SOLUTION METHODS OF MONOTONE INCLUSION PROBLEMS, EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS IN BANACH SPACES.

By

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DECLARATION

I declare that this dissertation has been composed by me and that no part of the thesis has formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship or any other similar title to me.

(**SOLOMON BEKELE ZEGEYE**)
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CERTIFICATE

This is to certify that this dissertation is a genuine record of the research work done by Mr. Solomon Bekele Zegeye in the Department of Mathematics, Addis Ababa University under our supervision.

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Dedicated to:

My mother Sheworki Aber Belhu

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Abstract

In this dissertation, we discuss four fixed point approximation methods that can be applied to solve optimization problems, differential equations, variational inequalities and equilibrium problems.

In the first main result of the dissertation, we propose an inertial algorithm for solving split equality of monotone inclusion and f-fixed point of Bregman relatively f-nonexpansive mapping problems in reflexive real Banach spaces and established strong convergence theorems for the algorithm.

Secondly, we establish a strong convergence theorem for approximating a common element of sets of solutions of a finite family of generalized mixed equilibrium problem, sets of semi-fixed points of a finite family of continuous semi-pseudocontractive mappings and sets of solutions of a finite family of variational inequality for a finite family of monotone and L-Lipschitz mappings in Banach spaces.

Thirdly, we constructed and proved a strong convergence of an algorithm for approximating a common element of the set of solutions of a finite family of generalized mixed equilibrium problems, the set of f-fixed points of a finite family of f-pseudocontractive mappings

and the set of solutions of a finite family of variational inequality problems for Lipschitz monotone mappings in real reflexive Banach spaces.

In the fourth main result of the dissertation, we introduce an iterative process which converges strongly to a common point of sets of solutions of a finite family of generalized equilibrium problems, fixed points of a finite family of continuous asymptotically quasi-ϕ-nonexpansive mapping in intermediate sense, and zeros of a finite family of γ-inverse strongly monotone operators in uniformly convex and uniformly smooth real Banach space.

We give numerical examples to demonstrate the behavior of the convergence of the algorithms analysed in each of the four main results of the thesis.

Chapter 1

Introduction

1.1 Background of the Study

Fixed point theory is a beautiful mixture of analysis, topology, and geometry. Fixed point theory, as an important branch of nonlinear analysis theory, has played very important roles in many different fields. We can find a lot of demonstrations in many areas such as optimization theory, differential equations, equilibrium theory, economics theory, image recovery, game theory, mechanics, quantum physics, control theory, and so forth. In the last century, fixed point theorems are developed for single-valued or multi-valued mappings of many types of spaces. The topic of approximation of fixed points of mappings is as useful as existence theorems for applied mathematics. Approximation methods can also be applied to prove the solvability of optimization problems, differential equations, variational inequalities and equilibrium problems (for more details refer [10, 13, 35, 38, 39, 42, 51, 68]).

In particular, the existence theory for nonlinear equations of evolution; that is, equations of the form:

$$\begin{cases} x'(t) + Bx(t) = V(x(t)), \quad t \geq 0, \\ x(0) = x_0 \end{cases}, \tag{1.1.1}$$

at equilibrium state, and setting $V = 0$ in equation (1.1.1), we obtain the following equation:

$$Bx = 0. \tag{1.1.2}$$

In many cases, where the mapping B is accretive, solutions of equation (1.1.2), represent the equilibrium state of the system described by equation (1.1.1). For solving equation (1.1.2), Browder [19] introduced a self mapping, $S =: I - B$, on a real Banach space, which he called a pseudocontractive mapping. Therefore, the fixed point of S is a solution of $Bx = 0$. Approximating zeros of accretive mappings is equivalent to approximating fixed points of pseudocontractive mappings, by assuming existence of such zeros. For earlier and more recent results on the approximation of fixed points of pseudocontractive mappings refer to [20, 27, 31, 33, 58, 59, 69, 103, 101].

Throughout this dissertation, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. We also assume that E is a real Banach space, E^* is the dual space of E, C is a nonempty closed convex subset of E, and $\langle\ .\ ,.\rangle$ is the dual pairing between E and E^*.

Let $g : E \to \mathbb{R} \cap \{+\infty\}$ be a proper convex function. The sub-differential of g, $\partial g : E \to 2^{E^*}$, is defined, for each $x \in E$, by $\partial g(x) = \{x^* \in E^* : g(y) - g(x) \geq \langle y - x, x^* \rangle, \forall y \in E\}$. It is shown in [**?**, 34] that ∂g is a monotone mapping, and that $0 \in \partial g(u)$ if and only if u is a minimizer of the function g. In particular, setting the sub-differential equivalently as B, where B is monotone, we have the fact that, $0 \in B(u)$, which reduces to $Bu = 0$, for the case where B is single-valued. Therefore, approximating zeros of such monotone mappings is equivalent to finding a minimizer of some convex function.

It is obvious that the fixed point technique introduced by Browder [19] for approximating zeros of accretive mappings, is not applicable when B from a real Banach space to its dual space is monotone. Hence, there is the need to develop techniques for approximating zeros of monotone mappings. This motivated the study of developing techniques for approximating zeros of monotone mappings in arbitrary normed linear spaces. In addition, a new notation of fixed points for mappings from a normed linear space to its dual space called semi-fixed points. The concept of a semi-fixed point was introduced by Zegeye [99] and has recently been studied by several authors, for example, [30] and [106]. Approximation methods for finding semi-fixed points of nonlinear mappings from C into E^* are very active area of researchs.

On the other hand, many mathematical models, originating in economics, physics, engineering, and statistics can be formulated as *generalized mixed equilibrium problem* (Ceng and Yao [24]) which is to find $x \in C$ such that

$$H(x,y) := F(x,y) + \varphi(y) - \varphi(x) + \langle y - x, Bx \rangle \geq 0, \forall y \in C. \tag{1.1.3}$$

where $F : C \times C \to \mathbb{R}$ is a bifunction, $\varphi : C \to \mathbb{R}$ is a real valued function, and $B : C \to E^*$ is a nonlinear mapping. The generalized mixed equilibrium problem is quite general it includes and unifies a wide class of problems, such as generalized mixed equilibrium problem, equilibrium problem, mixed varational inequality problem, varational inequality problem, optimization problem, saddle point problem, Nash equilibrium in non-cooperative game.

Moreover, constructing iterative algorithms for finding a common element of the set of fixed points of nonlinear mappings, the set of zeros of monotone mappings, set of solution of variational inequality and the set of solutions of equilibrium problem in different spaces have been also extensively studied by many authors due to its direct connection with applied sciences.

Let $f : E \to \mathbb{R}$ be a convex and smooth function and $g : E \to \mathbb{R}$ be a convex and lower semicontinuous function. Consider the following minimization problem:

$$\min_{x \in E} \{f(x) + g(x)\}. \tag{1.1.4}$$

By Fermat's rule, problem (1.1.4) is equivalent to the problem of finding a point $p \in E$ such that

$$0 \in (\nabla f + \partial g)(p), \tag{1.1.5}$$

where ∇f is the gradient of f and ∂g is the subdifferential of g.

The general form of problem (1.1.4) is called *monotone inclusion problem* which is to find an element $x \in E$ such that

$$0 \in (A + B)x, \tag{1.1.6}$$

where $A : E \to E^*$ is a monotone mapping and $B : E \to 2^{E^*}$ is a maximal monotone mapping.

This problem is also connected with fixed point problem. Thus, Taiwo et al. [84] introduced and studied a new generalization of the monotone inclusion problem in Hilbert spaces which is called the *split equality of monotone inclusions and fixed point problems*. More specifically, split equality of monotone inclusions and fixed point problems are applicable in the fields of machine learning, statistical regression, image processing and signal recovery(see, [46], [87], [88], [89], [90]). In addition, the problem includes the core of many mathematical problems, as special cases, such as: common solutions of monotone inclusion and fixed point problems, split equality monotone inclusion problem, split equality fixed point problem and many important optimization problems such as, split feasibility problems, split minimization problems, split equilibrium problems, split saddle-point problems (see for example, [45], [46], [79], [83], [84], [85], [87]). Split equality of monotone inclusions and fixed point problems are one of active areas of research in nonlinear analysis at present.

Moreover, to speed up the convergence of iterative algorithms has always been of great importance. In 1964, Polyak [61] proposed an *inertial algorithm* which can be seen as a discrete version of a second order time dynamical system to speed up convergence rate of smooth convex minimization problem. The main idea of this method is to make use of two previous iterates in order to update the next iterate, which results in speeding up the algorithm's convergence.

Our purpose in this dissertation is, therefore, to introduce and study iterative algorithms for approximating: a common point of sets of solutions of a finite family of generalized equilibrium problems, fixed points of a finite family of continuous asymptotically quasi-ϕ-nonexpansive mapping in intermediate sense, and zeros of a finite family of γ-inverse strongly monotone operators in uniformly convex and uniformly smooth real Banach spaces; a common element of sets of solutions of a finite family of generalized mixed equilibrium problem, sets of semi-fixed points of a finite family of continuous semi-pseudocontractive mappings and sets of solutions of a finite family of variational inequality for a finite family of monotone and L-Lipschitz mappings in Banach spaces; a common element of the set of solutions of a finite family of generalized mixed equilibrium problems, the set of f-fixed points of a finite family of f-pseudocontractive mappings and the set of solution of a finite family of variational inequality problems for Lipschitz monotone mappings in real reflexive Banach spaces and an for solving split equality of monotone inclusion and f-fixed point of Bregman relatively f-nonexpansive mapping problems in reflexive real Banach spaces and also using inertial method to speed up algorithm.

1.2 Preliminaries

In this section, we shall present some important definitions and preliminary results which will play a crucial role in the subsequent chapters. They can be found in any standard functional analysis book such as Alber and Ryazantseva[3], Chidume [?], Khamsi and Kirk [50].

Throughout this dissertation, we denote the strong convergence and weak convergence of a sequence $\{x_n\}$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Recall that the unit sphere in E is given by $S_E = \{x \in E : \|x\| = 1\}$.

1.2.1 Reflexivity and Weak Convergence

Definition 1.2.1. If E_1 and E_2 are real Banach spaces with duals E_1^* and E_2^*, respectively and $S : E_1 \to E_2$ is a bounded linear mapping, then the adjoint S^* is a linear mapping $S^* : E_2^* \to E_1^*$ satisfying the equation

$$\langle x, S^* y^* \rangle = \langle Sx, y^* \rangle \text{ for all } x \in E_1, y^* \in E_2^*.$$

The dual space of E^* is called the bidual of E and is denoted by E^{**}. Note that E^{**} is a Banach space.

Let $x \in E$ be given. A function $f_x : E^* \to \mathbb{R}$ defined by

$$f_x(j) = \langle x, j \rangle \text{ for all } j \in E^*.$$

Then, since j is linear bounded functional on E,

$$|f_x(j)| = |\langle x, j \rangle| \leq \|j\|\|x\| \text{ for all } j \in E^*,$$

and f_x is linear functional. Furthermore, as result of the Hahn-Banach theorem, $\|f_x\| = \|x\|$.

Definition 1.2.2. An injection mapping $\nu : E \to E^{**}$ is called canonical embedding if the following properties holds:

a) ν is linear: $\nu(\alpha x + \beta y) = \alpha\nu(x) + \beta\nu(y)$ for all $x, y \in E$, $\alpha, \beta \in \mathbb{R}$;

b) ν is isometry: $\|\nu(x)\| = \|x\|$ forall $x \in E$.

Generally, the canonical embedding mapping ν from E into E^{**} is not surjective. It is well known that a mapping $J : E \to E^{**}$ defined by $J(x) = f_x$ is canonical embedding. If J is surjective, then we have an important class of Banach spaces.

Definition 1.2.3. A Banach space E is said to be reflexive if the canonical embedding mapping J is surjective, that is, $J(E) = E^{**}$.

Hilbert spaces, L_p, ℓ_p and $W_p^m(\Omega)$ spaces for $1 < p < \infty$) are examples of reflexive Banach space, where $W_p^m(\Omega)$ denotes the usual Sobolev space (see, [17]).

Definition 1.2.4. A sequence $\{x_n\} \subset E$ is called weakly convergent sequence which converges weakly to $x \in E$ if for all $x^* \in E^*$

$$\langle x, x^* \rangle = \lim_{n\to\infty} \langle x_n, x^* \rangle.$$

Proposition 1.2.1 (Eberlein-Smulian)**.** *Every bounded sequence in reflexive Banach space E admits a weakly convergent subsequence.*

1.2.2 Geometry of Banach Spaces

Definition 1.2.5 ([2], [28])**.** A Banach space E is said to be strictly convex if for all $x, y \in S_E$ with $x \neq y$ imply $\|\lambda x + (1-\lambda)y\| < 1$ for all $\lambda \in (0,1)$.

Definition 1.2.6 ([2], [28])**.** A Banach space E is called uniformly convex if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\|$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$, we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Remark 1.2.1. Every uniformly convex is strictly convex Banach space. But, the converse is true if the dimension of the Banach space is finite.

The following is an interesting property of uniformly convex spaces which is due to Milman and Pettis [2].

Proposition 1.2.2. *Every uniformly convex Banach space is reflexive.*

Lemma 1.2.3. *[95] If E is a uniformly convex real Banach space, then there exists a continuous, strictly increasing and convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$, such that for all $x, y \in B_r(0)$ and for any $p > 1$, $\alpha \in [0,1]$, we have*

$$\|\alpha x + (1-\alpha)y\|^p \leq \alpha\|x\|^p + (1-\alpha)\|y\|^p - W_p(\alpha)g(\|x-y\|),$$

where $W_p(\alpha) = \alpha^p(1-\alpha) + \alpha(1-\alpha)^p$ and $B_r(0) = \{x \in E : \|x\| \leq 0\}$.

Definition 1.2.7. The modulus of convexity of E is a function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) = \inf\{1 - \|\frac{x+y}{2}\| : x, y \in S_E; \|x-y\| \geq \epsilon\}.$$

It is well known that E is uniformly convex if and only if $\delta_E(\epsilon) > 0$, for all $\in (0, 2]$.

Definition 1.2.8. Let $p > 1$ be a real number. Then, E is called p-uniformly convex if there is a constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in (0, 2]$.

It is known that a p-uniformly convex Banach space is uniformly convex.

Definition 1.2.9 ([2], [28])**.** The Banach space E is said to be smooth if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t},$$

exists for each $x, y \in S_E$. It is also said to be uniformly smooth if the above limit is attained uniformly for $x, y \in S_E$.

Definition 1.2.10. The modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(\lambda) = \sup\left\{ \frac{\|x + \lambda y\| + \|x - \lambda y\|}{2} - 1 : x, y \in S_E \right\}.$$

It is obvious that a space E is uniformly smooth if and only if $\lim_{\lambda\to 0^+} \frac{\rho_E(\lambda)}{\lambda} = 0$.

Definition 1.2.11 ([2], [28])**.** Let E be a real Banach space with its dual space E^*. The generalized duality mapping $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|, \|f\| = \|x\|^{p-1}\}, \text{ for all } x \in E.$$

If $p = 2$, then we write $J_2 = J$ and call it normalized duality mapping and $J_p(x) = \|x\|^{p-2}J(x)$. One can easily see the following facts:

a) For each $x \in E$, $J_p(x)$ is a nonempty, closed and convex subset of E^* for each $x \in E$.

b) J_p is monotone mapping. Moreover, J_p is strictly monotone if E is strictly convex Banach space.

c) If E^* is strictly convex Banach space, then J_p is single valued mapping.

d) If E is reflexive and E^* is strictly convex Banach spaces, then J_p is single valued monotone and demicontinuous mapping.

Proposition 1.2.4 ([2], [28])**.** *If E is a uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.*

It is well known that E is uniformly smooth if and only if E^* is uniformly convex (see, e.g., [2]).

1.2.3 Convex Functions and Bregman Distance

In this subsection various important definitions and basic concepts which are used in the subsequent chapters.

Let $f : E \to (-\infty, +\infty]$ be a function. We denote the domain of f by $domf$, that is, $domf = \{x \in E : f(x) < \infty\}$. A function f is said to be *proper* if $domf \neq \emptyset$. It is said to be *lower semi-continuous* if the set $\{x \in E : f(x) \leq r\}$ is closed for all $r \in \mathbb{R}$.

Definition 1.2.12. Let $f : E \to (-\infty, +\infty]$ be a function. Then,

a) f is called *convex* if $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$ for all $x, y \in E$ and $\alpha \in [0, 1]$.

b) f is called *uniformly convex* if there exists a continuous increasing function $\psi : [0, +\infty) \to \mathbb{R}$, $\psi(0) = 0$, such that $f(tx+(1-t)y) \leq tf(x)+(1-t)f(y) - t(1-t)\psi(\|x-y\|)$, for all $x, y \in domf$. The function ψ is called a *modulus of convexity* of f. It is called *strongly convex* if f is uniformly convex with the modulus of convexity $\psi(t) = ct^2$, $c > 0$.

(3) f is said to be *strongly coercive* if $\lim_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} = +\infty$.

Definition 1.2.13. Let $f : E \to (-\infty, +\infty]$ be function. The *Fenchel conjugate* of f is a function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x, x^* \rangle - f(x) : x \in E\}.$$

Definition 1.2.14. Let $f : E \to (-\infty, +\infty]$ be convex function. The *subdifferential* of f at x is defined by

$$\partial f(x) = \{x^* \in E^* : f(y) - f(x) \geq \langle y - x, x^* \rangle, \forall y \in E\}.$$

For any $x \in int(domf)$ and any $y \in E$, we denote by $f^0(x, y)$ the right-hand derivative of f at x in the direction of y, that is,

$$f^0(x, y) = \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{1.2.1}$$

The function f is called *Gâteaux differentiable* at x if the limit (1.2.1) exists for any $y \in E$. In this case, the gradient of f at x, $\nabla f(x)$, coincides with $f^0(x, y)$ for all $y \in E$. It is called *Gâteaux differentiable* if it is Gâteaux differentiable at every point $x \in int(domf)$. We note that if the subdifferential of f is single-valued, then $\partial f = \nabla f$. The function f is said to be *Fréchet differentiable* at x if the limit (1.2.1) is attained uniformly for every $y \in E$ with $\|y\| = 1$ and f is said to be *uniformly Fréchet differentiable* on a subset C of E if the limit (1.2.1) is attained uniformly for $x \in C$ and $\|y\| = 1$.

Proposition 1.2.5. *[60] If f is a uniformly convex and Gâteaux differentiable function in $(dom f)$ with modulus of convexity ψ, then*

$$\langle x - y, \nabla f(x) - \nabla f(y)\rangle \geq 2\psi(\|x - y\|), \forall x, y \in dom f,$$

or equivalently,

$$f(y) \geq f(x) + \langle y - x, \nabla f(x)\rangle + \psi(\|x - y\|), \forall x, y \in dom f.$$

Proposition 1.2.6. *[60] If a function f is strongly convex with constant $\mu > 0$ and Gâteaux differentiable in $(dom f)$, then*

$$\langle x - y, \nabla f(x) - \nabla f(y)\rangle \geq \mu\|x - y\|^2, \forall x, y \in dom f,$$

or equivalently,

$$f(y) \geq f(x) + \langle y - x, \nabla f(x)\rangle + \frac{\mu}{2}\|x - y\|^2, \forall x, y \in dom f.$$

Proposition 1.2.7 ([108])**.** *If f is strongly convex with constant μ, then f^* has a Lipschitz gradient with parameter $\frac{1}{\mu}$ and if f has a Lipschitz gradient with parameter L, then f^* is strongly convex with parameter $\frac{1}{L}$.*

Remark 1.2.2 ([60]). If E is a smooth and strictly convex Banach space, the function $f(x) = \|x\|^2, \forall x \in E$ is strongly convex with constant $\mu \in (0, 1]$.

Definition 1.2.15. A function $f : E \to (-\infty, +\infty]$ is called Legendre if it satisfies the following two properties:

(L1) the interior of the domain of f, $int(dom f)$, is nonempty, f is Gâteaux differentiable and $dom(\nabla f) = int(dom f)$;

(L2) the interior of the domain of f^*, $int(dom f^*)$, is nonempty, f^* is Gâteaux differentiable and $dom(\nabla f^*) = int(dom f^*)$;

Example 1.2.1. *(see, Bauschke [7] and Bauschke et al. [8]) One of the important and interesting Legendre function in a smooth and strictly convex Banach space is $f(x) = \frac{1}{p}\|x\|^p$ $(1 < p < +\infty)$ with its conjugate function $f^*(x^*) = \frac{1}{q}\|x^*\|^q$ $(1 < q < +\infty)$, where $\frac{1}{p} + \frac{1}{q} = 1$. In this case, the gradient of f, ∇f, coincides with the generalized duality mapping, J_p, of E, that is, $\nabla f = J_p$.*

Note that $\nabla f = I$ if $E = H$, a real Hilbert space and $f(x) = \|x\|^2$, where I is the identity mapping on H.

Lemma 1.2.8. *(see, Bonnans and Shapiro [15]) If the function f is a Legendre function and E is a reflexive Banach space, then $\nabla f^* = (\nabla f)^{-1}$.*

Definition 1.2.16. (see, Bregman [16]) Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable convex function. The function $D_f : domf \times int(domf) \to [0, +\infty)$, defined by

$$D_f(y, x) = f(y) - f(x) - \langle y - x, \nabla f(x)\rangle, \forall x, y \in E. \qquad (1.2.2)$$

is called the *Bregman distance* with respect to f.

The Bregman distance has the following two important properties (see, Reich and Sabach [70]), called the *three-point identity*: for any $x \in domf$ and $y, z \in int(domf)$,

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle x - y, \nabla f(z) - \nabla f(y)\rangle, \qquad (1.2.3)$$

and the *four-point identity*: for any $y, w \in domf$ and $x, z \in int(domf)$,

$$D_f(y, x) - D_f(y, z) - D_f(w, x) + D_f(w, z) = \langle y - w, \nabla f(z) - \nabla f(x)\rangle. \quad (1.2.4)$$

Lemma 1.2.9. *(Phelps [60]) If $f : E \to (-\infty, +\infty]$ is a proper, lower semi-continuous and convex function, then $f^* : E^* \to (-\infty, +\infty]$ is a proper, weak* lower semi-continuous and convex function and for any $x \in E$, $\{y_k\}_{k=1}^N \subseteq E$ and $\{c_k\}_{k=1}^N \subseteq (0, 1)$ with $\sum_{k=1}^N c_k = 1$ the following holds:*

$$D_f\left(x, \nabla f^*\left(\sum_{k=1}^N c_k \nabla f(y_k)\right)\right) \leq \sum_{k=1}^N c_k D_f(x, y_k)\right). \qquad (1.2.5)$$

If E is a smooth and strictly convex Banach space and $f(x) = \|x\|^2$ for all $x \in E$, then we have that $\nabla f = J$, where J is the normalized duality mapping from E into 2^{E^*} and the Bregman distance with respect to f, D_f, reduces to the Lyapunov functional $\phi : E \times E \to [0, +\infty)$ defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2, \forall x, y \in E. \qquad (1.2.6)$$

It is obvious from the definition of a function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \forall x, y \in E, \qquad (1.2.7)$$

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle, \forall x, y, z \in E, \qquad (1.2.8)$$

and

$$\phi(x, J^{-1}(\alpha y + (1-\alpha)z)) \leq \alpha\phi(x, y) + (1-\alpha)\phi(x, z), \qquad (1.2.9)$$

for all $x, y, z \in E$ and $\alpha \in (0, 1)$. Observe that, in a Hilbert space H, the equality reduces to $\phi(x, y) = \|x - y\|^2, x, y \in H$.

Let $f : E \to \mathbb{R}$ be a Legendre function. We make use of the function $V_f : E \times E^* \to \mathbb{R}$ defined by

$$V_f(x, x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*), \text{ for all } x \in E \text{ and } x^* \in E^*.$$

We note that V_f is a nonnegative function which satisfies (see, Senakka and Cholamjiak [77])

$$V_f(x, x^*) = D_f(x, \nabla f^*(x^*)) \text{ for all } x \in E \text{ and } x^* \in E^*, \qquad (1.2.10)$$

and

$$V_f(x, x^*) + \langle \nabla f^*(x^*) - x, y^* \rangle \leq V_f(x, x^* + y^*), \text{ for all } x \in E \text{ and } x^*, y^* \in E^*. \qquad (1.2.11)$$

Definition 1.2.17. Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable convex function. The Bregman projection of $x \in int(domf)$ onto the nonempty, closed and convex set $C \subset domf$ is the unique vector $P_C^f(x) \in C$ satisfying

$$D_f(P_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}.$$

If E is strictly convex and smooth Banach space and $f(x) = \|x\|^2$ for all $x \in E$, then the Bregman projection mapping P_C^f reduced to the generalized projection mapping $\Pi_C : E \to C$ which is defined as for any arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$; that is, $\Pi_C y = p$, where p is the solution to the minimization problem

$$\phi(p, x) = \min_{y \in C} \phi(y, x). \qquad (1.2.12)$$

In addition, if $f(x) = \|x\|^2$ for all $x \in E$ and $E = H$, where H is Hilbert space, then P_C^f reduced to a metric projection mapping P_C of H onto C. As we all know, if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and, consequently, it is not available in more general Banach spaces. The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J; see, for example, [2, 34].

1.2.4 Nonlinear Mappings

Definition 1.2.18. A mapping $T : C \to E$ is called asymptotically regular on C if for any bounded subset K of C,

$$\limsup_{n\to\infty}\{\|T^{n+1}x - T^n x\| : x \in K\} = 0.$$

Definition 1.2.19. Let C be a nonempty subset of E. A mapping $T : C \to E$ is said to be Lipschitz if there exists nonnegative real number L such that

$$\|Tx - Ty\| \leq L\|x - y\| \text{ for all } x, y \in C.$$

If $L < 1$, then T is called contraction mapping and it is called nonexpansive if $L = 1$.

Clearly, every Lipschitz mapping is uniformly continuous.

Definition 1.2.20. A point $x \in C$ is said to be a fixed point of a mapping $T : C \to E$ if $Tx = x$. The set of all fixed point of a mapping T is denoted by $F(T)$.

Definition 1.2.21. A mapping $T : C \to E$ is said to be

a) asymptotically regular on C if for any bounded subset K of C'

$$\limsup_{n\to\infty}\{\|T^{n+1}x - T^n x\| : x \in K\} = 0;$$

b) quasi-nonexpansive if $F(T) \neq \emptyset$, and

$$\|Tx - p\| \leq \|x - p\|, \forall x \in C, \forall p \in F(T).$$

c) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subseteq [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$, such that

$$\|T^n x - p\| \leq k_n\|x - p\|, \forall x \in C, \forall p \in F(T), n \in \mathbb{N}.$$

d) asymptotically quasi-nonexpansive in the intermediate sense if $F(T) \neq \emptyset$ and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{p\in F(T), x\in C} (\|T^n x - p\| - \|x - p\|) \leq 0.$$

Remark 1.2.3. a) The class of nonexpansive mappings is a subset of the class of quasi-nonexpansive mappings if the set of fixed points of nonexpansive mappong is nonempty set;

b) The class of quasi-nonexpansive $\subseteq$ the class of asymptotically quasi-nonexpansive $\subseteq$ the class of asymptotically quasi-nonexpansive in the intermediate mappings.

Definition 1.2.22. [71] A point p in C is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$.

Definition 1.2.23. [104] A point p in C is said to be a strong asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges strongly to p such that $\lim_{n\to\infty}(Tx_n - p) = 0$. The set of strong asymptotic fixed points of T will be denoted by $\hat{F}(T)$.

Definition 1.2.24. A mapping $T : C \to E$ is called

a) relatively nonexpansive [22, 26] if $\widetilde{F}(T) = F(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$ for all $x \in C$ and $p \in F(T)$.

b) relatively-ϕ-nonexpansive [22, 26] if $\widetilde{F}(T) = F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

c) relatively weak nonexpansive [104] if $\hat{F}(T) = F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

d) relatively-ϕ-asymptotically nonexpansive [1, 67] if $\widetilde{F}(T) = F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(p, T^n x) \leq k_n \phi(p, x)$ for all $x \in C, p \in F(T)$ and $n \geq 1$.

e) quasi-ϕ-nonexpansive [63] if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C, p \in F(T)$.

f) asymptotically quasi-ϕ-nonexpansive [62, 65, 109] if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(p, T^n x) \leq k_n \phi(p, x)$ for all $x \in C, p \in F(T)$ and $n \geq 1$.

g) accretive if for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that $\langle j(x - y), u - v\rangle \geq 0, \forall u \in Tx, v \in Ty$.

Remark 1.2.4. The class of asymptotically quasi-ϕ-nonexpansive mappings is more general than the class of relatively asymptotically nonexpansive mappings, which requires the restriction: $\widetilde{F}(T) = F(T)$.

Definition 1.2.25 ([68])**.** A mapping $T : C \to E$ is said to be asymptotically quasi-ϕ-nonexpansive in the intermediate sense if $F(T) \neq \emptyset$ and

$$\limsup_{n\to\infty} \sup_{p\in F(T), x\in C} (\phi(p, T^n x) - \phi(p, x)) \leq 0. \qquad (1.2.13)$$

Put

$$\xi_n = \max\{0, \sup_{p\in F(T), x\in C} (\phi(p, T^n x) - \phi(p, x))\}.$$

It follows that $\xi_n \to 0$ as $n \to \infty$. Then (1.2.13) is reduced to the following:

$$\phi(p, T^n x) \leq \phi(p, x) + \xi_n, \forall p \in F(T), \forall x \in C. \qquad (1.2.14)$$

From the definitions, we observe that the class of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense is a generalization of the class of asymptotically quasi-nonexpansive mappings in the intermediate sense in the framework of Banach spaces.

Example 1.2.2. *[56] Let $E = \mathbb{R}$ with the usual norm. For $n \in \mathbb{N}$, we define a mapping T_j on $\mathbb{R}$ by*

$$T_j x = \begin{cases} 0, & \text{if } x \leq \frac{1}{j^2}; \\ \frac{1}{j^2}, & \text{if } x > \frac{1}{j^2}, \end{cases} \qquad \forall x \in \mathbb{R}.$$

Then $\bigcap_{j=1}^{d} F(T_j) = \{0\}$ and $\phi(0, T_j^n x) = |T_j^n x - 0|^2 = 0 \leq |x|^2 = |x - 0|^2 = \phi(0, x)$ and
$\xi_n^j = \max\{0, \sup_{p\in F(T), x\in\mathbb{R}} (\phi(p, T^n x) - \phi(p, x))\} = 0, \forall n \in \mathbb{N}$, *that is T_j is asymptotically quasi-ϕ- nonexpansive mapping in intermediate sense, $j = 1, 2, \cdots, d$. But T_j is not relatively nonexpansive, because, let $\{x_n\}$ be a sequence define by $x_n = \frac{1}{j^2} + \frac{1}{n}$. Then $x_n \to \dfrac{1}{j^2}, x_n - T_j x_n = \dfrac{1}{n} \to 0$ as $n \to \infty$. This implies that $\frac{1}{j^2} \in \widetilde{F}(T_j)$ and $\frac{1}{j^2} \notin F(T_j)$.*

Definition 1.2.26. Let C be a nonempty subset a real Hilbert space H. A mapping $T : C \to H$ is called

a) pseudocontractive if $\langle x - y, Tx - Ty\rangle \leq \|x - y\|^2$ for all $x, y \in C$;

b) k-pseudocontractive if $\langle x - y, Tx - Ty\rangle \leq k\|x - y\|^2$ for all $x, y \in C$.

Definition 1.2.27. A mapping $A : D(A) \subset E \to E^*$, is said to be

a) monotone if for each $x, y \in D(A)$, the following inequality holds:

$$\langle x-y, Ax-Ay\rangle \geq 0;$$

b) γ-inverse strongly monotone if there exists a positive real number γ such that

$$\langle x-y, Ax-Ay\rangle \geq \gamma\|Ax-Ay\|^2;$$

c) semi-pseudocontractive mapping if for each $x, y \in C$, the following inequality holds:

$$\langle x-y, T(x)-T(y)\rangle \leq \langle x-y, J(x)-J(y)\rangle.$$

d) J-nonexpansive mapping if for each $x, y \in C$, the following inequality holds:

$$\|x-y\|\|Tx-Ty\| \leq \langle x-y, J(x)-J(y)\rangle.$$

Hence, every J-nonexpansive mapping is semi-pseudocontractive.

Proposition 1.2.10. *Let $A : D(A) \subset E \to E^*$ be monotone mapping. Then $T := J - A$ is semi-pseudocontractive mapping.*

Proof. Let $A : D(A) \subset E \to E^*$ be monotone mapping. Then, for all $x, y \in D(A)$,

$$\begin{aligned}\langle x-y, T(x)-T(y)\rangle &= \langle x-y, (J(x)-A(x))-(J(y)-A(y))\rangle \\ &= \langle x-y, J(x)-J(y)\rangle - \langle x-y, A(x)-A(y)\rangle \\ &\leq \langle x-y, J(x)-J(y)\rangle.\end{aligned}$$

Definition 1.2.28. [99] A point $p \in E$ is called a *semi-fixed point* (J-fixed point) of mapping $T : E \to E^*$, if there exists $Tp = Jp$, where J is the normalized duality mapping. The set of semi-fixed points of T is denoted by $F_s(T)$, that is , $F_s(T) = \{x^* \in C : Tx^* = Jx^*\}$.

Definition 1.2.29. [106] Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable convex function. A mapping $T : C \to E^*$ is called

a) f-pseudocontractive if for each $x, y \in E$, we have

$$\langle x-y, T(x)-T(y)\rangle \leq \langle x-y, \nabla f(x)-\nabla f(y)\rangle;$$

b) γ-strictly f-pseudocontractive if for all $x, y \in C$, there exists $\gamma > 0$ such that

$$\langle x-y, T(x)-T(y)\rangle \leq \langle x-y, \nabla f(x)-\nabla f(y)\rangle - \gamma\|(\nabla f(x)-\nabla f(y))-(Tx-Ty)\|^2.$$

Definition 1.2.30. A point $p \in C$ is called f-fixed point of T if $Tp = \nabla f(p)$. The set of f-fixed points of T is denoted by $F_f(T)$, that is, $F_f(T) = \{p \in C : Tp = \nabla f(p)\}$.

We remark that if E is smooth and strictly convex and $f(x) = \frac{1}{2}\|x\|^2$ for all $x \in E$, then $\nabla f = J$, where J is the normalized duality mapping from E into 2^{E^*}, and the notion of f-pseudocontractive mapping reduces to the notion of semi-pseudocontractive mapping and f-fixed point of T reduces to semi-fixed point of T. If, in addition, $E = H$, a real Hilbert space, then f-pseudocontractive mapping becomes pseudocontractive mapping. The mapping T is f-pseudocontractive if and only if $A = \nabla f - T$ is monotone and T is strictly f-pseudocontractive if and only if $A = \nabla f - T$ is γ-inverse strongly monotone. In this case, the zero of A corresponds to f-fixed point of T. In fact, if T and ∇f are continuous on E then A is maximal monotone and the set of zeros of A and hence the set of f-fixed points of an f-pseudocontractive mapping T is closed and convex (see, Zegeye and Wega [106]).

Definition 1.2.31. (Wega and Zegeye [93]) A mapping T is called a Bregman relatively f-nonexpansive if

$$D_f(p, \nabla f^*(Tx)) \leq D_f(p, x), \forall x \in E, p \in \widetilde{F_f}(T) \text{ and } F_f(T) = \widetilde{F_f}(T) \neq \emptyset.$$

We remark that if $f(x) = \frac{1}{2}\|x\|^2$, for all $x \in E$, then $\nabla f = J$ and $D_f(y, x) = \phi(y, x)$ for all $x, y \in E$ and hence the Bregman relatively f-nonexpansive mappings reduce to the ϕ-relatively J-nonexpansive mapping. Moreover, f-fixed point and f-asymptotic fixed point of T reduce to semi-fixed point and semi-asymptotic fixed point of T, respectively. If, in addition, $E = H$, a real Hilbert space, then Bregman relatively f-nonexpansive mappings become relatively nonexpansive mapping.

1.3 Generalized Mixed Equilibrium Problem

Let $F : C \times C \to \mathbb{R}$ be a bifunction, $\varphi : C \to \mathbb{R}$ be a real valued function, and $B : C \to E^*$ be a nonlinear mapping. The *Generalized Mixed Equilibrium Problem (GMEP)* (Ceng and Yao [24]) is to find $x \in C$ such that

$$H(x, y) := F(x, y) + \varphi(y) - \varphi(x) + \langle Bx, y - x\rangle \geq 0, \forall y \in C. \tag{1.3.1}$$

The set of solution of (1.3.1) is denoted by $GMEP(F, \varphi, B)$.
If $\varphi \equiv 0$, the problem (1.3.1) reduces to the *Generalized Equilibrium problem (GEP)* (Mouda and Thera [53]) which is to find $x \in C$ such that

$$\overline{H}(x, y) := F(x, y) + \langle Bx, y - x\rangle \geq 0, \forall y \in C. \tag{1.3.2}$$

The set of solutions of (1.3.2) is denoted by $GEP(F, B)$.
If in (1.3.1), we consider $F \equiv 0$, then problem (1.3.1) reduces to finding $x \in C$ such that

$$\varphi(y) - \varphi(x) + \langle Bx, y - x\rangle \geq 0, \forall y \in C, \tag{1.3.3}$$

which is called the *Mixed Variational Inequality of Browder type (MVI)* [18]. The set of solutions to (1.3.3) is denoted by $MVI(C, B, \varphi)$.
If $F \equiv 0$ and $\varphi(y) \equiv 0$ for all $y \in C$, problem (1.3.1) reduces to finding $x \in C$ such that

$$\langle Bx, y - x\rangle \geq 0, \forall y \in C, \tag{1.3.4}$$

which is the classical *Variational Inequality Problem (VIP)*. The set of solutions to (1.3.4) is denoted by $VI(C, B)$.
If in (1.3.2), $B \equiv 0$, then problem (1.3.2) reduces to the *Equilibrium problem (EP)* (Blum and Oettli [14]) which is to find $x \in C$ such that

$$F(x, y) \geq 0, \forall y \in C. \tag{1.3.5}$$

The set of solutions to (1.3.5) is denoted by $EP(F)$.

Some basic facts about generalized mixed equilibrium problem.

Lemma 1.3.1. *[107] Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E. Let be $B : C \to E^*$ be a continuous and monotone mapping, $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function and let $F : C \times C \to \mathbb{R}$ be a bi-function satisfying* ***Condition A****. For $r > 0$ and $x \in E$, define a mapping $T_r^H : E \to C$ as follows:*

$$T_r^H x = \{z \in C : H(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \text{ for all } y \in C\}, \tag{1.3.6}$$

for all $x \in E$, where $H(z, y) := F(z, y) + \varphi(y) - \varphi(z) + \langle y - z, Bz\rangle$. Then, the following hold:

(1) T_r^H is single-valued;

(2) $F(T_r^H) = GMEP(F, \varphi, B)$;

(3) $GMEP(F, \varphi, B)$ is closed and convex;

(4) T_r^H is quasi-ϕ-nonexpansive;

(5) $\phi(p, T_r^H x) + \phi(T_r^H x, x) \leq \phi(p, x), \forall p \in F(T_r^H)$.

Remark 1.3.1. If $\varphi \equiv 0$, then $H = \overline{H}$ and Lemma 1.3.1 hold.

Optimization: Let $\phi : C \to \mathbb{R}$ be a convex and lower semi-continuous function. The minimization problem is to find $x^* \in C$ such that

$$\phi(x^*) \leq \phi(y), \forall y \in C. \quad (1.3.7)$$

Setting $F(x,y) := \phi(y) - \phi(x)$, problem (1.3.7) coincides with (1.3.5).
Saddle Point Problem: Let $\varphi : C_1 \times C_2 \to \mathbb{R}$. Then $x^* = (x_1^*, x_2^*)$ is called saddle point of the function φ if and only if for $x^* = (x_1^*, x_2^*)$,

$$\varphi(x_1^*, y_2) \leq \varphi(y_1, x_2^*), \forall (y_1, y_2) \in C_1 \times C_2. \quad (1.3.8)$$

If $C := C_1 \times C_2$, and $F : C \times C \to \mathbb{R}$ is defined by

$$F((x_1, x_2), (y_1, y_2)) := \varphi(y_1, x_2) - \varphi(x_1, y_2),$$

then $x^* = (x_1^*, x_2^*)$ is a solution of (1.3.5) if and only if $x^* = (x_1^*, x_2^*)$ satisfies (1.3.8).
Nash Equilibrium in Non-cooperative Games: Let I be a finite set of players and let C_i be a strategy set of the i^{th} player, for each $i \in I$. Let $f_i : C := \prod_{i \in I} C_i \to \mathbb{R}$ be a loss function of the i^{th} player depending on the strategies of all players, for all $i \in I$. For $x = (x_i)_{i \in I} \in C$, we find $x_{-i} = (x_j)_{j \in I | j \neq i}$. The point $x^* = (x^*)_{i \in I} \in C$ is called Nash Equilibrium if for $i \in I$, the following holds:

$$f_i(x^*) \leq f_i(x_{-i}^*, y_i), \forall y_i \in C_i, \quad (1.3.9)$$

(that is, no player can reduce his loss by varying his strategy alone). If $F : C \times C \to \mathbb{R}$ given by

$$F(x,y) := \sum_{i \in I} (f_i(x_{-i}, y_i) - f_i(x)),$$

then $x^* \in C$ is a Nash equilibrium if and only if x^* satisfies (1.3.5).

1.4 Statement of the Problem

Several authors have studied algorithms for finding a common element of the set fixed points of nonlinear mappings and the set of solution of equilibrium problems in Banach spaces. In recent times, some authors proposed many iteration algorithms for finding fixed points and common fixed points of the nonexpansive mappings and the generalized equilibrium problem in the framework of Hilbert spaces and Banach spaces [49, 63, 86]. In 2017, Ibiam et. al. [42] also studied the strong convergence result for finding a common element of sets of solutions of a finite family of generalized equilibrium problems, sets of fixed points of a finite family of continuous relatively nonexpansive mappings and sets of zeros of a finite family of γ_i-inverse strongly monotone mappings in 2-uniformly convex and uniformly smooth Banach spaces. However, it is still open to extend the result of Ibiam et. al. [42] to more general nonlinear mappings than continuous relatively nonexpansive mappings and to more general spaces than 2-uniformly convex and uniformly smooth Banach spaces.

On the other hand, approximating zeros of these monotone mappings is equivalent to approximating Semi-fixed points of semi-pseudocontractive mappings, assuming the existence of such zeros, which is also equivalent to finding a minimizer of some convex functions. In recent times, some authors proposed the algorithm converges strongly to a common solution of a variational inequality, an equilibrium problem, and semi-fixed points of a continuous semi-pseudocontractive mapping in the framework of Hilbert spaces and Banach spaces.

In 2019, Shahzad and Zegeye [78] proved the following convergence theorem for a common solution of fixed point, equilibrium and variational inequality problems in Hilbert spaces.

Theorem 1.4.1. *Let C be a nonempty, closed, and convex subset of a real Hilbert spaces H. Let $A : C \to H$ be a Lipschitz monotone mapping with Lipschitz constant $L > 0$, $F : C \times C \to \mathbb{R}$ be a bi-functional satisfying **Condition A**, and $T : C \to H$ be a continuous pseudocontractive mapping with $\mathcal{F} := F(T) \bigcap VI(A, C) \bigcap EP(F) \neq \emptyset$. Let the sequence $\{x_n\}$ be generated by*

$$\begin{cases} u, x_0 \in C, \\ z_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)(\beta y_n + (1 - \beta) u_n), \end{cases} \tag{1.4.1}$$

where P_C is the metric projection from H onto C, $y_n = K^T_{r_n} T^F_{r_n} x_n$, where $T^F_{r_n}, K^T_{r_n} : H \to C$ are mappings defined as follows: for $x \in H$,

$$T^F_{r_n} x = \{z \in C : F(z, y) + \frac{1}{r_n} \langle y - z, z - x \rangle \leq 0, \forall y \in C\}$$

and

$$K^T_{r_n} x = \{z \in C : \langle y - z, Tz \rangle - \frac{1}{r_n} \langle y - z, (1 + r_n) z - x \rangle \leq 0, \forall y \in C\},$$

respectively, $\{r_n\} \subset [a, \infty)$*, for some* $a > 0$*,* $u_n = P_C(x_n - \lambda Az_n)$*,* $\lambda \in [a, b] \subset (0, \frac{1}{L})$ *and* $\{\alpha_n\} \subset (0, c] \subset (0, 1)$ *with* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum \alpha_n = \infty$*. Then, the sequence* $\{x_n\}$ *converges strongly to a point* $P_{\mathcal{F}}u$*.*

Recently, in Banach space settings, Bello and Nnakwe [9] studied a new Halpern-type subgradient extragradient iterative algorithm which converges strongly to a common solution of a variational inequality, an equilibrium problem, and a semi-fixed point of a continuous semi-pseudocontractive mapping in uniformly smooth and 2-uniformly convex real Banach spaces. They proved the following theorem.

Theorem 1.4.2. *Let* E^* *be the dual space of a uniformly smooth and 2-uniformly convex real Banach space* E*. Let* C *be a nonempty, closed, and convex subset of* E*. Let* $A : C \to E^*$ *be a monotone and L-Lipschitz mapping,* $F : C \times C \to \mathbb{R}$ *be a bi-functional satisfying* ***Condition A****, and* $T : E \to E^*$ *be a continuous semi-pseudocontractive mapping with* $\mathcal{F} := F_s(T) \bigcap VI(A, C) \bigcap EP(F) \neq \emptyset$*. Let the sequence* $\{x_n\}$ *be generated by*

$$\begin{cases} x_0 \in C, \\ z_n = \Pi_C J^{-1}(Jx_n - \lambda Ax_n) \\ T_n = \{w \in E : \langle w - z_n, Jx_n - \lambda Ax_n - Jz_n\rangle \leq 0\} \\ x_{n+1} = J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n)[\beta Jv_n + (1 - \beta)Jw_n]), \end{cases} \tag{1.4.2}$$

where $v_n = T^F_{r_n} K^T_{r_n} x_n$ *with* $T^F_{r_n}$ *and* $K^S_{r_n}$ *as the resolvent mappings for* F *and* T*, respectively,* $\{r_n\} \subset [a, \infty)$*, for some* $a > 0$*,* $w_n = \Pi_{T_n} J^{-1}(Jx_n - \lambda Az_n)$*,* $\lambda \in (0, 1)$ *with* $\lambda < \frac{c}{L}$ *and* $\{\alpha_n\} \subset (0, 1)$ *with* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum \alpha_n = \infty$*. Then the sequence* $\{x_n\}$ *converges strongly to a point* $\Pi_{\mathcal{F}} x_0$*.*

We observe that the strong convergence result of Bello and Nnakwe [9] is restricted to solution of a variational inequality, an equilibrium problem, and a semi-fixed point of a continuous semi-pseudocontractive mapping in uniformly smooth and 2-uniformly convex real Banach spaces.

Moreover, Moudafi [54] introduced and studied a new generalization of the monotone inclusion problem in Hilbert spaces. It is called *Split Monotone Inclusion Problem (SMIP)* which is defined as finding a point $p \in H_1$ such that

$$(p, S(p)) \in (A + B)^{-1}0 \times (C + D)^{-1}0, \tag{1.4.3}$$

where $A : H_1 \to H_1$ and $C : H_2 \to H_2$ are inverse strongly monotone mappings, $B : H_1 \to 2^{H_1}$ and $D : H_2 \to 2^{H_2}$ are maximal monotone mappings and $S : H_1 \to H_2$ is bounded linear mapping, where $H_i, i = 1, 2$ are Hilbert spaces. If $S = I$, then the SMIP reduces to the *Common Solution Monotone Inclusion Problem (CSMIP).* He proposed the following iterative algorithm for approximating the solution of SMIP and proved its weak convergence. For $x_1 \in H_1$, the sequence $\{x_n\}$ generated by

$$x_{n+1} = U(x_n + \gamma S^*(T - I)Sx_n), \tag{1.4.4}$$

where S^* is the adjoint mapping of S, $T = J^B_\lambda(I - \lambda A)$, and $U = J^D_\lambda(I - \lambda C)$, where $\lambda > 0$.

The Split Monotone Inclusion and Fixed Point Problem (SMIFPP) [84] is the generalization of SMIP which is defined as finding a point $(p, q) \in H_1 \times H_2$ such that

$$p \in F(T) \cap (A + B)^{-1}0, q \in F(G) \cap (C + D)^{-1}0 \text{ and } S(p) = K(q), \tag{1.4.5}$$

where $A : H_1 \to H_1$ and $C : H_2 \to H_2$ are inverse strongly monotone mappings, $B : H_1 \to 2^{H_1}$ and $D : H_2 \to 2^{H_2}$ are maximal monotone mappings, $T : H_1 \to H_1$ and $G : H_2 \to H_2$ are demi-contractive mappings, $S : H_1 \to H_3$ and $K : H_2 \to H_3$ are bounded linear mappings, where $H_i, i = 1, 2, 3$ are Hilbert spaces. If $C = 0 = D$, $T = G = I$, and $K = I$, where I is identity mapping, the SMIFPP reduces to the Split Monotone Inclusion Problem (SMIP).

To the best of our knowledge, many researchers have established and studied different algorithms for approximating the solution of the following problems: split monotone inclusion over the fixed point problems or split feasibility problems over the solution set of monotone inclusion problems in real Hilbert spaces (see, e.g., [45, 44, 46, 79, 83, 85, 87, 97] and the references therein).

1.5 Research Questions

Based on aforementioned research gaps lead us the following research questions.

1) Can we obtain a strong convergence theorem for finding a common element of set of solutions of a common finite family of generalized equilibrium problem, set of fixed points of a finite family of continuous asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense and set of zeros of a finite family γ-inverse strongly monotone mappings in Banach spaces?

2) Can we establish strong convergence theorem for finding a common element

of set of solutions of a common finite family of generalized mixed equilibrium problems, set of a common semi-fixed points of a finite family of continuous semi-pseudocontractive mappings and set of a common solutions of a finite family of variational inequality for a finite family of monotone and L-Lipschitz mappings in Banach spaces?

3) Can we construct an algorithm for approximating a common element of the set of solutions of a finite family of generalized mixed equilibrium problems, the set of a common f-fixed points of a finite family of f-pseudocontractive mappings and the set of a common solutions of a finite family of variational inequality problems for Lipschitz monotone mappings in real reflexive Banach spaces?

4) Can we obtain an inertial method for approximating a solution of split equality of monotone inclusion and f-fixed point problems in real Banach spaces?

1.6 Objectives

The main objectives of this study are:

1) to obtain strong convergence iterative algorithm for finding a common element of set of solutions of a a common finite family of generalized equilibrium problem, set of a common fixed points of a finite family of continuous asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense and set of a common zeros of a finite family γ-inverse strongly monotone mappings in uniformly smooth and uniformly convex Banach spaces.

2) to establish strong convergence of a hybrid iterative method for finding a common element of set of solutions of a common finite family of generalized mixed equilibrium problems, set of a common semi-fixed points of a finite family of continuous semi-pseudocontractive mappings and set of a common solutions of a finite family of variational inequality for a finite family of monotone and L-Lipschitz mappings in reflexive real Banach spaces.

3) to develop a Halpern type algorithm for approximating a common element of the set of solutions of a common finite family of generalized mixed equilibrium problems, the set of a common f-fixed points of a finite family of f-pseudocontractive mappings and the set of solutions of a common finite family of variational inequality problems for Lipschitz monotone mappings in reflexive real Banach spaces.

4) to introduce an inertial method for approximating a solution of split equality of monotone inclusion and f-fixed point problems in real Banach spaces.

6) to give applications of our results to solutions of other problems.

5) to give numerical example that justify our results.

7) to discuss the improvements against recent results in the literature.

1.7 Scope of the Study

The scope of this thesis is limited to:

- find method that approximate a common solution of a common solution of generalized equilibrium, zeros of Monotone Mapping and fixed points problems in uniformly convex and uniformly smooth real Banach space;
- construct an iterative process for solving generalized mixed equilibrium problems, semi-fixed (f-fixed) point problems and variational inequality problems in reflexive real Banach spaces;
- develop an inertial approximation method for solving split equality of monotone inclusion and the f-fixed point problems in Banach Spaces.

Chapter 2

Review of Literature

2.1 Common Solution of Generalized Equilibrium, Zero Point and Fixed Point Problems

In this section, we present some research methods that approximate a common element of solution of fixed point of nonlinear mappings, equilibrium problems and zeros of monotone mappings in the framework of Hilbert and Banach spaces.

In 2013, Hao [40] investigated fixed point problems of asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense based on hybrid projection methods. He established the following strong convergence theorem in a reflexive Banach space:

Theorem 2.1.1. *Let E be a strictly convex, smooth and reflexive Banach space such that both E and E^* have the Kadec-Klee property. Let C be a nonempty, closed, and convex subset of E. Let $T : C \to C$ be an asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense. Assume that T is asymptotically regular on C and closed, and $F(T)$ is nonempty and bounded. Then, the sequence $\{x_n\}$ defined by*

$$\begin{cases} x_0 \in C_1 = C, \\ x_1 = \Pi_{C_1} x_0 \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T^n x_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{2.1.1}$$

where $\xi_n = \max\{0, \sup_{p\in F(T), x\in C}(\phi(p, T^n x) - \phi(p, x))\}$ *and* $\{\alpha_n\}$ *satisfies the restriction* $\limsup_{n\to\infty} \alpha_n < 1$*, converges to a fixed point of* T*.*

On the other hand, Iiduka and Takahashi [43] established the following strong convergence theorem for finding a zero point of a monotone mapping A in a 2-uniformly convex and uniformly smooth Banach space E:

Theorem 2.1.2. *Let* E *be a 2-uniformly convex and uniformly smooth Banach space with dual* E^**. Let* $A : E \to E^*$ *be* γ*-inverse strongly monotone mapping with* $A^{-1}(0) \neq \emptyset$*. Then the sequence* $\{x_n\}$ *defined by*

$$\begin{cases} x_1 = x \in E, \\ y_n = J^{-1}(Jx_n - \lambda_n Ax_n), \\ C_n = \{z \in E : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in E : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n\cap Q_n} x, \end{cases} \tag{2.1.2}$$

where J *is the normalized duality mapping on* E *and* $\lambda_n \in (0, \frac{c^2\gamma_n}{2}]$ *for some* $c > 0$*, converge strongly to* $p \in A^{-1}(0)$*, where* $p = \Pi_{A^{-1}(0)} x$*.*

Notice that the theorem is given to approximate the zero point of γ-inverse strongly monotone mapping in 2-uniformly convex and uniformly smooth Banach spaces. In 2009, Zegeye and Shahzad [104] introduced an iterative algorithm to approximate a common element of a zero of monotone mapping and a fixed point of relatively weak nonexpansive mapping in a real uniformly convex and uniformly smooth Banach space and and proved the following strong convergence theorem:

Theorem 2.1.3. *Let* E *be a real uniformly convex and uniformly smooth Banach space with dual* E^**. Let* C *be a nonempty, closed and convex subset of* E*. Let* $A : E \to E^*$ *be a maximal monotone mapping with* $J^{-1}S$ *relatively weak nonexpansive, where* $Sx := Jx - Ax$*. Let* $T : E \to E$ *be a relatively weak nonexpansive mapping with* $\mathbb{A} := A^{-1}(0) \cap F(T) \neq \emptyset$*. Assume that* $0 < a \leq \lambda_n \leq b = \frac{c^2\gamma_n}{2}$*, where* c *is* *uniformly convexity constant. Then, the sequence* $\{x_n\}$ *defined by*

$$\begin{cases} x_1 = x \in C, \\ y_n = J^{-1}(Jx_n - \lambda_n A x_n), \\ z_n = T y_n, \\ C_0 = \{z \in C : \phi(z, z_0) \leq \phi(z, y_0) \leq \phi(z, x_0)\}, \\ Q_0 = E, \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, z_n) \leq \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x, \end{cases} \tag{2.1.3}$$

where J *is the normalized duality mapping on* E*, converge strongly to* $p = \Pi_{\mathbb{A}} x$*.*

In 2009, Takahashi and Zembayashi [86] introduced hybrid projection method for finding a common solution of an equilibrium problem and a fixed point problem for a relatively nonexpansive mapping in uniformly smooth and uniformly convex Banach spaces and proved the following theorem:

Theorem 2.1.4. *Let* E *be a uniformly smooth and uniformly convex Banach space. Let* C *be a nonempty, closed and convex subset of* E*. Let* $F : C \times C \to \mathbb{R}$ *be a bifunctional satisfying* ***Condition A****, and* $T : C \to C$ *be a relatively nonexpansive mapping with* $\mathbb{E} := F(T) \cap EP(F) \neq \emptyset$*. Let the sequence* $\{x_n\}$ *be generated by*

$$\begin{cases} x_0 \in C, \\ u_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T x_n), \\ z_n \in C \text{ such that } F(z_n, y) + \frac{1}{r_n} \langle y - z_n, J z_n - J u_n \rangle \geq 0, \forall y \in C, \\ H_n = \{z \in C : \phi(z, z_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, J x_0 - J x_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0 \end{cases} . \tag{2.1.4}$$

for every $n \geq 0$*, where* J^{-1} *is the inverse of* J *,* Π_C *is the generalized projection of* E *onto* C *(defined below),* $\{\alpha_n\}$ *is a sequence in* $[0, 1]$ *satisfying* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ *and* $r_n \subset [a, \infty)$ *for some* $a > 0$*. Then,* $\{x_n\}$ *converges strongly to* $p = \Pi_{\mathbb{E}} x_0$*.*

In 2016, Kazmi and Ali [49] also studied a strong convergence result for finding a common solution of an equilibrium problem and a fixed point problem for an asymptotically quasi-ϕ-nonexpansive mapping in intermediate sense in uniformly smooth and strictly convex Banach spaces as follows.

$$\begin{cases} x_0 \in C, \\ u_n = J^{-1}(\alpha_n J z_n + (1-\alpha_n) J T^n x_n), \\ z_{n+1} \in C \text{ s.t. } F(z_{n+1}, y) + \frac{1}{r_n}\langle y - z_{n+1}, Jz_{n+1} - Ju_n\rangle \geq 0, \forall y \in C, \\ H_n = \{z \in C : \phi(z, z_{n+1}) \leq \alpha_n \phi(z, z_n) + (1-\alpha_n)\phi(z, x_n) + \xi_n\}, \\ W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0 \end{cases} \tag{2.1.5}$$

where $\xi_n = \max\{0, \sup\limits_{p\in F(T), x\in C} (\phi(p, T^n x) - \phi(p, x))\}$.

Recently, Ibiam et. al. [42] studied a strong convergence result for finding a common element of set of solutions of a finite family of generalized equilibrium problems, set of fixed points of a finite family of continuous relatively nonexpansive mappings and set of zeros of a finite family of γ_i-inverse strongly monotone mappings in 2-uniformly convex and uniformly smooth Banach spaces, which gives by

$$\begin{cases} x_0 \in C_0 = C, \\ z_n = \Pi_C J^{-1}(Jx_n - \lambda_n A_{n+1} x_n) \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T_{n+1} z_n), \\ u_n, v_n \in C \text{ such that} \\ F_1(u_n, y) + \langle B_1 y_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \forall y \in C, \\ F_2(v_n, y) + \langle B_2 y_n, y - v_n\rangle + \frac{1}{r_n}\langle y - v_n, Jv_n - Jy_n\rangle \geq 0, \forall y \in C, \\ w_n = J^{-1}(\beta J u_n + (1-\beta) J v_n), \\ C_{n+1} = \{z \in C_n : \phi(z, w_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0. \end{cases} \tag{2.1.6}$$

Inspired and motivated by the above works, in Chapter 3, we proved a strong convergence theorem for finding a common element of set of a common solutions of a finite family of generalized equilibrium problem, set of a common fixed points of a finite family of continuous asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense and set of zeros of a finite family γ-inverse strongly monotone mappings in Banach spaces. Moreover, a numerical example is given to illustrate the implementability of our algorithm.

2.2 Approximation of Common Solutions of Non-linear problems

Approximating zeros of accretive mappings is equivalent to approximating fixed points of pseudocontractive mappings, by assuming existence of such zeros. For earlier and more recent results on the approximation of fixed points of pseudocontractive mappings refer to [20, 27, 31, 33, 58, 59, 69, 103, 101].

It is obvious that the fixed point technique introduced by Browder [19] for approximating zeros of accretive mappings, is not applicable when the mapping is monotone from a real Banach space to its dual space. Hence, there is the need to develop techniques for approximating zeros of monotone mapping. In 2008, Zegeye [99] introduced a new class of mappings, that is, semi-pseudocontractive and he proved that $T := J - A$ is semi-pseudocontractive mappings, where J is normalized duality mapping if and only if A is monotone mapping. Hence, approximating semi-fixed points of these semi-pseudocontractive mappings is equivalent to approximating zeros of monotone mappings , assuming the existence of such zeros, which is also equivalent to finding a minimizer of some convex functions.

In recent times, some authors proposed algorithms which converge strongly to a common solution of a variational inequality, an equilibrium problem, and semi-fixed points of a continuous semi-pseudocontractive mapping in the framework of Hilbert spaces and Banach spaces.

In 2011, Zegeye [101] studied a viscosity approximation method for finding a common fixed point of two continuous pseudocontractive mappings in a real Hilbert space. He proved the following theorem.

Theorem 2.2.1. *Let C be a nonempty, closed, and convex subset of a Hilbert space H. Let $T_i : C \to C$, $i = 1, 2$, be continuous pseudocontractive mapping such that $\mathcal{F} = \bigcap_{i=1}^{2} Fix(T_i) \neq \emptyset$. Let f be a self contraction on C, and let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n f(x) + (1 - \alpha_n) K_{r_n}^{T_1} K_{r_n}^{T_2} x_n \tag{2.2.1}$$

where $\{\alpha_n\} \subset [0, 1]$ with $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, and $\{r_n\} \subset (0, \infty)$ with $\liminf_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ and $K_{r_n}^{T_i} : H \to C$, $i = 1, 2$, are mappings defined as follows: for $x \in H$,

$$K_{r_n}^{T_i} x = \{z \in C : \langle y - z, T_i z\rangle - \frac{1}{r_n} \langle y - z, (1 + r_n)z - x\rangle \leq 0, \forall y \in C\}.$$

Then, $\{x_n\}$ converges strongly to $z = \Pi_{\mathcal{F}} f(z)$.

In 2019, Shahzad and Zegeye [78] proved the following convergence theorem for a common solution of fixed point, equilibrium and variational inequality problems in Hilbert spaces.

Theorem 2.2.2. *Let C be a nonempty, closed, and convex subset of a real Hilbert spaces H. Let $A : C \to H$ be a Lipschitz monotone mapping with Lipschitz constant $L > 0$, $F : C \times C \to \mathbb{R}$ be a bi-functional satisfying Condition A, and $T : C \to H$ be a continuous pseudocontractive mapping with $\mathcal{F} := F(T) \bigcap VI(A, C) \bigcap EP(F) \neq \emptyset$. Let the sequence $\{x_n\}$ be generated by*

$$\begin{cases} u, x_0 \in C, \\ z_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)(\beta y_n + (1 - \beta) u_n), \end{cases} \tag{2.2.2}$$

where P_C is the metric projection from H onto C, $y_n = K_{r_n}^T T_{r_n}^F x_n$ with $T_{r_n}^F$ and $K_{r_n}^S$ as the resolvent mappings for F and T, respectively, $\{r_n\} \subset [a, \infty)$, for some $a > 0$, $u_n = P_C(x_n - \lambda A z_n)$, $\lambda \in [a, b] \subset (0, \frac{1}{L})$ and $\{\alpha_n\} \subset (0, c] \subset (0, 1)$ with $\lim_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to a point $P_{\mathcal{F}} u$.

In 2020, Nnakwe and Okeke [57] constructed a new Halpern-type iterative algorithm which converges strongly to a common solution of two generalized equilibrium problems and a common semi-fixed point of two continuous semi-pseudocontractive mappings in a uniformly smooth and uniformly convex real Banach space. They proved the following theorem.

Theorem 2.2.3. *Let E^* be the dual space of a uniformly smooth and uniformly convex real Banach space E and C be a nonempty, closed, and convex subset of E. Let $B_i : C \to E^*$, $i = 1, 2$ be a continuous and monotone mapping, $F_i : C \times C \to \mathbb{R}$, $i = 1, 2$ be a bi-functional satisfying Condition A, and $T_i : C \to E^*$, $i = 1, 2$ be a continuous semi-pseudocontractive mapping with $\mathcal{F} := \bigcap_{i=1}^{2} (F_s(T_i) \bigcap GEP(F_i, B_i)) \neq \emptyset$. Let the sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_1 \in C, \\ z_n = T_{r_n}^{H_1} T_{r_n}^{H_2} x_n \\ x_{n+1} = J^{-1}(\alpha_n J x_1 + (1 - \alpha_n) J K_{r_n}^{T_1} K_{r_n}^{T_2} z_n]), \forall n \geq 1. \end{cases} \tag{2.2.3}$$

where $T_{r_n}^{H_i}$ and $K_{r_n}^{T_i}$ are resolvent mappings for H_i and T_i, $i = 1, 2$, respectively, and $\{\alpha_n\} \subset (0, 1)$ with $\lim_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to a point $\Pi_{\mathcal{F}} x_1$.

In 2021, Bello and Nnakwe [9] studied a new Halpern-type subgradient extragradient iterative algorithm which converges strongly to a common solution of a variational inequality problem, an equilibrium problem, and a semi-fixed point problem of a continuous semi-pseudocontractive mapping in uniformly smooth and 2-uniformly convex real Banach spaces. They proved the following theorem.

Theorem 2.2.4. *Let E^* be the dual space of a uniformly smooth and 2-uniformly convex real Banach space E. Let C be a nonempty, closed, and convex subset of E. Let $A : C \to E^*$ be a monotone and L-Lipschitz mapping, $F : C \times C \to \mathbb{R}$ be a bi-functional satisfying **Condition A**, and $T : C \to E^*$ be a continuous semi-pseudocontractive mapping with $\mathcal{F} := F_s(T) \bigcap VI(A, C) \bigcap EP(F) \neq \emptyset$. Let the sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_0 \in C, \\ z_n = \Pi_C J^{-1}(Jx_n - \lambda Ax_n) \\ T_n = \{w \in E : \langle w - z_n, Jx_n - \lambda Ax_n - Jz_n \rangle \leq 0\} \\ x_{n+1} = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)[\beta Jv_n + (1-\beta)Jw_n]), \end{cases} \tag{2.2.4}$$

where $v_n = T^F_{r_n} K^T_{r_n} x_n$ with $T^F_{r_n}$ and $K^S_{r_n}$ as the resolvent mappings for F and T, respectively, $\{r_n\} \subset [a, \infty)$, for some $a > 0$, $w_n = \Pi_{T_n} J^{-1}(Jx_n - \lambda Az_n)$, $\lambda \in (0, 1)$ with $\lambda < \frac{c}{L}$ and $\{\alpha_n\} \subset (0, 1)$ with $\lim_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then the sequence $\{x_n\}$ converges strongly to a point $\Pi_{\mathcal{F}} x_0$.

Inspired and motivated by the work given in [9, 57, 107], in chapter four of this thesis we constructed hybrid projection iterative algorithm for finding a common element of sets of solutions of a finite family of generalized mixed equilibrium problems, sets of semi-fixed points of a finite family of continuous semi-pseudocontractive mappings and sets of solutions of a finite family of variational inequality for a finite family of monotone and L-Lipschitz mappings in Banach spaces and proved strong convergence theorem.

Again we observe that, in all the above results, the mapping T is a continuous pseudocontractive mapping in Hilbert space or a continuous semi-pseudocontractive mapping in a uniformly smooth and 2-uniformly convex real Banach space. Moreover, Zegeye [101] proved convergence theorem for a common fixed points of two continuous pseudocontractive mappings. But, the algorithm can be extended to an algorithm for approximating a common element of the set of solutions of a finite family of generalized mixed equilibrium problems, the set of f-fixed points of a finite family of f-pseudocontractive mappings and the set of solutions of a finite family of variational inequality problems for Lipschitz monotone mappings in real reflexive Banach spaces. Thus, in chapter five of this thesis we introduced an algorithm and proved a strong convergence theorem to approximate the solution of this problem.

2.3 Inertial Method For a Solution of Split Equality Monotone Inclusion and Fixed Point Problems

In this section, we present some literature review on approximation of solutions of split equality monotone inclusion and fixed point problems.

Moudafi [54] introduced and studied *Split Monotone Inclusion Problem (SMIP)* which is defined as finding a point $p \in H_1$ such that

$$(p, S(p)) \in (A+B)^{-1}0 \times (C+D)^{-1}0, \tag{2.3.1}$$

where $A : H_1 \to H_1$ and $C : H_2 \to H_2$ are inverse strongly monotone mappings, $B : H_1 \to 2^{H_1}$ and $D : H_2 \to 2^{H_2}$ are maximal monotone mappings and $S : H_1 \to H_2$ is bounded linear mapping, where $H_i, i = 1, 2$ are Hilbert spaces. If $S = I$, then the SMIP reduces to the *Common Solution Monotone Inclusion Problem (CSMIP)*. He proposed the following iterative algorithm for approximating the solution of SMIP and proved its weak convergence. For $x_1 \in H_1$, the sequence $\{x_n\}$ generated by

$$x_{n+1} = U(x_n + \gamma S^*(T - I)Sx_n), \tag{2.3.2}$$

where S^* is the adjoint mapping of S, $T = J_\lambda^B(I - \lambda A)$, and $U = J_\lambda^D(I - \lambda C)$, where $\lambda > 0$.

The Split Monotone Inclusion and Fixed Point Problem (SMIFPP) [84] is the generalization of SMIP which is defined as finding a point $(p, q) \in H_1 \times H_2$ such that

$$p \in F(T) \cap (A+B)^{-1}0, q \in F(G) \cap (C+D)^{-1}0 \text{ and } S(p) = K(q), \tag{2.3.3}$$

where $A : H_1 \to H_1$ and $C : H_2 \to H_2$ are inverse strongly monotone mappings, $B : H_1 \to 2^{H_1}$ and $D : H_2 \to 2^{H_2}$ are maximal monotone mappings, $T : H_1 \to H_1$ and $G : H_2 \to H_2$ are demi-contractive mappings, $S : H_1 \to H_3$ and $K : H_2 \to H_3$ are bounded linear mappings, where $H_i, i = 1, 2, 3$ are Hilbert spaces. If $C = 0 = D$, $T = G = I$, and $K = I$, where I is identity mapping, the SMIFPP reduces to the Split Monotone Inclusion Problem (SMIP).

In 2021, Taiwo et al. [84] studied the split equality problem for systems of monotone inclusions and fixed point problems of set-valued demi-contractive mappings in real Hilbert spaces. They proposed a viscosity type algorithm and proved its strong convergence under some mild assumptions.
The need to speed up the convergence of iterative algorithms has always been of great importance. In 1964, Polyak [61] proposed an *inertial algorithm* which can be seen as a discrete version of a second order time dynamical system to speed up convergence rate of smooth convex minimization problem. The main idea of this method is to make use of two previous iterates in order to update the next iterate, which results in speeding up the algorithm's convergence. Very recently, some authors have proposed viscosity-type algorithm with different inertial parameters for solving equilibrium and fixed point problems; see for example [44, 97].

In 2021, Yao et al. [97] proposed the following iterative algorithm with inertial extrapolation step for approximating a solution of *SMIP* in real Hilbert spaces and proved weak convergence of the sequence generated by the proposed algorithm under some mild assumptions. Let $A : H_1 \to H_1$ and $C : H_2 \to H_2$ be inverse strongly monotone mappings, $B : H_1 \to 2^{H_1}$ and $D : H_2 \to 2^{H_2}$ be maximal monotone mappings. For arbitrary $x_0,\ x_1 \in H_1$, define the sequences $\{w_n\}$ and $\{x_n\}$ by

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}) \\ x_{n+1} = U(w_n + \gamma S^*(T - I)Sw_n), \end{cases} \tag{2.3.4}$$

where S^* is the adjoint mapping of S, $T = J_\lambda^B(I-\lambda A)$, and $U = J_\lambda^D(I-\lambda C)$, where $\lambda > 0$, $0 \leq \alpha_n \leq \overline{\alpha}_n$,where $\overline{\alpha}_n = \theta$ if $x_n = x_{n-1}$, otherwise $\overline{\alpha}_n = \min\{\theta, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\}$, where $\theta \in [0, 1)$ and $\{\varepsilon_n\} \subset \ell_1$ and $\gamma_n = \gamma > 0$ if $(T - I)Sw_n = 0$, otherwise $\gamma_n = \frac{\sigma_n\|(T-I)Sw_n\|^2}{\|S^*(T-I)Sw_n\|^2}$, where $0 < \sigma_n < 1$.

In 2020, Izuchukwu et al. [44] proposed and studied a new inertial extrapolation method for solving the split feasibility problems over the solution set of monotone inclusion problems in real Hilbert spaces. Let $A : H_1 \to H_1$ be Lipschitz monotone mapping, $T : H_1 \to H_1$ be nonexpansive mapping, $B : H_1 \to 2^{H_1}$ be maximal monotone mapping and $S : H_1 \to H_3$ be bounded linear mapping such that $\|S\| \neq$

0. For arbitrary x_0, $x_1 \in H_1$, define the sequences $\{u_n\}$, $\{w_n\}$, $\{y_n\}$ and $\{x_n\}$ by

$$\begin{cases} u_n = x_n + \alpha_n(x_n - x_{n-1}) \\ w_n = u_n - \gamma_n S^*(I - T)Su_n \\ y_n = (I + \lambda_n B)^{-1}(1 - \lambda_n A)w_n \\ z_n = y_n - \lambda_n(Ay_n - Aw_n) \\ x_{n+1} = (1 - \theta_n - \beta_n)w_n + \theta_n z_n, \end{cases} \tag{2.3.5}$$

where $0 \leq \alpha_n \leq \overline{\alpha}_n$,where $\overline{\alpha}_n = \frac{n-1}{n+\alpha-1}$ if $x_n = x_{n-1}$, otherwise $\overline{\alpha}_n = \min\{\frac{n-1}{n+\alpha-1}, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\}$, $0 \leq b \leq \gamma_n \leq c < \frac{1}{\|S\|^2}$, and $\lambda_{n+1} = \min\{\frac{\mu\|w_n - v_n\|}{\|Aw_n - Ay_n\|}, \lambda_n\}$ if $Aw_n \neq Ay_n$, otherwise λ_n, $\forall n > 0$ and $\{\beta_n\}$, $\{\theta_n\}$, and $\{\varepsilon_n\}$ are sequence of positive real numbers. They proved that the proposed method converges strongly to x^*, where $x^* \in (A+B)^{-1}(0)$ and $S(x^*) \in F(T)$, under some condition on the control parameters β_n, θ_n, and ε_n, provided that $\{p \in (A + B)^{-1}(0) : S(p) \in F(T)\} \neq \emptyset$.

All the results addressed above deal with either of the following: split monotone inclusion and fixed point problem or split feasibility problems over the solution set of monotone inclusion problems in real Hilbert spaces.

Based on these results, the following important question arises:

> *Can we obtain an inertial method for approximating a solution of split equality of monotone inclusion and f-fixed point problems in real Banach spaces?*

Answer to this question is given in chapter six in the affirmative.

Chapter 3

A common Solution of Generalized Equilibrium, Zeros of Monotone Mapping and Fixed Points problems

3.1 Introduction

We partially answer our first research problem of this dissertation. We propose an iterative algorithm to approximate a common solution of generalized equilibrium problem, zeros of monotone mapping, and fixed points problem of a finite family of continuous asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense. It is also proved that the proposed algorithm converges strongly to a common solution. We collect technical lemmas from literature to prove our main result in the second section of this chapter. We state the proposed algorithm in section 3 and prove its boundedness. The main result will be stated and proved in section 4. Finally, application of the main result is supported by numerical examples presented in the last section of the chapter.

3.2 Technical Tool Box

In this section, we collect technical results that will be useful throughout the chapter to prove our main theorems.

Lemma 3.2.1. *[100] Let E be a uniformly convex real Banach space and $B_R(0)$ be closed ball of E. Then there exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$, such that*

$$\|\alpha_0 x_0 + \alpha_1 x_1 + \cdots + \alpha_k x_k\|^2 \leq \sum_{i=0}^{k} \alpha_i \|x_i\|^2 + \alpha_s \alpha_t g(\|x_s - x_t\|),$$

for any $s, t \in \{0, 1, 2, \ldots, k\}$ and for $x_i \in B_R(0) := \{x \in : \|x\| \leq R\}$, $i = 0, 2, \ldots, K$ with $\sum_{i=0}^{k} \alpha_i = 1$ and $\alpha_i \in [0, 1]$.

Lemma 3.2.2. *Let E be a uniformly convex real Banach space. Then for all $x, y \in E$; we have*

$$\|x - y\| \leq \frac{1}{\mu}\|Jx - Jy\|, \tag{3.2.1}$$

where J is the normalized duality mapping of E and for some $\mu > 0$.

Proof. Note that the function $f(x) = ||x||^2$ is a strongly convex Gâteaux differentiable function with constant $\mu > 0$ and $\nabla f x = Jx$, where J is the normalized duality mapping (see, for example [11]). Thus, by the definition of strong convexity of f we have

$$\langle Jx - Jy, x - y\rangle \geq \mu||x - y||^2,$$

and hence

$$\|x - y\| \leq \frac{1}{\mu}\|Jx - Jy\|.$$

The proof is complete. □

Lemma 3.2.3. *[92] Let E be a reflexive and smooth real Banach space. Then for each $x, y \in E$, there is $\mu \in (0, 1]$ such that*

$$\mu\|x - y\|^2 \leq \phi(x, y), \forall x, y \in E.$$

Lemma 3.2.4. *[4] Let C be a nonempty, closed and convex subset of a reflexive, strictly convex, and smooth real Banach space E and let $x \in E$. Then for all $y \in C$,*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x).$$

Lemma 3.2.5. *[47] Let C be a nonempty, closed and convex subset of a smooth, uniformly convex Banach space E. Let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 3.2.6. *[4] Let C be a convex subset of a smooth real Banach space E. Let $x \in E$. Then*

$$x_0 = \Pi_C x \Longleftrightarrow \langle z - x_0, Jx_0 - Jx \rangle \geq 0, \forall z \in C.$$

Lemma 3.2.7. *[4] Let C be a nonempty, closed and convex subset of a reflexive, strictly convex, and smooth real Banach space E and let $x \in E$. Then $\forall y \in C$,*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x).$$

We make use of the function $V : E \times E^* \to \mathbb{R}$ defined by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2, \text{ for all } x \in E \text{ and } x^* \in E^*,$$

studied by Alber [4]. That is $V(x, x^*) = \phi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E^*$.

Lemma 3.2.8. *[4] Let E be a reflexive, strictly convex and smooth real Banach space with E^* as its dual. Then*

$$V(x, x^*) + 2\left\langle J^{-1}x^* - x, y^* \right\rangle \leq V(x, x^* + y^*),$$

for all $x \in E$ and $x^*, y^* \in E^*$.

We denote by $N_C(v)$ the normal cone for C at a point $v \in C$, that is $N_C(v) := \{x^* \in E^* : \langle v - y, x^* \rangle \geq 0 \text{ for all } y \in C\}$. In the sequel we shall use the following lemma.

Lemma 3.2.9. *[75] Let C be a nonempty, closed and convex subset of a real Banach space E and let A be a monotone and hemicontinuous opeartor of C into E^* with $C = D(A)$. Let $B : E \to 2^{E^*}$ be an operator defined as follows:*

$$Bv = \begin{cases} Av + N_C v, & \text{if } \ v \in C; \\ \emptyset, & \text{if } \ v \notin C \end{cases}$$

Then B is maximal monotone and $B^{-1}(0) = VI(A, C)$.

3.3 Iterative Algorithm

Let C be a nonempty, closed and convex subset of uniformly convex and uniformly smooth real Banach space E. Let F_1 and F_2 be bi-functions satisfying "**Condition A**" and $B_k : C \to E^*$ be monotone mappings where $k = 1, 2$. Let $T_j : C \to C, j = 1, 2, \ldots, d$ be a finite family of continuous asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense and $A_i : C \to E^*, i = 1, 2, \ldots, m$ be a finite family of γ_i-inverse strongly monotone mappings. Define a sequence $\{x_n\}$ by the iterative scheme:

$$\begin{cases} x_0 \in C_0 = C, \\ z_n = \Pi_C J^{-1}(Jx_n - \lambda_n A_{n+1} x_n) \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n) JT^n_{n+1} z_n), \\ u_n, v_n \in C \text{ such that} \\ F_1(u_n, y) + \langle B_1 y_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \forall y \in C, \\ F_2(v_n, y) + \langle B_2 y_n, y - v_n \rangle + \frac{1}{r_n} \langle y - v_n, Jv_n - Jy_n \rangle \geq 0, \forall y \in C, \\ w_n = J^{-1}(\beta Ju_n + (1-\beta) Jv_n), \\ C_{n+1} = \{ z \in C_n : \phi(z, w_n) \leq \phi(z, x_n) + \xi_n \}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0. \end{cases} \tag{3.3.1}$$

where $\xi_n = \sum_{j=1}^{d} \max\{0, \sup_{p \in F(T), x \in C} (\phi(p, T_j^n x) - \phi(p, x))\}$; $A_n = A_{n (\bmod m)}$, $T_n = T_{n (\bmod d)}$ and J is the normalized duality mapping on E; $\{r_n\} \subset [c_1, \infty)$ for some $c_1 > 0$, $\beta, \alpha_n \in (0, 1)$ for all $n \in \mathbb{N}$ such that $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$; and $\{\lambda_n\}$ is a sequence in $[a, b]$ for some $0 < a < b < \mu\gamma$, where $\gamma = \min_{1 \leq i \leq m} \gamma_i$.

With some assumptions on the parameters and operators, the stated algorithm is well-defined.

Lemma 3.3.1. *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth real Banach space E. Let F_1 and F_2 be bi-functions satisfying "**Condition A**" and $B_k : C \to E^*$ be monotone mappings where $k = 1, 2$. Let $T_j : C \to C, j = 1, 2, \ldots, d$ be a finite family of continuous asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense and $A_i : C \to E^*, i = 1, 2, \ldots, m$ be a finite family of γ_i-inverse strongly monotone operators. Assume that $F = \left[\bigcap_{j=1}^{d} F(T_j)\right] \cap \left[\bigcap_{i=1}^{m} A_i^{-1}(0)\right] \cap \left[\bigcap_{k=1}^{2} GEP(F_k, B_k)\right] \neq \emptyset$. Let $\{x_n\}$ be a* *sequence defined by (3.3.1). Assume that T_j is asymptotically regular on C and $F(T_j)$ is bounded for each j. Then, the sequence $\{x_n\}$ is well defined for each $n \geq 0$.*

Proof. Now, we divide the proof into two steps.

Step 1 We show that C_n is closed and convex for each $n \geq 0$.

It is obvious that $C_0 = C$ is closed and convex by assumption. Suppose that C_k is closed and convex for some $k \in \mathbb{N}$. Let $\{z_m\} \subseteq C_{k+1}$ be such that $z_m \to z$ as $m \to \infty$. Since $C_{k+1} \subseteq C_k$ and C_k is closed, $z \in C_k$ and

$$\begin{aligned} \phi(z, w_k) &= \|z\|^2 - 2\langle z, Jw_k\rangle + \|w_k\|^2 \\ &= \lim_{m\to\infty} [\|z_m\|^2 - 2\langle z_m, Jw_k\rangle + \|w_k\|^2] \\ &= \lim_{m\to\infty} \phi(z_m, w_k) \end{aligned}$$

$$\begin{aligned} &\leq \liminf_{m\to\infty} (\phi(z_m, x_k) + \xi_k) \\ &= \phi(z, x_k) + \xi_k. \end{aligned}$$

Then $z \in C_{k+1}$ and so C_{k+1} is closed.

Let $z_1, z_2 \in C_{k+1}$. Then $z_1, z_2 \in C_k$. It follow that $z = tz_1 + (1-t)z_2 \in C_k$ for each $t \in [0, 1]$. Notice that

$$\phi(z_1, w_k) \leq \phi(z_1, x_k) + \xi_k, \tag{3.3.2}$$

and

$$\phi(z_2, w_k) \leq \phi(z_2, x_k) + \xi_k. \tag{3.3.3}$$

The inequalities (3.3.2) and (3.3.3) are equivalent to

$$2\langle z_1, Jx_k - Jw_k\rangle \leq \|x_k\|^2 - \|w_k\|^2 + \xi_k, \tag{3.3.4}$$

and

$$2\langle z_2, Jx_k - Jw_k\rangle \leq \|x_k\|^2 - \|w_k\|^2 + \xi_k. \tag{3.3.5}$$

Multiplying t and $(1-t)$ on both sides of (3.3.4) and (3.3.5), respectively, we obtain

$$2\langle z, Jx_k - Jw_k\rangle \leq \|x_k\|^2 - \|w_k\|^2 + \xi_k.$$

That is,

$$\phi(z, w_k) \leq \phi(z, x_k) + \xi_k.$$

This implies $z \in C_{k+1}$ and hence C_{k+1} is closed and convex. Therefore, inductively C_n is closed and convex for all $n \geq 0$. Hence $\Pi_{C_{k+1}} x_0$ is well defined for all $n \geq 0$.

Step 2 We show that $F \subseteq C_n$ for each $n \geq 0$.
From the assumption, we see that $F \subseteq C = C_0$. Suppose that $F \subseteq C_k$ for some $k \in \mathbb{N}$. Now, we show that $F \subseteq C_{k+1}$. Let $w \in F$. Then by Lemma 3.2.1, we get

$$\begin{aligned}
\phi(w, y_k) &= \phi(w, J^{-1}(\alpha_k J x_k + (1-\alpha_k) J T_{k+1}^k z_k)) \\
&= \|w\|^2 - 2\left\langle w, \alpha_k J x_k + (1-\alpha_k) J T_{k+1}^k z_k \right\rangle \\
&\quad + \|\alpha_k J x_k + (1-\alpha_k) J T_{k+1}^k z_k\|^2 \\
&\leq \|w\|^2 - 2\alpha_k \langle w, J x_k \rangle - 2(1-\alpha_k)\left\langle w, J T_{k+1}^k z_k \right\rangle \\
&\quad + \alpha_k \|x_k\|^2 + (1-\alpha_k)\|T_{k+1}^k z_k\|^2 \\
&\leq \alpha_k \phi(w, x_k) + (1-\alpha_k)\phi(w, T_{k+1}^k z_k) \\
&\leq \alpha_k \phi(w, x_k) + (1-\alpha_k)\phi(w, z_k) + (1-\alpha_k)\xi_k.
\end{aligned}$$

Then

$$\phi(w, y_k) \leq \alpha_k \phi(w, x_k) + (1-\alpha_k)\phi(w, z_k) + \xi_k. \tag{3.3.6}$$

And by Lemma 3.2.2, Lemma 3.2.7 and Lemma 3.2.8, we get

$$\begin{aligned}
\phi(w, z_k) &= \phi(w, \Pi_C J^{-1}(J x_k - \lambda_k A_{k+1} x_k)) \\
&\leq \phi(w, J^{-1}(J x_k - \lambda_k A_{k+1} x_k)) \\
&\leq V(w, J x_k - \lambda_k A_{k+1} x_k) \\
&\leq V(w, (J x_k - \lambda_k A_{k+1} x_k) + \lambda_k A_{k+1} x_k) \\
&\quad - 2\left\langle J^{-1}(J x_k - \lambda_k A_{k+1} x_k) - w, \lambda_k A_{k+1} x_n \right\rangle \\
&\leq V(w, J x_k) - 2\lambda_k \left\langle J^{-1}(J x_k - \lambda_k A_{k+1} x_k) - w, A_{k+1} x_k \right\rangle
\end{aligned}$$

$$\begin{aligned}
&= \phi(w, x_k) - 2\lambda_k \langle x_k - w, A_{k+1} x_k \rangle \\
&\quad + 2\left\langle J^{-1}(J x_k - \lambda_k A_{k+1} x_k) - x_k, -\lambda_k A_{k+1} x_k \right\rangle \\
&\leq \phi(w, x_k) - 2\lambda_k \langle x_k - w, A_{k+1} x_k \rangle \\
&\quad + 2\lambda_k \|J^{-1}(J x_k - \lambda_k A_{k+1} x_k) - x_k\| \|A_{k+1} x_k\| \\
&\leq \phi(w, x_k) - 2\lambda_k \langle x_k - w, A_{k+1} x_k \rangle + 2\frac{\lambda_k^2}{\mu}\|A_{k+1} x_k\|^2 \\
&\leq \phi(w, x_k) - 2\lambda_k \gamma \|A_{k+1} x_k\|^2 + 2\frac{\lambda_k^2}{\mu}\|A_{k+1} x_k\|^2 \\
&= \phi(w, x_k) + 2\lambda_k(\frac{\lambda_k}{\mu} - \gamma)\|A_{k+1} x_k\|^2.
\end{aligned}$$

Then

$$\phi(w, z_k) \leq \phi(w, x_k) + 2\lambda_k(\frac{\lambda_k}{\mu} - \gamma)\|A_{k+1} x_k\|^2. \tag{3.3.7}$$

Therefore, from (3.3.7) and $\lambda_k \in (0, \mu\gamma)$ we obtain that

$$\phi(w, z_k) \le \phi(w, x_k). \tag{3.3.8}$$

Substituting (3.3.8) into (3.3.6), we have

$$\phi(w, y_k) \le \phi(w, x_k) + \xi_k,$$

and so $w \in C_{k+1}$. This implies, by induction, that $F \subseteq C_n$ and the sequence $\{x_n\}$ generated by (3.3.1) is well defined for all $n \ge 0$.

3.4 Strong Convergence Theorem

Theorem 3.4.1. *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth real Banach space E. Let F_1 and F_2 be bi-functions satisfying "**Condition A**" and $B_k : C \to E^*$ be monotone mappings where $k = 1, 2$. Let $T_j : C \to C, j = 1, 2, \ldots, d$ be a finite family of continuous asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense and $A_i : C \to E^*, i = 1, 2, \ldots, m$ be a finite family of γ_i-inverse strongly monotone operators. Assume that $F = \left[\bigcap_{j=1}^{d} F(T_j)\right] \cap \left[\bigcap_{i=1}^{m} A_i^{-1}(0)\right] \cap \left[\bigcap_{k=1}^{2} GEP(F_k, B_k)\right] \ne \emptyset$. Let $\{x_n\}$ be a sequence defined by (3.3.1). Assume that T_j is asymptotically regular on C and $F(T_j)$ is bounded for each j. Then, the sequence $\{x_n\}$ converges to some element of F.*

Proof. We divide the proof into four steps.

Step 1. We show that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} w_n = x$, $\lim_{n\to\infty} y_n = x$ and $\lim_{n\to\infty} z_n = x$ for some point $x \in C$.

In view of $\Pi_{C_n} x_0$, we see from Lemma 3.2.6 that

$$\langle x_n - z, Jx_0 - Jx_n \rangle \ge 0, \forall z \in C_n.$$

The fact that $F \subseteq C_n$ implies that

$$\langle x_n - w, Jx_0 - Jx_n \rangle \ge 0, \forall w \in F. \tag{3.4.1}$$

From Lemma 3.2.7, we have

$$\begin{aligned} \phi(x_n, x_0) &= \phi(\Pi_{C_n} x_0, x_0) \\ &\le \phi(\Pi_F x_0, x_0) - \phi(\Pi_F x_0, x_n) \\ &\le \phi(\Pi_F x_0, x_0). \end{aligned}$$

This implies that the sequence $\{\phi(x_n, x_0)\}$ is bounded. Since $x_n = \Pi_{C_n}x_0$ and $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1} \subseteq C_n$, we have that

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \forall n \geq 0,$$

which implies that $\{\phi(x_n, x_0)\}$ is increasing and bounded sequence and so $\lim_{n\to\infty} \phi(x_n, x_0)$ exists. Furthermore, for any positive integer l, the inequality in Lemma 3.2.7 provides

$$\begin{aligned}\phi(x_{n+l}, x_n) &= \phi(x_{n+l}, \Pi_{C_n}x_0) \\ &\leq \phi(x_{n+l}, x_0) - \phi(\Pi_{C_n}x_0, x_0) \\ &\leq \phi(x_{n+l}, x_0) - \phi(x_n, x_0).\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \phi(x_{n+l}, x_n) = 0, \text{ for any integer } l > 0. \quad (3.4.2)$$

Thus by Lemma 3.2.5 implies that

$$\lim_{n\to\infty} \|x_{n+l} - x_n\| = 0, \quad (3.4.3)$$

and hence $\{x_n\}$ is Cauchy. Therefore, there exists a point $x \in C$ such that $x_n \to x$ as $n \to \infty$.

Since $x_{n+1} \in C_{n+1}$, we get

$$\phi(x_{n+1}, w_n) \leq \phi(x_{n+1}, x_n) + \xi_n,$$

this with (3.4.2) implies that

$$\lim_{n\to\infty} \phi(x_{n+1}, w_n) = 0. \quad (3.4.4)$$

Thus, by Lemma 3.2.5, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - w_n\| = 0. \quad (3.4.5)$$

and hence

$$\|x_n - w_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - w_n\| \to 0 \text{ as } n \to \infty, \quad (3.4.6)$$

which implies that $w_n \to x$ as $n \to \infty$. Furthermore, the uniform continuity of J on bounded sets, gives that

$$\lim_{n\to\infty} \|Jx_n - Jw_n\| = 0. \quad (3.4.7)$$

Thus, the uniform convexity of E and Lemma 1.2.3 we have, for all $p \in F$, that

$$\begin{aligned} \phi(p,y_n) &= \phi(p, J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT^n_{n+1}z_n)) && (3.4.8)\\ &= \|p\|^2 - 2\left\langle p, \alpha_n Jx_n + (1-\alpha_n)JT^n_{n+1}z_n\right\rangle \\ &\quad + \|\alpha_n Jx_n + (1-\alpha_n)JT^n_{n+1}z_n\|^2 \\ &\leq \|p\|^2 - 2\alpha_n\langle p, Jx_n\rangle - 2(1-\alpha_n)\left\langle p, JT^n_{n+1}z_n\right\rangle + \alpha_n\|x_n\|^2 && (3.4.9)\\ &\quad + (1-\alpha_n)\|T^n_{n+1}z_n\|^2 - \alpha_n(1-\alpha_n)g(\|Jx_n - JT^n_{n+1}z_n\|) \\ &= \alpha_n\phi(p,x_n) + (1-\alpha_n)\phi(p,T^n_{n+1}z_n) - \alpha_n(1-\alpha_n)g(\|Jx_n - JT^n_{n+1}z_n\|) \\ &\leq \alpha_n\phi(p,x_n) + (1-\alpha_n)\phi(p,z_n) + \xi_n && (3.4.10)\\ &\quad - \alpha_n(1-\alpha_n)g(\|Jx_n - JT^n_{n+1}z_n\|). \end{aligned}$$

Then

$$\phi(p,y_n) \leq \alpha_n\phi(p,x_n) + (1-\alpha_n)\phi(p,z_n) + \xi_n - \alpha_n(1-\alpha_n)g(\|Jx_n - JT^n_{n+1}z_n\|). \tag{3.4.11}$$

Thus from (3.3.7) and (3.4.11) we have that

$$\begin{aligned} \phi(p,y_n) \leq \phi(p,x_n) + 2(1-\alpha_n)\lambda_n\left(\frac{\lambda_n}{\mu} - \gamma\right)\|A_{n+1}x_n\|^2 + \xi_n \quad (3.4.12)\\ -\alpha_n(1-\alpha_n)g(\|Jx_n - JT^n_{n+1}z_n\|). \end{aligned}$$

On the other hand from Remark 1.3.1 we get that

$$\begin{aligned} \phi(p,w_n) &= \phi(p, J^{-1}(\beta Ju_n + (1-\beta)Jv_n)) && (3.4.13)\\ &\leq \beta\phi(p,u_n) + (1-\beta)\phi(p,v_n) \\ &= \beta\phi(p,T_{1,r}y_n) + (1-\beta)\phi(p,T_{2,r}y_n) \\ &\leq \phi(p,y_n). \end{aligned}$$

Substituting (3.4.12) into (3.4.13) we get that

$$\begin{aligned} \phi(p,w_n) &\leq \phi(p,x_n) + 2(1-\alpha_n)\lambda_n\left(\frac{\lambda_n}{\mu} - \gamma\right)\|A_{n+1}x_n\|^2 + \xi_n \quad (3.4.14)\\ &\quad -\alpha_n(1-\alpha_n)g(\|Jx_n - JT^n_{n+1}z_n\|). \end{aligned}$$

Now, using the fact $\lambda_n < \mu\gamma$ and inequality (3.4.14) imply that

$$\begin{aligned} \alpha_n(1-\alpha_n)g(\|Jx_n - JT^n_{n+1}z_n\|) &\leq \phi(p,x_n) - \phi(p,w_n) + \xi_n \\ &= \|x_n\|^2 - \|w_n\|^2 + 2\langle p, Jw_n - Jx_n\rangle + \xi_n \\ &\leq (\|x_n\| - \|w_n\|)(\|x_n\| + \|w_n\|) \\ &\quad + 2\|p\|\|Jw_n - Jx_n\| + \xi_n \\ &\leq M_0(\|x_n\| - \|w_n\|) + 2\|p\|\|Jw_n - Jx_n\| + \xi_n, \end{aligned}$$

where $M_0 = \sup_{n\in\mathbb{N}} (\|x_n\| + \|w_n\|)$. Thus, from (3.4.6), (3.4.7) and the fact that $\lim_{n\to\infty} \xi_n = 0$ we obtain

$$\lim_{n\to\infty} g\left(\|Jx_n - JT^n_{n+1}z_n\|\right) = 0.$$

Therefor, from the property of g we get that $\|Jx_n - JT^n_{n+1}z_n\| \to 0$ as $n \to \infty$. Furthermore, since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we see that

$$\lim_{n\to\infty} \|x_n - T^n_{n+1}z_n\| = 0. \tag{3.4.15}$$

Moreover, from (3.4.14) and (3.4.15) we have that

$$2(1-\alpha_n)\lambda_n\left(\gamma - \frac{\lambda_n}{\mu}\right)\|A_{n+1}x_n\|^2 \leq \phi(p,x_n) - \phi(p,w_n) + \xi_n,$$

which yields

$$\lim_{n\to\infty} \|A_{n+1}x_n\| = 0. \tag{3.4.16}$$

Now, from Lemma 3.2.7, 3.2.8 and $0 < a < \lambda_n < b < \mu\gamma$ imply that

$$\begin{aligned}
\phi(x_n, z_n) &= \phi(x_n, \mathit{\Pi}_C J^{-1}(Jx_n - \lambda_n A_{n+1}x_n)) \\
&\leq \phi(x_n, J^{-1}(Jx_n - \lambda_n A_{n+1}x_n)) \\
&\leq V(x_n, Jx_n - \lambda_n A_{n+1}x_n) \\
&\leq V(x_n, (Jx_n - \lambda_n A_{n+1}x_n) + \lambda_n A_{n+1}x_n) \\
&\quad -2\left\langle J^{-1}(Jx_n - \lambda_n A_{n+1}x_n) - x_n, \lambda_n A_{n+1}x_n\right\rangle \\
&= \phi(x_n, x_n) + 2\lambda_n\left\langle J^{-1}(Jx_n - \lambda_n A_{n+1}x_n) - x_n, -A_{n+1}x_n\right\rangle \\
&\leq \phi(x_n, x_n) + 2\lambda_n\|J^{-1}(Jx_n - \lambda_n A_{n+1}x_n) - x_n\|\| - A_{n+1}x_n\|
\end{aligned}$$

$$\begin{aligned}
&\leq \phi(x_n, x_n) + 2\frac{\lambda_n^2}{\mu}\|A_{n+1}x_n\|^2 \\
&\leq \phi(x_n, x_n) + 2\frac{b^2}{\mu}\|A_{n+1}x_n\|^2.
\end{aligned}$$

Then

$$\phi(x_n, z_n) \leq 2\frac{b^2}{\mu}\|A_{n+1}x_n\|^2. \tag{3.4.17}$$

It follows from (3.4.16) and (3.4.17) that $\phi(x_n, z_n) \to 0$ as $n \to \infty$. Then by Lemma 3.2.5 we get

$$\lim_{n\to\infty} \|x_n - z_n\| = 0, \tag{3.4.18}$$

and hence $z_n \to x$ as $n \to \infty$. Furthermore, from (3.3.6), we have

$$\phi(x, y_n) \leq \phi(x, x_n) + \phi(x, z_n) + \xi_n.$$

Thus, we obtain $\phi(x, y_n) \to 0$ as $n \to \infty$ since $\phi(x, x_n) \to 0$, $\phi(x, z_n) \to 0$, and $\xi_n \to 0$ as $n \to \infty$ and by Lemma 3.2.5, we obtain that $\lim_{n\to\infty} \|y_n - x\| = 0$ and so $y_n \to x$ as $n \to \infty$.

Step 2. We show that $x \in \bigcap_{j=1}^{d} F(T_j)$. Observe that from (3.4.15) and (3.4.18), we obtain

$$\|T_{n+1}^n z_n - z_n\| \leq \|T_{n+1}^n z_n - x_n\| + \|x_n - z_n\| \to 0, \text{ as } n \to \infty.$$

Thus

$$\lim_{n\to\infty} T_{n+1}^n z_n = x. \tag{3.4.19}$$

Let $\{n_{l_1}\}_{l_1} \subset \mathbb{N}$ be such that $T_{n_{l_1}+1} = T_1$ for all $l_1 \in \mathbb{N}$. In view of asymptotic regularity of T_{n+1} and (3.4.19) we get

$$\begin{aligned} \|T_1^{n_{l_1}+1} z_{n_{l_1}+1} - x\| &= \|T_{n_{l_1}+1}^{n_{l_1}+1} z_{n_{l_1}+1} - x\| \qquad (3.4.20)\\ &\leq \|T_{n_{l_1}+1}^{n_{l_1}+1} z_{n_{l_1}+1} - T_{n_{l_1}+1}^{n_{l_1}} z_{n_{l_1}+1}\| + \|T_{n_{l_1}+1}^{n_{l_1}} z_{n_{l_1}+1} - x\| \\ &= \|T_1^{n_{l_1}+1} - T_1^{n_{l_1}}\| + \|T_1^{n_{l_1}} z_{n_{l_1}+1} - x\| \to 0, \text{ as } l \to \infty. \end{aligned}$$

Hence,

$$\lim_{l\to\infty} T_1^{n_{l_1}+1} z_{n_{l_1}} = x. \tag{3.4.21}$$

Since $z_{n_{l_1}} \to x$ as $l_1 \to \infty$, we obtain from (3.4.21), using continuity of T_1 and (3.4.19), that

$$x = \lim_{l_1\to\infty} T_{n_{l_1}+1}\left(T_{n_{l_1}+1}^{n_{l_1}} z_{n_{l_1}}\right) = \lim_{l_{1_1}\to\infty} T_1\left(T_{n_{l_1}+1}^{n_{l_1}} z_{n_{l_1}}\right) = T_1 x.$$

Similarly, if $\{n_{l_j}\}_{l_j} \subset \mathbb{N}$ is such that $T_{n_{l_j}+1} = T_j$ for all $l_j \in \mathbb{N}$, then we have again

$$x = \lim_{l_j\to\infty} T_{n_{l_j}+1}\left(T_{n_{l_j}+1}^{n_{l_j}} z_{n_{l_j}}\right) = \lim_{l_j\to\infty} T_j\left(T_{n_{l_j}+1}^{n_{l_j}} z_{n_{l_j}}\right) = T_j x,$$

where $j = 2, 3, \ldots, d$. Hence $x \in \bigcap_{j=1}^{d} F(T_j)$.

Step 3. We show that $x \in \bigcap_{i=1}^{m} A_i^{-1}(0)$.

Since A_i is γ_i-inverse strongly monotone for $i = 1, 2, \ldots, m$, we have that A_i is $\frac{1}{\gamma}$-Lipschitz monotone continuous. Thus

$$\|A_{n+1}x_n - A_{n+1}x\| \leq \frac{1}{\gamma}\|x_n - x\| \to 0 \text{ as } n \to \infty. \tag{3.4.22}$$

Hence, from 3.3.7 and 3.4.22, we obtain that

$$\|A_{n+1}x\| \leq \|A_{n+1}x_n - A_{n+1}x\| + \|A_{n+1}x_n\| \to 0 \text{ as } n \to \infty. \tag{3.4.23}$$

Consequently, we get that

$$\lim_{n\to\infty} A_{n+1}x = 0. \tag{3.4.24}$$

Let $\{n_{s_i}\}, s_i \geq 1 \subseteq \mathbb{N}$ be such that $A_{n_{s_i}+1} = A_i$ for all $s_i \in \mathbb{N}$ where $i = 1, 2, \ldots, m$. Then

$$A_i x = \lim_{s_i\to\infty} A_{n_{s_i}+1}x = 0, \tag{3.4.25}$$

where $i = 1, 2, \ldots, m$. Thus $x \in \bigcap_{i=1}^{m} A_i^{-1}(0)$.

Step 4. We show that $x \in \bigcap_{k=1}^{2} GEP(F_k, B_k)$. From $u_n = T_{1,r_n}y_n$, Remark 1.3.1 , the fact that $x_n \to x$ and $z_n \to x$ as $n \to \infty$, we obtain

$$\begin{aligned} \phi(x, u_n) &\leq \phi(x, y_n) \\ &\leq \alpha_n\phi(x, x_n) + (1-\alpha_n)\phi(x, z_n) \\ &\leq \phi(x, x_n) + \phi(x, z_n) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.4.26}$$

Thus, by Lemma 3.2.5 and (3.4.26), we have $u_n \to x$ and $y_n \to x$ as $n \to \infty$. These imply that, $\lim_{n\to\infty} \|u_n - y_n\| = 0$. Consequently, $\lim_{n\to\infty} \|Ju_n - Jy_n\| = 0$. Hence,

$$\lim_{n\to\infty} \frac{\|u_n - y_n\|}{r_n} = 0. \tag{3.4.27}$$

But from $(A2)$ we note that

$$\langle B_1 y_n, v - u_n\rangle + \frac{1}{r_n}\langle v - u_n, Ju_n - Jy_n\rangle \geq -F_1(u_n, v) = F_1(v, u_n), \forall v \in C.$$

and hence

$$\langle B_1 y_n, v - u_n\rangle + \left\langle v - u_n, \frac{Ju_n - Jy_n}{r_n}\right\rangle \geq F_1(v, u_n), \forall v \in C. \tag{3.4.28}$$

Put $z_t = tv + (1-t)x$ for all $t \in [0, 1]$ and $v \in C$. Since C is convex, $z_t \in C$. Since B_1 is monotone, we have that $\langle B_1 z_t - B_1 u_n, z_t - u_n\rangle \geq 0$. Then, from (3.4.28), we have

$$\begin{aligned} \langle B_1 z_t, z_t - u_n\rangle &\geq F_1(z_t, u_n) - \left\langle z_t - u_n, \frac{Ju_n - Jy_n}{r_n}\right\rangle \\ &+ \langle B_1 u_n - B_1 y_n, z_t - u_n\rangle + \langle B_1 z_t - B_1 u_n, z_t - u_n\rangle \\ &\geq F_1(z_t, u_n) - \left\langle z_t - u_n, \frac{Ju_n - Jy_n}{r_n}\right\rangle + \langle B_1 u_n - B_1 y_n, z_t - u_n\rangle. \end{aligned} \tag{3.4.29}$$

By the continuity of B_1 and the fact that $u_n \to x$, $y_n \to x$ as $n \to \infty$, we obtain that

$$\lim_{n\to\infty} (B_1 u_n - B_1 y_n) = 0. \tag{3.4.30}$$

Using (3.4.28), and (3.4.30), it follows from (A4) and (3.4.29) that

$$\begin{aligned} F_1(z_t, x) &\le \liminf_{n\to\infty} \left(F_1(z_t, u_n) - \left\langle z_t - u_n, \frac{Ju_n - Jy_n}{r_n} \right\rangle + \langle B_1 u_n - B_1 y_n, z_t - u_n \rangle \right) \\ &\le \lim_{n\to\infty} \langle B_1 z_t, z_t - u_n \rangle = \langle B_1 z_t, z_t - x \rangle . \end{aligned} \tag{3.4.31}$$

Now, from (3.4.31), (A1) and (A4) we get that

$$\begin{aligned} 0 = F_1(z_t, z_t) &\le tF_1(z_t, v) + (1-t)F_1(z_t, x) \\ &\le tF_1(z_t, v) + (1-t) \langle B_1 z_t, z_t - x \rangle \\ &\le tF_1(z_t, v) + (1-t)t \langle B_1 z_t, v - x \rangle . \end{aligned}$$

and hence

$$F_1(z_t, v) + (1-t) \langle B_1 z_t, v - x \rangle \ge 0.$$

Letting $t \to 0$, we have

$$F_1(x, v) + \langle B_1 x, v - x \rangle \ge 0.$$

This implies that $x \in GEP(F_1, B_1)$. Similarly, considering $v_n = T_{2,r_n} y_n$, the same argument gives that $x \in GEP(F_2, B_2)$. Therefore, $x \in \bigcap_{k=1}^{2} GEP(F_k, B_k)$.

Finally, we prove that $x = \Pi_F(x_0)$. From $x_n = \Pi_{C_n}(x_0)$, we have

$$\langle Jx_0 - Jx_n, x_n - z \rangle \ge 0, \forall z \in C_n.$$

Since $F \subseteq C_n$, we also have that

$$\langle Jx_0 - Jx_n, x_n - z \rangle \ge 0, \forall z \in F. \tag{3.4.32}$$

By taking limits in (3.4.32), we get by the uniform continuity of J on bounded sets

$$\langle Jx_0 - Jx, x - z \rangle \ge 0, \forall z \in F.$$

Now, by Lemma 3.2.6 we have that $x = \Pi_F(x_0)$. This completes the proof.

If $A_i \equiv 0, i = 1, 2, \dots, m$, then strong convergence theorem for approximating a common element of sets of solutions of two generalized equilibrium problems and the sets of fixed points of finite family of asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense in Banach spaces may not require that E is a uniformly convex real Banach space and Theorem 3.4.1 is reduced to the following.

Theorem 3.4.2. *Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex Banach space such that both E and E^* have the Kadec-Klee property. Let F_1 and F_2 be bi-functions satisfying "**Condition A**" and $B_k : C \to E^*$ be monotone mappings where $k = 1, 2$. Let $T_j : C \to C, j = 1, 2, \ldots, d$ be a finite family of continuous asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense. Assume that $F = \left[\bigcap_{j=1}^{d} F(T_j)\right] \cap \left[\bigcap_{k=1}^{2} GEP(F_k, B_k)\right] \neq \emptyset$ and Assume that T_j is asymptotically regular on C and $F(T_j)$ is bounded for each j. Let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} x_0 \in C_0 = C, \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T_{n+1}^n x_n), \\ u_n, v_n \in C \text{ such that} \\ F_1(u_n, y) + \langle B_1 y_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, J u_n - J y_n \rangle \geq 0, \forall y \in C, \\ F_2(v_n, y) + \langle B_2 y_n, y - v_n \rangle + \frac{1}{r_n} \langle y - v_n, J v_n - J y_n \rangle \geq 0, \forall y \in C, \\ w_n = J^{-1}(\beta J u_n + (1-\beta) J v_n), \\ C_{n+1} = \{z \in C_n : \phi(z, w_n) \leq \phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0. \end{cases} \tag{3.4.33}$$

where $\xi_n = \sum_{j=1}^{d} \max\{0, \sup_{p \in F(T), x \in C} (\phi(p, T_j^n x) - \phi(p, x))\}$; $T_n = T_{n(\bmod d)}$ and J is the normalized duality mapping on E;$\{r_n\} \subseteq [c_1, \infty)$ for some $c_1 > 0$,$\beta, \alpha_n \in (0, 1)$ for all $n \in \mathbb{N}$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Then, the sequence $\{x_n\}$ converges to some element of F.

Proof. We get that $z_n = x_n$ and the method of proof of Theorem 3.4.1 gives the required assertion without the requirement that E is a uniformly convex real Banach space.

If $B_1 \equiv B_2 \equiv 0$ in Theorem 3.4.2,then we get the following corollary.

Corollary 3.4.3. *Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex Banach space such that both E and E^* have the Kadec-Klee property. Let F_1 and F_2 be bi-functions satisfying "**Condition A**." Let $T_j : C \to C, j = 1, 2, \ldots, d$ be a finite family of continuous asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense. Assume that $F = \left[\bigcap_{j=1}^{d} F(T_j)\right] \cap$*

$\left[\bigcap_{k=1}^{2} EP(F_k)\right] \neq \emptyset$ *and Assume that* T_j *is asymptotically regular on* C *and* $F(T_j)$ *is bounded for each* j. *Let* $\{x_n\}$ *be a sequence defined by*

$$\begin{cases} x_0 \in C_0 = C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT_{n+1}^n x_n), \\ u_n, v_n \in C \text{ such that} \\ F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \forall y \in C, \\ F_2(v_n, y) + \frac{1}{r_n}\langle y - v_n, Jv_n - Jy_n\rangle \geq 0, \forall y \in C, \\ w_n = J^{-1}(\beta Ju_n + (1-\beta)Jv_n), \\ C_{n+1} = \{z \in C_n : \phi(z, w_n) \leq \phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0. \end{cases} \tag{3.4.34}$$

where $\xi_n = \sum_{j=1}^{d} \max\{0, \sup_{p\in F(T), x\in C}(\phi(p, T_j^n x) - \phi(p, x))\}$; $T_n = T_{n(\bmod d)}$ *and* J *is the normalized duality mapping on* E; $\{r_n\} \subseteq [c_1, \infty)$ *for some* $c_1 > 0$, $\beta, \alpha_n \in (0,1)$ *for all* $n \in \mathbb{N}$ *such that* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. *Then, the sequence* $\{x_n\}$ *converges to some element of* F.

If $F_1 \equiv F_2 \equiv 0$ in Theorem 3.4.2, then we get the following corollary.

Corollary 3.4.4. *Let* C *be a nonempty, closed and convex subset of a uniformly smooth and strictly convex Banach space such that both* E *and* E^* *have the Kadec-Klee property. Let* $B_k : C \to E^*$ *be monotone mappings where* $k = 1, 2$. *Let* $T_j : C \to C, j = 1, 2, \ldots, d$ *be a finite family of continuous asymptotically quasi-*ϕ*-nonexpansive mapping in the intermediate sense. Assume that* $F = \left[\bigcap_{j=1}^{d} F(T_j)\right] \cap \left[\bigcap_{k=1}^{2} VI(B_k, C)\right] \neq \emptyset$ *and Assume that* T_j *is asymptotically regular on* C *and* $F(T_j)$

is bounded for each j. Let $\{x_n\}$ be a sequence defined by

$$\begin{cases} x_0 \in C_0 = C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT_{n+1}^n x_n), \\ u_n, v_n \in C \text{ such that} \\ \langle B_1 y_n, y-u_n\rangle + \frac{1}{r_n}\langle y-u_n, Ju_n - Jy_n\rangle \geq 0, \forall y \in C, \\ \langle B_2 y_n, y-v_n\rangle + \frac{1}{r_n}\langle y-v_n, Jv_n - Jy_n\rangle \geq 0, \forall y \in C, \\ w_n = J^{-1}(\beta Ju_n + (1-\beta)Jv_n), \\ C_{n+1} = \{z \in C_n : \phi(z, w_n) \leq \phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0. \end{cases} \tag{3.4.35}$$

where $\xi_n = \sum_{j=1}^{d} \max\{0, \sup_{p\in F(T),x\in C} (\phi(p, T_j^n x) - \phi(p, x))\}$; $T_n = T_{n(\mod d)}$ and J is the normalized duality mapping on E; $\{r_n\} \subseteq [c_1, \infty)$ for some $c_1 > 0$, $\beta, \alpha_n \in (0, 1)$ for all $n \in \mathbb{N}$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Then, the sequence $\{x_n\}$ converges to some element of F.

Theorem 3.4.5. *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth real Banach space E. Let F_1 and F_2 be bi-functions satisfying "**Condition A**" and $B_k : C \to E^*$ be monotone mappings where $k = 1, 2$. Let $T_j : C \to C, j = 1, 2, \ldots, d$ be a finite family of continuous asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense and $A_i : C \to E^*, i = 1, 2, \ldots, m$ be a finite family of γ_i-inverse strongly monotone operators. Assume that $F = \left[\bigcap_{j=1}^{d} F(T_j)\right] \cap \left[\bigcap_{i=1}^{m} VI(A_i, C)\right] \cap \left[\bigcap_{k=1}^{2} GEP(F_k, B_k)\right] \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by (3.3.1). Assume that T_j is asymptotically regular on C and $F(T_j)$ is bounded for each j, and $\|A_i x\| \leq \|A_i x - A_i p\|, \forall x \in C, p \in F$ for each i. Then, the sequence $\{x_n\}$ converges to some element of F.*

Proof. Let $p \in F$. Then by assumption $\|A_i x\| \leq \|A_i x - A_i p\|, \forall x \in C$ and in particular $\|A_i p\| \leq \|A_i p - A_i p\| = 0$ which implies that $A_i p = 0$ and so $p \in A_i^{-1}(0)$. Therefore, the conclusion follows from Theorem 3.4.1.

3.5 Numerical Example

We give an example of a finite family of continuous asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense, a finite family of γ_i-inverse strongly monotone, a bi-functions satisfying "**Condition A**" and monotone mappings satisfying Theorem 3.2 and some numerical experiment result to explain the conclusion of theorem as follows: Let $H = \mathbb{R}$ with Euclidean norm. Let $C = [-1, 2]$ and $T_j : C \to C$ where $j = 1, 2, 3, 4, 5$ be defined by

$$T_j x = \begin{cases} \frac{1}{2j} x \text{ if } x \in [-1, 0] \\ \frac{1}{4j} x \text{ if } x \in [0, 2] \end{cases}, \tag{3.5.1}$$

$A_k : C \to \mathbb{R}$ where $k = 1, 2, 3, 4$ be defined by

$$A_k x = \frac{1}{2k} x, \forall x \in C,$$

$F_i : [-1, 2] \times [-1, 2] \to \mathbb{R}$ where $i = 1, 2$ be defined by

$$F_i(x, y) = x^2 - y^2, \forall x, y \in C,$$

and $B_1, B_2 : C \to \mathbb{R}$ be defined by

$$B_1 x = \begin{cases} 0 \text{ if } x \in [-1, 1] \\ (x-2)^2 \text{ if } x \in (1, 2] \end{cases}, \tag{3.5.2}$$

and

$$B_2 x = \begin{cases} 2x & \text{if } x \in [-1, 0] \\ 0 & \text{if } x \in [0, \frac{1}{2}] \\ x - \frac{1}{2} & \text{if } x \in [\frac{1}{2}, 2] \end{cases}. \tag{3.5.3}$$

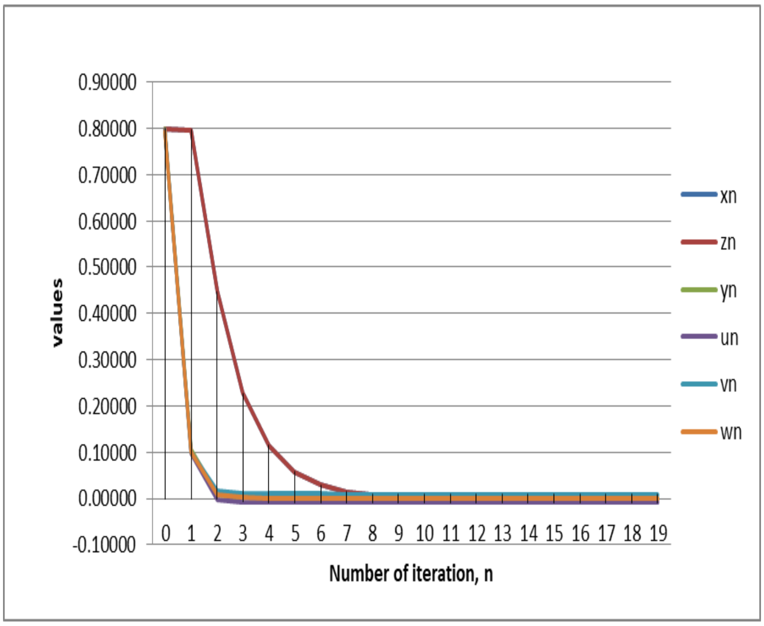
Then T_j, A_k, F_i and B_i are continuous asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense, γ_i-inverse strongly monotone, a bi-functions satisfying "**Condition A**" and monotone mappings respectively for each j, k, i. It is clear that $F = \left[\bigcap_{j=1}^{4} F(T_j)\right] \cap \left[\bigcap_{i=1}^{5} A_i^{-1}(0)\right] \cap \left[\bigcap_{k=1}^{2} GEP(F_k, B_k)\right] = \{0\}$. Now, if we take, $\lambda_n = 0.001 + \frac{1}{n+200}, \alpha_n = r_n = \frac{1}{n+100}$, and $\beta = 0.5$, we observe that the conditions of Theorem 3.2 are satisfied and for $x_0 = 0.8 \in C_0 = C$ Scheme 3.1 reduces to

$$\begin{cases} z_n = \begin{cases} -1, & \text{if } \frac{2((n \mod 4)+1)-\lambda_n}{2((n \mod 4)+1)} x_n \leq -1 \\ \frac{2((n \mod 4)+1)-\lambda_n}{2((n \mod 4)+1)} x_n, & \text{if } \frac{2((n \mod 4)+1)-\lambda_n}{2((n \mod 4)+1)} x_n \in C \\ 2, & \text{if } \frac{2((n \mod 4)+1)-\lambda_n}{2((n \mod 4)+1)} x_n \geq 2, \end{cases}, \\ y_n = \begin{cases} \alpha_n x_n + \frac{1}{(2((n \mod 5)+1))^n}(1-\alpha_n) z_n & \text{if } z_n \in [-1,0] \\ \alpha_n x_n + \frac{1}{(4((n \mod 5)+1))^n}(1-\alpha_n) z_n & \text{if } z_n \in (0,2], \end{cases}, \\ u_n = \begin{cases} \min\{2, \frac{2r_n+y_n}{1-r_n}, \max\{-1, \frac{y_n-r_n}{1-r_n}\}\} & \text{if } y_n \in [-1,1] \\ \min\{2, \frac{-r_n(y_n-2)^2+y_n+2r_n}{1-r_n}, \max\{-1, \frac{-r_n(y_n-2)^2+y_n-r_n}{1-r_n}\}\} & \text{if } y_n \in (1,2], \end{cases}, \end{cases}$$

$$\begin{cases} v_n = \begin{cases} \min\{2, \frac{2r_n-(2r_n-1)y_n}{1-r_n}, \max\{-1, \frac{r_n-(2r_n-1)y_n}{1-r_n}\}\} & \text{if } y_n \in [-1,0] \\ \min\{2, \frac{2r_n+y_n}{1-r_n}, \max\{-1, \frac{y_n-r_n}{1-r_n}\}\} & \text{if } y_n \in [0, \frac{1}{2}] \\ \min\{2, \frac{5r_n}{2(1-r_n)} + y_n, \max\{-1, \frac{-r_n}{2(1-r_n)} + y_n\}\} & \text{if } y_n \in [\frac{1}{2}, 2], \end{cases}, \\ w_n = \frac{u_n+v_n}{2}, \\ C_{n+1} = \begin{cases} \{z \in C_n : \frac{x_n+w_n}{2} \leq z\} & \text{if } w_n \geq x_n \\ \{z \in C_n : \frac{x_n+w_n}{2} \geq z\} & \text{if } w_n \leq x_n \end{cases}, \\ x_{n+1} = P_{C_{n+1}} x_0. \end{cases} \tag{3.5.4}$$

Then the scheme (5.5.1) converges strongly to 0. See the following table and figure.

n	xn	zn	yn	un	vn	wn
0	0.8	0.7976	0.797624	0.7955798	0.79257349	0.794076646
1	0.79703832	0.79584772	0.10638747	0.09745135	0.10138747	0.099419411
2	0.44822887	0.44778434	0.00747353	-0.0023535	0.01730052	0.007473528
3	0.2278512	0.22768241	0.00226719	-0.0075145	0.01204889	0.002267194
4	0.1150592	0.11471966	0.00110705	-0.0085909	0.01080504	0.001107049
5	0.05808312	0.05799777	0.00060927	-0.0090003	0.0102188	0.000609272
6	0.0293462	0.02931756	0.00027696	-0.0092442	0.00979813	0.000276962
7	0.01481158	0.01480078	0.00013843	-0.0092942	0.00957108	0.000138426
8	0.007475	0.0074533	6.9213E-05	-0.0092759	0.00941436	6.9213E-05
9	0.00377211	0.00376665	3.4606E-05	-0.0092243	0.00929355	3.46065E-05
10	0.00190336	0.00190153	1.7305E-05	-0.0091568	0.00919146	1.7305E-05
11	0.00096033	0.00095964	8.6516E-06	-0.0090822	0.00909948	8.65163E-06
12	0.00048449	0.00048311	4.3258E-06	-0.0090046	0.0090133	4.32582E-06
13	0.00024441	0.00024406	2.1629E-06	-0.0089264	0.00893072	2.16291E-06
14	0.00012329	0.00012317	1.0815E-06	-0.0088485	0.00885063	1.08145E-06
15	6.2184E-05	6.214E-05	5.4073E-07	-0.0087714	0.00877247	5.40727E-07
16	3.1362E-05	3.1274E-05	2.7036E-07	-0.0086954	0.00869592	2.70363E-07
17	1.5816E-05	1.5794E-05	1.3518E-07	-0.0086206	0.00862082	1.35182E-07
18	7.9757E-06	7.9683E-06	6.7591E-08	-0.0085469	0.00854708	6.75909E-08
19	4.0217E-06	4.0189E-06	3.3795E-08	-0.0084745	0.00847461	3.37954E-08



Remark 3.5.1. We provided a numerical example to demonstrate that for any choice of initial values the sequence generated by Scheme 3.1 converges to a common solution of the problem indicated.

Chapter 4

Approximation of Common Solutions of Nonlinear Problems in Banach spaces

4.1 Introduction

In this chapter, we discuss the second research problem of our dissertation. We propose hybrid projection iterative algorithm for finding a common element of solution set of a finite family of generalized mixed equilibrium problem, semi-fixed point set of a finite family of continuous semi-pseudocontractive mappings and solution set of a finite family of variational inequality for a finite family of monotone and L-Lipschitz mappings in Banach spaces and proved strong convergence theorem. We collect technical lemmas from literature to prove our main result in the second section of this chapter. We state the proposed algorithm in section 3 and prove its boundedness. The main result will be stated and proved in section 4. Finally, we give a numerical example to demonstrate the behavior of the convergence of the algorithm in the last section of the chapter.

4.2 Technical Tool Box

We shall need the following known results in the sequel of this chapter.

Lemma 4.2.1. *[57, 107] Let E^* be the dual space of a smooth and strictly convex real Banach space E and C be a nonempty, closed and convex subset of E. Let*

$T : E \to E^*$ be a continuous semi-pseudocontractive mapping. Let $r > 0$ and $x \in E$. Define a mapping $K_r^T : E \to 2^E$ by

$$K_r^T x = \{z \in C : \langle y - z, Tz\rangle - \frac{1}{r}\langle y - z, (1 + r)Jz - Jx\rangle \leq 0, \forall y \in C\}, x \in E.$$

Then, $K_r^T x \neq \emptyset, \forall x \in E$, and the following hold:

(1) K_r^T is single-valued;

(2) $F(K_r^T) = F_s(T)$, where $F_s(T)$ denotes the set of semi-fixed points of T;

(3) K_r^T is quasi-ϕ-nonexpansive;

(4) $\phi(p, K_r^T x) + \phi(K_r^T x, x) \leq \phi(p, x), \forall p \in F(K_r^T)$, $x \in E$.

4.3 Iterative Algorithm

Let C be a nonempty, closed and convex subset of uniformly convex and uniformly smooth real Banach space E. Let $B_k : C \to E^*$, $k = 1, 2, \cdots, M$ be a finite family of continuous and monotone mappings, $F_k : C \times C \to \mathbb{R}$, $k = 1, 2, \cdots, M$ be a finite family of bi-functionals satisfying **Condition A**, $A_i : C \to E^*$ be a finite family of L_i-Lipschitz monotone mappings with Lipschitz constants L_i, for $i = 1, 2, \ldots, K$, and $T_j : C \to E^*$, $j = 1, 2, \cdots, N$ be a finite family of continuous semi-pseudocontractive mappings with

$$\mathcal{F} := \left[\bigcap_{j=1}^{N} F_s(T_j)\right] \cap \left[\bigcap_{i=1}^{K} VI(C, A_i)\right] \cap \left[\bigcap_{k=1}^{M} GMEP(F_k, \varphi_k, B_k)\right] \neq \emptyset.$$

Let $\{x_n\}$ be the sequence generated by the iterative scheme:

$$\begin{cases} x_0 \in C_0 = C, \\ z_n = \Pi_C J^{-1}(Jx_n - \lambda_n A_{n+1} x_n) \\ u_n = \Pi_C J^{-1}(Jx_n - \lambda_n A_{n+1} z_n) \\ y_n = J^{-1}(\alpha_{n_0} Jx_n + \sum_{j=1}^{N} \alpha_{n_j} JK_{r_n}^{T_i} z_n), \\ w_n = J^{-1}(\beta_{n_0} Ju_n + \sum_{k=1}^{M} \beta_{n_k} JT_{r_n}^{H_k} y_n), \\ C_{n+1} = \{z \in C_n : \phi(z, w_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{4.3.1}$$

where $A_n = A_{n(\mod K)}$ and J is the normalized duality mapping on E; $\{r_n\} \subset [c_1, \infty)$ for some $c_1 > 0$, $\alpha_{n_j}, \beta_{n_k} \in [e, 1) \subset (0,1)$ such that $\sum_{j=0}^{N} \alpha_{n_j} = 1$, $\sum_{k=0}^{M} \beta_{n_k} = 1$, and $\{\lambda_n\}$ is a sequence in $[a, b]$ for some $0 < a < b < \frac{\mu}{L}$, where $L = \max_{1\le i\le K} L_i$.

Lemma 4.3.1. *Let C be a nonempty, closed and convex subset of a uniformly convex and smooth real Banach space E. Let $B_k : C \to E^*$, $k = 1, 2, \cdots, M$ be continuous and monotone mappings, $F_k : C \times C \to \mathbb{R}$, $k = 1, 2, \cdots, M$ be bi-functionals satisfying **Condition A**, $\varphi_k : C \to \mathbb{R}$, $k = 1, 2, \cdots, M$ be real valued functions, $A_i : C \to E^*$ be L_i-Lipschitz monotone mappings with Lipschitz constants L_i, for $i = 1, 2, \ldots, K$, and $T_j : E \to E^*$, $j = 1, 2, \cdots, N$ be continuous semi-pseudocontractive mappings with*

$$\mathcal{F} := \left[\bigcap_{j=1}^{N} F_s(T_j)\right] \cap \left[\bigcap_{i=1}^{K} VI(C, A_i)\right] \cap \left[\bigcap_{k=1}^{M} GMEP(F_k, \varphi_k, B_k)\right] \neq \emptyset.$$

Let $\{x_n\}$ be a sequence defined by (4.3.1). Then, the sequence $\{x_n\}$ is well defined for each $n \ge 0$.

Proof. Now, we divide the proof into two steps.

Step 1. We show that C_n is closed and convex for each $n \ge 0$. It is obvious that $C_0 = C$ is closed and convex by assumption. Suppose that C_k is closed and convex for some $k \in \mathbb{N}$.

Let $\{v_m\} \subseteq C_{k+1}$ be such that $v_m \to v$ as $m \to \infty$. Since $C_{k+1} \subseteq C_k$ and C_k is closed, $v \in C_k$ and

$$\begin{aligned}
\phi(v, w_k) &= \|v\|^2 - 2\langle v, Jw_k\rangle + \|w_k\|^2 \\
&= \lim_{m\to\infty} [\|v_m\|^2 - 2\langle v_m, Jw_k\rangle + \|w_k\|^2] \\
&= \lim_{m\to\infty} \phi(v_m, w_k) \\
&\le \liminf_{m\to\infty} \phi(v_m, x_k) = \phi(v, x_k).
\end{aligned}$$

Then $v \in C_{k+1}$ and so C_{k+1} is closed.

Let $v_1, v_2 \in C_{k+1}$ and $t \in [0, 1]$. Then $v_1, v_2 \in C_k$ and so that $v = tv_1 + (1-t)v_2 \in C_k$. Notice that

$$\phi(v_1, w_k) \le \phi(v_1, x_k) \text{ and } \phi(v_2, w_k) \le \phi(v_2, x_k). \tag{4.3.2}$$

The inequalities (4.3.2) are equivalent to

$$2\langle v_1, Jx_k - Jw_k\rangle \le \|x_k\|^2 - \|w_k\|^2 \tag{4.3.3}$$

and

$$2\langle v_2, Jx_k - Jw_k\rangle \leq \|x_k\|^2 - \|w_k\|^2. \tag{4.3.4}$$

Multiplying t and $(1-t)$ on both sides of (4.3.3) and (4.3.4), respectively, we obtain

$$2\langle v, Jx_k - Jw_k\rangle \leq \|x_k\|^2 - \|w_k\|^2.$$

That is,

$$\phi(v, w_k) \leq \phi(v, x_k).$$

This implies $v \in C_{k+1}$ and hence C_{k+1} is closed and convex. Therefore, inductively C_n is closed and convex for all $n \geq 0$. Hence $\Pi_{C_{n+1}} x_0$ is well defined for all $n \geq 0$.

Step 2. We show that $\mathcal{F} \subseteq C_n$ for each $n \geq 0$.

From the assumption, we see that $\mathcal{F} \subseteq C = C_0$. Suppose that $\mathcal{F} \subseteq C_k$ for some $k \in \mathbb{N}$. Now, we show that $\mathcal{F} \subseteq C_{k+1}$. Let $p \in \mathcal{F}$. Then by Lemma 3.2.1 and Lemma 4.2.1, we get

$$\begin{aligned}
\phi(p, y_k) &= \phi(p, J^{-1}(\alpha_{k_0} Jx_k + \sum_{i=1}^{N} \alpha_{k_i} JK_{r_k}^{T_i} z_k)) \qquad (4.3.5)\\
&= \|p\|^2 - 2\left\langle p, \alpha_{k_0} Jx_k + \sum_{i=1}^{N} \alpha_{k_i} JK_{r_k}^{T_i} z_k \right\rangle + \|\alpha_{k_0} Jx_k + \sum_{i=1}^{N} \alpha_{k_i} JK_{r_k}^{T_i} z_k\|^2\\
&\leq \|p\|^2 - 2\alpha_{k_0}\langle p, Jx_k\rangle - 2\sum_{i=1}^{N} \alpha_{k_i}\left\langle p, JK_{r_k}^{T_i} z_k\right\rangle\\
&\quad + \alpha_{k_0}\|x_k\|^2 + \sum_{i=1}^{N} \alpha_{k_i}\|K_{r_k}^{T_i} z_k\|^2 - \alpha_{k_0}\alpha_{k_j} g(\|Jx_k - JK_{r_k}^{T_j} z_k\|)\\
&\leq \alpha_{k_0}\left[\|p\|^2 - 2\langle p, Jx_k\rangle + \|x_k\|^2\right]\\
&\quad + \sum_{i=1}^{N} \alpha_{k_i}\left[\|p\|^2 - 2\left\langle p, JK_{r_k}^{T_i} z_k\right\rangle + \|K_{r_k}^{T_i} z_k\|^2\right]\\
&\quad - \alpha_{k_0}\alpha_{k_j} g(\|Jx_k - JK_{r_k}^{T_j} z_k\|)\\
&= \alpha_{k_0}\phi(p, x_k) + \sum_{i=1}^{N} \alpha_{k_i}\phi(p, K_{r_k}^{T_i} z_k) - \alpha_{k_0}\alpha_{k_j} g(\|Jx_k - JK_{r_k}^{T_j} z_k\|)\\
&\leq \alpha_{k_0}\phi(p, x_k) + \sum_{i=1}^{N} \alpha_{k_i}\phi(p, z_k) - \alpha_{k_0}\alpha_{k_j} g(\|Jx_k - JK_{r_k}^{T_j} z_k\|). \qquad (4.3.6)
\end{aligned}$$

Then

$$\phi(p, y_k) \leq \alpha_{k_0}\phi(p, x_k) + \sum_{i=1}^{N} \alpha_{k_i}\phi(p, z_k). \tag{4.3.7}$$

And by Lemma 3.2.4 and monotonicity of A_{k+1}, we have

$$\begin{aligned}
\phi(p, z_k) &= \phi(p, \Pi_C J^{-1}(Jx_k - \lambda_k A_{k+1}x_k)) \qquad (4.3.8)\\
&\leq \phi(p, J^{-1}(Jx_k - \lambda_k A_{k+1}x_k)) - \phi(z_k, J^{-1}(Jx_k - \lambda_k A_{k+1}x_k))\\
&= \|p\|^2 - 2\langle p, Jx_k - \lambda_k A_{k+1}x_k\rangle + \|Jx_k - \lambda_k A_{k+1}x_k)\|^2\\
&\quad -[\|Jx_k - \lambda_k A_{k+1}x_k\|^2 - 2\langle z_k, Jx_k - \lambda_k A_{k+1}x_k\rangle + \|z_k\|^2]\\
&= \|p\|^2 - 2\langle p, Jx_k\rangle + \|x_k\|^2 - \|x_k\|^2 + 2\langle z_k, Jx_k\rangle - \|z_k\|^2\\
&\quad +2\langle p - z_k, \lambda_k A_{k+1}x_k\rangle\\
&= \phi(p, x_k) - \phi(z_k, x_k) + 2\langle p - z_k, \lambda_k A_{k+1}x_k\rangle\\
&= \phi(p, x_k) - \phi(z_k, x_k) + 2\lambda_k\langle p - z_k, A_{k+1}x_k - A_{k+1}p\rangle\\
&\quad +2\lambda_k\langle p - z_k, A_{k+1}p\rangle\\
&\leq \phi(p, x_k).
\end{aligned}$$

and also,

$$\begin{aligned}
\phi(p, u_k) &= \phi(p, \Pi_C J^{-1}(Jx_k - \lambda_k A_{k+1}z_k))\\
&\leq \phi(p, J^{-1}(Jx_k - \lambda_k A_{k+1}z_k)) - \phi(u_k, J^{-1}(Jx_k - \lambda_k A_{k+1}z_k))
\end{aligned}$$

$$\begin{aligned}
&= \|p\|^2 - 2\langle p, Jx_k - \lambda_k A_{k+1}z_k\rangle + \|Jx_k - \lambda_k A_{k+1}z_k)\|^2 \qquad (4.3.9)\\
&\quad -[\|Jx_k - \lambda_k A_{k+1}z_k\|^2 - 2\langle u_k, Jx_k - \lambda_k A_{k+1}z_k\rangle + \|u_k\|^2]\\
&= \|p\|^2 - 2\langle p, Jx_k\rangle + \|x_k\|^2 - \|x_k\|^2 + 2\langle u_k, Jx_k\rangle - \|u_k\|^2\\
&\quad +2\langle p - u_k, \lambda_k A_{k+1}z_k\rangle\\
&= \phi(p, x_k) - \phi(u_k, x_k) + 2\langle p - u_k, \lambda_k A_{k+1}z_k\rangle\\
&= \phi(p, x_k) - \phi(u_k, x_k) + 2\lambda_k\langle p - z_k, A_{k+1}z_k\rangle + 2\lambda_k\langle z_k - u_k, A_{k+1}z_k\rangle\\
&= \phi(p, x_k) - \phi(u_k, x_k) + 2\lambda_k\langle p - z_k, A_{k+1}z_k - A_{k+1}p\rangle\\
&\quad +2\lambda_k\langle p - z_k, A_{k+1}p\rangle + 2\lambda_k\langle z_k - u_k, A_{k+1}z_k\rangle\\
&\leq \phi(p, x_k) - \phi(u_k, x_k) + 2\lambda_k\langle z_k - u_k, A_{k+1}z_k\rangle
\end{aligned}$$

and from (1.2.8), we obtain

$$\phi(u_k, x_k) = \phi(u_k, z_k) + \phi(z_k, x_k) + \langle u_k - z_k, Jz_k - Jx_k\rangle \qquad (4.3.10)$$

Thus, from (4.3.9) and (4.3.10) we get

$$\begin{aligned}
\phi(p, u_k) \quad &\leq \phi(p, x_k) - \phi(u_k, z_k) - \phi(z_k, x_k) \qquad (4.3.11)\\
&\quad + \langle z_k - u_k, \lambda_k A_{k+1}z_k - Jx_k + Jz_k\rangle
\end{aligned}$$

Moreover, by Lemma 3.2.6, we have that

$$\begin{aligned}\langle z_k - u_k, \lambda_k A_{k+1} z_k - Jx_k + Jz_k\rangle &= \langle u_k - z_k, \lambda_k A_{k+1} x_k - \lambda_k A_{k+1} z_k\rangle \\ &\quad + \langle u_k - z_k, Jx_k - \lambda_k A_{k+1} x_k - Jz_k\rangle \\ &\leq \lambda_k \langle u_k - z_k, A_{k+1} x_k - A_{k+1} z_k\rangle . \quad (4.3.12)\end{aligned}$$

Then, since A_{k+1} is Lipschitz, by (4.3.11) and (4.3.12), we have that,

$$\begin{aligned}\phi(p, u_k) &\leq \phi(p, x_k) - \phi(u_k, z_k) - \phi(z_k, x_k) + 2\lambda_k \langle u_k - z_k, A_{k+1} x_k - A_{k+1} z_k\rangle \\ &\leq \phi(p, x_k) - \phi(u_k, z_k) - \phi(z_k, x_k) + 2\lambda_k \|u_k - z_k\| \|A_{k+1} x_k - A_{k+1} z_k\| \\ &\leq \phi(p, x_k) - \phi(u_k, z_k) - \phi(z_k, x_k) + 2\lambda_k L \|u_k - z_k\| \|x_k - z_k\| \\ &\leq \phi(p, x_k) - \phi(u_k, z_k) - \phi(z_k, x_k) \quad (4.3.13) \\ &\quad + \lambda_k L[\|u_k - z_k\|^2 + \|x_k - z_k\|^2]\end{aligned}$$

Hence, from (4.3.13) and Lemma 3.2.3 we have

$$\begin{aligned}\phi(p, u_k) &\leq \phi(p, x_k) - \mu \|u_k - z_k\|^2 - \mu \|z_k - x_k\|^2 \\ &\quad + \lambda_k L[\|u_k - z_k\|^2 + \|x_k - z_k\|^2] \\ &\leq \phi(p, x_k) + (\lambda_k L - \mu)[\|u_k - z_k\|^2 + \|x_k - z_k\|^2]. \quad (4.3.14)\end{aligned}$$

Therefore, from (4.3.14) and $\lambda_k \in (0, \frac{\mu}{L})$ we obtain that

$$\phi(p, u_k) \leq \phi(p, x_k) \quad (4.3.15)$$

Thus, by (4.3.7), (4.3.8), (4.3.15) and Lemma 3.2.1, we obtain

$$\begin{aligned}\phi(p, w_k) &= \phi(p, J^{-1}(\beta_{k_0} Ju_k + \sum_{s=1}^{M} \beta_{k_s} JT_{r_k}^{H_s} y_k)) \\ &= \|p\|^2 - 2 \left\langle p, \beta_{k_0} Ju_k + \sum_{s=1}^{M} \beta_{k_s} JT_{r_k}^{H_s} y_k \right\rangle \\ &\quad + \|\alpha_{k_0} Ju_k + \sum_{s=1}^{M} \beta_{k_s} JT_{r_k}^{H_s} y_k\|^2 \\ &\leq \|p\|^2 - 2\beta_{k_0} \langle p, Jx_k\rangle - 2 \sum_{s=1}^{M} \beta_{k_s} \langle p, JT_{r_k}^{H_s} y_k\rangle + \beta_{k_0} \|u_k\|^2 \\ &\quad + \sum_{s=1}^{M} \beta_{k_s} \|T_{r_k}^{H_s} y_k\|^2 - \beta_{k_0} \beta_{k_t} g(\|Ju_k - JT_{r_k}^{H_t} y_k\|)\end{aligned}$$

$$\begin{aligned}
&\le \beta_{k_0}\left[\|p\|^2 - 2\langle p, Ju_k\rangle + \|u_k\|^2\right] - \beta_{k_0}\beta_{k_t} g(\|Ju_k - JT_{r_k}^{H_t} y_k\|) \\
&\quad + \sum_{s=1}^{M} \beta_{k_s}\left[\|p\|^2 - 2\left\langle p, JT_{r_k}^{H_s} y_k\right\rangle + \|T_{r_k}^{H_s} y_k\|^2\right] \\
&= \beta_{k_0}\phi(p, u_k) - \beta_{k_0}\beta_{k_t} g(\|Ju_k - JT_{r_k}^{H_t} y_k\|) + \sum_{s=1}^{M} \beta_{k_s}\phi(p, T_{r_k}^{H_s} y_k) \\
&\le \beta_{k_0}\phi(p, u_k) + \sum_{s=1}^{M} \beta_{k_s}\phi(p, y_k) - \beta_{k_0}\beta_{k_t} g(\|Ju_k - JT_{r_k}^{H_t} y_k\|) \qquad (4.3.16) \\
&\le \phi(p, x_k). \qquad (4.3.17)
\end{aligned}$$

Then $p \in C_{k+1}$. This implies, by induction, that $\mathcal{F} \subseteq C_n$ and the sequence $\{x_n\}$ generated by (4.3.1) is well defined for all $n \ge 0$.

4.4 Strong Convergence Theorem

Theorem 4.4.1. *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth real Banach space E. Let $B_k : C \to E^*$, $k = 1, 2, \cdots, M$ be continuous and monotone mappings, $F_k : C \times C \to \mathbb{R}$, $k = 1, 2, \cdots, M$ be bi-functionals satisfying **Condition A**, $\varphi_k : C \to \mathbb{R}$, $k = 1, 2, \cdots, M$ be real valued functions, $A_i : C \to E^*$ be L_i-Lipschitz monotone mappings with Lipschitz constants L_i, for $i = 1, 2, \ldots, K$, and $T_j : E \to E^*$, $j = 1, 2, \cdots, N$ be continuous semi-pseudocontractive mappings with*

$$\mathcal{F} := \left[\bigcap_{j=1}^{N} F_s(T_j)\right] \cap \left[\bigcap_{i=1}^{K} VI(C, A_i)\right] \cap \left[\bigcap_{k=1}^{M} GMEP(F_k, \varphi_k, B_k)\right] \ne \emptyset.$$

Let $\{x_n\}$ be a sequence defined by (4.3.1). Then, the sequence $\{x_n\}$ converges to some element of $\mathcal{F}$.

Proof. We divide the proof into four steps.

Step 1. We show that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} w_n = x$, $\lim_{n\to\infty} y_n = x$, $\lim_{n\to\infty} u_n = x$, $\lim_{n\to\infty} z_n = x$, $\lim_{n\to\infty} K_{r_n}^{T_j} y_n = x, \forall j = 1, 2, \ldots, N$ and $\lim_{n\to\infty} T_{r_n}^{H_k} y_n = x, \forall k = 1, 2, \ldots, M$, for some point $x \in C$.

In view of $\Pi_{C_n} x_0$, we see from Lemma 3.2.6 that

$$\langle x_n - z, Jx_0 - Jx_n\rangle \ge 0, \forall z \in C_n.$$

The fact that $\mathcal{F} \subseteq C_n$ implies that

$$\langle x_n - p, Jx_0 - Jx_n\rangle \ge 0, \forall p \in \mathcal{F}. \qquad (4.4.1)$$

From Lemma 3.2.7, we have

$$\begin{aligned}\phi(x_n, x_0) &= \phi(\Pi_{C_n}x_0, x_0)\\ &\leq \phi(\Pi_{\mathcal{F}}x_0, x_0) - \phi(\Pi_{\mathcal{F}}x_0, x_n)\\ &\leq \phi(\Pi_{\mathcal{F}}x_0, x_0).\end{aligned}$$

This implies that the sequence $\{\phi(x_n, x_0)\}$ is bounded. Since $x_n = \Pi_{C_n}x_0$ and $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1} \subseteq C_n$, we have that

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \forall n \geq 0$$

which implies that $\{\phi(x_n, x_0)\}$ is increasing and bounded sequence and so $\lim_{n\to\infty} \phi(x_n, x_0)$ exists. Furthermore, for any positive integer l, the inequality in Lemma 3.2.7 provides

$$\begin{aligned}\phi(x_{n+l}, x_n) &= \phi(x_{n+l}, \Pi_{C_n}x_0)\\ &\leq \phi(x_{n+l}, x_0) - \phi(\Pi_{C_n}x_0, x_0)\\ &\leq \phi(x_{n+l}, x_0) - \phi(x_n, x_0).\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \phi(x_{n+l}, x_n) = 0, \text{ for any integer } l > 0. \tag{4.4.2}$$

Thus, it follows from Lemma 3.2.5 that

$$\lim_{n\to\infty} \|x_{n+l} - x_n\| = 0. \tag{4.4.3}$$

and hence $\{x_n\}$ is a Cauchy sequence. Therefore, there exists a point $x \in C$ such that $x_n \to x$ as $n \to \infty$. Since $x_{n+1} \in C_{n+1}$, we get

$$\phi(x_{n+1}, w_n) \leq \phi(x_{n+1}, x_n),$$

this with (4.4.2) implies that

$$\lim_{n\to\infty} \phi(x_{n+1}, w_n) = 0. \tag{4.4.4}$$

Thus, by Lemma 3.2.5, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - w_n\| = 0, \tag{4.4.5}$$

and hence

$$\|x_n - w_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - w_n\| \to 0 \text{ as } n \to \infty, \tag{4.4.6}$$

which implies that $w_n \to x$ as $n \to \infty$.

Now, from (4.3.17), (4.3.14), (4.3.6) and (4.3.8), we get

$$\begin{aligned} \phi(p, w_n) &\leq \phi(p, x_n) + \beta_{n0}(\lambda_n L - \mu)\left[\|u_n - z_n\|^2 + \|x_n - z_n\|^2\right] \\ &- \alpha_{n0}\alpha_{n_j}(1-\beta_{n0})g(\|Jx_n - JK_{r_n}^{T_j} z_n\|) - \beta_{n0}\beta_{n_t} g(\|Ju_n - JT_{r_n}^{H_t} y_n\|) \end{aligned} \tag{4.4.7}$$

Thus, from (4.4.7) and the fact that $\alpha_{n_j}, \beta_{n_t} \geq e > 0$, and $\lambda_n \in (0, \frac{\mu}{L})$, for all $n \geq 0$, we have

$$\lim_{n\to\infty} \|u_n - z_n\| = 0, \lim_{n\to\infty} \|x_n - z_n\| = 0, \tag{4.4.8}$$

and

$$\lim_{n\to\infty} g(\|Jx_n - JK_{r_n}^{T_j} z_n\|) = 0 \text{ and } \lim_{n\to\infty} g(\|Ju_n - JT_{r_n}^{H_t} y_n\|) = 0. \tag{4.4.9}$$

Therefore, from (4.4.9) and the property of g, we get that

$$\lim_{n\to\infty} \|Jx_n - JK_{r_n}^{T_j} z_n\| = 0 \text{ and } \lim_{n\to\infty} \|Ju_n - JT_{r_n}^{H_t} y_n\| = 0. \tag{4.4.10}$$

As E^* is uniformly smooth, J^{-1} is uniformly norm-to-norm continuous on bounded subsets of E^* we obtain

$$\lim_{n\to\infty} \|x_n - K_{r_n}^{T_j} z_n\| = 0 \text{ and } \lim_{n\to\infty} \|u_n - T_{r_n}^{H_t} y_n\| = 0. \tag{4.4.11}$$

Now

$$\begin{aligned} \phi(x_n, y_n) &\leq \phi(x_n, J^{-1}(\alpha_{n0} Jx_n + \sum_{j=1}^{N} \alpha_{n_j} JK_{r_n}^{T_i} z_n)) \\ &\leq \sum_{j=1}^{N} \alpha_{n_j} \phi(x_n, K_{r_n}^{T_i} z_n). \end{aligned} \tag{4.4.12}$$

From (4.4.11) and (4.4.12), we obtain

$$\liminf_{n\to\infty} \phi(x_n, y_n) = 0$$

and so, by Lemma 3.2.5 we have

$$\liminf_{n\to\infty} \|x_n - y_n\| = 0 \tag{4.4.13}$$

Since $x_n \to x$ as $n \to \infty$, and from (4.4.8), (4.4.11), and (4.4.13), we obtain $z_n \to x$, $u_n \to x$, $y_n \to x$, $K_{r_n}^{T_j} z_n \to x$, $T_{r_n}^{H_t} y_n \to x$ as $n \to \infty$, for each $j = 1, 2, \dots, N$ and $t = 1, 2, \dots, M$.

Step 2. We show that $x \in \bigcap_{i=1}^{K} VI(C, A_i)$.

Since A_i is Lipschitz monotone countinuous for $i = 1, 2, \dots, K$, we have

$$\|A_{n+1}u_n - A_{n+1}z_n\| \leq L\|u_n - z_n\| \to 0 \text{ as } n \to \infty. \tag{4.4.14}$$

Now, Let

$$B_i v = \begin{cases} A_i v + N_C v \text{ if } v \in C; \\ \emptyset \text{ if } v \notin C \end{cases},$$

where N_C is the normal cone to C at $v \in C$ given by $N_C = \{w \in E^* : \langle v - u, w\rangle \geq 0, \forall u \in C\}$. Then, by Lemma 3.3.1, B_i is maximal monotone and $B_i^{-1} = VI(A_i, C)$. Let $w \in B_i v$. Then, we have $w \in A_i v + N_C v$ and hence $w - A_i v \in N_C v$. Thus, we get $\langle v - u, w - A_i v\rangle \geq 0, \forall u \in C$. Let $\{n_{s_i}\}_{s_i \geq 1} \subseteq \mathbb{N}$ be such that $A_{n_{s_i}+1} = A_i$ for all $s_i \in \mathbb{N}$ where $i = 1, 2, \dots, K$. Then $\langle v - u, w - A_{n_{s_i}+1}v\rangle \geq 0, \forall u \in C$. On the other hand, since $u_{n_{s_i}} = \Pi_C J^{-1}(Jx_{n_{s_i}} - \lambda_{n_{s_i}} A_{n_{s_i}+1} z_{n_{s_i}})$, and $v \in C$, we have

$$\left\langle v - u_{n_{s_i}}, Jd_{n_{s_i}} - (Jx_{n_{s_i}} - \lambda_n A_{n_{s_i}+1} z_{n_{s_i}})\right\rangle \geq 0,$$

and so

$$\left\langle v - u_{n_{s_i}}, \frac{Ju_{n_{s_i}} - Jx_{n_{s_i}}}{\lambda_{n_{s_i}}} + A_{n_{s_i}+1} z_{n_{s_i}}\right\rangle \geq 0.$$

Thus as $u_{n_{s_i}} \in C$, the above imply that

$$\begin{aligned} \left\langle v - u_{n_{s_i}}, w\right\rangle &\geq \left\langle v - u_{n_{s_i}}, A_{n_{s_i}+1} v\right\rangle \\ &\geq \left\langle v - u_{n_{s_i}}, A_{n_{s_i}+1} v\right\rangle - \left\langle v - u_{n_{s_i}}, \frac{Ju_{n_{s_i}} - Jx_{n_{s_i}}}{\lambda_{n_{s_i}}} + A_{n_{s_i}+1} z_{n_{s_i}}\right\rangle \\ &= \left\langle v - u_{n_{s_i}}, A_{n_{s_i}+1} v - A_{n_{s_i}+1} u_{n_{s_i}}\right\rangle \\ &\quad + \left\langle v - u_{n_{s_i}}, A_{n_{s_i}+1} u_{n_{s_i}} - A_{n_{s_i}+1} z_{n_{s_i}}\right\rangle \\ &\quad - \left\langle v - u_{n_{s_i}}, \frac{Ju_{n_{s_i}} - Jx_{n_{s_i}}}{\lambda_{n_{s_i}}}\right\rangle \\ &= \left\langle v - u_{n_{s_i}}, A_{n_{s_i}+1} u_{n_{s_i}} - A_{n_{s_i}+1} z_{n_{s_i}}\right\rangle \\ &\quad - \left\langle v - u_{n_{s_i}}, \frac{Ju_{n_{s_i}} - Jx_{n_{s_i}}}{\lambda_{n_{s_i}}}\right\rangle. \end{aligned} \tag{4.4.15}$$

Therefore, since E is uniformly smooth we have J is uniformly continuous and hence $Ju_{n_{s_i}} - Jz_{n_{s_i}} \to 0$ as $s_i \to \infty$. Thus, from (4.4.15) we get that $\langle v - x, w\rangle \geq 0$ as $n \to \infty$. Then, maximality of B_i gives that $x \in B_i^{-1}(0) = VI(A_i, C)$ for each i. Therefore, $x \in \bigcap_{i=1}^{m} VI(C, A_i)$.

Step 3. We show that $x \in \bigcap_{j=1}^{N} F_J(T_j)$. Let $v_n^i = K_{r_n}^{T_i} x_n$. By Lemma 4.2.1(2), we get that

$$\left\langle y - v_n^i, T_i v_n^i \right\rangle - \frac{1}{r_n} \left\langle y - v_n^i, (1 + r_n) J v_n^i - J x_n \right\rangle \geq 0, \forall y \in C$$

Since C is convex, $y_\lambda = \lambda y + (1 - \lambda)x \in C$, where $\lambda \in (0, 1]$. Then

$$\begin{aligned}
\left\langle v_n^i - y_\lambda, T_i y_\lambda \right\rangle &\geq \left\langle v_n^i - y_\lambda, T_i y_\lambda \right\rangle + \left\langle y_\lambda - v_n^i, T_i v_n^i \right\rangle \\
&\quad - \frac{1}{r_n} \left\langle y_\lambda - v_n^i, (1 + r_n) J v_n^i - J x_n \right\rangle \\
&= \left\langle v_n^i - y_\lambda, T_i y_\lambda - T_i v_n^i \right\rangle - \frac{1}{r_n} \left\langle y_\lambda - v_n^i, (1 + r_n) J v_n^i - J x_n \right\rangle \\
&\geq \left\langle v_n^i - y_\lambda, J y_\lambda - J v_n^i \right\rangle - \frac{1}{r_n} \left\langle y_\lambda - v_n^i, (1 + r_n) J v_n^i - J x_n \right\rangle \\
&= \left\langle v_n^i - y_\lambda, J y_\lambda \right\rangle - \frac{1}{r_n} \left\langle y_\lambda - v_n^i, J v_n^i - J x_n \right\rangle \\
&\geq \left\langle v_n^i - y_\lambda, J y_\lambda \right\rangle - \|y_\lambda - v_n^i\| \frac{\|J v_n^i - J x_n\|}{r_n} \\
&\geq \left\langle v_n^i - y_\lambda, J y_\lambda \right\rangle - W \frac{\|J v_n^i - J x_n\|}{r_n}, \qquad (4.4.16)
\end{aligned}$$

where $W = \max_{1 \leq i \leq N} \sup_{n \geq 0} \|y_\lambda - v_n^i\|$.

Taking $n \to \infty$ on both sides in (4.4.16) , we get that

$$\langle x - y_\lambda, T_i y_\lambda \rangle \geq \langle x - y_\lambda, J y_\lambda \rangle \qquad (4.4.17)$$

Then, we obtain

$$\langle x - y, T_i(x + \lambda(y - x)) \rangle \geq \langle x - y, J(x + \lambda(y - x)) \rangle, \forall y \in C. \qquad (4.4.18)$$

Letting $\lambda \downarrow 0$ and using the fact that T_i is continuous, we have from inequality (4.4.18) that

$$\langle x - y, T_i x \rangle \geq \langle x - y, J x \rangle, \forall y \in C \Leftrightarrow 0 \geq \langle x - y, J x - T_i x \rangle, \forall y \in C. \qquad (4.4.19)$$

Setting $y = J^{-1}(T_i x)$. Since E^* is strictly convex and J^{-1} is monotone, we get that

$$\left\langle x - J^{-1}(T_i x), J x - T_i x \right\rangle = 0 \qquad (4.4.20)$$

which implies that $T_i x = J x$. Hence $x \in F_J(T_i)$, for each $i = 1, 2, \ldots, N$. Therefore, $x \in \bigcap_{j=1}^{N} F_J(T_j)$.

Step 4. We show that $x \in \bigcap_{k=1}^{M} GMEP(F_k, \varphi_k, B_k)$.

Let $d_n^i = T_{r_n}^{H_i} y_n$. Then, by lemma 1.3.1 we have

$$H_i(d_n^i, y) + \frac{1}{r_n}\left\langle y - d_n^i, Jd_n^i - Jy_n \right\rangle \geq 0, \forall y \in C.$$

Thus, by **"Condition A"**, we have

$$\begin{aligned} H_i(y, d_n^i) = -H_i(d_n^i, y) &\leq \frac{1}{r_n}\left\langle y - d_n^i, Jd_n^i - Jy_n \right\rangle \\ &\leq \|y - d_n^i\| \frac{\|Jd_n^i - Jy_n\|}{r_n} \\ &\leq P\frac{\|Jd_n^i - Jy_n\|}{r_n}, \end{aligned} \tag{4.4.21}$$

where $P = \max_{1\leq i\leq N} \sup_{n\geq 0} \|y - d_n^i\|$. Since $y \mapsto H(x, y)$ is convex and lower semi-continuous and from (4.4.21), we obtain

$$H_i(y, x) \leq 0, \forall y \in C. \tag{4.4.22}$$

Setting $y_\lambda = \lambda y + (1-\lambda)x, \lambda \in (0, 1]$, we get $y_\lambda \in C$ and so, from (4.4.22) we obtain $H_i(y_\lambda, x) \leq 0$. By **Condition A**, we have that

$$\begin{aligned} 0 &= H_i(y_\lambda, y_\lambda) \\ &= \lambda H_i(y_\lambda, y) + (1-\lambda)H_i(y_\lambda, x) \\ &\leq H_i(x + \lambda(x - y), y). \end{aligned} \tag{4.4.23}$$

Letting $\lambda \downarrow 0$ and using **"Condition A"**, we have from inequality (4.4.23) that

$$H_i(x, y) \geq 0, \forall y \in C.$$

Hence, $x \in GMEP(F_i, \varphi_i, B_i)$, for each $i = 1, 2, \ldots\ldots, M$. Therefore, we obtain that

$$x \in \bigcap_{i=1}^{M} GMEP(F_i, \varphi_i, B_i).$$

Finally, we prove that $x = \Pi_{\mathcal{F}} x_0$. From $x_n = \Pi_{C_n} x_0$, we have

$$\langle Jx_0 - Jx_n, x_n - z\rangle \geq 0, \forall z \in C_n.$$

Since $\mathcal{F} \subseteq C_n$, we also have that

$$\langle Jx_0 - Jx_n, x_n - z\rangle \geq 0, \forall z \in \mathcal{F}.$$

Therefore, Now, by Lemma 3.2.6 we obtain $x = \Pi_{\mathcal{F}} x_0$. This is completes the proof. If in Theorem 4.4.1, we assume that $F_k = 0$, $B_k = 0$, and $\varphi_k = 0$ for $k = 1, 2, \cdots, M$, then we get the following corollary.

Corollary 4.4.2. *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth real Banach space E. Let $A_i : C \to E^*$ be a finite family of L_i-Lipschitz monotone mappings with Lipschitz constants L_i, for $i = 1, 2, \ldots, K$, and $T_j : E \to E^*$, $j = 1, 2, \cdots, N$ be a a finite family of continuous semi-pseudocontractive mappings with*

$$\mathcal{F} := \left[\bigcap_{j=1}^{N} F_s(T_j)\right] \cap \left[\bigcap_{i=1}^{K} VI(C, A_i)\right] \neq \emptyset.$$

Let $\{x_n\}$ be the sequences generated by the iterative scheme:

$$\begin{cases} x_0 \in C_0 = C, \\ z_n = \Pi_C J^{-1}(Jx_n - \lambda_n A_{n+1} x_n) \\ u_n = \Pi_C J^{-1}(Jx_n - \lambda_n A_{n+1} z_n) \\ y_n = J^{-1}(\alpha_{n_0} Jx_n + \sum_{j=1}^{N} \alpha_{n_j} JK_{r_n}^{T_i} z_n), \\ w_n = J^{-1}(\beta_{n_0} Ju_n + (1 - \beta_{n_0}) y_n), \\ C_{n+1} = \{z \in C_n : \phi(z, w_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{4.4.24}$$

where $A_n = A_{n(\bmod K)}$ and J is the normalized duality mapping on E; $\{r_n\} \subset [c_1, \infty)$ for some $c_1 > 0$, $\alpha_{n_j}, \beta_{n_0} \in [e, 1) \subset (0, 1)$ such that $\sum_{j=0}^{N} \alpha_{n_j} = 1$, and $\{\lambda_n\}$ is a sequence in $[a, b]$ for some $0 < a < b < \frac{1}{L}$, where $L = \max_{1 \leq i \leq K} L_i$. Then, the sequence $\{x_n\}$ converges to some element of $\mathcal{F}$.

If in Theorem 4.4.1, we assume that $A_i = 0$ for $i = 1, 2, \cdots, K$, then we get the following corollary.

Corollary 4.4.3. *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth real Banach space E. Let $B_k : C \to E^*$, $k = 1, 2, \cdots, M$ be continuous and monotone mappings, $F_k : C \times C \to \mathbb{R}$, $k = 1, 2, \cdots, M$ be bi-functionals satisfying **Condition A**, $\varphi_k : C \to \mathbb{R}$, $k = 1, 2, \cdots, M$ be real valued functions and $T_j : E \to E^*$, $j = 1, 2, \cdots, N$ be continuous semi-pseudocontractive mappings with*

$$\mathcal{F} := \left[\bigcap_{j=1}^{N} F_s(T_j)\right] \cap \left[\bigcap_{k=1}^{M} GMEP(F_k, \varphi_k, B_k)\right] \neq \emptyset.$$

Let $\{x_n\}$ be the sequences generated by the iterative scheme:

$$\begin{cases} x_0 \in C_0 = C, \\ y_n = J^{-1}(\alpha_{n_0} Jx_n + \sum_{j=1}^{N} \alpha_{n_j} JK_{r_n}^{T_i} x_n), \\ w_n = J^{-1}(\beta_{n_0} Jx_n + \sum_{k=1}^{M} \beta_{n_k} JT_{r_n}^{H_k} y_n), \\ C_{n+1} = \{z \in C_n : \phi(z, w_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{4.4.25}$$

where $\{r_n\} \subset [c_1, \infty)$ for some $c_1 > 0$, $\alpha_{n_j}, \beta_{n_k} \in [e, 1) \subset (0, 1)$ such that $\sum_{j=0}^{N} \alpha_{n_j} = 1$, $\sum_{k=0}^{M} \beta_{n_k} = 1$. Then, the sequence $\{x_n\}$ converges to some element of $\mathcal{F}$.

If in Theorem 4.4.1, we assume that $A_i = 0$ for $i = 1, 2, \cdots, K$ and $\varphi_k = 0$ for $k = 1, 2, \cdots, M$, then we get the following corollary.

Corollary 4.4.4. *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth real Banach space E. Let $B_k : C \to E^*$, $k = 1, 2, \cdots, M$ be continuous and monotone mappings, $F_k : C \times C \to \mathbb{R}$, $k = 1, 2, \cdots, M$ be bi-functionals satisfying **Condition A** and $T_j : E \to E^*$, $j = 1, 2, \cdots, N$ be continuous semi-pseudocontractive mappings with $\mathcal{F} := \left[\bigcap_{j=1}^{N} F_s(T_j)\right] \cap \left[\bigcap_{k=1}^{M} GEP(F_k, B_k)\right] \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by (4.4.25) with $\varphi_k = 0$ for $k = 1, 2, \cdots, M$. Then, the sequence $\{x_n\}$ converges to some element of $\mathcal{F}$.*

If in Theorem 4.4.1, we assume that $B_i = 0$ for $i = 1, 2, \cdots, K$ and $\varphi_k = 0$ for $k = 1, 2, \cdots, M$, then we get the following corollary.

Corollary 4.4.5. *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth real Banach space E. Let $F_k : C \times C \to \mathbb{R}$, $k = 1, 2, \cdots, M$ be bi-functionals satisfying **Condition A**, $A_i : C \to E^*$ be L_i-Lipschitz monotone mappings with Lipschitz constants L_i, for $i = 1, 2, \ldots, K$, and $T_j : E \to E^*$, $j = 1, 2, \cdots, N$ be continuous semi-pseudocontractive mappings with $\mathcal{F} := \left[\bigcap_{j=1}^{N} F_s(T_j)\right] \cap \left[\bigcap_{i=1}^{K} VI(C, A_i)\right] \cap \left[\bigcap_{k=1}^{M} EP(F_k)\right] \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by (4.3.1) with $B_i = 0$ for $i = 1, 2, \cdots, K$ and $\varphi_k = 0$ for $k = 1, 2, \cdots, M$. Then, the sequence $\{x_n\}$ converges to some element of $\mathcal{F}$.*

If in Theorem 4.4.1, we assume that $F_k = 0$, $B_k = 0$, $A_i = 0$ for $i = 1, 2, \cdots, K$ and $\varphi_k = 0$ for $k = 1, 2, \cdots, M$, then we get the following corollary.

Corollary 4.4.6. *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth real Banach space E. Let $T_j : E \to E^*$, $j = 1, 2, \cdots, N$ be continuous semi-pseudocontractive mappings with $\mathcal{F} := \bigcap_{j=1}^{N} F_s(T_j) \neq \emptyset$. Let $\{x_n\}$ be the sequences generated by the iterative scheme:*

$$\begin{cases} x_0 \in C_0 = C, \\ y_n = J^{-1}(\alpha_{n_0} Jx_n + \sum_{j=1}^{N} \alpha_{n_j} JK_{r_n}^{T_i} x_n), \\ w_n = J^{-1}(\beta_{n_0} Jx_n + (1-\beta_{n_0}) y_n), \\ C_{n+1} = \{z \in C_n : \phi(z, w_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{4.4.26}$$

where $\{r_n\} \subset [c_1, \infty)$ for some $c_1 > 0$, $\alpha_{n_j}, \beta_{n_0} \in [e, 1) \subset (0, 1)$ such that $\sum_{j=0}^{N} \alpha_{n_j} = 1$. Then, the sequence $\{x_n\}$ converges to some element of $\mathcal{F}$.

4.5 Numerical Experiments

In this section, we provide a numerical experiment which shows the implementation of the proposed algorithm.

Example 4.5.1. *Let $E = L_2^{\mathbb{R}}([0, 1])$ with norm $\|x\|_{L_2^{\mathbb{R}}} = (\int_0^1 |x(t)|^2 dt)^{\frac{1}{2}}$, for $x \in E$ and inner product $\langle z, w \rangle = \int_0^1 z(t)w(t)dt$, for $z, w \in E$ and let $C = \{x \in E : \|x\|_{L_2^{\mathbb{R}}} \leq 1\}$ be nonempty, closed and convex subset of E. Let A, T, $B : C \to E^*$ be defined by $A(x)(t) = J(x)(t)$; $T(x)(t) = -tJ(x)(t)$ and $B(x)(t) = \frac{3}{5}J(x)(t)$, for all $x(t) \in C, t \in [0, 1]$, respectively. Let $F : C \times C \to \mathbb{R}$ be defined by $F(x, y) = \frac{2}{5}\langle y - x, J(x) \rangle, \forall x, y \in C$. Then A is Lipschitz monotone mapping with $VI(C, A) = \{0\}$; T is continuous semi-pseudocontractive mapping with $F_s(T) = \{0\}$; B is continuous monotone mapping, and F is bi-function satisfying* ***Condition A****. Thus, a solution of the generalized mixed equilibrium problem is $GMEP(F, \varphi, B) = \{0\}$, where $\varphi \equiv$ constant and hence $\mathcal{F} := F_s(T) \cap GMEP(F, \varphi, B) \cap VI(C, A) = \{0\}$. For implementation, we choose $N = M = K = 1$, $\alpha_{n_0} = \frac{1}{n_0+10}$, $\beta_{n_0} = 0.1$, $r_n = 0.5$ and $\lambda_n = 0.001 + \frac{1}{100n}$, for $n \geq 0$, and we* *compute the we compute the* $(n+1)^{th}$ *iteration as follows:*

$$\begin{cases} v_n(t) = x_n(s) - \lambda_n(1+2t)x_n(t), \\ z_n(t) = \min\{1, \frac{1}{\|v_n\|_{L_2^{\mathbb{R}}}}\}v_n(t), \\ h_n(t) = x_n(s) - \lambda_n(1+2t)z_n(t), \\ u_n(t) = \min\{1, \frac{1}{\|h_n\|_{L_2^{\mathbb{R}}}}\}h_n(t), \\ y_n(t) = \alpha_{n_0}x_n + (1-\alpha_{n_0})\frac{1}{r_nt+r_n+1}u_n(t), \\ w_n(s) = \beta_{n_0}u_n(t) + (1-\beta_{n_0})\frac{1}{1+r_n}y_n(t), \\ C_{n+1} = \{z \in C_n : \langle z, x_n - w_n\rangle \leq \frac{1}{2}\langle w_n + x_n, x_n - w_n\rangle\}, \\ x_{n+1}(t) = \begin{cases} x_0(t) - \frac{\langle a_n, x_0\rangle - b_n}{\|a_n\|^2}a_n(t) & \text{if} \quad \langle a_n, x_0\rangle > b_n \\ x_0(t) & \text{if} \quad \langle a_n, x_0\rangle \leq b_n \end{cases} \end{cases} \tag{4.5.1}$$

where $a_n(t) = x_n(t) - w_n(t)$ *and* $b_n = \frac{1}{2}\langle w_n + x_n, x_n - w_n\rangle$.

Now, taking different initial points, $x_0 = t^{10}$, $x_0 = t^{15}$, $x_0 = t^{20}$ and $x_0 = t^{30}$ from C, we obtain Table 4.1 of numerical results provides that the sequence $\{\|x_n - p\|\}$ approaches zero as $n \to \infty$ and described in Figure 1 below. In this case, we observe that the sequence $\{x_n\}$ converges faster when the power of t gets large.

Table 4.1: Table of numerical results for Example 5.5.1

x_0	0	1	5	6	7	8
t^{10}	0.600000	0.218218	0.052004	0.036432	0.025545	0.019613
t^{15}	0.600000	0.179605	0.042661	0.029862	0.020915	0.015413
t^{20}	0.600000	0.156174	0.037032	0.025910	0.018138	0.013113
t^{30}	0.600000	0.128037	0.030307	0.021196	0.014829	0.010546

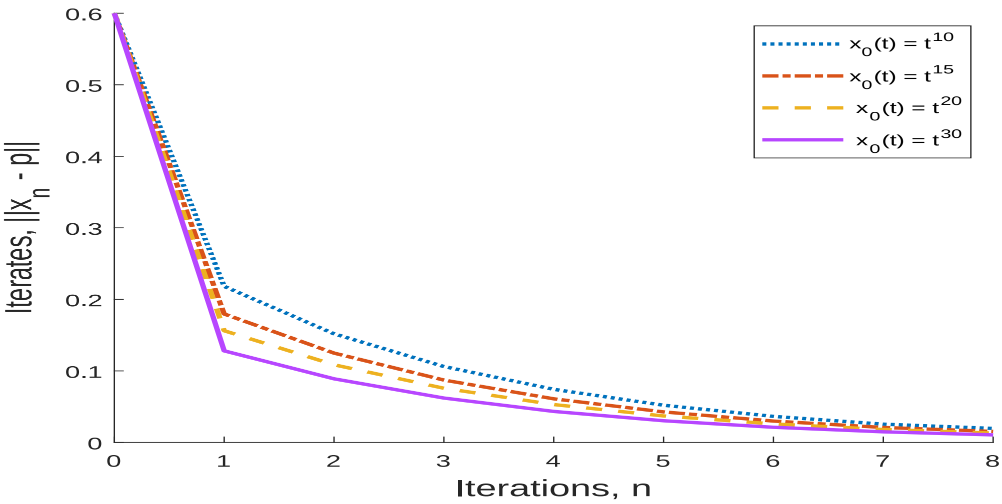


Figure 4.1: Convergence of the sequence $\{||x_n - p||\}$ as n gets large.

Chapter 5

Approximating a Common Solution of f-fixed point, Variational Inequality and Generalized Mixed Equilibrium Problems

5.1 Introduction

The third main result of this dissertation is discussed in this chapter. An algorithm for finding a common element of the set of solutions of a finite family of generalized mixed equilibrium problems, the set of f-fixed points of a finite family of f-pseudocontractive mappings and the set of solutions of a finite family of variational inequality problems for Lipschitz monotone mappings in real reflexive Banach spaces is proposed. The convergence of the proposed algorithm is discussed. Practicality of the main result is supported by numerical examples.

5.2 Preliminaries

Some of the results in the literature that provide technical tools to prove the main result are mentioned below.

Lemma 5.2.1. *([21]) Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function.*

Then the following properties are satisfied:

(i) $\nabla f : E \to E^*$ *is one-to-one, onto and norm-to-weak continuous;*

(ii) *the set* $\{x \in E : D_f(x, y) \leq r\}$ *is bounded for all* $y \in E$ *and* $r > 0$*;*

(iii) $dom f^* = E^*$, f^* *is Gâteaux differentiable and* $\nabla f^* = (\nabla f)^{-1}$.

Lemma 5.2.2. *([23]) Let* $f : E \to \mathbb{R}$ *be a totally convex and Gâteaux differentiable function, and* $x \in E$*. Let* C *be a nonempty, closed and convex subset of* E*. The Bregman projection* P_C^f *from* E *onto* C *has the following properties:*

(i) $z = P_C^f(x)$ *if and only if* $\langle y - z, \nabla f(x) - \nabla f(z)\rangle \leq 0$, $\forall y \in C$*;*

(ii) $D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y, x)$, $\forall y \in C$.

Lemma 5.2.3. *([73]) Let* $f : E \to \mathbb{R}$ *be a Gâteaux differentiable and totally convex function. If* $x \in E$ *and the sequence* $D_f(x_n, x)$ *is bounded, then the sequence* $\{x_n\}$ *is also bounded.*

Lemma 5.2.4. *([73]) Let* $f : E \to \mathbb{R}$ *be a Gâteaux differentiable function which is uniformly convex on bounded subset of* E*. Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in* E*. Then, the following assertions are equivalent*

(i) $\lim_{n\to\infty} D_f(x_n, y_n) = 0$*;*

(ii) $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 5.2.5. *([98]) If* f *is a convex and lower semi-continuous function on a Banach space* E*, then the following statements are equivalent:*

(i) f *is uniformly convex;*

(ii) *for all* $(x, x^*), (y, y^*) \in Gph(\partial f)$ *there exists modulus* g *such that*

$$f(y) \geq f(x) + \langle y - x, x^*\rangle + g(\|x - y\|);$$

(iii) $dom f^* = E^*$, f^* *is Frèchet differentiable and* ∇f^**is uniformly continuous.*

Lemma 5.2.6. *([?]) Let* f *be a strongly convex function with constant* $\mu > 0$*. Then, for all* $y \in dom f$ *and* $x \in int(dom f)$*,*

$$D_f(y, x) \geq \frac{\mu}{2}\|x - y\|^2,$$

where $D_f(y, x)$ *is a Bregman distance with respect to* f *.*

Lemma 5.2.7. *([37]) Let $f : E \to (-\infty, +\infty]$ be a coercive and Gâteaux differentiable function. Let C be a closed and convex subset of a real reflexive Banach space E. Assume that $B : C \to E^*$ is a continuous and monotone mapping, $\varphi : C \to \mathbb{R}$ is a lower semi-continuous and convex function and let $F : C \times C \to \mathbb{R}$ be a bi-function satisfying **Condition A**. For $r > 0$ and $x \in E$, define a mapping $T_H^{f,r} : E \to C$ as follows:*

$$T_H^{f,r}x = \{z \in C : H(z,y) + \frac{1}{r}\langle y - z, \nabla f(z) - \nabla f(x)\rangle \geq 0, \forall y \in C\}, \tag{5.2.1}$$

where $H(z,y) := F(z,y) + \varphi(y) - \varphi(z) + \langle y - z, Bz\rangle$. Then, $T_H^{f,r}(x) \neq \emptyset$, and the following hold:

(1) $T_H^{f,r}$ is single-valued;

(2) $F(T_H^{f,r}) = GMEP(F, \varphi, B)$;

(3) $GMEP(F, \varphi, B)$ is closed and convex;

(4) $T_H^{f,r}$ is quasi-Bregman nonexpansive;

(5) $D_f(p, T_H^{f,r}x) + D_f(T_H^{f,r}x, x) \leq D_f(p, x), \forall p \in F(T_H^{f,r})$.

Lemma 5.2.8. *Let $f : E \to (-\infty, +\infty]$ be a coercive and Gâteaux differentiable function. Let E^* be the dual space of a real reflexive Banach space E and C be a closed and convex subset E. Let $T : C \to E^*$ be a continuous f-pseudocontractive mapping. For $r > 0$ and $x \in E$, define a mapping $K_T^{f,r} : E \to C$ as follows:*

$$K_T^{f,r}x = \{z \in C : \langle y - z, T(z)\rangle - \frac{1}{r}\langle y - z, (1 + r)\nabla f(z) - \nabla f(x)\rangle \leq 0, \forall y \in C\}. \tag{5.2.2}$$

Then, $K_T^{f,r}(x) \neq \emptyset$, and the following hold:

(1) $K_T^{f,r}$ is single-valued;

(2) $F(K_T^{f,r}) = F_f(T)$

(3) $F_f(T)$ is closed and convex;

(4) $K_T^{f,r}$ is quasi-Bregman nonexpansive;

(5) $D_f(p, K_T^{f,r}x) + D_f(K_T^{f,r}x, x) \leq D_f(p, x), \forall p \in F(K_T^{f,r})$.

Proof. Let $B := \nabla f - T$. Then, B is monotone and continuous. Putting $F \equiv 0$ and $\varphi \equiv 0$ in Lemma 5.2.7. Then, there exists $z \in C$ such that

$$\langle y - z, B(z)\rangle + \frac{1}{r}\langle y - z, \nabla f(z) - \nabla f(x)\rangle \geq 0, \forall y \in C,$$

Equivalently,

$$\langle y - z, T(z)\rangle - \frac{1}{r}\langle y - z, (1+r)\nabla f(z) - \nabla f(x)\rangle \leq 0, \forall y \in C.$$

Furthermore, applying Lemma 5.2.7, we get the results (1)-(5) of Lemma 5.2.8. This completes the proof.

Lemma 5.2.9. *(Xu [94]) Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n, n \geq n_0$$

where $\{\alpha_n\} \subset (0,1)$ and $\{b_n\} \subset \mathbb{R}$ satisfying the following conditions: $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\limsup_{n\to\infty} b_n \leq 0$, or $\sum_{n=1}^{\infty} |\alpha_n b_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 5.2.10. *([52]) Suppose $\{s_n\}$ is a sequence of real numbers such that there exists a subsequence $\{s_i\}$ of $\{n\}$ such that $s_{n_i} < s_{n_i+1}$ for all $i \in \mathbb{N}$. Let the sequence of $\{m_k\}$ be defined by $m_k = \max\{j \leq k : s_j < s_{j+1}\}$. Then, $\{m_k\}$ is a nondecreasing sequence satisfying $m_k \to \infty$ as $k \to \infty$ and the following properties hold:*

$$s_{m_k} \leq s_{m_k+1} \text{ and } s_k \leq s_{m_k+1},$$

for all $k \geq N_0$, for some $N_0 > 0$.

5.3 Basic Assumptions and Proposed Algorithm

The following assumptions will be used in the sequel.
Assumption 5

(B1) Let C be a nonempty, closed and convex subset of a reflexive real Banach space E with its dual E^*;

(B2) Let $T_i : E \to E^*$, $i = 1, 2, \cdots, N$ be continuous f-pseudocontractive mappings;

(B3) Let $B_t : C \to E^*$, $t = 1, 2, \cdots, M$ be continuous monotone mappings;

(B4) Let $F_t : C \times C \to \mathbb{R}$, $t = 1, 2, \cdots, M$ be bi-functionals satisfying **Condition A**;

(B5) Let $\varphi_t : C \to \mathbb{R}$, $t = 1, 2, \cdots, M$ be real valued functions;

(B6) Let $A_j : C \to E^*$ be Lipschitz monotone mappings with Lipschitz constants L_j, for $j = 0, 1, 2, \ldots, K$.

(B7) Let the common set of solutions, denoted by $\mathcal{F}$, be nonempty, that is

$$\mathcal{F} := \left[\bigcap_{i=1}^{N} F_f(T_i)\right] \cap \left[\bigcap_{j=0}^{K} VI(C, A_j)\right] \cap \left[\bigcap_{t=1}^{M} GMEP(F_t, \varphi_t, B_t)\right] \neq \emptyset.$$

(C1) Let f be a strongly coercive, bounded and uniformly Fréchet differentiable Legendre function which is strongly convex with constant $\mu > 0$ on bounded subsets of E.

Let $\{x_n\}$ be the sequence generated by the iterative scheme:

$$\begin{cases} u, x_0 \in C, \\ z_n = P_C^f \nabla f^*(\nabla f(x_n) - \lambda_n A_n x_n), \\ d_n = P_C^f \nabla f^*(\nabla f(x_n - \lambda_n A_n z_n), \\ u_n = T_{H_M}^{f,r_n} \circ T_{H_{N-1}}^{f,r_n} \circ \cdots \circ T_{H_2}^{f,r_n} \circ T_{H_1}^{f,r_n} x_n, \\ v_n = K_{T_N}^{f,r_n} \circ K_{T_{N-1}}^{f,r_n} \circ \cdots \circ K_{T_2}^{f,r_n} \circ K_{T_1}^{f,r_n} u_n, \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + \theta_n \nabla f(x_n) + \beta_n \nabla f(d_n) + \gamma_n \nabla f(v_n)). \end{cases} \tag{5.3.1}$$

where $A_n = A_{n \bmod (K+1)}$ and ∇f is the gradient of f on E; $\{r_n\} \subset [c_1, \infty)$ for some $c_1 > 0$, $\alpha_n, \theta_n, \beta_n, \gamma_n \in (0, 1)$, $\forall n \geq 0$ such that $\alpha_n + \theta_n + \beta_n + \gamma_n = 1$, $\lim\limits_{n\to\infty} \alpha_n = 0$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\beta_n, \gamma_n \in [c, 1)$ for some $c > 0$, and $d_n = P_C^f \nabla f^*(\nabla f(x_n - \lambda_n A_n z_n)$, $0 < a \leq \lambda_n \leq b < \frac{\mu}{L}$, for $L = \max\limits_{0 \leq i \leq K} L_i$.

The following lemma assures that the proposed iterative algorithm yields a bounded sequence.

Lemma 5.3.1. *Assume that Conditions $(B1) - (B7)$, and $(C1)$ hold. Then, the sequence $\{x_n\}$ generated by (5.3.1) is bounded.*

Proof. Let $a_0 = b_0 = I$, where I is the identity mapping on E, $a_i = K_{T_i}^{f,r_n} \circ K_{T_{i-1}}^{f,r_n} \circ \cdots \circ K_{T_2}^{f,r_n} \circ K_{T_1}^{f,r_n}$ for $i = 1, 2, \ldots, N$, and , $b_t = T_{H_t}^{f,r_n} \circ T_{H_{t-1}}^{f,r_n} \circ \cdots \circ T_{H_2}^{f,r_n} \circ T_{H_1}^{f,r_n}$ for $t = 1, 2, \ldots, M$. Let $p \in \mathcal{F}$. Then, by Lemma 5.2.7 and 5.2.8, we get

$$\begin{aligned} D_f(p, u_n) &\leq D_f(p, b_{M-1}(x_n)) - D_f(u_n, b_{M-1}(x_n)) \\ &\leq D_f(p, b_{M-2}(x_n)) - D_f(b_{M-1}(x_n), b_{M-2}(x_n)) - D_f(u_n, b_{M-1}(x_n)), \end{aligned}$$

and, by induction we obtain

$$D_f(p, u_n) \leq D_f(p, x_n) - \sum_{t=0}^{M-1} D_f(b_{t+1}(x_n), b_t(x_n)). \tag{5.3.2}$$

Similarly,

$$D_f(p, v_n) \leq D_f(p, u_n) - \sum_{t=0}^{N-1} D_f(a_{t+1}(u_n), a_t(u_n)). \tag{5.3.3}$$

Thus, from (5.3.2), (5.3.3) and Lemma 5.2.6, we obtain

$$\begin{aligned} D_f(p, v_n) &\leq D_f(p, x_n) - \sum_{t=0}^{M-1} D_f(b_{t+1}(x_n), b_t(x_n)) - \sum_{i=0}^{N-1} D_f(a_{i+1}(u_n), a_i(u_n)) \\ &\leq D_f(p, x_n) \qquad (5.3.4) \\ &\quad -\frac{\mu}{2}\left(\sum_{t=0}^{M-1} \|b_{t+1}(x_n) - b_t(x_n)\|^2 + \sum_{i=0}^{N-1} \|a_{i+1}(u_n) - a_i(u_n)\|^2\right) \\ &\leq D_f(p, x_n). \qquad (5.3.5) \end{aligned}$$

Let $w_n = \nabla f^*(\nabla f(x_n) - \lambda_n A_n z_n)$. By Lemma 5.2.2 and the fact that $\lambda_n \leq \frac{\mu}{L}$, we get

$$\begin{aligned} D_f(p, d_n) &= D_f(p, P_C^f w_n) \leq D_f(p, w_n) - D_f(d_n, w_n) \\ &= f(p) - f(w_n) - \langle p - w_n, \nabla f(w_n)\rangle \\ &\quad - [f(d_n) - f(w_n) - \langle d_n - w_n, \nabla f(w_n)\rangle] \\ &= f(p) - \langle p - d_n, \nabla f(w_n)\rangle - f(d_n) \\ &= f(p) - \langle p - d_n, \nabla f(x_n) - \lambda_n A_n z_n\rangle - f(d_n) \\ &= f(p) - \langle p - d_n, \nabla f(x_n)\rangle + \langle p - d_n, \lambda_n A_n z_n\rangle - f(d_n) \\ &= f(p) - \langle p - x_n, \nabla f(x_n)\rangle - f(x_n) \\ &\quad - [f(d_n) - \langle d_n - x_n, \nabla f(x_n)\rangle - f(x_n)] + \langle p - d_n, \lambda_n A_n z_n\rangle \end{aligned}$$

$$\begin{aligned}
&= D_f(p, x_n) - D_f(d_n, x_n) + \langle p - d_n, \lambda_n A_n z_n \rangle \\
&= D_f(p, x_n) - D_f(d_n, x_n) + \langle p - z_n, \lambda_n A_n z_n \rangle + \langle z_n - d_n, \lambda_n A_n z_n \rangle \\
&= D_f(p, x_n) - D_f(d_n, x_n) + \lambda_n \langle p - z_n, A_n z_n - A_n p \rangle \\
&\quad + \lambda_n \langle p - z_n, A_n p \rangle + \langle z_n - d_n, \lambda_n A_n z_n \rangle \\
&\leq D_f(p, x_n) - D_f(d_n, x_n) + \langle z_n - d_n, \lambda_n A_n z_n \rangle . \qquad (5.3.6)
\end{aligned}$$

Now, from (1.2.3), we obtain

$$D_f(d_n, x_n) = D_f(d_n, z_n) + D_f(z_n, x_n) + \langle d_n - z_n, \nabla f(z_n) - \nabla f(x_n) \rangle . \qquad (5.3.7)$$

Thus, from (5.3.6), (5.3.7) and Lemma 5.2.6, we get

$$\begin{aligned}
D_f(p, d_n) &\leq D_f(p, x_n) - D_f(d_n, z_n) - D_f(z_n, x_n) \\
&\quad + \langle z_n - d_n, \lambda_n A_n z_n + \nabla f(z_n) - \nabla f(x_n) \rangle \\
&\leq D_f(p, x_n) - \frac{\mu}{2} \left[\|d_n - z_n\|^2 + \|x_n - z_n\|^2 \right] \qquad (5.3.8) \\
&\quad + \langle z_n - d_n, \lambda_n A_n z_n + \nabla f(z_n) - \nabla f(x_n) \rangle
\end{aligned}$$

Using the fact that A_i is Lipschitz monotone for $i = 0, 1, 2, \ldots, K$ and Lemma 5.2.2, we have that

$$\begin{aligned}
\langle z_n - d_n, \lambda_n A_n z_n + \nabla f(z_n) - \nabla f(x_n) \rangle &= \langle d_n - z_n, \lambda_n A_n x_n - \lambda_n A_n z_n \rangle \\
&\quad + \langle d_n - z_n, \nabla f(x_n) - \lambda_n A_n x_n - \nabla f(z_n) \rangle \\
&\leq \lambda_n \langle d_n - z_n, A_n x_n - A_n z_n \rangle \\
&\leq \lambda_n \|d_n - z_n\| \|A_n x_n - A_n z_n\|
\end{aligned}$$

$$\begin{aligned}
&\leq L\lambda_n \|d_n - z_n\| \|x_n - z_n\| \\
&\leq \frac{1}{2} L\lambda_n \left[\|d_n - z_n\|^2 + \|x_n - z_n\|^2 \right] . \qquad (5.3.9)
\end{aligned}$$

Thus, from (5.3.9), (5.3.9) and the fact that $\lambda_n \leq \frac{\mu}{L}$, we get

$$\begin{aligned}
D_f(p, d_n) &\leq D_f(p, x_n) - \frac{1}{2}(\mu - L\lambda_n) \left[\|d_n - z_n\|^2 + \|x_n - z_n\|^2 \right] \quad (5.3.10) \\
&\leq D_f(p, x_n). \qquad (5.3.11)
\end{aligned}$$

By (5.3.4), (5.3.10), $\lambda_n \leq \frac{\mu}{L}$ and Lemma 1.2.9, we obtain

$$\begin{aligned} D_f(p, x_{n+1}) &= D_f(p, \nabla f^*(\alpha_n \nabla f(u) + \theta_n \nabla f(x_n) + \beta_n \nabla f(d_n) + \gamma_n \nabla f(v_n))) \\ &\leq \alpha_n D_f(p, u) + \theta_n D_f(p, x_n) + \beta_n D_f(p, d_n) + \gamma_n D_f(p, v_n) \\ &\leq \alpha_n D_f(p, u) + (1 - \alpha_n) D_f(p, x_n) \qquad (5.3.12) \\ &\quad - \frac{1}{2}\beta_n(\mu - L\lambda_n)\left[\|d_n - z_n\|^2 + \|x_n - z_n\|^2\right] \\ &\quad - \gamma_n \frac{\mu}{2}\left[\sum_{t=0}^{M-1} \|b_{t+1}(x_n) - b_t(x_n)\|^2 + \sum_{i=0}^{N-1} \|a_{i+1}(u_n) - a_i(u_n)\|^2\right] \\ &\leq \alpha_n D_f(p, u) + (1 - \alpha_n) D_f(p, x_n) \\ &\leq \max\{D_f(p, u), D_f(p, x_n)\}. \qquad (5.3.13) \end{aligned}$$

Therefore, by induction, we get

$$D_f(p, x_n) \leq \max\{D_f(p, u), D_f(p, x_0)\}, \text{ for all } n \geq 0. \qquad (5.3.14)$$

This implies that $\{D_f(p, x_n)\}$ is bounded. Therefore, by Lemma 5.2.3 we have, $\{x_n\}$ is bounded and also the sequences $\{z_n\}$, $\{d_n\}$, $\{u_n\}$ and $\{v_n\}$ are bounded.

5.4 Approximating a Common Solution of Nonlinear Problems

Theorem 5.4.1. *Assume that Conditions $(B1)-(B7)$ and $(C1)$ of* ***Assumption 5*** *hold. Then, the sequence $\{x_n\}$ generated by (5.3.1) converges strongly to a common soluion p in $\mathcal{F}$ which is nearest to u with respect to the Bregman distance.*

Proof. Let $p = P^f_{\mathcal{F}} u$. From (1.2.10), (1.2.11), (5.3.4), (5.3.10) and Lemma 1.2.9, we obtain

$$\begin{aligned}
D_f(p,x_{n+1}) &= D_f(p,\nabla f^*\left(\alpha_n\nabla f(u)+\theta_n\nabla f(x_n)+\beta_n\nabla f(d_n)+\gamma_n\nabla f(v_n)\right)) \\
&= V_f(p,\alpha_n\nabla f(u)+\theta_n\nabla f(x_n)+\beta_n\nabla f(d_n)+\gamma_n\nabla f(v_n)) \\
&\le V_f(p,\alpha_n\nabla f(p)+\theta_n\nabla f(x_n)+\beta_n\nabla f(d_n)+\gamma_n\nabla f(v_n)) \\
&\quad -\alpha_n\left\langle x_{n+1}-p,\nabla f(p))-\nabla f(u)\right\rangle \\
&= D_f(p,\nabla f^*\left(\alpha_n\nabla f(p)+\theta_n\nabla f(x_n)+\beta_n\nabla f(d_n)+\gamma_n\nabla f(v_n)\right)) \\
&\quad -\alpha_n\left\langle x_{n+1}-p,\nabla f(p)-\nabla f(u)\right\rangle \\
&\le \alpha_n D_f(p,p)+\theta_n D_f(p,x_n)+\beta_n D_f(p,d_n)+\gamma_n D_f(p,v_n) \\
&\quad -\alpha_n\left\langle x_{n+1}-p,\nabla f(p)-\nabla f(u)\right\rangle \\
&= (1-\alpha_n)D_f(p,x_n)-\frac{1}{2}\beta_n(\mu-L\lambda_n)\left[\|d_n-z_n\|^2+\|x_n-z_n\|^2\right] \\
&\quad -\gamma_n\frac{\mu}{2}\left[\sum_{t=0}^{M-1}\|b_{t+1}(x_n)-b_t(x_n)\|^2+\sum_{i=0}^{N-1}\|a_{i+1}(u_n)-a_i(u_n)\|^2\right] \\
&\quad +\alpha_n\left\langle x_{n+1}-p,\nabla f(u)-\nabla f(p)\right\rangle \qquad (5.4.1) \\
&\le (1-\alpha_n)D_f(p,x_n)+\alpha_n\left\langle x_n-p,\nabla f(u)-\nabla f(p)\right\rangle \\
&\quad +\alpha_n\left\langle x_{n+1}-x_n,\nabla f(u)-\nabla f(p)\right\rangle
\end{aligned}$$

$$\begin{aligned}
&\le (1-\alpha_n)D_f(p,x_n)+\alpha_n\left\langle x_n-p,\nabla f(u)-\nabla f(p)\right\rangle \qquad (5.4.2) \\
&\quad +\alpha_n\|x_{n+1}-x_n\|\|\nabla f(u)-\nabla f(p)\|.
\end{aligned}$$

Now, we divide the rest of the proof into two parts as follows.

Case 1. Assume that there exists $n_0\in\mathbb{N}$ such that $\{D_f(p,x_n)\}$ is decreasing for all $n\ge n_0$. It then follows that $\{D_f(p,x_n)\}$ is convergent and hence $D_f(p,x_n)-D_f(p,x_{n+1})\to 0$ as $n\to\infty$. Thus, from (5.4.1) and the conditions on α_n, β_n, γ_n, and λ_n, we get

$$\lim_{n\to\infty}\|d_n-z_n\|^2+\|x_n-z_n\|^2=0, \qquad (5.4.3)$$

and

$$\lim_{n\to\infty}\left[\sum_{t=0}^{M-1}\|b_{t+1}(x_n)-b_t(x_n)\|^2+\sum_{i=0}^{N-1}\|a_{i+1}(u_n)-a_i(u_n)\|^2\right]=0, \qquad (5.4.4)$$

which imply

$$\lim_{n\to\infty}\|d_n-z_n\|=\lim_{n\to\infty}\|x_n-z_n\|=0,\ \text{and hence,}\ \lim_{n\to\infty}\|x_n-d_n\|=0, \qquad (5.4.5)$$

$$\lim_{n\to\infty}\|b_{t+1}(x_n)-b_t(x_n)\|=0, t=0,1,\dots,M-1,\ \text{and hence,}\ \lim_{n\to\infty}\|u_n-x_n\|=0, \qquad (5.4.6)$$

and

$$\lim_{n\to\infty} \|a_{i+1}(u_n) - a_i(u_n)\| = 0, \qquad 0 \le i \le N-1, \text{ and hence, } \lim_{n\to\infty} \|v_n - u_n\| = 0. \tag{5.4.7}$$

Now,

$$\begin{aligned}
&\|\nabla f(x_{n+1}) - \nabla f(x_n)\| \\
&= \| \left(\alpha_n \nabla f(u) + \theta_n \nabla f(x_n) + \beta_n \nabla f(d_n) + \gamma_n \nabla f(v_n)\right) - \nabla f(x_n)\| \\
&\le \alpha_n \|\nabla f(u) - \nabla f(x_n)\| + \beta_n \|\nabla f(d_n) - \nabla f(x_n)\| \\
&\quad + \gamma_n \|\nabla f(v_n) - \nabla f(x_n)\|,
\end{aligned} \tag{5.4.8}$$

and from (5.4.5), (5.4.6), (5.4.7), the fact that $\alpha_n \to 0$ as $n \to \infty$ and uniform continuity of ∇f, we get $\|\nabla f(x_{n+1}) - \nabla f(x_n)\| \to 0$ as $n \to \infty$. Moreover, the uniform continuity of ∇f^* implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{5.4.9}$$

Now, for $j = 0, 1, \ldots, K$, we have

$$\|d_{n+j} - x_n\| \le \|d_{n+j} - x_{n+j}\| + \sum_{l=n}^{n+j-1} \|x_{l+1} - x_l\| \tag{5.4.10}$$

Then, from (5.4.5), (5.4.9) and (5.4.10), we obtain that

$$\lim_{n\to\infty} \|d_{n+j} - x_n\| = 0, \text{ for } j = 0, 1, \ldots, K. \tag{5.4.11}$$

Since $\{x_n\}$ is bounded in E, there exists $q \in E$ and a subsequence $\{x_{n_s}\}$ of $\{x_n\}$ such that $x_{n_s} \rightharpoonup q$ and

$$\limsup_{n\to\infty} \langle x_n - p, \nabla f(u) - \nabla f(p)\rangle = \lim_{s\to\infty} \langle x_{n_s} - p, \nabla f(u) - \nabla f(p)\rangle. \tag{5.4.12}$$

Then, from (5.4.6), (5.4.7) and (5.4.11), we have that $b_t(x_{n_s}) \rightharpoonup q$, $a_i(u_{n_s}) \rightharpoonup q$, $d_{n_s+j} \rightharpoonup q$ for $t \in \{1, 2, \ldots, M\}$, $i \in \{1, 2, \ldots, N\}$ and $j \in \{1, 2, \ldots, K\}$. Now, we show that $q \in \mathcal{F}$.

Step 1. First we show that $q \in \bigcap_{j=0}^{K} VI(C, A_j)$.

Let

$$B_j v = \begin{cases} A_j v + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{otherwise }, \end{cases}$$

where N_C is the normal cone to C at $v \in C$ given by $N_C = \{w \in E^* : \langle v - x, w\rangle \ge 0, \forall x \in C\}$. Then, by Lemma 3.2.9, B_j is maximal monotone and $B_j^{-1}(0) =$

$VI(C, A_j)$. Let $w \in B_j v$. Then, we have $w \in A_j v + N_C v$ and hence $w - A_j v \in N_C v$. Thus, we obtain that

$$\langle v - x, w - A_j v\rangle \geq 0, \forall x \in C. \tag{5.4.13}$$

Let $\{n_s + j\}, s \geq 1$ be such that $A_{n_s+j} = A_j$ for all $s \in \mathbb{N}$ where $j = 0, 1, 2, \ldots, K$. Then, since $d_{n_s+j} = P_C^f \nabla f^*(\nabla f(x_{n_s+j}) - \lambda_{n_s+j} A_j z_{n_s+j})$, and $v \in C$, we have

$$\langle v - d_{n_s+j}, \nabla f(d_{n_s+j}) - (\nabla f(x_{n_s+j}) - \lambda_{n_s+j} A_j z_{n_s+j})\rangle \geq 0,$$

and so

$$\left\langle v - d_{n_s+j}, \frac{\nabla f(d_{n_s+j}) - \nabla f(x_{n_s+j})}{\lambda_{n_s+j}} + A_j z_{n_s+j} \right\rangle \geq 0. \tag{5.4.14}$$

From (5.4.13), (5.4.14) and A_j is monotone mapping, we get that

$$\begin{aligned}
\langle v - d_{n_s+j}, w\rangle &\geq \langle v - d_{n_s+j}, A_j v\rangle \\
&\geq \langle v - d_{n_s+j}, A_j v\rangle \\
&\quad - \left\langle v - d_{n_s+j}, \frac{\nabla f(d_{n_s+j}) - \nabla f(x_{n_s+j})}{\lambda_{n_s+j}} + A_j z_{n_s+j} \right\rangle \\
&= \langle v - d_{n_s+j}, A_j v - A_j d_{n_s+j}\rangle + \langle v - d_{n_s+j}, A_j d_{n_s+j} - A_j z_{n_s+j}\rangle \\
&\quad - \left\langle v - d_{n_s+j}, \frac{\nabla f(d_{n_s+j}) - \nabla f(x_{n_s+j})}{\lambda_{n_s+j}} \right\rangle \\
&\geq \langle v - d_{n_s+j}, A_j d_{n_s+j} - A_j z_{n_s+j}\rangle \\
&\quad - \left\langle v - d_{n_s+j}, \frac{\nabla f(d_{n_s+j}) - \nabla f(x_{n_s+j})}{\lambda_{n_s+j}} \right\rangle \\
&\geq \langle v - d_{n_s+j}, A_j d_{n_s+j} - A_j z_{n_s+j}\rangle \\
&\quad - \|v - d_{n_s+j}\| \frac{\|\nabla f(d_{n_s+j}) - \nabla f(x_{n_s+j})\|}{\lambda_{n_s+j}} \\
&\geq \langle v - d_{n_s+j}, A_j d_{n_s+j} - A_j z_{n_s+j}\rangle \\
&\quad - R \frac{\|\nabla f(d_{n_s+j}) - \nabla f(x_{n_s+j})\|}{\lambda_{n_s+j}},
\end{aligned} \tag{5.4.15}$$

where $R = \max_{0 \leq j \leq K} \sup_{s \geq 0} \|v - d_{n_s+j}\|$. Taking limits on both sides of the inequality (5.4.15) as $s \to \infty$ and using the fact that $\lambda_n \geq a > 0$, for all $n \geq 0$, ∇f is uniformly continuous, and (5.4.5), we get that $\langle v - q, w\rangle \geq 0$ as $s \to \infty$ for each j. Therefore, the maximality of B_j gives that $q \in B_j^{-1}(0) = VI(C, A_j)$ for each j. Therefore, $q \in \bigcap_{j=0}^{K} VI(C, A_j)$.

Step 2. We show that $q \in \bigcap_{j=1}^{N} F_f(T_j)$. Let $a_i(u_{n_s}) = K_{T_i}^{f,r_{n_s}} a_{i-1}(u_{n_s})$. By Lemma 5.2.8 (2), we get that

$$\langle y - a_i(u_{n_s}), T_i a_i(u_{n_s})\rangle - \frac{1}{r_{n_s}} \langle y - a_i(u_{n_s}), (1 + r_{n_s})\nabla f(a_i(u_{n_s})) - \nabla f(a_{i-1}(u_{n_s}))\rangle \leq 0,$$

for all $y \in C$.

Since C is convex, $y_\lambda = \lambda y + (1 - \lambda)q \in C$, where $\lambda \in [0, 1]$ and $y \in C$. Thus,

$$\begin{aligned}
\langle a_i(u_{n_s}) - y_\lambda, T_i y_\lambda\rangle &\geq \langle a_i(u_{n_s}) - y_\lambda, T_i y_\lambda\rangle + \langle y_\lambda - a_i(u_{n_s}), T_i a_i(u_{n_s})\rangle \\
&\quad - \frac{1}{r_{n_s}} \langle y_\lambda - a_i(u_{n_s}), (1 + r_{n_s})\nabla f(a_i(u_{n_s})) - \nabla f(a_{i-1}(u_{n_s}))\rangle
\end{aligned}$$

$$\begin{aligned}
&= \langle a_i(u_{n_s}) - y_\lambda, T_i y_\lambda - T_i a_i(u_{n_s})\rangle \\
&\quad - \frac{1}{r_{n_s}} \langle y_\lambda - a_i(u_{n_s}), (1 + r_{n_s})\nabla f(a_i(u_{n_s})) - \nabla f(a_{i-1}(u_{n_s}))\rangle \\
&\geq \langle a_i(u_{n_s}) - y_\lambda, \nabla f(y_\lambda) - \nabla f(a_i(u_{n_s}))\rangle \\
&\quad - \frac{1}{r_{n_s}} \langle y_\lambda - a_i(u_{n_s}), (1 + r_{n_s})\nabla f(a_i(u_{n_s})) - \nabla f(a_{i-1}(u_{n_s}))\rangle \\
&= \langle a_i(u_{n_s}) - y_\lambda, \nabla f(y_\lambda)\rangle \\
&\quad - \frac{1}{r_{n_s}} \langle y_\lambda - a_i(u_{n_s}), \nabla f(a_i(u_{n_s})) - \nabla f(a_{i-1}(u_{n_s}))\rangle \\
&\geq \langle a_i(u_{n_s}) - y_\lambda, \nabla f(y_\lambda)\rangle \\
&\quad - \|y_\lambda - a_i(u_{n_s})\| \frac{\|\nabla f(a_i(u_{n_s})) - \nabla f(a_{i-1}(u_{n_s}))\|}{r_{n_s}} \\
&\geq \langle a_i(u_{n_s}) - y_\lambda, \nabla f(y_\lambda)\rangle \\
&\quad - W \frac{\|\nabla f(a_i(u_{n_s})) - \nabla f(a_{i-1}(u_{n_s})\|}{r_{n_s}},
\end{aligned} \tag{5.4.16}$$

where $W = \max_{1 \leq i \leq N} \sup_{s \geq 0} \|y_\lambda - a_i(u_{n_s})\|$. From the facts that $a_i(u_{n_s}) \rightharpoonup q$, ∇f is uniformly continuous, (5.4.7), $r_n \geq c_1$, for all $n \geq 0$ and taking the limits on both sides of the inequality (5.4.16) as $s \to \infty$, we obtain that

$$\langle q - y_\lambda, T_i y_\lambda\rangle \geq \langle q - y_\lambda, \nabla f(y_\lambda)\rangle. \tag{5.4.17}$$

Thus, from inequality (5.4.17), we obtain

$$\langle q - y, T_i(q + \lambda(y - q))\rangle \geq \langle q - y, \nabla f(q + \lambda(y - q))\rangle, \forall y \in E. \tag{5.4.18}$$

Using the fact that T_i is continuous and ∇f is uniformly continuous on bounded subset of E and letting $\lambda \downarrow 0$, we have from inequality (5.4.18) that

$$\langle q-y, T_iq\rangle \geq \langle q-y, \nabla f(q)\rangle, \forall y \in C \Leftrightarrow 0 \geq \langle q-y, \nabla f(q)-T_iq\rangle, \forall y \in E. \tag{5.4.19}$$

Now, set $y = \nabla f^*(T_iq)$. Since E is reflexive and ∇f^* is monotone, we get that

$$\langle q-\nabla f^*(T_iq), \nabla f(q)-T_iq\rangle = 0, \tag{5.4.20}$$

which implies that $T_iq = \nabla f(q)$. Hence $q \in F_f(T_i)$, for each $i = 1, 2, \ldots, N$ and $q \in \bigcap_{i=1}^{N} F_f(T_i)$.

Step 3. We show that $q \in \bigcap_{t=1}^{M} GMEP(F_t, \varphi_t, B_t)$.

Set $b_t(x_{n_s}) = T_{H_t}^{f,r_{n_s}} b_{t-1}(x_{n_s})$. Then,

$$H_t(b_t(x_{n_s}), y) + \frac{1}{r_{n_s}} \langle y - b_t(x_{n_s}), \nabla f(b_t(x_{n_s})) - \nabla f(b_{t-1}(x_{n_s}))\rangle \geq 0, \forall y \in C.$$

Thus, by Condition (A2), we have

$$\begin{aligned} H_t(y, b_t(u_{n_s})) \leq -H_t(b_t(x_{n_s}), y) &\leq \frac{1}{r_{n_s}} \langle y - b_t(x_{n_s}), \nabla f(b_t(x_{n_s})) - \nabla f(b_{t-1}(x_{n_s})\rangle \\ &\leq \|y - b_t(x_{n_s})\| \frac{\|\nabla f(b_t(x_{n_s})) - \nabla f(b_{t-1}(x_{n_s}))\|}{r_{n_s}} \\ &\leq P \frac{\|\nabla f(b_t(x_{n_s})) - \nabla f(b_{t-1}(x_{n_s})\|}{r_{n_s}}, \end{aligned} \tag{5.4.21}$$

where $P = \max_{1\leq t\leq M} \sup_{s\geq 0} \|y - b_t(x_{n_s})\|$. From the facts that $b_t(x_{n_s}) \rightharpoonup q$, **Condition A** (A4), $r_n \geq c_1$, for all $n \geq 0$ and taking limits on both sides of the inequality (5.4.21) as $s \to \infty$, we obtain that

$$H_t(y, q) \leq 0, \forall y \in C. \tag{5.4.22}$$

Set $y_\lambda = \lambda y + (1-\lambda)q, \lambda \in (0, 1]$ and $y \in C$. Consequently, we get $y_\lambda \in C$. From (5.4.22) and **Condition A** (A1), we obtain

$$\begin{aligned} 0 &= H_t(y_\lambda, y_\lambda) \leq \lambda H_t(y_\lambda, y) + (1-\lambda)H_t(y_\lambda, q) \\ &\leq H_t(q + \lambda(q-y), y) \end{aligned} \tag{5.4.23}$$

If $\lambda \downarrow 0$, using **Condition A** (A3) we have

$$H_t(q, y) \geq 0, \forall y \in C.$$

Hence, $q \in GMEP(F_t, \varphi_t, B_t)$, for each $t = 1, 2, \ldots\ldots, M$. Therefore, $q \in \bigcap_{t=1}^{M} GMEP(F_t, \varphi_t, B_t)$.

Finally, we show that $\{x_n\}$ converge strongly to the point p.

From (5.4.12) and Lemma 5.2.2, we obtain that

$$\begin{aligned} \limsup_{n\to\infty} \langle x_n - p, \nabla f(u) - \nabla f(p)\rangle &= \lim_{s\to\infty} \langle x_{n_s} - p, \nabla f(u) - \nabla f(p)\rangle \quad (5.4.24) \\ &= \langle q - p, \nabla f(u) - \nabla f(p)\rangle \leq 0. \end{aligned}$$

Thus, using (5.4.2), (5.4.9), (5.4.24) and Lemma 6.2.2, we conclude that

$$\lim_{n\to\infty} D_f(p, x_n) = 0.$$

Hence, Lemma 5.2.4 implies that $x_n \to p$ as $n \to \infty$.

Case 2. Suppose that there exists $\{n_s\}$ of $\{n\}$ such that $D_f(p, x_{n_s}) < D_f(p, x_{n_s+1})$, for all $s \geq 0$. It follows from Lemma 5.2.10 that there exists a nondecreasing sequence$\{k_s\} \subset \mathbb{N}$ such that $k_s \to \infty$ as $s \to \infty$ and

$$\max\{D_f(p, x_{k_s}), D_f(p, x_s)\} < D_f(p, x_{k_s+1}), \tag{5.4.25}$$

for all $s \geq 0$. Thus, from (5.4.1) and the conditions on α_n, β_n, γ_n, and λ_n, we get

$$\lim_{n\to\infty} \|d_{k_s} - z_{k_s}\|^2 + \|x_{k_s} - z_{k_s}\|^2 = 0, \tag{5.4.26}$$

and

$$\lim_{s\to\infty} \left[\sum_{t=0}^{M-1} \|b_{t+1}(x_{k_s}) - b_t(x_{k_s})\|^2 + \sum_{i=0}^{N-1} \|a_{i+1}(u_{k_s}) - a_{i-1}(u_{k_s})\|^2\right] = 0. \tag{5.4.27}$$

Then

$$\lim_{s\to\infty} \|d_{k_s} - z_{k_s}\| = \lim_{s\to\infty} \|x_{k_s} - z_{k_s}\| = 0 \text{ and hence } \lim_{s\to\infty} \|x_{k_s} - d_{k_s}\| = 0, \tag{5.4.28}$$

$$\lim_{s\to\infty} \|b_{t+1}(x_{k_s}) - b_t(x_{k_s})\| = 0, \qquad 0 \leq t \leq M-1, \lim_{s\to\infty} \|u_{k_s} - x_{k_s}\| = 0, \tag{5.4.29}$$

and

$$\lim_{s\to\infty} \|a_i(u_{k_s}) - a_{i-1}(u_{k_s})\| = 0, \qquad 0 \leq i \leq N-1, \lim_{s\to\infty} \|v_{k_s} - u_{k_s}\| = 0. \tag{5.4.30}$$

Moreover, following the methods used in ***Case 1***, we get

$$\limsup_{s\to\infty} \langle x_{k_s} - p, \nabla f(u) - \nabla f(p)\rangle \leq 0. \tag{5.4.31}$$

Therefore, from (5.4.2), (5.4.9), (5.4.31) and Lemma 6.2.2, we obtain that

$$\lim_{s\to\infty} D_f(p, x_{k_s}) = 0. \tag{5.4.32}$$

This together with (5.4.2) imply that

$$\lim_{s\to\infty} D_f(p, x_{k_s+1}) = 0. \tag{5.4.33}$$

Thus, from (5.4.25), and (5.4.33) we have that

$$\lim_{s\to\infty} D_f(p, x_s) = 0.$$

This together with Lemma 5.2.4 imply that $x_s \to p$ as $s \to \infty$. Therefore, from ***Case 1*** and ***Case 2***, we can conclude that $\{x_n\}$ converges strongly to the point p in $\mathcal{F}$. The proof is complete.

We note that the method of proof of Theorem 6.4.1 provides the following theorem for approximating a common solution of f-fixed point, variational inequality and generalized mixed equilibrium problems in real Banach spaces.

Theorem 5.4.2. *Assume that Conditions $(B1)-(B7)$ and $(C1)$ are satisfied with $N = K = M = 1$. Then, the sequence $\{x_n\}$ generated by (5.3.1) with $N = K = M = 1$ converges strongly to p in $\mathcal{F}$ which is nearest to u with respect to the Bregman distance.*

If, in Theorem 6.4.1, we assume that $A_j \equiv 0$, for $j = 0, 1, 2, \ldots, K$, then Theorem 6.4.1 provides the following corollary.

Corollary 5.4.3. *Assume that Conditions $(B1)-(B5)$, and $(C1)$ hold. Let $\mathcal{F} := \left[\bigcap_{i=1}^N F_f(T_i)\right] \cap \left[\bigcap_{t=1}^M GMEP(F_t, \varphi_t, B_t)\right] \neq \emptyset$. Let $\{x_n\}$ be a sequence generated from arbitrary $u_0, x_0 \in C$ by*

$$\begin{cases} u_n = T_{H_M}^{f,r_n} \circ T_{H_{M-1}}^{f,r_n} \circ \cdots \circ T_{H_2}^{f,r_n} \circ T_{H_1}^{f,r_n} x_n, \\ v_n = K_{T_N}^{f,r_n} \circ K_{T_{N-1}}^{f,r_n} \circ \cdots \circ K_{T_2}^{f,r_n} \circ K_{T_1}^{f,r_n} u_n, \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + \theta_n \nabla f(x_n) + \gamma_n \nabla f(v_n)), \end{cases} \tag{5.4.34}$$

where ∇f is the gradient of f on E; $\{r_n\} \subset [c_1, \infty)$ for some $c_1 > 0$, $\alpha_n, \theta_n, \gamma_n \in (0,1)$, $\forall n \geq 0$ such that $\alpha_n + \theta_n + \gamma_n = 1$, $\lim\limits_{n\to\infty} \alpha_n = 0$ with $\sum_{n=1}^\infty \alpha_n = \infty$ and $\gamma_n \in [c, 1)$ for some $c > 0$. Then, the sequence $\{x_n\}$ converges strongly to p in $\mathcal{F}$ which is nearest to u with respect to the Bregman distance.

If, in Corollary 5.4.3, we assume that $F_i \equiv 0$, for $i = 1, 2, \dots, K$, then Corollary 6.4.1 provides the following corollary for approximating the common solution of a finite family of mixed variational inequality of Browder type problems for continuous monotone mappings and f-fixed point problems for continuous f-pseudocontractive mapping in a reflexive real Banach space.

Corollary 5.4.4. *Let $\{x_n\}$ be a sequence generated from arbitrary $u_0, x_0 \in C$ by*

$$\begin{cases} u_n = T_{H_M}^{f,r_n} \circ T_{H_{M-1}}^{f,r_n} \circ \cdots \circ T_{H_2}^{f,r_n} \circ T_{H_1}^{f,r_n} x_n, \\ v_n = K_{T_N}^{f,r_n} \circ K_{T_{N-1}}^{f,r_n} \circ \cdots \circ K_{T_2}^{f,r_n} \circ K_{T_1}^{f,r_n} u_n, \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + \theta_n \nabla f(x_n) + \gamma_n \nabla f(v_n)), \end{cases} \tag{5.4.35}$$

where ∇f is the gradient of f on E; $\{r_n\} \subset [c_1, \infty)$ for some $c_1 > 0$, $\alpha_n, \theta_n, \gamma_n \in (0,1)$, $\forall n \geq 0$ such that $\alpha_n + \theta_n + \gamma_n = 1$, $\lim\limits_{n\to\infty} \alpha_n = 0$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\gamma_n \in [c, 1)$ for some $c > 0$. If the Conditions $(B1) - (B3)$, $(B5)$ and $(C1)$ are satisfied and $\mathcal{F} := \left[\bigcap_{i=1}^{N} F_f(T_i)\right] \cap \left[\bigcap_{t=1}^{M} VI(B_t, \varphi_t, C)\right] \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to p in $\mathcal{F}$ which is nearest to u with respect to the Bregman distance.

If we assume that E is smooth and strictly convex, then $f(x) = \frac{1}{2}\|x\|^2$ is strongly coercive, bounded and uniformly Fréchet differentiable Legendre function which is strongly convex with constant $\mu = 1$ and conjugate $f^*(x^*) = \frac{1}{2}\|x^*\|^2$. In this case, we have $\nabla f = J$, $\nabla f^* = J^{-1}$ and for $r > 0$ and $x \in E$, we have

$$T_H^r x = \{z \in C : H(z,y) + \frac{1}{r}\langle y - z, J(z) - J(x)\rangle \geq 0, \forall y \in C\}, \tag{5.4.36}$$

where $H(z,y) := F(z,y) + \varphi(y) - \varphi(z) + \langle y - z, Bz\rangle$, and

$$K_T^r x = \{z \in C : \langle y - z, T(z)\rangle - \frac{1}{r}\langle y - z, (1+r)J(z) - J(x)\rangle \leq 0, \forall y \in C\}. \tag{5.4.37}$$

In this case, Theorem 6.4.1 reduces to the following corollary:

Corollary 5.4.5. *Let C be nonempty, closed and convex subset of a smooth and strictly convex reflexive real Banach space E with its dual E^*. Assume that Conditions $(B1)-(B7)$ hold. Let $\{x_n\}$ be a sequence generated from arbitrary $u_0, x_0 \in C$*

by

$$\begin{cases} z_n = \Pi_C J^{-1}(J(x_n) - \lambda_n A_n x_n) \\ d_n = \Pi_C J^{-1}(J(x_n - \lambda_n A_n z_n), \\ u_n = T^{r_n}_{H_M} \circ T^{r_n}_{H_{N-1}} \circ \cdots \circ T^{r_n}_{H_2} \circ T^{r_n}_{H_1} x_n, \\ v_n = K^{r_n}_{T_N} \circ K^{r_n}_{T_{N-1}} \circ \cdots \circ K^{r_n}_{T_2} \circ K^{r_n}_{T_1} u_n, \\ x_{n+1} = J^{-1}(\alpha_n J(u) + \theta_n J(x_n) + \beta_n J(d_n) + \gamma_n J(v_n)), \end{cases} \tag{5.4.38}$$

where $A_n = A_{n \bmod (K+1)}$, *and* Π_C *is the generalized metric projection from* E *onto* C; $\{r_n\} \subset [c_1, \infty)$ *for some* $c_1 > 0$, $\alpha_n, \theta_n, \beta_n, \gamma_n \in (0, 1)$, $\forall n \geq 0$ *such that* $\alpha_n + \theta_n + \beta_n + \gamma_n = 1$, $\lim\limits_{n\to\infty} \alpha_n = 0$ *with* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\beta_n, \gamma_n \in [c, 1)$ *for some* $c > 0$, *and* $0 < a \leq \lambda_n \leq b < \frac{1}{L}$, *for* $L = \max\limits_{0\leq i\leq K} L_i$. *Then, the sequence* $\{x_n\}$ *converges strongly to* p *in* $\mathcal{F}$ *which is nearest to* u *with respect to the generalized metric projection.*

If, in Corollary 5.4.5, we assume that $E = H$, a real Hilbert space, and $f(x) = \frac{1}{2}\|x\|^2$, then we have $\nabla f = J = I$ and $\nabla f^* = J^{-1} = I$, were I is identity mapping on H. Moreover, f-pseudocontractive mapping reduces to pseudocontractive mapping. In this case, for $r > 0$ and $x \in E$, we have

$$T^r_H x = \{z \in C : H(z, y) + \frac{1}{r} \langle y - z, z - x\rangle \geq 0, \forall y \in C\}, \tag{5.4.39}$$

where $H(z, y) := F(z, y) + \varphi(y) - \varphi(z) + \langle y - z, Bz\rangle$, and

$$K^r_T x = \{z \in C : \langle y - z, T(z)\rangle - \frac{1}{r} \langle y - z, (1 + r)z - x\rangle \leq 0, \forall y \in C\}. \tag{5.4.40}$$

Thus, we have the following corollary.

Corollary 5.4.6. *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H *and let* $T_i : H \to H$, $i = 1, 2, \cdots, N$ *be continuous pseudocontractive mappings. Let* $\{x_n\}$ *be a sequence generated from an arbitrary* $u, x_0 \in C$ *by*

$$\begin{cases} z_n = P_C(x_n - \lambda_n A_n x_n) \\ d_n = P_C(x_n - \lambda_n A_n z_n), \\ u_n = T^{r_n}_{H_M} \circ T^{r_n}_{H_{M-1}} \circ \cdots \circ T^{r_n}_{H_2} \circ T^{r_n}_{H_1} x_n, \\ v_n = K^{r_n}_{T_N} \circ K^{r_n}_{T_{N-1}} \circ \cdots \circ K^{r_n}_{T_2} \circ K^{r_n}_{T_1} u_n, \\ x_{n+1} = \alpha_n u + \theta_n x_n + \beta_n d_n + \gamma_n v_n, \end{cases} \tag{5.4.41}$$

where $A_n = A_{n \bmod (K+1)}$, *and* P_C *is metric projection of* H *onto* C; $\{r_n\} \subset [c_1, \infty)$ *for some* $c_1 > 0$, $\alpha_n, \theta_n, \beta_n, \gamma_n \in (0, 1)$, $\forall n \geq 0$ *such that* $\alpha_n + \theta_n + \beta_n + \gamma_n = 1$,

$\lim_{n\to\infty} \alpha_n = 0$ *with* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\beta_n, \gamma_n \in [c, 1)$ *for some* $c > 0$*, and* $0 < a \leq \lambda_n \leq b < \frac{1}{L}$*, for* $L = \max_{0\leq i\leq K} L_i$*. If Conditions* $(B3) - (B7)$ *are satisfied, then the sequence* $\{x_n\}$ *converges strongly to* p *in* $\mathcal{F}$ *which is nearest to* u *with respect to the metric projection.*

5.5 Numerical Example

In this section, we present an example to illustrate the main result of our paper.

Example 5.5.1. *Let* $E = L_2^{\mathbb{R}}([0,1])$ *with norm* $\|x\|_{L_2^{\mathbb{R}}} = (\int_0^1 |x(s)|^2)^{\frac{1}{2}}$*, for* $x \in E$ *and* $C = \{x \in E : \|x\|_{L_2^{\mathbb{R}}} \leq 1\}$*. Define* $f : E \to \mathbb{R}$ *by* $f(x) = \frac{1}{2}\|x\|^2_{L_2^{\mathbb{R}}}$*, then* $\nabla f = J = I$ *and* $\nabla f^* = J = I$*, where* I *is identity mapping on* E*. Let* A_j*,* T_i*,* $B_t : C \to E$ *be defined by* $A_j(x)(s) = (1 + j)\nabla f(x)(s)$*,* $j = 0, 1, \ldots, K$*;* $T_i(x)(s) = -s^i\nabla f(x)(s)$*,* $i = 1, \ldots, N$ *and* $B_t(x)(s) = \frac{t+1}{2t+1}\nabla f(x)(s)$*,* $t = 1, \ldots, M$*, for all* $x(s) \in C, s \in [0, 1]$*, respectively. Let* $F_t : C \times C \to \mathbb{R}$ *be defined by* $F_t(x, y) = \frac{t}{2t+1} \langle y - x, \nabla f(x)\rangle, \forall x, y \in C$*. Then* A_j*, for* $j = 0, 1, \ldots, K$ *are Lipschitz monotone mappings with* $\bigcap_{j=0}^{K} VI(C, A_j) = \{0\}$*;* T_i*, for* $i = 1, \ldots, N$ *are continuous* f*-pseudocontractive with* $\bigcap_{i=1}^{N} F_f(T_i) = \{0\}$*;* B_t*, for* $t = 1, \ldots, M$ *are continuous monotone mappings, and* F_t*, for* $t = 1, \ldots, M$ *are bi-function satisfying* ***Condition A****. Thus, a common solution set of the generalized equilibrium problems is* $\bigcap_{t=1}^{M} GMEP(F_t, \varphi_t, B_k) = \{0\}$*, where* $\varphi_t \equiv$ *constant. Now, for implementation, we choose* $K = 0$*,* $N = M = 1$*,* $r_n = 1$*,* $\theta_n = \beta_n = \gamma_n = \frac{1}{3}(1 - \alpha_n)$*,* $\lambda_n = 0.00001 + \frac{1}{100n}$*, for* $n \geq 0$ *and we compute the* $(n + 1)^{th}$ *iteration as follows:*

$$\begin{cases} z_n(s) = \min\{1, \frac{1}{\|w_n\|_{L_2^{\mathbb{R}}}}\}w_n(s), \\ d_n(s) = \min\{1, \frac{1}{\|h_n\|_{L_2^{\mathbb{R}}}}\}h_n(s), \\ u_n(s) = \frac{1}{r_n+1}x_n(s), \\ v_n(s) = \frac{1}{1+r_n(1+s)}u_n, \\ x_{n+1}(s) = \alpha_n u(s) + \theta_n x_n(s) + \beta_n d_n(s) + \gamma_n v_n(s), \end{cases} \tag{5.5.1}$$

where $w_n(s) = x_n(s) - \lambda_n(1 + s)x_n(s)$ *and* $h_n(s) = x_n(s) - \lambda_n(1 + s)z_n(s)$*.*

Now, taking different initial points, $x_0(s) = 2s$, $x_0(s) = 2s^5$, $x_0(s) = 2s^{10}$ and fixed $u_0(s) = 2s^2$ in C and $\alpha_n = \frac{1}{10000n+10}$, the numerical experiment result provides that the sequence $\{\|x_n - p\|\}$ approaches zero as $n \to \infty$ (see, Figure 1 below), where $p = 0$. In this case, we observe that the sequence $\{x_n\}$ converges faster when the power of s gets large.

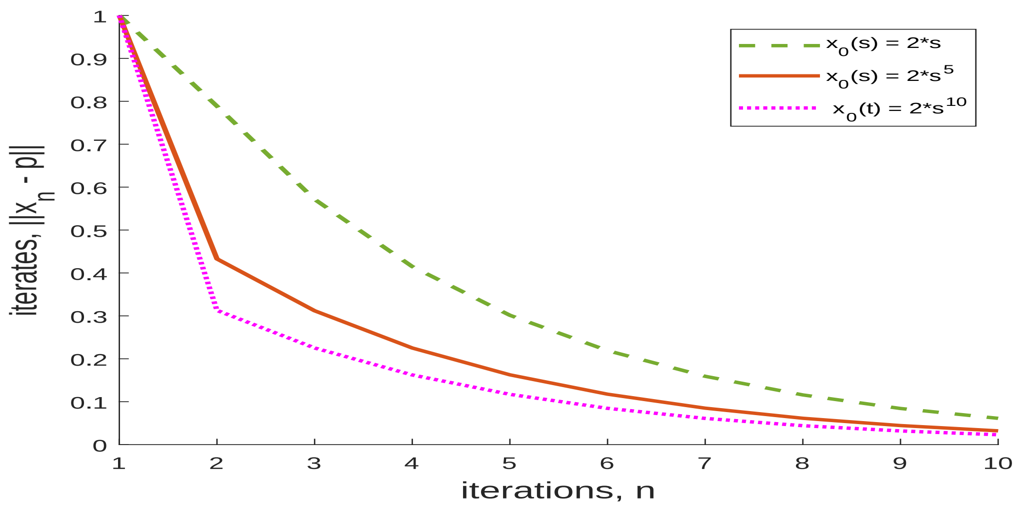


Figure 5.1: Figure 1: Convergence of the sequence $\{||x_n - p||\}$ as n gets large.

Next, we obtain the same numerical tests of algorithm 5.5.1 by taking initial points $u_0(s) = 2s^2$, $x_0(s) = 2s^{10}$ and different control parameters, $\alpha_n = \frac{1}{100n+10}$, $\alpha_n = \frac{1}{(100)^2 n+10}$, $\alpha_n = \frac{1}{(100)^3 n+10}$. In this case, we observe that the rate of convergence looks the same through out (see, Figure 2).

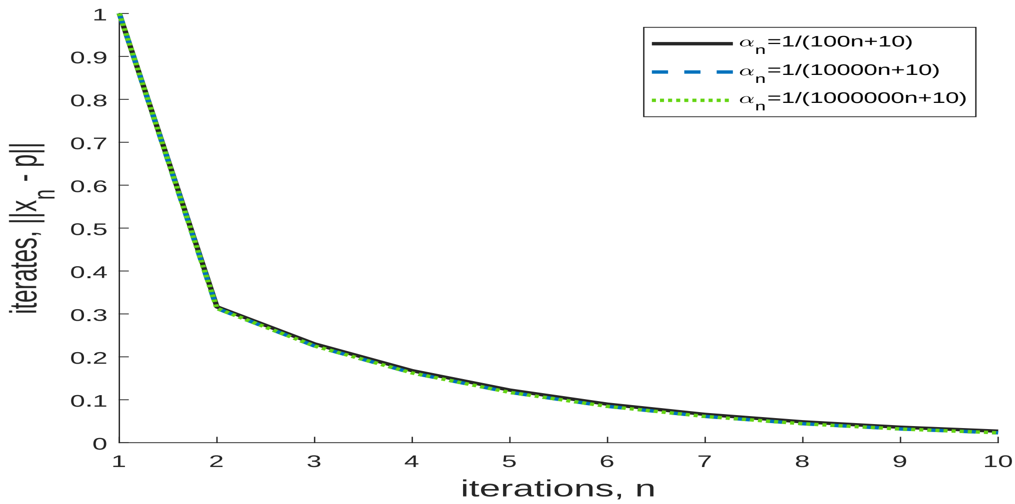


Figure 5.2: Figure 2: Convergence of the sequence $\{||x_n - p||\}$ as n gets large.

Chapter 6

Inertial method for a solution of Split Equality of Monotone Inclusion and the f-Fixed Point Problems in Banach Spaces

6.1 Introduction

In this chapter, we discussed the last research problem of our dissertation. We propose an inertial algorithm for solving split equality of monotone inclusion and f-fixed point of Bregman relatively f-nonexpansive mapping problems in reflexive real Banach spaces. Using the Bregman distance function, we prove a strong convergence theorem for the algorithm produced by the method in real reflexive Banach spaces. In addition, we provide some applications of our method and give numerical results to demonstrate the applicability and efficiency of the proposed method. In the second section of this chapter, we collect technical lemmas from literature to prove our main result. We state the proposed algorithm in section 3 and prove its boundedness. The main result will be stated and proved in section 4. Finally, we provide some applications of our method and give a numerical example to to demonstrate the applicability and efficiency of the proposed method.

6.2 Technical Tool Box

In this section, we collect technical results that will be useful throughout the chapter to prove our main theorems.

Lemma 6.2.1. *(Wega and Zegeye [93]) If $T : E \to E^*$ is a Bregman relatively f-nonexpansive mapping, then $F_f(T)$ is closed and convex.*

Lemma 6.2.2. *(Saejung and Yotkaew [76]) Let $\{b_n\} \subset \mathbb{R}$ and let $\{a_n\}$ be a sequence in $(0, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n, n \geq 1.$$

If for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $\liminf_{k\to\infty} (a_{n_k+1} - a_{n_k}) \geq 0$ we have $\limsup_{k\to\infty} b_{n_k} \leq 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 6.2.3. *(Barbu [6]) Let $A : E \to E^*$ be a monotone, hemicontinuous and bounded mapping, and $B : E \to 2^{E^*}$ be a maximal monotone mapping. Then $A+B$ is maximal monotone.*

Let E_1 and E_2 be reflexive real Banach spaces with duals E_1^* and E_2^*, respectively. Let $E = E_1 \times E_2$ with dual $E^* = E_1^* \times E_2^*$ and duality pairing

$$\langle x, y^* \rangle = \langle x_1, y_1^* \rangle + \langle x_2, y_2^* \rangle,$$

where $x = (x_1, x_2) \in E$, $y^* = (y_1^*, y_2^*) \in E^*$.

Let $h : E = E_1 \times E_2 \to (-\infty, +\infty]$ be defined by $h(x_1, x_2) = f(x_1) + g(x_2)$, $\forall (x_1, x_2) \in E_1 \times E_2$, where $f : E_1 \to (-\infty, +\infty]$ and $g : E_2 \to (-\infty, +\infty]$ are proper, lower semi-continuous and convex functions. Then h is a proper, lower semi-continuous and convex function and the subdifferential of h at $x = (x_1, x_2)$ is the convex set given by

$$\begin{aligned} \partial h(x) &= \{x^* \in E^* : h(y) - h(x) \geq \langle y - x, x^* \rangle, \forall y \in E\} \\ &= \{(x_1^*, x_2^*) \in E_1^* \times E_2^* : x_1^* \in \partial f(x_1) \text{ and } x_2^* \in \partial g(x_2)\}. \end{aligned}$$

If $f : E_1 \to (-\infty, +\infty]$ and $g : E_2 \to (-\infty, +\infty]$ are Gâteaux differentiable convex functions, then h is Gâteaux differentiable convex function and $\nabla h(x_1, x_2) = (\nabla f(x_1), \nabla g(x_2))$, $\forall (x_1, x_2) \in E_1 \times E_2$.

Definition 6.2.1. *The Split Equality of Monotone Inclusion and f-Fixed Point Problems (SEMIfFPP)* is defined as finding a point $(p, q) \in E_1 \times E_2$ such that

$$(p, q) \in (F_f(T) \cap (A + B)^{-1}0) \times (F_g(G) \cap (C + D)^{-1}0) \text{ and } S(p) = K(q), \quad (6.2.1)$$

where $T : E_1 \to E_1^*$ and $G : E_2 \to E_2^*$ are Bregman relatively f-nonexpansive and Bregman relatively g-nonexpansive mappings, respectively, $A : E_1 \to E_1^*$ and $C : E_2 \to E_2^*$ are monotone mappings, and $B : E_1 \to 2^{E_1^*}$ and $D : E_2 \to 2^{E_2^*}$ are maximal monotone mappings, $S : E_1 \to E_3$ and $K : E_2 \to E_3$ are bounded linear mappings with adjoints $S^* : E_3^* \to E_1^*$ and $K^* : E_3^* \to E_2^*$, respectively.

6.3 Iterative Algorithm

In this section, we give the assumptions that will be used in the sequel first. Next, we state the proposed algorithm and prove its boundedness.

Assumptions

(A1) Let $E_i, i = 1, 2, 3$ be reflexive real Banach spaces with their respective duals E_i^*, $i = 1, 2, 3$;

(A2) Let $f : E_1 \to \mathbb{R}$ and $g : E_2 \to \mathbb{R}$ be strongly coercive, lower semi-continuous, strongly convex, bounded and uniformly Fr$\acute{e}$chet differentiable Legendre functions on bounded subsets with strongly convex conjugate f^* and g^*, respectively. Let the strong convexity constants of f and g be μ_1 and μ_2, respectively, and let $\mu = \min\{\mu_1, \mu_2\}$;

(A3) Let $T : E_1 \to E_1^*$ and $G : E_2 \to E_2^*$ be Bregman relatively f-nonexpansive and Bregman relatively g-nonexpansive mappings, respectively;

(A4) Let $A : E_1 \to E_1^*$ and $C : E_2 \to E_2^*$ be uniformly continuous monotone mappings;

(A5) Let $B : E_1 \to 2^{E_1^*}$ and $D : E_2 \to 2^{E_2^*}$ be maximal monotone mappings;

(A6) Let $S : E_1 \to E_3$ and $K : E_2 \to E_3$ be bounded linear mappings with adjoints $S^* : E_3^* \to E_1^*$ and $K^* : E_3^* \to E_2^*$, respectively;

(A7) Let $\Omega = \{(a, b) \in (F_f(T) \cap (A + B)^{-1}(0)) \times (F_g(G) \cap (C + D)^{-1}(0)) : S(a) = K(b)\} \neq \emptyset$.

(A8) Let $\{\alpha_n\} \subset (0, 1)$ be such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(A9) Let $\{r_n\}$ be sequence in $(0, \frac{\mu}{2})$ such that $\lim_{n\to\infty} \frac{r_n}{\alpha_n} = 0$;

(A10) Let J_{E_3} be a normalized duality mapping on E_3.

Algorithm 6.1

Initialization: Choose $(x, w), (x_0, w_0), (x_1, w_1) \in E_1 \times E_2$, $\beta \in (0, 1)$, $\theta \in (0, \mu)$, $0 < \sigma, \rho, \eta, \lambda_1, \delta_1$. Define the algorithm as follows:

Step 0: Choose σ_n such that $0 \leq \sigma_n \leq \overline{\sigma}_n$ where $\psi_n = \|\nabla f(x_n) - \nabla f(x_{n-1})\| + \|\nabla g(w_n) - \nabla g(w_{n-1})\|$ and

$$\overline{\sigma}_n = \begin{cases} \min\left\{\frac{r_n}{\psi_n}, \sigma\right\} & \text{if } x_n \neq x_{n-1} \ \ \& \ \ w_n \neq w_{n-1} \\ \sigma & \text{otherwise} \end{cases} \tag{6.3.1}$$

Step 1: Compute

$$\begin{aligned} a_n &= \nabla f^*(\nabla f(x_n) + \sigma_n(\nabla f(x_n) - \nabla f(x_{n-1}))), \\ b_n &= \nabla g^*(\nabla g(w_n) + \sigma_n(\nabla g(w_n) - \nabla g(w_{n-1}))). \end{aligned} \tag{6.3.2}$$

Step 2: Choose γ_n such that $\rho \leq \gamma_n \leq \rho_n$ for $S(a_n) \neq K(b_n)$ otherwise $\gamma_n = \rho$, for some $\rho > 0$, where $\omega_n = \|S^* J_{E_3}(S(a_n)) - K(b_n)\|^2 + \|K^* J_{E_3}(K(b_n)) - S(a_n)\|^2$ and

$$\rho_n = \min\left\{\rho + 1, \frac{\mu\|S(a_n) - K(b_n)\|^2}{2\omega_n}\right\}. \tag{6.3.3}$$

Step 3: Compute

$$\begin{aligned} d_n &= \nabla f^*(\nabla f(a_n) - \gamma_n S^* J_{E_3}(S(a_n) - K(b_n))), \\ e_n &= \nabla g^*(\nabla g(b_n) - \gamma_n K^* J_{E_3}(K(b_n) - S(a_n))), \end{aligned} \tag{6.3.4}$$

Step 4: Compute

$$\begin{aligned} y_n &= J^B_{\lambda_n} \nabla f^*(\nabla f(d_n) - \lambda_n A(d_n)), \\ z_n &= J^D_{\lambda_n} \nabla g^*(\nabla g(e_n) - \delta_n C(e_n)). \end{aligned}$$

Step 5: Compute

$$\begin{aligned} u_n &= \nabla f^*(\nabla f(y_n) - \lambda_n(A(y_n) - A(d_n))), \\ v_n &= \nabla g^*(\nabla g(z_n) - \delta_n(C(z_n) - C(e_n))), \\ x_{n+1} &= \nabla f^*(\alpha_n \nabla f(x) + (1 - \alpha_n)\left[\beta \nabla f(u_n) + (1 - \beta)T(u_n)\right]), \quad (6.3.5) \\ w_{n+1} &= \nabla g^*(\alpha_n \nabla g(w) + (1 - \alpha_n)\left[\beta \nabla g(v_n) + (1 - \beta)G(v_n)\right]). \quad (6.3.6) \end{aligned}$$

Step 6: Choose λ_{n+1} and δ_{n+1} such that $\eta \leq \lambda_{n+1} \leq \overline{\lambda_{n+1}}$ and $\eta \leq \delta_{n+1} \leq \overline{\delta_{n+1}}$, for some $\eta > 0$, where

$$\overline{\lambda_{n+1}} = \begin{cases} \min\left\{\lambda_n, \frac{\theta\|y_n - d_n\|}{\|A(y_n) - A(d_n)\|}\right\}, & \text{if } A(y_n) \neq A(d_n), \\ \lambda_n, & \text{otherwise}, \end{cases} \tag{6.3.7}$$

and

$$\overline{\delta_{n+1}} = \begin{cases} \min\left\{\delta_n, \frac{\theta\|z_n - e_n\|}{\|C(z_n)-C(e_n)\|}\right\} & \text{if } C(z_n) \neq C(e_n), \\ \delta_n, & \text{otherwise.} \end{cases} \tag{6.3.8}$$

Set $n := n+1$ and go to **Step 0**.

Remark 6.3.1. We note that if A and C are Lipschitz monotone mappings with Lipschitz constants L_1 and L_2, respectively, then following the method in [80], we obtain $\eta = \min\left\{\frac{\theta}{L_1}, \lambda_1, \frac{\theta}{L_2}, \delta_1\right\}$ and hence $\lim_{n\to\infty}\lambda_n = \lambda$ and $\lim_{n\to\infty}\delta_n = \delta$, where $\lambda, \delta \geq \eta$.

Lemma 6.3.1. *Suppose that the assumptions (A1)- (A10) hold. Then the sequences $\{x_n\}$ and $\{w_n\}$ generated by Algorithm 6.1 are bounded.*

Proof. Let $(p, q) \in \Omega$. By the definition of the Bregman distance, (6.3.7) and Lemma 5.2.6, we have

$$\begin{aligned} D_f(p, u_n) &= D_f(p, \nabla f^*(\nabla f(y_n) - \lambda_n(A(y_n) - A(d_n)))) \\ &= f(p) - f(u_n) - \langle p - u_n, \nabla f(y_n)\rangle + \langle p - u_n, \lambda_n(A(y_n) - A(d_n))\rangle \\ &= f(p) - f(y_n) - \langle p - y_n, \nabla f(y_n)\rangle \\ &\quad -[f(u_n) - f(y_n) - \langle u_n - y_n, \nabla f(y_n)\rangle] \\ &\quad + \langle p - u_n, \lambda_n(A(y_n) - A(d_n))\rangle \\ &= D_f(p, y_n) - D_f(u_n, y_n) + \langle p - u_n, \lambda_n(A(y_n) - A(d_n))\rangle \\ &= D_f(p, y_n) - D_f(u_n, y_n) + \langle p - y_n, \lambda_n(A(y_n) - A(d_n))\rangle \\ &\quad + \langle y_n - u_n, \lambda_n(A(y_n) - A(d_n))\rangle \\ &\leq D_f(p, y_n) - D_f(u_n, y_n) + \langle p - y_n, \lambda_n(A(y_n) - A(d_n))\rangle \\ &\quad + \lambda_n\|A(y_n) - A(d_n)\|\|y_n - u_n\| \\ &\leq D_f(p, y_n) - D_f(u_n, y_n) + \langle p - y_n, \lambda_n(A(y_n) - A(d_n))\rangle \\ &\quad + \left(\frac{\lambda_n}{\lambda_{n+1}}\right)\theta\|y_n - d_n\|\|y_n - u_n\| \end{aligned}$$

$$
\begin{aligned}
&\le D_f(p,y_n) - D_f(u_n,y_n) + \langle p-y_n, \lambda_n(A(y_n)-A(d_n))\rangle \\
&\quad + \frac{\theta\lambda_n}{2\lambda_{n+1}}\|y_n-d_n\|^2 + \frac{\theta\lambda_n}{2\lambda_{n+1}}\|y_n-u_n\|^2 \\
&\le D_f(p,y_n) - D_f(u_n,y_n) + \langle p-y_n, \lambda_n(A(y_n)-A(d_n))\rangle \\
&\quad + \frac{\theta\lambda_n}{2\lambda_{n+1}}\frac{2}{\mu}D_f(y_n,d_n) + \frac{\theta\lambda_n}{2\lambda_{n+1}}\frac{2}{\mu}D_f(u_n,y_n) \\
&\le D_f(p,y_n) - \left(1-\frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right)D_f(u_n,y_n) + \frac{\theta\lambda_n}{\mu\lambda_{n+1}}D_f(y_n,d_n) \qquad (6.3.9)\\
&\quad + \langle p-y_n, \lambda_n(A(y_n)-A(d_n))\rangle .
\end{aligned}
$$

From (1.2.3), we have

$$
D_f(p,y_n) = D_f(p,d_n) - D_f(y_n,d_n) + \langle p-y_n, \nabla f(d_n) - \nabla f(y_n)\rangle . \qquad (6.3.10)
$$

Furthermore, from (1.2.10) and (1.2.11), we obtain

$$
\begin{aligned}
D_f(p,d_n) &= D_f(p, \nabla f^*(\nabla f(a_n) - \gamma_n S^* J_{E_3}(S(a_n)-K(b_n)))) \\
&= V_f(p, \nabla f(a_n) - \gamma_n S^* J_{E_3}(S(a_n)-K(b_n))) \\
&\le V_f(p, \nabla f(a_n)) - \gamma_n \langle d_n - p, S^* J_{E_3}(S(a_n)-K(b_n))\rangle \\
&= D_f(p,a_n) - \gamma_n \langle S(d_n)-S(p), J_{E_3}(S(a_n)-K(b_n))\rangle . \qquad (6.3.11)
\end{aligned}
$$

Substituting (6.3.11) into (6.3.10) gives

$$
\begin{aligned}
D_f(p,y_n) &\le D_f(p,a_n) - \gamma_n \langle S(d_n)-S(p), J_{E_3}(S(a_n)-K(b_n))\rangle \\
&\quad - D_f(y_n,d_n) + \langle p-y_n, \nabla f(d_n)-\nabla f(y_n)\rangle \\
&= D_f(p,a_n) - D_f(y_n,d_n) + \langle p-y_n, \nabla f(d_n)-\nabla f(y_n)\rangle \qquad (6.3.12)\\
&\quad - \gamma_n \langle S(d_n)-S(p), J_{E_3}(S(a_n)-K(b_n))\rangle .
\end{aligned}
$$

Again, from (1.2.3), we have

$$
D_f(p,a_n) = D_f(p,x_n) - D_f(a_n,x_n) + \langle p-a_n, \nabla f(x_n) - \nabla f(a_n)\rangle . \qquad (6.3.13)
$$

Now, from (6.3.1), (6.3.2) and Lemma 5.2.6 we obtain that

$$
\begin{aligned}
\langle p-a_n, \nabla f(x_n)-\nabla f(a_n)\rangle &\le \|\nabla f(x_n)-\nabla f(a_n)\|\|p-a_n\| \\
&= \sigma_n\|\nabla f(x_n)-\nabla f(x_{n-1})\|\|p-a_n\| \\
&\le \frac{\sigma_n}{2}\|\nabla f(x_n)-\nabla f(x_{n-1})\|\left[\|p-a_n\|^2+1\right] \\
&\le \sigma_n\|\nabla f(x_n)-\nabla f(x_{n-1})\|\left[\|p-x_n\|^2+\|x_n-a_n\|^2\right] \\
&\quad + \frac{\sigma_n}{2}\|\nabla f(x_n)-\nabla f(x_{n-1})\| \\
&\le \frac{2r_n}{\mu}D_f(p,x_n) + \frac{2r_n}{\mu}D_f(a_n,x_n) + \frac{r_n}{2}. \qquad (6.3.14)
\end{aligned}
$$

Combining (6.3.9), (6.3.12), (6.3.13) and (6.3.14), we find

$$\begin{aligned}
D_f(p,u_n) &\leq D_f(p,y_n) - \left(1-\frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right) D_f(u_n,y_n) + \frac{\theta\lambda_n}{\mu\lambda_{n+1}} D_f(y_n,d_n) \\
&\quad + \langle p-y_n, \lambda_n(A(y_n)-A(d_n))\rangle \\
&\leq D_f(p,a_n) - D_f(y_n,d_n) + \langle p-y_n, \nabla f(d_n)-\nabla f(y_n)\rangle \\
&\quad -\gamma_n \langle S(d_n)-S(p), J_{E_3}(S(a_n)-K(b_n))\rangle \\
&\quad -\left(1-\frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right) D_f(u_n,y_n) + \frac{\theta\lambda_n}{\mu\lambda_{n+1}} D_f(y_n,d_n) \\
&\quad + \langle p-y_n, \lambda_n(A(y_n)-A(d_n))\rangle \\
&= D_f(p,a_n) - D_f(y_n,d_n) - \left(1-\frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right) D_f(u_n,y_n) \\
&\quad + \frac{\theta\lambda_n}{\mu\lambda_{n+1}} D_f(y_n,d_n) - \gamma_n \langle S(d_n)-S(p), J_{E_3}(S(a_n)-K(b_n))\rangle \\
&\quad + \langle p-y_n, \nabla f(d_n)-\nabla f(y_n)+\lambda_n(A(y_n)-A(d_n))\rangle
\end{aligned}$$

$$\begin{aligned}
&= D_f(p,x_n) - D_f(a_n,x_n) + \langle p-a_n, \nabla f(x_n)-\nabla f(a_n)\rangle \\
&\quad -D_f(y_n,d_n) - \left(1-\frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right) D_f(u_n,y_n) + \frac{\theta\lambda_n}{\mu\lambda_{n+1}} D_f(y_n,d_n) \\
&\quad -\gamma_n \langle S(d_n)-S(p), J_{E_3}(S(a_n)-K(b_n))\rangle \\
&\quad + \langle p-y_n, \nabla f(d_n)-\nabla f(y_n)+\lambda_n(A(y_n)-A(d_n))\rangle \\
&\leq D_f(p,x_n) - D_f(a_n,x_n) + \frac{2r_n}{\mu} D_f(p,x_n) + \frac{2r_n}{\mu} D_f(a_n,x_n) + \frac{r_n}{2} \\
&\quad -\left(1-\frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right) D_f(y_n,d_n) - \left(1-\frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right) D_f(u_n,y_n) \\
&\quad -\gamma_n \langle S(d_n)-S(p), J_{E_3}(S(a_n)-K(b_n))\rangle \\
&\quad + \langle p-y_n, \nabla f(d_n)-\nabla f(y_n)+\lambda_n(A(y_n)-A(d_n))\rangle \\
&= \left(1+\frac{2r_n}{\mu}\right) D_f(p,x_n) - \left(1-\frac{2r_n}{\mu}\right) D_f(a_n,x_n) \qquad (6.3.15) \\
&\quad -\left(1-\frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right) D_f(y_n,d_n) - \left(1-\frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right) D_f(u_n,y_n) \\
&\quad -\gamma_n \langle S(d_n)-S(p), J_{E_3}(S(a_n)-K(b_n))\rangle + \frac{r_n}{2} \\
&\quad + \langle p-y_n, \nabla f(d_n)-\nabla f(y_n)+\lambda_n(A(y_n)-A(d_n))\rangle .
\end{aligned}$$

By the definition of y_n, we have $\nabla f(d_n) - \lambda_n A(d_n) \in \nabla f(y_n) + \lambda_n B(y_n)$. Since B is maximal monotone, there exists $h_n \in B(y_n)$ such that

$$\nabla f(d_n) - \lambda_n A(d_n) = \nabla f(y_n) + \lambda_n h_n.$$

This implies that

$$h_n = \frac{1}{\lambda_n}(\nabla f(d_n) - \nabla f(y_n) - \lambda_n A(d_n)) \in B(y_n). \qquad (6.3.16)$$

Since $0 \in (A+B)(p)$, $A(y_n)+h_n \in (A+B)(y_n)$ and $A+B$ is monotone, we get

$$\langle p-y_n, A(y_n)+h_n\rangle \leq 0. \tag{6.3.17}$$

From (6.3.16) and (6.3.17), we have

$$\langle p-y_n, \nabla f(d_n)-\nabla f(y_n)+\lambda_n(A(y_n)-A(d_n))\rangle \leq 0. \tag{6.3.18}$$

Thus, from (6.3.15) and (6.3.18) we get

$$\begin{aligned} D_f(p,u_n) \ \leq\ & \left(1+\frac{2r_n}{\mu}\right)D_f(p,x_n)-\left(1-\frac{2r_n}{\mu}\right)D_f(a_n,x_n) \\ & -\left(1-\frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right)D_f(y_n,d_n)-\left(1-\frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right)D_f(u_n,y_n) \\ & -\gamma_n\left\langle S(d_n)-S(p), J_{E_3}(S(a_n)-K(b_n))\right\rangle+\frac{r_n}{2}. \end{aligned} \tag{6.3.19}$$

Furthermore, from (6.3.5), (6.3.19) and the Bregman relatively f-nonexpansiveness of T, we have

$$\begin{aligned} D_f(p,x_{n+1}) &= D_f(p,\nabla f^*(\alpha_n\nabla f(x)+(1-\alpha_n)\left[\beta\nabla f(u_n)+(1-\beta)T(u_n)\right])) \\ &\leq \alpha_n D_f(p,x)+(1-\alpha_n)D_f(p,\nabla f^*(\beta\nabla f(u_n)+(1-\beta)T(u_n))) \\ &\leq \alpha_n D_f(p,x)+(1-\alpha_n)\left[\beta D_f(p,u_n)+(1-\beta)D_f(p,\nabla f^*(T(u_n)))\right] \\ &\leq \alpha_n D_f(p,x)+(1-\alpha_n)D_f(p,u_n) \\ &\leq \alpha_n D_f(p,x)+(1-\alpha_n)\left(1+\frac{2r_n}{\mu}\right)D_f(p,x_n) \\ &\quad -(1-\alpha_n)\left(1-\frac{2r_n}{\mu}\right)D_f(a_n,x_n) \\ &\quad -(1-\alpha_n)\left(1-\frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right)D_f(y_n,d_n) \\ &\quad -(1-\alpha_n)\left(1-\frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right)D_f(u_n,y_n) \\ &\quad -(1-\alpha_n)\gamma_n\left\langle S(d_n)-S(p), J_{E_3}(S(a_n)-K(b_n))\right\rangle+\frac{r_n}{2}(1-\alpha_n). \end{aligned} \tag{6.3.20}$$

Similarly, we have

$$\begin{aligned} D_g(q,w_{n+1}) &\leq \alpha_n D_g(q,w)+(1-\alpha_n)\left(1+\frac{2r_n}{\mu}\right)D_g(q,w_n) \\ &\quad -(1-\alpha_n)\left(1-\frac{2r_n}{\mu}\right)D_g(b_n,w_n) \\ &\quad -(1-\alpha_n)\left(1-\frac{\theta\delta_n}{\mu\delta_{n+1}}\right)D_g(z_n,e_n) \\ &\quad -(1-\alpha_n)\left(1-\frac{\theta\delta_n}{\mu\delta_{n+1}}\right)D_g(v_n,z_n) \\ &\quad -(1-\alpha_n)\gamma_n\left\langle K(e_n)-K(q), J_{E_3}(K(b_n)-S(a_n))\right\rangle+\frac{r_n}{2}(1-\alpha_n). \end{aligned} \tag{6.3.21}$$

Since $\lim_{n\to\infty}\lambda_n$ and $\lim_{n\to\infty}\delta_n$ exist and $\theta \in (0,\mu)$, we obtain that

$$\lim_{n\to\infty}\left(1-\frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right) = \lim_{n\to\infty}\left(1-\frac{\theta\delta_n}{\mu\delta_{n+1}}\right) = 1-\frac{\theta}{\mu} > 0. \tag{6.3.22}$$

Take $\epsilon \in (0, \frac{\mu}{2})$. Then, from (A9) and (6.3.22), there exists $N \in \mathbb{N}$ such that

$$\frac{2r_n}{\mu} < \alpha_n\epsilon, \quad \left(1-\frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right) > 0 \text{ and } \left(1-\frac{\theta\delta_n}{\mu\delta_{n+1}}\right) > 0, \forall n \geq N. \tag{6.3.23}$$

Now, from (6.3.20), (6.3.21) and (6.3.23), we have

$$\begin{aligned} D_f(p,x_{n+1}) + D_g(q,w_{n+1}) \leq\ & \alpha_n(D_f(p,x) + D_g(q,w)) \\ & +(1-\alpha_n)(D_f(p,x_n) + D_g(q,w_n)) \\ & +\alpha_n\epsilon(D_f(p,x_n) + D_g(q,w_n)) + \alpha_n\epsilon\frac{\mu}{2} \\ & -(1-\alpha_n)\gamma_n\langle K(e_n) - S(d_n), J_{E_3}(K(b_n) - S(a_n))\rangle, \end{aligned} \tag{6.3.24}$$

for all $n \geq N$.

But,

$$\begin{aligned} -\langle K(e_n) - S(d_n), J_{E_3}(K(b_n) - S(a_n))\rangle &= -\langle K(b_n) - S(a_n), J_{E_3}(K(b_n) - S(a_n))\rangle \\ &\quad -\langle K(e_n) - K(b_n), J_{E_3}(K(b_n) - S(a_n))\rangle \\ &\quad -\langle S(a_n) - S(d_n), J_{E_3}(K(b_n) - S(a_n))\rangle \\ &= -\|K(b_n) - S(a_n)\|^2 \\ &\quad -\langle e_n - b_n, K^*J_{E_3}(K(b_n) - S(a_n))\rangle \\ &\quad -\langle a_n - d_n, S^*J_{E_3}(K(b_n) - S(a_n))\rangle \\ &\leq -\|K(b_n) - S(a_n)\|^2 \\ &\quad +\|e_n - b_n\|\|K^*J_{E_3}(K(b_n) - S(a_n))\| \\ &\quad +\|a_n - d_n\|\|S^*J_{E_3}(K(b_n) - S(a_n))\|. \end{aligned}$$

From the fact that ∇g^* is a Lipschitz mapping with constant $\frac{1}{\mu}$ and the definition of e_n, we obtain that

$$\begin{aligned} \|e_n - b_n\| &= \|\nabla g^*(\nabla g(b_n) - \gamma_n K^*J_{E_3}(K(b_n) - S(a_n))) - b_n\| \\ &\leq \frac{\gamma_n}{\mu}\|K^*J_{E_3}(K(b_n) - S(a_n)\|. \end{aligned} \tag{6.3.25}$$

Similarly, by the Lipschitz property of ∇f^* and the definition of d_n gives

$$\|d_n - a_n\| \leq \frac{\gamma_n}{\mu}\|S^*J_{E_3}(S(a_n) - K(b_n))\|. \tag{6.3.26}$$

Then, from 6.3.3, (6.3.25), (6.3.25) and (6.3.26), we get

$$\begin{aligned}
-\gamma_n \langle K(e_n) - S(d_n), J_{E_3}(K(b_n) - S(a_n))\rangle &\leq -\gamma_n\|K(b_n) - S(a_n)\|^2 \\
&\quad +\frac{\gamma_n^2}{\mu}\|K^* J_{E_3}(K(b_n) - S(a_n))\|^2 \\
&\quad +\frac{\gamma_n^2}{\mu}\|S^* J_{E_3}(S(a_n) - K(b_n))\|^2 \\
&\leq -\gamma_n\|K(b_n) - S(a_n)\|^2 \\
&\quad +\frac{\gamma_n}{2}\|K(e_n) - S(d_n)\|^2 \\
&= -\frac{\gamma_n}{2}\|K(b_n) - S(a_n)\|^2 \\
&\leq -\frac{\rho}{2}\|K(b_n) - S(a_n)\|^2. \qquad (6.3.27)
\end{aligned}$$

Thus, from (6.3.24) and (6.3.27), we obtain for all $n \geq N$

$$\begin{aligned}
D_f(p, x_{n+1}) + D_g(q, w_{n+1}) &\leq \alpha_n(D_f(p, x) + D_g(q, w)) \\
&\quad +(1 - \alpha_n)(D_f(p, x_n) + D_g(q, w_n)) \\
&\quad +\alpha_n\epsilon(D_f(p, x_n) + D_g(q, w_n)) + \alpha_n\epsilon\frac{\mu}{2} \\
&\quad -\frac{\rho}{2}(1 - \alpha_n)\|K(b_n) - S(a_n)\|^2 \\
&\leq \alpha_n(D_f(p, x) + D_g(q, w)) \\
&\quad +(1 - \alpha_n(1 - \epsilon))(D_f(p, x_n) + D_g(q, w_n)) + \alpha_n\epsilon\frac{\mu}{2} \\
&\leq (1 - \alpha_n(1 - \epsilon))(D_f(p, x_n) + D_g(q, w_n)) \\
&\quad +\alpha_n(1 - \epsilon)\left[\frac{1}{1 - \epsilon}(D_f(p, x) + D_g(q, w)) + \frac{\mu\epsilon}{2(1 - \epsilon)}\right] \\
&\leq \max\{D_f(p, x_n) + D_g(q, w_n), \\
&\quad \frac{1}{1 - \epsilon}(D_f(p, x) + D_g(q, w)) + \frac{\mu\epsilon}{2(1 - \epsilon)}\}. \qquad (6.3.28)
\end{aligned}$$

Therefore, by induction, for all $n \geq N$, we have that

$$\begin{aligned}
D_f(p, x_n) + D_g(q, w_n) &\leq \max\{D_f(p, x_N) + D_g(q, w_N), \\
&\quad \frac{1}{1 - \epsilon}(D_f(p, x) + D_g(q, w)) + \frac{\mu\epsilon}{2(1 - \epsilon)}\},
\end{aligned}$$

and hence $\{D_f(p, x_n) + D_g(q, w_n)\}$ is bounded which implies that the sequences $\{D_f(p, x_n)\}$ and $\{D_g(q, w_n)\}$ are bounded. Furthermore, by Lemma 5.2.3, we have $\{x_n\}$ and $\{w_n\}$ are bounded. $\square$

6.4 Strong Convergence Theorem

Theorem 6.4.1. *Suppose that assumption (A1)- A(10) are satisfied. Then, the sequence* $\{(x_n, w_n)\}$ *generated by Algorithm 6.1 converges strongly to* (p, q) *in* Ω*, where* $(p, q) = P^h_\Omega(x, w)$*, where* $h : E_1 \times E_2 \to \mathbb{R}$ *is given by* $h(x, y) = f(x) + g(y)$*.*

Proof. Let $(p, q) = P^h_\Omega(x, w)$. Then, from Algorithm 6.1, Lemma 1.2.9, (1.2.10), and (1.2.11), we obtain

$$\begin{aligned}
D_f(p, x_{n+1}) &= D_f(p, \nabla f^* (\alpha_n \nabla f(x) + (1 - \alpha_n)[\beta \nabla f(u_n) + (1 - \beta)T(u_n)])) \\
&= V_f(p, \alpha_n \nabla f(x) + (1 - \alpha_n)[\beta \nabla f(u_n) + (1 - \beta)T(u_n)]) \\
&\leq V_f(p, \alpha_n \nabla f(p) + (1 - \alpha_n)[\beta \nabla f(u_n) + (1 - \beta)T(u_n)]) \\
&\quad -\alpha_n \langle x_{n+1} - p, \nabla f(p) - \nabla f(x)\rangle \\
&= D_f(p, \nabla f^* (\alpha_n \nabla f(p) + (1 - \alpha_n)[\beta \nabla f(u_n) + (1 - \beta)T(u_n)])) \\
&\quad -\alpha_n \langle x_{n+1} - p, \nabla f(p) - \nabla f(x)\rangle
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n D_f(p, p) + (1 - \alpha_n)D_f(p, \nabla f^* (\beta \nabla f(u_n) + (1 - \beta)T(u_n))) \\
&\quad -\alpha_n \langle x_{n+1} - p, \nabla f(p) - \nabla f(x)\rangle \\
&= (1 - \alpha_n)D_f(p, \nabla f^* (\beta \nabla f(u_n) + (1 - \beta)T(u_n))) \\
&\quad -\alpha_n \langle x_{n+1} - p, \nabla f(p) - \nabla f(x)\rangle \\
&= (1 - \alpha_n)V_f(p, (\beta \nabla f(u_n) + (1 - \beta)T(u_n))) \\
&\quad +\alpha_n \langle x_{n+1} - x_n, \nabla f(x) - \nabla f(p)\rangle + \alpha_n \langle x_n - p, \nabla f(x) - \nabla f(p)\rangle \\
&\leq (1 - \alpha_n)V_f(p, \beta \nabla f(u_n) + (1 - \beta)T(u_n)) \qquad (6.4.1) \\
&\quad +\alpha_n \|x_{n+1} - x_n\| \|\nabla f(x) - \nabla f(p)\| + \alpha_n \langle x_n - p, \nabla f(x) - \nabla f(p)\rangle .
\end{aligned}$$

Thus, from the definition of V_f, uniformly convexity of f^* and the Bregman relatively f-nonexpansiveness of T, we obtain that

$$\begin{aligned}
V_f(p, \beta \nabla f(u_n) + (1 - \beta)T(u_n)) &= f(p) - \langle p, \beta \nabla f(u_n) + (1 - \beta)T(u_n)\rangle \\
&\quad +f^*(\beta \nabla f(u_n) + (1 - \beta)T(u_n)) \\
&\leq f(p) - \beta \langle p, \nabla f(u_n)\rangle - (1 - \beta) \langle p, T(u_n)\rangle \\
&\quad +\beta f^*(\nabla f(u_n)) + (1 - \beta)f^*(T(u_n)) \\
&\quad -\beta(1 - \beta)\psi_1(\|\nabla f(u_n) - T(u_n)\|) \\
&= \beta V_f(p, \nabla f(u_n)) + (1 - \beta)V_f(p, T(u_n)) \\
&\quad -\beta(1 - \beta)\psi_1(\|\nabla f(u_n) - T(u_n)\|) \\
&= \beta D_f(p, u_n) + (1 - \beta)D_f(p, \nabla f^*(T(u_n))) \\
&\quad -\beta(1 - \beta)\psi_1(\|\nabla f(u_n) - T(u_n)\|)
\end{aligned}$$

$$
\begin{aligned}
&= D_f(p,u_n) \\
&\quad -\beta(1-\beta)\psi_1(\|\nabla f(u_n)-T(u_n)\|), \qquad (6.4.2)
\end{aligned}
$$

where ψ_1 is the modulus of convexity of f.
From (6.3.19), (6.4.1) and (6.4.2), we obtain that

$$
\begin{aligned}
D_f(p,x_{n+1}) &\leq (1-\alpha_n)D_f(p,u_n)-\beta(1-\beta)(1-\alpha_n)\psi_1(\|\nabla f(u_n)-T(u_n)\|) \\
&\quad +\alpha_n\|x_{n+1}-x_n\|\|\nabla f(x)-\nabla f(p)\|+\alpha_n\left\langle x_n-p,\nabla f(x)-\nabla f(p)\right\rangle \\
&\leq (1-\alpha_n)\left(1+\frac{2r_n}{\mu}\right)D_f(p,x_n)-(1-\alpha_n)\left(1-\frac{2r_n}{\mu}\right)D_f(a_n,x_n) \\
&\quad -(1-\alpha_n)\left(1-\frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right)D_f(y_n,d_n) \\
&\quad -(1-\alpha_n)\left(1-\frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right)D_f(u_n,y_n) \\
&\quad -\gamma_n(1-\alpha_n)\left\langle S(d_n)-S(p),J_{E_3}(S(a_n)-K(b_n))\right\rangle+(1-\alpha_n)\frac{r_n}{2} \\
&\quad -\beta(1-\beta)(1-\alpha_n)\psi_1(\|\nabla f(u_n)-T(u_n)\|) \\
&\quad +\alpha_n\|x_{n+1}-x_n\|\|\nabla f(x)-\nabla f(p)\| \\
&\quad +\alpha_n\left\langle x_n-p,\nabla f(x)-\nabla f(p)\right\rangle, \qquad (6.4.3)
\end{aligned}
$$

and hence from (6.3.23) and (6.4.3), we get

$$\begin{aligned}
D_f(p, x_{n+1}) &\leq (1-\alpha_n)D_f(p, x_n) + \frac{2r_n}{\mu}D_f(p, x_n) + \frac{r_n}{2} \\
&\quad +\alpha_n \langle x_n - p, \nabla f(x) - \nabla f(p)\rangle + \alpha_n\|x_{n+1} - x_n\|\|\nabla f(x) - \nabla f(p)\| \\
&\quad -(1-\alpha_n)\left(1 - \frac{2r_n}{\mu}\right) D_f(a_n, x_n) \\
&\quad -(1-\alpha_n)\left(1 - \frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right) D_f(y_n, d_n) \\
&\quad -(1-\alpha_n)\left(1 - \frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right) D_f(u_n, y_n) \\
&\quad -\gamma_n(1-\alpha_n)\langle S(d_n) - S(p), J_{E_3}(S(a_n) - K(b_n))\rangle \\
&\quad -\beta(1-\beta)(1-\alpha_n)\psi_1(\|\nabla f(u_n) - T(u_n)\|) \\
&\leq (1-\alpha_n)D_f(p, x_n) + \alpha_n\epsilon D_f(p, x_n) + \frac{r_n}{2} \\
&\quad +\alpha_n \langle x_n - p, \nabla f(x) - \nabla f(p)\rangle + \alpha_n\|x_{n+1} - x_n\|\|\nabla f(x) - \nabla f(p)\| \\
&\quad -(1-\alpha_n)(1 - \alpha_n\epsilon) D_f(a_n, x_n) \\
&\quad -(1-\alpha_n)\left(1 - \frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right) D_f(y_n, d_n) \\
&\quad -(1-\alpha_n)\left(1 - \frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right) D_f(u_n, y_n) \\
&\quad -\gamma_n(1-\alpha_n)\langle S(d_n) - S(p), J_{E_3}(S(a_n) - K(b_n))\rangle \\
&\quad -\beta(1-\beta)(1-\alpha_n)\psi_1(\|\nabla f(u_n) - T(u_n)\|) \\
&= (1-\alpha_n(1-\epsilon))D_f(p, x_n) \qquad (6.4.4) \\
&\quad +\alpha_n(1-\epsilon)\left[\frac{1}{1-\epsilon}\langle x_n - p, \nabla f(x) - \nabla f(p)\rangle\right] \\
&\quad +\frac{r_n}{2} + \alpha_n\|x_{n+1} - x_n\|\|\nabla f(x) - \nabla f(p)\| \\
&\quad -(1-\alpha_n)(1 - \alpha_n\epsilon) D_f(a_n, x_n) - (1-\alpha_n)\left(1 - \frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right) D_f(y_n, d_n) \\
&\quad -(1-\alpha_n)\left(1 - \frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right) D_f(u_n, y_n) \\
&\quad -\beta(1-\beta)(1-\alpha_n)\psi_1(\|\nabla f(u_n) - T(u_n)\|) \\
&\quad -\gamma_n(1-\alpha_n)\langle S(d_n) - S(p), J_{E_3}(S(a_n) - K(b_n))\rangle.
\end{aligned}$$

Similarly, we have

$$\begin{aligned} D_g(q, w_{n+1}) \;\le\; & (1-\alpha_n(1-\epsilon))D_g(q,w_n) \qquad (6.4.5)\\ & +\alpha_n(1-\epsilon)\left[\frac{1}{1-\epsilon}\langle w_n - q, \nabla g(w) - \nabla g(q)\rangle\right]\\ & +\frac{r_n}{2} + \alpha_n\|w_{n+1}-w_n\|\|\nabla g(w) - \nabla g(q)\|\\ & -(1-\alpha_n)(1-\alpha_n\epsilon)D_g(b_n,w_n) - (1-\alpha_n)\left(1-\frac{\theta\delta_n}{\mu\delta_{n+1}}\right)D_g(v_n,z_n)\\ & -(1-\alpha_n)\left(1-\frac{\theta\delta_n}{\mu\delta_{n+1}}\right)D_g(z_n,e_n)\\ & -\beta(1-\beta)(1-\alpha_n)\psi_2(\|\nabla g(v_n) - G(v_n)\|)\\ & -(1-\alpha_n)\gamma_n\langle K(e_n) - K(q), J_{E_3}(K(b_n) - S(a_n))\rangle, \end{aligned}$$

where ψ_2 is the modulus of convexity of g.
From the fact that $\{r_n\} \subset (0, \frac{\mu}{2})$, $\theta \in (0,\mu)$, (A7), (6.4.4) and (6.4.5) we obtain that

$$\begin{aligned} D_f(p,x_{n+1}) + D_g(q,w_{n+1}) \;\le\; & (1-\alpha_n(1-\epsilon))(D_f(p,x_n)+D_g(q,w_n)) \qquad (6.4.6)\\ & +\alpha_n(1-\epsilon)\frac{1}{1-\epsilon}\langle x_n - p, \nabla f(x) - \nabla f(p)\rangle\\ & +\alpha_n(1-\epsilon)\frac{1}{1-\epsilon}\langle w_n - q, \nabla g(w) - \nabla g(q)\rangle\\ & +\alpha_n D(\|x_{n+1}-x_n\| + \|w_{n+1}-w_n\|)\\ & +r_n - (1-\alpha_n)(1-\epsilon\alpha_n)(D_f(a_n,x_n) + D_g(b_n,w_n))\\ & -(1-\alpha_n)\left(1-\frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right)D_f(y_n,d_n)\\ & -(1-\alpha_n)\left(1-\frac{\theta\delta_n}{\mu\delta_{n+1}}\right)D_g(z_n,e_n)\\ & -(1-\alpha_n)\left(1-\frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right)D_f(u_n,y_n) \end{aligned}$$

$$
\begin{aligned}
& -(1-\alpha_n) + \left(1 - \frac{\theta\delta_n}{\mu\delta_{n+1}}\right) D_g(v_n, z_n) \\
& -M(1-\alpha_n)\psi_1(\|\nabla f(u_n) - T(u_n)\|) \\
& -M(1-\alpha_n)\psi_2(\|\nabla g(v_n) - G(v_n)\|) \\
& -(1-\alpha_n)\frac{\rho}{2}\|S(a_n) - K(b_n)\|^2 \\
\leq\ & (1-\alpha_n(1-\epsilon))(D_f(p, x_n) + D_g(q, w_n)) \qquad (6.4.7) \\
& +\alpha_n(1-\epsilon)\frac{1}{1-\epsilon}\langle x_n - p, \nabla f(x) - \nabla f(p)\rangle \\
& +\alpha_n(1-\epsilon)\frac{1}{1-\epsilon}\langle w_n - q, \nabla g(w) - \nabla g(q)\rangle \\
& +r_n + \alpha_n D(\|x_{n+1} - x_n\| + \|w_{n+1} - w_n\|),
\end{aligned}
$$

where $D = \max\{\|\nabla f(x) - \nabla f(p)\|, \|\nabla g(w) - \nabla g(q)\|\}$ and $M = \beta(1-\beta)$.

Suppose that $\{D_f(p, x_{n_k}) + D_g(q, w_{n_k})\}$ is a subsequence of $\{D_f(p, x_n) + D_g(q, w_n)\}$ such that

$$\liminf_{k\to\infty} [(D_f(p, x_{n_k+1}) + D_g(q, w_{n_k+1})) - (D_f(p, x_{n_k}) + D_g(q, w_{n_k}))] \geq 0. \quad (6.4.8)$$

Then, from (6.4.6) and the fact that $\{\alpha_n\} \subset (0,1)$, $r_n \leq \frac{\mu}{2}$ and $\theta \leq \mu$ for all $n \geq 0$, $\alpha_n \to 0$ and $r_n \to 0$ as $n \to \infty$, we obtain

$$
\begin{aligned}
D_f(a_{n_k}, x_{n_k}) \to 0, D_g(b_{n_k}, w_{n_k}) \to 0 \text{ as } k \to \infty, \quad & (6.4.9) \\
D_f(y_{n_k}, d_{n_k}) \to 0, D_g(z_{n_k}, e_{n_k}) \to 0 \text{ as } k \to \infty, \quad & (6.4.10) \\
D_f(u_{n_k}, y_{n_k}) \to 0, D_g(v_{n_k}, z_{n_k}) \to 0, \text{ as } k \to \infty, \quad & (6.4.11) \\
\psi_1(\|\nabla f(u_{n_k}) - T(u_{n_k})\|) \to 0, \psi_2(\|\nabla g(v_{n_k}) - G(v_{n_k})\|) \to 0 \text{ as } k \to \infty, \quad & (6.4.12)
\end{aligned}
$$

and

$$\|S(a_{n_k}) - K(b_{n_k})\| \to 0 \text{ as } k \to \infty. \quad (6.4.13)$$

Moreover, from (6.4.9), (6.4.10) (6.4.11),(6.4.12), Lemma 5.2.4 and the property of ψ_1 and ψ_2, we have

$$
\begin{aligned}
\|a_{n_k} - x_{n_k}\| \to 0, \|y_{n_k} - d_{n_k}\| \to 0, \|u_{n_k} - y_{n_k}\| \to 0 \text{ as } k \to \infty, \quad & (6.4.14) \\
\|b_{n_k} - w_{n_k}\| \to 0, \|z_{n_k} - e_{n_k}\| \to 0, \|v_{n_k} - z_{n_k}\| \to 0 \text{ as } k \to \infty, \quad & (6.4.15)
\end{aligned}
$$

and

$$\|\nabla f(u_{n_k}) - T(u_{n_k})\| \to 0, \|\nabla g(v_{n_k}) - G(v_{n_k})\| \to 0 \text{ as } k \to \infty. \quad (6.4.16)$$

From (6.3.5), (6.3.6), (6.4.16) and the fact that $\alpha_{n_k} \to 0$ as $k \to \infty$, we have

$$
\begin{aligned}
\lim_{k\to\infty} \|\nabla f(x_{n_k+1}) - \nabla f(u_{n_k})\| \leq\ & \lim_{k\to\infty} \alpha_{n_k}\|\nabla f(x) - \nabla f(u_{n_k})\| \qquad (6.4.17) \\
& +(1-\beta)\lim_{k\to\infty}\|\nabla f(u_{n_k}) - T(u_{n_k})\| = 0
\end{aligned}
$$

and

$$\lim_{k\to\infty} \|\nabla g(w_{n_k+1}) - \nabla g(v_{n_k})\| = 0. \tag{6.4.18}$$

Now, from (6.3.4) and (6.4.13) we obtain that

$$\begin{aligned} \|\nabla f(a_{n_k}) - \nabla f(d_{n_k})\| &= \gamma_n \|S^* J_3(S(a_{n_k}) - K(b_{n_k}))\| \\ &\leq (\rho+1)\|S^*\|\|J_3(S(a_{n_k}) - K(b_{n_k}))\| \to 0 \text{ as } k \to \infty. \end{aligned} \tag{6.4.19}$$

Similarly, we get

$$\|\nabla g(b_{n_k}) - \nabla g(e_{n_k})\| \to 0 \text{ as } k \to \infty. \tag{6.4.20}$$

From (6.4.17), (6.4.18), (6.4.19), (6.4.20) and the uniformly continuity of ∇f^* and ∇g^*, we have

$$\begin{aligned} &\|x_{n_k+1} - u_{n_k}\| \to 0, \text{ and } \|w_{n_k+1} - v_{n_k}\| \to 0 \text{ as } k \to \infty, \\ &\|a_{n_k} - d_{n_k}\| \to 0 \text{ and } \|b_{n_k} - e_{n_k}\| \to 0 \text{ as } k \to \infty. \end{aligned} \tag{6.4.21}$$

Then, from (6.4.14), (6.4.15) and (6.4.21), we obtain that

$$\begin{aligned} \lim_{k\to\infty} \|x_{n_k+1} - x_{n_k}\| &\leq \lim_{k\to\infty} \|x_{n_k+1} - u_{n_k}\| + \lim_{k\to\infty} \|u_{n_k} - y_{n_k}\| \\ &\quad + \lim_{k\to\infty} \|y_{n_k} - d_{n_k}\| + \lim_{k\to\infty} \|d_{n_k} - a_{n_k}\| + \lim_{k\to\infty} \|a_{n_k} - x_{n_k}\| = 0. \end{aligned} \tag{6.4.22}$$

Similarly, we have

$$\lim_{k\to\infty} \|w_{n_k+1} - w_{n_k}\| = 0. \tag{6.4.23}$$

Now, since $\{(x_{n_k}, w_{n_k})\}$ is bounded in $E_1 \times E_2$, there exists $(x^*, w^*) \in E_1 \times E_2$ and a subsequence $\{(x_{n_{k_j}}, w_{n_{k_j}})\}$ of $\{(x_{n_k}, w_{n_k})\}$ such that $(x_{n_{k_j}}, w_{n_{k_j}}) \rightharpoonup (x^*, w^*)$ and

$$\begin{aligned} &\limsup_{k\to\infty} \langle (x_{n_k}, w_{n_k}) - (p,q), (\nabla f(x), \nabla g(w)) - (\nabla f(p), \nabla g(q)) \rangle \\ &= \lim_{j\to\infty} \langle (x_{n_{k_j}}, w_{n_{k_j}}) - (p,q), (\nabla f(x), \nabla g(w)) - (\nabla f(p), \nabla g(q)) \rangle. \end{aligned} \tag{6.4.24}$$

But $(x_{n_{k_j}}, w_{n_{k_j}}) \rightharpoonup (x^*, w^*)$ implies that $x_{n_{k_j}} \rightharpoonup x^*$ and $w_{n_{k_j}} \rightharpoonup w^*$. Moreover, from (6.4.14), (6.4.15) and (6.4.21), we have

$$a_{n_{k_j}} \rightharpoonup x^*, d_{n_{k_j}} \rightharpoonup x^*, y_{n_{k_j}} \rightharpoonup x^*, u_{n_{k_j}} \rightharpoonup x^* \text{ as } j \to \infty, \tag{6.4.25}$$

and

$$b_{n_{k_j}} \rightharpoonup w^*, e_{n_{k_j}} \rightharpoonup w^*, z_{n_{k_j}} \rightharpoonup w^*, v_{n_{k_j}} \rightharpoonup w^* \text{ as } j \to \infty. \tag{6.4.26}$$

In addition, from (6.4.16), (6.4.25), (6.4.26), the fact that T is Bregman relatively f-nonexpansive and G is Bregman relatively g-nonexpansive mapping, we conclude

that $x^* \in \widetilde{F}_f(T)$ and $w^* \in \widetilde{F}_g(G)$, and hence $x^* \in F_f(T)$ and $w^* \in F_g(G)$. Next, we need to show that $(x^*, w^*) \in \Omega$. Let $s \in (A+B)(e)$. Then, there exists $h \in B(e)$ such that $s = A(e) + h$.
Thus, from (6.3.16) and monotonicity of A and B, we have

$$\begin{aligned}
\langle e - y_{n_{k_j}}, s\rangle &= \langle e - y_{n_{k_j}}, A(e) + h\rangle \\
&= \langle e - y_{n_{k_j}}, A(e) - A(y_{n_{k_j}})\rangle + \langle e - y_{n_{k_j}}, A(y_{n_{k_j}}) - A(d_{n_{k_j}})\rangle \\
&\quad + \langle e - y_{n_{k_j}}, A(d_{n_{k_j}}) + h_{n_{k_j}}\rangle + \langle e - y_{n_{k_j}}, h - h_{n_{k_j}}\rangle \\
&\geq \langle e - y_{n_{k_j}}, A(y_{n_{k_j}}) - A(d_{n_{k_j}})\rangle \\
&\quad + \langle e - y_{n_{k_j}}, A(d_{n_{k_j}}) + \frac{1}{\lambda_{n_{k_j}}}(\nabla f(d_{n_{k_j}}) - \nabla f(y_{n_{k_j}}) - \lambda_{n_{k_j}} A(d_{n_{k_j}}))\rangle \\
&= \langle e - y_{n_{k_j}}, A(y_{n_{k_j}}) - A(d_{n_{k_j}})\rangle \qquad (6.4.27) \\
&\quad + \frac{1}{\lambda_{n_{k_j}}}\langle e - y_{n_{k_j}}, \nabla f(d_{n_{k_j}}) - \nabla f(y_{n_{k_j}}))\rangle
\end{aligned}$$

Taking limits on both sides of the inequality (6.4.27) as $j \to \infty$ and using the fact that ∇f and A are uniformly continuous, (6.4.14) and (6.4.25), we have

$$\langle e - x^*, s\rangle \geq 0. \qquad (6.4.28)$$

Then, by the maximal monotonicity of $A+B$, we get $0 \in (A+B)x^*$, that is, $x^* \in (A+B)^{-1}(0)$. Similarly we obtain that, $w^* \in (C+D)^{-1}(0)$. Now, from (6.4.13), (6.4.25), (6.4.26) and the fact that S and K are bounded linear mappings we have $Sx^* = Kw^*$ and hence $(x^*, w^*) \in \Omega$. Therefore, from (6.4.24) and Lemma 5.2.2, we obtain that

$$\begin{aligned}
&\limsup_{k\to\infty}\langle (x_{n_k}, w_{n_k}) - (p,q), (\nabla f(x), \nabla g(w)) - (\nabla f(p), \nabla g(q))\rangle \\
&= \lim_{j\to\infty}\langle (x_{n_{k_j}}, w_{n_{k_j}}) - (p,q), (\nabla f(x), \nabla g(w)) - (\nabla f(p), \nabla g(q))\rangle \\
&= \langle (x^*, w^*) - (p,q), (\nabla f(x), \nabla g(w)) - (\nabla f(p), \nabla g(q))\rangle \leq 0. \qquad (6.4.29)
\end{aligned}$$

Let $s_n = \alpha_n(1-\epsilon)$ and

$$\begin{aligned}
b_n &= \frac{1}{1-\epsilon}\left[\langle x_n - p, \nabla f(x) - \nabla f(p)\rangle + \langle w_n - q, \nabla g(w) - \nabla g(q)\rangle\right] \\
&\quad + \frac{r_n}{\alpha_n(1-\epsilon)} + \frac{1}{1-\epsilon} D\left[\|x_{n+1} - x_n\| + \|w_{n+1} - w_n\|\|\right].
\end{aligned}$$

Then, inequality (6.4.7) implies that

$$\{D_f(p, x_{n+1}) + D_g(q, w_{n+1})\} \leq (1 - s_n)\{D_f(p, x_n) + D_g(q, w_n)\} + s_n b_n \qquad (6.4.30)$$

From (6.4.22), (6.4.23), (6.4.29) and using the fact that $\alpha_n \to 0$ and $\frac{r_n}{\alpha_n} \to 0$ as $n \to \infty$, we obtain $\lim_{n\to\infty} s_n = 0$, and $\limsup_{k\to\infty} b_{n_k} \leq 0$. From Lemma 6.2.2, we conclude that $\lim_{n\to\infty} (D_f(p, x_n) + D_g(q, w_n)) = 0$ and so $\lim_{n\to\infty} D_f(p, x_n) = 0$ and $\lim_{n\to\infty} D_g(q, w_n) = 0$. Hence, by Lemma 5.2.4 we obtain $\lim_{n\to\infty} x_n = p$ and $\lim_{n\to\infty} w_n = q$.

Therefore, the sequence $\{(x_n, w_n)\}$ generated by Algorithm 6.1 converges strongly to $(p, q) = P^h_\Omega(x, w)$ and this completes the proof. □

If, in Theorem 6.4.1, we assume that $E_1 = E_2 = E_3 = E$, $C = 0 = D$, $S = 0 = K$ and $G = \nabla g$, then we obtain the following theorem.

Theorem 6.4.2. *Let E be a reflexive real Banach space with its dual space E^*. Let $f : E \to \mathbb{R}$ be a strongly coercive, lower semi-continuous, strongly convex, bounded and uniformly Fréchet differentiable Legendre function on bounded subsets with strongly convex conjugate f^*. Let $g : E \to \mathbb{R}$ be a strongly convex and Gâteaux differentiable function with the strong convexity constant of g be μ_2. Let $T : E \to E^*$ be Bregman relatively f-nonexpansive mapping. Let $A : E \to E^*$ and $B : E \to 2^{E^*}$ be uniformly continuous monotone and maximal monotone mappings, respectively. If the assumptions (A2),(A8), (A9) and (A10) are satisfied and $\Omega = \{a \in E : a \in (F_f(T) \cap (A + B)^{-1}(0))\} \neq \emptyset$, then the sequence $\{x_n\}$ generated by Algorithm 6.1 with $E_1 = E_2 = E_3 = E$, $C = 0 = D$, $S = 0 = K$, $G = \nabla g$ and $w = w_0 = w_1$ converges strongly to $p = P^f_\Omega(x)$.*

If E_i, $i = 1, 2, 3$ are strictly convex and smooth Banach spaces with their respective, duals E^*_i, $i = 1, 2, 3$ and $g(x) = f(x) = \frac{1}{2}\|x\|^2$, then $\nabla f = \nabla g = J$, $\nabla f^* = \nabla g^* = J^{-1}$, $P^f_\Omega = \Pi_\Omega = P^g_\Omega$ and the Bregman relatively f-nonexpansive mapping T and the Bregman relatively g-nonexpansive mapping G reduce to relatively semi-nonexpansive mappings. Thus, we obtain the following theorem.

Theorem 6.4.3. *Let E_i, $i = 1, 2, 3$ be strictly convex and smooth reflexive real Banach spaces with their respective, duals E^*_i, $i = 1, 2, 3$ and let $T : E_1 \to E^*_1$ and $G : E_2 \to E^*_2$ be relatively semi-nonexpansive mappings. Suppose that the assumptions (A4)- (A6) and (A8)-(A10) are satisfied. If $\Omega = \{(a, b) \in (F_s(T) \cap (A + B)^{-1}(0)) \times (F_s(G) \cap (C + D)^{-1}(0)) : S(a) = K(b)\} \neq \emptyset$, then the sequence $\{(x_n, w_n)\}$ generated by Algorithm 6.1 with $\nabla f = J = \nabla g$, $\nabla f^* = J^{-1} = \nabla g^*$ and $P^f_\Omega = \Pi_\Omega = P^g_\Omega$ converges strongly to (p, q) in Ω, where $(p, q) = \Pi_\Omega(x, w)$.*

If E_i, $i = 1, 2, 3$ are real Hilbert spaces and $g(x) = f(x) = \frac{1}{2}\|x\|^2$ then $\nabla f = \nabla g = I$, $\nabla f^* = \nabla g^* = I$, $P^f_\Omega = P_\Omega = P^g_\Omega$, $J_{E_3} = I$ and the Bregman relatively f-nonexpansive mapping T and the Bregman relatively g-nonexpansive mapping

G reduce to relatively nonexpansive mappings. Thus, we obtain the following corollary.

Corollary 6.4.4. *Let H_i, $i = 1, 2, 3$ be real Hilbert spaces, $T : H_1 \to H_1$ and $G : H_2 \to H_2$ be relatively nonexpansive mappings. Suppose that the assumptions (A4)- (A6) and (A8)-(A9) are satisfied. If $\Omega = \{(a, b) \in (F(T) \cap (A + B)^{-1}(0)) \times (F(G) \cap (C + D)^{-1}(0)) : S(a) = K(b)\} \neq \emptyset$, then the sequence $\{(x_n, w_n)\}$ generated by Algorithm 6.1 with $\nabla f = I = \nabla g$, $\nabla f^* = I = \nabla g^*$, $J_{E_3} = I$ and $P^f_\Omega = P_\Omega = P^g_\Omega$ converges strongly to $(p, q) = P_\Omega(x, w)$.*

6.5 Application

6.5.1 Split Monotone Inclusion and f-fixed Point Problems

If $E_i = E$, $i = 1, 2, 3$, $K = I$, where I is identity mapping, then *SEMIfFPP* reduces to the split monotone inclusion and f, g-fixed point problems, which is defined as finding $p \in (F_f(T) \cap (A + B)^{-1}(0))$ such that $Sp \in (F_g(G) \cap (C + D)^{-1}(0))$.

Denote $\Psi = \{p \in (F_f(T) \cap (A + B)^{-1}(0)) : S(p) \in (F_g(G) \cap (C + D)^{-1}(0))\}$.

Corollary 6.5.1. *Assume that conditions (A1)- (A6) and (A8)- (A10) are satisfied with $E_i = E$, $i = 1, 2, 3$ and $K = I$. If $\Psi \neq \emptyset$, then the sequence $\{(x_n, w_n)\}$ generated by Algorithm 3.1 converges strongly to $(p, S(p)) = P^h_\Psi(x, w)$.*

6.5.2 Common Solutions of Monotone Inclusion and f-fixed Point Problems

Let $E_i = E$, $i = 1, 2, 3$, $S = K = I$. Then, *SEMIfFPP* reduces to a common solution of two monotone inclusion and f, g-fixed point problems, which is defined as finding a point $p \in E$ such that $p \in (F_f(T) \cap (A + B)^{-1}(0)) \cap (F_g(G) \cap (C + D)^{-1}(0))$.

Denote $\Gamma = \{p \in E : p \in (F_f(T) \cap (A + B)^{-1}(0)) \cap (F_g(G) \cap (C + D)^{-1}(0))\}$.

Corollary 6.5.2. *Assume that conditions (A1)-(A5) and (A8)-(10) are satisfied with $E_i = E$, $i = 1, 2, 3$. If $\Gamma \neq \emptyset$, then the sequence $\{(x_n, w_n)\}$ generated by Algorithm 6.1 with $S = K = I$ converges strongly to $(p, p) = P^h_\Gamma(x, w)$.*

6.5.3 Split Equality Monotone Inclusion Problem

If $T = \nabla f$ and $G = \nabla g$, then *SEMIfFPP* reduces to the split equality of monotone inclusion problems, which is finding a point $(p,q) \in E_1 \times E_2$ such that $p \in (A+B)^{-1}(0)$, $q \in (C+D)^{-1}(0)$ and $Sp = Kq$.

$$\text{Denote } \Lambda = \{(p,q) \in (A+B)^{-1}(0) \times (C+D)^{-1}(0) : Sp = Kq\}.$$

Corollary 6.5.3. *If conditions (A1),(A2), (A4)-(A6) and (A8)-(A10) are satisfied and $\Lambda \neq \emptyset$, then the sequence $\{(x_n, w_n)\}$ generated by Algorithm 6.1 with $T = \nabla f$ and $G = \nabla g$ converges strongly to (p,q) in Λ, where $(p,q) = P_\Lambda^h(x,w)$.*

6.5.4 Split Equality f-Fixed Point Problem

If $A = B = 0 = C = D$, then *SEMIfFPP* reduces to the split equality of f,g-fixed point problems, which is finding a point $(p,q) \in E_1 \times E_2$ such that $p \in F_f(T)$, $q \in F_g(G)$ and $Sp = Kq$.

$$\text{Denote } \Sigma = \{(p,q) \in E_1 \times E_2 : p \in F_f(T), q \in F_g(G) \text{ and } Sp = Kq\}.$$

Corollary 6.5.4. *If conditions (A1)-(A3),(A6) and (A8)-(A10) are satisfied and $\Sigma \neq \emptyset$, then the sequence $\{(x_n, w_n)\}$ generated by Algorithm 6.1 with $A = B = 0 = C = D$ converges strongly to $(p,q) = P_\Sigma^h(x,w)$.*

6.5.5 Optimization Problem

Let $E_i = E$, $i = 1,2,3$, be reflexive real Banach spaces. Let $f_i : E_i \to \mathbb{R}$ be convex smooth functions and $g_i : E_i \to \mathbb{R}$ be convex, lower semicontinuous functions, $i = 1,2$. We consider the following minimization problem: Find $(p,q) \in E_1 \times E_2$ such that

$$p \text{ solves} \quad \min_{x \in E_1} \{f_1(x) + g_1(x) : (\nabla f_1 - T)(x) = 0\}, \tag{6.5.1}$$

$$q \text{ solves} \quad \min_{y \in E_2} \{f_2(y) + g_2(y) : (\nabla f_2 - G)(y) = 0\} \tag{6.5.2}$$

and

$$Sp = Kq, \tag{6.5.3}$$

where $T : E_1 \to E_1^*$ and $G : E_2 \to E_2^*$ are Bregman relatively f-nonexpansive mappings and $S : E_1 \to E_3$ and $K : E_2 \to E_3$ are bounded linear mappings.

$$\text{Denote } \Delta = \{(z,v) \in E_1 \times E_2 : z \text{ solve } 6.5.1, v \text{ solve } 6.5.2 \text{ and } Sz = Kv\}.$$

This problem is equivalent, by Fermat's rule, to the problem of finding $(p, q) \in E_1 \times E_2$ such that

$$(p, q) \in \left[F_{f_1}(T) \cap (\nabla f_1 + \partial g_1)^{-1}(0)\right] \times \left[F_{f_2}(G) \cap (\nabla f_2 + \partial g_2)^{-1}(0)\right] \text{ and } S(p) = K(q), \tag{6.5.4}$$

where ∇f_i are gradient of f_i and ∂g_i are subdifferential of g_i, $i = 1, 2$. Note that ∇f_i and ∂g_i are monotone and maximal monotone mappings, respectively.

Corollary 6.5.5. *Let $f_i : E_i \to \mathbb{R}$ be convex smooth functions and $g_i : E_i \to \mathbb{R}$ be convex, lower semicontinuous functions, $i = 1, 2$. Assume that the conditions (A1),(A3), (A6) and (A8) - (A10) are satisfied. If $\Delta \neq \emptyset$, then the sequence $\{(x_n, w_n)\}$ generated by Algorithm 6.1 with $A = \nabla f_1$, $C = \nabla f_2$, $B = \partial g_1$ and $D = \partial g_2$, converges strongly to (p, q) in Δ, where $(p, q) = P^h_\Delta(x, w)$, where $h = f_1 + f_2$.*

6.6 Numerical Experiment

In this section, we provide some numerical examples to clearly exhibit the behavior of the convergence of the proposed method.

Example 6.6.1. *Consider the minimization problem: find $(p, q) \in \mathbb{R}^4 \times \mathbb{R}^3$ such that*

$$p \text{ solve} \quad \min_{x \in \mathbb{R}^4} \{\|x\|_1 + \frac{1}{2}\|x\|_2^2 + (3, 4, -2, 5)^t x + 3 : T(x) = (-2, -3, 1, -4)\} \tag{6.6.1}$$

$$q \text{ solve} \quad \min_{y \in \mathbb{R}^3} \{\|y\|_1 + \|y\|_2^2 + (0, -5, 3)^t y + 2 : G(y) = (0, 2, -1)\} \tag{6.6.2}$$

and

$$Sp = Kq, \tag{6.6.3}$$

where $S(x_1, x_2, x_3, x_4) = (x_1 + x_2, x_3)$, $T(x_1, x_2, x_3, x_4) = (\frac{1}{2}x_1 - 1, \frac{2}{3}x_2 - 1, x_3, \frac{3}{4}x_4 - 1)$, $\forall (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ and $K(y_1, y_2, y_3) = (-2y_2 + y_3, -y_3)$, $G(y_1, y_2, y_3) = (y_1, \frac{1}{4}y_2 + \frac{3}{2}, -\frac{1}{2}y_3 - \frac{3}{2})$, $\forall (y_1, y_2, y_3) \in \mathbb{R}^3$.

By Fermat's rule, this problem is equivalent to the problem of finding a point $(p, q) \in \mathbb{R}^4 \times \mathbb{R}^3$ such that

$$(p, q) \in \Omega = \{(x, y) \in \left[F_f(T) \cap (A + B)^{-1}(0)\right] \times \left[F_g(G) \cap (C + D)^{-1}(0)\right] : S(x) = K(y)\}, \tag{6.6.4}$$

where $A(x) = \nabla(\frac{1}{2}\|x\|_2^2 + (3, 4, -2, 5)^t x + 3) = x + (3, 4, -2, 5)$, $B(x) = \partial(\|x\|_1)$ and $f(x) = \frac{1}{2}\|x\|_2^2$, $\forall x \in \mathbb{R}^4$ and $C(y) = \nabla(\|y\|_2^2 + (0, -5, 3)^t y + 2) = 2y +$

$(0, -5, 3)$, $D(y) = \partial(\|y\|_1)$ and $g(y) = \frac{1}{2}\|y\|_2^2$, $y \in \mathbb{R}^3$. We note that the mappings $A : E_1 \to E_1$ and $C : E_2 \to E_2$ are monotone mappings, $B : E_1 \to E_1$ and $D : E_2 \to E_2$ are maximal monotone mappings, $S : E_1 \to E_3$ and $K : E_2 \to E_3$ are bounded linear mappings with adjoints $S^*(x_1, x_2) = (x_1, x_1, x_2, 0)$ and $K^*(x_1, x_2) = (0, -2x_1, x_1, -x_2)$, $(x_1, x_2) \in \mathbb{R}^2$, respectively, where $E_1 = \mathbb{R}^4$, $E_2 = \mathbb{R}^3$ and $E_3 = \mathbb{R}^2$. Moreover, we observe that $\nabla f(x) = x, \nabla g(y) = y$, $J_{\mathbb{R}^2} = I$, where I is identity mapping on $\mathbb{R}^2$, and the mapping $T : E_1 \to E_1$ is a Bregman relatively f-nonexpansive and $G : E_2 \to E_2$ is a Bregman relatively g-nonexpansive mapping and $\Omega = \{((-2, -3, 1, -4), (0, 2, -1))\} \neq \emptyset$.

Now, we present the numerical experiment results for testing and comparing the performance of the control parameter by taking different values in the Algorithm 3.1. All experiments are performed by using MATLAB R2021b.

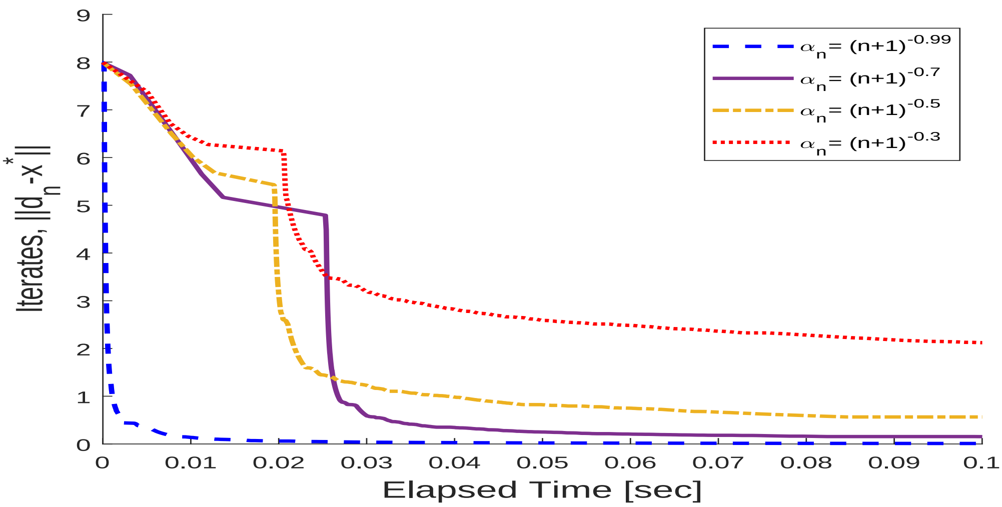


Figure 6.1: Algorithms 3.1. with $\beta = 0.5$, $\lambda_1 = 1 = \delta_1$ $\mu = 0.99$, $\theta = 0.1$, $\gamma = 10^{-4}$ and different α_n.

From FIGURE 6.1 and 6.2 we observe that the sequence $d_n = (x_n, w_n)$ converges faster to $x^* = ((-2, -3, 1, 4), (0, 2, -1))$, when the power a of the control parameters $\alpha_n = (n+1)^{-a}$ gets closer to one while the initial point and all other parameters are kept fixed, and the control parameter β gets closer to zero while the initial point and all other parameters are kept fixed, respectively.

Example 6.6.2. *Let $E_1 = E_2 = E_3 = L_2([0, 1])$ and let $Q = W = \{x \in L_2([0, 1] : \|\|_{L_2} \leq 1\}$. Consider the minimization problem: find $(p, q) \in E_1 \times E_2$ such that*

$$p \text{ solve} \qquad \min_{x \in Q}\{\frac{1}{2}\|x\|_{L_2}^2 + 1 : T(x) = \nabla f(x)\}, \tag{6.6.5}$$

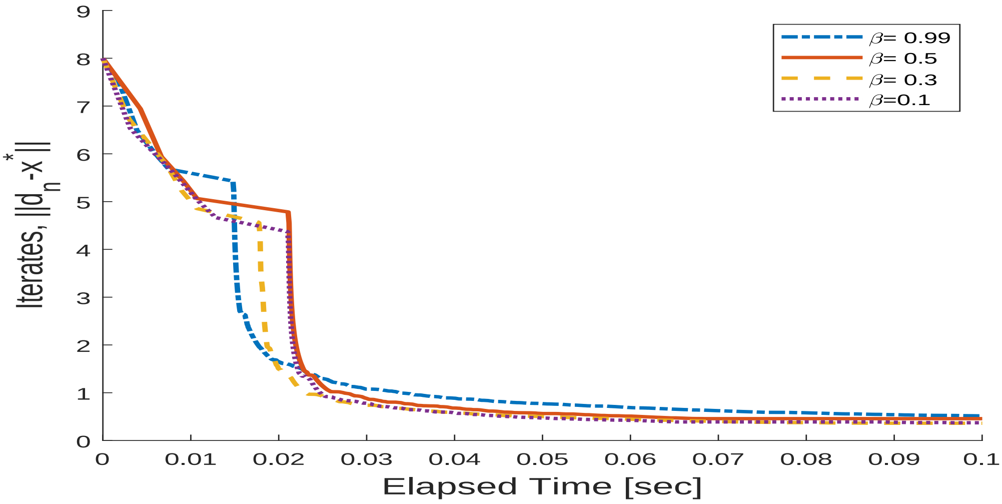


Figure 6.2: Algorithms 3.1. with $\alpha_n = (n+1)^{-0.5}$, $\lambda_1 = 1 = \delta_1$, $\mu = 0.99$, $\theta = 0.1$, $\gamma = 10^{-5}$ and different β.

$$q \ solve \qquad \min_{y \in W} \{\frac{1}{2}\|y\|_{L_2}^2 + 3 : G(x) = \nabla g(y)\}, \tag{6.6.6}$$

and

$$Sp = Kq, \tag{6.6.7}$$

where $S(x) = 2x$ *and* $f(x) = \|x\|_{L_2}^2, \forall x \in L_2([0,1])$ *and* $K(y) = y$ *and* $g(y) = \|y\|_{L_2}^2, \forall y \in L_2([0,1])$.

Thus, this problem is equivalent to the following problem:

$$p \text{ solve} \qquad \min_{x \in E_1} \{i_Q(x) + \frac{1}{2}\|x\|_{L_2}^2 + 1 : T(x) = \nabla f(x)\}, \tag{6.6.8}$$

$$q \text{ solve} \qquad \min_{y \in E_2} \{i_W(y) + \frac{1}{2}\|y\|_{L_2}^2 + 3 : G(x) = \nabla g(y)\}, \tag{6.6.9}$$

and

$$Sp = Kq, \tag{6.6.10}$$

where i_Q and i_W are indicator functions of Q and W, respectively, $S(x) = 2x$ and $K(y) = y$.

By Fermat's rule, this problem is equivalent to the problem of finding a point $(u, v) \in E_1 \times E_2$ such that

$$(u,v) \in \Upsilon = \{(x,y) \in \big[F_f(T) \cap (A+B)^{-1}(0)\big] \times \big[F_g(G) \cap (C+D)^{-1}(0)\big] : S(x) = K(y)\}, \tag{6.6.11}$$

where $A(x) = \nabla[\frac{1}{2}\|x\|^2_{L_2}+1]$ and $B(x) = \partial(i_Q(x))$, $\forall x \in E_1$ and $C(y) = \nabla[frac12\|y\|^2_{L_2}+3]$ and $D(y) = \partial(i_W(y))$, $\forall y \in E_2$. We note that the mappings $A : E_1 \to E_1^*$ and $C : E_2 \to E_2^*$ are monotone mappings, $B : E_1 \to 2^{E_1^*}$ and $D : E_2 \to 2^{E_2^*}$ are maximal monotone mappings, $S : E_1 \to E_3$ and $K : E_2 \to E_3$ are bounded linear mappings $S^*(x) = 2x$ and $K^*(x) = x, x \in L_2([0,1])$, respectively. Moreover we observe that $\nabla f = I = \nabla g$, where I is identity mapping on $L_2([0,1])$, and the mapping $T : E_1 \to E_1^*$ is a Bregman relatively f-nonexpansive mapping and $G : E_2 \to E_2^*$ is a Bregman relatively g-nonexpansive mapping and $\Upsilon = \{(0,0)\} \neq \emptyset$.

Now, we present the numerical experiment results for testing and comparing the performance of the control parameter by taking different values in Algorithm 3.1. All experiments are performed by using MATLAB R2021b.

From Figure 6.3, we observe that the sequence (x_n, w_n) converges strongly to $(u, v) = (0,0)$, and the convergence is faster when a in the control parameter $\alpha_n = \frac{1}{(n+1)^a}$ gets closer to one while the initial point and all other parameters are kept fixed.

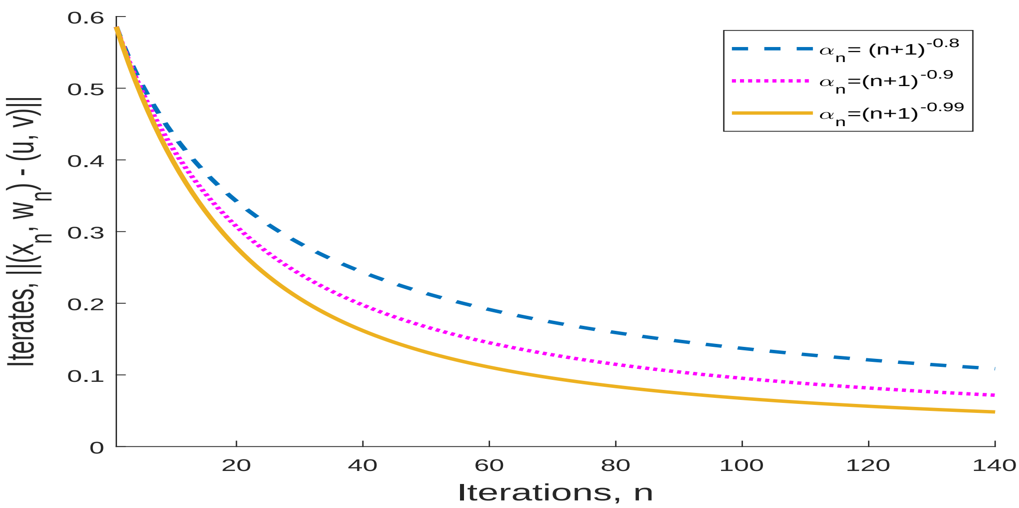


Figure 6.3: Algorithms 3.1. with $x(t) = t^2$, $w(t) = t^2$, $x_0(t) = t^2$, $w_0(t) = t^3$ $\beta = 0.5$, $\lambda_1 = 1 = \delta_1$ $\mu = 0.99$, $\theta = 0.1$, $\gamma = 10^{-4}$ and different α_n.

Chapter 7

Discussion, Conclusion and Recommendation

7.1 Discussion

In this dissertation, we established iterative algorithms that convergence strongly to common solutions of nonlinear problems. The nonlinear problems discussed in this work mainly focus on the fixed point problems of nonlinear mappings, the generalized equilibrium problems, zeros of monotone mappings, and varitional inequality problems for Lipschitz monotone mappings. All supporting numerical experiments were presented to indicate the worthiness of these results to practical problems in nonlinear analysis. The research works are entirely new in the literature as manifested by the publication of these results by four different highly reputable peer-reviewed international journals indexed by archives such as Web of Science, Scopus and Mathematical Reviews.

7.2 Conclusion

The theorems obtained in this dissertation generalize, extend and unify so many results in the literature. The main results of the dissertation are stated below to illustrate this.

• In Chapter 3, we have proposed an algorithm for solving a common element of sets of solutions of a finite family of generalized equilibrium problem, sets of fixed points of a finite family of continuous asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense and sets of zeros of a finite family γ-inverse strongly monotone mappings in uniformly convex and uniformly smooth real Banach spaces. We also gave some consequences of our results to a common solution of some problems in Banach spaces. Our results extend and improve the results of Kazmi and Ali [49], Ibiam et. al. [42] and Hao [40] to a common solution of a more general problems that involves a common solution of a finite family continuous asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense in Banach spaces. The results obtained in Section 3.2 have been published; see "**S. B. Zegeye**, M. G. Sangago and H. Zegeye, A common solution of generalized equilibrium, zeros of monotone mapping and fixed point problems. *J. Analysis*, 30(2)(2022), 569595. https://doi.org/10.1007/s41478-021-00359-w.".

• In Chapter 4, we constructed a hybrid projection type algorithm whose sequence converges strongly to a common element of sets of solutions of a finite family of generalized mixed equilibrium problem, sets of semi-fixed points of a finite family of continuous semi-pseudocontractive mappings and sets of solutions of a finite family of variational inequality for a finite family of monotone and L-Lipschitz mappings in Banach spaces is proved. Our results in this chapter generalizes some of known results in the literature. As a result, we obtain strong convergence results for a common semi-fixed point of a finite family of continuous f-pseudocontractive mappings and for a common solution of a finite family of varitional inequality problems for Lipschitz monotone mappings in Banach spaces. We remark that the theorem proved is applicable in L_p, l_p or $W_p^m(\Omega)$ spaces, $1 < p < \infty$, where $W_p^m(\Omega)$ denotes the usual Sobolev space. Our results provide an affirmative answer to the question raised. The results obtained in this chapter appeared in "Computational and Applied Mathematics 41, 200 (2022). https://doi.org/10.1007/s40314-022-01907-1."

• In Chapter 5, we constructed a new algorithm to approximate a common element of the set of solutions of a finite family of generalized mixed equilibrium problems, the set of f-fixed points of a finite family of f-pseudocontractive mappings and the set of solutions of a finite family of variational inequality problems for a finite family of Lipschitz monotone mappings in reflexive real Banach spaces. We proved a strong convergence theorem for the developed algorithm in reflexive real Banach spaces. In addition, a numerical example is given to illustrate the implementability of our algorithm. Specifically, the result of our method improve the result obtained by Shahzad and Zegeye [78] from a *Hilbert spaces* to a *reflexive Banach spaces*, from continuous pseudocontractive to *continuous f-pseudocontractive* and from *equilibrium problem* to *generalized mixed equilibrium problem*. In addition, Theorem 6.4.1 extends Theorem 3.1 of Bello and Nnakwe [9] from *2-uniformly convex and uniformly smooth spaces* to *reflexive Banach spaces*, from *continuous semi-pseudocontractive* to *continuous f-pseudocontractive* and from *equilibrium problem* to *generalized mixed equilibrium problem.*

• In Chapter 6, we proposed an inertial type algorithm to approximate the solution of the split equality of monotone inclusion and f, g-fixed point of Bregman relatively f, g-nonexpansive mapping problems. We proved a strong convergence theorem for the developed algorithm in reflexive real Banach spaces. The main result of our method improves the result obtained by Izuchukwu et al. [44] from the split feasibility over the solution set of monotone variational inclusion problems in real Hilbert spaces to the split equality of monotone inclusion and f, g-fixed point of Bregman relatively f, g-nonexpansive mapping problems in reflexive real Banach spaces. As an application, we provided several applications of our method and provided a numerical result to demonstrate the behavior of the convergence of the algorithm. A numerical example is also provided to illustrate the behavior of the proposed algorithm.

7.3 Recommendation

To the best of our knowledge the following problems are still open for investigations.

• In Chapter 3, we have obtained strong convergence results for solving a common element of sets of solutions of a finite family of generalized equilibrium problem, sets of fixed points of a finite family of continuous asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense and sets of zeros of a finite family γ-inverse strongly monotone mappings in uniformly convex and uniformly smooth real Banach spaces. The following questions can be addressed in the future:

a) Can we extend these results to Banach spaces more general than uniformly convex and uniformly smooth real Banach spaces?

b) Can we extend these results to zeros of a finite family pseudo-monotone mappings more general than zeros of a finite family γ-inverse strongly monotone mappings?

c) Can we introduce a simpler iteration scheme which is efficient and easy to implement?

• In Chapter 4, we have proved a strong convergence theorem for approximating a common element of sets of solutions of a finite family of generalized mixed equilibrium problem, sets of semi-fixed points of a finite family of continuous semi-pseudocontractive mappings and sets of solutions of a finite family of variational inequality for a finite family of monotone and L-Lipschitz mappings in uniformly convex and uniformly smooth real Banach spaces. Can we obtain sets of semi-fixed points results for a class of mappings more general than a finite family of continuous semi-pseudocontractive mappings in spaces more general than uniformly smooth real Banach spaces?

• In Chapter 5, we have introduced an algorithm for approximating a common element of the set of solutions of a finite family of generalized mixed equilibrium problems, the set of f-fixed points of a finite family of f-pseudocontractive mappings and the set of solutions of a finite family of variational inequality problems for Lipschitz monotone mappings in real reflexive Banach spaces. For further work related with this research the following questions are open.

1. Can we extend these result to the set of f-fixed points results for a class of mappings more general than a finite family of continuous f-pesudocontractive mappings in spaces more general than reflexive Banach spaces?
2. Can we extend these result to the set of f-fixed points results for a class of mappings more general than a finite family of continuous f-pesudocontractive mappings in spaces more general than reflexive Banach spaces?
3. Can we extend these result to the set of solutions of a finite family of variational inequality problems for pseudo-monotone mappings more general than Lipschitz monotone mappings?

• In Chapter 6, we have developed an inertial algorithm for solving split equality of monotone inclusion and f-fixed point of Bregman relatively f-nonexpansive mapping problems in reflexive real Banach spaces. Can we extended this result to a finite family of maximal pseudo-monotone mappings in Banach spaces?

List of Papers Published/ Accepted

- **Solomon Bekele Zegeye**, Mengistu Goa Sangago and Habtu Zegeye, A common solution of generalized equilibrium, zeros of monotone mapping and fixed point problems. *The Journal of Analysis*, 30(2)(2022), 569595.

- **Solomon Bekele Zegeye**, Habtu Zegeye, Mengistu Goa Sangago and Oganeditse A. Boikanyo, A convergence theorem for a common solution of f-fixed point, variational inequality and generalized mixed equilibrium problems in Banach spaces. *International Journal of Nonlinear Analysis and Applications*, 13(2)(2022), 1069-1087.

- **Solomon Bekele Zegeye**, Mengistu Goa Sangago and Habtu Zegeye, Approximation of common solutions of nonlinear problems in Banach spaces. *Computational and Applied Mathematics*, 41(200)(2022).

- **Solomon Bekele Zegeye**, Habtu Zegeye, Mengistu Goa Sangago, Oganeditse A. Boikanyo and Sebsibe Teferi Woldeamanuel, An inertial method for a solution of split equality of monotone inclusion and the f-fixed point problems in Banach spaces. *Mathematical Methods in the Applied Sciences*, 46(2)(2023), 2884-2905.

Bibliography

[1] R.P. Agarwal, Y.J. Cho, and X. Qin, *Generalized projection algorithms for nonlinear operators.* Numer. Func. Anal. Optimi. 28, no. 11-12 (2007): 1197-1215.

[2] R.P. Agarwal, D. O'Regan, and D.R. Sahu, *Fixed point theory for Lipschitzian-type mappings with applications.* Vol. 6. New York: Springer, 2009.

[3] Y. Alber, I. Ryazantseva, *Nonlinear ill-posed problems of monotone type.* London: Springer, 2006.

[4] Y. Alber I, *Metric and generalized projection operators in Banach spaces: properties and applications.* arXiv preprint funct-an/9311001 (1993).

[5] A. Barbagallo, *Existence and regularity of solutions to nonlinear degenerate evolutionary variational inequalities with applications to dynamic network equilibrium problems*, Appl. Math. Comput., 208 (2009), 1-13.

[6] V. Barbu, *Nonlinear differential equations of monotone types nonlinear differential in Banach spaces*, Springer, New York, 2010.

[7] H.H. Bauschke,J.M. Borwein, *Legendre functions and the method of random Bregman projections.* J. Convex Anal. 4, 27-67 (1997)

[8] H.H. Bauschke,J.M. Borwein, P.L. Combettes,*Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces.* Commun. Contemp. Math. 3, 615-647 (2001)

[9] A.U. Bello and M.O. Nnakwe, *An algorithm for approximating a common solution of some nonlinear problems in Banach spaces with an application.* Adv. Differ.Equ., 2021(1), pp.1-17. https://doi.org/10.1186/s13662-021-03268-1

[10] V. Berinde, *Iterative Approximation of Fixed Points.* Springer-Verlag Berlin Heidelberg 2007.

[11] D. Bertsekas, *Convex Analysis and Optimization Contributors*, Angelia Nedic and Asuman E. Ozdaglar. Athena Scientific. pp. 72 (2003), ISBN 9781886529458.

[12] D. Bertsekas, *Convex Analysis and Optimization Contributors.* Angelia Nedic and Asuman E. Ozdaglar, Athena Scientific, (2003)

[13] E. Blum, *From optimization and variational inequalities to equilibrium problems.* Math. stud. 63 (1994): 123-145.

[14] E. Blum and W. Oettli,*From optimization and variational inequalities to equilibrium problems*, Math. Stud., 63 (1994), 123-145.

[15] F. J. Bonnans, A. Shapiro, *Perturbation Analysis of Optimization Problem*, Springer, New York, 2000.

[16] L.M. Bregman, *The relaxation method for finding common points of convex sets and its application to the solution of problems in convex programming.* USSR Comput. Math. Math. Phys. 1967 (7), 200-217.

[17] H. Brezis, H. Brézis, *Functional analysis, Sobolev spaces and partial differential equations*, Vol. 2, no. 3. New York: Springer, 2011.

[18] F. E. Browder,*Existence and approximation of solutions of nonlinear variational inequalities.* Proc. Natl. Acad. Sci. USA 56(4)(1966), 1080-1086.

[19] F.E. Browder, *Nonlinear mappings of nonexpansive and accretive-type in Banach spaces.* Bull. Amer. Math. Soc. 73(1967), 875-882.

[20] R.E. Bruck, *A strongly convergent iterative method for the solution of* $0 \in Ux$ *for a maximal monotone operator* U *in Hilbert space*, J. Math. Anal. Appl. 48(1974), 114-126.

[21] D. Butnariu, A.N. Iusem, *Totally Convex Functions for Fixed Points Computation and Infnite Dimentional Optimization.* vol. 40. Klumer Academic, Dodrecht (2000)

[22] D. Butnariu, R. Simeon, and J.Z. Alexander, *Weak convergence of orbits of nonlinear operators in reflexive Banach spaces.* Numer. Func. Anal. Optimi. 24, no. 5-6 (2003): 489-508.

[23] D. Butnariu, E. Resmerita, *Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces*, Abstr. Appl. Anal. 2006: 1-39, 2006.

[24] L.C. Ceng, J.C. Yao, *A hybrid iterative scheme for mixed equilibrium problems and fixed point problems.* J.Comput. Appl. Math., 214(2008), 186-201.

[25] Y. Censor, A. Lent, *An iterative row-action method for interval convex programming.* J. Optim. Theory Appl. 34(1981), 321-353.

[26] Y. Censor, and S. Reich, *Iterations of paracontractions and firmaly nonexpansive operators with applications to feasibility and optimization.* Optimization 37, no. 4 (1996): 323-339.

[27] C. E. Chidume, *Iterative approximation of fixed points of Lipschitz pseudocontractive mappings.* Proc. Amer. Math. Soc. 129, 2245-2251(2001).

[28] C. E. Chidume, *Geometric Properties of Banach Spaces and Nonlinear Iterations.* Springer-Verlag London Limited, 2009.

[29] C.E. Chidume, C.G. Ezea, *New algorithms for approximating zeros of inverse strongly monotone maps and J-fixed points.* Fixed Point Theory and Appl. (2020)(1), 1-16.

[30] C.E. Chidume, K.O. Idu, *Approximation of zeros of bounded maximal monotone maps, solutions of Hammerstein integral equations and convex minimization problemss.* Fixed Point Theory Appl. (2016).

[31] C.E. Chidume, S.A. Mutangadura, *An example on the Mann iteration method for Lipschitz pseudocontrations.* Proc. Amer. Math. Soc. 129, 8, 2359-2363 (2001).

[32] C.E. Chidume, M.O. Nnakwe, *A new Halpern-type algorithm for a generalized mixed equilibrium problem and a countable family of generalized nonexpansive-type maps.* Carpathian J. Math. 34 (2018) (2), pp.191-198.

[33] C. E. Chidume, H. Zegeye, *Approximate fixed point sequences and convergence theorems for Lipschitz pseudocontractive maps.* Proc. Amer. Math. Soc., 132 (2004), 831-840.

[34] I. Cioranescu, *Geometry of Banach spaces, duality mappings and nonlinear problems. Vol. 62. Springer Science & Business Media, 2012.*

[35] P. L. Combettes, The convex feasibility problem in image recovery, in Advanced in Imaging and Electron Physcis, P. Hawkes, Ed., vol. 95, pp. 155-270, Academic Press, New York, NY, USA, 1996.

[36] S. Dafermos, A. Nagurney, A network formulation of market equilibrium problems and variational inequalities, Oper. Res. Lett., 3 (1984), 247-250.

[37] *V. Darvish, Strong Convergence Theorem for Generalized Mixed Equilibrium Problems and Bregman Nonexpansive Mapping in Banach Spaces. Mathematica Moravica, Vol. 20:1 (2016), 69-87.*

[38] *R. Dautray and J. L. Lions, Mathematical Analysis and Numerical Methods for Science and Technology, vol.16, Springer, New York, NY, USA, 1988-1993.*

[39] *H. O. Fattorini, Infinite-Dimensional Optimization and Control Theory, vol. 62 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, UK, 1999.*

[40] *Y. Hao, Some results on a modified Mann iterative scheme in a reflexive Banach space. Fixed Point Theory Appl. 2013, no. 1 (2013): 227.*

[41] *R. He, Coincidence theorem and existence theorems of solutions for a system of Ky Fan type minimax inequalities in FC-spaces, Adv. Fixed Point Theory, 2 (2012), 47-57.*

[42] *O.I. Ibiam, L.O. Madu, E.U. Ofoedu, C.E. Onyi, and H. Zegeye. Hybrid Algorithm for Nonlinear Equilibrium, Variational Inequality and Fixed Point Problems. arXiv preprint arXiv:2012.00087 (2020).*

[43] *H. Iiduka, W. Takahashi, Strong convergence studied by Hybrid type method for monotone operators in a Banach space, Nonlinear Anal. 68 (2008) 36793688.*

[44] *C. Izuchukwu, G. N. Ogwo, O. T. Mewomo, An inertial method for solving generalized split feasibility problems over the solution set of monotone variational inclusions, Optimization, 2020, 1-29, https://doi.org/10.1080/02331934.2020.1808648.*

[45] *C. Izuchukwu, C. C. Okeke, F. O. Isiogugu, A viscosity iterative technique for split variational inclusion and fixed point problems between a Hilbert space and a Banach space, J. Fixed Point Theory Appl. 20(157), 2018.*

[46] *L. O. Jolaoso, F. U. Ogbuisi, and O. T. Mewomo, On split equality variation inclusion problems in Banach spaces without operator norms, Int. J. Nonlinear Anal. Appl., (12)2021: 425-446.*

[47] *S. Kamimura and W. Takahashi. Strong convergence of a proximal-type algorithm in a Banach space. SIAM J.Optim. 13, no. 3 (2002): 938-945.*

[48] *T. Kato, Nonlinear semigroups and evolution equations. J. Math. Soc. Japan. 19(1967), 511-520.*

[49] *K.R. Kazmi and R. Ali. Common solution to an equilibrium problem and a fixed point problem for an asymptotically quasi-ϕ-nonexpansive mapping in intermediate sense. Rev. R. Acad. Cienc. Exactas. Fipfs. Nat. Ser. A Math. RACSAM 111, no. 3 (2017): 877-889.*

[50] *M. A. Khamsi and W. A. Kirk. An Introduction to Metric Spaces and Fixed Point Theory. John Wiley & Sons, Inc. 2001*

[51] *M. A. Khan and N. C. Yannelis, Equilibrium Theory in Infinite Dimensional Spaces, Springer, New York, NY, USA, 1991.*

[52] *P.E. Maingé, Strong convergence of projected subgradiant method for nonsmooth and nonstrictily convex minimization, Set-Valued Anal. (2008)16: 899-912, 2008.*

[53] *A. Mouda, M. Thera, Proximal and dynamical approaches to equilibrium problems, in Lecture notes in Economics and Mathematical Systems, 477, spinger (1999), 187-201.*

[54] *A. Moudafi, Split monotone variational inclusions, J. Optim. Theory Appl. 150(2011): 275-283, 2011.*

[55] *A. Moudafi and M. Thŕa. Proximal and dynamical approaches to equilibrium problems. In Ill-Posed Variational Problems and Regularization Techniques, pp. 187-201. Springer, Berlin, Heidelberg, 1999.*

[56] *W. Nilsrakoo and S. Saejung . Strong convergence to common fixed points of countable relatively quasi-nonexpansive mappings. Fixed Point Theory Appl., (2008): 312-454.*

[57] *M.O. Nnakwe, C.C. Okeke, A common solution of generalized equilibrium problems and fixed points of pseudo-contractive-type maps. J. Appl. Math. Comput. (2020). https://doi.org/10.1007/s12190-020-01457-x*

[58] *E.U. Ofoedu, Strong convergence theorem for uniformly L-Lipschitzian asymptotically pseudocontractive mapping in real Banach spaces. J. Math. Anal. Appl. 321 (2006), 722-728.*

[59] *E.U. Ofoedu, J. Odumegwu, H. Zegeye, N. Shahzad, An algorithm for finding common solutions of various problems in nonlinear operator theory. Fixed Point Theory Appl. 2014, 9 (2014).*

[60] *R.P. Phelps, Convex functions, monotone operators, and differentiability. Lecture Notes in Mathematics, vol. 1364, 2nd edn. Springer, Berlin (1993).*

[61] *B. T. Polyak, Some methods of speeding up the convergence of iteration methods, USSR Comput. Math. Math. Phy. 4(5)(1964): 1-17.*

[62] *X. Qin, and R.P. Agarwal. Shrinking projection methods for a pair of asymptotically quasi-ϕ-nonexpansive mappings. Numer. Funct. Anal. Optim. 31, no. 9 (2010): 1072-1089.*

[63] *X. Qin, Y.J. Cho, and S.M. Kang. Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces.J. Comput. Appl. Math. 225, no. 1 (2009): 20-30.*

[64] *X. Qin, S.Y. Cho, and S.M. Kang. On hybrid projection methods for asymptotically quasi-ϕ-nonexpansive mappings. Appl. Math. Comput. 215, no. 11 (2010): 3874-3883.*

[65] *X. Qin, S. Y. Cho, S. M. Kang, Strong convergence of shrinking projection methods for quasi-image-nonexpansive mappings and equilibrium problems, J. Comput. Appl. Math., 234 (2010), 750-760.*

[66] *X. Qin, S. Y. Cho, L. Wang, A regularization method for treating zero points of the sum of two monotone operators, Fixed Point Theory Appl., 2014 (2014), 10 pages.*

[67] *X. Qin, Y. Su, C. Wu, and K. Liu. Strong convergence theorems for nonlinear operators in Banach spaces. Commun. Appl. Nonlinear Anal. 14 (2007): 35-50.*

[68] *X. Qin, and L. Wang. On asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense. Abstr. Appl. Anal., vol. 2012.*

[69] *S. Reich, Extension problems for accretive sets in Banach spaces. J. Functional Analysis. 26, 378-395 (1977)*

[70] *S. Reich, Product formulas, nonlinear semigroups, and accretive operators. J. Function. Anal. 36(1980), 147-168.*

[71] *S. Reich. A weak convergence theorem for the alternating method with Bregman distances. Theory and applications of nonlinear operators of accretive and monotone type (1996).*

[72] *S. Reich and S. Sabach, strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces, J. Nonlinear Convex Anal. 10(3)(2009): 471-485.*

[73] *S. Reich and S.Sabach, Two strong convergence theorems for a proximal method in reflexive Banach spaces, Numer. Funct. Anal. Optim. 31(1): 22-44, 2010.*

[74] *S. Reich, S. Sabach, A projection method for solving nonlinear problems in reflexive Banach spaces. J. Fixed Point Theory Appl. 9 (2011) 101-116*

[75] *R.T. Rockafellar. Monotone operators and the proximal point algorithm. SIAM: J. Control Optim. 14, No. 5 (1976): 877-898.*

[76] *S. Saejung, P. Yotkaew, Approximation of zeros of inverse strongly monotone operators in Banach spaces, Nonlinear Anal. 75(2): 742-750, 2012.*

[77] *P. Senakka, P. Cholamjiak, Approximation method for solving fixed point problem of Bregman strongly nonexpansive mappings in reflexive Banach spaces, Ric. Mat. 65 (2016), 209-220.*

[78] *N. Shahzad, H. Zegeye, Convergence theorems of common solutions for fixed Point, variational inequality and equilibrium problems. J. Nonlinear Var. Anal. 3 (2019), No. 2, pp. 189-203,https://doi.org/10.23952/jnva.3.2019.2.08.*

[79] *Y. Shehu, P. Cholamjiak, Another look at the split common fixed point problem for demicontractive operators, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM 110(1): 201-218, 2016.*

[80] *Y. Shehu, O. S. Iyiola, J-C. Yao, New projection methods with inertial steps for variational inequalities, Optimization, 2021, 1-32, https://doi.org/10.1080/02331934.2021.1964079.*

[81] *T. V. Su, Second-order optimality conditions for vector equilibrium problems. J. Nonlinear Funct. Anal., 2015 (2015), 31 pages.*

[82] P. Sunthrayuth, N. Pholasa, P. Cholamjiak, Mann-type algorithms for solving the monotone inclusion problem and the fixed point problem in reflexive Banach spaces, Ricerche di Matematica, 1-28, 2021, https://doi.org/10.1007/s11587-021-00596-y.

[83] A. Taiwo, L. O. Jolaoso, O. T. Mewomo, Inertial-type algorithm for solving split common fixed point problems in Banach spaces, J Sci Comput 86(12): 1-30, 2021.

[84] A. Taiwo, A.O.-E. Owolabi, L.O. Jolaoso, O.T. Mewomo, A. Gibali, A new approximation scheme for solving various split inverse problems, Afr. Mat. 2021(32): 369-401, 2021.

[85] A. Taiwo, T.O. Alakoya, O.T. Mewomo, Halpern-type iterative process for solving split common fixed point and monotone variational inclusion problem between Banach spaces, Numer. Algor. 2021(86): 1359-1389. 2021.

[86] W. Takahashi and K.Zembayashi. Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces. Nonlinear Anal. 70, No. 1 (2009): 45-57.

[87] Y. Tang, A. Gibali, New self-adaptive step size algorithms for solving split variational inclusion problems and its applications, Numer Algorithms. 2020(83): 305-331, 2020.

[88] D. V. Thong, A. Gibali, P. T. Vuong, An explicit algorithm for solving monotone variational inequalities, Appl. Numer. Math. 2022(171): 408-425, 2021.

[89] R. Tibshirani, Regression shrinkage and selection via the Lasso, J. R. Stat. Soc. B Methodol. 58(1): 267-288, 1996.

[90] A. R. Tufa, H. Zegeye, M. Thuto, Convergence theorems for non-self mappings in CAT(0) spaces, Numer. Funct. Anal. Optim. 38(6): 705-722, 2017.

[91] N. Van Dung and N.T. Hieu. A new hybrid projection algorithm for equilibrium problems and asymptotically quasi-ϕ-nonexpansive mappings in Banach spaces. Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math. RACSAM 113, no. 3 (2019): 2017-2035.

[92] G.B. Wega, H. Zegeye, Convergence results of Forward-Backward method for a zero of the sum of maximally monotone mappings in Banach spaces, Computational and Applied Mathematics, Computational and Applied Mathematics, (2020) 39:223 https://doi.org/10.1007/s40314-020-01246-z.

[93] G.B. Wega, H. Zegeye, Convergence theorems of common solutions of variational inequality and f-fixed point problems in Banach spaces. Appl. Set-Valued Anal. Optim. 3(2021)(1), 55-73.

[94] H.K. Xu, Viscosity approximation methods for nonexpansive mappings. J. Math. Anal. Appl. 298(1) (2004), 279-291.

[95] H.K. Xu. Inequalities in Banach spaces with applications. Nonlinear Anal. 16, No. 12 (1991): 1127-1138.

[96] Y. Yao, M. Postolache, Y.C. Liou, Extragradient-type method for monotone variational inequalities. Fixed Point Theory Appl. 2013, 185 (2013).

[97] Y. Yao, Y. Shehu, X.H. Li, Q.L. Dong, A method with inertial extrapolation step for split monotone inclusion problems, Optimization, 70(4)(2021): 741-761.

[98] C. Zalinescu, Convex Analysis in General Vector Spaces, World Scientifc, River Edge, 2002.

[99] H. Zegeye, Strong convergence theorems for maximal monotone mappings in Banach spaces. J. Math. Anal. Appl. 343, 663-671 (2008)

[100] H. Zegeye, A hybrid iteration scheme for equilibrium problems, variational inequality problems and common fixed point problems in Banach spaces, Nonlinear Anal. 72 (2010) 2136-2146.

[101] H. Zegeye, An iterative approximation for a common fixed point of two pseudocontractive mappings. ISRN Math. Anal. 2011, 14.

[102] H. Zegeye, O. A. Boikanyo, A common solution of f-fixed point and variational inequality problems in Banach spaces, Optimization, (2021): DOI: 10.1080/02331934.2021.1982939.

[103] H. Zegeye, E. Prempeh, Strong convergence of approximants to fixed points of Lipschitzian pseudocontractive maps. Comput. Math. Appl., 44 (2002), 339-346.

[104] H. Zegeye, N. Shahzad, Strong convergence theorems for monotone mappings and relatively weak nonexpansive mappings, Nonlinear Anal. 70(2009), 27072716.

[105] H. Zegeye, N. Shahzad, M.A. Alghamdi, Strong convergence theorems for a common point of solution of variational inequality, solutions of equilibrium and fixed point problems. Fixed Point Theory Appl.,(2012), 1-17.

[106] H. Zegeye, G.B. Wega, Approximation of a common f-fixed point of f-pseudocontractive mappings in Banach spaces. Rendiconti del Circolo Matematico di Palermo Series 2, 2020.

[107] S.S. Zhang,Generalized mixed equilibrium problem in Banach spaces. Appl. Math. Mech. -Engl. Ed. 30(9), 1105-1112 (2009) DOI: 10.1007/s10483-009-0904-6

[108] X. Zhou, On the fenchel duality between Strong convexity and Lipschitz continuous gradient, arXiv:1803.06573 [math.OC], 2018.

[109] H. Zhou,G. Gao, and B. Tan. Convergence theorems of a modified hybrid algorithm for a family of quasi-ϕ-asymptotically nonexpansive mappings. J. Appl. Math. Comput. 32, no. 2 (2010): 453-464.