MUON DECAY IN A LINEARLY POLARIZED LASER FIELD

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By

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Abstract

We make use of the light cone coordinators for studies on muon decay. We derive the Volkov wave function in a linearly polarized laser field. In this work we examine the process of muon decay in a linearly polarized laser field $\mu^- \rightarrow e^- + \bar{\nu}_{e^-} + \nu_\mu$. A detailed calculation of this process is carried out, using Volkove wave function for the initial muon and electron.

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Chapter 1

Introduction

All hadrons and leptons experience weak interaction, and hence can undergo weak decays. Leptons do not participate in strong interactions. The weak decay of $\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu$ which is the dominant decay of the muon is interesting. In principle, the muon decays electromagnetically via $\mu^- \rightarrow e^- + \gamma$ [1]. Muon decay involves neutrinos. Neutrinos are unique in that they can interact by weak interaction. They are colorless, electrically neutral, and massless. For muon decay, the left handed masselss neutrinos are designated by spinor:

$$(1 - \gamma_5) u_\nu(p) \tag{1.0.1}$$

and, the right handed anti-neutrino spinors are designated by spinor:

$$(1 + \gamma_5) v_\nu(p). \tag{1.0.2}$$

We calculated lowest-order graphs for weak interaction, where the momentum exchanged due to the weak current satisfies $q^2 << M_W$. The existance of $W^\pm$ simply leads to a reinterpretation of the Fermi coupling G. Weak currents involve massive vector bosons ($W^\pm, Z^0$) according to the “Standard model” for electroweak interactions. These weak bosons were first discovered by Weinberg and Salam.

We use Glashow-Salam-Weinberg(GWS) theory of weak and electromagnetic interactions among lepton and study its properties. The main features of GWS model for weak interaction are:

1. There exist charged and neutral weak currents.

2. The charged currents contain only couplings between left-handed leptons.

3.The bosons W^+, W^- and Z^0 mediating the weak interaction must be very massive.

4. This gauge bosons receive masses through the Higgs mechanism.

In this thesis we have introduced the light cone coordinates in terms of the usual space-time coordinate as:

$$a_+ = \frac{1}{2}(a_0 + a_3) \tag{1.0.3}$$

$$a_- = \frac{1}{2}(a_0 - a_3) \tag{1.0.4}$$

$$a_\perp = (a_1, a_2). \tag{1.0.5}$$

With this ,we can write the dot product of two-four-vectors as

$$a.b = 2a_+b_- + 2a_-b_+ - a_\perp b_\perp. \tag{1.0.6}$$

The light cone Dirac matrices are:

$$\gamma_+ = \gamma^0 + \gamma^3 \tag{1.0.7}$$

$$\gamma_- = \gamma^0 - \gamma^3 \tag{1.0.8}$$

$$\gamma_\perp = (\gamma_1, \gamma_2). \tag{1.0.9}$$

The Feynman slash matrix is then,

$$\not{a} = \gamma.a = \gamma_+a_- - \gamma_-a_+ - \gamma_\perp a_\perp \tag{1.0.10}$$

with the property

$$\gamma_\pm\gamma_\pm = 0 \tag{1.0.11}$$

$$\gamma_0\gamma_\pm = \gamma_\mp\gamma_0 \tag{1.0.12}$$

$$\gamma_{\mp}\gamma_{\pm} = 2\gamma_0\gamma_{\mp} \tag{1.0.13}$$

$$\gamma_{\pm}\gamma_{\perp} = -\gamma_{\perp}\gamma_{\pm}. \tag{1.0.14}$$

The gradient in terms of the light cone coordinates have been expressed as:

$$\partial_{+} = \partial_0 + \partial_3 \tag{1.0.15}$$

$$\partial_{-} = \partial_0 - \partial_3 \tag{1.0.16}$$

$$\partial_{\perp} = (\partial_1, \partial_2) \tag{1.0.17}$$

Through out the thesis, natural units are used, where $\hbar = c = 1$ with $\hbar$ being the reduced Planck's constant and c the speed of light in vacuum and $\mu,\ \nu$ taking values (0,1,2,3). The thesis is organized as follows; Volkove wave function of free muon in a laser field are desired in Chapter 2. As a mathematical preliminary, the study of free muon decay through the process $\mu^- \to e + \nu_\mu + \bar{\nu}_e$ is done in Chapter 3. In Chapter 4 we study muon decay in an intense laser field using Volkove wave function for both initial muon and electron. we calculate decay width for this process. Chapter 5 deals with summary and conclusion.

Chapter 2

Intermediate Vector Boson (IVB) and Volkov States

2.1 Intermediate Vector Bosons

The weak interactions are mediated by the exchange of intermediate vector bosons just as electromagnetic interactions are mediated by photons. The intermediate vector bosons $W^{\pm}$, Z^0 are massive in contrast to photons which is massless [2].

2.1.1 Massive Vector Boson field

At this point we need to develop the field theory of a free (non-interacting) massive vector particle. Let‘s begin with a vector field W_μ which we take to be complex, since we want to describe charged particles.

The field strength tensor for W_μ given by:

$$G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu \tag{2.1.1}$$

and, the free Lagrangian for a massive W boson field is,

$$L = -\frac{1}{2} G^*_{\mu\nu} G^{\mu\nu} + m_W^2 W^*_\mu W^\mu \tag{2.1.2}$$

where $G_{\mu\nu}$ is the field strength tensor of $W^{\pm}$ bosons.

$$G_{\mu\nu} = \partial_{\nu} W_{\mu} - \partial_{\mu} W_{\nu} \tag{2.1.3}$$

$$G^{*}_{\mu\nu} = \partial_{\nu} W^{*}_{\mu} - \partial_{\mu} W^{*}_{\nu} \tag{2.1.4}$$

Under Gauge transformation $G_{\mu\nu}$ and $G^{*}_{\mu\nu}$ are invariant.

The equation of motion for W_{μ} by using the corresponding Euler-Lagrange equations as:

$$\frac{\partial L}{\partial W^{*}_{\mu}} - \partial v(\frac{\partial L}{\partial(\partial v W^{*}_{\mu})}) = 0. \tag{2.1.5}$$

where L is given in Eq.(2.1.2),

$$\frac{\partial L}{\partial W^{*}_{\mu}} = m_W^2 W_{\mu} \tag{2.1.6}$$

$$(\frac{\partial L}{\partial(\partial_{\nu} W_{\mu})}) = -\partial_{\nu} W_{\mu} + \partial_{\mu} W_{\nu} \tag{2.1.7}$$

$$\partial_{\nu}(\frac{\partial L}{\partial(\partial_{\nu} W^{*}_{\mu})}) = \partial_{\nu}(-\partial_{\nu} W_{\mu} + \partial_{\mu} W_{\nu}) \tag{2.1.8}$$

Under Lorentz gauge condition,

$$\partial_{\nu} W^{\nu} = 0, \tag{2.1.9}$$

we obtain the equation of motion of W-boson field W_{μ} is,

$$(\Box + m_W^2) W_{\mu}(x) = 0 \tag{2.1.10}$$

This is the equation of motion for $W_{\mu}(x)$.

Similarly the equation of motion for W^{*}_{μ} field as,

$$(\Box + m_W^2) W^{*}_{\mu}(x) = 0 \tag{2.1.11}$$

There are three independent polarization vectors that satisfy the Lorentz condition $\epsilon.k$=0 with $k^2 = m_{\mu}^2$ for a W-boson moving in the +z direction with

$$k^{\mu} = (w, 0, 0, w); w^2 = k^2 + m_W^2 \tag{2.1.12}$$

A convenient basis of polarization vector is,

$$\epsilon^{\pm} = \frac{1}{\sqrt{2}}(0, 1, \pm\imath, 0) \tag{2.1.13}$$

$$\epsilon^{0} = \frac{1}{\sqrt{m_W}}(k, 0, 0.k) \tag{2.1.14}$$

which are the two transverse polarization vectors and the longitudinal polarization vector

The transverse polarizations have helicity ±1, while the longitudinal polarization has helicity 0. These obey the orthogonality / completeness relations

$$\sum_{\mu} \epsilon^{\imath *}_{\mu} \epsilon^{\jmath \mu} = -\delta \imath \jmath \tag{2.1.15}$$

$$\sum_{\imath} \epsilon^{\imath}_{\mu} \epsilon^{\imath *}_{\nu} = -g_{\mu\nu} + \frac{k_{\mu} k_{\nu}}{m_W^2} \tag{2.1.16}$$

To reproduce the successes of 4-Fermi theory the $W^{\pm}$ must have some unusual properties.

1,It must be massive, with $m_W = (\frac{1}{\sqrt{G_F}})$,so that at energies $<< m_W$ we recover the point-like interaction of 4-fermi theory.

2,It must carry ±1 unit of electric charge, so that electric charge is conserved at each vertex in the invariant amplitude diagram .

2.2 Volkov States

In this section we want to solve the Dirac equation

$$\Big(i\gamma_\mu\partial_\mu - e\gamma_\mu\mathcal{A}_\mu - m\Big)\psi = 0 \tag{2.2.1}$$

in an external electromagnetic field with potential $\mathcal{A}_\mu$, moving with the velocity of light in a fixed direction, specified by the wave vector $\overrightarrow{k}$. The potential A_μ is assumed to depend on the space-time coordinates x through the scalar product $\varphi = k.x$

$$\mathcal{A}_\mu = \mathcal{A}_\mu(k.x) \tag{2.2.2}$$

Moreover, we will extend the general Volkov solution to the special case of linearly polarized laser field solution.

We can rewrite Eq.(2.2.1), by separating the space and time parts as [3].

$$\Big[i\gamma_0\partial_0 + i\gamma_i\partial_i - e(\gamma_0\mathcal{A}_0 - \gamma_i\mathcal{A}_i) - m\Big]\psi = 0 \tag{2.2.3}$$

We now introduce the concept of the technique of light-cone variables, which is useful in problems involving motion at the speed of light [3].

Thus, in light-cone coordinates Eq.(2.2.3) becomes

$$\Big\{i\left[\frac{1}{2}(\gamma_+ + \gamma_-)\frac{1}{2}(\partial_+ + \partial_-) + \frac{1}{2}(\gamma_+ + \gamma_-)\frac{1}{2}(\partial_+ - \partial_-) + \gamma_\perp\partial_\perp\right]$$

$$+ e\left[\frac{1}{2}(\gamma_+ + \gamma_-)(\mathcal{A}_+ + \mathcal{A}_-) - \frac{1}{2}(\gamma_+ - \gamma_-)(\mathcal{A}_+ - \mathcal{A}_-) - \gamma_\perp\mathcal{A}_\perp\right] - m\Big\}\psi = 0 \tag{2.2.4}$$

Simplifying terms we get,

$$\left[i\frac{1}{2}\gamma_+\partial_+ + i\frac{1}{2}\gamma - \partial_- + i\gamma_\perp\partial_\perp - e(\gamma_+\mathcal{A}_- + \gamma_-\mathcal{A}_+ - \gamma_\perp\mathcal{A}_\perp) - m\right]\psi = 0 \tag{2.2.5}$$

We can now assume without loss of generality that the wave moves in the x_3 direction. Thus,

$$k_\mu = \omega(1,0,0,1), \tag{2.2.6}$$

which implies that

$$k_+ = \omega \neq 0 \tag{2.2.7}$$

and

$$k_- = k_\perp = 0 \tag{2.2.8}$$

The Lorentz gauge condition is given by

$$k.\mathcal{A} = 2k_+\mathcal{A}_- + 2k_-\mathcal{A}_+ - k_\perp\mathcal{A}_\perp = 0 \tag{2.2.9}$$

which immediately implies that

$$\mathcal{A}_- = 0 \tag{2.2.10}$$

The Dirac equation then reads

$$\left[i\frac{1}{2}\gamma_+\partial_+ + i\frac{1}{2}\gamma_-\partial_- + i\gamma_\perp\nabla_\perp - e\gamma_-\mathcal{A}_+ + e\gamma_\perp\mathcal{A}_\perp - m\right]\psi = 0 \tag{2.2.11}$$

The potential is assumed to move with the velocity of light in the x_3 direction and will only depend on the variable x_- [3]. Thus,

$$\mathcal{A}_+ = \mathcal{A}_+(x_-) \quad and \quad \mathcal{A}_- = \mathcal{A}_-(x_-) \tag{2.2.12}$$

The motion of the electron in the x_+ and $x_\perp$ directions can be described by a plane wave

$$\psi = \psi(x_+, x_-, x_\perp) = N_p\ exp(-ip.x)\phi(x_-) \tag{2.2.13}$$

where

$$p.x = 2p_+x_- + 2p_-x_+ - p_\perp x_\perp \tag{2.2.14}$$

Upon substituting Eq.(2.2.13) and Eq(14) into Eq.(2.2.11), there follows

$$\left[i\frac{1}{2}\gamma_+\partial_+ + i\frac{1}{2}\gamma_-\partial_- + i\gamma_\perp\nabla_\perp - e\gamma_-\mathcal{A}_+ + e\gamma_\perp\mathcal{A}_\perp - m\right]$$

$$exp\ i(2p_+x_- + 2p_-x_+ - p_\perp x_\perp)\ \phi(x_-) = 0, \tag{2.2.15}$$

which leads to

$$\Big[-\frac{1}{2}\gamma_+(2p_-) - \gamma_-(2p_+) + e\gamma_-\mathcal{A}_+ + e\gamma_\perp.\mathcal{A}_\perp - m\Big]$$

$$exp\ i(2p_+x_- + 2p_-x_+ - p_\perp x_\perp)\ \phi(x_-)$$

$$+\Big[i\frac{1}{2}\gamma_-\partial_-\phi(x_-)\ exp\ i(2p_+x_- + 2p_-x_+ - p_\perp x_\perp)\Big] = 0 \tag{2.2.16}$$

Simplifying terms we finally obtain Dirac equation in light cone coordinates as,

$$\Big[i\frac{1}{2}\gamma_-\partial_- - \gamma_+p_- - \gamma_-p_+ - \gamma_\perp p_\perp - e\gamma_-\mathcal{A}_+ + e\gamma_\perp\mathcal{A}_\perp - m\Big]\phi(x_-) = 0 \tag{2.2.17}$$

Splitting the wave function into its light-cone projections we have

$$\phi = \frac{1}{2}(\phi_+ + \phi_-) \quad with \quad \phi_\pm = \gamma_0\gamma_\pm\phi, \tag{2.2.18}$$

satisfying we get,

$$\gamma_\pm\phi_\mp = 0 \quad and \quad \gamma_\pm\phi_\pm = 2\gamma_0\phi_\pm. \tag{2.2.19}$$

The Dirac Eq.(2.2.17) upon substitution of Eq.(2.2.18) and Eq.(2.2.19) becomes

$$\Big[\gamma_+p_- - \gamma_\perp(p_\perp - e\mathcal{A}_\perp) - m\Big]\phi_+$$

$$+\Big[i\frac{1}{2}\gamma_-\partial_- + \gamma_-(p_+ - e\mathcal{A}_+) - \gamma_\perp(p_\perp - e\mathcal{A}_\perp) - m\Big]\phi_- = 0 \tag{2.2.20}$$

Multiplying Eq.(2.2.20) from left by γ_- and applying the relations in Eq.(1.0.11), Eq.(1.0.12), Eq.(1.0.13) and Eq.(1.0.14) we obtain

$$\Big[2\gamma_0\gamma_+p_-\Big]\phi_+ + \Big[2\gamma_\perp\gamma_0(p_\perp - e\mathcal{A}_\perp) - 2\gamma_0 m\Big]\phi_- = 0 \tag{2.2.21}$$

This can in turn be expressed as

$$\Big[2\gamma_0\gamma_+p_-\Big]\phi_+ = 2\gamma_0\Big[\gamma_\perp(p_\perp - e\mathcal{A}_\perp) + m\Big]\phi_-. \tag{2.2.22}$$

We can write ϕ_+ in terms of ϕ_- as

$$\phi_+ = \frac{\gamma_0}{2p_-}\Big[\gamma_\perp(p_\perp - e\mathcal{A}_\perp) + m\Big]\phi_-. \tag{2.2.23}$$

Now we multiply Eq(2.2.20) from left by γ_+ as

$$\Big[i\partial_- + 2(p_+ - e\mathcal{A}_+)\Big]\phi_- + \Big[\gamma_\perp(p_\perp - e\mathcal{A}_\perp)\gamma_0. - m\gamma_0\Big]\frac{\gamma_0}{2p_-}\Big[\gamma_\perp(p_\perp - e\mathcal{A}_\perp) + m\Big]\phi_- = 0. \tag{2.2.24}$$

This is reduced to

$$\Big[i\partial_- - \Big(- 4p_+p_- + 4e\mathcal{A}_+p_- + (p_\perp - e\mathcal{A}_\perp)^2 + m^2\Big)\Big]\phi_- = 0, \tag{2.2.25}$$

where we used the relations

$$\gamma_\perp\gamma_0 = -\gamma_0\gamma_\perp \quad and \quad (\gamma_\perp.\mathcal{A}_\perp)^2 = -\mathcal{A}_\perp^2. \tag{2.2.26}$$

Now the square of the field is given by

$$\mathcal{A}^2 = 4\mathcal{A}_+\mathcal{A}_- - \mathcal{A}_\perp^2 = -\mathcal{A}_\perp^2. \tag{2.2.27}$$

Thus,

$$\Big[i\partial_- - \frac{1}{2p_-}\Big(- 4p_+p_- + 4e\mathcal{A}_+p_- + p_\perp^2 + e^2\mathcal{A}_\perp^2 - 2ep_\perp\mathcal{A}_\perp + m^2\Big)\Big]\,\phi_- = 0 \tag{2.2.28}$$

Now we write

$$p^2 = 4p_+p_- - p_\perp^2. \tag{2.2.29}$$

Thus Eq.(2.2.28) becomes

$$\Big[i\partial_- - \frac{1}{2p_-}(4e\mathcal{A}_+p_- - 2ep_\perp\mathcal{A}_\perp - e^2\mathcal{A}^2)\Big]\,\phi_- = 0, \tag{2.2.30}$$

where we have used the condition

$$p^2 = m^2 \tag{2.2.31}$$

Substituting

$$2\mathcal{A}.p = 4\mathcal{A}_+p_- - 2\mathcal{A}_\perp p_\perp \tag{2.2.32}$$

we arrive at the most simplified Dirac equation

$$\left[i\partial_- - \frac{1}{2p_-}(2e\mathcal{A}.p - e^2\mathcal{A}^2)\right]\phi_- = 0 \tag{2.2.33}$$

Solution of Eq.(2.2.33) is

$$\phi_-(x_-) = \phi_0 \; exp\,(-i\Phi(x_-)) \tag{2.2.34}$$

with the phase

$$\Phi_-(x_-) = \int_0^{x_-} \left(\frac{e\mathcal{A}.p}{p_-} - \frac{e^2\mathcal{A}^2}{2p_-}\right) \tag{2.2.35}$$

and ϕ_0 is a constant spinor satisfying

$$\gamma_+\phi_0 = 0. \tag{2.2.36}$$

We can choose it to be

$$\phi_0 = \gamma_0\gamma_- u(p), \tag{2.2.37}$$

$u(p)$ being a unit spinor satisfying the free Dirac equation in$(x_0, ...x_3)$ notation x_0=ct.

$$(\gamma.p + m)u(p) = 0 \tag{2.2.38}$$

We substitute Eq.(2.2.23) and Eq.(2.2.34) into Eq.(2.2.18) and obtain

$$\phi = \frac{1}{2}\left[1 + \frac{1}{2p_0}\gamma_0\gamma_\perp(p_\perp - e\mathcal{A}_\perp) + m\right]\gamma_0\gamma_- u(p)exp(-i\Phi)$$

$$= \frac{1}{2}\left[\gamma_0\gamma_- + \frac{1}{2p_-}\gamma_0\left(\gamma_\perp\gamma_0\gamma_-(p_\perp - e\mathcal{A}_\perp) + \gamma_0\gamma_- m\right)\right]u(p)exp(-i\Phi)$$

$$= \frac{1}{2}\left[\gamma_0\gamma_- + \frac{1}{2p_-}\left(-\gamma_\perp\gamma_-(p_\perp - e\mathcal{A}_\perp) + \gamma_- m\right)\right]u(p)exp(-i\Phi)$$

$$= \frac{1}{2}\left[\gamma_0\gamma_- + \frac{1}{2p_-}\left(\gamma_-\gamma_\perp(p_\perp - e\mathcal{A}_\perp) + \gamma_- m\right)\right]u(p)exp(-i\Phi) \tag{2.2.39}$$

Commuting $\gamma_0\gamma_-$ to the left and using the Dirac equation in the form

$$(\gamma_\perp p_\perp + m)u(p) = (\gamma_- p_+ + \gamma_+ p_-)u(p), \tag{2.2.40}$$

we find

$$\phi = \frac{1}{2}\left[\gamma_0\gamma_- + \frac{1}{2p_-}\gamma_-\left(\gamma_+ p_- + \gamma_- p_+ - e\gamma_\perp \mathcal{A}_\perp\right)\right] u(p)exp(-i\Phi) \tag{2.2.41}$$

which in turn be expressed as

$$\phi = \frac{1}{2}\left[\gamma_0\gamma_- + \gamma_0\gamma_+ - \frac{1}{2p_-}e\gamma_-\gamma_\perp \mathcal{A}_\perp\right] u(p)exp(-i\Phi) \tag{2.2.42}$$

Thus,

$$\phi = \frac{1}{2}\left[1 - \frac{1}{2p_-}e\gamma_-\gamma_\perp \mathcal{A}_\perp\right] u(p)exp(-i\Phi) \tag{2.2.43}$$

where $\gamma_0 = \gamma_- + \gamma_+$.

Now it is possible to show that

$$\gamma_-[\gamma.\mathcal{A}] = \gamma_-[\gamma_+\mathcal{A}_- - \gamma_-\mathcal{A}_+ - \gamma_\perp\mathcal{A}_\perp] = -\gamma_-\gamma_\perp\mathcal{A}_\perp \tag{2.2.44}$$

Therefore, we finally arrive at the solution

$$\phi = \left[1 - \frac{1}{2p_-}e\gamma_-\gamma_\perp \mathcal{A}_\perp\right] u(p)exp(-i\Phi) \tag{2.2.45}$$

Finally, the solution for the Dirac equation in an external laser field becomes

$$\psi = Nexp(-ip.x)\phi(x_-). \tag{2.2.46}$$

It then follows that

$$\psi_e(x) = N\left(1 + \frac{1}{2p_-}e\gamma_-\not{\mathcal{A}}\right) u(p)exp\left(-ip.x - i\Phi\right) \tag{2.2.47}$$

The solution found can be immediately generalized to an arbitrary direction of the vector k by replacing p_- and γ_- as,

$$\Phi(x_-) = \int_0^{x_-} dx_-^{'}\left(\frac{e\mathcal{A}.p}{p_-} - \frac{e^2\mathcal{A}^2}{2p_-}\right) \tag{2.2.48}$$

$$= \int_0^{\varphi} d\varphi\left(\frac{e\mathcal{A}.p}{k.p} - \frac{e^2\mathcal{A}^2}{2k.p}\right), \tag{2.2.49}$$

where

$$2\omega p_- = k.p$$

$$\omega\gamma_{-} = \gamma.k = \not{k} \tag{2.2.50}$$

With the help of Eq.(2.2.50) we arrive at the Volkov solution

$$\psi = N\left(1 + \frac{e\not{k}\not{\mathcal{A}}}{2k.p}\right) u(p)\ exp(-ip.x - i\Phi). \tag{2.2.51}$$

The electron bispinor has to be normalized by demanding that

$$\bar{u}_{p,s}u_{p,s'} = 2m\delta_{ss'} \tag{2.2.52}$$

This leads to the conclusion that the normalizer above is

$$N = \sqrt{\frac{m}{EV}}, \tag{2.2.53}$$

where V is the normalization volume.

Therefore, the complete solution of the Dirac equation for an electron in an electromagnetic field is given by:

$$\psi = \sqrt{\frac{m}{EV}}\left(1 + \frac{e\not{k}\not{\mathcal{A}}}{2k.p}\right) u(p)\ exp(\pm if) \tag{2.2.54}$$

with f the phase term

$$f = -px - \int_0^{\varphi} d\varphi \left(\frac{e(p\mathcal{A})}{(kp)} - \frac{e^2\mathcal{A}^2}{2(kp)}\right) \tag{2.2.55}$$

and $\varphi = k.x$ and E denotes the energy of the particle in the field.

Here we note that the exact solution of the Dirac equation for an electron Eq.(54) in the absence of an electromagnetic field $A_\mu = 0$ is reduced to the free-field solution

$$\psi = \sqrt{\frac{m}{EV}}u(p)\ e^{-ip.x}$$

The the effective mass m^* also reduces to the electron mass m.

For a linearly polarized plane wave laser field we write the potential in the form

$$\mathcal{A}(x) = a\ cos(\varphi), \tag{2.2.56}$$

where a indicates the amplitude of the vector potential and the polarization is chosen as $\epsilon = (0, 1, 0, 0)$.

Substituting the $\mathcal{A}(x)$ into the phase term we obtain

$$f = -px - \int_0^{\varphi} d\varphi \left(\frac{eap\ cos(\varphi)}{(kp)} - \frac{e^2a^2\ cos^2(\varphi)}{2(kp)} \right) \tag{2.2.57}$$

$$f = -px - \frac{1}{2kp} \int_0^{\varphi} d\varphi \Big(2eap\ cos(\varphi) - e^2a^2\ cos^2(\varphi) \Big) \tag{2.2.58}$$

$$= -px - \frac{1}{2kp} \left[2eap\ sin(\varphi) - e^2a^2 \left(\frac{1}{4} sin(2\varphi) + \frac{1}{2}\varphi \right) \right], \tag{2.2.59}$$

where we have used the integrals

$$\int_0^{\varphi} d\varphi\ cos(\varphi) = sin(\varphi)$$

$$\int_0^{\varphi} d\varphi\ cos^2(\varphi) = \frac{1}{4} sin(2\varphi) + \frac{1}{2}\varphi \tag{2.2.60}$$

Rearranging terms we obtain

$$f = -qx - \frac{eap\ sin(\varphi)}{2(kp)} + \frac{e^2a^2\ sin(2\varphi)}{(8kp)}, \tag{2.2.61}$$

where

$$q = p - \frac{e^2a^2k}{(4kp)} \tag{2.2.62}$$

is the effective four-momentum of the particle in the laser field.

Now the full Volkov solution for a linearly polarized plane wave laser field is written as

$$\psi(\varphi) = \sqrt{\frac{m}{EV}} \left(1 + \frac{e\not{k}\not{\mathcal{A}}}{2(kp)} \right) u(p)\ exp \left[-iqx - \frac{ieap\ sin(\varphi)}{(kp)} + \frac{ie^2a^2\ sin(2\varphi)}{(8kp)} \right], \tag{2.2.63}$$

These are the eigen wave functions of the particles in the laser field.

Chapter 3

Free Muon Decay

3.1 The Decay of Muon

Free Muon decay through the process $\mu^- \rightarrow e^-\bar{\nu}_e\nu_\mu$, was first studied using the original Fermi theory of weak interaction. The decay implies a change in the state of the muon,because the interaction that causes it is weak. It can be described in the framework of Fermi theory for point interaction as in Fig 3.1

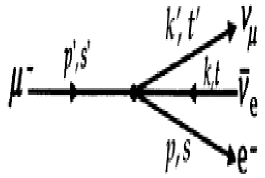

Figure 3.1: Momenta and spins for muon decay

The S-matrix element S_{fi} for a transition from an initial state $|i\rangle$ to a final state $|f\rangle$ can be expressed as:

$$S_{fi} = -\imath \int_{-\infty}^{\infty} dt H_{int}^{L}(\mu^- \to e^- \bar{\nu}_e \nu_\mu), \tag{3.1.1}$$

and the interaction Hamiltonian(H_{int}^{L}) is given by:

$$H_{int}^{L} = \frac{G}{\sqrt{2}} \int d^3(x)[\bar{u}_{\nu\mu}(x)\gamma_\mu(1-\gamma_5)u_\mu(x)][\bar{u}_e(x)\gamma_\mu(1-\gamma_5)u_{\nu e}(x)], \tag{3.1.2}$$

where, $|f\rangle \neq |i\rangle$, and G is weak interaction coupling constant.

For this first-order approximation we may choose free wave functions to describe the four particles with four-momenta $p, p^{'}, k, k^{'}$ and spins s, s , t, t respectively. According to the Feynman rules the (outgoing) anti neutrino is represented by an (incoming) wave function with negative energy (see Fig. 3.1). We employ the form of the plane wave solution,

$$u_\mu(x) = (2E_\mu V)^{-1/2} u_\mu(p^{'}, s^{'}) \exp(-\imath p^{'}_{\mu} x_\mu) \tag{3.1.3}$$

$$u_e(x) = (2E_e V)^{-1/2} u_e(p, s) \exp(-\imath p_\mu x^\mu) \tag{3.1.4}$$

$$u_{\bar{\nu} e}(x) = (2E_{\bar{\nu} e} V)^{-1/2} v_{\nu e}(k, t) \exp(\imath k_\mu x_\mu) \tag{3.1.5}$$

$$u_{\nu\mu(x)} = (2E_{\nu\mu} V)^{-1/2} u_{\nu\mu}(k^{'}, t^{'}) \exp(-\imath k^{'}_{\mu} x \mu) \tag{3.1.6}$$

$u_\mu(x), u_e(x), u_{\bar{\nu} e}(x)$ and $u_{\nu\mu}(x)$ are the free wave function of muon,electron, muon neutrino and electron anti neutrino

where

$$E_\mu = p^{'0}, E_e = p^0, E_{\nu e} = k^0, E^{\nu\mu} = k^{'0} \tag{3.1.7}$$

and $u(p, s), v(p, s)$ denotes the spinor part(E positive)

$$u(p, s) = (E+m)^{1/2} \begin{pmatrix} \chi \\ \frac{\sigma . p \chi}{E+m} \end{pmatrix} \tag{3.1.8}$$

$$\upsilon(p,s) = (E+m)^{1/2}\begin{pmatrix}\frac{\sigma.p\chi}{E+m}\\ \chi_s\end{pmatrix} \tag{3.1.9}$$

with the two- component unit spinors χ_s.

Substituting this expression in to the matrix element, Eq.(3.1.1) yields the S-matrix element

$$S(\mu^- \to e^-\bar{\nu}_e\nu_\mu) = \frac{-\imath G}{\sqrt{2}}\int d^4x \frac{\exp[\imath(k'_\mu - p'_\mu + p_\mu + k_\mu)x_\mu]}{[16k'^0Vp'^0Vp^0Vk^0V]^{1/2}} \tag{3.1.10}$$

$$\times\,[\bar{u}_{\nu\mu}(k',t')\gamma_\mu(1-\gamma_5)u_\mu(p',s')],\ \times[\bar{u}_e(p,s)\gamma_\mu(1-\gamma_\mu)\upsilon_{\nu e}(k,t)]$$

where $\int d^4x \exp[\imath(k'_\mu - p'_\mu + p_\mu + k_\mu).x_\mu] = (2\pi)^4\delta^4(p+k+k'-p')$

$$S(\mu^- \to e^-\bar{\nu}_e\nu_\mu) = -\imath(2\pi)^4\frac{G\delta^4(p+k+k'-p')}{\sqrt{2}[16V^4k'^0p'^0p^0k^0]^{1/2}} \tag{3.1.11}$$

$$\times\,[\bar{u}_{\nu\mu}(k',t')\gamma_\mu(1-\gamma_5)u_\mu(p',s')]\times[\bar{u}_e(p',s)\gamma_\mu(1-\gamma_5)\upsilon_\nu e(k,t)]$$

$$= \frac{-\imath G}{\sqrt{2}}(2\pi)^4\delta^4(p+k+k'-p')[\bar{u}_{\nu\mu}(k',t')\gamma_\mu(1-\gamma_5)u_\mu(p',s')][\bar{u}_e(p',s)\gamma_\mu(1-\gamma_5)\upsilon_{\nu e}(k,t)] \tag{3.1.12}$$

we can write,

$$S(\mu^- \to e^-\bar{\nu}_e\nu_\mu) = -\imath(2\pi)^4\delta^4(p+k+k'-p')M \tag{3.1.13}$$

$$S(\mu^- \to e^-\bar{\nu}_e\nu_\mu) = -\imath(2\pi)^4\delta^4(p_f-p_i)M \tag{3.1.14}$$

where

$$M = \frac{G}{\sqrt{2}}[\bar{u}_{\nu\mu}(k',t')\gamma_\mu(1-\gamma_5)u_\mu(p',s')][\bar{u}_e(p',s')\gamma_\mu(1-\gamma_5)\upsilon_{\nu e}(k,t)], \tag{3.1.15}$$

is invariant amplitude of free muon decay through the process $\mu^- \to e^-\bar{\nu}_e\nu_\mu$.

3.2 Decay Rate and Life Time

The muon decays practically 100 percent of the time to an electron and two neutrinos. Here we investigate its decay rate and see what it tells us[5].

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu. \tag{3.2.1}$$

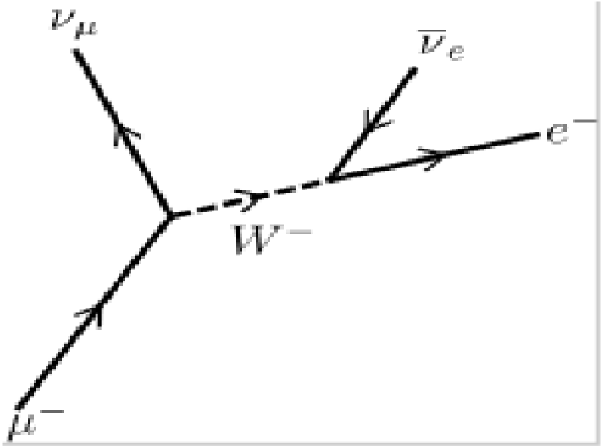

Figure 3.2: Muon decay interms of the vector bosons and the momentum transfer in the vector boson.

The decay rate of muon with polarization $S^{'}$ and an electron with polarization S is [4].

$$d\Gamma = \frac{G^2 d^3p}{(2\pi)^5 2p'^0 2p^0}\int \frac{d^3k}{2k^0}\int \frac{d^3k^{'}}{2k'^0}\delta^4(p+k^{'}+k-p^{'})\sum_{t,t'}|M|^2, \tag{3.2.2}$$

where

$$M = [\bar{u}_{\nu_\mu}\gamma^\mu(1-\gamma_5)u_\mu][\bar{u}_e\gamma_\mu(1-\gamma_5)v_{\nu_e}] \tag{3.2.3}$$

$|M|^2$ consists of two similar factors for the muonic and electronic transition currents if we write $M = M^\mu E_\mu$ with

$$M^\mu = [\bar{u}_{\nu_\mu}\gamma^\mu(1-\gamma^5)u_\mu] \tag{3.2.4}$$

$$E_\mu = [\bar{u}_e\gamma_\mu(1-\gamma_5)u_{\nu_e}] \tag{3.2.5}$$

and

$$\sum_{t,t'}|M|^2 = \sum_{t,t'}(M^\mu E_\mu)(M^\nu E_\nu)^+ = \sum_{t,t'}(M^\mu M^{\nu+})(E_\mu E_\nu^+). \tag{3.2.6}$$

Let us first focus on the muonic factor,

$$X^{\mu\nu}(\mu) = M^\mu M^{\nu+} \tag{3.2.7}$$

$$X^{\mu\nu}(\mu) = \sum_{t'}[\bar{u}_{\nu_\mu}(k^{'},t^{'})\gamma^\mu(1-\gamma^5)u_\mu(p^{'},s^{'})]\times[\bar{u}_{\nu_\mu}(k^{'},t^{'})\gamma^\nu(1-\gamma^5)u_\mu(p^{'},s^{'})]^+ \tag{3.2.8}$$

where

$[\bar{u}_{\nu_\mu}(k^{'},t^{'})\gamma^\nu(1-\gamma^5)u_\mu(p^{'},s^{'})]^+ = [\bar{u}_\mu(p^{'},s^{'})\gamma^\mu(1-\gamma^5)u_\mu(k^{'},t^{'})]$ Thus,

$$X^{\mu\nu}(\mu) = \sum_{t'}\bar{u}_{\nu_\mu}(k^{'},t^{'})\gamma^\mu(1-\gamma^5)u_\mu(p^{'},s^{'})\bar{u}_\mu(p^{'},s^{'})\gamma^\nu(1-\gamma_5)u_{\nu_\mu}(k^{'},t^{'}). \tag{3.2.9}$$

Now using the spin sum formula

$$\sum_{t'}u_{\nu_\mu}(k^{'},t^{'})_\alpha\bar{u}_{\nu_\mu}(k^{'},t^{'})_\beta = (\not{k}^{'}+m_\nu)_{\alpha\beta} = \not{k}^{'}_{\alpha\beta} \tag{3.2.10}$$

where α, β denote the spinor indices and $m_\nu = 0$

$$u_\mu(p^{'}, s^{'})_\alpha \bar{u}_\mu(p^{'}, s^{'})_\beta = [(\not{p}^{'} + m_\mu)(\frac{1+\gamma^5 \ \not{S}^{'}}{2})]_{\alpha\beta} \tag{3.2.11}$$

Substitution of Eq.(3.2.10) and Eq.(3.2.11) into Eq.(3.2.9), we get the muonic factor

$$X^{\mu\nu}(\mu) = \sum_{t^{'}} \gamma^\mu_{\pi\varrho}(1-\gamma^5)_{\varrho\tau}[(\not{p}^{'}+m_\mu)(\frac{1+\gamma_5 \ \not{S}^{'}}{2})]_{\alpha\beta}\bar{u}_{\mu_\nu}(k^{'}, t^{`\prime})_\pi \gamma^\nu_{\beta\alpha}(1-\gamma^5)_{\sigma\tau}u_{\nu_\mu}(k^{'}, t^{'})_\tau \tag{3.2.12}$$

Thus,

$$X^{\mu\nu}(\mu) = \gamma^\mu_{\pi\varrho}(1-\gamma^5)_{\varrho\alpha}[(\not{p}^{'} + m_\mu)(\frac{1+\gamma_5 \ \not{S}^{'}}{2})]_{\alpha\beta}\gamma^\nu_{\beta\sigma}(1-\gamma_5)_{\sigma\tau} \ \not{k}^{'}_{\tau\pi}. \tag{3.2.13}$$

which can in turn be expressed as,

$$X^{\mu\nu}(\mu) = Tr[\gamma^\mu(1-\gamma^5)(\not{p}^{'} + m_\mu)(\frac{1+\gamma_5 \ \not{S}^{'}}{2})\gamma^\nu(1-\gamma^5) \ \not{k}^{'}] \tag{3.2.14}$$

Since $\gamma^\mu\gamma_5 = -\gamma_5\gamma^\mu$ this yields

$$X^{\mu\nu}(\mu) = \frac{1}{2}Tr[(\not{p}^{'} + m_\mu)(1 + \gamma_5 \ \not{S}^{'})\gamma^\nu \ \not{k}^{'}(1-\gamma_5)\gamma^\mu(1-\gamma_5)] \tag{3.2.15}$$

Now we make use of the property that any trace of a product of an odd number of γ matrix vanishes. Further more it holds that $(1-\gamma_5)^2 = 2(1-\gamma_5)$ so that Eq.(3.2.15) becomes

$$X^{\mu\nu}(\mu) = Tr[(\not{p}^{'} + m_\mu)(1 + \gamma_5 \ \not{S}^{'})\gamma^\nu \ \not{k}^{'}\gamma\gamma^\mu(1-\gamma_5)] \tag{3.2.16}$$

$$= Tr[\not{p}\gamma^\nu \ \not{k}^{'}\gamma^\mu(1-\gamma_5) + \ \not{p}\gamma_5 \ \not{S}^{'}\gamma^\nu \ \not{k}^{'}\gamma_\mu(1-\gamma_5) \tag{3.2.17}$$

$$+ m_\mu\gamma^\nu \ \not{k}^{'}\gamma^\mu(1-\gamma_5) + m_\mu\gamma_5 \ \not{S}^{'}\gamma^\nu \ \not{k}^{'}\gamma^\mu(1-\gamma_5)]$$

Obviously the second and the third terms are odd: due to which they do not contribute. The remaining terms contribute. Taking into account that $\gamma_5(1-\gamma_5) = -(1-\gamma_5)$, we can express,

$$X^{\mu\nu}(\mu) = Tr[\not{p}'\gamma^{\nu}\ \not{k}'\gamma^{\mu}(1-\gamma_5) - m_{\mu}\ \not{S}'\gamma^{\nu}\ \not{k}'\gamma^{\mu}(1-\gamma_5)] \tag{3.2.18}$$

Thus,

$$X^{\mu\nu}(\mu) = Tr[(\not{p}' - m_{\mu}\ \not{S}')\gamma^{\nu}\ \not{k}'\gamma^{\mu}(1-\gamma_5)] \tag{3.2.19}$$

Using the trace technique, the final result for the muonic part is,

$$X^{\mu\nu}(\mu) = 4[(p' - m_{\mu}S')k' - (p' - m_{\mu}S')k'g^{\mu\nu} + (p' - m_{\mu}S')k' + \imath\varepsilon^{\alpha\nu\beta\mu}(p' - m_{\mu}S')k'. \tag{3.2.20}$$

The electronic contribution is evaluated in a similar manner, which gives

$$X_{\mu\nu}(e) = E_{\mu}E^{+}_{\nu} \tag{3.2.21}$$

$$X_{\mu\nu}(e) = \sum_{t}[\bar{u}_e(p,s)\gamma_{\mu}(1-\gamma_5)v_{\nu e}(k,t)][\bar{u}_e(p,s)\gamma_{\nu}(1-\gamma_5)v_{\nu e}(k,t)]^{+} \tag{3.2.22}$$

where

$[\bar{u}_e(p,s)\gamma_{\nu}(1-\gamma_5)v_{\nu e}(k,t)]^{+} = [\bar{v}_{\nu_e}(k,t)\gamma_{\mu}(1-\gamma_5)u_e(p,s)]$

$$X_{\mu\nu}(e) = Tr[(\not{p} - m_e\ \not{S})\gamma_{\mu}\ \not{k}\gamma_{\nu}(1-\gamma_5) \tag{3.2.23}$$

$$= 4[(p - m_eS)k - (p - m_eS)kg_{\mu\nu} + (p - m_eS)_k - \imath\varepsilon_{\alpha\mu\beta\nu}(p - m_eS)k. \tag{3.2.24}$$

The final result for the squared invariant matrix element in Eq.(3.2.6) becomes

$$\sum_{t,t'}|M|^2 = X^{\mu\nu}(\mu)X_{\mu\nu}(e) \tag{3.2.25}$$

$$\sum_{t,t'}|M|^2 = 64(p' - m_{\mu}S')^{\alpha}k'_{\alpha}(p - m_eS)^{\beta}k_{\beta} \tag{3.2.26}$$

Substituting Eq.(3.2.26) into Eq.(3.2.2) we get,

$$d\Gamma = \frac{G^2 64 d^3p}{(2\pi)^5 2p'^0 2p^0}(p' - m_{\mu}S')^{\alpha}(P - m_eS)^{\beta}\int\frac{d^3k}{2k^0}\frac{d^3k'}{2k'^0}kk'\delta^4(p+k'+k-p') \tag{3.2.27}$$

We may perform the integration over the neutrino momenta by using the well known relation .

$$\int \frac{d^3k}{2k^0}\frac{d^3k^{'}}{2k^{'0}}kk^{'}\delta^4(p+k^{'}+k-p^{'}) = \frac{\pi}{24}(q^2g_{\alpha\beta}+2q_\alpha q_\beta)\Theta(q^0)\Theta(q^2) \tag{3.2.28}$$

where $q = p - p^{'}$

Upon substituting Eq.(3.2.28) into Eq.(3.2.27) we get,

$$d\Gamma = \frac{G^2\pi d^3p}{3(2\pi)^5p^{'0}p^0}[(p^{'}-p)^2(p^{'}-m_\mu S^{'})^\alpha(p-m_eS)_\alpha + \tag{3.2.29}$$

$$2(p^{'}-p)_\alpha(p^{'}-m_\mu S^{'})^\alpha(p^{'}-p)_\beta(p-m_eS)^\beta] \times \Theta(p^{'0}-p^0)(\Theta(p^{'}-p)^2)$$

$$(p^{'}-p)^2 = -2pp^{'}+p^2+p^{'2} \tag{3.2.30}$$

where $p = m_e = p^0, p^{'} = m_\mu$

$$(p^{'}-p)^2 = -2p^0m_\mu + m_\mu^2 + m_e^2 \tag{3.2.31}$$

The condition $(p^{'}-p)^2 > 0$ for a non-vanishing $d\Gamma$ yields,the restriction

$$p^0 < p^0_{max} = (\frac{m_\mu^2+m_e^2}{2m_\mu}) \tag{3.2.32}$$

which consequently requires $p^{'0} - p^0 > 0$ since

$$p^{'0} - p^0 = m_\mu - p^0 > m_\mu - p^0_{max} = (\frac{m_\mu^2 - m_e^2)}{2m_\mu} > 0 \tag{3.2.33}$$

The condition $p^0 < p^0_{max}$ Eq.(3.2.29) gives,

$$d\Gamma = \frac{G^2\pi d^3p}{3(2\pi)^5p^0}[(m_\mu - p^{'0})^2 - p^{'2}][(p^0 - m_eS^0)+ \quad (3.2.34)$$
$$S^{'}.(p - m_e\tilde{S})] + 2[m_\mu - p^{'0} - S^{'}.p^{'}][(m_\mu - p^{'0})(p^0 - m_eS^0)+$$
$$p.(p - m_e\tilde{S})] \times \Theta(p^0_{max} - p^0),$$

where

$$d^3\vec{p} = 4\pi p_0^2 dp_o. \quad (3.2.35)$$

Here $\tilde{S} = \frac{S+(ps)p}{m_e(p^0+m_e)}$ is the space component of the electron spin vector.

Integrating Eq.(3.2.34) over dp_0 we get,

$$\Gamma_\mu = \frac{3G^2}{3(2\pi)^3}\int dp^0\sqrt{(p^0)^2 - m_e^2}[-4m_\mu(p^0)^2 + 2p^0(m_\mu^2 + m_e^2) - 2m_\mu m_e^2]. \quad (3.2.36)$$

Thus,

$$\Gamma_\mu = \frac{G^2m_\mu^5}{192\pi^3}[1 - 8y + 8y^2 - y^4 - 12y^2\ln y \quad (3.2.37)$$

where $y = \frac{m_e^2}{m_\mu^2}$. The contribution of $8(y^2)$ and terms is negligible in comparison to $8(y)$ terms due to $m_e << m_\mu$ and hense those terms can be dropped.

Thus,

$$\Gamma_\mu = \frac{G^2m_\mu^5}{192\pi^3}(1 - 8y) \quad (3.2.38)$$

From Eq.(3.2.37), it is obvious that the decay rate would vanish if $y = 1$. In that case the muon would be stable. So that there would be no phase space for the final-state electron. The result in Eq.(3.2.38) is calculated at tree level (which gives the maximum contribution) so does not include the so-called radiative corrections.

Therefore, the total decay rate of muon is equal to

$$\Gamma_{\mu^-} = \frac{G_f^2m_\mu^5}{192\pi^3}[1 - \frac{8m_e^2}{m_\mu^2}]. \quad (3.2.39)$$

Due to the use of free muon decay through the process $\mu^- \to e^-\bar{\nu}_e\nu_\mu$, we drive the decay width. The life time of muon decay as,

$$\tau_{\mu^-} = \frac{1}{\Gamma_{\mu^-}} \quad (3.2.40)$$

3.3 Differential Cross Section of Muon Decay

Golden rule for scattering;- tells us if particle 1 and 2 collide producing particle 3,4,.....n,,..

i.e,

$$1 + 2 \to 3 + 4 + ... + n, \tag{3.3.1}$$

the cross section is given by general formula[7],

$$d\sigma = \frac{|M|^2}{4\sqrt{(p_1.p_2)^2 - (m_1 m_2)^2}}\left[\frac{d^3p_3}{(2\pi)^3 2E_3}\frac{d^3p_4}{(2\pi)^3 2E_4}....\frac{d^3p_n}{(2\pi)^3 2E_n}\right] \tag{3.3.2}$$

$$\times (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4...........p_n)$$

In the center of mass frame $p_2 = -p_1$ and $p_1.p_2 = E_1E_2 + p_1^2$.

$$\sqrt{(p_1.p_2)^2 - (m_1 m_2)^2} = (E_1 + E_2)|p_1| \tag{3.3.3}$$

Thus.

$$d\sigma = (\frac{1}{8\pi})^2 \frac{|M|^2 d^3p_3 d^3p_4}{(E_1 + E_2)|p_1|E_3E_4}\delta^4(p_1 + p_2 - p_3 - p_4) \tag{3.3.4}$$

$$\delta^4(p_1 + p_2 - p_3 - p_4) = \delta(E_1 + E_2 - E_3 - E_4)\delta^3(-\vec{p}_3 - \vec{p}_4) \tag{3.3.5}$$

The out going energy in terms of p_3 and p_4 is $E_i = \sqrt{m_i^2 + p_i^2},$. We carry out p_4 integration where $p_3 = -p_4$. This gives,

$$d\sigma = (\frac{1}{8\pi})^2 \frac{|M|^2}{(E_1 + E_2)|p_1^{'}|} \times \frac{\delta(E_1 + E_2 - \sqrt{m_3^2 + p_3} - \sqrt{m_4^2 + p_3^2}}{\sqrt{m_3^2 + p_3^2\sqrt{m_4^2 + p_3^2}}}d^3p_3 \tag{3.3.6}$$

where,

$$d^3p_3 = \rho^2 d\rho d\Omega; \rho = |p_3| \tag{3.3.7}$$

and $d\Omega = \sin\Theta d\Theta d\Phi$

Substituting Eq.(3.3.7) into Eq.(3.3.6) we get,

$$\frac{d\sigma}{d\Omega} = (\frac{1}{8\pi})^2 \frac{1}{(E_1 + E_2)|p_1|}\int_0^\infty |M|^2 \times \frac{\delta(E_1 + E_2) - \sqrt{m_3^2 + \rho^2} - \sqrt{m_4^2 + \rho^2}}{\sqrt{m_3^2 + \rho^2\sqrt{m_4^2 + \rho^2}}}\rho^2 d\rho \tag{3.3.8}$$

The integral over ρ and $m_1 \rightarrow (E_1 + E_2)$ is then

$$\frac{d\sigma}{d\Omega} = (\frac{1}{8\pi})^2 \frac{|M|^2_{p_3=-p_4}}{(E_1 + E_2)^2} \frac{|p_i|}{|p_f|} \tag{3.3.9}$$

where $|p_f|$ is the magnitude of either out going momentum and $|p_i|$ is the magnitude of either in coming momentum.

$$\frac{|p_f|}{|p_i|} = [1 - (\frac{m_\mu}{2E})^2] \tag{3.3.10}$$

The invariant amplitude of muon decay is,

$$|M|^2 = 2G_f^2(p_1.p_2)(p_3 p_4) \tag{3.3.11}$$

where $p_3 = -p_4$

$$|M|^2 = 2G_f^2(p_1.p_2)(-p_4^2) \tag{3.3.12}$$

If we now go to the center of mass frame and neglect the mass of the electron, we can write,

$$|M|^2 = 2G_f^2(p_1.p_2)(p_3.p_4) \tag{3.3.13}$$

$$p_1.p_2 = [(p_1 + p_2)^2 - p_1^2 - p_2^2]/2 \tag{3.3.14}$$

$$= [(2E)^2 - 0 - 0]/2 \tag{3.3.15}$$

$$= 2E^2 \tag{3.3.16}$$

$$p_3.p_4 = [(p_3 + p_4)^2 - p_3^2 - p_4^2]/2 \tag{3.3.17}$$

$$= [(p_1 + p_2)^2 - 0 - m_\mu^2]/2 \tag{3.3.18}$$

$$= [4E^2 - m_\mu]/2 \tag{3.3.19}$$

$$= 2E^2[1 - \frac{(m_\mu}{2E})^2] \tag{3.3.20}$$

$$|M|^2 = 8G_f^2 E^2[1 - (\frac{m_\mu}{2E})^2]. \tag{3.3.21}$$

The differential scattering cross section of muon in Eq.(3.3.9) is

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} G_f^2 \frac{1}{(4\pi)^2} E^2[1 - (\frac{m_\mu}{2E})^2]^2 \tag{3.3.22}$$

$$\sigma = \int \frac{1}{2} G_f^2 \frac{1}{(4\pi)^2} E^2 [1 - (\frac{m_\mu}{2E})^2]^2 d\Omega \tag{3.3.23}$$

$d\Omega = \sin\Theta d\Theta d\Phi,\ \int \sin\Theta d\Theta d\Phi = 4\pi$

And the total cross section is

$$\sigma = \frac{1}{8\pi} G_f^2 E^2 [1 - (\frac{m_\mu}{3E})^2]^2 \tag{3.3.24}$$

Chapter 4

Muon Decay in A Linearly Polarized Laser Field

Previously, some attempts have been made to find the change in the decay rate of the muon whenever a strong laser field is present. Liu, Li, and Berakdar (LLB)[8]. Tried to calculate this change for a strong linearly polarized laser, using an approximated electron wave function combined with numerical calculations[6]. They found a large modification of the life time, as much as an order of magnitude. Narozhny and Fedotov challenged this result. Recently,the full analytical calculations of the decay rate of the muon in the presence of a strong circularly polarized laser field, was finding only small (explicit) corrections to the unperturbed decay rate. Although one might not intuitively expect major alterations of this conclusion once a laser with a different polarization is used, it proves to be worth going through the calculations for a strong linearly polarized laser. The electron wave function will involve two different exponents of the sine function, resulting in triple summations of the Bessel function. In contrast with the circularly polarized case, there is only one exponent of the sine function, and consequently, only one summation of the Bessel functions . Also, this calculation will settle any possible disagreement among the community regarding the difference in laser polarization.

4.1 Laser Assisted Muon Decay

We assume the decay of muon to occur in the presence of a monochromatic, linearly polarized, spatially homogeneous laser field. The final state electron is treated relativistically[8]. The electromagnetic field is described by the classical four potential $A_\mu(x) = a_\mu \cos(k.x)$ that satisfies the Lorenz condition. $k.\varepsilon = 0$ and $a_\mu = (0, \frac{\varepsilon_0}{\omega})$ is a constant four vector and ε_0 is the amplitude of laser electric field. We choose the photons to propagate along the z-direction $k = (\omega, 0, 0, \omega)$ follows from the laser frequency ω and wave number k.

4.1.1 The S-Matrices Element for the Laser Assisted Muon Decay

$$S_{fi} = -\imath \int_{-\infty}^{\infty} dt H_{int}(\mu^- \to e^- \bar{\nu}_e \nu_\mu), \tag{4.1.1}$$

where H_{int} is the Hamiltonian of the weak interaction in-ducing the decay process.

$$H_{int} = \frac{G}{\sqrt{2}} \int d^3x [\bar{\psi}_{\nu_\mu}(x)\gamma_\mu(1-\gamma_5)\psi_\mu(x)][\bar{\psi}_e(x)\gamma_\mu(1-\gamma_5)\psi_{\bar{\nu}_e}(x)], \tag{4.1.2}$$

Substituting Eq.(4.1.2) into Eq.(4.1.1) gives,

$$S_{fi} = \frac{-iG}{\sqrt{2}} \int d^4x [\bar{\psi}_{\nu_\mu}(x)\gamma_\mu(1-\gamma_5)\psi_\mu(x)][\bar{\psi}_e(x)\gamma_\mu(1-\gamma_5)\psi_{\bar{\nu}_e}(x)], \tag{4.1.3}$$

Here G is the constant of the weak interaction and x stands for the spatial coordinates. $\psi_\mu, \bar{\psi}_{\nu_\mu}, \bar{\psi}_e$, and $\psi_{\bar{\nu}_e}$ are, respectively, wave function of the muon, the muonic neutrino, the electron, and the electronic anti neutrino. The neutrinos are treated as massless particles described by Dirac spinors. For the laser intensities considered here the state of the electron and intial muon in the laser field is represented by the Dirac- Volkove function(normalized in a large volume V). The full Volkove solution for a linearly polarized plane wave laser field it has the form.

$$\psi(\varphi) = \sqrt{\frac{m}{EV}} \left(1 + \frac{e\not{k}\not{\mathcal{A}}}{2(kp)}\right)\ u(p)exp\left[-iqx - \frac{ieap\ sin(\varphi)}{(kp)} + \frac{ie^2a^2\ sin(2\varphi)}{(8kp)}\right], \tag{4.1.4}$$

where $\varphi = k.x$, p is the laser free four momentum, q is the effective momentum of the particle in the laser field and m is mass. $q = p - \frac{e^2a^2k}{(4kp)}$,$m^2 = m_0^2 - \frac{e^2a^2}{2}$.

Thus,

$$\psi_e = \sqrt{\frac{m_e}{E_eV}} \left(1 + \frac{e\not{k}\not{\mathcal{A}}}{2(kq)}\right)\ u(q)exp\left[-iqx - \frac{ieaq\ sin(\varphi)}{(kq)} + \frac{ie^2a^2\ sin(2\varphi)}{(8kq)}\right], \tag{4.1.5}$$

$$\bar{\psi}_e = \sqrt{\frac{m_e}{E_eV}}\bar{u}(q) \left(1 + \frac{e\not{\mathcal{A}}\not{k}}{2(kq)}\right)\ exp\left[iqx + \frac{ieaq\ sin(\varphi)}{(kq)} - \frac{ie^2a^2\ sin(2\varphi)}{(8kq)}\right], \tag{4.1.6}$$

$$\psi_\mu = \sqrt{\frac{m_\mu}{E_\mu V}} \left(1 + \frac{e\not{k}\not{\mathcal{A}}}{2(kp)}\right)\ u(p)exp\left[-ipx - \frac{ieap\ sin(\varphi)}{(kp)} + \frac{ie^2a^2\ sin(2\varphi)}{(8kp)}\right], \tag{4.1.7}$$

$$\psi_{\nu_\mu} = \sqrt{\frac{1}{E_{\nu_\mu}V}}u(q_2)\exp(-iq_2.x) \tag{4.1.8}$$

$$\bar{\psi}_{\nu_\mu} = \sqrt{\frac{1}{E_{\nu_\mu}V}}\bar{u}(q_2)\exp(iq_2.x) \tag{4.1.9}$$

$$\psi_{\bar{\nu}_e} = \sqrt{\frac{1}{E_{\nu_e}V}}v(q_1)\exp(iq_1.x) \tag{4.1.10}$$

Upon substituting Eq.(4.1.10),Eq.(4.1.9),Eq.(4.1.7) and Eq.(4.1.6) into Eq.(4.1.3) gives the S-matrices elements for the laser assisted μ^- decay,

$$S_{fi} = \frac{-iG}{\sqrt{2}}\int d^4x\sqrt{\frac{1}{E_{\nu_\mu}V}}\bar{u}(q_2)\exp(iq_2.x)\gamma_\mu(1-\gamma_5)\sqrt{\frac{m_\mu}{E_\mu V}}\left(1 + \frac{e\not{k}\not{A}}{2(kp)}\right)u(p) \tag{4.1.11}$$

$$exp\left[-ip - \tfrac{ieap\ sin(\varphi)}{(kp)} + \tfrac{ie^2a^2\ sin(2\varphi)}{(8kp)}\right]\sqrt{\tfrac{m_e}{E_eV}}\bar{u}(q)\left(1 + \tfrac{e\not{A}\not{k}}{2(kq)}\right)exp\left[iqx + \tfrac{ieaq\ sin(\varphi)}{(kq)} - \tfrac{ie^2a^2\ sin(2\varphi)}{(8kq)}\right]$$

$$\gamma_\mu(1-\gamma_5)\sqrt{\tfrac{1}{E_{\nu_e}V}}v(q_1)\exp(iq_1.x)$$

.

$$S_{fi}=-i\frac{G}{\sqrt{2}}\sqrt{\frac{m_\mu m_e}{E_\mu E_e E_{\nu_\mu} E_{\bar{\nu}_e}}}\frac{1}{V^2}\int d^4x[\bar{u}(q_2)e^{iq_2.x}\gamma_\mu(1-\gamma_5)\left(1+\frac{e\not{k}\not{A}}{2(kp)}\right)u(p) \tag{4.1.12}$$

$$exp\left(-ipx-\frac{ieap\ sin(\varphi)}{(kp)}+\frac{ie^2a^2\ sin(2\varphi)}{(8kp)}\right)]\ [\bar{u}(q)\left(1+\frac{e\not{A}\not{k}}{2(kq)}\right)exp\left(iqx+\frac{ieaq\ sin(\varphi)}{(kq)}-\frac{ie^2a^2\ sin(2\varphi)}{(8kq)}\right)$$
$$\gamma_\mu(1-\gamma_5)v(q_1)\exp(iq_1.x)]$$

.

$$S_{fi}=-i\frac{G}{\sqrt{2}}\sqrt{\frac{m_\mu m_e}{E_\mu E_e E_{\nu_\mu} E_{\bar{\nu}_e}}}\frac{1}{V^2}\int d^4x[e^{i(q_2-p).x}\bar{u}(q_2)\gamma_\mu(1-\gamma_5)[exp\left(-\frac{ieap\ sin(\varphi)}{(kp)}+\frac{ie^2a^2\ sin(2\varphi)}{(8kp)}\right) \tag{4.1.13}$$

$$+\frac{e\not{k}\not{A}}{2k.p}exp\left(\frac{-ieap\ sin(\varphi)}{(kp)}+\frac{ie^2a^2\ sin(2\varphi)}{(8kp)}\right)]u(p)][\bar{u}(q)e^{i(q+q_1).x}\gamma_\mu(1-\gamma_5)\ [exp\left(\frac{ieaq\ sin(\varphi)}{(kq)}-\frac{ie^2a^2\ sin(2\varphi)}{(8kq)}\right)$$
$$+\frac{e\not{A}\not{k}}{2k.q}exp\left(\frac{ieaq\ sin(\varphi)}{(kq)}-\frac{ie^2a^2\ sin(2\varphi)}{(8kq)}\right)]v(q_1)]$$

where E_e, E_μ, E_{ν_μ} and $E_{\bar{\nu}_e}$ are, respectively, the energies of μ, ν_ν and $\bar{\nu}_e, e^-$. Furthermore, p,q, q_1 and q_2 are the four momenta of μ, ν_μ and $\bar{\nu}_e, e^-$, and $\bar{u}(q_2), u(p), \bar{u}(q), v(q_1)$, are, respectively, the free Dirac spinors.

$$S_{fi}=-i\frac{G}{\sqrt{2}}\sqrt{\frac{m_\mu m_e}{E_\mu E_e E_{\nu_\mu} E_{\bar{\nu}_e}}}\frac{1}{V^2}\int d^4x e^{i(q_2-p+q_1+q_2).x}[\bar{u}(q_2)\gamma_\mu(1-\gamma_5) \tag{4.1.14}$$

$$[exp\left(\frac{-ieap\ sin(\varphi)}{(kp)}+\frac{ie^2a^2\ sin(2\varphi)}{(8kp)}\right)+\frac{e\not{k}\not{A}}{2k.p}exp\left(\frac{-ieap\ sin(\varphi)}{(kp)}+\frac{ie^2a^2\ sin(2\varphi)}{(8kp)}\right)]u(p)]\ [\bar{u}(q)\gamma_\mu(1-$$
$$\gamma_5)[exp\left(\frac{ieaq\ sin(\varphi)}{(kq)}-\frac{ie^2a^2\ sin(2\varphi)}{(8kq)}\right)+\frac{e\not{A}\not{k}}{2k.q}exp\left(\frac{ieaq\ sin(\varphi)}{(kq)}-\frac{ie^2a^2\ sin(2\varphi)}{(8kq)}\right)]v(q_1)].$$

Therefore

$$S_{fi}=-i\frac{G}{\sqrt{2}}\sqrt{\frac{m_\mu m_e}{E_\mu E_e E_{\nu_\mu} E_{\bar{\nu}_e}}}\frac{1}{V^2}\int d^4x e^{i(q_2-p+q_1+q_2+2L+l).x}[\bar{u}(q_2)\gamma_\mu(1-\gamma_5)u(p)] \tag{4.1.15}$$

$$[\bar{u}(q)\gamma_\mu(1-\gamma_5)v(q_1)],$$

where

$$\int d^4x e^{i(q-p+q_1+q_2+2L+l).x}=(2\pi)^4\delta^4(q+q_1+q_2-p+2L+l). \tag{4.1.16}$$

$$S_{fi} = -i\frac{G}{\sqrt{2}}(2\pi)^4\delta^4(q + q_1 + q_2 - p + 2L + l)[\bar{u}(q_2)\gamma_\mu(1 - \gamma_5)f^{\mu'}u(p)] \qquad (4.1.17)$$

$$\times\, [\bar{u}(q)\gamma_\mu(1 - \gamma_5)f^{\mu}v(q)].$$

Thus,

$$M_{L,l} = \frac{G}{\sqrt{2}}[\bar{u}(q_2)\gamma_\mu(1 - \gamma_5)u(p)f^{\mu^{'}}][\bar{u}(q)f^{\mu}\gamma^{\mu}(1 - \gamma^5)v(q_1)] \qquad (4.1.18)$$

Therefore, the S-matrices element of Eq.(4.1.17) gives,

$$S_{fi} = -i(2\pi)^4\delta^4(q + q_1 + q_2 - p + (2L + l))M_{L,l} \qquad (4.1.19)$$

where $M_{L,l}$ is the invariant amplitude of muon decay in a linearly polarized laser field.

4.1.2 The Decay Rate of Muon in the presence of Laser Field

Muon decay is represented by,

$$\mu^-(p) \to e^-(q) + \bar{\nu}_e(q_1) + \nu_\mu(q_2), \tag{4.1.20}$$

where the arguments label the associated momenta. The momentum conservation is then

$$p^\mu + (2L+l)k^\mu = q^\mu + q_1^\mu + q_2^\mu. \tag{4.1.21}$$

Not that L has a prefatory 2. Following the standard procedure of determining the decay rate, there are four summation indices, L, $L^{'}$,l, and $l^{'}$. One of the summation indices,L, $L^{'}$, l, and $l^{'}$. One of the two delta function, however, takes care of one of the summation indices

$$l^{'} = 2(L - L^{'}) + l, \tag{4.1.22}$$

A quantity of interest to be calculated for this process is the rate of production . Physically, the rate of production is the probability per unit time for the muon to emit electron, muon neutrinos, electron anti neutrinos, in the presence of the laser field. We begin by finding the probability p defined by

$$p = \sum_f \int \frac{d^3p_f}{(2\pi)^3 2E_f (2\pi)^3 2m_\mu} \sum_{spin} |\bar{S}_{fi}|^2 \tag{4.1.23}$$

where we have taken summation over the spin state of the final muon neutrino,electron anti neutrinos,and electron and average over the spin state of the incoming muon. The phase space integral over the final momentum q, q_1 and q_2 ,has been simplified in to the form.

$$\sum_f \int \frac{d^3p_f}{2E_f(2\pi)^3} = \int \frac{d^3q}{2E_e(2\pi)^3} \int \frac{d^3q_1}{(2\pi)^3 2q_1^0} \int \frac{d^3q_2}{(2\pi)^3 2q_2^0}. \tag{4.1.24}$$

$$|\bar{S}_{fi}|^2 = \frac{1}{2} \sum_{spin} |S_{fi}|^2 \tag{4.1.25}$$

The S-matrix for this process the normal 4-dimensional δ-function.

$$S_{fi} = -\imath \sum_{l=-\infty}^{\infty} (2\pi)^4 \delta^4(q - p + q_1 + q_2 + (2L + l)k) M_{L,l} \tag{4.1.26}$$

Eq(4.1.26) suggest conservation of momentum and energy. The (2L+l)k is the number of photon, that are pulled from the laser field, we have observed the coupling spinors,Bessel function, in to the scattering matrix $M_{L,l}$. The probability of the interaction are

$$p = \sum_f \int \frac{d^3p_f}{(2\pi)^3 2E_f (2\pi)^3 2m_\mu} \times \frac{1}{2} \sum_{spin} \sum_{l=-\infty}^{\infty} (2\pi)^8 [\delta^4(q-p+q_1+q_2+(2L+l)k))]^2 M^*_{L',2(L-L')+l} M_{L,l} \tag{4.1.27}$$

where $l^{'} = 2(L - L^{'}) + l$,

$$p = \sum_f \int \frac{d^3p_f}{(2\pi)^3 2E_f 2m_\mu} \times \frac{1}{2} \sum_{spin} \sum_{l=-\infty}^{\infty} (2\pi)^5 [\delta^4(q-p+q_1+q_2+(2L+l)k))]^2 M^*_{L',l'} M_{L,l} \tag{4.1.28}$$

One of the two 4-dimensional δ- function imply that for there to be any contribution to the summation either is no incoming photon energy (E_γ=0,L,$l = L^{'}l^{'}$. Therefore replace $M^*_{L,l} M_{L',l'}$ with the square of scattering amplitude $|M_{L,l}|^2$ eliminate the sum over $L^{'}$,$l^{'}$.

We can write the probability of interaction in Eq.(4.1.28) as,

$$p = \sum_f \int \frac{d^3p_f}{(2\pi)^3 2E_f 2m_\mu} \times \sum_{l=-\infty}^{\infty} \frac{1}{2} (2\pi)^5 [\delta^4(q-p+q_1+q_2+(2L+l)k))]^2 \frac{1}{2} \sum_{spin} |M_{L,l}|^2, \tag{4.1.29}$$

where

$$|\bar{M}_{L,l}|^2 = \frac{1}{2} \sum_{s_\mu, s_e, s_{\nu_\mu}, s_{\nu_e}} |M_{L,l}|^2. \tag{4.1.30}$$

Which can be worked out using Dirac trace theromes.

Next we can use three of the δ-function.

$$p=\sum_f\int\frac{d^3p_f}{(2\pi)^3 2E_f 2m_\mu}\times\sum_{l=-\infty}^{\infty}(2\pi)^5\delta^4(q-p+q_1+q_2+(2L+l)k)(\delta(E_e+E_{\nu_\mu}+E_{\bar{\nu}_e}-E_\mu+(2L+l)k))^2|\bar{M}_{L,l}|^2 \tag{4.1.31}$$

where p is the probability of the interaction

$$p=\sum_f\int\frac{d^3p_f}{(2\pi)^3 2E_f 2m_\mu}\times\sum_{l=-\infty}^{\infty}(2\pi)^5\delta^3(q-p+q_1+q_2+(2L+l)k)(\delta(E_e+E_{\bar{\nu}_e}+E_{\nu_\mu}-E_\mu+(2L+l)k))^2|\bar{M}_{Ll}|^2 \tag{4.1.32}$$

where

$$p\propto\sum_{l=\infty}^{\infty}\delta(E_e+E_{\bar{\nu}_e}+E_{\nu_\mu}-E_\mu+(2L+l)k)\frac{1}{2\pi}\lim_{T\to\infty}\int_{T/2}^{-T/2}dt\exp\imath(E_e+E_{\bar{\nu}_e}+E_{\nu_\mu}-E_\mu+(2L+l)k)t \tag{4.1.33}$$

And use the other δ-function to reduce the integrand to unity.

$$p\propto\sum_{l=-\infty}^{\infty}\delta(E_e+E_{\bar{\nu}_e}+E_{\nu_\mu}-E_\mu+(2L+l)k)\frac{1}{2\pi}\lim_{T\to\infty}\int_{-T/2}^{T/2}dt \tag{4.1.34}$$

$$p\propto\sum_{l=-\infty}^{\infty}\delta(E_e+E_{\bar{\nu}_e}+E_{\nu_\mu}-E_\mu+(2L+l)k)\frac{1}{2\pi}\lim_{T\to\infty}T \tag{4.1.35}$$

Substitute Eq.(4.2.35) in to Eq.(4.2.32) we get the probability of the interaction as,

$$p=\lim_{T\to\infty}\sum_f\int\frac{d^3p_f}{(2\pi)^3 2E_f 2m_\mu}T\sum_{l=-\infty}^{\infty}(2\pi)^4\delta^4(q-p+q_1+q_2+(2L+l)k)|\bar{M}_{L,l}|^2 \tag{4.1.36}$$

Therefore total production rate is of Γ is

$$\Gamma=\frac{dp}{dt}=\frac{p}{T} \tag{4.1.37}$$

$$\Gamma=\sum_f\int\frac{d^3p_f}{(2\pi)^3 2E_f 2m_\mu}\sum_{l=-\infty}^{\infty}(2\pi)^4\delta^4(q-p+q_1+q_2+(2L+l)k)|\bar{M}_{L,l}|^2 \tag{4.1.38}$$

hence,we are left with three summation indices. The total decay rate is,

$$\Gamma=\sum_{L,L',l,}\Gamma_{L,L',l} \tag{4.1.39}$$

$$\Gamma_{L,L',l} = \sum_f \int \frac{d^3p_f}{(2\pi)^3 2E_f 2m_\mu}(2\pi)^4\delta^4(q-p+q_1+q_2+(2L+l)k)|\bar{M}_{L,l}|^2 \quad (4.1.40)$$

$$\sum_f \int \frac{d^3p_f}{2E_f(2\pi)^3} = \int \frac{d^3q}{2E_e(2\pi)^3}\int \frac{d^3q_1}{(2\pi)^3 2q_1^0}\int \frac{d^3q_2}{(2\pi)^3 2q_2^0}. \quad (4.1.41)$$

$$\Gamma_{L,L',l} = \int \frac{d^3q}{2E_e(2\pi)^3}\frac{d^3q_1}{(2\pi)^3 2q_1^0}\frac{d^3q_2}{(2\pi)^3 2q_2^0}(2\pi)^4\delta^4(q-p+q_1+q_2+(2L+l)k)|\bar{M}_{L,l}|^2 \quad (4.1.42)$$

$$\Gamma_{L,L',l} = \int \frac{d^3q}{2E_e(2\pi)^9}\frac{d^3q_1}{2q_1^0}\frac{d^3q_2}{2q_2^0}(2\pi)^4\delta^4(q-p+q_1+q_2+(2L+l)k)|\bar{M}_{L,l}|^2 \quad (4.1.43)$$

where,

$$d^3q = 4\pi dEq^2 \quad (4.1.44)$$

and

We may perform the integration over the neutrino momenta by using the well-known relation

$$\int \frac{d^3q_1}{2q_1^0}\frac{d^3q_2}{2q_2^0}\delta^4(Q+q_1+q_2)q_1^\alpha q_2^\beta = \frac{\pi}{24}(Q^2g^{\alpha\beta}+2Q^\alpha Q^\beta)\Theta(Q^2), \quad (4.1.45)$$

with $Q = q - p + (2L+l)k$.

Substituting Eq.(4.1.45) and Eq.(4.1.44) into Eq.(4.1.43) these then reduced in muon rest frame to,

$$\Gamma_{L,L',l} = \frac{1}{3072\pi^3 M}\int dE|q|\int dz\Theta(Q^2)T_{L,L',l} \quad (4.1.46)$$

where M is the muon rest mass, and E is q. The object $T_{L,L',l}$ is the square of the matrix element in Eq.(4.1.18), summed over the spin and integrated over the neutrino momenta whose value can be worked out using trace theorems. The unaltered decay rate of the muon is,

$$\Gamma^0 = \frac{G^2M^5}{192\pi^3}. \quad (4.1.47)$$

The limits of integration are determined by the Θ function, and the integration separates into two parts

$$\int dE \int dz \Theta(Q^2) = \int_m^{M/2} dE \int_{-1}^{1} dz + \int_{M/2}^{M/2+(2L+l)\omega} dE \int_{Z_L(E)}^{1} dz, \tag{4.1.48}$$

with $z_L(E)$ coming from the condition $Q^2 \geqslant 0$

$$z_L(E) = \frac{-M^2 + 2M(2L+l)\omega - 2E[M + (2L+l)\omega]}{2(2L+l)\omega E}. \tag{4.1.49}$$

Since the first terms of Eq.(4.1.48) is independent of the summation indices, substitution of Eq.(4.1.48) into Eq.(4.1.46) allowed us to sum before integrating over this first terms. In order to perform the summation per index of $T_{L,L',l}$, we use $\sum_{n=-\infty}^{\infty} J_n(z) J_{n+k}(z) = J_k(0)$, the recursion relation for Bessel function, $nJ_n(z) = \frac{z}{2}(J_{n+1}(z) + J_{n-1}(z)$, and $J_{-n}(z) = (-1)^2 J_n(z)$. We not that $(L - L^{'})$ can only equal integer values; this notion force many of the summation terms to be zero[6]. The summations of the decay width of the muon, the integration may be performed solving,

$$\Gamma = \Gamma^0[1 + \frac{8e^2a^2}{M^2}(-\frac{5}{3} + \ln\frac{M}{m}) - \frac{e^4a^4}{4M^4}(6 - 10\ln\frac{M}{m} + \frac{M^2}{m^2})]. \tag{4.1.50}$$

The second integration term can also tackled in the same way. The indices of the Bessel function are limited by $l \leq D$, and $L \leq B$, as Bessel function becomes very small once the index exceeds the argument. Therefore, given typical values such as $ea \sim 10^{-4} M_e V$ and $\omega \sim 1eV$, $(2l + L)\omega)$ is always much less than $1M_eV$. Hence, the correction from this integration term are small, as the range of integration is small, after keeping only the first non-zero term, the correction becomes.

$$\Gamma^C = \Gamma^0 \frac{4\omega}{M^5} \int_{-1}^{1} dz \sum_{L,L',l=\infty}^{\infty} (2L+l)\tilde{T}_{L,L',l} \tag{4.1.51}$$

where $\tilde{T}_{L,L',l}$ is the same as $T_{L,L',l}$. Not that $z_L(\frac{M}{2}) = -1$, from Eq.(4.1.49), and hence, once again we are allowed to switch the order of summation and integration.

Thus, the decay rate of muon gives,

$$\Gamma^C = \Gamma^0[\frac{8e^2a^2}{M^2}(\frac{\omega^2}{M^2} - \frac{2\omega^2}{M^2}\ln\frac{M}{m}) - \frac{e^4a^4}{4M^4}(3 - 2\ln\frac{M}{m} - \frac{M^2}{m^2}] \tag{4.1.52}$$

The total change of the decay rate is then the sum of Eq.(4.1.50) and Eq.(4.1.52) gives,

$$\Gamma = \Gamma^0[1 + \frac{8e^2a^2}{M^2}((1 - \frac{2\omega^2}{M^2})\ln\frac{M}{m} - \frac{5}{3} + \frac{\omega^2}{M^2}) - \frac{e^4a^4}{4M^4}(9 - 12\ln\frac{M}{m}]. \tag{4.1.53}$$

Due to the use of an intense laser field, we have calculated muon decay width taking Volkvoe wave functions for both electron and initial muon in the presence of laser field. However in the calculation done in[6], muon was taken as a free particle whose wave function was taken to be a plane wave function.

Chapter 5

Conclusion and Summary

In this thesis as a preliminary exercise before solving to main problem, we first calculated free muon decay. We calculate the decay width for free muon decay through the process $\mu^- \rightarrow \bar{e} + \nu_\mu + \bar{\nu}_{e^-}$, in Chapter 3 which proceeds through the weak interaction. We then studied Volkov wave function in a linearly polarized laser field. The first full calculation of muon decay in a strong laser field was done in Chapter 4.

In Chapter 2 we reviewed the intermediate vector bosons and Volkov state in an intense laser field. Through the process $\mu^- \rightarrow \bar{e} + \nu_\mu + \bar{\nu}_{e^-}$ we have calculated in detail the decay rate, S-matrix and the differential cross section of free muon decay as in Chapter 3. As it was presented in Chapter 4, muon decay in a linearly polarized laser field a long with laser assisted muon decay we show that the S-matrices element and the decay width of muon in a strong laser field. Due to the use of an intense laser field, we have calculated muon decay width using Volkvoe wave functions for both electron and initial muon in the presence of laser field. However in the calculation done in[6], muon was taken as a free particle whose wave function was taken to be a plane wave type and they employed Volkove wave function only for the electron.

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Declaration

This thesis is my original work, has not been presented for a degree in any other University and that all the sources of material used for the thesis have been dully acknowledged.

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