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On Subharmonic Functions and Potentials

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ADDIS ABABA UNIVERSITY

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On Subharmonic Functions and Potentials

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PERMISSION

This is to certify that this thesis is written by Esubalew Feleke in the Department of Mathematics, Addis Ababa University, under my supervision. I hereby also confirm that the thesis can be submitted for evaluation by examiners and eventual defense.

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ABSTRACT

Potential Theory in the Complex Plane

I'll discuss some of the main definitions, theorems, and techniques on subharmonic functions and potential theory in the complex plane. This includes the origin of the subject in physics, connections with complex analysis, harmonic and subharmonic functions, logarithmic potentials, equilibrium potentials, Dirichlet problem, thin sets, polar sets, capacity.

Chapter1.

Preliminaries

1.1 Introduction

A Convex function f is called Sublinear in the following sense: if a linear function $l \geq f$ at the boundary points of an interval, then $l \geq f$ in the interior of that interval also. If we replace the terms interval and linear function by the term domain and harmonic function, we obtain a statement which expresses the characteristic property of subharmonic function of two or more variables. This generalization was formulated and developed by F.Riesz.This immediately attracted the attention of many mathematicians both on account of its intrinsic interests and on account of the wide range of its application

If f is an analytic function of the complex variable $z = x + iy$ then $|f(z)|$ is subharmonic. The potential of a negative mass distribution is subharmonic .In differential geometry, surfaces of negative curvature and minimal surface can be characterized in terms of subharmonic function. The idea of subharmonic function leads to significant applications and interpretation in the fields just referred to, and conversely every one of these fields is an apparently in exhaustible sources of new theorems on subharmonic function either by analogy or direct implication.

1.2 Notations and definitions

In this project much of the discussion takes on a domain (i.e non-empty ,connected ,open set) in the complex plane.

- We let $\mathbb{C}$ denote the set of Complex number. By $\mathbb{C}_\infty$ we mean $\mathbb{C} \cup \{\infty\}$.
- For a subset $\Omega \subseteq \mathbb{C}\ or\ \mathbb{C}_\infty$, $\partial\Omega$ denote the boundary of Ω will always taken relative to $\mathbb{C}_\infty$.

We use the symbol $\Delta(\omega, \rho)$ for open disc and $\bar{\Delta}(\omega, \rho)$ for closed disc, given $\omega \in \mathbb{C}$ and $\rho > 0$, we write

$$\Delta(\omega, \rho) := \{z \in \mathbb{C} : |z - \omega| < \rho\}$$

$$\bar{\Delta}(\omega, \rho) \coloneqq \{ z \in \mathbb{C} : |z - \omega| \leq \rho \}$$

Although Δ is the standard symbol for the Laplacian, namely

$$\Delta f \coloneqq f_{xx} + f_{yy}$$

no confusion should arise since Δ followed by a bracket and the other by a function.

- Let μ be a Borel measure We define supports of μ, as the set
$$supp\mu = \{x \in X: \mu(U) > 0 for\ each\ open\ neighbourhood\ U\ of\ \ x\}$$

1.2 .Harmonic function and its properties

Harmonic function, namely solution of the Laplace's equation, exhibit many properties reminiscent of those of holomorphic function. We begin with the formal definition.

Definition 1.3.1: let U be an open subset of ℂ. A function h : **U**→ ℝ is called harmonic if

h∈ C^2(U) and Δh = 0 on U .

Example1. h(z) = log|z| is harmonic function on ℂ .

Example 2. Let h(x+iy) = $e^x(x\cos y - y\sin y)$ is harmonic on ℂ.

Example 3 .h(z) =$|z|^2 = x^2 + y^2$ is not harmonic anywhere on ℂ as $h_{xx} + h_{yy} = 4$

The following basic result not only furnishes numerous examples of harmonic function, but also provides a useful tool in deriving their elementary properties from those of holomorphic function.

Theorem 1.3.2 Let D be a domain in ℂ.

(a) If f is holomorphic on D and h = Ref, then h is harmonic in D

(b) If h is harmonic in D, and if D is simply connected, then h = Ref for some f holomorphic on D, Moreover f is unique up to adding a constant.

Proof:

(a) Let f (z) = h + ik be holomorphic function in D where

h = Ref. Then a function f is differentiable at z_o. Then, from Cauchy-Riemann Equations, we have, $h_x = k_y$ and $h_y = -k_x$.Then differentiating h_x with respect to x and h_y with respect to y Then, $h_{xx} = k_{yx}$ and $h_{yy} = -k_{xy}$. From this, $h_{xx} = -k_{yy}$, (since $k_{yx} = k_{xy}$) this implies that

$h_{xx} + h_{yy} = \Delta h = 0$. Hence, h is harmonic in D.

(b) If h = Ref for some holomorphic function f, say f = h + ik, then

$f' = h_x + ik_x = h_x - ih_y$.(since, from CRE, $-h_y = k_x$) . -----------------------------------(1.1)

Thus if f exists, then f' is completely determined by h, and hence f is unique up to adding a constant.

Equation (1.1) also suggests how we might construct such a function f.

Define $g: D \to \mathbb{C}$ by

$$g = h_x - ih_y$$

Then $g \in C^1(D)$ and g satisfies the Cauchy –Riemann equation because

$$h_{xx} = -h_{yy} \ and \ h_{xy} = h_{yx} \ .$$

Therefore g is holomorphic on D.

Fix $z_0 \in D,$ and define $f: D \to \mathbb{C}$ by

$$f(z) = h(z_0) + \int_{z_o}^{z} g(\omega)d\omega,$$

The integral being taken over any path in D from $z_o \ to \ z$ .As D is simply connected, Cauchy's theorem ensures that is independent of the particular path chosen. Then f is holomorphic on D and $f' = g = h_x - ih_y$ Writing $\widetilde{h}$ = Ref, we have

$$\tilde{h}_x - i\tilde{h}_y = f' = h_x - ih_y,$$

So that $\left(\tilde{h} - h\right)_x \equiv 0$ and $\left(\tilde{h} - h\right)_y \equiv 0$ .It follows that $\tilde{h} - h$ is constant on D, and putting $z = z_0$ shows that the constant is zero. Thus indeed h = Ref.

Theorem 1.3.3 Let f be holomorphic and non-zero on a simple connected domain D in $\mathbb{C}$. Then there exists a holomorphic function g on D such that f = e^g.

Proof. Put h=log |f| on D. Because h is locally the real part of a holomorphic function ,namely abranch of log f, it is harmonic, by Theorem1.3.2(a) Theorem 1.3.2(b) there exist g holomorphic on D such that h=Reg there, in other words, $|fe^{-g}|$=1 on D.By the maximum principle for holomorphic function , fe^{-g} is a constant C. Adding a suitable constant to g, we can suppose that C=1, and so f = e^g.

Theorem 1.3.4 (Mean-value property):

Let u be a harmonic function on an open neighborhood of the disc $\bar{\Delta}(\omega, \rho)$. Then

$$u\ (\omega) = \frac{1}{2\pi}\int_0^{2\pi} u(\omega + \rho e^{i\theta})d\theta$$

Proof.

Since u is harmonic function on the disc $\bar{\Delta}(\omega, \rho)$, then there is an analytic function f in the disc such that Ref = u. choose $\rho_0 > \rho$ so that u is harmonic on $\Delta(\omega,\rho_0)$ Applying Theorem 1.3.2(b) there exists f analytic function on $\Delta(\omega,\rho_0)$ such that u = Ref there. This implies that, f is differentiable at ω. By Cauchy's integral formula, we have,

$$f(\omega) = \frac{1}{2\pi i}\int_{|z-\omega|=\rho}\frac{f(z)dz}{z-\omega} = \frac{1}{2\pi}\int_0^{2\pi} f(\omega+\rho e^{i\theta})d\theta$$

Where $z = \omega + \rho e^{i\theta}$ and $z - \omega = \rho e^{i\theta}$ and $0 \le \theta \le 2\pi$

The result follows upon taking real parts of both sides. i.e

$f(\omega) = u + iv$. This implies that

$f(\omega + \rho e^{i\theta}) = u(\omega + \rho e^{i\theta}) + iv(\omega + \rho e^{i\theta})$. Hence,

$u(\omega) = \frac{1}{2\pi}\int_0^{2\pi} u(\omega + \rho e^{i\theta})d\theta$. This show that harmonic function have mean value property.

Theorem 1.3.5 (Identity principle for harmonic function):

Let h and k be harmonic functions on a domain D in $\mathbb{C}$.

If h = k on a non-empty open subset U of D. Then h = k throughout D.

Proof.

We can suppose, without loss of generality, that k = 0. Set $g = h_x - ih_y$. Then as in the proof of Theorem 1.3.2, g is holomorphic on D , and also g = 0 on U since h = 0 there by the identity principle for holomorphic function it follows that g = 0 throughout D, and hence that $h_x = 0$ and $h_y = 0$ on D there fore h is constant on D, and since h = 0 on U, this constant must be zero.

Theorem 1.3.6 (Maximum Principle)

Let h be a harmonic function on a domain D in $\mathbb{C}$.

(a) If h attains a local maximum on D, then h is constant.

(b) If h extends continuously to $\overline{D}$ and $h \le 0$ on ∂D, then $h \le 0$ on D.

Proof:

(a) Suppose that h attain a local maximum at a point $\omega_0 \in D$. Then for some $r > 0$ we have

$h \le h(\omega_0)$ on $\Delta(\omega, r)$, where $\Delta(\omega, r)$ is a disc with center ω and radius r by theorem 1.3.2(b)there exists a function f holomorphic on $\Delta(\omega, r)$, such that

h = Ref there .Then, by Theorem 1.3.3. There exists a holomorphic function g on D such that

$f = e^g$. Then $|e^g|$ attains a local maximum at ω_0 so, e^g must be constant. Therefore h is constant on $\Delta(\omega, r)$ and hence on the whole of D by the identity principle.

(b) As $\overline{\mathrm{D}}$ is compact (connected and bounded), h must attain a maximum at some point ω ∈ $\overline{\mathrm{D}}$. If ω∈∂D, then h(w) ≤ 0 by assumption ,and so h ≤ 0 on D. If ω ∈D, then by part (a) h is a constant on D, hence on $\overline{\mathrm{D}}$, and so once again h ≤ 0 on D.

1.4. The Dirchelet problem on the Disc

The Dirichlet problem is to find a harmonic function on a domain with prescribed boundary value s. It is one of the great advantages of harmonic function over holomorphic ones that for nice domains, a solution always exists. This is a power full tool with many applications. Here is the formal statement of the problem.

Definition 1.4.1 Let D be a subdomain of complex $\mathbb{C}$, and let $\emptyset: \partial \mathrm{D} \to \mathrm{R}$ be a continuous function. The Dirichelt problem is to find a harmonic function h on D such that

$$\lim_{\mathrm{z}\to \mathrm{z}_0} \mathrm{h(z)} = \emptyset(\mathrm{z}_0) \text{ for all } z_0 \in \partial D$$

Example. Solve the Dirchlet problem for the disc D = $\Delta(0,2)$ with boundary data

$$\emptyset(z) = x^2 + 2xy^2$$

We set $z = 2e^{i\theta}$

$$h(z) = 4\cos^2\theta + 16cos\theta \sin^2\theta$$

$$= 2(1+\cos(2\,\theta) + 8\cos\theta\,(1-\cos(2\theta)$$

$$= 2 + 8\cos\theta + 2\cos(2\theta) - 8\cos(2\theta)\cos\theta$$

$$= 2 + 8\cos\theta + 2\cos(2\theta) - 4\cos(3\theta) - 4\cos\theta$$

$$= 2 + 4\cos\theta + 2\cos(2\theta) - 4\cos(3\theta)$$

Define $h: \mathbb{C} \to \mathcal{R}\ by$

$h\left(re^{i\theta}\right) = 2 + 4a_1 r\cos\theta + 2a_2 r^2\cos 2\theta - 4a_3 r^3 \cos 3\theta$ is harmonic

When r = 2, then $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{4}$, $a_3 = \frac{1}{8}$. Thus the solution of the Dirchlet problem is thus proved by

$$h(z) = 2 + 2r\cos\theta + \frac{r^2}{2}\cos(2\theta) - \frac{r^3}{2}\cos(3\theta)$$

$$h(z) = Re(2 + 2z + \frac{z^2}{2} - \frac{z^3}{2})$$

$$h(z) = 2 + 2x + \frac{x^2}{2} - \frac{y^2}{2} - \frac{x^3}{2} + \frac{3xy^2}{2}$$

Theorem1.4.2 (Uniqueness Theorem) with the notation of definition above there is at most one Solution h to the Dirichelet problem.

Proof. Suppose that h_1 and h_2 are both solutions. Then $h_1 - h_2$ is harmonic on D, extends continuously to $\overline{D}$, and is zero on ∂D. Applying the maximum principle theorem to $\pm(h_1 - h_2)$, we conclude that $h_1 - h_2 = 0$

Definition 1.4.3

(a) The Poisson kernel P: $\Delta(\mathbf{0},\mathbf{1}) \times \boldsymbol{\partial}\Delta(\mathbf{0},\mathbf{1}) \rightarrow \mathbb{R}$ is defined by

(A) $P(z,\varsigma) := \mathrm{Re}(\left(\frac{\varsigma+z}{\varsigma-z}\right) = \frac{1-|z|^2}{|\varsigma-z|^2}$ $(|\varsigma|=1,\ |z| < 1)$.

(B) If $\Delta = \Delta(\omega,\rho)$ and $\emptyset: \partial\Delta \longrightarrow \mathbb{R}$ *is* a Lebesgue integrable function, then its Poisson integral $P_\Delta\,\emptyset: \Delta \rightarrow \mathbb{R}$ *is* defined by

$$P_\Delta\,\emptyset(Z) := \frac{1}{2\pi}\int_0^{2\pi} P\left(\frac{z-\omega}{\rho}, e^{i\theta}\right)\emptyset\left(\omega + \rho e^{i\theta}\right)d\theta \quad (z \in \Delta).$$

More explicitly, if $r < \rho$ and $0 \le t \le 2\pi$, then

$$P_\Delta\,\emptyset\left(\omega + re^{i\theta}\right) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2\rho r\cos(\theta - t) + r^2}\emptyset(\omega + \rho e^{i\theta})d\theta$$

The following result is fundamental

Theorem 1.4.4 (Properties of Poisson integral)

(a) $P_\Delta\,\emptyset$ harmonic on Δ;
(b) if $\emptyset$ is continuous at $\varsigma_0 \in \partial\Delta$, then $\lim_{z\to\varsigma_0} P_\Delta\,\emptyset(z) = \emptyset(\varsigma_0)$

In particular, if $\emptyset$ is continuous on the whole of $\partial\Delta$, then $h := P_\Delta\,\emptyset$ solves the Dirichlet problem on Δ.

Proof.(a) making an affine change of variable if necessary,we can suppose that ω = 0 and $\rho = 1$, so that $\Delta = \Delta(0,1)$. then

$$P_\Delta\,\emptyset(Z) := \frac{1}{2\pi}\int_0^{2\pi} P\left(\frac{z-\omega}{\rho}, e^{i\theta}\right)\emptyset\left(\omega + \rho e^{i\theta}\right)d\theta \quad (z \in \Delta).$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{P}(\mathrm{z}, \mathrm{e}^{\mathrm{i}\theta})\emptyset(\omega + \rho \mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta \quad (\mathrm{z} \in \Delta). \text{ where } \omega = 0 \textit{ and } \rho = 1$$

$$\mathrm{P}_\Delta\, \emptyset(\mathrm{z}) = \mathrm{Re}\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}\theta}+\mathrm{z}}{\mathrm{e}^{\mathrm{i}\theta}-\mathrm{z}}\emptyset(\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta\right) \quad (\mathrm{z} \in \Delta),$$

So that $\mathrm{P}_\Delta\, \emptyset$ is the real part of a holomorphic function of z. Hence it is harmonic on Δ.

(b), it is convenient first to prove a lemma about the Poisson kernel

Lemma 1.4.5 **The Poisson kernel P** satisfies:

(i) $\mathrm{P}(\mathrm{z}, \varsigma) > 0 \quad (|\mathrm{z}| < 1, |\varsigma| = 1)$;

(ii) $\frac{1}{2\pi}\int_0^{2\pi} \mathrm{P}(\mathrm{z}, \mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta = 1 \quad (|\mathrm{z}| < 1)$;

(iii) $\sup_{|\varsigma-\varsigma_0|\geq\delta}\mathrm{P}(\mathrm{z}, \varsigma) \to 0$ as $\mathrm{z} \to \varsigma_0 \quad (|\varsigma_0| = 1, \delta > 0)$

Proof.

(i) this is clear from the definition of $\mathrm{P}(\mathrm{z}, \varsigma)$.

ii) expresing the given integral as the contour integral and using the cauchy integral formula, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} P(z, e^{i\theta})d\theta = Re\left(\frac{1}{2\pi i}\int_{|\varsigma|=1} \frac{\varsigma + z}{\varsigma - z}\frac{d\varsigma}{\varsigma}\right) \quad (Since\ \ \varsigma = e^{i\theta} \ \ and\ d\varsigma = ie^{i\theta}d\theta)$$

$$= \mathrm{Re}\left(\frac{1}{2\pi \mathrm{i}}\int_{|\varsigma|=1} \frac{2}{\varsigma-\mathrm{z}} - \frac{1}{\varsigma}\right)\mathrm{d}\varsigma$$

$$= \mathrm{Re}\left(\frac{1}{2\pi \mathrm{i}}\int_{|\varsigma|=1} \frac{2}{\varsigma-\mathrm{z}}\mathrm{d}\varsigma - \int_{|\varsigma|=1} \frac{1}{\varsigma}\mathrm{d}\varsigma\right)$$

$$= \mathrm{Re}\ (2\text{-}1) = 1$$

(iii) if $|\mathrm{z} - \varsigma_0| < \delta$ then $\sup_{|\varsigma-\varsigma_0|\geq\delta}\mathrm{P}(\mathrm{z}, \varsigma) \leq \frac{1-|\mathrm{z}|^2}{(\delta-|\varsigma_0-\mathrm{z}|^2}$, and the result follows easily from this

Now the prove of the above theorem (b) is

Once again, we may suppose that $\Delta = \Delta(0,1)$. Using lemma (i) and (ii) we have

$$|\mathrm{P}_\Delta\, \emptyset(\mathrm{z}) - \emptyset(\varsigma_0)| = \left|\frac{1}{2\pi}\int_0^{2\pi} \mathrm{P}(\mathrm{z}, \mathrm{e}^{\mathrm{i}\theta})\left(\emptyset(\mathrm{e}^{\mathrm{i}\theta}) - \emptyset(\varsigma_0)\right)\mathrm{d}\theta\right|$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} P(z, e^{i\theta})| \left(\emptyset(e^{i\theta}) - \emptyset(\varsigma_0)\right) |d\theta$$

Let $\varepsilon > 0$ if $\emptyset$ is continuous at ς_0, then there exists $\delta > 0$ such that

$$|\varsigma - \varsigma_0| < \delta \Rightarrow |\emptyset(\varsigma) - \emptyset(\varsigma_0)| < \varepsilon.$$

Hence using lemma (i) and (ii) again it follows that

$$\frac{1}{2\pi}\int_{|e^{i\theta}-\varsigma_0|<\delta} P(z, e^{i\theta})| \left(\emptyset(e^{i\theta}) - \emptyset(\varsigma_0)\right) |d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} P(z, e^{i\theta})\epsilon\, d\theta = \varepsilon. \quad \text{----*}$$

also from Lemma (iii), there exists $\delta' > 0$ such that

$$|z - \varsigma_0| < \delta' \quad \Rightarrow \quad \sup_{|\varsigma-\varsigma_{0|\geq\delta}|} P(z, \varsigma) < \varepsilon.$$

Hence if $|z - \varsigma_0| < \delta'$ then $\frac{1}{2\pi}\int_{|e^{i\theta}-\varsigma_0|\geq\delta} P(z, e^{i\theta})| \left(\emptyset(e^{i\theta}) - \emptyset(\varsigma_0)\right) |d\theta$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} \epsilon| \left(\emptyset(e^{i\theta}) - \emptyset(\varsigma_0)\right) |d\theta$$

$$\leq \epsilon\left(\frac{1}{2\pi}\int_0^{2\pi} | \left(\emptyset(e^{i\theta})|d\theta + |\emptyset(\varsigma_0)\right) |d\theta\right) \text{---------**}$$

Combing these facts, we deduce that if $|z - \varsigma_0| < \delta'$ then

$$|P_\Delta\, \emptyset(z) - \emptyset(\varsigma_0)| \leq \epsilon\left(1 + \frac{1}{2\pi}\int_0^{2\pi} | \left(\emptyset(e^{i\theta})|d\theta + |\emptyset(\varsigma_0)\right) |d\theta\right)$$

Corollary 1.4.6 (Poisson integral formula)

If h is harmonic on an open neighborhood of the disc $\bar{\Delta}(\omega, \rho)$, then for $r < \rho$ and $0 \leq t \leq 2\pi$.

$$h(\omega + re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2\rho\cos(\theta - t) + r^2} h(\omega + \rho e^{i\theta})d\theta$$

Proof.

Consider the Dirchlet problem on $\Delta \coloneqq \Delta(\omega, \rho)$with $\emptyset = h\backslash\, \partial\Delta$. by theorem 1.4.4 h and $P_\Delta h$ are both solutions, so by uniqness theorem(1.4.2), h = $P_\Delta h$ on Δ. Note that this result is a generalization of the mean value property, which is just the case r = 0.

It allows us to recapture the value s of h everywhere on Δ from knowledge of h on $\partial\Delta$.

Theorem 1.4.7 (Converse to Mean –value property)

Let h: U→ $\mathbb{R}$ *be* acontinuous function on an open subset U of complex $\mathbb{C}$, and suppose that it possesses the local mean –value property, i.e. given $\omega \in U$, there exsits $\rho > 0$ such that

$$h(\omega) = \frac{1}{2\pi}\int_0^{2\pi} h\left(\omega + re^{it}\right)dt \qquad (0 \leq r < \rho).$$

Then h is harmonic on D.

Proof.

It is enough to show that h is harmonic on each open disc Δ with $\bar{\Delta} \subset U$.

Fix such a Δ, and define K: $\bar{\Delta}\rightarrow \mathbb{R}$ by

$$K=\begin{cases} h - P_\Delta h & \text{on } \Delta, \\ 0 & \text{on } \partial\Delta \end{cases}$$

then k is continuous on $\bar{\Delta}$ and has the local mean –value property on Δ. As $\bar{\Delta}$ is Compact, k attains a maximum value M at some point of $\bar{\Delta}$.

Define $A = \{z\in \Delta: k(z) < M\}$ and

$$B = \{z \in \Delta: k(z) = M\}$$

Then A is open, since K is continuous. Also B is open, for if k(w) = M, then the local mean-value property forces k to be equal to M on all sufficiently small circles around w. As A and B partition the connected set Δ, either $A = \Delta$, in which case k attains its maximum on $\partial\Delta$ and so $M = 0$ or else

$B = \Delta$ in which case $k \equiv M$ and again $M = 0$. thus $k \leq 0$,And asimilar argument shows that $k \geq 0$. hence h = $P_\Delta h$ on Δ, and since $P_\Delta h$ is harmonic there, so is h.

Thorem1. 4. 8 (Reflection Principle)

Let $\Delta = \Delta(0, R)$, and write $\Delta^+ = \{z \in \Delta: \operatorname{Im} z > 0\}$, and $I = \{z \in \Delta: \operatorname{Im} z = 0\}$.

Suppose that f is holomorphic function on Δ^+ such that Ref extends continuously to $\Delta^+ \cup I$ with Ref = 0 on I. Then f extends holomorphically to the whole of Δ.

Note that no assumption is made about continuity of imf on I—this comes for free.

Proof.

Define h: $\Delta \rightarrow \mathbb{R}$ by

$$h(z) = \begin{cases} \mathrm{Re} f(z), & z \in \Delta^{+} \\ 0, & z \in I \\ -\mathrm{Re} f(\bar{z}), & \bar{z} \in \Delta^{+}, \end{cases}$$

then h is continuous on Δ and has the local mean-value property there, so by the converse mean-value property theorem it is harmonic on Δ from theorem 1.3.2 (b), there exists a holomorphic function $\tilde{f}$ on Δ such that $h = \mathrm{Re}\tilde{f}$. Now $f - \tilde{f}$ is holomorphic on Δ^{+} and takes only imaginary values, so it is constant there. Adjusting $\tilde{f}$ appropriately,we can make this constant zero. Then $\tilde{f}$ provides the promised holomorphic extension of f to the whole of Δ.

Chapter 2 Properties of Subharmonic Function

There are two standards approach to define Subharmonic function. One is to require that $u_{xx} + u_{yy} \geq 0$ in the sense of distributive property, matching the characteristic property of the harmonic function. And the other one is via a sub mean property of harmonic function. We shall follow the second approach. Recall that harmonic function are continuous, However sub harmonic functions are not required to be continuous for if they were that would be too restrictive as continuity is not preserved when taking limits of the function with $u_{xx} + u_{yy} \geq 0$ Obviously, one need to require some kind of regularity to have a meaning full theory and it appears that semi continuity suffices. subharmonic function is going to be upper semi continuous so before making this definition we take a brief look at upper semicontinuous function [Ra95].

2.1 Upper Semicontinuous Functions

Recall that a function f on a metric space X is called continuous at x if for any given positive ϵ one can find a open ball $\Delta(x, r)$ of radius r about x, $\Delta(x, r) = \{y: dis(x, y) < r\}$ such that

$f(x) - \epsilon \leq f(y) \leq f(x) + \epsilon$ (y$\in \Delta(x, r)$ the inequality is two sided. if only one side of the inequality holds then the function is said to be semi continuous.

Definition. 2.1.1 A function $f: X \to [-\infty, \infty)$ is called upper semicontinuous

(at $x \in X$) if for any given positive ϵ one can find a positive δ such that

$$f(y) \leq f(x) + \epsilon \qquad y \in \Delta\,(x, \delta)\ .$$

Note that upper semicontinuous function are allowed to take value $-\infty$.

An equivalent definition for function f to be upper semicontinuous on X is to require that if $\lim_{y \to x} \sup f(y) \leq f(x)\ for\ all\ \ x \in X$

Yet another equivalent definition is to require the set $\{x \in D: f(x) < \alpha\}$ be open in D for every $\alpha \in \mathfrak{R}$. Obviously, Every Continuous function is also upper semicontinuous.

Below are few examples of function that are upper semi continuous but not continuous.

1) $f(z) = \ln|z|, z \in \mathbb{C}$ f is not continuous at z = 0

Generally a continuous real valued function is clearly upper semicontinuous.

Theorem 2.1.2 Let u be an upper semi continuous function on a topological space X and let K be a compact subset of X. Then u is bounded above on K and attains its bound.

Proof. The sets $\{x \in X: u(x) < n\}$ $(n \geq 1)$ form an open cover of K, so have a finite subcover. Hence u is bounded above on K.

Let $M = \sup_k u$. Then the open sets $\left\{x \in X: u(x) < M - \frac{1}{n}\right\}$ $(n \geq 1)$ cannot cover k, because they have no finite subcover. Hence $u(x) = M$ for at least one $x \in k$.

2.2 Subharmonic Function

Definition2.2.1 Let U be an open subset of $\mathbb{C}$.

A function $u: U \to [-\infty, \infty)$ is called subharmonic in U

a) If u is upper semi continuous and not identically $-\infty$

b) For each $\omega \in U$ there is a ball B $(\omega, \rho) \leq U$ satisfies the local submeans in equality,

i.e. given $\omega \in U$, there exists $\rho > 0$ such that

$$u(\omega) \leq \frac{1}{2\pi}\int_0^{2\pi} u(\omega + re^{it)}\, dt \qquad 0 \leq r < \rho \quad 2.1$$

Also $u: U \to [-\infty, \infty)$ is superharmonic if $-u$ is subharmonic. If $u = -\infty$ the set of points in which a subharmonic function takes the value $-\infty$ is so small that all our integrals are finite.

Example .A harmonic function is subharmonic by mean value property.

Theorem 2.2.2 If f is holomorphic on an open set U in $\mathbb{C}$ then $\log|f\,|$ is subharmonic on U.

Proof.

Evidently u: = $\log|f|$ is upper semicontinuous. Also it satisfies the local submean inequality at each $\omega \in U$ for which $u(\omega) > -\infty$ because near such a point $\log|f|$ is actually harmonic. On the other hand, if $u(\omega) = -\infty$, then 2.1 is obvious any way.

2.3 The maximum principle

As a consequence of Mean value property and Converse to mean value property the local mean value property implies the (global) mean value property. To make much further progress with the subharmonic function, we need a corresponding result for the submean inequality. As with harmonic function, we shall deduce this via a maximum principle. The importance of the maximum principle lies in the fact that from local assumption it derives a global conclusion. Such results are usually very powerful, and the maximum principle is no exception. Since it will feature prominently in what follows, we shall digress slightly in order to study it in a little more detail, returning to the submean inequality in the next section.

Theorem 2.3.1 (Maximum Principle)

Let u be a subharmonic function on a domain D in $\mathbb{C}$

A) If u attains a global maximum on D, then u is constant.

B) If D is bounded and $\lim_{z\to\varsigma}\sup u(z) \leq 0$ for all $\varsigma \in \partial D$, then u ≤ 0 on D.

Proof (A) Supposes that u attains a maximum value M on D.

Define A= $\{z \in D: u(z) < M\}$ and

B =$\{z \in D: u(z) = M\}$ then

The set A is open because u is upper semicontinuous. Also the set B is open, because of the local submean property for subharmonic function.

(any, sufficient small circles about z $\in$ B must lie in B, for if not then there is a circle that intersects with A, and since A is open the intersection will contain a segment of finite length hence the mean value integral will be $< 2\pi M$ in the violation of the local submean property. By assumption, A and B partition on D since D is connected; one of the two sets must be empty. The set B is non-empty by assumption, therefore $A = \emptyset$ and part (a) is proved.

b) Let us extend u to the ∂D by defining $u(\varsigma)= \lim_{z\to\varsigma}\sup u(z)$ for all $\varsigma \in \partial D$.Then u is upper semicontinuous on $\overline{D}$, since $\overline{D}$, which is compact so by Theorem 2.1.2 u attains a maximum at some $\omega \in \overline{D}$. If the maximum point is in D then u = 0 on $\overline{D}$, by part (a) if the maximum point is at the boundary of D, then $u \leq 0$ *on* $\overline{D}$,

Comparison to the maximum principle for harmonic and subharmonic function

- local maximum for harmonic, global for subharmonic
- *minimum or* maximum for harmonic function, only maximum for subharmonic.

- The subharmonic function $u = \max(Rez, 0)$ attains a local maximum and a global minimum but is not constant.

The following theorem explains the name for subharmonic functions.

Theorem2.3.2 (Harmonic majoration)

Let D be abounded domain in ℂ and suppose that u is subharmonic in D and h is harmonic there. Then

$\lim_{z\to\varsigma}\sup(u-h)(z) \le 0\ for\ all\ \varsigma \epsilon\ \partial D.$ This implies that $u \le h\ on\ D$.

Proof.

The function $u - h$ is subharmonic, hence the result follows from the maximum principle for subharmonic functions, Part(b) of Thm2.3.1
Recall that harmonic functions can be represented via the Poisson integral formula correspondingly, subharmonic functions are bounded from above by the Poisson integral.

2.4 Criteria of Subharmonic Function

We can prove that subharmonic function satisfies the global submean inequality. In fact more is true : they also obey an inequality corresponding to the Poisson integral formula, as shown by the following theorem.

Theorem 2.4.1 Let U be an open subset of ℂ, and

let u: U→ $[-\infty,\infty)$ be an upper semicontinuous function. Then the following are equivalent.

A) The function u is subharmonic on U

B) Whenever $\bar{\Delta}(\omega,\rho) \subset u$, then for $r < \rho$ and $0 \le t \le 2\pi$

$$u(\omega + re^{it}) \le \frac{1}{2\pi}\int_0^{2\pi}\frac{\rho^2 - r^2}{\rho^2 - 2\rho r\cos(\theta - t) + r^2}\, u(\omega + \rho e^{i\theta})d\theta$$

C) Whenever D is relatively compact subdomain of U, and h is harmonic function on D satisfying $\lim_{z\to\varsigma}\sup(u-h)(z) \le 0 \quad (\varsigma\epsilon\,\partial D)$, then $u \le h$ on D

Proof

(a) → (c); given D and h as in (c), the function u-h is subharmonic on D,so the result follows by the maximum principle, (theorem 2.3.1.b).

(C)→ (b);

Suppose that $\bar{\Delta} := \bar{\Delta}(\omega, \rho) \subset$ u by Theorem 2.1.3 there exist continuous functions

$\emptyset_n : \partial\Delta \to$ R Such that $\emptyset_n \downarrow$ u on $\partial\Delta$.

By Theorem 1.5.4 each $p_\Delta \emptyset_n$ is harmonic on Δ. Also $\lim_{z\to\varsigma} p_\Delta \emptyset_n(z) = \emptyset_n(\varsigma)$ for all $\varsigma \in \partial\Delta$ and hence

$$\lim_{z\to\varsigma} \sup(u - p_\Delta \emptyset_n)(z) \leq u(\varsigma) - \emptyset_n(\varsigma) \leq 0 \quad (\varsigma \in \partial\Delta)$$

From(C) it follows that $u \leq p_\Delta \emptyset_n$ on Δ. Letting $n \to \infty$ gives the desired inequality

Theorem 2.4.2 Let (Ω, μ) be a measure space with $\mu(\Omega) < \infty$, let U be an open subset of $\mathbb{C}$, and let $u: U \times \Omega \to [-\infty, \infty)$ be a function such that :

A) U is measureable on $U \times \Omega$

B) $z \mapsto u(z,\omega)$ is subharmonic on U for each $\omega \in \Omega$;

C) $z \mapsto sup_{\omega\in\Omega} u(z,\omega)$ is locally bounded above on U.

Then $u(z) := \int_\Omega u(z, \omega)\, d\mu(\omega)$is subharmonic on U.

Proof:

It is sufficient to prove that u is subharmonic on each relatively compact subdomain D of U.

Fix such a D. Then (c) implies that $sup_{\omega\in\Omega} u(z,\omega)$ is bounded above on D, so by subtracting a constant, if necessary, we can suppose that $u \leq 0$ on $D \times \Omega$. This legitimizes the use of Fatou's lemma and Fubini's theorem in what follows.

Whenever $\omega_n \to \omega$ in D, then by Fatou's lemma

$$\lim_{n\to\infty} \sup u(\omega_n) \leq \int_\Omega \lim_{n\to\infty} \sup u(\omega_n, \omega) d\mu(\omega)$$

$$\leq \int_\Omega u(w, \omega) d\mu(\omega) = u(w)$$

It follows that u is upper semicontinuous on D.

Also, if $\bar{\Delta}(\omega,\rho) \subset$ D, then by Fubini's theorem

$$\frac{1}{2\pi}\int_0^{2\pi} u(\omega + \rho e^{i\theta}) d\theta = \int_\Omega \left(\frac{1}{2\pi}\int_0^{2\pi} u(\omega + \rho e^{i\theta}, \omega) d\theta\right) d\mu(\omega)$$

$\geq \int_\Omega u(w, \omega) d\mu(\omega) = u(w)$, so that u satisfies the submean inequality on D.

Chapter 3 Potential Theory

3.1. Potentials

Potentials plays at least two roles [9,10]. Firstly they provide an important source of examples of subharmonic functions, giving us the means, for instance, of constructing such functions with various prescribed properties .Secondly, despite their apparently rather special nature, which makes them comparatively easy to study, we shall see that potentials turn out to be almost as general as arbitrary subharmonic function, and for many purposes the two classes are equivalent [1,2,9,10]

In this section we define potentials for finite measure of compact support ,so before making this definition we present Borel measure, its support and Regularity.

Defintion3.1.1 Borel measure is a positive measure on the Borel σ-algebera of a topological space. In the Borel measurable space all continuous functions are measurable.

Definition 3.1.2 (Supports),

Let μ be a Borel measure on a topological space X.

The $support\ of \mu, denoted\ by\ supp\mu, is\ the\ set\ \ of\ x \in X\ such\ that\ \mu(U) > 0$ for each open neighborhood $U\ of\ x$.

Definition 3.1.3 (Regularity).

Let μ be a Borel measure on a topological space X. Then μ is called regular if each Borel set B has the following property; given $\varepsilon > 0,$ there exsit an open set U and a closed set F such that $F \subset B \subset U$ and $\mu\left({}^{U}/_{F}\right) < \varepsilon$

Theorem3.1.4 (Weak Identity Principle)

Suppose that u and v are sub harmonic function on an open set U and in ℂ such that u = v a.e on u then u ≡ v on U.

Definition 3.1.5

Let μ be a finite Borel measure ℂ with compact support.

Its potential is the function $p_{\mu}: \mathbb{C} \longrightarrow [-\infty, \infty)$ defined by $p_{\mu}(z) = \int_{\mathbb{C}} \log|z - \omega| d\mu(\omega)$

Definition3.1.6 (Newtonian Potential) Let $n \geq 3$ and consider μ being a finite Borel measure on R^n with compact support. The Newtonian potential of μ denoted by p_μ is a function from

$$p_\mu : \mathbb{R}^n \longrightarrow [-\infty, \infty) \text{ defined as}$$

$$p_\mu(X) = -\int_{R^n} \frac{1}{|w-x|^{n-2}} d\mu(w)$$

Example. Suppose for this example that we are working in R^n and let us try to find

a harmonic function on the annuls $B\,(0,r_1,r_2) = \{x\in R^n : r_1 < \|x\| < r_2\}$

First we assume that u is radial and write $u_*(r)= u(x)$ for $|x| = r$.

We know that u must be differentiable if it exists and its Laplacian is given by a simple calculation as

$$\Delta u\,(X) = \frac{d^2u_*}{dr^2}(|x| + \frac{n-1}{r}\frac{du_*}{dr}(|x|)$$

Hence starting from a partial differential equation, Find u such that $\Delta u = 0$ we have obtained an ordinary second order differential equation for u_*

Note that the geometry of the domain plays a crucial role in this set up. In any cases the ODE can easily be solved to give,

$$U(x) = \begin{cases} A\log\frac{1}{|x|} + B \text{ if } n = 2 \\ \frac{1}{|x|^{n-2}} + B \text{ if } n \leq 3 \end{cases}$$

from the example we observe the special role of logarithmic of the absolute value in dimension 2.

It turns out that the analogue for $n \geq 3$ for the potential is not as straight forward as all result in the previous section.

Theorem 3.1.8 The Logarithmic potential p_μ is subharmonic on complex $\mathbb{C}$ and harmonic on complex $\mathbb{C} \setminus \text{supp}\mu$.

$Also \;\; p_\mu(z) = \mu(\mathbb{C})\, log\,|z| + o(|z|^{-1})$ as $z\rightarrow \infty$

Proof

Set K = supp μ, so μ can be regarded as a measure on K. Applying theorem 2.4.2 with

u(z, ω) = $log\ |z - \omega|$ on $\mathbb{C} \times K$, We see that p_μ is subharmonic on complex ℂ.

Applying the same theorem, but with u(z, ω) = $-log\ |z - \omega|$ on $(\mathbb{C}\backslash K) \times K$,

we also find that p_μ is superharmonic on $\mathbb{C}\backslash K$, and hence harmonic there**.**

For the last part, observe that for z$\neq$ 0,

$p_\mu(z) = \mu(\mathbb{C})\ log\ |z| + \int log(1 - \frac{\omega}{z})\ d\mu(\omega)$.

As μ has compact support, the final term is O ($|z|^{-1}$) as z→ ∞

Potential enjoy several properties over and above those displayed by general subharmonic function. Let us see the continuity principle and the minimum principle.

Theorem 3.1.9 (Continuity Principle)

Let μ be a finite Borel measure on ℂ with compact support K

a) If $\varsigma_0 \in k$, then $lim_{z\to\varsigma_0} inf p_{\mu(Z)} = lim_{\varsigma\to\varsigma_0} inf p_{\mu(\varsigma)}$ for $\varsigma \in k$.
b) If further $\varsigma \in k$, $lim_{\varsigma\to\varsigma_0} p_{\mu(\varsigma)} = p_\mu(\varsigma_0)$ then $lim_{z\to\varsigma_0} p_{\mu(Z)} = p_\mu(\varsigma_0)$

Proof

a) If $p_\mu(\varsigma_0) = -\infty$, then by upper semicontinuity $lim_{z\to\varsigma_0} p_\mu(z) = -\infty$ and the result is clear. Thus we can suppose that $p_\mu(\varsigma_0) > -\infty$.
Then necessarily $\mu(\{\varsigma_0\}) = 0$ and
so,given $\varepsilon > 0$ there exists r> 0 such that $\mu(\Delta(\varsigma_0, r)) < \varepsilon$. Given $\varsigma \in \mathbb{C}$, choose $\varsigma \in k$ minimizing $|\varsigma - z|$. Then for all $w \in k$,
$$\frac{|\varsigma - w|}{|z - w|} \leq \frac{|\varsigma - z| + |z - w|}{|z - w|} \leq 2$$
Therefore
$$p_\mu(z) = p_{\mu(\varsigma)} - \int_k log\ |\frac{\varsigma - w}{z - w}|\ d\mu(\omega)$$
$$\geq p_{\mu(\varsigma)} - \varepsilon\ log\ 2 - \int_{k\backslash\Delta(\varsigma_0, r)} log\ |\frac{\varsigma - w}{z - w}|\ d\mu(\omega)$$
As z→ ς_0 in ℂ, the corresponding ς→ ς_0 in k, and hence
$$\lim_{z\to\varsigma_0} inf p_{\mu(Z)} \geq \ \varsigma \in k,\ \lim_{\varsigma\to\varsigma_0} inf p_{\mu(\varsigma)} - \varepsilon\ log\ 2 - 0$$
since ε is arbitrary, the results follows
b) if p_μ satisfies the premise of (b), then by part(a)
$lim_{z\to\varsigma_0} inf p_{\mu(Z)} = p_\mu(\varsigma_0)$.
Also as p_μ is upper semicontinuous,

$lim_{z\to\varsigma_0}\, supp_{\mu(z)} \leq p_{\mu(\varsigma_0)}$). Combing these observations, we deduce that p_μ satisfies the conclusion of (b)

Theorem 3.1.10 (Minimum principle)

Let μ be a finite Borel measure on $\mathbb{C}$ with compact support k. If $p_\mu \geq M$ on k, then $p_\mu \geq M$ on the whole of $\mathbb{C}$.

Proof

Put U = $-p_\mu$ on $\mathbb{C}$ \k. Then u is subharmonic on $\mathbb{C}$ \k and (assuming that $\mu \neq 0$)

u(z) → $-\infty\ as\ z \to \infty$. Also if $z_0 \in \partial k,$ then by Theorem 3.1.9(A)

$$z \in \mathbb{C}\backslash k, \lim_{z\to z_0} sup\ p_\mu \leq \lim_{z\to z_0} inf p_{\mu(Z)} = z\epsilon k, -\ lim_{z\to z_0} inf p_{\mu(Z)} \leq -M.$$

Applying the maximum principle to u on each component of $\mathbb{C}\backslash k$ *, we get* $u \leq -M$ there. Hence $p_\mu \geq M$ on $\mathbb{C}$.

3.2. Polar Set

Polar sets play the role of negligible sets in potential theory, much as sets of measure zero do in measure theory. To define polar sets, we need first to introduce the concept of energy associated with measure (actually, anti-energy, due to our choice of sign of the logarithmic potential, so that minimization of physical energy would correspond maximization of $I(\mu)$ as defined below.)

Definition 3.2.1 Let μ be a finite Borel measure on $\mathbb{C}$ with compact support.

Its energy $I(\mu)$ is given by

$I(\mu)$: = $\iint log\,|z-w| d_\mu$(z)d$\mu(w)$ =$\int p_\mu$(z)d$\mu(z)$.

To explain this terminology, think of μ as being a charge distribution on complex .then p_μ(z) represents the potential energy at z due to μ , and so the total energy of μ is just $\int p_\mu$(z)d$\mu(z)$ in other words $I(\mu)$.

Actually, since like charges repel, most physicists would define the energy as $-I(\mu)$, but definition 3.2.1 will be more convent for us.

It is possible that $I(\mu)$ = $-\infty$ indeed some sets only support measures of infinite energy. These are important enough to deserve a name.

Definition 3.2.2

(A) A subset E of complex $\mathbb{C}$ is called polar if $I(\mu) = -\infty$ for every finite Borel measure $\mu \neq 0$ for which suppμ is a compact subset of E.

B) A Property is said to hold nearly everywhere (n.e) on a subset S of $\mathbb{C}$, if it holds everywhere on S\E for some Borel polar set E.

Clearly singleton sets are polar. Also every subset of a polar set is polar. In the other direction if a set is nonpolar, then it contains a compact subset which is nonpolar (namely suppμ for some measure μ $with$ $I(\mu) > -\infty$. It is easy to see that the measure of finite energy can have no atoms in fact, more generally ,they do not charge polar sets.

Theorem3.2.3 Let μ be a finite Borel measure on $\mathbb{C}$ with Compact Support, and Suppose that I(μ) $> -\infty$. Then $\mu(E) = 0$ for every Borel polar.

Proof.

Let E be a Borel set such that $\mu(E) > 0.$ we shall show that E is not polar. By regularity of μ, we can choose a compact subset k of E with $\mu(k) > 0$)

$$Set\ \tilde{\mu} = \mu \backslash k\ and\ \ d = diam(supp\mu).$$

$then$ $\tilde{\mu}$ is a finite nonzero measure whose support is a compact support of E, and

$$I(\tilde{\mu}) = \int_k \int_k log \frac{|z-w|}{d}\, d\mu(z) d\mu(w) + \mu(k^2) log\, d$$

$$\geq \int_{\mathbb{C}} \int_{\mathbb{C}} log \left| \frac{z-w}{d} \right| d\mu(z) d\mu(w) + \mu(k^2) log\, d \geq -\infty.$$

Hence E is nonpolar.

Corollary 3.2.4

Every Borel Polar set has Lebesgue measure zero.

Proof It is enough to show that, for $\rho > 0,$ the measure d$\mu \coloneqq dA \backslash \Delta(0, \rho)$

has energy $I(\mu) > -\infty$. For then by Theorem 3.2.3, every Borel polar set E has μ measure zero,

$i.e. E \cap \Delta(0, \rho)\ has\ lebesgue\ measure\ zero,$ and the result then follows by letting $\rho \to \infty$. Accordingly,

Fix $\rho > 0$ and let d$\mu = dA \backslash \Delta(0, \rho). Then\ \ for\ z \in \Delta(0, \rho),$

$$p_\mu(z) = \int_{\Delta(0,\rho)} log \left| \frac{z-w}{2\rho} \right| dA(w) + \pi\rho^2 \log(2\rho)$$

$$\geq \int_{t=0}^{2\pi} \int_{r=0}^{2\rho} log \left(\frac{r}{2\rho} \right) \mathrm{rdrdt} + \pi\rho^2\ \ log(2\rho)$$

$$= -2\pi\rho^2 + \pi\rho^2 \, log(2\rho).$$

Hence we have

$$I(\mu) = \int_{\Delta(0,\rho)} p_\mu(z) d\mu(z) \geq (-2\pi\rho^2 + \pi\rho^2 \, log(2\rho)) \, \pi\rho^2 \; - \infty,$$

as desired.

Corollary 3.2.5 Accountable union of Borel polar sets is polar. In particular, every countable subset of $\mathbb{C}$ is polar.

Proof Suppose that $(E_n)n \geq 1$ are Borel polar sets and that E=$U_n E_{n.}$. Let $\mu \; be$ a finite Borel

$measure \; on \; \mathbb{C} \; whose \; support \; is \; acompact \; subset \; of \; E$.

If $I(\mu) > -\infty,$ then by theorem above $\mu(E_n) = 0$ for each n, so $\mu(E) = 0,$ and hence $\mu = 0.$ This shows that E is polar.

We conclude by remarking that, through every countable set is polar, not every polar set is countable.

3.3 Equilibrium Measures

In physics, a charge placed upon a conductor will distribute itself so as to minimize the energy. This suggests looking at probability measure μ on a compact set K which maximize $I(\mu)$.

Definition 3.3.1 Let K be a compact subset of $\mathbb{C}$, and denote by P(K) the collection of all Borel probability measure on K . If there exists $v \in p(K)$ such that

$I(v)$ = $supI(\mu)_{\mu \in p(K),}$ then v is called an equilibrium measure for k.

Theorem 3.3.2 Every Compact set K in $\mathbb{C}$, has an equilibrium measure.

Proof.

Let put M = $supI(\mu)_{\mu \in p(K)}$, and choose a sequence (μ_n) in P(K)

such that $I(\mu_n) \to M$ as n $\to \infty$. By weak *convergent theorem, there is a subsequence μ_{nK} which is $weak^*$ -convergent to some v $\in p(K)$.

By lemma 3.3.3 I(V) $\geq$ $lim_{k\to\infty} \, supI(\mu_{nK})$ = M

So v is an equilibrium measure for K.

Lemma 3.3.3 if $\mu_n \overset{w*}{\rightarrow} \mu$ in p(K), then $lim_{n\to\infty} supI(\mu_n) \leq I(\mu)$.

Proof.

Given continuous function $\phi, \psi \; on \; K$ the definition of $weak^*$ convergence implies that, as n→ ∞,

$$\int_k \int_k \phi(z)\psi(w)d\mu_n(z)d\mu_n(w) \to \int_k \int_k \phi(z)\psi(w)d_\mu(z)d_\mu(w)$$

Now using the Stone –Weierstrass theorem, one can show that every continuous function x(z,w) on KxK can be uniformly approximated by finite sum of the form $\sum_j \phi_j(z)\psi_j(w)$, where the ϕ_j, ψ_j, are continuous function on K.

It follows that for every such x,

$$\int_k \int_k \chi(z,w)d\mu_n(z)d\mu_n(w) \to \int_k \int_k \chi(z,w)d\mu(z)d\mu(w) \;\; as\; n \to \infty.$$

Applying this with x(z,w): = max $(log|z-w|, -m)$, where m ≥ 1, we get

$$\begin{aligned} lim_{n\to\infty} supI(\mu_n) &= lim_{n\to\infty} sup \int_k \int_k log|z-w|d\mu_n(z)d\mu_n(w). \\ &\leq lim_{n\to\infty} \sup \int_k \int_k max(\,log|z-w|, -m)\; d\mu_n(z)d\mu_n(\mathrm{w}) \\ &= \int_k \int_k max(\,log|z-w|, -m)\; d\mu_n(z)d\mu_n(\mathrm{w}). \end{aligned}$$

The result follows upon letting m → ∞ and using the monotone convergence theorem.

Theorem 3.3.4 Frostman's Theorem

Let K be a compact set in ℂ, and let v be an equilibrium measure for K. Then

(a) $p_v \geq I(v) \; on \; ℂ$;

(b) $p_v = I(v)$ on K\E, where E is an F_σ is polar subset of ∂K.

$proof. If I(v) = -\infty\; (i.e, K\; is\; polar\;)\; then\; the\;\; result\;\; is\; obvious, so$

We may as well assume that $I(v) > -\infty$ it is sufficient to proof that.

I. $K_n = \left\{z \in K : p_v(z) \geq I(v) + \frac{1}{n}\right\}$ is polar for each n ≥ 1, and

II. l_n:={z ∈ $suppv$: $p_v(z) < I(V) - \frac{1}{n}$} is empty for each n ≥ 1.

For (ii) then implies that $p_v(z) \geq I(v)$ on $suppv$, and so by the minimum principle we get $p_v \geq I(v)$ on ℂ, which gives (a). Also, if we put E = $u_n K_{n,}$ then (i) together with Corollary 3.2.5 implies that E is an F_σ polar set. Since

$p_v \le I(V) on\ K\setminus E.$ This gives (b), apart from the assertion that E ⊂ ∂K.This last is proved by observing that as E is Polar it must have Lebesgue measure zero. So p_v = I(V) a.e on K, and hence by weak identity principle p_v=I(v) every where on $int(K)$.

It thus remains to prove (i) and (ii), we shall proof (i) by contradiction. Suppose, if possible, that some k_n is non polar choose $\mu \in p(K_n)$ with $I(\mu) > -\infty$.

Since I(v) = $\int p_v\, dv,$ there exists $z_0 \in\ supp\ v$ such that $p_v(z_0) \le I(V)$. By upper continuity, there exists r > 0 such that $p_v < I(V) + {}^1\!/_{2n}$ on $\bar{\Delta}(z_{0,}\ r)$. In particular, $\bar{\Delta}(z_{0,}\ r) \cap K_n$= ∅. As $z_0 \in suppv$ the number a: = v ($\bar{\Delta}(z_{0,}\ r)$) is strictly positive. Define a signed measure $\sigma\ on\ K\ by.$

$$\sigma = \begin{cases} \mu\ onk_n \\ -{}^{v}\!/_{a} \qquad on\ \bar{\Delta}\left(z_{0,}r\right), \\ 0\ other\ wise. \end{cases}$$

Then for each t ∈ $(0, a)$ the measure $v_t := v + t\sigma$ is positive, and there fore $v(t) \in P(K)$

Also, nothing that $I(\mu) > -\infty\ implies\ I(|\sigma|) > -\infty, we\ have$

$$I(v_t) - I(v)$$

$$=2t \int\int log|z - w| dv(w) d\sigma(Z) + t^2 \int\int log|z - w| d\sigma(w) d\sigma(Z)$$

$$=2t\int p_v\,(z) d\sigma(Z) + o(t^2)$$

$$=2t\left(\int_{Kn} p_v(z) d\mu(z) - \int_{\bar{\Delta}(z_0,r)} p_v(z) dv(z)/a + o(t)\right)$$

$$\ge 2t\left(\left(I(v) + \frac{1}{n}\right) - (I(v) + \frac{1}{2n}) + o(t)\right).$$

Therefore $I(v_t) > I(v)$ if t is sufficiently small, contradicting the assumption that v is an equilibrium measure. Hence each $k_n\ is\ polar\ which\ proves\ (i)$.

We shall also prove (ii) by Contradiction. Suppose, if possible, that some L_n is non empty. Pick $z_1 \in L_n$. By upper semi continuity, there exists

Such that $p_v < I(v) - \frac{1}{n}$ on $\bar{\Delta}(z_1, s)$.

As $z_1 \in suppv,$ the number b: = v($\bar{\Delta}(z_{1,}s)$ is strictly positive. Now by (i) and Corollary3.3.4 $v(k_n)$ = 0 for each n, and so $p_v \le I(v)\ v -$ almost everywhere on k.hence

$$I(v)=\int_k p_v dv$$

$$= \int_{\bar{\Delta}(z_{1,S})} p_{v\ dv} + \int_{k\setminus\bar{\Delta}(z_{1,s})} p_v dv \ \le \left(I(v) - \frac{1}{n}\right) b + I(v)(1 - b)$$

$< I(V)$

Which is obviously a Contradiction, Hence each L_n is empty, which gives (ii) and completes the proof.

3.4. Minus Infinity Sets

If u is subharmonic on a domain and $u \not\equiv \infty$, then the set where $u \neq -\infty$ has Lebesgue measure zero. We are now measure in a position to prove a much stronger result.

Theorem 3.4.1 Let u be a subharmonic function on a domain D in $\mathbb{C}$,with u $\neq -\infty$, then

E := $\{z \in D: u(z) = -\infty\}$ is a G_σ polar set.

Proof. Since E=$\bigcap_n \{z: u(z) < -n\}$, it is certainly a$G_\sigma$ set. To show it is Polar, put $v = lim_{n\to\infty}({}^{u}/_{n})$, so that

$$v(z) = \begin{cases} 0, z \in D\backslash E, \\ -\infty, z \in E. \end{cases}$$

v^* is sub harmonic on D, since it evidently attains a maximum value 0 there, it follows that $v^* \equiv 0$ on D . Also $v^* = \mathrm{v}\ n.e\ on\ D$. Therefore v = 0 n.e on D, and E is indeed Polar.

Theorem 3.4.2 Let E be an Fσ is polar set, and Let F be an Fσ set disjoint from E. Then there exists a sub harmonic function U: $\mathbb{C} \to [-\infty, \infty)$ such that

$u = -\infty\ on\ E\ and\ u > -\infty\ on\ F.$

Lemma 3.4.3 Let E be a compact Polar set, and Let F be a compact set disjoint from E. then there exists a Borel Probability measure $\mu\ on\ \mathbb{C}$ with compact support such that

E = $\{z \in \mathbb{C}: p_\mu(z) = -\infty\ and\ supp\mu \cap F = \emptyset\}$.

Proof.

Let (K_n) n$\geq$ 1 be Sequence of compact sets , with $K_{n+1} \subset int(K_n)$ for all n , such that $\cap_n k_n = E$ and $K_1 \cap F = \emptyset$. for each n, Let v_n be an equilibrium measure for $K_{n.}$ note that $I(v_n) > -\infty\ since\ int(k_n) \neq \emptyset. now\ v_n \in p(K_1)$for all n, so by weak*convergent theorem of the preliminary there is a subsequence of the (v_n) (which , by relabeling, we may assume to be the whole sequence) that is $weak^* - convergent\ to\ some\ v \in p(K_1)$. In fact, since $suppv_n \subset k_n$ for each n, we must have supp$v \subset E$ as E is Polar, it follows that

I(v) $= -\infty$ Hence by lemma 3.3.3 $I(v_n) \to -\infty$) as n→ ∞ $and\ so,$ replacing v_n by further subsequence , we can Suppose that $I(v_n) < -2^n$ for each n .put $\mu = \sum_1^\infty 2^{-n} v_n$ then $\mu \in p(k_1),$ so supp$\mu \cap F = \emptyset, and$ to finish the proof we shall show that $p_\mu(z) = -\infty\ if\ and\ only\ if z \in E.$

First Suppose that z ∈ $E\ then\ z \in int(K_n)$ for each n, so $p_{v_n}(z) = I(v_n\ ) < -2^n$.hence

$p_v(z) = \sum_1^\infty 2^{-n} pv_n(z) \leq \sum_1^\infty 2^{-n}(-2^{\text{n}}) = -\infty$

Now suppose that z ∉ E.choose n_0 such that z ∉ kn_0 and put $\sigma = dist(z, Kn_0)$. for all n ≥ $n_{0.}$

$pv_n(z) \geq \int log\ \delta dv_n = \ log\ \delta$ And also by theorem 3.4.4(A)

$pv_n(z) \geq I(v_n) > -\infty\ for\ every\ n.$

$$hence\ \ p_v(z) = \sum_1^\infty 2^{-n} pv_n(z) \geq \sum_1^{n_0-1} 2^{-n} I(v_n) + \sum_{n_0}^\infty -2^n\ log\ \delta > -\infty$$

This completes the proof of lemma.

The proof of the above theorem

Write E as U_nE_n and F as U_nF_n , where (E_n) and (F_n) are increasing sequences of compact sets. By the above lemma for each n there exsits a Borel probability measure μ_n with compact support such that

$E_n = \{z: p\mu_n(z) = -\infty\}$ And supp$\mu_n \cap F_n = \emptyset$ then $p\mu_n$ is bounded above on Δ(0,n) and below on $F_{n,}$ so we can choose constants $\alpha_n > 0$ and $\beta_{n\ \epsilon R}$ such that $u_n := \alpha_n p_{\mu n} + \beta_n$ satisfies $\underset{\Delta(0,n)}{sup\ u_n} < 0$ and $\underset{F_n}{inf\ u_n} > -2^{-n}$

Put U = $\sum_1^\infty u_{n.}$ then on any bounded set, the sequence of partial sums is eventually decreasing, and U is subharmonic on complex$\mathbb{C}$. Also if z ∈ E,then

U_n (z) $> -\infty$

For Some n, and so u (Z) $= -\infty$. Finally if z∈ $F, then$ u_n (Z) $> -\infty$ for all sufficiently large n, whence u (Z) $> -\infty$

3.5. Removable Singularities

It is thus of interest to determine in what ways polar sets are negligible. The key to this is the following removable singularity

Theorem3.5.1 (Removable Singularity Theorem).

Let U be an open subset of ℂ, Let E be a closed Polar set , and Let U be a Subharmonic function on U\E. Suppose that each point of U$\cap E$ has a neighborhood N such that u is bounded above on N\E. Then u has a unique Subharmonic extension to the whole of U.

Proof. Uniqueness follows immediately from the weak identity principle theorem Since E has a measure zero. To construct the extension, we define U on U$\cap E$ by $\underset{z\epsilon u\backslash E}{lim_{z\to w}} supu(z)$ $(w\epsilon U\cap E)$.

The boundedness condition ensures that u $<\infty$ everywhere, and so u is upper semi continuous on u. To check that it is subharmonic, we shall use theorem 2.4.2(c). Let D be relatively Compact sub domain of U, and let h be a harmonic function on D such that

$lim_{z\to\varsigma} sup(u-h)(z)\leq 0$ for all $\varsigma\in\partial D.$

We need to show that u $\leq h$ on D. now by Corollary 3.5.4 there exists a Subharmonic function u on ℂ such that E=$\{z\colon u(z)=-\infty\}.for\ each\ \varepsilon>0$,the function u$-h+\varepsilon u$ is certainly sub harmonic on D\E, and equals$-\infty$ on E, so in fact it is subharmonic on the whole of D. Therefore by the maximum principle $u-h+\varepsilon u\leq \underset{\partial D}{sup(\varepsilon u)} onD$

Letting $\varepsilon\to 0$ we deduce that u $\leq h$ on D\E .From the way that U is defined on D$\cap E$,it follows that u $\leq h$ there too. Hence u $\leq h\ on$ D, as required.

Corollary 3.5.2 Let U be an open subset of ℂ ,Let E be a closed polar set , and Let h be a harmonic function on u\E. Suppose that each point of U$\cap E$ has a neighborhood N such that h is bounded on N\E. then h has a unique harmonic extension to the whole of U.

Proof

Uniqueness is clear. For the existence, apply theoerem3.6.1 to $\pm h$ to obtain function u and v which are subharmonic on U , which agree respectively with h and $-h$ on U\E. then u+v is sub harmonic on U and u+v = 0 on U\E, So by the weak identity principle u+v = 0 on the whole of U. therefore u is Superharmonic on U as well as being Subharmonic, and hence it is harmonic there. Thus it is the desired extension of h.

The removable singularity theorem can used to demonstrate a further sense in which Polar sets are small.

Theorem 3.5.3 Let D be a domain in ℂ and Let E be closed Polar set .then D\E is still connected.

Proof .Suppose that D\E = A∪B, where A and B are disjoint open sets.

Define u: D\E→ $[-\infty,\infty)\ by$ U= $\begin{cases} 0\ \ on\ A, \\ -\infty\ on\ B. \end{cases}$ by theorem 3.5.1 U has a subharmonic extension to the whole of D .it then follows, if B$\neq\emptyset$ then U $\equiv -\infty$ on D, and so A $=\emptyset$. Hence D\E is connected.

Corollary3.5.4 Every closed polar set E is totally disconnected.

Proof: we need to show that if ω $\in E$, then its component in E is just {ω}.

Without loss of generality, we may assume that ω = 0. Let $\varepsilon > 0\ and\ set$

$$\Delta = \Delta(0,\varepsilon), \qquad \Delta^+ = \Delta\backslash[0,\varepsilon), \quad \Delta^- = \Delta\backslash(-\varepsilon,0].$$

Choose $\omega_1,\omega_2 \in \Delta\backslash E\ with\ Im(\omega_1)\ > 0and\ Im(\omega_2) < 0$. theorem3.5.3 both Δ^+\E and Δ^-\E are connected , so we can join ω_1 to ω_2 by path γ^+ in $\Delta^+\backslash E$, and ω_2 to ω_1 by path $\gamma^- in\ \Delta^-\backslash E. then\ \gamma := \gamma^+ \cup \gamma^-$ is a closed path in $\Delta\backslash E$

Which winds once around 0. It must therefore also wind once around every point in the same component lies inside the disc $\Delta(0,\varepsilon)$, and as ε is arbitrary ,the component is just $\{0\}$.

Theorem 3.5.5(Rado-Stout Theorem) Let D be a domain in complex ℂ ,let E be a closed polar set ,and let f:D → ℂ be acontinuous function which is holomorphic on D\ (E).

Then f is holomorphic on the whole of D.

Proof .If f(D) ⊂ E then ,as f(D) is connected and E is totally disconnected ,it follows that f is constant ,in which case the result is obvious. From now on, we suppose that f(D) ⊄ E.

By Corollary 3.4.4 there exist a subharmonic function u on ℂ such that E = $\{z: u(z) = -\infty\}$ then u **o** f is subharmonic on D\$f^{-1}(E)$, and equals$-\infty$ on $f^{-1}(E)$, so it is subharmonic on the whole of D,so by theorem 3.4.1 $f^{-1}(E)\ is\ polar. we\ can\ now\ apply\ corollary$ 3.5.2 to Ref and Imf to deduce that they are in fact harmonic on D, and hence that f ∈ $ℂ^\infty(D)$.

Since f satisfies the Cauchy-Riemann equations on D\E, by continuity it must also do so on E, and hence it is holomorphic on D

Theorem3.5.6 (Extended liouville Theorem)

Let E be a closed polar subset of ℂ and Let u be a subharmonic function on ℂ \E which is bounded above. Then u is constant.

Proof: by Theorem 3.5.1. U extends to be subharmonic on the whole of ℂ .

Moreover, if M = sup **C**\E$^{\mathrm{u}}$, then max (u, M) on $\mathbb{C}$ \E, and hence everywhere on complex $\mathbb{C}$ by weak identity Principle theorem in the preliminary. Therefore u is bounded above on complex $\mathbb{C}$, and so applying by **(**Liouville's theorem) we deduce that u is constant.

Corollary3.5.7 Let E be Closed Polar subset of $\mathbb{C}$, and Let f be holomorphic function on $\mathbb{C}$ \E such that $\mathbb{C}$ \f($\mathbb{C}$) is non polar. Then f is constant.

Proof .

Choose a compact non non polar set K such that F($\mathbb{C}$ \E)⊂ $\mathbb{C}$ \K, and let v be an equilibrium measure for K .then $\mathrm{p_v}$ is harmonic and bounded below on $\mathbb{C}$ \K ,so $-p_v$ of harmonic and bounded above on $\mathbb{C}$ \E. hence by theorem 3.5.6$-\mathrm{p_v}$ of f is Constant. as $\lim_{z\to\infty} \mathrm{p_v}(Z) = \infty$ this implies that f is bounded on $\mathbb{C}$ \E. applying theorem 3.6.5 again ,this time to Ref and Imf, we deduce that f is constant.

Theorem 3.5.8 (Extended Maximum Principle)

Let D be a domain in Complex $\mathbb{C}$, and Let u be a subharmonic function on D which is bounded above.

(a) If ∂D is polar , then u is constant.
(b) if ∂D is non –polar and $\lim_{z\to\varsigma}\sup(z) \leq 0$ for n.e. $\varsigma \in \partial$D, then $u \leq 0$ on D.

Proof.

(a) Put E = ∂D\{∞}.Then E is a closed polar subset of $\mathbb{C}$, so

By theorem 3.5.3 $\mathbb{C}$ \ **E** is connected. Since D is a component of $\mathbb{C}$ \ E, it follows that in fact

D= $\mathbb{C}$ \E**.** The result is now an immediate consequence of theorem 3.5.6

(b) Given $\varepsilon > 0$, define $E_\varepsilon = \{\varsigma \epsilon \partial D \backslash \{\infty\} : \lim_{z\to\varsigma} sup u(z) \geq \varepsilon\}$

Then E_ε is a Closed polar subset of $\mathbb{C}$. Define v on $\mathbb{C}\backslash \mathrm{E}_\varepsilon$ by

$$\mathrm{v} = \begin{cases} \max(\mathrm{u}, \varepsilon) \text{ on D}, \\ \varepsilon \text{ on C } (\mathrm{D} \cup \mathrm{E}_\varepsilon\ ). \end{cases}$$

v is sub harmonic on $\mathbb{C}$ \E$_\varepsilon$, and it is clearly bounded above there, so by theorem 3.6.5 it is constant. Since $\mathrm{v} = \varepsilon$ on $\partial\mathrm{D}\backslash(\mathrm{E}_\varepsilon \cup \{\infty\})$ which is non empty, it follows that $\mathrm{v} \equiv \varepsilon$.

Hence $\mathrm{u} \leq \varepsilon$ on D , and letting $\varepsilon \to 0$ we deduce that $\mathrm{u} \leq 0$ on D.

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