left(\frac{\partial L}{\partial v_1}\right) + \sum_{k=2}^{\infty} \frac{(-1)^k}{} \frac{d^k}{dx_0^k}\left(\frac{\partial L}{\partial \overset{k}{x}_1}\right) = 0$$

Differentiating the Lagrangian with respect to $\bar{x}_1$ and $\bar{v}_1$ and substituting into this equation at once gives

$$\frac{d}{dx_0}\left[\frac{m_1 c^2 \bar{\vartheta}_1}{\sqrt{1-\vartheta_1^2}}\right] = \frac{e_1}{2}\left[\bar{E}^{2+} + \bar{v}_1 \times \bar{H}^{2+}\right] + \frac{e_2}{2}\left[-\frac{\partial \phi^{1+}}{\partial \bar{x}_1} + \frac{\partial}{\partial \bar{x}_1}(\bar{v}_2 \cdot \bar{A}^{1+})\right]$$

$$+ \frac{e_2}{2}\frac{d}{dt}\left[\frac{\partial \phi^{1+}}{\partial \bar{v}_1} - \frac{\partial}{\partial \bar{v}_1}(\bar{v}_2 \cdot \bar{A}^{1+})\right] + \sum_{k=2}^{\infty} \frac{(-1)^k}{} \frac{d^k}{dx_0^k}\left(\frac{\partial L}{\partial \overset{k}{x}_1}\right) \qquad (3.17)$$

Now let us consider the term which contains higher derivatives of the Lagrangian:

$$\frac{\partial L}{\partial \overset{k}{x}_1} = \frac{e_2}{2}\left[-\frac{\partial \phi^{1+}}{\partial \overset{k}{x}_1} + \frac{\partial}{\partial \overset{k}{x}_1}(\bar{v}_2 \cdot \bar{A}^{1+})\right]$$

$$\frac{\partial \phi^{1+}}{\partial \overset{k}{x}_1} = \frac{\partial}{\partial \overset{k}{x}_1}\Phi_0^{1+} = 2e_1 \int \theta(\tau)\, \frac{\partial f}{\partial \overset{k}{x}}\delta'(f)d\tau \qquad (3.18)$$

Since

$$\frac{\partial f}{\partial \overset{k}{x}_1} = \frac{\partial}{\partial \overset{k}{x}_1}\left[(\bar{R} + \tau\bar{\vartheta}_1 + \tau\bar{g})^2 - \tau^2\right] = 2(\bar{R} + \tau\bar{\vartheta}_1 + \tau\bar{g})\,\tau\,\frac{\partial \bar{g}}{\partial \overset{k}{x}_1} \quad ; \; k \geq 2$$

$$\frac{\partial \bar{g}}{\partial \overset{k}{x}_1} = \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{(\ell+1)!}\tau^\ell \delta_{\ell,k-1} = \frac{(-1)^{k-1}}{k!}\tau^{k-1} \quad ,$$

and

$$\frac{\partial f}{\partial \bar{x}_1} = -2\,(R + \tau\bar{\vartheta}_1 + \tau\bar{g}) = -\frac{\partial f}{\partial \bar{x}_2}$$, we may write the above

integral in (3.18) as

$$\frac{\partial \phi^{1+}}{\partial \bar{x}_1^k} = \frac{\partial}{\partial \bar{x}_1}\left\{ 2e_1 \int \theta(\tau)\, \frac{(-1)^k}{k!}\, \tau^k\, \delta[f(t,\tau)]d\tau \right\}$$

then

$$\frac{e_2}{2}\sum_{k=2}^{\infty}(-1)^k \frac{d^k}{dx_0^k}\left(\frac{\partial \phi^{1+}}{\partial \bar{x}_1^k}\right) = \frac{\partial}{\partial \bar{x}_1}\left\{ e_1 e_2 \int \theta(\tau) \sum_{k=0}^{\infty}\frac{\tau^k}{k!}\, \frac{d^k}{dt^k}\, \delta[f(t,\tau)]d\tau \right\}$$

$$+ \frac{e_2}{2}\left[-\frac{\partial \phi^{1+}}{\partial \bar{x}_1} + \frac{d}{dt}\left(\frac{\partial \phi^{1+}}{\partial \bar{v}_1}\right)\right]$$

$$= \frac{e_1}{2}\frac{\partial}{\partial \bar{x}_1}\left\{ 2e_2 \int \theta(\tau)\; \delta[f(t+\tau,\tau)]d\tau \right\} + \frac{e_2}{2}\left[-\frac{\partial \phi^{1+}}{\partial \bar{x}_1} + \frac{d}{dt}\left(\frac{\partial \phi^{1+}}{\partial \bar{v}_1}\right)\right]$$

Here again the expression in the curled bracket is the scalar potential at the positon of particle 1 due to particle 2.

Thus

$$\frac{e_2}{2}\sum_{k=2}^{\infty}(-1)^k \frac{d^k}{dx_0^k}\left(\frac{\partial \phi^{1+}}{\partial \bar{x}_1^k}\right) = \frac{e_1}{2}\frac{\partial \phi^{2-}}{\partial \bar{x}_1} + \frac{e_2}{2}\left[-\frac{\partial \phi^{1+}}{\partial \bar{x}_1} + \frac{d}{dt}\left(\frac{\partial \phi^{1+}}{\partial \bar{v}_1}\right)\right]$$

Using the expression for the retarded vector potential $\vec{A}^{1+}$

$$\left(\bar{\vartheta}_2\cdot\vec{A}^{1+}\right) = \bar{\vartheta}_2(t)\cdot 2e_1 \int \theta(\tau)\bar{\vartheta}_1(t-\tau)\delta[f(t,\tau)]d\tau$$

$$\frac{\partial}{\partial \overset{k}{x}_1}(\bar\vartheta_2\cdot\bar A^{1+}) = \bar\vartheta_2\left\{ 2e_1\int\theta(\tau)\,\frac{(-1)^{k-1}}{(k-1)!}\tau^{k-1}\,\delta[f(t,\tau)]d\tau\right.$$

$$\left. + 2e_1\int\theta(\tau)\bar\vartheta_1(t-\tau)\,\frac{(-1)^k}{k!}\tau^k\frac{\partial}{\partial\bar x_1}\delta[f(t,\tau)]d\tau\right\}$$

the first term on the right hand side of this equation can be found by differentiating $\bar\vartheta_1(t-\tau)$ with respect to $\overset{k}{x}_1$ (or $\overset{k-1}{\bar\vartheta}$) after being expanded in powers of τ.

$$\frac{e_2}{2}\sum_{k=2}^{\infty}(-1)^k\frac{d^k}{dx_0^k}\left[\frac{\partial}{\partial\overset{k}{x}_1}(\bar\vartheta_2\cdot\bar A^{1+})\right] = \sum_{k=0}^{\infty}(-1)^k\frac{d^k}{dt^k}\left\{\bar\vartheta_2\cdot e_1e_2\int\theta(\tau)\frac{(-1)^{k-1}}{(k-1)!}\tau^{k-1}\delta(f)d\tau\right\}$$

$$+\sum_{k=0}^{\infty}(-1)^k\frac{d^k}{dt^k}\left\{\bar\vartheta_2\cdot e_1e_2\int\theta(\tau)\bar\vartheta_1(t-\tau)\,\frac{(-1)^k}{k!}\tau^k\frac{\partial}{\partial\bar x_1}\delta[f(t,\tau)]d\tau\right\}$$

$$+\frac{e_2}{2}\left\{-\frac{\partial}{\partial\bar x_1}(\bar\vartheta_2\cdot\bar A^{1+}) + \frac{d}{dt}(\bar\vartheta_2\cdot\bar A^{1+})\right\}$$

Using Leibnitz differentation formula for the k-th derivative of a product of functions the first term on the right hand side of the above last equation becomes

$$-\frac{e_1}{2}\frac{d}{dt}\left\{2e_2\int\theta(\tau)\sum_{k,n=0}^{\infty}\frac{\tau^{k+n}}{k!n!}\overset{n}{\bar\vartheta}_2\frac{d^k}{dt^k}\delta[f(t,\tau)]d\tau\right\} = -\frac{e_1}{2}\frac{d\bar A^{2-}}{dt}$$

and the second term

$$\frac{e_1}{2}\frac{\partial}{\partial\bar{x}_1}\left\{2e_2\int\theta(\tau)\sum_{k,n=0}^{\infty}\frac{\tau^{k+n}}{k!n!}\overset{n}{\bar{\vartheta}}_2\frac{d^k}{dt^k}\left[\bar{\vartheta}_1(t-\tau)\delta[f(t,\tau)]\right]d\tau\right\} = \frac{e_1}{2}\frac{\partial}{\partial\bar{x}_1}(\bar{\vartheta}_1\cdot\bar{A}^{2-})$$

with these we have

$$\frac{e_2}{2}\sum_{k=2}^{\infty}(-1)^k\frac{d^k}{dx_0^k}\left[\frac{\partial}{\partial\bar{x}_1^k}(\bar{\vartheta}_2\cdot\bar{A}^{1+})\right] = \frac{e_1}{2}\left[-\frac{d\bar{A}^{2-}}{dt}+\frac{\partial}{\partial\bar{x}_1}(\bar{\vartheta}_1\cdot\bar{A}^{2-})\right]$$

$$+\frac{e_2}{2}\left\{-\frac{\partial}{\partial\bar{x}_1}(\bar{\vartheta}_2\cdot\bar{A}^{1+})+\frac{d}{dt}\left[\frac{\partial}{\partial\bar{v}_1}(\bar{\vartheta}_2\cdot\bar{A}^{1+})\right]\right\} \tag{3.19}$$

Combining the results we have obtained in (3.17) and (3.19), we get

$$\sum_{k=2}^{\infty}\frac{(-1)^k}{}\frac{d^k}{dx_0^k}\left(\frac{\partial L}{\partial\bar{x}_1^k}\right) = \frac{e_1}{2}\left[\bar{E}^{2-}+\bar{\vartheta}_1\times\bar{H}^{2-}\right]-\frac{e_2}{2}\frac{\partial}{\partial\bar{x}_1}\left[\phi^{1+}+(\bar{\vartheta}_2\cdot\bar{A}^{2+})\right]$$

$$+\frac{e_2}{2}\frac{d}{dt}\left\{\frac{\partial}{\partial\bar{v}_1}\left[-\phi^{1+}+(\bar{\vartheta}_2\cdot\bar{A}^{1+})\right]\right\}$$

Substituting this expression into (3.17) at once gives the equation of motion given in (3.15) for particle 1. Similar equation of motion can be found for the other particle 2 without difficulty.

Conclusion

1. *We have shown that the correct retarded Green function gives the field of a moving charged particle at any point in space including the particle's position. The advanced part of this Green function becomes very important when the field of a point particle at its own position is considered.*

2. *We have demonstrated that the Lagrange expansion of retarded fields and Taylor expansion of Green function are equivalent approaches to the simultaneous expansion of the electromagnetic fields of a relativistic charged particle in arbitrary motion.*

3. *We have made two different but exact calculations of the self-action force without making any assumption and have obtained (in both ways) similar expressions for this force that coincide with the conventional expression for the radiation reaction force given by Dirac. This shows that mass renormalization in classical electrodynamics is unnecessary.*

4. *We have shown that an exact Lagrangian description of a system of relativistic charged particles with a common laboratory time may be obtained with no independent field variables but higher derivatives of velocities of the particles instead.*

5. *We have demonstrated that the appearance of advanced interaction in the Lagrangian description of a system of relativistic charged particles (the Lagrangian of which does not contain advanced interactions) is the direct result of the simultaneous expansions for potentials. An intrinsic relation between retarded and advanced fields of two interacting particles have been obtained by means of Green function expansion.*

6. *It appears to be possible to obtain the Hamiltonian formalism for a system of relativistic charged particle with a universal laboratory time for all particles by using the simultaneous expansions for potentials in the equation*

$$\frac{d}{dt}\left[\bar{p} + \frac{e}{c}\bar{A}\right] = -e\,\bar{\nabla}\left[\phi - (\bar{\beta}\cdot\bar{A})\right]$$

but this remains a problem for future research.

Appendix A

A. Incorrect And Correct Retarded Green Functions.

Let us start with the usual "*incorrect*" retarded Green function " G_r^u " given in (1.9a)

$$G_r^u(R,\tau) = 2c\theta(\tau)\delta\left[\tau^2 - \frac{R^2}{c^2}\right] = \frac{\theta(\tau)}{R}\left[\delta(\tau - \frac{R}{c}) + \delta(\tau + \frac{R}{c})\right]$$

and the Green function equation in (1.5)

$$\Box G(R,\tau) = \left[\nabla^2 - \partial^2/(c\partial t)^2\right] G(R,\tau) = -4\pi\delta(\bar{R})\delta(\tau)$$

The above Green function can be written as

$$G_r^u(R,\tau) = \theta(\tau)\frac{\mathcal{E}(\sqrt{\bar{R}^2})}{\sqrt{\bar{R}^2}}\left[\delta\left(\tau - \frac{\sqrt{\bar{R}^2}}{c}\right) + \delta\left(\tau + \frac{\sqrt{R^2}}{c}\right)\right]$$

where $\mathcal{E}(R) = R/|R|$

Now Let us apply the d'Alembert's operator to G_r^u:

$$\bar{\nabla}G_r^u = -\theta(\tau)\,\mathcal{E}(\sqrt{\bar{R}^2})\left\{\frac{1}{(\bar{R}^2)^{3/2}}\left[\delta\left(\tau - \frac{\sqrt{\bar{R}^2}}{c}\right) + \delta\left(\tau + \frac{\sqrt{\bar{R}^2}}{c}\right)\right]\right.$$

$$\left. + \frac{1}{c\,\bar{R}^2}\left[\delta'\left(\tau - \frac{\sqrt{\bar{R}^2}}{c}\right) - \delta'\left(\tau + \frac{\sqrt{\bar{R}^2}}{c}\right)\right]\right\}\bar{R}$$

We have used the relations $d\mathcal{E}(R)/dR = 2\delta(R)$ and $R\delta(R) = 0$.

Taking the divergence of this expression,

$$\nabla^2 G_r^u = -2\theta(\tau)\left\{\frac{1}{\bar{R}^2}\left[\delta(\tau - \frac{R}{c}) + \delta(\tau + \frac{R}{c})\right] + \frac{1}{cR}\left[\delta'(\tau - \frac{R}{c}) - \delta'(\tau + \frac{R}{c})\right]\right\}$$

$$+ \frac{\theta(\tau)}{Rc^2}\left[\delta''(\tau - \frac{R}{c}) + \delta''(\tau + \frac{R}{c})\right]$$

with $\quad \theta(\tau)\delta(R)\delta(\tau \pm \frac{R}{c}) = \theta(\tau)\delta(R)\delta(\tau) = \frac{1}{2}\,\delta(R)\delta(\tau),$

$$\theta(\tau)\delta'(\tau \pm \frac{R}{c}) = 0, \;\; \delta'(\tau) = -\delta(\tau)/\tau \;\; \text{and} \;\; \delta(\bar{R}) = \delta(R)/2\pi R^2.$$

The expression for $\nabla^2 G_r^u$ becomes

$$\nabla^2 G_r^u = -4\pi\delta(\bar{R})\delta(\tau)(1 - \frac{R}{c\tau}) + \frac{\theta(\tau)}{Rc^2}\left[\delta''(\tau - \frac{R}{c}) + \delta''(\tau + \frac{R}{c})\right] \tag{A.1}$$

It remains to differentiate G_r^u twice with respect to time t.

Since $\tau = t - t'$, $\partial G_r^u(R,\tau)/\partial t = \partial G_r^u(R,\tau)/\partial\tau$

$$\frac{\partial}{\partial t} G_r^u = \frac{\delta(\tau)}{R}\left[\delta(\tau - \frac{R}{c}) + \delta(\tau + \frac{R}{c})\right] + \frac{\theta(\tau)}{Rc^2}\left[\delta'(\tau - \frac{R}{c}) + \delta'(\tau + \frac{R}{c})\right]$$

where $\partial\theta(\tau)/\partial t = \delta(\tau)$.

again with $\delta(\tau)\delta(\tau - \frac{R}{c}) = \delta'(\tau)\delta(\tau - \frac{R}{c}) = c\delta(\tau)\delta(R)$ and $R\delta(R) = 0$.

we have

$$\frac{\partial}{\partial t} G_r^u = \frac{\delta(\tau)}{R}\left[\delta'(\tau - \frac{R}{c}) + \delta'(\tau + \frac{R}{c})\right]$$

Differentiating this expression once more with respect ot the same variable and using the above relations for δ – functions we get

$$\frac{1}{c^2}\frac{\partial^2 G_r^u}{\partial t^2} = \frac{\theta(\tau)}{Rc^2}\left[\delta''(\tau - \frac{R}{c}) + \delta''(\tau + \frac{R}{c})\right] \tag{A.2}$$

Hence, the results obtained in (A.1) and (A.2) gives

$$\Box G_r^u = -\,4\pi\delta(\bar{R})\delta(\tau)(1 - \frac{R}{c\tau}) \tag{A.3}$$

Thus the commonly used retarded "*Green function*" on substituting it into the equation it supposed to satisfy gives an expression different from (1.5). That clearly shows its failure of being a solution for (1.5) at all points in space. Particularly it is out of consideration for the point at R = 0. Note also that due to the product of delta functions in (A.3) the ratio $R/c\tau$ is indeterminate.

Let us check our correct retarded Green function given in (1.7a):

$$G_r(R,\tau) = \frac{\theta(\tau)}{R}\left[\delta(\tau - \frac{R}{c}) - \delta(\tau + \frac{R}{c})\right] \qquad (1.7a)$$

We may write this Green function in an equivalent form as

$$G_r(R,\tau) = \frac{\theta(\tau)}{\sqrt{\bar{R}^2}}\left[\delta\left[\tau - \frac{\sqrt{\bar{R}^2}}{c}\right] - \delta\left[\tau + \frac{\sqrt{\bar{R}^2}}{c}\right]\right] \qquad (A.4)$$

For $\sqrt{\bar{R}^2} = R$, eq. (A.4) is the same as (1.7a) and when $\sqrt{\bar{R}^2} = -R$ the delta functions will interachange their argument but the whole expression in (A.4) remains unchanged. We can use the relations given for delta functions above to get the following results of differentation.

$$\bar{\nabla}G_r = -\theta(\tau)\left\{\frac{1}{(\bar{R}^2)^{3/2}}\left[\delta\left[\tau - \frac{\sqrt{\bar{R}^2}}{c}\right] - \delta\left[\tau + \frac{\sqrt{\bar{R}^2}}{c}\right]\right]\right.$$

$$\left. + \frac{1}{c\,\bar{R}^2}\left[\delta'\left[\tau - \frac{\sqrt{\bar{R}^2}}{c}\right] + \delta'\left[\tau + \frac{\sqrt{\bar{R}^2}}{c}\right]\right]\right\}\bar{R}$$

then

$$\nabla^2 G_r = \frac{\theta(\tau)}{Rc^2}\left[\delta''(\tau - \frac{R}{c}) - \delta''(\tau + \frac{R}{c})\right] \qquad (A.5)$$

and

$$\frac{\partial}{\partial t}G_r = \frac{\theta(\tau)}{R}\left[\delta'(\tau - \frac{R}{c}) - \delta'(\tau + \frac{R}{c})\right]$$

$$\frac{1}{c^2}\frac{\partial^2 G_r}{\partial t^2} = 4\pi\delta(\bar{R})\delta(\tau) + \frac{\theta(\tau)}{Rc^2}\left[\delta''(\tau - \frac{R}{c}) - \delta''(\tau + \frac{R}{c})\right] \qquad (A.6)$$

Hence, with (A.5) and (A.6), we have

$$\Box G_r = -4\pi\delta(\bar{R})\delta(\tau) ,$$

Which is Eq. (1.5) identifying the correct retarded function to be G_r given by (1.7a).

Appendix B

B. Lagrange's theorem

Let f(z) and g(z) be functions of z analytic on and inside a contour c surrounding the point z = t and let ϵ satisfy the inequality

$$|\epsilon g(z)| < |z-t|$$

for all z on the perimeter of c. then the equation

$$t' = t + \epsilon g(t')$$

regarded as an equation for t' has precisely one root in the interior or c and further any function analytic on and inside c can be expanded as a power series in ϵ in the form

$$f(t') = f(t) + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \frac{d^{n-1}}{dt^{n-1}} \left\{ f'(t)\ [g(t)]^n \right\} \tag{B.1}$$

This expansion is known as the Lagrange expansion.

To derive the Lagrange expansion (B.1), we may write f(t') as

$$f(t') = \frac{1}{2\pi i} \int_c f(\omega) \frac{(1 - \epsilon g'(\omega))}{\omega - t - \epsilon g(\omega)}\, d\omega \tag{B.2}$$

where c is the contour mentioned above.

with $$\frac{1}{\omega - t - \epsilon g(\omega)} = \sum_{n=0}^{\infty} \frac{[\epsilon g(\omega)]^n}{(\omega - t)^{n+1}}$$

the integral in (B.2) becomes

$$f(t') = \sum_{n=0}^{\infty} \frac{\epsilon^n}{2\pi i} \int_c f(\omega) \frac{[g(\omega)]^n}{(\omega - t)^{n+1}}\, d\omega - \sum_{n=1}^{\infty} \frac{\epsilon^n}{2\pi i} \int_c f(\omega) \frac{[g(\omega)]^{n-1} g'(\omega)}{(\omega - t)^n}\, d\omega \tag{B.3}$$

The integrand of the second integral in (B.3) can be written as

$$\frac{f(\omega)[g(\omega)]^{n-1}\, g'(\omega)}{(w - t)^{n}} = \frac{1}{n}\, f(\omega)\, \frac{d}{d\omega} \left[\frac{g(\omega)}{(\omega - t)}\right]^{n} + \frac{f(\omega)\ [g(\omega)]^{n}}{(\omega - t)^{n+1}}$$

Substituting this into (B.3) and integrating by parts

$$f(t') \;=\; f(t) \;+ \sum_{n=1}^{\infty} \frac{\epsilon^{n}}{n!}\ \frac{1}{2\pi i} \int_{c} \frac{f'(\omega)\ [g(\omega)]^{n}}{(\omega - t)^{n}} d\omega$$

With the help of Cauchy's integral formula this expression can be written as in (B.1)

$$f(t') \;=\; f(t) \;+ \sum_{n=1}^{\infty} \frac{\epsilon^{n}}{n!}\ \frac{d^{n-1}}{dt^{n-1}} \left\{ f'(t)\ [g(t)]^{n} \right\}.$$

The retarded Lienard-Wiechert potentials with equation (1.12) can be examples of $f(t')$. In this case we have from (1.12) $\epsilon = (-\frac{1}{c})$ and $g(t) = R(t)$.

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