College of Natural Science
Department of Mathematics

Graduate Project Report on Method of Characteristics on Wave and Burger's Equations

Submitted in partial fulfilment of the requirements for the Degree of Master of Science in Mathematics

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Contents

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The undersigned hereby certify that they have read and recommend to the school of graduate studies for acceptance of a project entitled **Method of Characteristics on Wave and Burger's Equations** by Tigistu Mehari in partial fulfillment of the requirements for the degree of master of Science.

Date: August, 2014

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Notation

$\mathbf{R}$	The set of all real numbers.
$\mathbf{R}^+$	The set of all non-negative real numbers.
$\mathbf{C^1}(\mathbf{R})$	Continuously differentiable function on $\mathbf{R}$.
$\mathbf{C^2}(\mathbf{R})$	Twice continuously differentiable function on $\mathbf{R}$.
$\mathbf{C^1}(\mathbf{R^2})$	Continuously differentiable function on $\mathbf{R^2}$
$\mathbf{C^2}(\mathbf{R^2})$	Twice continuously differentiable function on $\mathbf{R^2}$
$\mathbf{\Delta x}$	The local distance between adjacent points.
$\mathbf{u} = \mathbf{u}(\mathbf{x}, \mathbf{t})$	The vertical displacement of a vibrating string.
$\mathbf{c}$	The speed of wave on a string.
$\|.\|$	Norm of a function.
$\mathbf{s}$	Shock curve.
$\dot{\mathbf{s}} = \mathbf{u}_+ + \mathbf{u}_- = \mathbf{U}$	Shock speed.
$[\mathbf{u}] = \mathbf{u}_+ - \mathbf{u}_-$	The jump of $\mathbf{u}$.

Abstract

This project provides how the method of characteristics is applied to derive the fundamental solution of linear homogeneous wave equation and how shock waves and shock solutions are formed by a discontinuous initial condition. Moreover, it provides how the solution of the linear non-homogeneous wave equation is determined by combining the split equations, the homogeneous wave equation with non-zero initial data and the non-homogeneous wave equation with zero initial data using method of characteristics.

Introduction

Method of characteristic is used to solve first order linear and quasilinear PDEs. This method is based on finding the characteristic curve of the PDE. We will also show how to generalize this method for a second order constant coefficients wave equation. The method of characteristics can be used for hyperbolic problems which possess the right number of characteristic families. However, for the second order parabolic problems we have only one family of characteristics and for elliptic PDEs no real characteristic curves exist.
In this project, homogeneous and non-homogeneous wave equations and Burger's equation are considered, so that we will see the following.

- The fundamental solution of a linear homogeneous wave equation

$$u_{tt} = c^2 u_{xx}, x \in R, t \geq 0.$$

- Solution of initial-boundary value problem of the wave equation in a semi-infinite string

$$u_{tt} = c^2 u_{xx}, u(x,0) = f(x), u_t(x,0) = g(x), u(0,t) = h(t), x \in R^+, t \geq 0.$$

- Shock waves and shock solution of Burger's equation

$$u_t + (u^2)_x = 0, x \in R, t \geq 0.$$

- Solution of a non-homogeneous wave equation with initial data

$$u_{tt} - c^2 u_{xx} = f, u(x,0) = \phi(x), u_t(x,0) = \psi(x), x \in R, t \geq 0.$$

All the above equations are going to be solved using the **Method of characteristics** based on finding the characteristic curves.

Chapter 1

Preliminaries

1.1 Wave equation

Wave equation is a second order hyperbolic partial differential equation (PDE) for the description of waves as they occur in physics such as sound wave, light wave, water wave, and string waves. A wave equation in its simplest form concerns with a time variable t, and a spatial variable x and a scalar function

$$u = u(x,t)$$

which values could be modeled in the displacement of wave solution of

$$u_{tt} - c^2 u_{xx} = f$$

The above equation is homogeneous wave equation if $f = 0$ otherwise non-homogeneous wave equation. Here $t \geq 0$ and $x \in U$,where $U \subseteq R$ is open and the unknown u is $u : \overline{U}X[0,\infty) \rightarrow R$, where $u(x,t)$ represents the displacement in some direction of the point x at time $t \geq 0$.
Homogeneous wave equation are initially zero outside some restricted region propagate out from the region at a fixed speed in a spatial directions, as do physical waves from a localized disturbance. The constant c is identified with the propagation speed of the wave. A PDE is linear if it is linear in the highest order derivatives of the unknown function and its derivatives with coefficients depending only on the independent variable and non-linear if it is not linear. A PDE is quasilinear if it is linear in the highest order derivatives with coefficients depending on the independent variables, the unknown function and its derivatives of order less than the order of the equation.

1.2 A PDE in two independent variables

$$A(x,y)u_{xx}+B(x,y)u_{xy}+C(x,y)u_{yy}+D(x,y)u_x+E(x,y)u_y+F(x,y)u = G(x,y)$$

where the coefficients $A, B, C, D, E, F, G \in C^2(U), U \subseteq R^2$ and $A^2 + B^2 + C^2 \neq 0$ in U. A PDE is called homogeneous if the above equation does not contain a term independent of the unknown function and its derivatives.
For example:- The equation is homogeneous if $G(x, y) = 0$.
But the equation is non-homogeneous if it is not homogeneous.
Define a discriminant $\Delta(x, y)$ by

$$\Delta(x_0, y_0) = B^2(x_0, y_0) - 4A(x_0, y_0)C(x_0, y_0)$$

Definition 1.2.1. *An equation is called hyperbolic at the point* (x_0, y_0) *if* $\Delta(x_0, y_0) > 0$, *it is parabolic at the point* (x_0, y_0) *if* $\Delta(x_0, y_0) = 0$ *and elliptic if* $\Delta(x_0, y_0) < 0$.

Definition 1.2.2. *A Jacobian is the functional determinant used for the inversion of the transformation of one plane to another plane by computing the determinant of functions.*

Suppose also that the Jacobian of the transformation defined by

$$J = \frac{\partial(\eta, \xi)}{\partial(x, y)} = \begin{vmatrix} \eta_x & \xi_x \\ \eta_y & \xi_y \end{vmatrix}$$

is non-zero. This assumption is necessary to ensure that one can make the transformation back to the original variables x, y. Use the chain rule to obtain all the partial derivatives required in the above equation.

Definition 1.2.3. *A differential equation is said to be well-posed if the following conditions satisfied*

- *There exist a solution*
- *The solution is unique*
- *The solution is stable, the stability of a solution means that small variations of the initial data yields small variation on the corresponding solutions. This is also referred to as continuity of the function. The meaning of small variation is made precise in terms of the topology suggested by the problem. A problem that does not satisfy any one of these conditions is called ill-posed.*

1.3 Canonical forms

Canonical form is a particular simple choices of the coefficients of the second partial derivatives of the unknown function in order to reduce either the order or the number of terms of the equation. To obtain the canonical form, we have to transform the PDE which in turn will require the knowledge of characteristic curves. Recall that a linear second order PDE in two variables is given by

$$A(x,y)u_{xx}+B(x,y)u_{xy}+C(x,y)u_{yy}+D(x,y)u_x+E(x,y)u_y+F(x,y)u=G(x,y) \tag{1.1}$$

where the coefficients $A, B, C, D, E, F, G \in C^2(R), U \subseteq R^2$ and $A^2+B^2+C^2 \neq 0$ in U.
Define a dicriminant $\Delta(x,y)$ by

$$\Delta(x_0,y_0) = B^2(x_0,y_0) - 4A(x_0,y_0)C(x_0,y_0) \tag{1.2}$$

To transform the equation into a canonical form, we first show how a general transformation affects equation (1.1). Let ξ,η be twice continuously differentiable functions of x,y such that

$$\xi = \xi(x,y) \tag{1.3}$$

$$\eta = \eta(x,y) \tag{1.4}$$

Suppose

$$J = \frac{\partial(\eta,\xi)}{\partial(x,y)} = \begin{vmatrix} \eta_x & \xi_x \\ \eta_y & \xi_y \end{vmatrix} \tag{1.5}$$

Use the chain rule to obtain all the partial derivatives required in (1.1). It is easy to see that

$$u_x = u_\xi \xi_x + u_\eta \eta_x \tag{1.6}$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y \tag{1.7}$$

The second partial derivatives can be obtained as follows:

$$u_{xy} = (u_x)_y = (u_\xi)_y \xi_x + u_\xi \xi_{xy} + (u_\eta)_y \eta_x + u_\eta \eta_{xy}$$

Now use (1.7)

$$u_{xy} = u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_{x\eta y} + u_\xi\xi_{xy} + u_\eta\eta_{xy}. \tag{1.8}$$

In a similar fashion we get u_{xx},u_{yy}

$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi\xi_{xx} + u_\eta\eta_{xx}. \tag{1.9}$$

$$u_{yy} = u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi\xi_{yy} + u_\eta\eta_{yy}. \tag{1.10}$$

introducing these into (1.1) one finds after collecting like terms

$$A^*u_{\xi\xi} + B^*u_{\xi\eta} + C^*u_{\xi\xi} + D^*u_\xi + E^*u_\eta + F^*u = G^* \tag{1.11}$$

where all the coefficients are now functions of ξ,η and

$$A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \tag{1.12}$$

$$B^* = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \tag{1.13}$$

$$C^* = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \tag{1.14}$$

$$D^* = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \tag{1.15}$$

$$E^* = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \tag{1.16}$$

$$F^* = F \tag{1.17}$$

$$G^* = G \tag{1.18}$$

The resulting equation (1.11) is in the same form as the original one. The type of the equation (hyperbolic, parabolic or elliptic) will not change under this transformation. The reason for this is that

$$\Delta^* = (B^*)^2 - 4A^*C^* = J^2(B^2 - 4AC) = J^2\Delta \tag{1.19}$$

and since $J \neq 0$, the sign of Δ^* is the same as that of Δ. The classification depends only on the coefficients of the second derivative terms and thus we write (1.1) and (1.11) respectively as

$$Au_{xx} + Bu_{xy} + Cu_{yy} = H(x, y, u, u_x, u_y) \tag{1.20}$$

$$A^*u_{\xi\xi} + B^*u_{\xi\eta} + C^*u_{\eta\eta} = H^*(\xi, \eta, u, u_\xi, u_\eta) \tag{1.21}$$

We now consider specific choices for the functions ξ,η. This will be done in such a way that some of the coefficients A^*,B^*,and C^* in (1.21) become zero. Note that A^*,C^* are similar and can be written as

$$A\zeta_x^2 + B\zeta_x\zeta_y + C\zeta_y^2 \tag{1.22}$$

in which ζ stands for either ξ or η.
Suppose we try to choose ξ,η such that $A^* = C^* = 0$.
This is of course possible only if the equation is hyperbolic. (Recall that $\Delta^* = (B^*)^2 - 4A^*C^*$) and for this choice $\Delta^* = (B^*)^2 > 0$. Since the type does not change under the transformation, we must have a hyperbolic PDE).
In order to annihilate A^* and C^* we have to find ζ such that

$$A\zeta_x^2 + B\zeta_x\zeta_y + C\zeta_y^2 = 0 \tag{1.23}$$

Dividing by ζ_y^2, the above equation becomes

$$A(\frac{\zeta_x}{\zeta_y})^2 + B(\frac{\zeta_x}{\zeta_y}) + C = 0 \tag{1.24}$$

along the curve

$$\zeta(x, y) = constant \tag{1.25}$$

We have

$$d\zeta = \zeta_x dx + \zeta_y dy = 0 \tag{1.26}$$

Therefore,

$$\frac{\zeta_x}{\zeta_y} = -\frac{dy}{dx} \tag{1.27}$$

and equation (1.24) becomes

$$A(\frac{dy}{dx})^2 - B(\frac{dy}{dx}) + C = 0 \tag{1.28}$$

This is a quadratic equation for $\frac{dy}{dx}$ and its roots are

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} \tag{1.29}$$

These equations are called characteristic equations and are ordinary diffential equations for families of curves in x,y plane along which $\zeta = constant$. The solutions are called characteristic curves. Notice that the discriminant is under the radical in (1.29) and since the problem is hyperbolic,$B^2 - 4AC > 0$, there are two distinct characteristic curves. We can choose one to be $\xi(x, y)$ and the other $\eta(x, y)$. Solving the ODEs (1.29), we get

$$\phi_1(x, y) = c_1 \tag{1.30}$$

$$\phi_2(x, y) = c_2 \tag{1.31}$$

Thus the transformation

$$\xi = \phi_1(x, y) \tag{1.32}$$

$$\eta = \phi_2(x, y) \tag{1.33}$$

will lead to $A^* = C^* = 0$ and the canonical form is

$$B^* u_{\xi\eta} = H^* \tag{1.34}$$

or after division by B^*

$$u_{\xi\eta} = \frac{H^*}{B^*} \tag{1.35}$$

This is called the first canonical form of the hyperbolic equation. Sometimes we find another canonical form for hyperbolic PDEs which is obtained by making a transformation

$$\alpha = \xi + \eta \tag{1.36}$$

$$\beta = \xi - \eta \tag{1.37}$$

Using (1.27)-(1.29) for this transformation one has

$$u_{\alpha\alpha} - u_{\beta\beta} = H^{**}(\alpha, \beta, u, u_\alpha, u_\beta) \tag{1.38}$$

This is called the second canonical form of the hyperbolic equation.

Chapter 2

Homogeneous Wave Equation

2.1 Wave equation in one dimension

Suppose we have a tightly stretched elastic string of length L. We imagine that the ends are tied down in some way. We describe the motion of the string as a result of disturbing it from equilibrium at time $t = 0$.
We assume that the slope of the string is small and thus the horizontal displacement can be neglected. Consider a small segment of string between x and $x + \Delta x$. The forces acting on this segment are along the string (Tension) and vertical (gravity). But in many situations, the force of gravity is negligible relative to the tensile force. Let $T(x, t)$ be the tension at point x at time t, if we assume the string is flexible then the tension is in the direction tangent to the string.

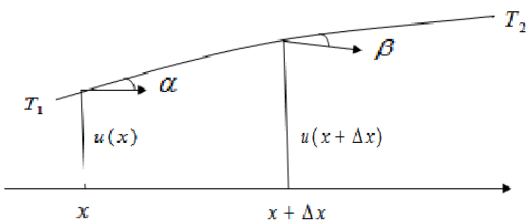

Figure A. The force acting on a segment of the string

We consider an elastic string of length L. The components of force are

$$F_V = T_2 \sin\beta - T_1 \sin\alpha$$

$$F_H = T_2 \cos\beta - T_1 \cos\alpha$$

Since the slope of the string is very small, thus the horizontal displacement can be neglected. Hence, the vibration is only in the vertical planes that is, there is no transverse vibration. Then

$$F_H = 0 \Rightarrow T_2 \cos\beta = T_1 \cos\alpha := T$$

Applying Newton's second law of motion we have

$$F_V \alpha u_{tt} \Rightarrow F_V = ma$$
$$\Rightarrow T_2 \sin\beta - T_1 \sin\alpha = \sigma_0(x)\Delta x u_{tt}$$

$\sigma_0(x)$ is the mass density, then

$$\begin{aligned}
\frac{T_2 \sin\beta - T_1 \sin\alpha}{T} &= \frac{\sigma_0(x)\Delta x u_{tt}}{T} \\
\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} &= \frac{\sigma_0(x)\Delta x u_{tt}}{T} \\
\Rightarrow \tan\beta - \tan\alpha &= \frac{\sigma_0(x)\Delta x utt}{T} \\
\Rightarrow \frac{\tan\beta - \tan\alpha}{\Delta x} &= \frac{\sigma_0(x) u_{tt}}{T} \\
\Rightarrow \frac{u_x(x+\Delta x) - u_x(x)}{\Delta x} &= \frac{\sigma_0(x) u_{tt}}{T} \\
\Rightarrow \lim_{\Delta x \to 0} \frac{u_x(x+\Delta x) - u_x(x)}{\Delta x} &= \frac{\sigma_0(x) u_{tt}}{T} \\
\Rightarrow u_{xx} &= \frac{\sigma_0(x) u_{tt}}{T} \\
\Rightarrow u_{tt} &= \frac{T}{\sigma_0(x)} u_{xx}
\end{aligned}$$

Setting, $\frac{T}{\sigma_0(x)} = c^2$
we get

$$u_{tt} = c^2 u_{xx}$$

Which is equation of the motion and is homogeneous wave equation. Here $t \geq 0$ and $x \in U$, where $U \in R$ is open and the unknown u is $u : \overline{U} X[0, \infty) \to R$, where $u(x,t)$ represents the displacement in some direction of the point x at time $t \geq 0$.

2.2 First order wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{2.1}$$

$$\Rightarrow (\partial_t + c\partial_x)(\partial_t - c\partial_x)u = 0 \tag{2.2}$$

$$\begin{aligned}
\Rightarrow (\partial_t + c\partial_x)(\partial_t - c\partial_x)u^2 = 0 \\
\Rightarrow (\partial_t + c\partial_x)(2uu_t - 2cuu_x) = 0 \\
\partial_t(2uu_t - 2cuu_x) = c\partial_x(2cuu_x - 2uu_t) \\
u_t^2 = c^2 u_x^2.
\end{aligned}$$

Hence

$$(u_t - cu_x)(u_t + cu_x) = 0 \tag{2.3}$$

If we let

$$v = u_t - cu_x \tag{2.4}$$

Then

$$\begin{aligned}
v_t + cv_x = u_{tt} - u_{xt} + u_{tx} - c^2 u_{xx} \\
\Rightarrow v_t + cv_x = u_{tt} - c^2 u_{xx} = 0
\end{aligned}$$

Then (2.2)becomes

$$v_t + cv_x = 0 \tag{2.5}$$

Similarly (2.2) yields

$$w_t - cw_x = 0 \tag{2.6}$$

If we let

$$w = u_t + cu_x \tag{2.7}$$

Here we reduced a second order PDE into first order PDEs. The only difference between (2.5) and (2.6) is the sign of the second term. We particularly consider equation (2.5), which is called first order wave equation or advection equation (in meteorology). To solve (2.5) we apply characteristic method. The corresponding system of characteristic is

$$\frac{dt}{ds} = 1, \frac{dx}{ds} = c, \frac{dv}{ds} = 0.$$

and the characteristic curves are given by

$$t = s, x = cs + x(0), v = v(x(0), 0) \tag{2.8}$$

If we eliminate s we obtain

$$x = ct + x_0, v = v(x(0), 0) \tag{2.9}$$

The characteristic ground curves are

$$x - ct = x(0) = constant \tag{2.10}$$

The characteristic ground curves are straight lines $x - ct = constant$, the characteristic curves are parallel to the ground curves and lie inside the surface of the solution: the solution is constant along the curves $x - ct$, i.e.

$$v = v_0 = v(x(0), 0) = f(x(0))$$

We note that $v = constant$ along the characteristic ground curve

$$x(t) = x(0) + ct.$$

But the constant $f(x(0))$ can be determined from (2.10), $x(0) = x - ct$ Therefore, the general solution is

$$v(x(t), t) = f(x(t) - ct) \tag{2.11}$$

Where f is any arbitrary continuously differentiable function. Notice that f is a function of one variable that is of $x - ct$.

2.3 Second order wave equation

Consider the wave equation

$$u_{tt} - c^2 u_{xx} = 0 \tag{2.12}$$

This is a homogeneous wave equation with constant coefficients, $A = 1, B = 0, C = -c^2$ then the discriminant D is given by

$$D = B^2 - 4AC = 4c^2 > 0$$

Thus (2.12) is a hyperbolic equation. The characteristic form is

$$\begin{aligned} \frac{dx}{dt} &= \frac{B \pm \sqrt{B^2 - 4AC}}{2A} \\ &= \pm c \end{aligned}$$

These are two ordinary differential equations, integrating both sides we get

$$x = ct + c_1, x = -ct + c_2$$

These are two families of real characteristics. Let η and ξ be twice continuously differentiable functions of x and t such that

$$\eta = \eta(x,t) = x - ct, \xi = \xi(x,t) = x + ct.$$

To insure the coordinate transformation we need the Jacobian

$$J = \frac{\partial(\eta,\xi)}{\partial(x,t)} = \begin{vmatrix} \eta_x & \xi_x \\ \eta_t & \xi_t \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -c & c \end{vmatrix} = 2c \neq 0$$

Thus, inversion of coordinate transformation is possible.
Now set $u(x,t) = v(\eta,\xi)$, then

$$u_t = -cv_\eta + cv_\xi$$

$$u_{tt} = c^2(v_{\eta\eta} - 2v_{\eta\xi} + v_{\xi\xi}) \tag{2.13}$$

$$u_x = v_\eta + v_\xi$$

$$u_{xx} = v_{\eta\eta} + 2v_{\eta\xi} + v_{\xi\xi} \tag{2.14}$$

Combining equations (2.12),(2.13) and (2.14) implies that

$$-4c^2 v_{\eta\xi} = 0 \Rightarrow v_{\eta\xi} = 0$$

This is a second order PDE in $\eta\xi - plane$. Since it has only one term as opposed to the original PDE that involve two terms, we call it the canonical form of the wave equation in one-dimension. Hence integrating with respect to ξ and η with η and ξ fixed respectively, we have

$$\begin{aligned} v_\eta &= f(\eta) \Rightarrow \partial v = f(\eta)\partial\eta \\ \Rightarrow \int \partial v &= \int f(\eta)\partial\eta + G(\xi) \\ \Rightarrow v &= F(\eta) + G(\xi) \\ v(\eta,\xi) &= F(\eta) + G(\xi). \end{aligned}$$

Therefore,

$$u(x,t) = F(x - ct) + G(x + ct) \tag{2.15}$$

is the general solution of (2.12), where F and G are arbitrary functions of x and t, and twice continuously differentiable functions. Here

$$u(x,t) = F(x - ct) + G(x + ct)$$

represents the sum of two waves, one represented by $F(x - ct)$ propagates from left to right without change of form at speed c and one represented by $G(x + ct)$ from right to left at speed c. To see this, consider the solution

$$y = F(x - ct),$$

and simply note that on the path $x = ct + c_1$, $F(x - ct)$ is constant. Similarly $G(x + ct)$ is constant on the path $x = -ct + c_2$.
In general, the wave equation being a constant coefficient second order PDE of hyperbolic type posses two families of characteristic lines, which correspond to constant value of respective characteristic variables. Using these variables the equation can be treated as a pair of successive ODEs integrating which leads to the general solution. The general solution was used to arrive at the d'Alembert's solution.

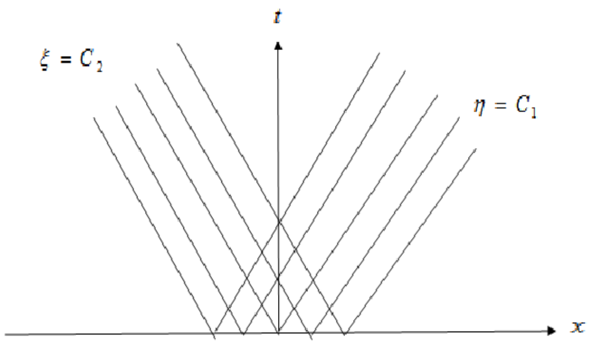

Figure B. Characteristics for the wave equation

Chapter 3

Initial-boundary value problem

3.1 Initial-value problem

It is to mean that the above wave equation is subjected to the initial displacement and initial velocity. That is the equation

$$u_{tt} = c^2 u_{xx}$$

is subject to

$$u(x,0) = f(x), x \in R \tag{3.1}$$
$$u_t(x,0) = g(x), x \in R \tag{3.2}$$

Where $f \in C^2(R)$ and $g \in C^1(R)$, these conditions will specify the arbitrary functions F and G. Combining the conditions with (2.15). We have initially with time $t = 0$ is

$$u(x,0) = F(x) + G(x) = f(x)$$
$$\Rightarrow F(x) + G(x) = f(x) \tag{3.3}$$

Now differentiating (2.15) with respect to time t and using chain rule on F and G we have

$$u_t(x,t) = -cF'(x-ct) + cG'(x+ct)$$

Then at time $t = 0$

$$u_t(x,0) = -c\frac{dF}{dx} + c\frac{dG}{dx} = g(x)$$

then

$$-c\frac{dF}{dx} + c\frac{dG}{dx} = g(x) \tag{3.4}$$

Equation (3.3) and (3.4) are two equations for the two arbitrary functions F and G. In order to solve the system we first have to integrate (3.4), so that

$$-dF(x) + dG(x) = \frac{1}{c} g(x)dx$$

$$\int -dF(x) + \int dG(x) = \frac{1}{c} \int_0^x g(\xi)d\xi$$

$$-F(x) + G(x) = \frac{1}{c} \int_0^x g(\xi)d\xi \tag{3.5}$$

$$F(x) + G(x) = f(x) \tag{3.6}$$

Solving the above two equations simultaneously we have

$$G(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_0^x g(\xi)d\xi$$

$$G(x+ct) = \frac{1}{2} f(x+ct) + \frac{1}{2c} \int_0^{x+ct} g(\xi)d\xi \tag{3.7}$$

.
And

$$F(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_0^x g(\xi)d\xi.$$

Then

$$F(x-ct) = \frac{1}{2} f(x-ct) - \frac{1}{2c} \int_0^{x-ct} g(\xi)d\xi \tag{3.8}$$

.
Now combining the expressions (3.8) and (3.9) with (2.15) we have

$$\begin{aligned} u(x,t) &= F(x-ct) + G(x+ct) \\ &= \frac{1}{2} f(x-ct) - \frac{1}{2c} \int_0^{x-ct} g(\xi)d\xi + \frac{1}{2} f(x+ct) + \frac{1}{2c} \int_0^{x+ct} g(\xi)d\xi. \end{aligned}$$

Hence,

$$u(x,t) = \frac{1}{2} [f(x-ct) + f(x+ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\xi)d\xi \tag{3.9}$$

Where $u \in C^2(RXR^+), f \in C^2(R)$ and $g \in C^1(R)$ is the solution of the initial value problem. Moreover, (3.9) is the d'Alembert's solution to (2.12) subject to (3.1) and (3.2). We derived formula (3.9) assuming u is a solution of (2.12). To check that this is really a solution we have a theorem.

Theorem 3.1.1. *Assuming $f \in C^2(R)$ and $g \in C^1(R)$ and define u by d'Alembert's formula then*
i.$u \in C^2(RX[0,\infty))$
ii.$u_{tt} - c^2 u_{xx} = 0$, *in* $RX(0,\infty)$ *and*
iii.$\lim_{(x,t)\to(x^0,0)} u(x,t) = f(x^0)$, $\lim_{(x,t)\to(x^0,0)} u_t(x,t) = g(x^0)$, *for* $t > 0$ *at each point* $x^0 \in R$.

Note that the solution u at a point (x_0, t_0) depends on f and g in the interval $(x_0 - ct_0, x_0 + ct_0)$ which is cut out of the initial line by the two characteristics.

$$x - ct = c_1, x + ct = c_2$$

with slope $\pm(\frac{1}{c})$ passing through the point (x_0, t_0). The interval $(x_0 - ct_0, x_0 + ct_0)$ on the time $t = 0$, is called the **domain of dependence** of the solution at the point (x_0, t_0).
The domain of dependence is obtained by drawing the two characteristics

$$x - ct = c_1, x + ct = c_2$$

through the point (x_0, t_0) as shown in the figure below.

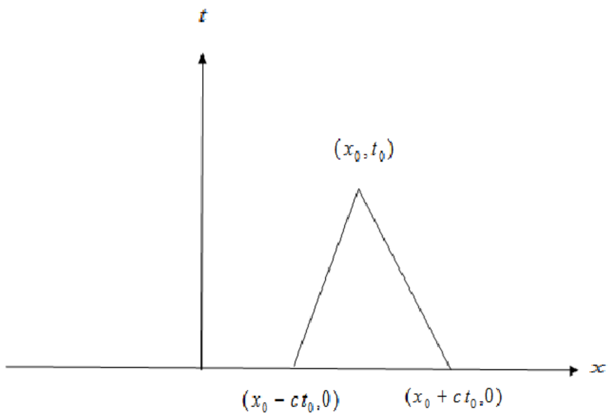

Figure C. Domain of dependence of a point (x_0, t_0)

Moreover, for a given $T > 0$ with $0 \leq t \leq T$ and for all $\epsilon > 0$ we need to show $\exists\delta > 0$ such that

$$\begin{aligned}
|u(x,t)| &= |\frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)d\xi| \leq \|u\| \\
&\leq \frac{1}{2}(\|f\| + \|f\|) + \frac{1}{2c}\|g\|\int_{x-ct}^{x+ct} d\xi \\
&= \|f\| + \frac{1}{2c}\|g\|\int_{x-ct}^{x+ct} d\xi \\
&\leq \|f\| + \frac{1}{2c}\|g\|2cT \\
&= \|f\| + \|g\|T.
\end{aligned}$$

Then for any $\epsilon > 0$ there exists $\delta \in (0, \frac{\epsilon}{1+T})$ if $\|f\| < \delta$ and $\|g\| < \delta$ it follows that $\|u\| < \epsilon$, which proves the continuous dependence.

Example 3.1.1.

$$u_{tt} - 4u_{xx} = 0$$

Subject to the initial conditions

$$\begin{aligned}
u(x,0) &= cosx, x \in R \\
u_t &= 0, x \in R
\end{aligned}$$

The solution can be determined by using the d'Alembert's formula

$$\begin{aligned}
u(x,t) &= \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)d\xi \\
&= \frac{f(x-2t) + f(x+ct)}{2} + 0 \\
&= \frac{f(x-2t) + f(x+ct)}{2} \\
&= \frac{\cos(x-2t) + \cos(x+2t)}{2} \\
&= cosxcos2t.
\end{aligned}$$

3.2 The wave equation in a semi-infinite string

$$u_{tt} - c^2 u_{xx} = 0, x \geq 0, t \geq 0 \tag{3.10}$$

Subject to the initial conditions

$$u(x, 0) = f(x), x \in [0, \infty) \tag{3.11}$$
$$u_t(x, 0) = g(x), x \in [0, \infty) \tag{3.12}$$

and the boundary condition

$$u(0, t) = h(t), t \in [0, \infty) \tag{3.13}$$

Note that f and g are defined only for nonnegative values of x. Therefore, (3.9) the d'Alembert's solution holds only if the arguments of f and g are nonnegative.
That is

$$x - ct \geq 0, x + ct \geq 0 \tag{3.14}$$

As can be seen in figure D, the first quadrant must be divided to two sectors by the characteristic

$$x - ct = 0.$$

In the lower sector I the solution (3.9) holds, in the other sector one should note that a characteristic

$$x - ct = constant$$

will cross the negative $x - axis$ and the positive $t - axis$.

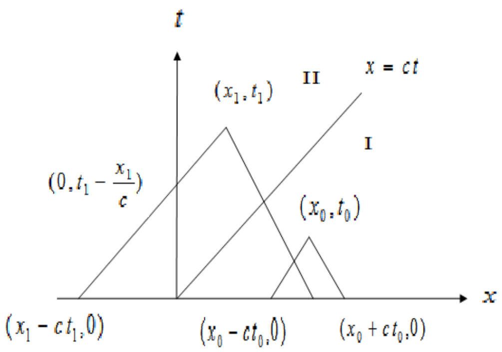

Figure D. The characteristic $x-ct=0$ divides the first quadrant

The solution at point (x_1, t_1) must depend on the boundary condition $h(t)$. We will show how the dependence presents itself.
For $x - ct < 0$, we proceed as follows. Combine (3.13) with the general solution (2.15) at $x = 0$.

$$h(t) = F(-ct) + G(ct) \tag{3.15}$$

Since $x-ct < 0$,and since F is evaluated at this negative value, we use (3.15)

$$F(-ct) = h(t) - G(ct) \tag{3.16}$$

Now let $z = -ct < 0$, then

$$F(z) = h(\frac{-z}{c}) - G(-z) \tag{3.17}$$

So,F for negative values is computed by (3.17) which requires G at positive values. Since z arbitrary, we can take $x - ct$ as z to get

$$F(x - ct) = h(\frac{ct - x}{c}) - G(ct - x) \tag{3.18}$$

Now combine (3.18) with the formula (3.7) for G we have

$$F(x - ct) = h(\frac{ct - x}{c}) - [\frac{1}{2}f(ct - x) + \frac{1}{2c}\int_0^{ct-x} g(\xi)d\xi]$$

The solution in sector II is then

$$u(x,t) = h(\frac{ct - x}{c})-[\frac{1}{2}f(ct-x)+\frac{1}{2c}\int_0^{ct-x} g(\xi)d\xi]+\frac{1}{2}f(ct+x)+\frac{1}{2c}\int_0^{ct+x} g(\xi)d\xi$$

Therefore, the solution of the initial / boundary value problem is

$$u(x,t) = \frac{f(x + ct) + f(x - ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)d\xi, x - ct \geq 0$$

$$u(x,t) = h(\frac{ct - x}{c}) + \frac{f(ct + x) - f(ct - x)}{2} + \frac{1}{2c}\int_{ct-x}^{ct+x} g(\xi)d\xi, x - ct < 0 \tag{3.19}$$

Example 3.2.1. *Solve*

$$\begin{aligned} u_{tt} &= 4u_{xx}, x \geq 0 \\ u(x,0) &= \sin x, x \geq 0 \\ u_t(x,0) &= 0, x \geq 0 \\ u(0,t) &= exp(-t), t \geq 0. \end{aligned}$$

3.3 Shock Waves in Scalar Conservation Law

3.3.1 Shock waves

If the initial solution is discontinuous, but the value to the left is larger than that to the right, one will see intersecting characteristics.
Consider the IVP

$$u_t + uu_x = 0 \tag{3.20}$$

$$u(x,0) = \begin{cases} 2, x < 1 \\ 1, x > 1 \end{cases} \tag{3.21}$$

The corresponding system of characteristics is

$$\frac{dt}{ds} = 1, \frac{dx}{ds} = u, \frac{du}{ds} = 0$$

then the characteristic ground curve is given by

$$x(t) = u(x(0),0)t + x(0) \tag{3.22}$$

$$u(x,t) = u(x(0),0) = constant \tag{3.23}$$

Thus the solution u is constant along the characteristic ground curve.
The characteristic ground curve along the regions is given by

$$x(t) = \begin{cases} 2t + x(0), x(0) < 1 \\ t + x(0), x(0) > 1 \end{cases} \tag{3.24}$$

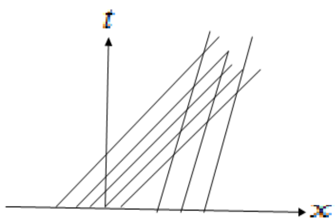

Figure E. Intersecting characteristics

Since there are two characteristics through a point, one cannot tell on which characteristic to move back to $t = 0$, to obtain the solution. In other words, at points of intersection the solution u is multi-valued. This situation happens whenever the speed along the characteristic on the left is larger than the one along the characteristic on the right, and thus catching up with it. We say in this case to have a shock wave.
In general, consider the equation

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

or

$$\frac{\partial u}{\partial x} + f'(u)\frac{\partial u}{\partial x} = 0$$

with $u(x, 0) = u_0(x)$, which is a **scalar conservation law**
The characteristic system is

$$\frac{dt}{ds} = 1, \frac{dx}{ds} = f'(u) = h(u), \frac{du}{ds} = 0$$

which yields

$$u = u_0(x_0), x(t) = h(u_0(x_0))t + x_0.$$

Hence the solution u is constant along the characteristic ground curve

$$x(t) = h(u_0(x_0))t + x_0$$

The characteristic ground curves and the characteristic curves are straight lines, but the slope depends on the initial value $u_0(x_0)$. In order to determine $u(x, t)$, we have to solve the equation

$$h(u_0(x_0))t + x_0 = x \tag{3.25}$$

for x_0. But this is in general not possible for all t.

Remark 3.3.1. *Because $u(x, t) = u_0(x_0)$ along the characteristic ground curve, one may even consider the equation*

$$u(x, t) = u_0(x - h(u(x, t))t)$$

which has to be solved for $u(x, t)$.
Now consider a classical example $h(u) = u$, which leads to the (inviscid) Burger's equation

$$u_t + (\frac{u^2}{2})_x = 0, u(x, 0) = u_0(x)$$

The characteristic ground curve is

$$x = u_0(x_0)t + x_0 \tag{3.26}$$

and depends on u_0. If we assume

$$u_0(x) = \begin{cases} 1, x < 0 \\ 1 - x, x \in [0, 1] \\ 0, x > 1 \end{cases}$$

we obtain from (3.26) the ground curves

$$x(t) = \begin{cases} t + x_0, x_0 < 0 \\ (1 - x_0)t + x_0, x_0 \in [0, 1] \\ x_0, x_0 > 1 \end{cases}$$

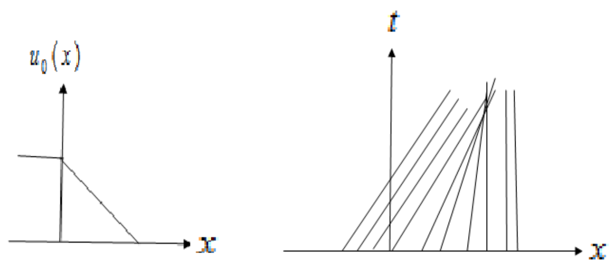

Figure F. Initial condition and corresponding Characteristic Ground Curves

For $0 \leq t \leq x < 1$ we have

$$\begin{aligned} x &= (1 - x_0)t + x_0 = t - x_0 t + x_0 \\ x - t &= x_0(1 - t) \\ \Rightarrow x_0 &= \frac{x - t}{1 - t} \end{aligned}$$

and therefore

$$u(x, t) = u_0(x_0) = 1 - x_0 = 1 - \frac{x - t}{1 - t} = \frac{1 - x}{1 - t}.$$

Since equation (3.25) is not solvable if $t \geq 1$. On the other hand, for $t \in [0,1)$ we obtain a classical solution given by

$$u(x,t) = \begin{cases} 1, x < t \\ \frac{1-x}{1-t}, 0 \leq t \leq x < 1 \\ 0, x > 1 \end{cases} \tag{3.27}$$

What happens with the solution (3.27) as $t \to 1$ is shown in (figure G1) there arises a so-called compression wave.
We should investigate, whether there exists a **weak** solution as $t \geq 1$, which will be done later. First we consider the same problem with an initial condition, which is an increasing function, $u_0 \uparrow$. If u_0 is smooth, the characteristics covers the whole $(x,t)-plane$ (figure G2); especially equation (3.25) is solvable for all $t \geq 0$ and the method of characteristics work everywhere.

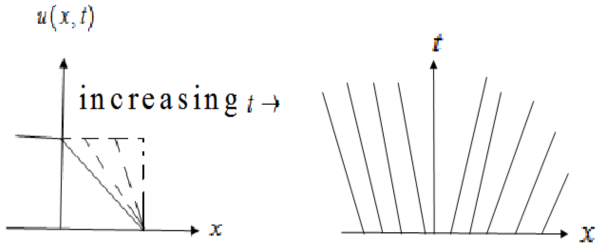

Figure G. Formation of a Compression Wave and Characteristic Ground Curves for $u_0 \uparrow$

If the initial condition contains a jump

$$u_0(x) = \begin{cases} c_1, x \leq 0 \\ c_2, x > 0 \end{cases}$$

The system of characteristics is given by

$$\frac{dt}{ds} = 1, \frac{dx}{ds} = u, \frac{du}{ds} = 0$$

The characteristic curves are given by

$$t = s, x = us + x_0 = u_0(x_0)s + x_0, u = u_0$$

then, the characteristic ground curves are

$$x = u_0(x_0)t + x_0 \tag{3.28}$$

we obtain from (3.28) the ground curves along the regions

$$x(t) = \begin{cases} c_1 t + x_0, x_0 \leq 0 \\ c_2 t + x_0, x_0 > 0 \end{cases} \tag{3.29}$$

Therefore,

$$u(x,t) = \begin{cases} c_1, x \leq c_1 t \\ c_2, x > c_2 t \end{cases} \tag{3.30}$$

but the $(x,t)-plane$ contains a region, where the solution is undefined,because non characteristic enters the region (figure H2)

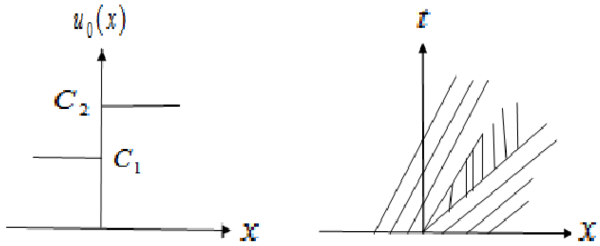

Figure H. Initial Condition with Jump and Characteristic Ground Curves

This phenomenon is called **rarefaction wave**.

3.3.2 Construction of weak solutions

For Burger's equation we say, a classical solution may not exists for all times t and we try to derive a weak solution for the problem.
For $x_1, x_2 \in R$ with $x_1 < x_2$, we have

$$\int_{x_1}^{x_2} [u_t + (f(u)_x)]dx = 0$$

implies that

$$\frac{d}{dt}\int_{x_1}^{x_2} u(\xi,t)d\xi = f(u(x_1,t)) - f(u(x_2,t)) \tag{3.31}$$

Here, we are looking for a special weak solution, the so-called shock solution.

Definition 3.3.1. *A shock solution is a weak solution of a scalar conservation law, for which a* ***shock front*** $x = s(t)$, $s \in C^1$ *exists, such that* u *is a classical solution in* $x < s(t)$ *and* $x > s(t)$ *respectively and that* u *has a jump* $[u](t) = u(s^+(t), t) - u(s^-(t), t)$ *at* $x = s(t)$*. We denote by* $\dot{s}(t)$ *the shock speed.*

The height of the jump $[u]$ is determined by the fact that $u(x,t)$ should be a weak solution. That is $u(x,t)$ should fulfill condition (3.31).
Let $x_1 < s(t) < x_2$, t fixed, then equation (3.31) reads

$$\frac{d}{dt}(\int_{x_1}^{s(t)} u(\xi,t)d\xi + \int_{s(t)}^{x_2} u(\xi,t)d\xi) = f_1 - f_2 \tag{3.32}$$

where $f_1 = f(u(x_1,t))$ and $f_2 = f(u(x_2,t))$. Computing the time derivatives in (3.33) yields

$$\int_{x_1}^{s(t)} \frac{\partial u}{\partial t} d\xi + \dot{s}u(s^-(t),t) + \int_{s(t)}^{x_2} \frac{\partial u}{\partial t} d\xi - \dot{s}u(s^+(t),t) + f_2 - f_1 = 0$$

Thus,

$$\lim_{x_1 \to s^-(t)} [\int_{x_1}^{s(t)} \frac{\partial u}{\partial t} d\xi + \dot{s}u(s^-(t),t) + \int_{s(t)}^{x_2} \frac{\partial u}{\partial t} d\xi - \dot{s}u(s^+(t),t)] = \lim_{x_1 \to s^-(t)} [f_1 - f_2]$$
$$\dot{s}u(s^-(t),t) - \dot{s}u(s^+(t),t) = f(u(s^-(t),t)) - f_2$$

and

$$\lim_{x_2 \to s^+(t)} [\dot{s}u(s^-(t),t) - \dot{s}u(s^+(t),t)] = \lim_{x_2 \to s^+(t)} [f(u(s^-(t),t)) - f_2]$$
$$\dot{s}u(s^-(t),t) - \dot{s}u(s^+(t),t) = f(u(s^-(t),t)) - f(u(s^+(t),t))$$

Hence,

$$-\dot{s}[u(s^+(t),t) - u(s^-(t),t)] = -[f(u(s^+(t),t)) - f(u(s^-(t),t))]$$
$$\Rightarrow \dot{s}[u] = [f]$$

Where $u(s^-(t),t)$ and $u(s^+(t),t)$ are the values of u as $x \to s$ from below and above respectively.

Theorem 3.3.1. *If* $x = s(t)$ *is the shock front of a shock solution of* $u_t + (f(u))_x = 0$*, we have for the shock speed* $\dot{s}(t) = U(t)$

$$U[u] = [f] \tag{3.33}$$

Equation (3.33) is the so-called **Rankine-Hugoniot condition**.

Remark 3.3.2. *Especially, we obtain a weak solution if and only if the shock solution fulfills the* ***Rankine-Hugoniot condition****.*

.

From the previous (inviscid) Burger's equation:

$$u_t + uu_x = 0$$

and compute with $f(u) = \frac{u^2}{2}$

$$[f] = \frac{u_+^2 - u_-^2}{2} = \frac{(u_+ + u_-)(u_+ - u_-)}{2} = \frac{(u_+ + u_-)[u]}{2}$$

Hence, the shock speed is given by $U = \frac{u_+ + u_-}{2}$. with the initial condition

$$u_0(x) = \begin{cases} 1, x < 0 \\ 1 - x, x \in [0, 1] \\ 0, x > 1 \end{cases}$$

We obtain the classical solution

$$u(x,t) = \begin{cases} 1, x < t \\ \frac{1-x}{1-t}, 0 \le t \le x < 1 \\ 0, x > 1 \end{cases}$$

as long as $t < 1$. For $t \ge 1$ we try the shock solution

$$u(x,t) = \begin{cases} 1, x \le s(t) \\ 0, x > s(t) \end{cases}$$

Since, $\dot{s}(t) = U(t) = \frac{u_+ + u_-}{2} = \frac{1}{2}$ we get

$$\frac{ds}{dt} = \frac{1}{2}$$
$$\Rightarrow \int_{s(1)}^{s(t)} dl = \frac{1}{2} \int_1^t dm$$
$$\Rightarrow s(t) - s(1) = \frac{1}{2}(t - 1)$$
$$s(t) = \frac{1}{2}t + \frac{1}{2}$$

Because at $t = 1$, the shock front starts at the point $s(1) = 1$.
Hence, the shock solution for $t \ge 1$ is given by

$$u(x,t) = \begin{cases} 1, x \le \frac{1}{2}t + \frac{1}{2} \\ 0, x > \frac{1}{2}t + \frac{1}{2}. \end{cases}$$

Chapter 4

Non-homogenous Wave Equation

4.1 Solution of non-homogeneous wave equation

Let $f \in C^1(R^2)$ and consider the initial value problem

$$u_{tt} - c^2 u_{xx} = f, x \in R, t > 0 \tag{4.1}$$

$$u(x,0) = \phi(x), x \in R$$
$$u_t(x,0) = \psi(x), x \in R$$

Where $f : UX[0,\infty) \to R$
In solving the above non-homogeneous wave equation we have to split the equation into two equations.
One the homogeneous wave equation with non- zero initial data

$$u_{tt} - c^2 u_{xx} = 0, x \in R, t > 0 \tag{4.2}$$

$$u(x,0) = \phi(x), x \in R$$

$$u_t(x,0) = \psi(x), x \in R$$

The other non-homogeneous wave equation with zero initial data

$$u_{tt} - c^2 u_{xx} = f, x \in R, t > 0 \tag{4.3}$$

$$u(x,0) = 0, x \in R$$
$$u_t(x,0) = 0, x \in R$$

Now, if we let u_1 and u_2 are solutions of the homogeneous and non-homogeneous wave equations respectively, then

$$u(x,t) = u_1(x,t) + u_2(x,t) \tag{4.4}$$

is the solution of the non-homogeneous wave equation. Since $u_1(x,t)$ is the solution of the initial value problem, then the solution $u_1(x,t)$ is the d'Alembert's formula. Therefore,

$$u_1(x,t) = \frac{\phi(x+ct) + \phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\xi)d\xi. \tag{4.5}$$

But to determine the solution $u_2(x,t)$ we apply the characteristic method. The characteristic form is

$$\frac{dx}{dt} = \pm c$$

integrating which, we have two families of real characteristics

$$x - ct = c_1, x + ct = c_2$$

Now by making of change of variable we set

$$\eta = x - ct, \xi = x + ct \tag{4.6}$$

Where η and ξ are twice continuously differentiable functions of x and t. To have the coordinate transformation we need to see the functional determinant (Jacobian)

$$J = \frac{\partial(\eta,\xi)}{\partial(x,t)} = \begin{vmatrix} \eta_x & \xi_x \\ \eta_t & \xi_t \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -c & c \end{vmatrix} = 2c \neq 0$$

Therefore, inversion of coordinate transformation is possible. Now we set

$$u_2(x,t) = U(\eta,\xi) \tag{4.7}$$

$$(u_2(x,t))_x = U_\eta + U_\xi$$
$$(u_2(x,t))_{xx} = U_{\eta\eta} + U_{\eta\xi} + U_{\xi\eta} + U_{\xi\xi}$$
$$(u_2(x,t))_{xx} = U_{\eta\eta} + 2U_{\xi\eta} + U_{\xi\xi} \tag{4.8}$$

And

$$(u_2(x,t))_t = -cU_\eta + cU_\xi$$
$$(u_2(x,t))_{tt} = c(cU_{\eta\eta} - 2cU_{\xi\eta} + cU_{\xi\xi})$$
$$(u_2(x,t))_{tt} = c^2(U_{\eta\eta} - 2U_{\xi\eta} + U_{\xi\xi}) \tag{4.9}$$

Thus combining equations (4.1), (4.8) and (4.9) to give

$$-4c^2U_{\xi\eta} = F(\eta, \xi)$$

Which is the canonical form, thus we transform (4.1) in to the form

$$U_{\xi\eta} = \frac{-1}{4c^2}F(\eta, \xi). \tag{4.10}$$

Since the initial conditions in(4.3) are all zero we have

$$U(\xi, \xi) = U_\eta(\xi, \xi) = U_\xi(\xi, \xi) = 0 \tag{4.11}$$

Where

$$U(\eta, \xi) = u_2(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2c})$$
$$F(\eta, \xi) = f(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2c})$$

Integrating (4.10) with respect to η, we have

$$\partial U_\xi = \frac{-1}{4c^2}F(\eta, \xi)\partial\eta \Rightarrow \int_{U_\xi(\eta,\xi)}^{U_\xi(\xi,\xi)} dl = \frac{-1}{4c^2}\int_\eta^\xi F(s, \xi)ds$$

$$U_\xi(\xi, \xi) - U_\xi(\eta, \xi) = \frac{-1}{4c^2}\int_\eta^\xi F(s, \xi)ds \tag{4.12}$$

Thus by (4.11)

$$-U_\xi(\eta, \xi) = \frac{-1}{4c^2}\int_\eta^\xi F(s, \xi)ds$$

$$U_\xi(\eta, \xi) = \frac{1}{4c^2}\int_\eta^\xi F(s, \xi)ds \tag{4.13}$$

Again integrating (4.13) with respect to ξ we have

$$\partial U(\eta, \xi) = [\frac{1}{4c^2}\int_\eta^\xi F(s, \xi)ds]\partial\xi$$
$$\Rightarrow \int_{U(\eta,\eta)}^{U(\eta,\xi)} dm = \frac{1}{4c^2}\int_\eta^\xi \int_\eta^z F(s, z)dsdz$$
$$\Rightarrow U(\eta, \xi) - U(\eta, \eta) = \frac{1}{4c^2}\int_\eta^\xi \int_\eta^z F(s, z)dsdz$$

In the same way by (4.11) we have

$$U(\eta,\xi) = \frac{1}{4c^2}\int_{\eta}^{\xi}\int_{\eta}^{z} F(s,z)dsdz \tag{4.14}$$

Let us make change of variables and we set

$$s = \delta - c\tau, z = \delta + c\tau$$

Which has Jacobian

$$J = \frac{\partial(s,z)}{\partial(\delta,\tau)} = \begin{vmatrix} s_\delta & z_\delta \\ s_\tau & z_\tau \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -c & c \end{vmatrix} = 2c$$

Thus the last change transforms the domain of integration from

$$D = \{(s,z) : \eta \le s \le z, \eta \le z \le \xi\}$$

$$to$$

$$D' = \{(\delta,\tau) : x - c(t-\tau) \le \delta \le x + c(t-\tau), 0 \le \tau \le t\}.$$

By the fact that $\eta \le s \le z \le \xi$, then we have

$$x - ct \le \delta - c\tau \le \delta + c\tau \le x + ct$$

Then it follows that

$$0 \le 2c\tau \le 2ct \Leftrightarrow 0 \le \tau \le t$$

$$And$$

$$x - c(t-\tau) \le \delta \le x + c(t-\tau)$$

Hence the solution of the non-homogeneous wave equation with zero initial data is

$$u_2(x,t) = U(\eta,\xi) = \frac{1}{4c^2}\int_{\eta}^{\xi}\int_{\eta}^{z} F(s,z)dsdz$$

$$u_2(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\delta,\tau)d\delta d\tau = \frac{1}{2c}\int\int_{\Delta(x,t)} f(\delta,\tau)d\delta d\tau \tag{4.15}$$

Where $\Delta(x,t)$ is the characterisic triangle. Therefore, the solution of (4.1) is

$$u(x,t) = u_1(x,t) + u_2(x,t)$$

$$u(x,t) = \frac{\phi(x+ct) + \phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\xi)d\xi + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\delta,\tau)d\delta d\tau$$

$$u(x,t)=\frac{\phi(x+ct)+\phi(x-ct)}{2}+\frac{1}{2c}[\int_{x-ct}^{x+ct}\psi(\xi)d\xi+\int_0^t\int_{x-c(t-\tau)}^{x+c(t-\tau)}f(\delta,\tau)d\delta d\tau] \tag{4.16}$$

Then (4.16) is the solution of the non-homogeneous wave equation.
Moreover, for $0\leq t\leq\tau$, and for all $\epsilon>0$ we need to show $\exists\delta>0$ such that

$$\begin{aligned}|u(x,t)|&=|\frac{\phi(x+ct)+\phi(x-ct)}{2}+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(\xi)d\xi+\frac{1}{2c}\int_0^t\int_{x-c(t-\tau)}^{x+c(t-\tau)}f(\delta,\tau)d\delta d\tau|\leq\|u\|\\&\leq\frac{1}{2}[\|\phi\|+\|\phi\|]+\frac{1}{2c}\|\psi\|2c\tau+\frac{1}{2c}\|f\|\int\int_{\Delta(x,t)}d\delta d\tau\\&=\|\phi\|+\frac{1}{2c}\|\psi\|2c\tau+\frac{1}{2c}\|f\|\int\int_{\Delta(x,t)}d\delta d\tau\\&\leq\|\phi\|+\|\psi\|\tau+\frac{\tau^2}{2}\|f\|, since\int\int_{\Delta(x,t)}d\delta d\tau\leq\frac{2c\tau.\tau}{2}\\&=\|\phi\|+\|\psi\|\tau+\frac{\tau^2}{2}\|f\|\end{aligned}$$

Then for $\epsilon>0,\exists\delta\in(0,\frac{2\epsilon}{\tau^2+2\tau+2})$ such that if $\|\phi\|<\delta,\|\psi\|<\delta,\|f\|<\delta$ it follows $\|u\|<\epsilon$. Therefore, u is well-posed.

Example 4.1.1. *Solve*

$$u_{tt}-u_{xx}=\sin x, x\in R, t>0 \tag{4.17}$$

$$\begin{aligned}u(x,0)&=\cos x, x\in R\\u_t(x,0)&=x, x\in R\end{aligned}$$

Solution

$$\begin{aligned}u(x,t)&=\frac{\phi(x+ct)+\phi(x-ct)}{2}+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(\xi)d\xi+\frac{1}{2c}\int_0^t\int_{x-c(t-\tau)}^{x+c(t-\tau)}f(\delta,\tau)d\delta d\tau\\&=\frac{1}{2}[\cos(x+ct)+\cos(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}\xi d\xi+\frac{1}{2c}\int_0^t\int_{x-c(t-\tau)}^{x+c(t-\tau)}(\sin\delta)d\delta d\tau\\&=\cos x\cos t+\frac{1}{2}[\frac{(x+t)^2-(x-t)^2}{2}]+\frac{1}{2}\int_0^t[\cos[x-(t-\tau)]-\cos[x+(t-\tau)]]d\tau\\&=\cos x\cos t+xt+(1-\cos t)\sin x\end{aligned}$$

Summary

In second order linear homogeneous wave equation the solution is the sum of two waves, one represented by $F(x-ct)$ propagates from left to right without change of form at speed c, and one represented by $G(x+ct)$ from right to left at speed c. And the scalar conservation law of the inviscid Burger's equation cannot be solved for all time $t \in [0, \infty)$ but **classical solution** can be determined for $0 \leq t < 1$ and **weak solution** for $t \geq 1$. The solution of a linear non-homogeneous wave equation is the sum of the general homogeneous solution and the non-homogeneous solution with zero initial data. And its solution is obtained by the method of characteristics through a successive integration in terms of the characteristic variables.

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