Graduate Seminar Report

On

Convergence of Fourier series

[In partial fulfillment of the M.SC degree in Mathematics]

By

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Dedicated to

My Father

ACNOWLEDGMENT

This report actually stars by giving some historical background and the importance of Fourier series, so as to present some sort of motivation for mathematician on the subject matter.

The main objective of this seminar report is to set the sufficient condition for the convergence of Fourier series of functions. In fact it also delivers a counter example which displays the converse of the statement is not necessarily true.

Explaining this much about the objective of this seminar report,First of all I wish to express my thank to God. After that my deepest gratitude goes to my lecturers in the department. Especially to **Dr. Seid Mohammed** , my advisor, with whom I had many consultations and from whom I have had many valuable comments and suggestions.

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Lastly but not the least, it is my pleasure to thanks my family, **Worke**, **Feleke**, **Hiwot, Israel** and **Etagu** and I want to say I love you all.

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SECTION I

INTRODUCTION

)7, Joseph Fourier, a mathematician and an engineer discovered his series in connection heory of heat conduction. Fourier claimed that an arbitrary function, defined in a finite al by an arbitrary capricious graph, can always be resolved in to a sum of pure sine and e function. But further fundamental investigation on Fourier's discovery were made by mathematicians and finally in 1829, L. Dirichlet circumscribed the function which allows sion in Fourier series and set up the theorem of sufficient condition for the convergence of er series.

ieory of representation of functions of real variable by means of Fourier series is of highest tance not only on account of facts that such mode of representation is at present an ensible tool in the various branches of mathematical physics, but also because this theory periences the most far reaching influence up on the development of modern mathematical sis. It is a significant fact that the theory of this mode of representation of function by a ometric series had its own origin in the attempt to investigate the form of a stretching string ate of vibration.

nometric and Fourier series constitutes one of the oldest parts of analysis. They arose, for ce in classical studies of heat and wave equations. Today they play a central role in the of sound, heat condition, electromagnetic waves, mechanical vibrations, signal processing nage analysis and compression.

er series can be used to represent very broad class of functions. For instance trigonometric is one of the forms:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n Cos\, nx + b_n Sin\, nx)$$

hall be concerned with three main questions in this seminar paper report:

Given a function f, how do we calculate the coefficients a_n *and* b_n?

Once the series of f has been obtained, when does the series converges and it converges to the function value?

Is there a continuous function which is continuous whose Fourier series diverges?

dition to all those questions listed above, we will also discuss some basic properties of er coefficients of functions. And the result will be very important on answering the last two ions.

to answer the above three basic questions, let us start by revising and developing some :ntary concepts from the real analysis in the coming section.

SECTION II

REVISION ON REAL ANALYSIS

Periodic functions

nction f issaid to have a period T>0, or simply periodic with period T>0, if

$$f(x+T) = f(x)$$

ach x in the domain of the definition of f. (It is understood that both x an x + T lies in the in)

nstance, $2\pi, 4\pi, 6\pi, ...$ are periods of the function $f(x) = Sinx$ and also $\frac{2\pi}{n}$, n>0 is a period $n\, nx \;\; and \;\; Cos\, nx$

arks

. If T is a period of f, then obviously, $f(x+nT) = f(x)$ for all $n \in Z$ and for all x in the domain of f.
. If we plot a periodic function $y = f(x)$ of period T on a closed interval $a \le x \le a+T$, we can obtain the entire graph of f by a corrospondingperiodic repetition of the portion of the graph corresponding to $a \le x \le a+T$
. If $f \;\; and \;\; g$ are periodic of period T, then their sum $f+g$ is also of period T.

rem 2.1.1

 be a periodic function of period T. Suppose that for some t_0, the integral $\int_{t_0}^{t_0+T} f(t)dt$ s. Then for every real number r, the integral $\int_{t_0}^{t_0+T} f(t+r)dt$ exists and for every t_1, the ral $\int_{t_1}^{t_1+T} f(t)dt$ exists and these integrals are with the same value.

f

 to show this WLOG, let us assume that $0 < r < T$, and then we have:

$$\int_{t_0}^{t_0+T} f(t)dt = \int_{t_0}^{t_0+r} f(t)dt + \int_{t_0+r}^{t_0+T} f(t)dt = \int_{t_0}^{t_0+r} f(t+T)dt + \int_{t_0+r}^{t_0+T} f(t)dt$$

considering $x = t + T - r$ *and* $x = t - r$ in the last two respective integrals, we have

$$\int_{t_0}^{t_0+r} f(t+T)dt = \int_{t_0+T-r}^{t_0+T} f(x+r)dx \text{ and}$$

$$\int_{t_0+r}^{t_0+T} f(t)dt = \int_{t_0}^{t_0+T-r} f(x+r)dx$$

:fore

$$\int_{t_0}^{t_0+T} f(t)dt = \int_{t_0}^{t_0+T-r} f(x+r)dx + \int_{t_0+T-r}^{t_0+T} f(x+r)dx$$

$$= \int_{t_0}^{t_0+T} f(x+r)dx$$

to show the second statement

sidering $r = t_1 - t_0$, in the above result we get

$$\int_{t_0}^{t_0+T} f(x + (t_1 - t_0)dx = \int_{1}^{t_1+T} f(t)dt$$

we have the result to be all the same.

Periodic extension

sider a function f which is defined on an interval$[a, b]$, where nothing has been said about dicity of the function f. Here we can extend f by periodicity from the interval $[a, b]$ in to vhole of the x-axis. This leads to a periodic function which coincides with f on the val$[a, b]$. Here this function, which is obtained by extending the function $f$on to the whole e x-axis by extension, is called periodic extension of f along the x-axis.

nnection with the problem of extending f by periodicity from the interval$[a, b]$ on to the e of x-axis, the following two cases have to be considered.

l: $f(a) = f(b)$

s case there is no difficulty in making the periodic extension of f. We can simply obtain the dic extension by just repeating the graph of f on the interval$[a, b]$. (see Example 2.2.1, the igure)

; continuous on$[a, b]$, then its extension is continuous on the whole of x-axis.

2: $f(a) \neq f(b)$

s case, we can not accomplish the required periodic extension of f without changing the of $f(a)$ *and* $f(b)$, since periodicity requires the equality of the two values.

e to have the periodic extension of f, we may consider two approaches as given below:

Completely avoid considering the values at the end points.

Suitably modifying the values of the function at a *and* b so that $f(a) = f(b)$,(see Example 2.2.1, the second figure)

on while we discuss on the issue of Fourier series of functions, we will encounter this ement is necessary and these two modifications on the function will not affect on ating the Fourier coefficients of the function.

iple2.2.1

der the following graphs of functions on the interval $[-\pi, \pi]$ and their corresponding le periodic extensions. Here the right side shows the function whereas the left shows the ponding periodic possible periodic extension.

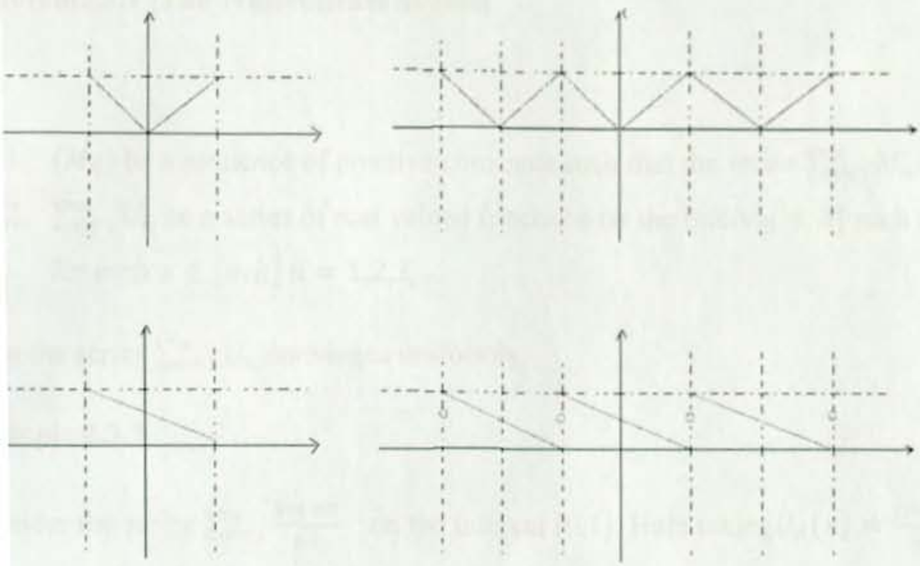


Figure 2.2.1

Revision on real analysis

nition 2.3.1 [Uniform convergence]

ıider an infinite series of real valued function$\sum_{k=1}^{\infty} f_k(x)$ on the interval$[a,b]$. Such a series d to converge for a given value of x on the interval, if its partial sum

$$s_n(x) = \sum_{k=1}^{n} f_k(x) \ \ (n = 1,2,3, \dots)$$

finite limit $s(x) = \lim_{n\to\infty} s_n(x)$

series converges for each$x \in [a,b]$, then its sum $s(x)$ is defined on the whole of the ᐟal$[a,b]$.

eries is said to converge to $s(x)$ uniformly on$[a,b]$, if and only if given $\varepsilon > 0$, there exist a ve integer $N = N(\varepsilon)$,such that $|S_n(x) - s(x)| < \varepsilon$,for each $n \geq N$ and $x \in [a,b]$

we give the following two basic theorems without proof, where one can get their proof different analysis text books. (Refer the bibliography at the end of this report.)

:orem2.3.1 [The Weierestrass M-test]

1. $\{M_n\}$ be a sequence of positive constants such that the series $\sum_{n=1}^{\infty} M_n$ converges
2. $\sum_{n=1}^{\infty} U_n$ be a series of real valued functions on the interval$[a, b]$ such that $|U_n(x)| \leq M_n$ for each $x \in [a, b]$ $n = 1,2,3, \ldots$

ı the series $\sum_{n=1}^{\infty} U_n$ converges uniformly.

mple 2.3.1

sider the series $\sum_{n=1}^{\infty} \frac{Sin\, nx}{n^2}$ on the interval [0,1]. Here taking$U_n(x) = \frac{Sin\, nx}{n^2}$, then since $x)| = \left|\frac{Sin\, nx}{n^2}\right| \leq \frac{1}{n^2}$ and the series$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, then by the Weierestrass M-test the s$\sum_{n=1}^{\infty} \frac{Sin\, nx}{n^2}$ converges uniformly on the interval [0,1]

ırem2.3.2

ider the infinite series $\sum_{n=1}^{\infty} U_n$ on the interval [a, b], where U_n are real valued functions. If :rms of the series are continuous on [a, b] and if the series converges uniformly on [a, b],

. The sum of the series is also continuous on [a, b] and
The sum can be integrated term by term
i.e. $\int_a^b \left(\sum_{n=1}^{\infty} U_n(x)\right) dx = \sum_{n=1}^{\infty} \int_a^b U_n(x) dx$

ition 2.3.2 [Smooth and piecewise smooth]

ction f is said to be smooth on [a, b], if it has a continuous derivative on [a, b]. In other :, the derivative of the function changes continuously without jump.

ınction f is said to be piecewise smooth on [a, b], if either f and its derivative are both ıuous on [a, b] or they have finite number of jump discontinuities on [a, b]. i.e. It is easy to at the graph of piecewise smooth function is either a continuous or discontinuous curve has finite number of corners at which the derivatives have jump.

mple 2.3.2

sider the following graphs on the interval [a, b]

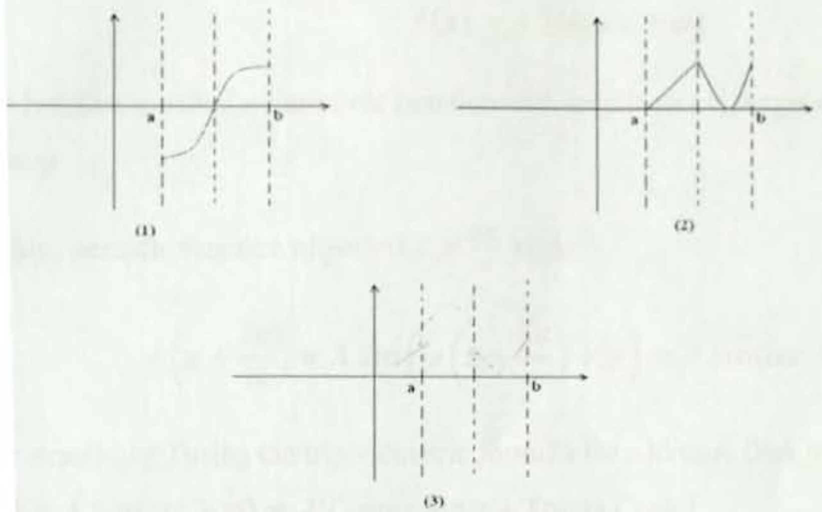


Figure 2.3.1

: above figure, we can observe that (1) is a smooth graph whereas (2) & (3) are piecewise th graphs. Moreover, (2) is continuous piecewise smooth, whereas (3) is discontinuous wise smooth.

ark

tinuous or discontinuous function defined on the whole of the x-axis is piecewise smooth, piecewise smooth on every interval of finite length.

ple 2.3.3

der a function f on the interval$[-\pi, \pi]$. Here

If$f(-\pi) = f(\pi)$, then the periodic extension of f on the whole of x-axis is continuous piecewise smooth.

If $f(-\pi) \neq f(\pi)$, then the periodic extension of f on the whole of x-axis is discontinuous piecewise function.

Harmonic and Trigonometric series

: simplest periodic function and one of the greatest important for application is the function

$$f(x) = A\,Sin(\omega x + \varphi)$$

s function is called a Harmonic function with amplitude $|A|$, angular frequency ω and initial se φ

also periodic function of period $T = \frac{2\pi}{\omega}$ since

$$f\left(x + \frac{2\pi}{\omega}\right) = A\,Sin\left(\omega\left(x + \frac{2\pi}{\omega}\right) + \varphi\right) = A\,Sin(\omega x + \varphi) = f(x)$$

/ expanding f using the trigonometric formula for addition, then we have

$) = A\,Sin(\omega x + \varphi) = A[Cos\omega x\,Sin\varphi + Sin\omega x\,Cos\varphi]$

$$= A\,Sin\varphi\,Cos\,\omega x + A\,Cos\varphi\,Sin\omega x$$

$$a = A\,Sin\varphi \quad and \quad b = A\,Cos\varphi$$

; we have

$$f(x) = aCos\omega x + bSin\omega x$$

:rally, we consider ourselves that every harmonic function can be represented in the form of

$$a\,Cos\omega x + b\,Sin\omega x.$$

if we let period of the harmonic function is $T = 2l$, then $\omega = \frac{\pi}{l}$

e harmonic function of period $2l$ is given by the form

$$a\,Cos\frac{\pi x}{l} + b\,Sin\frac{\pi x}{l}$$

now $T = 2l$ and consider the sequence of harmonics

$$\left\{a_k \, Cos\frac{\pi kx}{l} + b_k Sin\frac{\pi kx}{l}\right\}, k = 1,2,3, \dots$$

he above harmonics, the frequency of each harmonics is given by $\omega_k = \frac{\pi k}{l}$ and hence the od of each harmonics is thus

$$T_k = \frac{2\pi}{\omega_k} = \frac{2l}{k}$$

refore

$$T_k = \frac{T}{k}$$

s

$$T = kT_k$$

:e $T = 2l$ is the period of each harmonics as T is an integral multiple of period of each ıonics. (see the remarks in chapter2.1)

let's consider the finite series given by the form

$$S_n(x) = A + \sum_{k=1}^{n}\left(a_k \, Cos\frac{\pi kx}{l} + b_k Sin\frac{\pi kx}{l}\right), \text{ where } A \text{ is constant.}$$

·ly S_n is a periodic function with period $T = 2l$. Because A is constant and also the others are : same period T and hence their sum. (for the same reason)

'unction S_n is called a trigonometric polynomial of order n and period $2l$.

over, the infinite series

$$A + \sum_{k=1}^{\infty}\left(a_k \, Cos\frac{\pi kx}{l} + b_k Sin\frac{\pi kx}{l}\right)$$

ed an infinite trigonometric series or simply a trigonometric series.

re the main question that we need to answer at the end of this seminar is: "Can any given ction of period $T = 2l$ *be represented as a sum of trigonometric series?"*

shall see later that for very wide class of functions such kind of series representation is sible.

v consider f belongs to this class. Hence we have:

$$f(x) = A + \sum_{k=1}^{\infty}\left(a_k\, Cos\frac{\pi kx}{l} + b_k Sin\frac{\pi kx}{l}\right)$$

$$x = \frac{lt}{\pi}$$

s we get

$$f\left(\frac{lt}{\pi}\right) = A + \sum_{k=1}^{\infty}(a_k\, Cos\, kt + b_k Sin\, kt)$$

set

$$\varphi(t) = f\left(\frac{lt}{\pi}\right)$$

clearly, φ is of period 2π whereas f is of period $2l$

The Trigonometric system, Auxiliary integrals and Orthogonality

e basic trigonometric system we mean the system of functions

$s\ x, Sin\ x, Cos\ 2x, Sin\ 2x, Cos\ 3x, Sin\ 3x, \ldots.$

e following auxiliary integrals can be justified quite easily using the basic integration
hniques and it is just left as an exercise for the reader to check.

1. $\int_{-\pi}^{\pi} Cos\, nx\, dx = 0 \quad \int_{-\pi}^{\pi} Sin\, nx\, dx = 0$
2. $\int_{-\pi}^{\pi} Cos^2 nx\, d = \pi \quad \int_{-\pi}^{\pi} Sin^2 nx\, d = \pi$
3. $\int_{-\pi}^{\pi} Cosnx\, Cosmx\, dx = 0 \quad \int_{-\pi}^{\pi} Sin\, nx\, Sin\, mx\, dx = 0$

   For $n \neq m$
4. $\int_{-\pi}^{\pi} Sin\, nx\, Cosmx\, dx = 0$

ce the formulas (1),(3)&(4) shows that the product of any two different function from the
onometric system above vanishes.

Now we shall agree to call two functions$\varphi(x)$ & $\gamma(x)$orthogonal on the interval [a, b], if

$$\int_a^b \varphi(x)\, \gamma(x) dx = 0.$$

s with this definition, we can say that the functions of the trigonometric system above are
wise orthogonal on the interval$[-\pi,\pi]$. Or more precisely, the system above is orthogonal
$-\pi,\pi]$.

ark

e know from Theorem2.1.1, the integral of periodic functions is the same over any interval
e same length to the period. Therefore formulas (1) up to (4) are valid not only over $[-\pi,\pi]$
lso for any interval of the form$[a, a+2\pi]$. Thus the system is orthogonal on every such
val.

SECTION III

FOURIER SERIES [EXPANSION]

w here on this chapter we will have some sort of discussion on the issue of Fourier series of ctions with period2π,$2T$ and also their complex form. In this chapter we only focus on ırier series of functions, and we do not raise anything about the criteria that when this series l converges to the value of the function. This question will be answered in the coming pters.

Fourier series of functions of period 2π

sider a function $f \in L[-\pi,\pi]$ and with a period of2π. Then from the trigonometric series ıt of view (see chapter 2.4.1), set

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\,Cos\,kx + b_k Sin\,kx) \qquad (1)$$

' let us determine the coefficients $a_{0,}\ a_k\ and\ b_k$ by assuming that the series converges ɔrmly to the limit function f and hence term by term integration is possible.

ɟrating both sides from $-\pi$ to π and using the auxiliary integrals on chapter 2.5, we have

$$\int_{-\pi}^{\pi} f(x)dx = \frac{a_0}{2}\int_{-\pi}^{\pi} dx + \sum_{k=1}^{\infty}\left(a_k\int_{-\pi}^{\pi} Cos\,kx\,dx + b_k\int_{-\pi}^{\pi} Sin\,kx\,dx\right)$$

e

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \qquad (2)$$

ı multiplying (1) by $Cos\,nx$ and Integrating both sides from $-\pi$ to π we have

$f(x)Cos\ nxdx$

$$= \frac{a_0}{2}\int_{-\pi}^{\pi} Cos\ nxdx + \sum_{k=1}^{\infty}\left(a_k \int_{-\pi}^{\pi} Cos\ nxCos\ kx\ dx + b_k \int_{-\pi}^{\pi} Cos\ nxSin\ kx\ dx\right)$$

nce using the result in (chapter2.5, auxiliary formulas) we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Cos\ nxdx \quad n = 1,2,3 \dots \tag{3}$$

e again multiplying once again (1) with $Sin\ nx$ and Integrating both sides from $-\pi$ to π we e

$$f(x)Sin\ nxdx = \frac{a_0}{2}\int_{-\pi}^{\pi} Sin\ nxxdx$$

$$+ \sum_{k=1}^{\infty}\left(a_k \int_{-\pi}^{\pi} Sin\ nx\ Cos\ kx\ dx + b_k \int_{-\pi}^{\pi} Sin\ nx\ Sin\ kx\ dx\right)$$

s we get

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Sin\ nxdx \quad n = 1,2,3 \dots \tag{4}$$

efore combining the three results, we get

$$\begin{cases} a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Cos\ nxdx & n = 0,1,2,3 \dots \\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Sin\ nxdx & n = 1,2,3 \dots \end{cases} \tag{5}$$

finition 3.1.1

f $f \in L[-\pi,\pi]$ and with period 2π then the series in (1) with the coefficient in (5) is called the ırier series of the function f. And the coefficients in (5) are called the Fourier coefficients of function.

tation

$f \in L[-\pi,\pi]$ and with period 2π. If we form the Fourier series of f without deciding in ance whether it converges to $f(x)$, we write

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \, Cos\, kx + b_k Sin\, kx) \qquad (6)$$

this notation (6) only means that the Fourier series written on the right side corresponds to unction f. The sign "$\sim$" can be replaced by " $=$ " only if we succeed in proving that the :s converges and the sum equal to $f(x)$.

ıark

lentally, we note that the formula in (5) involves integrating the function of period 2π. Thus ıterval of integration $[-\pi,\pi]$ can be replaced by any other interval of length 2π so that her with the formula in (5) we have

$$\begin{cases} a_n = \dfrac{1}{\pi}\displaystyle\int_a^{a+\pi} f(x) Cos\, nx dx & n = 0,1,2,3 \ldots \\ b_n = \dfrac{1}{\pi}\displaystyle\int_a^{a+\pi} f(x) Sin\, nx dx & n = 1,2,3 \ldots \end{cases} \qquad (3)$$

rem 3.1.1

ınction $f \in L[-\pi,\pi]$ and with period 2π can be expanded in a trigonometric series which :rges uniformly on the whole of real axis, then the series is the Fourier series of f.

oof

: have $f$expressed in trigonometric series (1), that means:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \, Cos\, kx + b_k Sin\, kx) \quad (1)$$

nce since the trigonometric series converges uniformly over the whole real axis, then f is tinuous and term by term integration is possible.

s we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

he other hand multiplying the series in (1) by $Cosnx$

$$(x)Cos\, nx = \frac{a_0}{2} Cos\, nx + \sum_{k=1}^{\infty}(a_k \, Cos\, nxCos\, kx + b_k Cos\, nxSin\, kx) \quad (8)$$

we want to show that the series on the right of (8) converges uniformly to the limit function e left.

ever since (1) converges uniformly, putting the partial sum

$$S_m(x) = \frac{a_0}{2} + \sum_{k=1}^{m} (a_k \, Cos\, kx + b_k Sin\, kx)$$

$> 0, \exists\, N = N(\varepsilon)$ such that

$$|f(x) - S_m(x)| \le \varepsilon, for\ all\, n \ge N \ \ and\, x \in R$$

we have

$$|Cos\, nx||f(x) - S_m(x)| \le |Cos\, nx|\varepsilon \le \varepsilon$$

fore

$$|f(x)Cos\, nx - S_m(x)Cos\, nx| \le \varepsilon$$

e the series in (8) converges uniformly. Which means again term by term integration is ble. Thus we get once again

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Cos\ nxdx \quad n = 1,2,3 \ldots$$

ıilar ways, multiplying the series in (1) by that of $Sin\ nx$ and for the same argumentation, ɔtain

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Sin\ nxdx \quad n = 1,2,3 \ldots$$

fore, the trigonometric series in (1) is precisely the Fourier series of the function f.

Fourier series of functions on interval of length 2π

der a function $f \in L[-\pi,\pi]$. Since there is nothing has been said about periodicity of f on erval, we use its periodic extension (see chapter 2.2) of f over the whole of x-axis, where ves us a periodic function which coincides with that of the function f on the indicated ıl and which has a Fourier series identical to that of the function f.

rk

ve seen (in the section of periodic extension, chapter 2.2) that If

$\neq f(\pi)$, then the periodic extension of f needs some necessary modification (see the two Here it is important to notice that, Since changing values of the function at finite number ts or even failing to define it at finite number of points in the interval do not change the f then integrals in (5). Thus in both cases the Fourier series or coefficients will have the alue as before.

The Cosine and the sine series

$L[-l,l]$, then recall that

If f is even, then $\int_{-l}^{l} f(x)dx = 2\int_0^l f(x)dx$

If f is odd, then $\int_{-l}^{l} f(x)dx = 0$

w consider $f \in L[-l,l] and$ is an even function, then clearly $f(x)Cos\ nx$ is even and x)$Sin\ nx$ is odd.

us the Fourier coefficients of f thereby given as

$_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Cos\ nxdx = \frac{2}{\pi}\int_{0}^{\pi} f(x)Cos\ nxdx \quad n = 0,1,2,3 \ldots$, and $b_n = 0 \ , n = 1,2,3, \ldots$

reover the Fourier series of f is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k\ Cos\ kx) \qquad (9)$$

call series (9) as the cosine series.

ilarly, if $f \in L[-l,l] and$ is an odd function, then the Fourier series of the function is Sine es

$$f(x) \sim \sum_{k=1}^{\infty} (b_k Sin\ kx) \qquad (10)$$

re

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Sin\ nxdx = \frac{2}{\pi}\int_{0}^{\pi} f(x)Sin\ nxdx \qquad n = 1,2,3 \ldots$$

let us illustrate how to construct Fourier series of some functions through the following nples. Later on we will observe complex approach on Fourier series representation of ions.

nple 3.3.1

s expand the function $f(x) = x^2, \ -\pi \leq x \leq \pi$ in Fourier series

to expand f in Fourier series we use the periodic extension of$f$. Since f is even, then the dic extension is also an even function. We can observe the graph of f and its periodic sion below. (Observe also that the periodic extension is also continuous and piecewise th)

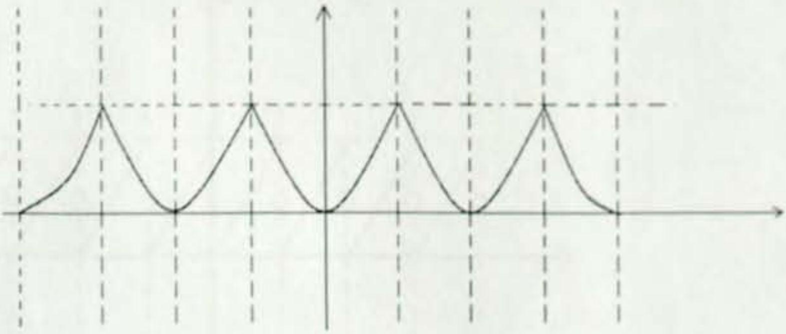


Figure 3.3.1

since the function is even, the Fourier series is a cosine series

ve have

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 Cos\, nxdx \qquad n = 0,1,2,3\ldots$$

we get $a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2\pi^2}{3}$ and for the others, using integration by parts, we have

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 Cos\, nxdx = \frac{2}{\pi}\left[\frac{x^2}{n} Sin\, nx + \frac{2x}{n^2} Cos\, nx - \frac{2}{n^3} Sin\, nx\right]_0^{\pi} = (-1)^n \frac{4}{n^2}$$

the Fourier series of the function f on the interval $[-\pi, \pi]$ is given by:

$$f(x) \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2}\, Cos\, nx$$

ıple 3.3.2

s now expand the function $f(x) = x,\ 0 < x < 2\pi$ in Fourier series

s case the function is neither even nor odd. Now considering the periodic extension of f (as

ı below),

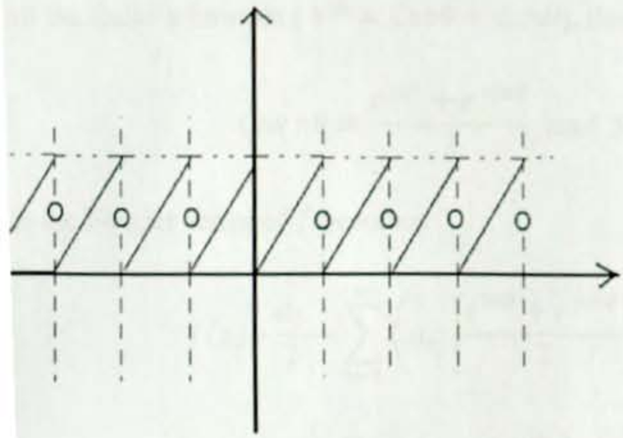


Figure 3.3.2

ɜ we can observe that the periodic extension of the function f is discontinuous, however ewise smooth.

/ let us determine the Fourier series of the function. Here since the function is neither even ɔdd, we have to compute both $a_n \& b_n$. After a sort of integration, we get:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x\, dx = 2\pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x \; Cos\; nx dx = 0, \qquad n = 1,2,3, \ldots \text{ and}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x \; Sin\; nx dx = \frac{-2}{n}, \qquad n = 1,2,3, ..$$

e the Fourier expansion of the function on the interval $0 < x < 2\pi$ is thus given by

$$f(x) \sim \pi + \sum_{n=1}^{\infty} \left(\frac{-2}{n}\right) Sin\; nx$$

The complex form of Fourier series of functions

$\in L[-\pi, \pi]$ and the Fourier series of f is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \; Cos\; nx + b_n Sin\; nx) \qquad (6)$$

m the Euler's formula ($e^{i\theta} = Cos\theta + iSin\theta$), then we have

$$Cos\, n\theta = \frac{e^{in\theta} + e^{-in\theta}}{2} \quad and \quad Sin\, n\theta = \frac{e^{in\theta} - e^{-in\theta}}{2i}$$

is the Fourier series of f becomes

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \left[\frac{e^{in\theta} + e^{-in\theta}}{2} \right] + b_n \left[\frac{-ie^{in\theta} + ie^{-in\theta}}{2} \right] \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n - ib_n}{2} \right) e^{in\theta} + \left(\frac{a_n + ib_n}{2} \right) e^{-in\theta}$$

v put

$$c_0 = \frac{a_0}{2}, c_n = \frac{a_n - ib_n}{2} \quad and \; c_{-n} = \frac{a_n + ib_n}{2} \quad , n = 1,2,3, \ldots$$

vith this notation, the Fourier series of the function f can be set in complex form as follows

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{11}$$

re

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \; , \quad n = 0, \pm 1, \pm 2 \ldots$$

ark

n^{th} partial sum of the previous series (11) is given by

$$S_m(x) = \sum_{n=-m}^{m} c_n e^{inx} \tag{12}$$

e the convergence of the series in (11) must be understood to mean the existence of the as $m \to \infty$ of the partial sum (12).

ample 3.4.1

nsider a function f given by $f(x) = e^x$ on the interval $-\pi < x < \pi$. Now let us determine the nplex Fourier series of the function.

ng the formula obtained above then the Fourier coefficient of f is hence given by:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-in)x}dx = \frac{1}{2\pi}\left[\frac{e^{(1-in)x}}{1-in}\right]_{-\pi}^{\pi} = \frac{(-1)^n Sin\, n\pi}{(1-in)\pi}$$

ıs we have

$$f(x) \sim \sum_{n=-\infty}^{n=\infty} \frac{(-1)^n Sin\, n\pi}{(1-in)\pi}\, e^{inx}$$

Fourier series of functions of period $2l$

sider a function $f \in L[-l, l]$ and with period $2l$.

$x = \frac{lt}{\pi}$ and define a function φ as

$$\varphi(t) = f\left(\frac{lt}{\pi}\right) \tag{13}$$

:

$$\varphi(t + 2\pi) = f\left(\frac{l(t+2\pi)}{\pi}\right) = f\left(\frac{lt}{\pi} + 2l\right) = f\left(\frac{lt}{\pi}\right) = \varphi(t)$$

;

$$\varphi(t + 2\pi) = \varphi(t)$$

efore, φ is periodic with period 2π, thus the Fourier series of φ is given by

$$\varphi(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n Cos\, nt + b_n Sin\, nt \tag{14}$$

ıere

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(t) Cos\, ntdt = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lt}{\pi}\right) Cos\, ntdt \quad and$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(t) Sin\, ntdt = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lt}{\pi}\right) Sin\, ntdt$$

ν putting $x = \frac{lt}{\pi}$, we have $\frac{1}{l}dx = \frac{1}{\pi}dt$

ıs

$$a_n = \frac{1}{l}\int_{-\pi}^{\pi} f(x) Cos \frac{n\pi}{l} xdx \quad, n = 0,1,2,3, \dots$$

$$b_n = \frac{1}{l}\int_{-\pi}^{\pi} f(x) Sin \frac{n\pi}{l} xdx \quad, n = 1,2,3, \dots\ and$$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n Cos \frac{n\pi}{l} x + b_n\, Sin \frac{n\pi}{l} x \qquad (5)$$

ıark

: note that everything that we have done for the case of period 2π is also applicable for the of period $2l$.

nstance, if f is even, then the Fourier series is cosine series and the coefficients are obtained $_{\iota} = \frac{2}{l}\int_0^{\pi} f(x) Cos \frac{n\pi}{l} xdx \ , n = 0,1,2,3, \dots$ whereas $b_n = 0$ and likewise for odd functions.

:over, we can also write the Fourier series of the function in complex form as:

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{\frac{inx}{l}}$$

·e

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-\frac{inx}{l}} dx \qquad n = 0, \pm 1, \pm 2 \dots$$

SECTION IV

SOME BASIC PRELIMINARY PROPERTIES

have not yet established anything in concern with convergence of at a point x of the Fourier es of function $f \in L[-\pi,\pi]$.

rder to accomplish that, let us now discuss on some basic properties on the Fourier series of ctions, especially for those functions belonging in $L^2[-\pi,\pi]$. First let us begin with the lling the following properties on trigonometry, for which they are very much helpful in esenting Fourier series of functions using other approach.

Dirichlet's kernel

ma 4.1.1

1. $\frac{1}{2}+\sum_{k=1}^{n} Cos\, kx = \frac{Sin\left(n+\frac{1}{2}\right)x}{2Sin\frac{x}{2}}$

2. $\sum_{k=0}^{n-1} Sin\left(k+\frac{1}{2}\right)x = \frac{Sin^2\left(\frac{nx}{2}\right)}{Sin\frac{x}{2}}$

f

ave the trigonometric identity $Sin(y+a) - Sin(y-a) = 2Cosy\, Sina$

$= kx \text{ and } a = \frac{x}{2}$, then we have

$$Sin\left(k+\frac{1}{2}\right)x - Sin\left(k-\frac{1}{2}\right)x = 2Sin\frac{x}{2}Coskx$$

adding for $k = 0,1,2,3,\ldots,n$, we get

$$Sin\left(n+\frac{1}{2}\right)x - Sin\left(\frac{-x}{2}\right) = 2Sin\left(\frac{x}{2}\right)[1+Cosx+Cos2x+\cdots+Cosnx]$$

$$Sin\left(n+\frac{1}{2}\right)x + Sin\left(\frac{x}{2}\right) = 2Sin\left(\frac{x}{2}\right)[1+Cosx+Cos2x+\cdots+Cosnx]$$

erefore

$$Sin\left(n+\frac{1}{2}\right)x = 2Sin\left(\frac{x}{2}\right)\left[\frac{1}{2}+Cosx+Cos2x+\cdots+Cosnx\right]$$

at means

$$\frac{1}{2}+\sum_{k=1}^{n} Cos\,kx = \frac{Sin\left(n+\frac{1}{2}\right)x}{2Sin\frac{x}{2}} \qquad (1)$$

nilarly, to show the second trig identity, we have that

$$2Sin\left(\frac{x}{2}\right)Sin\left(k+\frac{1}{2}\right)x = Coskx - Cos(k+1)x$$

s

$$2Sin\left(\frac{x}{2}\right)\sum_{k=0}^{n-1} Sin\left(k+\frac{1}{2}\right)x = \sum_{k=0}^{n-1}[Coskx - Cos(k+1)x] = 1 - Cosnx = 2Sin^2\left(\frac{nx}{2}\right)$$

ce

$$\sum_{k=0}^{n-1} Sin\left(k+\frac{1}{2}\right) = \frac{Sin^2\left(\frac{nx}{2}\right)}{Sin\frac{x}{2}} \qquad (2)$$

der to see if the Fourier series of a function $f \in L[-\pi,\pi]$actually converge to the value of some point$x \in [-\pi,\pi]$, we must investigate whether or not$\lim_{n\to\infty} S_n(x) = f(x)$, where) is the n^{th} partial sum of the Fourier series of$f$.

let us try to express $S_n(x)$in a more manageable form, so that we can study the convergence ria so easily.

$$) = \frac{a_0}{2}+\sum_{k=1}^{n} a_k Coskx + b_k Sinkx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \frac{1}{\pi}\sum_{k=1}^{n}\left[Coskx\int_{-\pi}^{\pi} f(t)cosktdt + Sinkx\int_{-\pi}^{\pi} f(t)Sinktdt\right]$$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{k=1}^{n} Coskx\, Coskt + SinkxSinkt\right] dt$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{k=1}^{n} Cos(x-t)k\right] dt$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_n(x-t)dt$

ere using the result of (1)

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} Cos\, kt = \frac{Sin\left(n+\frac{1}{2}\right)t}{2Sin\frac{t}{2}}, -\infty < t < \infty \tag{3}$$

re at $t = 2k\pi\, , k \in Z$ the value $D_n(t) = n + \frac{1}{2}$),

inition 4.1.1 [Dirichlet's kernel]

call D_n in (3) to be the *Dirichlet's kernel.*

put $u = x - t$, then we get

$$S_n(x) = \frac{1}{\pi}\int_{x-\pi}^{x+\pi} f(x-u)D_n(u)\, du$$

both f *and* D_n are periodic with period 2π, we have

$$) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-u)D_n(u)\, du = \frac{1}{\pi}\int_{-\pi}^{0} f(x-u)D_n(u)\, du + \frac{1}{\pi}\int_{0}^{\pi} f(x-u)D_n(u)\, du$$

$= \frac{1}{\pi}\int_{0}^{\pi} f(x+t)D_n(t)\, dt + \frac{1}{\pi}\int_{0}^{\pi} f(x-t)D_n(t)\, dt$

$= \frac{1}{\pi}\int_{0}^{\pi}[f(x+t) + f(x-t)]D_n(t)\, dt$

ng $f(x) = 1$, we have $a_0 = 2, a_k = b_k = 0, k = 1,2,3 \ldots$

$$S_n(x) = 1, for\ all\ n$$

ıce

$$1 = \frac{2}{\pi}\int_0^{\pi} D_n(t)dt$$

refore for any arbitrary $f \in L[-\pi, \pi]$

$$S_n(x) - f(x) = \frac{1}{\pi}\int_0^{\pi}[f(x+t) + f(x-t)]D_n(t)\,dt - \frac{2}{\pi}\int_0^{\pi} f(x)D_n(t)dt$$

conclude that the Fourier series of f at x will converge to $f(x)$ if and only if

$_{\to\infty} S_n(x) - f(x)$

$$= \lim_{n\to\infty}\frac{2}{\pi}\int_0^{\pi}\left[\frac{f(x+t) + f(x-t)}{2} - f(x)\right]D_n(t)\,dt = 0 \qquad (4)$$

Fejèr's kernel

: again to see the Fourier series of a function $f \in L[-\pi, \pi]$ is $(C, 1)$ summable at$x \in$ $\pi]$, we must investigate that

$$\lim_{n\to\infty} \delta_n(x) = f(x)$$

·e

$$\delta_n(x) = \frac{S_0(x) + S_1(x) + \cdots + S_{n-1}(x)}{n} = \frac{1}{n}\sum_{k=0}^{n-1} S_k(x)$$

ere using the previous result of Dirichlet's kernel, we have

$= \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{\pi}\int_0^{\pi}[f(x+t) + f(x-t)]D_k(t)\,dt = \frac{1}{\pi}\int_0^{\pi}[f(x+t) + f(x-t)]\frac{1}{n}\sum_{k=0}^{n-1} D_k(t)dt$

ɔutting

$$K_n(t) = \frac{1}{n}\sum_{k=0}^{n-1} D_k(t) = \frac{1}{2nSin\left(\frac{t}{2}\right)}\sum_{k=0}^{n-1} Sin\left(k + \frac{1}{2}\right)t$$

ısing (2) from the trigonometric identities discussed earlier,

$$K_n(t) = \frac{Sin^2\left(\frac{nt}{2}\right)}{2nSin\frac{t}{2}} \tag{5}$$

nition 4.2.1

call $K_n(t)$ in (5) to be the *Fejèr's kernel.*

take $f(x) = 1$, then $S_0(x) = S_1(x) = \cdots = S_{n-1}(x) = 1$

;

$$1 = \frac{2}{\pi}\int_0^{\pi} K_n(t)dt$$

efore

$$\delta_n(x) - f(x) = \frac{2}{\pi}\int_0^{\pi}\left[\frac{f(x+t)+f(x-t)}{2} - f(x)\right]K_n(t)\,dt$$

$f\ is\ (C,1)$ summable at $x \in [-\pi,\pi]$ if and only if

$\delta_n(x) - f(x)$

$$= \lim_{n\to\infty}\frac{2}{\pi}\int_0^{\pi}\left[\frac{f(x+t)+f(x-t)}{2} - f(x)\right]K_n(t)\,dt = 0 \tag{6}$$

rem 4.2.1

$L[-\pi,\pi]$and if f is continuous at $x \in [-\pi,\pi]$, then $f$is $(C,1)$ summable.

ix $\varepsilon > 0$, then to prove the statement we need to find N such that

$$|\delta_n(x) - f(x)| = \left|\frac{2}{\pi}\int_0^{\pi}\left[\frac{f(x+t)+f(x-t)}{2} - f(x)\right]K_n(t)\,dt\right| < \varepsilon\,, n \geq \text{N}$$

ince $f$is continuous at$x$, then there exist $\delta > 0, for\ 0 < \delta < \pi$ such that

$$|f(y) - f(x)| < \frac{\varepsilon}{2} \quad on\ |y - x| < \delta$$

ıce for $0 \leq t < \delta$

$$\left|\frac{f(x+t) + f(x-t) - 2f(x)}{2}\right| \leq \frac{1}{2}|f(x+t) - f(x)| + |f(x-t) - f(x)|$$

$\frac{1}{2}\left(\frac{\varepsilon}{2} + \frac{\varepsilon}{2}\right) = \frac{\varepsilon}{2}\ for\ n \in N$

ce we have

$$\left|\frac{2}{\pi}\int_0^{\delta}\left[\frac{f(x+t) + f(x-t) - 2f(x)}{2}\right]K_n(t)\,dt\right| \leq \frac{\varepsilon}{2}\frac{2}{\pi}\int_0^{\delta}K_n(t)dt < \frac{\varepsilon}{2}$$

$\geq \delta$, then $K_n(t) \leq \frac{1}{2nSin^2\left(\frac{\delta}{2}\right)}$, thus

$$\int^{\pi}\left[\frac{f(x+t) + f(x-t) - 2f(x)}{2}\right]K_n(t)\,dt\right|$$

$$\leq \frac{2}{4n\pi Sin^2\left(\frac{\delta}{2}\right)}\int_{\delta}^{\pi}|f(x+t)| + |f(x-t)| + 2|f(x)|dt \to 0$$

e for some N natural number

$$\left|\frac{2}{\pi}\int_{\delta}^{\pi}\left[\frac{f(x+t)+f(x-t)-2f(x)}{2}\right]K_n(t)\,dt\right| < \frac{\varepsilon}{2} \quad for\ n \geq N$$

ːfore

$$\left|\frac{2}{\pi}\int_0^{\pi}\left[\frac{f(x+t) + f(x-t) - 2f(x)}{2}\right]K_n(t)\,dt\right| < \varepsilon$$

e

$$\lim_{n\to\infty} \delta_n(x) - f(x) = 0$$

e f is $(C, 1)$ summable x.

Some basic theories in $L^2[-\pi,\pi]$

e in this subtopic, some basic theories with regard to the Fourier series of functions which are are integrable on the indicated interval will be discussed. The results of these theories are so ch helpful in proving the basic criterion for the convergence of Fourier series.

orem 4.3.1

$\in L^2[-\pi,\pi]$ and T_n be any trigonometric polynomial of degree n, then $\|f-S_n\| \leq$ $- T_n\|$, where S_n is the n^{th} partial sum of the Fourier series of f.

of

$$T_n = A_0 + \sum_{k=1}^{n} [A_k Coskt + B_k Sinkt]$$

$$J = \frac{1}{\pi}\int_{-\pi}^{\pi}[f(t) - T_n(t)]^2.$$

here we need to show that J is minimum when $T_n = S_n$

ow this, we have

$$J = \frac{1}{\pi}\int_{-\pi}^{\pi}[f(t)]^2 - \frac{2}{\pi}\int_{-\pi}^{\pi} f(t)T_n(t) + \frac{1}{\pi}\int_{-\pi}^{\pi}[T_n(t)]^2 \quad (7)$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)T_n(t) = \frac{A_0}{\pi}\int_{-\pi}^{\pi} f(t)dt + \sum_{k=1}^{n}\frac{A_k}{\pi}\int_{-\pi}^{\pi} f(t)Cosktdt + \frac{B_k}{\pi}\int_{-\pi}^{\pi} f(t)Sinktdt$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)T_n(t) = A_0 a_0 + \sum_{k=1}^{n} A_k a_k + B_k b_k \quad (8)$$

the other hand

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[T_n(t)]^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[A_0 + \sum_{k=1}^{n}[A_k Coskt + B_k Sinkt]\right]^2 dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\{A_0{}^2 + \sum_{k=1}^{n} A_0 A_k Coskt + \sum_{k=1}^{n} A_0 B_k Sinkt + \sum_{k=1}^{n} A_k{}^2 Cos^2kt +$$

$$+ \sum_{k=1}^{n} B_k{}^2 Sin^2kt + \sum_{k=1}^{n} A_k B_k Coskt\ Sinkt + \sum_{p\neq q} B_p B_q Sinpt\ Sinqt$$

$$+ \sum_{p\neq q} A_p A_q Cospt\ Cosqt\}dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} A_0{}^2 dt + \sum_{k=1}^{n} A_k{}^2 \frac{1}{\pi}\int_{-\pi}^{\pi} Cos^2ktdt + \sum_{k=1}^{n} B_k{}^2 \frac{1}{\pi}\int_{-\pi}^{\pi} Sin^2ktdt$$

$$= 2A_0{}^2 + \sum_{k=1}^{n} A_k{}^2 + B_k{}^2$$

;

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[T_n(t)]^2 = 2A_0{}^2 + \sum_{k=1}^{n} A_k{}^2 + B_k{}^2 \qquad (9)$$

substituting (2) &(3) in to (1), we get

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(t)]^2 - 2A_0 a_0 - 2\sum_{k=1}^{n} A_k a_k + B_k b_k + 2A_0{}^2 + \sum_{k=1}^{n} A_k{}^2 + B_k{}^2$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}[f(t)]^2 + \frac{a_0{}^2}{2} + \sum_{k=1}^{n}[a_k{}^2 + b_k{}^2] - \frac{a_0{}^2}{2} - \sum_{k=1}^{n}[a_k{}^2 + b_k{}^2]$$

$$-2A_0 a_0 - 2\sum_{k=1}^{n}[A_k a_k + B_k b_k] + 2A_0{}^2 + \sum_{k=1}^{n}[A_k{}^2 + B_k{}^2]$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}[f(t)]^2 - \left(\frac{a_0{}^2}{2} + \sum_{k=1}^{n}[a_k{}^2 + b_k{}^2]\right)^2 + \left\{2\left(A_0 - \frac{a_0}{2}\right)^2\right.$$

$$\left. + \sum_{k=1}^{n}[(A_k - a_k)^2 + (B_k - b_k)^2]\right\} \qquad (10)$$

value in the brace is always non negative. Hence J will be minimum when $A_0 = \frac{a_0}{2}, A_k =$ $and\ B_k = b_k\ for\ k = 1,2,3, \ldots$

ther words $T_n = S_n$.

the following corollary result is very important in proving the basic Riemann Lebesgue rem which comes soon.

ollary 4.3.1 [The Bessel's Inequality]

$\in L^2[-\pi, \pi]$, with the Fourier coefficient $\{A_k\}$ and $\{B_k\}$, then

$+ \sum_{k=1}^{n}[{a_k}^2 + {b_k}^2] \leq \frac{1}{\pi}\int_{-\pi}^{\pi}[f(t)]^2 dt$

f

$= \frac{a_0}{2}, A_k = a_k\ and\ B_k = b_k\ for\ k = 1,2,3, \ldots$ in (4), we obtain

$$J = \frac{1}{\pi}\int_{-\pi}^{\pi}(f - S_n)^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2 - \frac{{a_0}^2}{2} + \sum_{k=1}^{n}[{a_k}^2 + {b_k}^2]$$

ne integral on the left is non negative.

we obtain that

$$\frac{{a_0}^2}{2} + \sum_{k=1}^{n}[{a_k}^2 + {b_k}^2] \leq \frac{1}{\pi}\int_{-\pi}^{\pi}[f(t)]^2 dt$$

this results the proof of the Bessel's inequality

over from this well known Bessel's inequality we say that if $f \in L^2[-\pi, \pi]$, then atically

$$\frac{{a_0}^2}{2} + \sum_{k=1}^{n}[{a_k}^2 + {b_k}^2] < \infty$$

/ next let us prove the converse of the Bessel's inequality with the following theorem.

orem 4.3.2 [Converse of the Bessel's inequality]

$_k\}$ and $\{b_k\}$ are arbitrary sequence of real numbers such then, $\frac{{a_0}^2}{2} + \sum_{k=1}^{n}[{a_k}^2 + {b_k}^2] < \infty$, there exist $f \in L^2[-\pi,\pi]$, such that $\{a_k\}$ and $\{b_k\}$ are precisely its Fourier coefficients.

f

$\in Z$, define a trigonometric polynomial of order n by S_n

$$S_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k Coskt + b_k Sinkt$$

for $m < n$, we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[S_n - S_m]^2 = \frac{1}{\pi}\int_{\pi}^{\pi}\left[\sum_{k=m+1}^{n} a_k Coskt + b_k Sinkt\right]^2$$

$$\frac{1}{\pi}\|S_n - S_m\|^2 = \sum_{k=m+1}^{n}[{a_k}^2 + {b_k}^2]$$

etting m very large, the sequence $\{S_n\}$ is a Cauchy sequence in $L^2[-\pi,\pi]$. However since $,\pi]$ is a complete by the Resize Fisher, hence there exist $f \in L^2[-\pi,\pi]$ such that $_\infty\|S_n - f\| = 0$

we have

$$\lim_{n\to\infty}\frac{1}{\pi}\int_{-\pi}^{\pi} S_n(t) Cosjt\, dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) Cosjt\, dt = a_j$$

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) Cosjt\, dt$$

ıce a_j is the Fourier coefficient of f and similarly forb_j.

v before we come to the basic The Riemann Lebesgue theorem, it would be necessary the owing important lemma.

ıma 4.3.1

$f \in L[-\pi,\pi]$, then for every$\varepsilon > 0$, there exist a bounded and measurable function$g$ on $,\pi]$, such that $\int_{-\pi}^{\pi}|f(x) - g(x)|d < \varepsilon$

ɔf

how we need to follow to cases:

:1:

f be none negative valued such that $f \in L[-\pi,\pi]$, then by definition

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} {}^{n}f = \int_{-\pi}^{\pi} f, \text{ where } {}^{n}f = \min\{f(x),0\}$$

given $\varepsilon > 0$, there exist N=N(ε) such that$\left|\int_{-\pi}^{\pi} f - {}^{n}f\right| < \varepsilon$ for n ≥N.

$g = {}^{N}f$ which is measurable and bounded and moreover

ever since ≥ g on $[-\pi,\pi]$. Thus

$$\int_{-\pi}^{\pi}|f(x) - g(x)|d < \varepsilon$$

2:

ɜt f be any arbitrary function such that$f \in L[-\pi,\pi]$. Putting

$$f = f^{+} - f^{-} \text{Where } f^{+}, f^{-} \in L[-\pi,\pi]$$

nce by the previous step there are bounded and measurable functions g_1 *and* g_2 such that

$$\int_{-\pi}^{\pi}|f(x) - g_1(x)|d < \frac{\varepsilon}{2} \text{ and } \int_{-\pi}^{\pi}|f(x) - g_2(x)|d < \frac{\varepsilon}{2}$$

e now $g = g_1 - g_2$ which is bounded and measurable on$[-\pi, \pi]$

have

$$- g| = |(f^+ - f^-) - (g_1 - g_2)| = |(f^+ - g_1) + ((-f^- + g_2))| \leq |f^+ - g_1| + |f^- - g_2|$$

ce we have

$$\int_{-\pi}^{\pi}|f - g| \leq \int_{-\pi}^{\pi}|f^+ - g_1| + \int_{-\pi}^{\pi}|f^- - g_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

orem 4.3.3 [The Riemann Lebesgue theorem]

$^f \in L[-\pi, \pi]$ and $\{a_k\}$ and $\{b_k\}$ are Fourier coefficients of$f$, then

$$a_k = \lim_{k\to\infty} b_k = 0$$

f

> 0, then from the given information about$f$, we have $\int_{-\pi}^{\pi} f(x)dx < \infty$, then there exist a ded and measurable function g on $[-\pi, \pi]$ such that

$$\int_{-\pi}^{\pi}|f(x) - g(x)|dx < \frac{\varepsilon}{2\pi} \text{ (Using the above lemma)}$$

:ver$g \epsilon L^2[-\pi, \pi]$, since $g$is bounded and measurable.

e

$$\sum_{k=1}^{\infty}[{A_k}^2 + {B_k}^2] < \infty,$$

nsidering $\{A_k\}$ and $\{B_k\}$ are Fourier coefficients of$g$.

at means

$$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)Cosktdt \quad and\ B_k = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)Sinktdt$$

wever since $\sum_{k=1}^{\infty} {A_k}^2 < \infty$, then

$$\lim_{k\to\infty} A_k = 0$$

s there exist N>0, such that $|A_k| < \frac{\varepsilon}{2}$ for $k \geq$N

for any k, we have

$$|a_k - A_k| = \left|\frac{1}{\pi}\int_{-\pi}^{\pi} |f(t) - g(t)|Cosktdt\right| \leq \frac{1}{\pi}\int_{-\pi}^{\pi} |f(t) - g(t)| < \frac{\varepsilon}{2}$$

efore

$$|a_k - A_k| < \frac{\varepsilon}{2}, \text{ for } k > \text{N}$$

put

$$a_k = (a_k - A_k) + A_k$$

$$|a_k| = |(a_k - A_k) + A_k| \leq |a_k - A_k| + |A_k| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ for\ k \geq \text{N}$$

e

$$\lim_{k\to\infty} a_k = 0$$

arly we can also show for $\lim_{k\to\infty} b_k = 0$

ollary 4.3.3

$\in L[0,\pi]$, then $\lim_{k\to\infty}\int_0^{\pi}\varphi(t)Sin\left(k+\frac{1}{2}\right)t\,dt=0$

of

ne

$$\varphi(t)=0\ for -\pi\le t\le 0$$

$\varphi(t)\in L[-\pi,\pi]$

e

$$Sin\left(k+\tfrac{1}{2}\right)t=Sinkt\ Cos\tfrac{t}{2}+Coskt\ Sin\tfrac{t}{2}$$

ien have

$$(t)Sin\left(k+\frac{1}{2}\right)t\,dt=\int_{-\pi}^{\pi}\left[\varphi(t)Cos\frac{t}{2}\right]Sinkt\,dt+\int_{-\pi}^{\pi}\left[\varphi(t)Sin\frac{t}{2}\right]Coskt\,dt$$

$(t)Cos\frac{t}{2}$ *and* $\varphi(t)Sin\frac{t}{2}$ are functions in $L[-\pi,\pi]$

by the Riemann Lebesgue theorem above the integrals on the right approach to as $k\to\infty$

rem4.3.4

$\in L^2[-\pi,\pi]$, then the n^{th} partial sum of the Fourier series of f, that is S_n converges to f espect to the norm in $L^2[-\pi,\pi]$.

ler $f\in L^2[-\pi,\pi]$, from real analysis, we have a continuous function f^* on $[-\pi,\pi]$ such $(-\pi)=f^*(\pi)$ and moreover $\|f-f^*\|<\frac{\varepsilon}{2}$

ince $f^*$is continuous, then the Casaro's partial sum $\{\delta_n{}^*\}$ of the function f^*converges nly.

nce we have

$$\lim_{n\to\infty} \|\delta_n{}^{*} - f^{*}\| = 0$$

ere here $\delta_n{}^{*} = \frac{S_0{}^{*}+S_1{}^{*}+\cdots+S_{n-1}{}^{*}}{n}$, and $S_n{}^{*}$ is the n^{th} partial sum of the Fourier series of the ction f^{*}.

s now, there exist N, for the selected ε such that

$$\|\delta_{n+1}{}^{*} - f^{*}\| < \frac{\varepsilon}{2} \ for\, n \geq N$$

s for this N, we have

$$\|f - f^{*}\| + \|\delta_{n+1}{}^{*} - f^{*}\| < \varepsilon$$

:e

$$\|f - \delta_{n+1}{}^{*}\| < \varepsilon$$

$\|f - S_n\| < \varepsilon$, since $\delta_{n+1}{}^{*}$ is the n^{th} order trigonometric polynomial and hence by rem 4.3.1

:fore

$$\lim_{n\to\infty} \|f - S_n\| = 0$$

llary 4.3.2 [The parseval's Equation]

nction $f \in L^2[-\pi,\pi]$, then $\frac{a_0{}^2}{2} + \sum_{k=1}^{n}\left[a_k{}^2 + b_k{}^2\right] = \frac{1}{\pi}\int_{-\pi}^{\pi}[f(t)]^2 dt$

: $\{a_k\}$ and $\{b_k\}$ are the Fourier coefficients of f.

ɔ illustrate the proof, from the construction of the proof of theorem4.3.1, we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(f - S_n)^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2 - \left(\frac{a_0{}^2}{2} + \sum_{k=1}^{n} a_k{}^2 + b_k{}^2\right)$$

v by the previous theorem since$\|f - S_n\| \to 0$, then we have

$$\frac{a_0{}^2}{2} + \sum_{k=1}^{\infty} a_k{}^2 + b_k{}^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx$$

mple 4.3.1

sider the function$f(x) = \frac{x^2}{4}$ $on\ [-\pi, \pi]$. Now let us justify $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$ using the pars 's equation.

the Fourier series of the function is given by the form

$$f(x) \sim \frac{\pi^2}{12} + \sum_{k=1}^{\infty} \frac{Cosk\pi}{k^2} Coskx$$

g the parse vale's equation, $\frac{a_0{}^2}{2} + \sum_{k=1}^{\infty} a_k{}^2 + b_k{}^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx$

ave $\frac{\pi^4}{72} + \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{40}$

$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$, thus the result.

SECTION V

CONVERGENCE OF FOURIER SERIES OF FUNCTION

: have not yet established anything about the convergence at a point x of the Fourier series of ınction $f \in L[-\pi,\pi]$

: condition which is sufficient for the convergence that shall be given involve the existence of left and right hand limits and the existence of the generalized left and right hand derivatives ᶠat x.

ⱳ let us get started with discussing on the following notations and definitions.

Notations and Definitions

:ation

any real valued function f on R, we denote the left and right hand limits for $x \in R$, as:

$$\lim_{t\to x^-} f(t) = f(x-) \ \ and \ \ \lim_{t\to x^+} f(t) = f(x+)$$

ectively, provided that the limit exists.

: that, we say the function has jump discontinuity at a point provided that two limits exist and :qual. On the other had the function is continuous at the point provided that they are equal.

inition 5.1.1 [Generalized left and right hand derivatives]

f be a real valued function and $x \in R$ such that $f(x+)$ exists, then we define the :ralized right hand derivative of f at x as

$$f_r'(x) = \lim_{t\to x^+} \frac{f(t)-f(x+)}{t-x} \qquad (1)$$

ⱳided that this limit exists.

ilarly with the existence of $f(x-)$, we define the generalized left hand derivative of f at x

$$f_l'(x) = \lim_{t\to x^-} \frac{f(t) - f(x-)}{t - x} \qquad (2)$$

ided that the limit exists.

g the substitution $h = t - x$, (1) and (2) can be rewritten as

$$f_r'(x) = \lim_{h\to 0^+} \frac{f(h+x)-f(x+)}{h} \text{ and } f_l'(x) = \lim_{h\to 0^-} \frac{f(h+x)-f(x-)}{h}$$

nple 5.1.1

der the function $f(t) = \begin{cases} t+1 & t>1 \\ 17 & t=1 \\ 3t^2 & t<1 \end{cases}$

$f(1+) = \lim_{t\to 1^+} f(t) = \lim_{t\to 1^+} t+1 = 2$ and

$) = \lim_{t\to 1^-} f(t) = \lim_{t\to 1^+} 3t^2 = 3$

both $f(1+)$and $f(1-)$ exist

$_r'(1) = \lim_{t\to 1^+} \frac{f(t)-f(1+)}{t-1} = \lim_{t\to 1^+} \frac{t+1-2}{t-1} = 1$ and moreover

$$= \lim_{t\to 1^+} \frac{f(t) - f(1-)}{t - 1} = \lim_{t\to 1^+} \frac{3t^2 - 3}{t - 1} = 6$$

otice that in the above example both $f_r'(1)$ and $f_l'(1)$ exist at$x = 1$ but $f$is not left and ontinuous at$x = 1$. Thus f does not have an ordinary derivative at the point.

ufficient condition for the convergence of Fourier series of function at a point

following basic theorem and its proof give us brief criteria for the convergence of Fourier s of function at a point.

rem 5.2.1

$\in L[-\pi,\pi]$ and x be any point in$[-\pi,\pi]$

$f(x+)$and $f(x-)$ exist

$f(x) = \frac{f(x+)+f(x-)}{2}$ and

$f_r'(x)$ and $f_l'(x)$ exist,

he Fourier series of $f$at x will converge to$f(x)$.

otice that, if$x = \pm\pi$, then $f(x+)$and $f(x-)$ are computed from the periodic extension function f(see chapter 2.2).

nt to prove this from the Dirichlet's kernel approach (see chapter 4.1). The Fourier series x converge to $f(x)$ provide that

$$\lim_{n\to\infty}[S_n(x) - f(x)] = \lim_{n\to\infty}\frac{2}{\pi}\int_0^{\pi}\left[\frac{f(x+t)+f(x-t)}{2} - f(x)\right]D_n(t)dt = 0$$

$$D_n(t) = \frac{Sin\left(n+\frac{1}{2}\right)t}{2Sin\left(\frac{t}{2}\right)} \ ,-\infty < t < \infty$$

view of (ii), form the hypothesises, we have to show that

$$\lim_{n\to\infty}\frac{1}{\pi}\int_0^{\pi}\{[f(x+t)-f(x+)]+[f(x-t)-f(x-)]\}D_n(t)dt \;= 0$$

imply

$$\lim_{n\to\infty}\frac{1}{\pi}\int_0^{\pi}\varphi(t)Sin\left(n+\frac{1}{2}\right)tdt \;= 0$$

re

$$\varphi(t)=\{[f(x+t)-f(x+)]+[f(x-t)-f(x-)]\}\frac{1}{2Sin\left(\frac{t}{2}\right)}\quad 0<t\leq\pi$$

writing

$$\varphi(t)=\left[\frac{f(x+t)-f(x+)}{t}+\frac{f(x-t)-f(x-)}{t}\right]\frac{t}{2Sin\left(\frac{t}{2}\right)}$$

ee that

$$\lim_{t\to 0^+}\varphi(t)=f_r'(x)-f_l'(x)$$

φ is bounded on the interval $(0,\delta]$ for some $\delta>0$.

: $\varphi\in L[0,\delta]$.

ver, since $f\epsilon[-\pi,\pi]$ and $\frac{1}{2Sin\left(\frac{t}{2}\right)}$ is bounded on $[\delta,\pi]$, then $\varphi\epsilon L[\delta,\pi]$,

fore $\varphi\epsilon L[0,\pi]$

y the corollary 4.3.3 of the Riemann Lebesgue theorem we have

$$\lim_{n\to\infty}\frac{1}{\pi}\int_0^{\pi}\varphi(t)Sin\left(n+\frac{1}{2}\right)tdt \;= 0$$

ce the statement is precisely satisfied. That means the Fourier series of the function f at x verges to the value $f(x)$.

ıark

ıally we have proved a little more than we have stated in the theorem stated above. The rem only states that if $f(x+), f(x-), f_r'(x)$ & $f_l'(x)$ exist and $f(x) = \frac{f(x+)+f(x-)}{2}$, then the ier series of f given as

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k Cos\, kx + b_k Sinkx = f(x) = \frac{f(x+)+f(x-)}{2} \qquad (3)$$

ever, the value of f at a single point does not affect a_k and b_k in (3).Hence it does not the series in (3). Thus can necessarily put (3) provided that $f(x) \neq \frac{f(x+)+f(x-)}{2}$.

ally we state the following corollary as follows.

lary 5.2.1

$L[-\pi, \pi]$, and let x be any point in $[-\pi, \pi]$. If $f(x+), f(x-), f_r'(x)$ & $f_l'(x)$, then the r series of f at x converges to the value $\frac{f(x+)+f(x-)}{2}$

ıeans $\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k Cos\, kx + b_k Sinkx = f(x) = \frac{f(x+)+f(x-)}{2}$

rks

he previous theorem, one can understand the following basic results

Let $f \in L[-\pi, \pi]$, then at every continuity point of f, where the left and right hand derivatives exist, then the Fourier series of f converges to value of $f(x)$. In particular, this is true at every point where $f(x)$ has derivative.

Let $f \in L[-\pi, \pi]$, then at every point of discontinuity, where the left and right hand side derivative exist, then the Fourier series of f converges to the value $\frac{f(x+)+f(x-)}{2}$

mple 5.2.1

ider the function $f(x) = \begin{cases} -\frac{\pi}{4} & -\pi < x < 0 \\ 0 & x = 0 \\ \frac{\pi}{4} & 0 < x < \pi \end{cases}$

for this function the Fourier series of f is given by

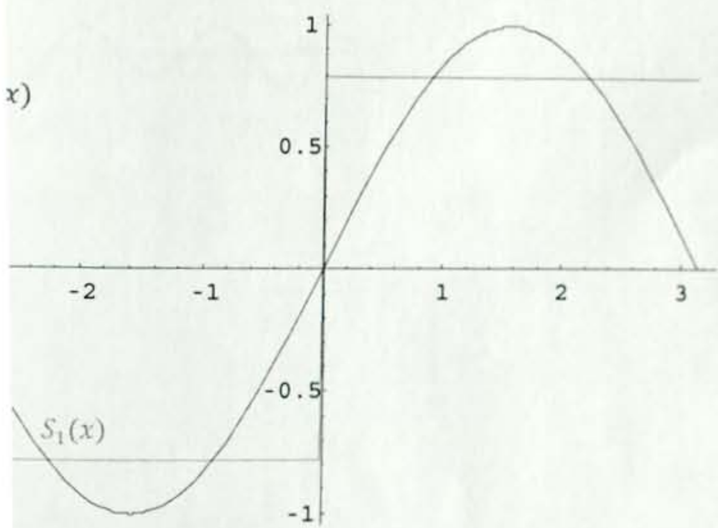
$$f(x) \sim \sum_{n=1}^{\infty} \frac{Sin(2n-1)x}{2n-1}$$

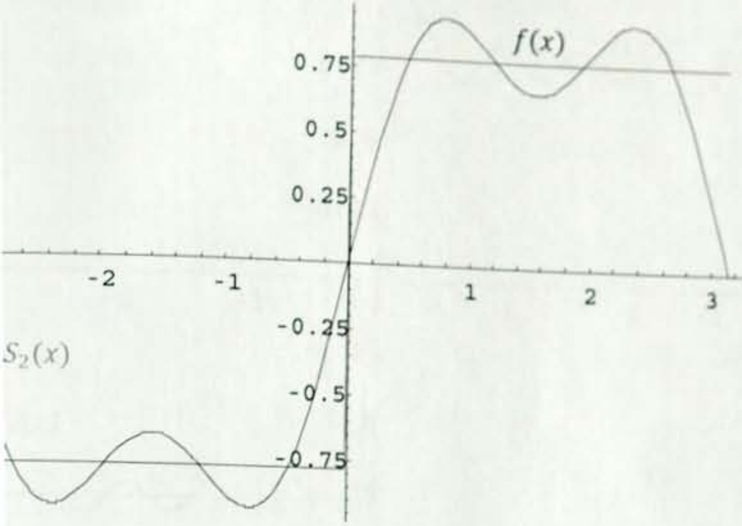
clearly f is continuous except at $x = 0$. Moreover at $x = 0$ $f(0) = \frac{f(0+)+f(0-)}{2}$ Thus by
evious remarks then the Fourier series of the function converge to the value of the function
ch x in the indicated interval.

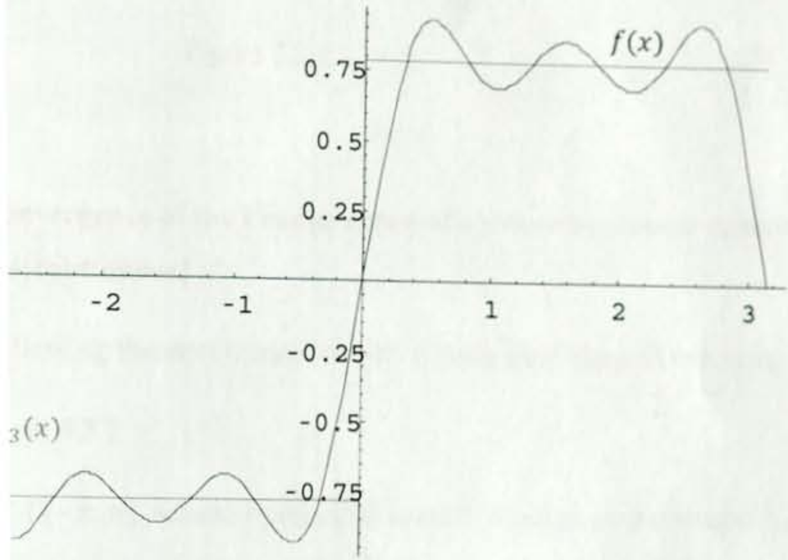
er words

$= \sum_{n=1}^{\infty} \frac{Sin(2n-1)x}{2n-1}$ on $-\pi < x < \pi$

/e some of the pictures of the Fourier series of the function and the graph of the function
indicated interval.




f(x)
$S_2(x)$
0.75
0.5
0.25
-0.25
-0.5
-0.75
-2
-1
1
2
3


f(x)
0.75
0.5
0.25
-0.25
-0.5
-0.75
-2
-1
1
2
3

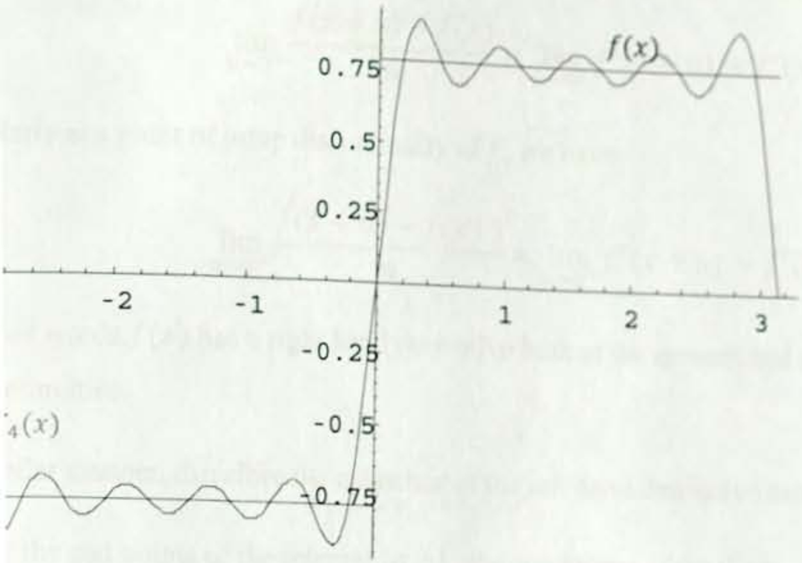


Figure 5.2.2

ɔnvergence of the Fourier series of a piecewise smooth function [Continuous or iscontinuous]

llowing theorem is an automatic consequence the previous remarks.

em 5.3.1

$L[-\pi, \pi]$, which is piecewise smooth function on the interval$[a, b]$, then for all$x \in (a, b)$, ırier series of$f$, converge to $f(x)$ at the point of continuity and to the value of $\frac{f(x+)+f(x-)}{2}$ ·oint of discontinuity. And the convergence may fail at$x = a$ *and* $x = b$.

orem is a simple consequence of the fact that a piecewise smooth function on $[a, b]$must eft and right hand derivative for all $x \in (a, b)$. So we need to apply the previous two ; then after.

ɔbvious at the point where $f(x)$ has a derivative.

orners of $f(x)$, We use the L'Hopital's rule.

$$\lim_{u\to 0^+}\frac{f(x+u)-f(x)}{u}=\lim_{u\to 0^+}f'(x+u)=f'(x+)$$

larly at a point of jump discontinuity of f, we have

$$\lim_{u\to 0^+}\frac{f(x+u)-f(x+)}{u}=\lim_{u\to 0^+}f'(x+u)=f'(x+)$$

ıer words, $f(x)$ has a right hand derivative both at the corners and at a point of jump ntinuities.

ıilar manner, therefore the existence of the left hand derivative can also be too.

r the end points of the interval $[a,b]$, the conditions of the theorem implies only that the ıand derivative at $x=a$ and left hand derivative at $x=b$. Therefore the criterions of the ›us remark at these points are not fully assured.

ver, if the interval $[a,b]$ is of length 2π, then $f(x)$ is piecewise smooth on the whole of x- ince $f(x)$ is periodic. Hence in that case the Fourier series converges everywhere to the ɔf f on the whole x-axis.

›solute and Uniform convergence of the Fourier series of a continuous piecewise ıooth function of period 2π

ext let us discuss on the issue of the absolute and uniform convergence of the Fourier ʼor the case of a continuous function f with the following theorem.

:m 5.4.1

e continuous piecewise smooth function on the interval $[-\pi,\pi]$.

$\}$ *and* $\{b_n\}$ are the Fourier coefficients of f.

$^{\infty}_{n=1}(|a_n|+|b_n|)<\infty$

s continuous piecewise smooth function on the interval $[-\pi,\pi]$, then $f'(x)$ exists ıere except at the corners of $f(x)$, and is bounded function.

efore applying the integration by parts, we have

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) Cosnx\, dx$

$= \frac{1}{n\pi}[f(x) Sin\, nx]_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x) Sin\, nx\, dx$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) Sin\, nx\, dx$

$= -\frac{1}{n\pi}[f(x) Cos\, nx]_{-\pi}^{\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x) Cos\, nx\, dx$

the values in the brace of the two expressions vanish.

if we put a_n' *and* b_n' as Fourier coefficients of $f'(x)$, which is square integrable as it is led, then we have

$$a_n = -\frac{b_n'}{n} \quad and \quad b_n = \frac{a_n'}{n} \tag{4}$$

$f' \in L^2[-\pi, \pi]$, then by the Bessel's inequality we have

$$\sum_{n=1}^{\infty} {a_n'}^2 + {b_n'}^2 < \infty$$

ver since

$\left(|a_n'| - \frac{1}{n}\right)^2 = {a_n'}^2 - \frac{2|a_n'|}{n} + \frac{1}{n^2} \geq 0$ and $\left(|b_n'| - \frac{1}{n}\right)^2 = {b_n'}^2 - \frac{2|b_n'|}{n} + \frac{1}{n^2} \geq 0$

e

$$\frac{|a_n'| + |b_n'|}{n} \leq \frac{1}{2}\left({a_n'}^2 + {b_n'}^2\right) + \frac{1}{n^2} \tag{5}$$

ice the right hand side is the general term of a convergent series, then the series

$$\sum_{n=1}^{\infty} \frac{|a_n'| + |b_n'|}{n} < \infty \tag{6}$$

s using (4) and (6)for any continuous piecewise smooth function

$$\sum_{n=1}^{\infty} |a_n| + |b_n| < \infty$$

let us consider a trigonometric series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n Cos\, nx + b_n\, Sin\, nx$$

h is not assumed to be the Fourier series of any function, and then we have the following em.

rem 5.4.2

series $\sum_{n=1}^{\infty} |a_n| + |b_n|$ converges, then the series converges absolutely and uniformly and Fourier series of a function.

.

ıow that

$$|a_n Cos\, nx + b_n\, Sin\, nx| \leq |a_n Cos\, nx| + |b_n\, Sin\, nx| \leq |a_n| + |b_n|$$

rm of the trigonometric series do not exceed the term of the convergent series.

by the Weierestrass M-test, we can see that, the associating trigonometric series ges uniformly.

by our previous justification (Theorem 3.1.1 of chapter3.1), the series is Fourier series ıverges uniformly and absolutely.

ıry 5.4.1

series of a continuous piecewise smooth function f on the interval $[-\pi, \pi]$converges ely and uniformly to$f(x)$.

of

proof of this corollary is an automatic consequence of the previous two theorems.

nple 5.4.1

ider the function $f(x) = |x|\ on - \pi \leq x \leq \pi$

$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{Cos(2k-1)x}{(2k-1)^2}$

aking the sequence of functions $u_k(x) = \frac{Cos(2k-1)x}{(2k-1)^2}$ on $-\pi \leq x \leq \pi$

bserve that $|u_k(x)| = \left|\frac{Cos(2k-1)x}{(2k-1)^2}\right| \leq \frac{1}{(2k-1)^2}$

consider $M_k = \frac{1}{(2k-1)^2}$

$\sum_{k=1}^{\infty} M_k = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} < \infty \quad (p - series)$

by the Weierestrass M-test $\sum_{k=1}^{\infty} \frac{Cos(2k-1)x}{(2k-1)^2}$ converges uniformly.

ore $\frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{Cos(2k-1)x}{(2k-1)^2}$ converges to $f(x)$ uniformly on the indicated interval.

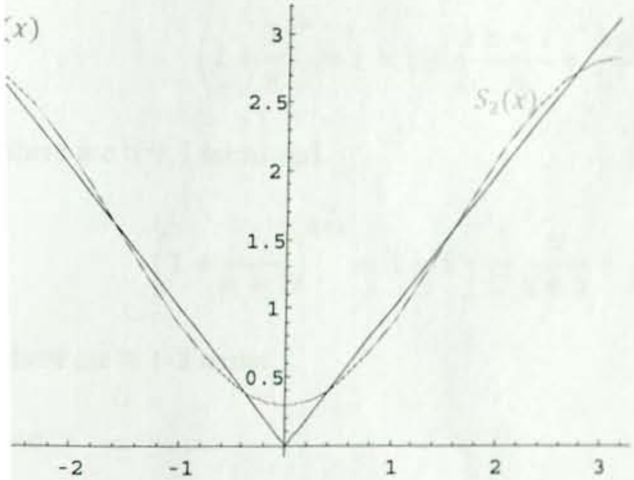


Figure 5.4.1

SECTION VI

THE EXISTANCE OF A CONTINOUS FUNCTION WHOSE FOURIER SERIES DIVERGES

ome preliminaries

im of this report is to produce a continuous function at a point and show that the ;ponding Fourier series diverges at that point. In order to make a suitable condition for the ·uction of the Fourier series for continuous function which diverges at a point, the /ing results are necessary.

are now building the necessary tools for the construction of our counter example.

ıa 6.1.1

quence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$is increasing whereas $\left\{\left(1+\frac{1}{n}\right)^{n+1}\right\}$is decreasing.

·e show only for the 1^{st}, and the second one can be done in similar ways.

ıe Binomial theorem, we have

$$\left(1+\frac{1}{n}\right)^n = 1+1+\frac{1}{2!}\frac{n-1}{n}+\frac{1}{3!}\frac{n-1}{n}\frac{n-2}{n}+\cdots$$

:here are $n+1$ terms and

$$\left(1+\frac{1}{n+1}\right)^{n+1} = 1+1+\frac{1}{2!}\frac{n}{n+1}+\frac{1}{3!}\frac{n}{n+1}\frac{n-1}{n+1}+\cdots$$

here are $n+2$ terms

ce

$$\frac{k}{n}<\frac{k+1}{n+1}, for\ k=1,2,3,\dots,n-1$$

lows that each term after the first two in the expression of $\left(1+\frac{1}{n+1}\right)^{n+1}$ exceeds the sponding terms in the expression of $\left(1+\frac{1}{n}\right)^{n}$. Moreover there is a positive term leftover in xpansion of $\left(1+\frac{1}{n+1}\right)^{n+1}$

$$\left(1+\frac{1}{n}\right)^{n} < \left(1+\frac{1}{n+1}\right)^{n+1}$$

: the sequence $\left\{\left(1+\frac{1}{n}\right)^{n}\right\}$ is increasing.

ıa 6.1.2

$< \ln\left(1+\frac{1}{n}\right) < \frac{1}{n}$

he result of the previous lemma,

$$\left(1+\frac{1}{n}\right)^{n} < e \text{ and } \left(1+\frac{1}{n}\right)^{n+1} > e$$

combining the two

$$\left(1+\frac{1}{n}\right)^{n} < e < \left(1+\frac{1}{n}\right)^{n+1}$$

ring ln on both sides, we get

$$n\ln\left(1+\frac{1}{n}\right) < 1 < (n+1)\ln\left(1+\frac{1}{n}\right)$$

re $\frac{1}{n+1} < \ln\left(1+\frac{1}{n}\right) < \frac{1}{n}$

ıma 6.1.3

Harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges

f

ȝ lemma 6.1.2, we have $\frac{1}{n} > \ln\left(\frac{n+1}{n}\right)$

e $1 > \ln 2,\ \frac{1}{2} > \ln\frac{3}{2},\ \frac{1}{3} > \ln\frac{4}{3}, \dots$

h imply

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \ln 2 + \ln\frac{3}{2} + \ln\frac{4}{3} + \cdots + \ln\left(\frac{n+1}{n}\right)$$

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \ln(n+1)$$

ıeans

$$\ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = x_n \qquad (1)$$

ıe sequence on the left is divergent hence, then the sequence on the right side is divergent

neans, the Harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges .

s the next lemma, which has the most important use to the construction of a continuous whose Fourier series diverges.

part of the lemma is on uniform boundedness of a certain function which depends ınd x. Whereas the second part of it is simply responsible to change the form of a in convenient one.

ıma 6.1.4

2 $Sin\ mx\ \sum_{k=1}^{n}\frac{Sinkx}{k}$ is uniformly bounded and is independent of x and the integers $m\&n$

$$2\ Sin\ mx\ \sum_{k=1}^{n}\frac{Sinkx}{k}=\frac{Cos(m-n)x}{n}+\frac{Cos(m-n+1)x}{n-1}+\cdots+\frac{Cos(m-1)x}{1}$$

$$-\frac{Cos(m+1)x}{1}-\cdots-\frac{Cos(m+n)x}{n}$$

f

ow that 2 $Sin\ mx\ \sum_{k=1}^{n}\frac{Sinkx}{k}$ is uniformly bounded, it is sufficient to show that $\sum_{k=1}^{n}\frac{Sinkx}{k}$

formly bounded

$$S_n(x)=\sum_{k=1}^{n}\frac{Sinkx}{k}\text{ at a point}x,\ 0<x\leq\pi$$

·onsider

$V=\left[\frac{1}{x}\right]$, that means $V$is the greatest integer such that $V\leq\frac{1}{x}<V+1$

the inequality$|Sin\ x|\leq|x|$, we have

$\leq V$

$$|S_n(x)|=\left|\sum_{k=1}^{n}\frac{Sinkx}{k}\right|\leq\sum_{k=1}^{n}\frac{kx}{k}=nx\leq Vx\leq 1 \qquad (2)$$

$\cdot V$

ıe result in (2), we obtain that

$$|S_n(x)|\leq|S_V(x)|+\left|\sum_{V+1}^{n}\frac{Sinkx}{k}\right|\leq 1+\frac{1}{V+1}\frac{1}{Sin\left(\frac{x}{2}\right)} \qquad (3)$$

last inequality is obtained using the property that

$|\sum_{k=m}^{n} \tau_k Sinkx| \leq \frac{\tau_m}{Sin(\frac{x}{2})}$ Where $\tau_k \xrightarrow[decrease]{} 0\ as\ k \to \infty$ (4)

'an be justified as

$$\left|\tau_k Sinkx\, Sin\left(\frac{x}{2}\right)\right| = \frac{1}{2}\left|\sum_{k=m}^{n} \tau_k \left[Cos\left(k - \frac{1}{2}\right)x - Cos\left(k + \frac{1}{2}\right)x\right]\right|$$

$$= \frac{1}{2}\left|\tau_m Cos\left(m - \frac{1}{2}\right)x + \sum_{k=m}^{n-1}(\tau_{k+1} - \tau_k)Cos\left(k + \frac{1}{2}\right)x - \tau_n Cos\left(n + \frac{1}{2}\right)x\right|$$

$$\leq \frac{1}{2}\left(\tau_m + \sum_{k=m}^{n-1}(\tau_k - \tau_{k+1}) + \tau_n\right) = \tau_m$$

$$\left|\sum_{k=m}^{n} \tau_k Sinkx\right| \leq \frac{\tau_m}{Sin\left(\frac{x}{2}\right)}$$

ince the function $Sin\ t$ is concave on $\left(0, \frac{\pi}{2}\right)$

ve $Sin\ t \geq \frac{2}{\pi} t$ on $0 \leq t \leq \frac{\pi}{2}$

$(V+1)Sin\left(\frac{x}{2}\right) \geq (V+1)\frac{x}{\pi} \geq \frac{1}{\pi}$, on $0 < x \leq \pi$ (5)

y (4) and (5)

$$|S_n(x)| \leq 1 + \pi = K \text{ for } n \in \boldsymbol{N}\ and\ 0 < x \leq \pi$$

ıce $S_n(x)$ is an odd function and of period 2π and furthermore $S_n(0) = 0$, we have

$\leq K$ for each $x \in \boldsymbol{R}$

re

$$\left|2\ Sin\ mx \sum_{k=1}^{n} \frac{Sinkx}{k}\right| \leq 2K \qquad (6)$$

s the uniform boundedness of $2\ Sin\ mx \sum_{k=1}^{n} \frac{Sinkx}{k}$ is true. And moreover the expression is pendent of $x, m\ and\ n$

v lastly let us prove the trigonometric identity.

:an put

$$\text{ː } mx \sum_{k=1}^{n} \frac{Sinkx}{k} = \sum_{k=1}^{n} \frac{2\ Sin\ mx\ Sin\ kx}{k}$$

$$= \sum_{k=1}^{n} \frac{1}{k} [Cos(m-k)x - COs(m+k)x]$$

$$= \frac{Cos(m-n)x}{n} + \frac{Cos(m-n+1)x}{n-1} + \cdots + \frac{Cos(m-1)x}{1}$$

$$- \frac{Cos(m+1)x}{1} - \cdots - \frac{Cos(m+n)x}{n}$$

ıt this moment we are ready to describe a continuous function where the Fourier series ;es for a given value of$x$.

jér's example of a continuous function with divergent Fourier series

preceding section we obtained some very general conditions which assure that a function ıl to the sum of its Fourier series. The question arises if every function or at least every ıous function is the sum of its Fourier series.

5, Du Bois Raymond found a function of period $2\pi$and continuous everywhere, but whose series diverges at a point and thus does not represent function there. Afterwards, several uthors found more such examples. But we shall now give an example which is issued by r.

n be positive integers and define

$$S(x, m, n) = 2\ Sin\ mx \sum_{k=1}^{n} \frac{Sinkx}{k}$$

ce by the previous lemma result, we have

$$x,m,n) = \frac{Cos(m-n)x}{n} + \frac{Cos(m-n+1)x}{n-1} + \cdots + \frac{Cos(m-1)x}{1} - \frac{Cos(m+1)x}{1} - \cdots$$

$$-\frac{Cos(m+n)x}{n} \quad (7)$$

also the function $S(x,m,n)$ is uniformly bounded. That is there exist $M_1 > 0$ such that

$$|S(x,m,n)| < M_1 \quad for\ each\ x, m\ and\ n$$

ı the expression in (7), the sum of the first n terms of $S(0,m,n)$ is

$$\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{2} + 1$$

e by (1), we can assign

$$\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{2} + 1 > \ln n \quad (8)$$

: consider the following three conditions

A positive sequence $\{a_k\}$ such that $\sum_{k=1}^{\infty} a_k < \infty$

A positive sequence $\{n_k\}$ such that $a_k \ln n_k$

does not converge to zero as $k \to \infty$

A positive $\{m_k\}$ such that sequence

$m_k + n_k < m_{k+1} - n_{k+1}$ $k = 1,2,3 \ldots$

ɜt us consider the series

$$\sum_{k=1}^{\infty} a_k S(x, m_k, n_k) \quad (9)$$

e observe that

$, m_k, n_k)| < a_k M_1$

/ the Weierestrass M-test the series in (9) converges uniformly.

efore since each terms of the sequence $\{a_k S(x, m_k, n_k)\}$ is continuous, then series (9) erges to the continuous limit function, say f. Here the continuity is on the whole of $\boldsymbol{R}$, and e at 0.

let as discuss a little bit about the Fourier series of the function f.

the cosine polynomials

$n, n)$ is an even function and has period 2π. Thus the Fourier series of f is a pure cosine . Hence

$$f(x) \sim \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m Cosmx \qquad (10)$$

rom the uniform convergence of series in (9),and term-by-term integration is possible, we

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) Cosmxdx = \frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{k=1}^{\infty} a_k S(x, m_k, n_k) Cosmxdx$$

$$= \sum_{k=1}^{\infty} a_k \frac{1}{\pi}\int_{-\pi}^{\pi} S(x, m_k, n_k) Cosmxdx$$

onsidering $S's$ as being cosine polynomial and using the some auxiliary integrals ed in section 2.5,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} S(x, m_k, n_k) Cosmxdx$$

l to the coefficient of $Cosmx$ in $S(x, m_k, n_k)$ and is zero if no such term occurs $n_k, n_k)$.

lition (3) above, two polynomials $S(x, m_k, n_k)$ with different subscript can not contain ith the same $Cosmx$.

e, we obtain the Fourier series of $f$by writing out the terms of each cosine polynomials in
nd multiplying by the corresponding coefficientsA_m. In other words (9) is nothing but the
er series of the function f.

let us display out that this Fourier series is divergent at the point$x = 0$.

out S_n is the n^{th} partial sum of the Fourier series (9). That means the n^{th} partial sum of the
keted series of (9). Hence using the result in (8), we have

$$0) - S_{m_{k-1}-n_{k-1}}(0)|$$

$$= a_k\left(\frac{1}{n_k} + \frac{1}{n_k - 1} + \cdots + \frac{1}{1}\right) > a_k \ln n_k \tag{11}$$

he condition (2), we know that the sequence $\{a_k \ln n_k\}$does not converge to zero
ugh $m_k, m_{k-1} - n_{k-1} \infty \rightarrow$) Hence the sequence $\{S_n(0)\}$ is not a Cauchy sequence. Thus
urier series does not converge. That means it diverge at zero.

we have constructed function, in which it is continuous at a point, however it's Fourier
s divergent.

le 6.2.1

er the sequences $a_k = \frac{1}{k^2}, n_k = 2^{k^2}$ *and* $m_k = n_k + 2^{k^3}$

sequences satisfy the conditions 1 up to 3 listed above. For the sake of convenience

$$\frac{1}{k^2} = \frac{1}{\pi^2} < \infty$$

$$n_k = \frac{1}{k^2} \ln 2^{k^2} = \ln 2 \quad \text{and}$$

$$+ n_k = 2^{k^2+1} + 2^{k^3} < 2^{(k+1)^3} = m_k - n_k$$

course implies the divergence at 0 of the Fourier series of the function which is
us at 0, where it defined in the form (9) above.

e have constructed a continuous function $f$at $x = 0$and yet the Fourier series diverges

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