Addis Ababa University

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Elliptic Problems and The Variational Form

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We, the under signed, hereby certify that we have read and examined this thesis, a thesis Elliptic Problems and the Vartiational Form which is done by Wubu Getahun in partial fulfillment of requirement for the degree of master of science in Mathematics and recommend to the school of graduate studies for acceptance of thesis.

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Declaration

I **Wubu Getahun** declare that this thesis has been composed by me and that no part of the thesis has formed the basis for the award of any degree, diploma, associateship, fellowship, or any other similar title to me.

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This is to certify that this thesis is written by Wubu Getahun in the Department of Mathematics, Addis Ababa University, under my supervision. I also confirm that you can submit this thesis for examiners' evaluation and eventual defense.

Tadesse Abdi (Ph.D)

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Abstract

In this thesis, given a bounded domain $\Omega \subset R^n$. we focus on the variational form

$$\int_{\Omega} \{\alpha \nabla u \cdot \nabla \phi + au\phi\} dx = \int_{\Omega} f\phi dx, \phi \in H_0^1(\Omega)$$

Of an elliptic problem,

$$-\alpha \Delta u + a(x)u = f, \qquad in\ \Omega$$

With various conditions (Dirichlet, Neumann, and Robin) on boundary $\partial\Omega$. We provide some general results on uniqueness and also find sub (supper)-solutions with in the frame work of variational setting.

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Introduction

The variational formulation is often the only effective method for formulating and solving problems while retaining their core structures. The first chapter of this thesis introduces essential tools from functional analysis that are necessary for a proper variational formulation of various boundary value problems. This section focuses on functional analysis in Hilbert (Sobolev) spaces, emphasizing the solvability of abstract variational problems, culminating in the Lax-Milgram theorem.

In the second chapter, the Laplace equation is frequently encountered in applied sciences, particularly in the analysis of steady-state phenomena, with solutions referred to as harmonic functions. For example, the equilibrium state of an elastic membrane is a harmonic function, as is the velocity potential of a homogeneous fluid. Additionally, the steady-state temperature in a homogeneous and isotropic body is also a harmonic function, representing the time-independent form of the diffusion equation.

Poisson's equation is significant in the study of conservative fields (such as electrical, magnetic, and gravitational fields), where the vector field is derived from the gradient of a potential. If u represents the position of a flexible membrane and f symbolizes an external distributed load (vertical force per surface area), a steady-state equation can be modeled.

This section discusses various boundary conditions: Dirichlet conditions fix the membrane's position at the boundary; Robin conditions describe elastic attachments; and homogeneous Neumann conditions correspond to free vertical motion at the boundary. These concepts are all examined within the context of variational formulation. Additionally, various solution concepts for these equations, using Poisson's equation as a reference, are briefly reviewed. Among classical, strong, distributional, and weak solutions, weak solutions are the most practical for implementing the Galerkin method (including finite and spectral elements), which is commonly used for numerical approximations of boundary value problems. The weak formulation efficiently addresses uniqueness, existence, and stability issues, naturally leading to the development of Galerkin-type numerical methods.

The weak formulation is constructed by selecting a space of smooth test functions, multiplying the differential equations by these test functions, integrating the divergence terms by parts, applying boundary conditions, and interpreting the resulting integral equation as an abstract variational problem. This process illustrates how a differential equation can be transformed into an integral equation, thus framing it as a variational problem.

Furthermore, various boundary conditions—Dirichlet, Robin, Neumann, and mixed (both homogeneous and non-homogeneous)—are analyzed through the lens of Poisson's problem's variational formulations. The Poisson problem, recognized as the simplest elliptic problem, is examined using the Lax-Milgram Theorem to verify the existence, uniqueness, and stability (well-posedness) of weak solutions. The continuity of bilinearity, linearity, and coercivity (ellipticity) is also assessed during this proof.

The third chapter focuses on the variational formulation of both linear and nonlinear (or semilinear) elliptic boundary value problems, starting with Poisson's equation and leading up to general second-order equations in divergence form. The results presented lay the theoretical groundwork for numerical methods, such as finite elements and, more broadly, Galerkin methods, enhancing the appeal of this theory. Advanced topics include broader solvability issues and the spectral properties of elliptic operators, formalizing concepts of eigenvalues and eigenfunctions.

Additionally, eigenvalues and eigenspaces are utilized in the separation of variables method and to demonstrate the asymptotic stability of steady-state solutions in dynamic systems. General elliptic equations in divergence form are employed to address boundary value problems involving elliptic operators with varying diffusion and transport terms. The discussion extends to the impact of regularity theorems on the smoothness of u.

Finally, the application of variational formulation theory is broadened to identify weak subsolutions or weak supersolutions, effectively serving as lower and upper barriers.

Chapter one

Function spaces, an Overview

1.1 introduction

This chapter presents important tools that make it easier to grasp and work with variational (weak) formulations of elliptic problems. The vector analysis formulas, primary Hilbert (Sobolev) spaces, and functional analysis theorems that can be used with the theorems later worked out in this thesis are all presented in this chapter.

1.2 Adapted function spaces and their properties

In this section we define certain function spaces that would be used to state certain fundamental results.

Definition (1):
let $\Omega \subset \mathbb{R}^n$ be a domain. The Sobolev spaces we will use most are the Hilbert spaces

$$H^1(\Omega):\{V \in L^2(\Omega) \ : \ \nabla V \in L^2(\Omega: \mathbb{R}^n)\} \tag{1.1}$$

And its closed subspace $H_0^1(\Omega) = \overline{D(\Omega)}^{H^1(\Omega)}$
$H_0^1(\Omega)$ Is a Hilbert subspace of $H^1(\Omega)$
Definition (2):
The space $H^m(\Omega), m > 1$
Let $\Omega \subset \mathbb{R}^N$ be a domain. Let $H = L^2(\Omega), z = L^2(\Omega: \mathbb{R}^N) \subset D^1(\Omega: \mathbb{R}^N)$ *and*
$L: L^2(\Omega) \longrightarrow D^1(\Omega: \mathbb{R}^N)$ given by

$$H^m(\Omega) = \{V \in L^2(\Omega): D^\alpha V \in L^2(\Omega); \forall_\alpha: |\alpha| \leq m\} \tag{1.2}$$

Definition (3):
Distributions:- the space $D(\Omega)$ is defined as

$$D(\Omega) = \{V: \Omega \subset \mathbb{R}^n \longrightarrow R, V \in C^\infty(\Omega), suppV \subset \Omega\} \tag{1.3}$$

Terminology: - the $D(\Omega)$ space is the space of functions C^∞ over Ω with a compact support strictly included in Ω

Definition (4):
The notion of support of V (noted as $suppV$) is introduced as the smallest closed subset containing all the points where a given function V is non-zero is another function space essential for the functional analysis of equations having partial derivations.

$$suppV = \overline{\{x \in \mathbb{R}^n \;/\; v(x) \neq 0\}}^{\mathbb{R}^n} \tag{1.4}$$

To illustrate the notion of support, consider a function defined as

$$H(x) = \begin{cases} 1, & given\ 0 < x < 1 \\ 0, & otherwise \end{cases}$$

In this case, the function H is non-zero in an open domain (0, 1), but has a closed interval [0, 1] as support.

$$suppH = \overline{\{x \in \mathbb{R}^n \;/\; H(x) \neq 0\}}^{\mathbb{R}^n} = [0,1]$$

Definition(5):

A distribution in Ω (generalized functions) is a linear continuous function in $D(\Omega)$. The set of distributions is denoted $by\ D'(\Omega)$. (1.5)

Theorem (1):

The space $D(\Omega)$ is dense in $L^2(\Omega)$

The closure of space $H^1(\Omega)\ in\ D(\Omega)$ is associated to the former and is noted as $H_0^1(\Omega)$

The following definition and property are thus obtained

Definition (6):

$$H_0^1(\Omega) = \overline{H^1}^{D(\Omega)} H_0^1(\Omega)\ is\ a\ closure\ of\ H^1 \tag{1.6}$$

Theorem (2)

$$H_0^1(\Omega) = \{V: \Omega \subset \mathbb{R}^n \longrightarrow R, V \in H^1(\Omega), V = o\ on\ \partial\Omega\} \tag{1.7}$$

The Green formula: (integration by parts formulas)

Definition (7):

Let $\Omega \subset \mathbb{R}^n$, be a bounded C^1-domain and

$$\boldsymbol{F}: \overline{\Omega} \longrightarrow \mathbb{R}^n$$

be a vector field with components $F_{j,}, J = 1, \dots, n,$ of class $C^1(\overline{\Omega})$; we write $\boldsymbol{F} \in C^1(\Omega: \mathbb{R}^n)$

The gauss divergence formula holds:

$$\int_\Omega divFdx = \int_{\partial\Omega} \mathrm{F}\, Vd\sigma, where\ divF = \sum_{j=1}^{n} \partial_{x_J} F_J \tag{1.8}$$

V- is outward normal unit vector to $\partial\Omega$ and $d\sigma$- is the 'surface' measure on $\partial\Omega$

A number of useful identities are derived from $\int_{\Omega} divFdx = \int_{\partial\Omega} \mathrm{F}\cdot Vd\sigma$ and recalling the identity$div(VF) = VdivF + \nabla V \cdot F$, we obtain the following integration by parts formula:

$$\int_{\Omega} \mathrm{V}\, divFdx = \int_{\partial\Omega} \mathrm{VF}\cdot Vd\sigma - \int_{\Omega} \nabla V \cdot Fdx \tag{1.9}$$

Choosing $F = \nabla u$ and $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, since $div\nabla u = \Delta u$ and $\nabla u \cdot V = \partial_v u$,
The following green's identity follows:

$$\int_{\Omega} V\Delta u dx = \int_{\partial\Omega} V\, \partial_{\mathrm{V}} \mathrm{u}\, \mathrm{d}\sigma - \int_{\Omega} \nabla \mathrm{V} \cdot \nabla \mathrm{u}\, \mathrm{dx} \tag{1.10}$$

In particular, the choice $V \equiv 1$ yields

$$\int_{\Omega} \Delta u dx = \int_{\partial\Omega} \partial_{\mathrm{V}} \mathrm{u} d\sigma \tag{1.11}$$

If also $V \in C^2(\Omega) \cap C^1(\overline{\Omega})$, inter changing the roles of u and v, we deserve a second green's identity

$$\int_{\Omega} (v\Delta u - u\Delta v)dx = \int_{\partial\Omega} (\mathrm{v}\, \partial_{\mathrm{V}} \mathrm{u} - \mathrm{u}\, \partial_{\mathrm{u}} \mathrm{v}) \tag{1.12}$$

1.3 A set of fundamental inequalities:-

We recall some fundamental inequalities that emerge from the analysis and that are used intensely in the frame work of functional analysis of equations having partial derivations.

Cauchy-Schwartz inequality

Theorem (3):

Let u and v be two functions belonging to $L^2(\Omega)$. The result obtained is:

$$|u, v| \leq \| u \| \| v \| = \int_{\Omega} u \cdot v d\Omega \leq \left[\int_{\Omega} u^2 d\Omega\right]^{\frac{1}{2}} \cdot \left[\int_{\Omega} v^2 d\Omega\right]^{\frac{1}{2}} \tag{1.13}$$

***Hölder's* Inequality:**

Theorem (4):

Let p and q be two conjugate real numbers that satisfy: $\frac{1}{p} + \frac{1}{q} = 1$. let u be a function belonging to $L^p(\Omega)$. The result obtained is:

$$\int_{\Omega} \mathrm{u} \cdot \mathrm{v} d\Omega \leq [\int_{\Omega} u^p d\Omega]^{\frac{1}{p}} \cdot [\int_{\Omega} v^q d\Omega]^{\frac{1}{q}} \quad (1.14)$$

Poicare's inequality:

Theorem (5)

Let Ω be an open bounded domain of $\mathbb{R}^n$ and u a function belonging to sobolev space $H_O^1(\Omega)$. under suitable hypotheses, the norm $||u||_{H^1(\Omega)}$ is equivalent to $||\nabla v||_{L^2(\Omega)}$. This means that there exists a constant C_p, independent of u, such that

$$||\nabla u||_{L^2(\Omega)} \leq \mathrm{C_p}||\nabla u||_{L^2(\Omega;\mathbb{R}^n)} \quad (1.15)$$

Trace inequality

Theorem (6):

Let Ω be a bounded Lipschitz domain (or half space) and set $\Gamma = \partial\Omega$. then there is a well defined trace operator

$\gamma_o: H^1(\Omega) \rightarrow L^2(\Gamma)$, which is linear, continuous and such that

1.$\gamma_o u = u/r$, if $u \in C(\overline{\Omega})$

2.$\|\gamma_o u\|_{L^2(\Gamma)} \leq C^* \| u \|_{H^1(\Omega)}$, with C^* is independent of u (1.16)

1.4 Some other definitions

The dual of $H_O^1(\Omega)$, denoted by $H^{-1}(\Omega)$ consists of elements that can be written as $f + div\boldsymbol{f}$, where

$$f \in L^2(\Omega; \mathbb{R}) \text{ and } \boldsymbol{f} \in L^2(\Omega; \mathbb{R}^n) \quad (1.17)$$

Definition (9):

Well-posedness

Well-posed problems are problems having the following features

a) there exists at least one solution
b) there exists at most one solution
c) the solution depends continuously on the data (data→solution is continuous means a small error on the data shows a small error on the solution) (1.18)

Definition (10):

Smooth and Lipschitz domains

- if Ω is a C^k- domain for every $K \geq 1$,it is said to be a C^∞domain. Then these domains are smooth domains (1.19)

- A bounded domain Ω is Lipschitz if $\partial\Omega$ can be described by a family of charts $(\varphi_J, N_J), J = 1, \dots, N$, where the functions $\varphi_J, J = 1, \dots, N$ are Lipschitz (1.20)
- We say that $u: \Omega \rightarrow \mathbb{R}$ is Lipschitz if there exists L such that

$$|u(x) - u(y)| \leq L|x - y| \quad (1.21)$$

for every,$x, y \in \Omega$. The number L is called Lipschitz constant of u

1.5 Lax-Milgram theorem and Inequalities

Lax-Milgram theorem

Theorem(7)

Let V be a (real) Hilbert space endowed with inner product$(\cdot,\cdot)$ and induced with norm $||\cdot||$. Let $a = a(u, v)$ be bilinear form in V, if:

i) a is continuous, i.e. there exists a constant M such that

$$|a(u, v)| \leq M\big||u|\big|\big||v|\big|, \forall u, v \in V \qquad (1.22)$$

ii) a is V-coercive, i.e. there exists a constant α such that
$\alpha(u, v) \geq \alpha||V||^2, \forall_v \in V$, Then there exists a unique solution $\bar{u} \in V! \, or \, V - elliptic$

If the following stability estimate holds

$$\| \bar{u} \| \leq \frac{1}{\alpha} \| F \|_{V^*} \qquad (1.23)$$

Chapter two

Elliptic problems

Introduction

In 2D, Poisson's equation $\Delta u = f$ is the is the simplest of all the elliptic equations. These kinds of equations describe a wide range of phenomena, many of which are stationary in nature. Diffusion and reaction models, regarded as stationary conditions, typically correspond to a stable state with no further temporal dependence. Elliptic equations also arise in the study of elastic structure vibration modes and in the theory of electrostatic and electromagnetic potentials (e.g., through the method of separation of variables for the wave equation).

2.1. Definition of elliptic problems

The definition elliptic equation in dimension n is. Let $\Omega \subset R^n$ be a domain, $A(x) = (a_{ij}(x))$ a square matrix of order n, where $i, j = 1, \dots, n$
$b(x) = (b_1, \dots, b_n)(x), c(x) = (c_1, \dots, c_n)(x)$ vector fields $in\ R^n$,
$a_0 = a_0(x)$ and $f = f(x)$ real functions. An equation of the form:

$$-\sum_{i=1}^{n} \partial_{x_i} b_i(x)u + \sum_{i=1}^{n} c_i(x)u_{x_i} + a_0(x) = f(x) \qquad (2.1)$$

or

$$\sum_{i=1}^{n} a_{ij}(x)u_{xixj} + \sum_{i=1}^{n} b_i(x)u_{x_i} + a_0(x)u = f(x) \qquad (2.2)$$

is said to be elliptic in Ω if A is positive in Ω, i.e. if the following ellipticity condition holds:

$$\sum_{i,j=1}^{n} a_{ij}(x)\varepsilon_i\varepsilon_j > 0, \forall_x \in \Omega, \forall_\varepsilon \in R^n, \varepsilon \neq 0$$

We say that (2.1) is in divergence form which can be written as:

$$-\mathrm{div}(A(x)\nabla u) + \mathrm{div}(b(x)u) + c(x) \cdot \nabla u + a_0(x)u = f(x) \qquad (2.3)$$

The flux function q, which is q = -A∇u. This represents the diffusion in heterogeneous or an isotropic media. In this case, u might stand for either a substance's concentration or temperature. Therefore, thermal or molecular diffusion is indicated by the word -div(A∇u). Anisotropic diffusion is indicated by matrix A, which is known as the diffusion matrix. Convection or transport is modeled by div(bu), which is equivalent to a flux function given by

$$q = \boldsymbol{b}u$$

The vector b has the dimensions of a velocity, such as the diffused odors carried by the wind from industry sites. The wind speed in this instance is b. div(bu) simplifies to b• ∇u, which is the same form as the third term, c• ∇u, if divb =0.

a_0u models reaction. When u is a substance's concentration, then a_0 denotes the rate of growth if ($a_0 < 0$) or decomposition if ($a_0 > 0$). Lastly, f stands for an external action that is dispersed throughout the domain, such as the rate at which an external source supplies heat per unit mass. If the entries a_{ij} of the matrix $\boldsymbol{A}$ and the component b_j of $\mathbf{b}$ are all differentiable, we may compute the divergence of both $\boldsymbol{A}\nabla u$ and bu, and reduce (2.1) to the non-divergence form

$$-\sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j} + \sum_{k=1}^{n} \tilde{b}_k(x)\, u_{xk} + \tilde{c}(x)\, u = f(x)$$

Where

$$\tilde{b}_k(\mathbf{x}) = \sum_{i=1}^{n} \partial_{x_i} a_{ik}(\mathbf{x}) + b_k(\mathbf{x}) + c_k\mathbf{x}) \text{ and } \tilde{c}(\mathbf{x}) = \operatorname{divb}(\mathbf{x}) + a_0(\mathbf{x})$$

In cases where either the a_{ij} or the b_j are not differentiable, it is necessary to maintain the divergence form and apply a suitably weak interpretation to the differential equation (2.3). Diffusion phenomena are also linked to a non-divergence form equation through stochastic processes, which generalize Brownian motion. we could move forward by examining a random walk in hZ^2, which is independently symmetric along each axis. By appropriately passing to the limit as h and the time step τ approach zero, we can derive an equation of the type

$$u_t = D_1(x,y)\, u_{xx} + D_2(x,y)\, u_{yy}$$

With diffusion matrix

$$\boldsymbol{A}(x,y) = \begin{pmatrix} D_1(x,y) & 0 \\ 0 & D_2(x,y) \end{pmatrix}$$

Where $D_1(x,y) > 0, D_2(x,y) > 0$. Thus, the steady state case is a solution of a non-divergence form equation.

2.2 .The Poisson Problem and various notions of solutions

2.2.1 The poisson problem

$Let\,\Omega \subset R^n$ is a domain, α is a positive constant and two real function $a_0, f\colon \Omega \to \mathbb{R}$. we want to fund a function u satisfying the equation

$$-\alpha\Delta u + a_0 u = f \text{ in } \Omega$$

and one of the boundary conditions of $\partial\Omega$.

- The final objective is crucial to resolving the poisson problem mentioned above. We must demonstrate the existence, uniqueness, and stability of the solutions, and then we wish to compute the answer using numerical analysis techniques based on these findings. A brief explanation of the many concepts of solutions using the poisson problem as a model are:

- ***Classical** solutions* are twice continuously differentiable functions ; the DE and boundary conditions are satisfied in point wise sense.
- ***Strong** solutions* belong to the sobolev space $H^2(\Omega)$. Thus, they possesses derivatives in $L^2(\Omega)$ up to the second order, in the sense of distributions.

The DE is satisfied in the point wise sense, a.e., with respect to lebesgue measure in Ω, condition is satisfied in the sense of traces.

- ***Distributional solutions*** belong to $L^1_{Loc}(\Omega)$ and the equation holds in the sense of distributions that is:

$$\int_\Omega \{-\alpha u \nabla\varphi + a_0(x)u\varphi\}dx = \int_\Omega f\varphi dx, \qquad \forall\varphi \in D(\Omega)$$

The boundary condition is satisfied in a very weak sense

- ***Weak or variational solutions*** belong to the Sobolev space $H^1(\Omega)$. The boundary value problem is recast within the framework of the abstract variational theory.

If all the data (domain, boundary data, forcing terms) and the solution are C^∞, all the above notions must be equivalent. Thus, the non-classical solutions constitute a generalization of the classical one

Let u be a non-classical solution of the Poisson problem for our purpose the most convenient notion of solution is the weak formulation, It leads to a quiet flexible formulation with a sufficiently high degree of generality and a basic theory relying on the Lax-Milgram theorem.

Finally, the varational formulation is the most natural to implement the Galerkin method (finite elements, spectral elements ,etc,…) widely used in the numerical approximation of the solutions of boundary value problems.

2.2.2 .Diffusion, Drift and Reaction

To present the main ideas behind the variational formulation we start from 1-dimensional problems with an equation slightly more general than Poisson's equation

-Let us derive the variational formulation of the problem (in 1-dimension)

$$\begin{cases} -(p(x)u')' + q(x)u + r(x) = f(x) & \text{in the interval } (a,b) \\ \text{Boundary conditions} & \text{at } x = a \text{ and } x = b \end{cases} \tag{2.4}$$

(2.4) can be understood as a stationary problem including diffusion, drift, and reaction. The primary steps for the weak formulation are as follows:

a) Choose a smooth test function space that satisfies the boundary criteria.

b) Integrate the differential equation over the domain Ω after multiplying it by using a test function and using the boundary conditions,
c) carry one of the derivatives in the divergence term onto the test function using an integration by parts to generate an integral equation.
d) The closure of the space of test functions indicates the appropriate Hilbert space in which to interpret the integral equation as an abstract variational problem.

2.2.3 Problems with homogeneous Dirichlet boundary conditions

we proceed by analyzing homogeneous Dirichlet conditions as:

$$\begin{cases} -(p(x)u')' + q(x)u' + r(x)u = f(x), & \text{in } (a,b) \\ u(a) = u(b) = 0 \end{cases} \tag{2.5}$$

Assume $p \in C^1([a,b])$ with $p > 0$, and $q, r, f \in C([a,b])$.Let $u \in C^2(a,b) \cap C([a,b])$ be a classical solution of (2.5). We select $C_0^1(a,b)$ as the space of test functions having a continuous derivative and compact support in (a,b). particularly, it vanishes at the end points.

Now we multiply the equation by an arbitrary $v \in C_0^1(a,b)$ and integrate over (a,b). We find:

$$\int_a^b (pu')' \, vdx + \int_a^b [qu' + ru]vdx = \int_a^b fvdx \tag{2.6}$$

From (2.6) we derive the integral equation

$$\int_a^b [pu'v' + qu'v + ruv]dx = \int_a^b fvdx, \qquad \forall v \in C_0^1(a,b). \tag{2.7}$$

Thus (2.5) implies (2.7)

On the other hand, assume that (2.7) is true. Integrating by parts in the reverse order, we recover (2.6), which can be written in the form

$$\int_a^b \{-(pu')' + q(x)u' + r(x)u - f(x)\} \, vdx = 0, \forall v \in C_0^1(a,b)$$

The arbitrariness of v shows

$$-(p(x)u')' + q(x)u' + r(x)u - f(x) = 0 \text{ in } (a,b)$$

i.e. the original differential equation.

Thus**, for classical solutions, the two formulations** (2.5) and (2.7) **are equivalent.** Equation (2.7) still makes perfect sense because it only includes one derivative of u. It has converted (2.5) into an integral equation that is applicable to test functions in an infinite-dimensional space. These result in the following functional setting: we identify a solution belonging to H_0^1 (a, b), in which the homogeneous Dirichlet requirements are already included; we enlarge the class of test functions to H_0^1 (a, b), which is the closure of C_0^1 (a, b) in H^1-norm.

Thus, the weak or variational formulation of problem (2.5) is:

Find $u \in H_0^1(a,b)$ such that

$$\int_a^b \{pu' v' + qu' v + ruv\}dx = \int_a^b fvdx \quad , \forall v \in H_0^1(a,b). \qquad (2.8)$$

The bilinear form

$$B(u,v) = \int_a^b \{pu' v' + qu' v + ruv\} dx$$

And the linear functional

$$LV = \int_a^b fv \, dx$$

Equation (2.8) can be recast as

$$B(U,V) = LV, \quad \forall v \in H_0^1(a,b).$$

Then existence, uniqueness and stability follow from the Lax-Milgram Theorem, under the hypotheses $a(x) \geq 0$ i.e in Ω. Recall that, by Poincare's inequality

$$||u||_0 \leq C_P || u'||_0 ,$$

choosing $H_0^1(a,b)$ the norm

$$||u||_1 = ||u'||_0$$

equivalent to $||u||_{1,2} = ||u||_0 + ||u'||_0$

Proposition 2.1. Assume that $p, q, q', r \in L^\infty(a,b) and\ f \in L^2(a,b)$.

$$\text{If } p(x) \geq \alpha > 0 \text{ and } -\frac{1}{2} q'(x) + r(x) \geq 0 \text{ a.e. in } (a,b) \qquad (2.9)$$

then (2.8) has a unique solution $u \in H_0^1$ (a, b). Moreover

$$||u||_0 \leq \frac{C_P}{\alpha} ||f||_0 \qquad (2.10)$$

Proof .We want to show that the hypotheses of the Lax-Milgram Theorem hold, with

$V = H_0^1(a,b)$. Continuity of the bilinear form B is:

$$|B(u,v)| \leq \int_a^b \{\| p \|_{L^\infty} |u'v'| + \| q \|_{L^\infty} |u'v| + \| r \|_{L^\infty} |uv|\}dx.$$

using Schwarz and Poincare's inequalities, we obtain

$$\begin{aligned} |B(u,v)| &\leq \| p \|_{L^\infty} \|u'\|_0 \|v'\|_0 + \| q \|_{L^\infty} \|u'\|_0 \| v \|_0 + \| r \|_{L^\infty} \| u \|_0 \| v \|_0 \\ &\leq (\| p \|_{L^\infty} + C_P \| q \|_{L^\infty} + C_P^2 \| r \|_{L^\infty}) \|u'\|_0 \|v'\|_0 \end{aligned}$$

so that B is continuous in V .

Coercivity of B:

$$\begin{aligned} B(u,u) &= \int_a^b \{p(u')^2 + qu'u + ru^2\}dx \\ &\geq \alpha \|u'\|_0^2 + \frac{1}{2}\int_a^b q(u^2)'dx + \int_a^b ru^2 dx \\ \text{(integrating by parts)} &= \alpha\|u'\|_0^2 + \int_a^b \left\{-\frac{1}{2}q' + r\right\}u^2 dx \end{aligned}$$

(from (2.9)) $\geq \alpha ||u'||_0^2$

and therefore B is V –coercive

Continuity of L in V . The Schwarz and Poincare's inequalities yield

$$|Lv| = \left|\int_a^b fv dx\right| \leq \| f \|_0 \| v \|_0 \leq C_P \| f \|_0 \|v'\|_0$$

so that $\| L \|_{V^*} \leq C_P \| f \|_0$

Then, the Lax-Milgram Theorem gives existence, uniqueness and the stability estimate (2.10). □

Remark 2.1. If $q = 0$, the bilinear form B is symmetric. In this case the weak solution minimizes in H_0^1 (a,b) the "energy functional"

$$J(u) = \int_a^b \{p(u')^2 + ru^2 - fu\}dx$$

Then, equation (2.8) coincides with the Euler equation of J:

$$J'(u)v = 0, \qquad \forall v \in H_0^1(a,b)$$

2.2.4 Neumann, Robin (mixed) conditions (non homogeneous)

The weak formulation of the Neumann problem can be derived as:

$$\begin{cases} -(p(x)u')' + q(x)u' + r(x)u = f(x), \text{in } (a,b) \\ -p(a)u'(a) = A, \qquad p(b)u'(b) = B. \end{cases} \tag{2.12}$$

Assume $p \in C^1([a,b])$, with $p > 0$, and $q, r, f \in C^0([a,b])$. A classical solution $u \in C^2(a,b) \cap C^1([a,b])$ is continuously differentiable up to the end points we choose a space of test functions $C^1([a,b])$. Multiply the equation by an arbitrary $V \in C^1([a,b])$, and integrating over (a,b), we find

$$\int_a^b (pu')' vdx + \int_a^b [qu' + ru]vdx = \int_a^b fvdx. \tag{2.13}$$

Integrating by parts the first term (divergence term) and using the Neumann conditions, we get

$$-\int_a^b (pu')'vdx = \int_a^b pu'v'dx - [pu'v]_a^b = \int_a^b pu'v'dx - v(b)B - v(a)A$$

Then (2.13) becomes

$$\int_a^b [pu'v' + qu'v + ruv]dx - v(b)B - v(a)A = \int_a^b fvdx, \tag{2.14}$$

for every $v \in C^1([a,b])$.
Thus, (2.12) implies (2.14). If the choice of the test functions is correct, we should be able to recover the classical formulation from (2.14). Let us start recovering the differential equation. Since

$$C_0^1(a,b) \subset C^1([a,b]),$$

(2.14) clearly holds for every $v \in C_0^1(a,b)$. Then, (2.14) reduces to (2.7) and we deduce as before,

$$-(pu')' + qu' + ru - f = 0, \text{in } (a,b). \tag{2.15}$$

Let us now use the test functions which do not vanish at the end points. Integrating by parts the first term in (2.14) we have:

$$\int_a^b pu'v'dx = -\int_a^b (pu')'vdx + p(b)v(b)u'(b) - p(a)v(a)u'(a).$$

Inserting this expression into (2.14) and taking into account (2.15) we find:

$$v(b)[p(b)u'(b) - B] - v(a)[p(a)u'(a) + A] = 0.$$

The arbitrariness of the values $v(b)$ *and* $v(a)$ forces

$$p(b)u'(b) = B, -p(a)u'(a) = A,$$

recovering the Neumann conditions as well. Thus, for classical solutions, the two formulations (2.12) and (2.14) are equivalent.
Enlarging the class of test functions to H^1(a, b), which is the closure of C^1([a, b]) in H^1-norm, we may state the weak or variational formulation of problem (2.12) as follows:
Figure out $u \in H^1(a,b)$ such that, $\forall v \in H^1(a,b)$,

$$\int_a^b \{pu' v' + qu' v + ruv\} dx = \int_a^b fv\, dx + v(b) B + v(b) A. \qquad (2.16)$$

The Neumann conditions are encoded in equation (2.16), rather than forced by the choice of the test functions, as in the Dirichlet problem.
The bilinear form is:

$$B(u,v) = \int_a^b \{pu'v' + qu'v + ruv\} dx$$

and the linear functional is:

$$Lv = \int_a^b fv dx + v(b)B + v(a)A$$

equation (2.16) can be recast in the abstract form

$$B(u,v) = Lv, \forall v \in H^1(a,b)$$

Again, existence, uniqueness and stability of a weak solution follow from the Lax-Milgram Theorem we get the trace inequality

$$v(x) \leq C^* ||v||_{1,2} \qquad , (2.17)$$

for every $x \in [a,b]$, withC $C^* = \sqrt{2}\max\{(b-a)^{-1/2}, (b-a)^{1/2}\}$.
Proposition 2.2. Assume that:
i) p, q, $r \in L^\infty(a,b)$ and $f \in L^2(a,b)$
ii) $p(x) \geq a_0 > 0, r(x) \geq c_0 > 0$ a.e. in (a,b) and

$$K_0 \equiv \min\{\alpha_0, c_0\} - \frac{1}{2} \| q||_{L^\infty} > 0$$

Then, (2.8) has a unique solution $u \in H^1(a,b)$. Furthermore

$$||u||_{1,2} \leq K_0^{-1} \{ ||f||_0 + C^*(|A| + |B|)\}. \qquad (2.18)$$

Proof. Let us check that the hypotheses of the Lax-Milgram Theorem hold, with $V = H^1(a,b)$.

Continuity of the bilinear form B. We have:

$$\mathrm{B}|(u,v)| \leq \int_{\mathrm{a}}^{\mathrm{b}} \{\| p \|_{L^\infty} |u'v'| + \| q \|_{L^\infty} |u'v| + \| r \|_{L^\infty} |uv|\} \, dx$$

Using Schwarz's inequality, we easily get

$$|B(u,v)| \leq (\| p \|_{L^\infty} + \| q \|_{L^\infty} + \| r \|_{L^\infty}) \| u \|_{1,2} \| v \|_{1,2}$$

so that B is continuous in V.

Coercivity of B. We have

$$B(u,u) = \int_a^b \{p(u')^2 + qu'u + ru^2\} dx$$

The Schwarz inequality gives

$$\left| \int_a^b qu'udx \right| \leq \| q \|_{L^\infty} \|u'\|_0 \| u \|_0 \leq \frac{1}{2} \| q \|_{L^\infty} \{\|u'\|_0^2 + \| u \|_0^2\}$$

Then, by ii of proposition 2.2

$$B(u,u) \geq \left(\alpha_0 - \frac{1}{2} \| q \|_{L^\infty}\right) \|u'\|_0^2 + \left(c_0 - \frac{1}{2} \| q \|_{L^\infty}\right) \| u \|_0^2 \geq K_0 \| u \|_{1,2}^2$$

so that B is V-coercive.
Continuity of L in V. Schwarz's inequality and (2.17) yield

$$\begin{aligned} |Lv| \quad & \leq \| f \|_0 \| v \|_0 + |v(b)B + v(a)A| \leq \\ & \leq \{\| f \|_0 + C^*(|A| + |B|)\} \| v \|_{1,2} \end{aligned}$$

whence $\| L \|_{V^*} \leq \| f \|_0 + C^*(|A| + |B|)$.
Then, the Lax-Milgram Theorem gives existence, uniqueness and the stability Estimate (2.18)

Remark 2.3. Suppose, $p = 1, q = r = 0$. The problem reduces to

$$\begin{cases} \quad u'' = & f \text{ in } (a,b) \\ -u'(a) = A, & u'(b) = B. \end{cases}$$

Proposition 2.2. Of (Hypothesis ii) is not satisfied (since $r = 0$). If u is a solution of the problem and $k \in \mathbb{R}, u + k$ is also a solution of the same problem. We cannot expect uniqueness. Not even we may prescribe f, A, B arbitrarily, if we want that a solution exists. In fact, integrating the equation $u'' = f$ over (a, b), we deduce that the Neumann data and f must satisfy the compatibility condition

$$\mathrm{B} + \mathrm{A} = \int_{\mathrm{a}}^{\mathrm{b}} \mathrm{f}(\mathrm{x})\mathrm{dx}. \tag{2.19}$$

If (2.19) does not hold the problem has no solution

Robin conditions(non homogeneous). Suppose that the boundary conditions in problem (2.12) are: −

$$-p(a)u'(a) = A, p(b)u'(b) + hu(b) = B(h > 0, \text{ constant })$$

The Robin condition is imposed at $x = b$ only. With small adjustments, we may repeat the same computations made for the Neumann conditions.

The weak formulation is:

Find out $u \in H^1(a, b)$ such that, $\forall v \in H^1(a, b)$,

$$\int_a^b \{\mathrm{pu'v'} + \mathrm{qu'v} + \mathrm{ruv}\}\,\mathrm{dx} + \mathrm{hu}(\mathrm{b})\,\mathrm{v}(\mathrm{b}) = \int_{\mathrm{a}}^{\mathrm{b}} \mathrm{fvdx} + \mathrm{v}(\mathrm{b})\mathrm{B} + \mathrm{v}(\mathrm{a})\mathrm{A}. \tag{2.20}$$

The bilinear form :

$$\tilde{B}(u, v) = \int_a^b \{pu'v' + qu'v + ruv\}dx + hu(b)v(b)$$

And the Linear functional is:

$$LV = \int_{\mathrm{a}}^{\mathrm{b}} \mathrm{fvdx} + \mathrm{v}(\mathrm{b})\mathrm{B} + \mathrm{v}(\mathrm{a})\mathrm{A}.$$

The problem in the abstract form can be written as

$$\tilde{\mathrm{B}}(\mathrm{u}, \mathrm{v}) = \mathrm{Lv}\ \forall_{\mathrm{v}} \in \mathrm{H}^1(\mathrm{a}, \mathrm{b})$$

Proposition 2.3. Assume that i) and ii) of Proposition 2.2 hold and that $h > 0$. Then (2.20) has a unique solution $u \in H^1(a, b)$. Furthermore

$$\| u \|_{1,2} \leq K_0^{-1}\{\| f \|_0 + C^*(|A| + |B|)\}$$

Proof. Let $V = H^1(a, b). since$

$$\tilde{B}(u, u) = B(u, u) + hu^2(b) \geq K_0 \| u \|_{1,2}^2$$

And

$$|\tilde{B}(u,v)| \leq |B(u,v)| + h|u(b)v(b)|$$
$$\leq (\| p \|_{L^\infty} + \| q \|_{L^\infty} + \| r \|_{L^\infty} + h(C^*)^2) \| u \|_{1,2} \| v \|_{1,2}$$

$\tilde{B}$ is continuous and V- coercive. the conclusion follows easily. □

Mixed conditions. The weak formulation of mixed problems does not present particular difficulties. The conditions

$$u\,(a) = 0, \qquad\qquad p\,(b)\,u'\,(b) = B$$

Thus, we have a mixed Dirichlet-Neumann problem. Since $u\,(a) = 0$, we have to choose $V = H_0^1$,a, the space of test functions $v \in H^1(a,b)$, vanishing at $x = a$. The Poincar´e inequality holds in H_0^1,a so that we may choose $||u'||_0$as the norm in H_0^1,a. Moreover, the inequality

$$v(x) \leq C^{**}||v||_0 \tag{2.21}$$

holds for every $x \in [a,b]$, with $C^{**} = (b-a)^{1/2}$.

The weak formulation is: Determine $u \in H_{0.a}^1$ such that

$$\int_a^b \{pu'\,v' + qu'\,v + ruv\}\,dx = \int_a^b fvdx + v\,(b)\,B, \forall_v \in H_0^1, a. \tag{2.22}$$

We have:

Proposition 2.4 Assume that i) and ii) of Proposition 2.2 hold. Then, (2.22) has a unique solution u $\in H_0^1,a$. Furthermore

$$\|u'\|_0 \leq K_0^{-1}\{C_P \| f \|_0 + C^{**}|B|\}$$

2.2.5 Variational Formulation of Poisson's Problem In $\boldsymbol{R^n}$

2.2.5.1 Dirichlet problem (homogeneous)

We now analyze the variational formulation of Poisson's problem in dimension n> 1, starting with dirichlet conditions.

Let $\Omega \subset R^n$be a bounded domain. The problem:

$$\begin{cases} -\alpha\Delta u + a_0(x)\,u = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{2.23}$$

Where $\alpha > 0$, constant. To examine and formulate a weak formulation.
Assume that $a_0$and f are smooth and that $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$is a classical solution of (2.23).

We select $C_0^1\,(\Omega)$ as the space of test functions, having continuous first derivatives and compact support in Ω. In particular, they vanish in a neighborhood of $\partial\Omega$.
Let v ∈ $C_0^1(\Omega)$ and multiply the Poisson equation by v. We get

$$\int_\Omega \{-\alpha\Delta u + a_0 u - f\} v d\mathbf{x} = 0. \qquad (2.24)$$

Integrating by parts and using the boundary condition, we obtain:

$$-\int_{\partial\Omega} \partial_\nu \mathrm{u} \cdot \mathrm{v} d\sigma + \int_\Omega \{\alpha\nabla u \cdot \nabla v + a_0 uv\} d\mathbf{x} = \int_\Omega fv d\mathbf{x},\ \forall v \in C_0^1(\Omega) \qquad (2.25)$$

Thus (2.23) implies (2.25). On the other hand, assume (2.25) is true. Integrating by parts in the reverse order we return to (2.24), which entails $-\alpha\Delta u + a_0 u - f = 0$ in Ω.
Thus, for classical solutions, the two formulations (2.23) and (2.25) are equivalent.
Observe that (2.25) only involves first order derivatives of the solution and of the test function. Then, enlarging the space of test functions to $H_0^1\,(\Omega)$ *in* H^1-*norm*, closure of $C_0^1(\Omega)$ in the norm
$\| u \|_1 = \| \nabla u \|_0$
The weak formulation of problem (2.25) is:
u ∈ $H_0^1\,(\Omega)$ such that

$$\int_\Omega \{\alpha\nabla u \cdot \nabla v + a_0 uv\} d\mathbf{x} = \int_\Omega fv dx, \forall v \in H_0^1(\Omega) \qquad (2.26)$$

The bilinear form

$$B(u,v) = \int_\Omega \{\alpha\nabla u \cdot \nabla v + a_0 uv\} d\mathbf{x}$$

and the linear functional

$$Lv = \int_\Omega fv d\mathbf{x}$$

Then, equation (2.26) corresponds to the abstract variational problem

$$B(u,v) = Lv, \forall v \in H_0^1(\Omega)$$

Then, the well-posedness of this problem follows the Lax-Milgram theorem under the hypothesis $a_0 \geq 0$. Precisely:

Theorem 2.1. Assume that $f \in L^2(\Omega)$ and that $0 \leq a_0(x) \leq \gamma_0 a.e. in\ \Omega$. Then, problem (2.26) has a unique solution $u \in H_0^1\,(\Omega)$. Moreover

$$\| \nabla u \|_0 \leq \frac{C_P}{\alpha} \| f \|_0.$$

Proof. We check that the hypotheses of the Lax-Milgram Theorem hold, with V = $H_0^1\,(\Omega)$.

Continuity of the bilinear form B. The Schwarz and Poincare's inequalities yield:

$$|B(u,v)| \leq \alpha \parallel \nabla u \parallel_0 \parallel \nabla v \parallel_0 + \gamma_0 \parallel u \parallel_0 \parallel v \parallel_0$$
$$\leq (\alpha + C_P^2 \gamma_0) \parallel \nabla u \parallel_0 \parallel \nabla v \parallel_0$$

So that B is continuous $in\ H_0^1\ (\Omega)$.
Coercivity of B.

$$B(u,u) = \int_\Omega \alpha|\nabla u|^2 d\mathbf{x} + \int_\Omega a_0 u^2 d\mathbf{x} \geq \alpha \parallel \nabla u \parallel_0^2$$

since $a_0 \geq 0$.
Continuity of L. The Schwarz and Poincare's inequalities give

$$|Lv| = \left|\int_\Omega fv d\mathbf{x}\right| \leq \parallel f \parallel_0 \parallel v \parallel_0 \leq C_P \parallel f \parallel_0 \parallel \nabla v \parallel_0.$$

Hence $L \in H^{-1}(\Omega)$ and $\parallel L \parallel_{H^{-1}(\Omega)} \leq C_P \parallel f \parallel_0$. The conclusions follow from the Lax-Milgram Theorem.
Remark 2.4. Assume that u is the elastic membrane's equilibrium location and that **c = 0.** The work performed by the elastic internal forces is then represented by B (u, v). However, Lv conveys the effort put in by outside forces. The weak formulation (2.26) claims that these two works are balanced, which is a variation on the virtual work principle.
Moreover, the Dirichlet functional solution u of the problem minimizes in $H_0^1(\Omega)$ because of B's symmetry.

$$E(u) = \int_\Omega \alpha|\nabla u|^2 d\mathbf{x} - \int_\Omega fu d\mathbf{x}$$

The first term on the right-hand side denotes internal elastic energy, and the second term denotes external potential energy. $E(u)$is the potential energy. It is the Euler equation for E represented by equation (2.26). Consequently u minimizes the potential energy in accordance with the virtual work principle.

2.2.5.2 Non homogeneous Dirichlet conditions.

Assume that u = g on $\partial\Omega$ is the Dirichlet condition. When g ∈ $H^{\frac{1}{2}}$ $(\partial\Omega)$ and Ω is a Lipschitz domain, then g represents the trace on $\partial\Omega$ of a (non unique) function $\widetilde{g} \in H^1$ (Ω), which is referred to as the extension of g to Ω. set

$$w = u - \widetilde{g}$$

We are reduced to homogeneous boundary conditions. Actually, $w \in H_0^1(\Omega)$ is a solution to the equation.

$$\int_\Omega \{\alpha\nabla w\cdot\nabla v dx + a_0 wv\}d\mathbf{x} = \int_\Omega Fvd\mathbf{x}, \forall v \in H_0^1(\Omega)$$

Where, $F = f - \alpha\nabla\tilde{g} - a_0\tilde{g} \in L^2(\Omega)$. The Lax-Milgram Theorem yields existence, uniqueness and the stability estimate.

$$\| \nabla w \|_0 \le \frac{C_P}{\alpha}\{\| f \|_0 + (\alpha + a_0) \| \tilde{g} \|_{1,2}\} \tag{2.27}$$

for any extension $\tilde{g}$ of g. Since $\| u \|_{1,2} \le \| w \|_{1,2} + \| \tilde{g} \|_{1,2}$ and recalling that

$$\| g \|_{H^{1/2}(\partial\Omega)} = \inf\{\| \tilde{g} \|_{1,2} : \tilde{g} \in H^1(\Omega), \tilde{g}_{|\partial\Omega} = g\}$$

taking the lowest upper bound with respect to $\tilde{g}$, from (2.27) we deduce, in terms of u:

$$\| u \|_{1,2} \le C(\alpha, \gamma_0, n, \Omega)\{\| f \|_0 + \| g \|_{H^{1/2}(\partial\Omega)}\}$$

2.2.5.3 Neumann, Robin (mixed) problems(non homogeneous)

Let $\Omega \subset R^n$ be a bounded, Lipschitz domain. We have

$$\begin{cases} -\alpha\Delta u + a_0(\mathbf{x})u = f \text{ in } \Omega \\ \partial_\nu u = g \text{ on } \partial\Omega \end{cases} \tag{2.28}$$

where v represents the outward normal unit vector to $\partial\Omega$ and $\alpha > 0$ is a constant. We assume that a_0, f, and g are smooth and that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a classical solution of (2.28) in order to construct a weak formulation. The space of test functions, with continuous first derivatives up to $\partial\Omega$, is selected as $C^1(\overline{\Omega})$. Let $v \in C^1(\overline{\Omega})$ be arbitrary, and multiply the Poisson equation by v. we obtain

$$\int_\Omega \{-\alpha\Delta u + a_0 u\}vd\mathbf{x} = \int_\Omega fvd\mathbf{x}. \tag{2.29}$$

Integration by parts gives

$$-\int_{\partial\Omega} \alpha\, \partial_\nu uvd\sigma + \int_\Omega \{\alpha\nabla u\cdot\nabla v + a_0 uv\}d\mathbf{x} = \int_\Omega fvd\mathbf{x}, \forall v \in C^1(\overline{\Omega}) \tag{2.30}$$

Using the Neumann condition we write

$$\int_\Omega \{\alpha\nabla u\cdot\nabla v + a_0 uv\}d = \int_\Omega fvd\mathbf{x} + \alpha\int_{\partial\Omega} gvd\sigma \forall v \in C^1(\overline{\Omega}). \tag{2.31}$$

Thus (2.28) implies (2.31).
Conversely, suppose that (2.31) is true. Integrating by parts in the reverse order, we find

$$\int_{\Omega} \{-\alpha\Delta u + a_0 u - f\} v d\mathbf{x} + \int_{\partial\Omega} \alpha\, \partial_\nu u v d\sigma = \alpha \int_{\partial\Omega} g v d\sigma \qquad (2.32)$$

for every $\forall v \in C^1(\Omega)$. Since $C_0^1\,(\Omega) \subset C^1(\Omega)$ we may insert any $v \in C_0^1\,(\Omega)$ into (2.32), to get

$$\int_{\Omega} \{-\alpha\Delta u + a_0 u - f\} v d\mathbf{x} = 0$$

The arbitrariness of $v \in C_0^1\,(\Omega)$ entails $-\alpha\Delta u + a_0 u - f = 0 \; in \; \Omega$.
Therefore (2.32) Becomes

$$\int_{\partial\Omega} \partial_\nu u v d\sigma = \int_{\partial\Omega} g v d\sigma \forall v \in C^1(\bar{\Omega})$$

And the arbitrariness of $v \in C_0^1\,(\bar{\Omega})$ forces $\partial_\nu u = g$, recover the Neumann conditions as well.

Thus, for classical solutions, the two formulations (2.28) and (2.31) are equivalent. Recall that, $C^1\,(\bar{\Omega})$ is dense in $H^1(\Omega)$, which therefore constitutes the natural Sobolev space for the Neumann problem. Then, enlarging the space of test functions to $H^1(\Omega)$, we may give the weak formulation of problem as follows:

Determine u $\in H^1(\Omega)$ such that

$$\int_{\Omega} \{\alpha\nabla u \cdot \nabla v + a_0 u v\} d\mathbf{x} = \int_{\Omega} f v d\mathbf{x} + \alpha \int_{\partial\Omega} g v d\sigma, \forall v \in H^1(\Omega). \qquad (2.33)$$

Here the Neumann condition is encoded in (2.33) and not explicitly expressed as in the case of Dirichlet boundary conditions.

The bilinear form

$$B(u,v) = \int_{\Omega} \{\alpha\nabla u \cdot \nabla v + a_0 u v\} d\mathbf{x} \qquad (2.34)$$

and the linear functions

$$Lv = \int_{\Omega} f v d\mathbf{x} + \alpha \int_{\partial\Omega} g v d\sigma, \qquad (2.35)$$

(2.33) the abstract variational problem

$$B(u,v) = Lv, \forall v \in H_0^1(\Omega)$$

The following states the well-possdness of this problem
Under the hypotheses on the data of trace inequality

$$\| v \|_{L^2(\partial\Omega)} \leq \bar{C}(n,\Omega) \| v \|_{1,2} . \qquad (2.36)$$

Theorem 2.2. Let $\Omega \subset R^n$ be a bounded, Lipschitz domain, $f \in L^2(\Omega), g \in L^2(\partial\Omega)$ and $0 < C_0 \leq a_0(x) \leq \gamma_0$ a.e. in Ω. Then, problem (2.33) has a unique solution $u \in H^1(\Omega)$. Moreover,

$$\| u \|_{1,2} \leq \frac{1}{\min\{\alpha, c_0\}} \left\{ \| f \|_0 + \bar{C}\alpha \| g \|_{L^2(\partial\Omega)} \right\}$$

Proof. suppose that the hypotheses of the Lax-Milgram Theorem hold, with $V = H^1(\Omega)$. Continuity of the bilinear form B. Using the Schwarz and Poincare's inequality we get:

$$|B(u,v)| \leq \alpha \| \nabla u \|_0 \| \nabla v \|_0 + \gamma_0 \| u \|_0 \| v \|_0$$

$$\leq (\alpha + \gamma_0) \| u \|_{1.2} \| v \|_{1.2}$$

so that B is continuous in $H^1(\Omega)$.

Continuity of L. From Schwarz's inequality and (2.36) we get:

$$\begin{aligned} |Lv| &\leq \left| \int_\Omega f v d\mathbf{x} \right| + \alpha \left| \int_{\partial\Omega} g v d\sigma \right| \leq \| f \|_0 \| v \|_0 + \alpha \| g \|_{L^2(\partial\Omega)} \| v \|_{L^2(\partial\Omega)} \\ &\leq \left\{ \| f \|_0 + \bar{C}\alpha \| g \|_{L^2(\partial\Omega)} \right\} \| v \|_{1,2} \end{aligned}$$

Therefore L is continuous in $H^1(\Omega)$ with

$$\| L \|_{H^1(\Omega)^*} \leq \| f \|_{L^2(\Omega)} + \bar{C}\alpha \| g \|_{L^2(\partial\Omega)}$$

Coercivity of B. It follows from

$$B(u,u) = \int_\Omega \alpha |\nabla u|^2 d\mathbf{x} + \int_\Omega a_0 u^2 d\mathbf{x} \geq \min\{\alpha, c_0\} \| u \|_{1,2}^2$$

since $a_0(\mathbf{x}) \geq c_0 > 0$ a.e. in Ω.

The conclusion follows from Lax-Milgram.□

Remark 2.5. As in the one-dimensional case, without the condition $a_0(x) \geq C_0 > 0$, neither the existence nor the uniqueness of a solution is guaranteed

For example, let $a_0 = 0$. Then two solutions of the same problem differ by a constant. A way to restore uniqueness is to select a solution with, e.g., zero mean value, that is

$$\int_\Omega u(\mathbf{x})d\mathbf{x} = 0.$$

The existence of a solution requires the following compatibility condition on the data f and g :

$$\int_\Omega f d\mathbf{x} + \alpha \int_{\partial\Omega} g d\sigma = 0, \qquad for \;\; v = 1 \tag{2.37}$$

$$\int_\Omega \alpha \nabla u \cdot \nabla v dx = \int_\Omega f v d\mathbf{x} + \alpha \int_{\partial\Omega} g v d\sigma$$

Note that, since Ω is bounded, the function v = 1 belongs to $H^1(\Omega)$.
If $a_0 = 0$ and (2.37) does not hold, problem (2.28) has no solution. If $g = 0$, (2.37) has a simple interpretation. Indeed problem (2.28) is a model for the equilibrium configuration of a membrane whose boundary is free to slide along a vertical guide. The compatibility condition $\int_\Omega f dx = 0$ expresses the obvious fact that, at equilibrium, the resultant of the external loads must vanish.

Robin problem. The same arguments leading to the weak formulation of the Neumann problem (2.28) may be used for the problem

$$\begin{cases} -\alpha \Delta u + a_0(\mathbf{x}) u & = f \quad \text{in } \Omega \\ \partial_\nu u + h u & = g \quad \text{on } \partial\Omega. \end{cases} \tag{2.38}$$

The weak formulation comes from (2.30), observing that

$$\partial_\nu u = -hu + g \text{ on } \partial\Omega$$

The **variational formulation:**
Find $u \in H^1(\Omega)$ such that

$$\int_\Omega \{\alpha \nabla u \cdot \nabla v + a_0 u v\} d\mathbf{x} + \alpha \int_{\partial\Omega} h u v d\sigma = \int_\Omega f v d\mathbf{x} + \alpha \int_{\partial\Omega} g d\sigma \forall v \in H^1(\Omega)$$

We have

Theorem 2.3. Let Ω, f, g and a_0 be as in Theorem 2.2 and $0 \le h(x) \le h_0$ a.e. on $\partial\Omega$. Then, problem (2.38) has a unique weak solution $u \in H^1(\Omega)$. Moreover

$$\| u \|_{1,2} \le \frac{1}{\min\{\alpha, c_0\}} \left\{ \| f \|_0 + \bar{C} \alpha \| g \|_{L^2(\partial\Omega)} \right\}$$

Proof. Introducing the bilinear form

$$\widetilde{B}(u,v) = B(u,v) + \alpha \int_{\partial\Omega} huvd\sigma$$

the variational formulation becomes

$$\tilde{B}(u,v) = Lv \; \forall v \in H^1(\Omega)$$

From the Schwarz inequality and trace inequality

$$\left| \int_{\partial\Omega} huvd\sigma \right| \leq h_0 \parallel u \parallel_{L^2(\partial\Omega)} \parallel v \parallel_{L^2(\partial\Omega)} \leq \bar{C}^2 h_0 \parallel u \parallel_{1,2} \parallel v \parallel_{1,2}.$$

Conversely, the positivity of α,a_0 and h entails that

$$\widetilde{B}(u,u) \geq B(u,u) \geq min\{\alpha, c_0\} \parallel u \parallel_{1,2}^2$$

The conclusion follow easily. □

Mixed (Dirichlet-Neumann) problem. Let Γ_D be a non empty relatively open subset of $\partial\Omega$. Set $\Gamma_N = \partial\Omega \setminus \Gamma_D$ and consider the problem

$$\begin{cases} -\alpha\Delta u + a_0(\mathbf{x})u & = f & \text{in } \Omega \\ u & = 0 & \text{on } \Gamma_D \\ \partial_\nu u & = g & \text{on } \Gamma_N. \end{cases}$$

The functional setting is $H^1_{0,\Gamma_D}(\Omega)$, i.e. the set of functions in $H^1(\Omega)$ with zero trace on Γ_D. the Poincare's inequality holds in $H^1_{0,\Gamma_D}(\Omega)$ and therefore we may choose the norm

$$\parallel u \parallel_{H^1_{0.\Gamma_n}(\Omega)} = \parallel \nabla u \parallel_0$$

From (2.29) and the Gauss formula, we obtain, since $u = 0$ on Γ_D,

$$-\int_{\Gamma_N} \alpha\, \partial_\nu uvd\sigma + \int_\Omega \{\alpha\nabla u \cdot \nabla v + a_0 uv\} d\mathbf{x} = \int_\Omega fvd\mathbf{x}, \forall v \in C^1(\bar{\Omega})$$

The Neumann condition on Γ_N, yields the following variational formulation: Determine $u \in H^1_{0,\Gamma_D}(\Omega)$ such that, $\forall v \in H^1_{0,\Gamma_D}(\Omega)$,

$$\int_\Omega \alpha\nabla u \cdot \nabla v dx + \int_\Omega a_0 uvd\mathbf{x} = \int_\Omega fvd\mathbf{x} + \alpha \int_{\Gamma_N} gv\, d\sigma.$$

Using the trace inequality

$$\| v \|_{L^2(\Gamma_N)} \leq \tilde{C} \| v \|_{1,2}, \qquad (2.39)$$

the proof of the next theorem follows the usual pattern.

Theorem 2.4. Let $\Omega \subset R^n$ be a bounded Lipschitz domain. Assume $f \in L^2(\Omega), g \in L^2(\Gamma_N)$ and $0 \leq a_0(x) \leq \gamma_0$ a.e. in Ω. Then the mixed problem has a unique solution $u \in H^1_{0,\Gamma_D}(\Omega)$. Moreover:

$$\| \nabla u \|_0 \leq \frac{1}{\alpha} \| f \|_0 + \tilde{C} \| g \|_{L^2(\Gamma_N)}$$

Chapter three :
Eigenvalues of the Laplacian

Introduction

This section shows that the availability of a basis of Eigenfunctions determines the efficiency of the separation of variables method for a specific situation. the spectral characteristics of uniformly elliptic operators are examined, with a particular focus on the Laplace operators. This spectrum of differential operators takes into account homogeneous boundary conditions.

3.1 Dirichlet Eigenfunctions for the Laplace operator

Let Ω be a domain. The problem

$$\begin{cases} -\Delta u = \lambda u \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.1}$$

A weak solution of problem (3.1) is a function $u \in H_0^1(\Omega)$ such that

$$a(u, v) \equiv (\nabla u, \nabla v)_0 = \lambda(u, v)_0 \qquad \forall v \in H_0^1(\Omega)$$

The bilinear form is H_0^1 (Ω) –coercive (V- coercive), If Ω is bounded

Theorem 3.1 Let Ω be a bounded domain. Then, there exists in $L^2(\Omega)$ an orthonormal basis $\{u_k\}_{k\geq 1}$ consisting of Dirichlet eigenfunctions for the Laplace operator. The corresponding eigenvalues $\{\lambda_k\}_{k\geq 1}$ are all positive and may be arranged in an increasing sequence.

The sequence $\left\{u_k/\sqrt{\lambda_k}\right\}_{k\geq 1}$ constitutes an orthonormal basis in H_0^1 (Ω), with respect to the scalar product $(u, v)_1 = (\nabla u, \nabla v)_0$

Remark 3.1. Let $u \in L^2(\Omega)$ and denote by $c_k = (u, v)_0$ the Fourier coefficients of u with respect to the Orthonormal basis $\{u_k\}_{k\geq 1}$ is written as:

$$u = \sum_{k=1}^{\infty} c_k u_k \text{ and } \| u \|_0^2 = \sum_{k=1}^{\infty} c_k^2$$

and

$$\|\nabla u_k\|_0^2 = (\nabla u_k, \nabla u_k)_0 = \lambda_k (u_k, u_k)_0 = \lambda_k$$

Thus, $u \in H_0^1$ (Ω) if and only if

$$\| \nabla u \|_0^2 = \sum_{k=1}^{\infty} \lambda_k c_k^2 < \infty \tag{3.2}$$

Moreover, (3.2) implies that, for every $u \in H_0^1(\Omega)$,

$$\| \nabla u \|_0^2 \geq \lambda_1 \sum_{k=1}^{\infty} c_k^2 = \lambda_1 \| u \|_0^2.$$

The variational principle for the first Dirichlet Eigenvalue:

$$\lambda_1 = \min \left\{ \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2} : u \in H_0^1(\Omega), u \text{ non identically zero.} \right\} \tag{3.3}$$

(3.3) is Raiyleigh's quotient. If the domain Ω is smooth, we can demonstrate that the corresponding normalized eigenvector u_1 is either strictly positive or strictly negative in Ω, and that λ_1 is a simple eigenvalue, meaning the related eigenspace has dimension 1. For instance, the non-trivial solutions to the problem are the Neumann Eigenfunctions for the Laplace operator in Ω.

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases}$$

Theorem 3.2. If Ω is a bounded Lipschitz domain, there exists in $L^2(\Omega)$ an orthonormal basis $\{u_k\}_{k\geq 1}$ consisting of Neumann eigenfunctions for the laplace operator. The corresponding eigenvalues form a non decreasing sequence $\{\mu_k\}_{k\geq 1}$, with $\mu_1 = 0$ and $\mu_k \to +\infty$

Moreover, the sequence $\{u_k/\sqrt{\mu_k + 1}\}_{k\geq 1}$ constitutes an orthonormal basis in in $H^1(\Omega)$, with respect to the scalar product $(u, v)_{1.2} = (u, v)_0 + (\nabla u, \nabla v)_0$

3.2 Asymptotic Stability

To prove the asymptotic stability of a steady-state solution of an evolution equation as time $t \to +\infty$, The results of the Eigenvalues of the Laplace operator may be used

Given the following heat equation Assume that $u \in C^{2,1}(\bar{\Omega} \times [O, +\infty)$ is the (unique) solution of

$$\begin{cases} u_t - \Delta u = f(\mathbf{x}) & \mathbf{x} \in \Omega, t > 0 \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}) & \mathbf{x} \in \Omega \\ u(\sigma, t) = 0 & \sigma \in \partial\Omega, t > 0 \end{cases}$$

where Ω is a smooth, bounded domain. Define $u_\infty = u_\infty(x)$ is the solution of the stationary problem

$$\begin{cases} -\Delta u_\infty = f & \text{in } \Omega \\ u_\infty = 0 & \text{on } \partial\Omega \end{cases}$$

Proposition 3.1. For $t \geq 0$, we have

$$\|u(\cdot,t) - u_\infty\|_0 \leq e^{-\lambda_1 t}\left\{C_P^2 \parallel f \parallel_0 + \|u_0\|_0\right\} \tag{3.4}$$

Proof. Set g (x) = u_0(x) − u_∞ (x). The function w (x,t) = u (x,t) − u_∞ (x) solves the problem

$$\begin{cases} w_t - \Delta w = 0 & \mathbf{x} \in \Omega, t > 0 \\ w(\mathbf{x},0) = g(\mathbf{x}) & \mathbf{x} \in \Omega \\ w(\sigma,t) = 0 & \sigma \in \partial\Omega, t > 0. \end{cases} \tag{3.5}$$

The method of separation of variables to find solutions of the form $w(x,t) = v(x) \cdot z(t)$ is:

$$\frac{z'(t)}{z(t)} = \frac{\Delta v(\mathbf{x})}{v(\mathbf{x})} = -\lambda$$

Thus, we are led to the Eigenvalue problem

$$\begin{cases} -\Delta v = \lambda v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

From Theorem 3.1, there exists in $L^2(\Omega)$ an orthonormal basis $\{u_k\}_{k\geq 1}$ consisting of eigenvectors, corresponding to a sequence of non decreasing eigenvalues $\{\lambda_k\}$, with $\lambda_1 > 0$ and $\lambda_k \to +\infty$. Then, if $g_k = (g, u_k)_0$, we can write

$$\text{g} = \sum_1^\infty g_k u_k \quad \text{and} \quad \parallel g \parallel_0^2 = \sum_{k=1}^\infty g_k^2$$

Consequently, we find $Z_k(t) = e^{-\lambda_k t}$, and finally

$$w(\mathbf{x},t) = \sum_1^\infty e^{-\lambda_k t} g_k u_k(\mathbf{x})$$

Thus,

$$\begin{aligned} \|u(\cdot,t) - u_\infty\|_0^2 &= \parallel w(\cdot,t) \parallel_0^2 \\ &= \sum_{k=1}^\infty e^{-2\lambda_k t} g_k^2 \end{aligned}$$

and since $\lambda_k > \lambda_1$ for every k, we deduce that

$$\|u(\cdot,t)-u_\infty\|_0^2 \le \sum_{k=1}^{\infty} e^{-2\lambda_1 t} g_k^2 = e^{-2\lambda_1 t} \parallel g \parallel_0^2$$

In particular

$$\|u_\infty\|_0 \le C_P^2 \parallel f \parallel_0$$

And hence

$$\begin{aligned} \parallel g \parallel_0 &\le \|u_0\|_0 + \|u_\infty\|_0 \\ &\le \|u_0\|_0 + C_P^2 \parallel f \parallel_0 \end{aligned}$$

Proposition 3.1. implies that the steady state u_∞ is asymptotically stable in $L^2(\Omega)-$ norm as $t \to +\infty$. The speed of convergence is exponential and it is determined by the first eigenvalue λ_1.

3.3. General Equations in Divergence Structure

Assumptions

In this section, boundary value problems for elliptic operators with general diffusion and transport terms are treated. Let $\Omega \subset R^n$ be a **bounded domain** and set

$$\mathcal{E}u = -\operatorname{div}(\mathbf{A}(\mathbf{x})\nabla u - \mathbf{b}(\mathbf{x})u) + \mathbf{c}(\mathbf{x}) \cdot \nabla u + a_0(\mathbf{x})u \tag{3.6}$$

The following hypotheses hold throughout this section.

1.The differential operator $\mathcal{E}$ is uniformly elliptic.

$$For\ \alpha, M, A > 0,\ \ \alpha|\xi|^2 \le \mathbf{A}(\mathbf{x})\xi \cdot \xi \le M|\xi|^2, \forall \xi \in \mathbb{R}^n,\ \text{a.e. in } \Omega. \tag{3.7}$$

2. The coefficients b, c and $a_0$are all bounded:

$$|\mathbf{b}(\mathbf{x})| \le \beta, |\mathbf{c}(\mathbf{x})| \le \gamma, |a_0(\mathbf{x})| \le \gamma_0,\ \text{a.e. in } \Omega. \tag{3.8}$$

3.3.1. Dirichlet problem

Consider the problem

$$\begin{cases} \mathcal{E}u = f + \operatorname{div} \mathbf{f} \text{ in } \Omega \\ \quad u = 0 \text{ on } \partial\Omega \end{cases} \tag{3.9}$$

where $f \in L^2(\Omega)$ and $\mathbf{f} \in L^2(\Omega; \mathbb{R}^n)$
since $H^{-1}(\Omega)$ is the dual of $H_0^1(\Omega)$. We know that every element of F which is an element of $F \in H^{-1}(\Omega)$ can be written with an element in $D^1(\Omega)$ of the form: $F = f + div\boldsymbol{f}$

Moreover,

$$\| F \|_{H^{-1}(\Omega)} \leq \| f \|_0 + \| \mathbf{f} \|_0 . \qquad (3.10)$$

Thus, the right hand side of (3.9) represents a generic element of $H^{-1}(\Omega)$. If all the coefficients and the data $f, \boldsymbol{f}$ are smooth, then we can formulate the variational form by multiplying the equation with a test function v $\in C_0^1(\Omega)$ and integrate over Ω:

$$\int_\Omega [-\mathrm{div}(\mathbf{A}\nabla u - \mathbf{b}u)v] d\mathbf{x} + \int_\Omega [\mathbf{c}\cdot\nabla u + a_0 u] v d\mathbf{x} = \int_\Omega [f + \mathrm{div}\mathbf{f}] v d\mathbf{x}$$

Integrating by parts, we find, since $v = 0$ on $\partial\Omega$:

$$\int_\Omega [-\mathrm{div}(\mathbf{A}\nabla u - \mathbf{b}u)v] d\mathbf{x} = \int_\Omega [\mathbf{A}\nabla u \cdot \nabla v - \mathbf{b}u \cdot \nabla v] d\mathbf{x}$$

and

$$\int_\Omega v \mathrm{div}\, \mathbf{f} d\mathbf{x} = -\int_\Omega \mathbf{f}\cdot\nabla v d\mathbf{x}$$

Thus, the resulting equation is

$$\int_\Omega \{\mathbf{A}\nabla u\cdot\nabla v - \mathbf{b}u\cdot\nabla v + \mathbf{c}v\cdot\nabla u + a_0 uv\} d\mathbf{x} = \int_\Omega \{fv - \mathbf{f}\cdot\nabla v\} d\mathbf{x} \qquad (\mathbf{3.11})$$

for every $v \in C_0^1(\Omega)$.

Thus, for classical solutions, the two formulations (3.9) and (3.11) are equivalent. Increasing the space of test functions to $H_0^1(\Omega)$ and using bilinear form:

$$\mathrm{B}(u, v) = \int_\Omega \{\mathbf{A}\nabla u\cdot\nabla v - \mathbf{b}u\cdot\nabla v + \mathbf{c}v\cdot\nabla u + a_0 uv\} d\mathbf{x}$$

and the linear functional

$$\mathrm{F}v = \int_\Omega \{fv - \mathbf{f}\cdot\nabla v\}$$

Then, the weak formulation of problem (3.9) is the following:

Find u $\in H_0^1(\Omega)$ such that

$$B(u, v) = Fv, \forall v \in H_0^1(\Omega) \qquad (3.12)$$

well-posedness of the problem is indicated in the following set of propositions.

Proposition 3.2. Assume that hypotheses (3.7) and (3.8) hold and that f $\in L^2(\Omega)$,

$\mathbf{f} \in L^2(\Omega;R^n)$. Then if b and c have Lipschitz components and

$$\frac{1}{2}\,\mathrm{div}\,(\mathrm{b}\ -\ \mathrm{c}) +\ \mathrm{a}_0 \geq\ 0, \mathrm{a.e.\,in}\ \Omega, \qquad (3.13)$$

(3.12) has a unique solution. Moreover, the following stability estimate holds:

$$||\mathrm{u}||_1\ \leq\ \frac{1}{\alpha}\ \{\ ||\mathrm{f}||\ +||\mathrm{f}||_0\}\ ) \qquad (3.14)$$

Proof. Using Lax-Milgram Theorem with $V\ =\ H_0^1\ (\Omega)$and Schwarz inequality
The continuity of B in V is

$$\left|\int_\Omega \mathbf{A}\nabla u \cdot \nabla v d\mathbf{x}\right|\ \leq M\int_\Omega |\nabla u||\nabla v|d\mathbf{x} \leq M \parallel \nabla u \parallel_0 \parallel \nabla v \parallel_0$$

Moreover, using Poincare's inequality as well, we get

and

$$\left|\int_\Omega a_0 uv d\mathbf{x}\right| \leq \gamma_0 \int_\Omega |u||v|d\mathbf{x} \leq \gamma_0 C_P^2 \parallel \nabla u \parallel_0 \parallel \nabla v \parallel_0$$

Thus, we can write

$$|B(u,v)| \leq \left(M + (\beta+\gamma)C_p + \gamma C_p^2\right) \parallel \nabla u \parallel_0 \parallel \nabla v \parallel_0$$

Shows that B is continuous.
The coercivity of B:

$$\mathrm{B}(u,u) = \int_\Omega \{\mathbf{A}\nabla u \cdot \nabla u - (\mathbf{b}-\mathbf{c})u \cdot \nabla u + a_0 u^2\} d\mathbf{x}$$

Observe that, since $u = 0$ on $\partial\Omega$, integrating by parts we obtain

$$\int_\Omega (\mathbf{b}-\mathbf{c})u \cdot \nabla u d\mathbf{x} = \frac{1}{2}\int_\Omega (\mathbf{b}-\mathbf{c}) \cdot \nabla u^2 d\mathbf{x} = -\frac{1}{2}\int_\Omega \mathrm{div}(\mathbf{b}-\mathbf{c})u^2$$

It follows that

$$\mathrm{B}(u,u) \geq \alpha \int_\Omega |\nabla u|^2 d\mathbf{x} + \int_\Omega \left[\frac{1}{2}\mathrm{div}(\mathbf{b}-\mathbf{c}) + a_0\right] u^2 d\mathbf{x} \geq \alpha \parallel \nabla u \parallel_0^2$$

so that B is V −coercive. Since it is known that $F \in H^{-1}(\Omega)$, the Lax-Milgram Theorem and (3.10) give existence, uniqueness and the stability estimate (3.14). □

Remark 3.1. If A is symmetric and **b = c = 0**, the solution u is a minimizer in $H_0^1(\Omega)$ for the "energy" functional

$$\mathrm{E}(u) = \int_\Omega \{\mathbf{A}\nabla u \cdot \nabla u + cu^2 - fu\}$$

the Euler equation for E.

Non-homogeneous Dirichlet conditions

Remark 3.2. If the Dirichlet condition is non homogeneous, i.e.

$$\mathrm{u} = \mathrm{g} \text{ on } \partial\Omega,$$

with $g \in H^{\frac{1}{2}}(\partial\Omega)$, we consider an extension $\tilde{g}$ of g in $H^1(\Omega)$ and set $w = u - \tilde{g}$ In this case Ω should be a Lipschitz domain to ensure the existence of $\tilde{g}$. Then $w \in H_0^1(\Omega)$ and solves the equation

$$\mathcal{E}w = f + \operatorname{div}(\mathbf{f} + \mathbf{A}\nabla\tilde{g} - \mathbf{b}\tilde{g}) - \mathbf{c} \cdot \nabla\tilde{g} - \mathrm{c}\tilde{g}$$

We have

$$\mathrm{c} \cdot \nabla\hat{g} + c\hat{g} \in L^2(\Omega) \text{ and } \mathbf{A}\nabla\hat{g} - \mathbf{b}\hat{g} \in L^2(\Omega; \mathbb{R}^n)$$

Therefore, the Lax-Milgram Theorem yields existence, uniqueness and the estimate

$$\| u \|_{1,2} \leq C(\alpha, n, M, \beta, \gamma, \gamma_0, \Omega)\left\{\| f \|_0 + \| \mathbf{f} \|_0 + \| g \|_{H^{1/2}(\partial\Omega)}\right\}$$

A different approach to the Dirichlet problem. It can be observed that (3.9) is a well-posed condition when $divb + a_0 \geq 0$, meaning that the coefficient c is not involved. This requirement is met, specifically, if $b(x) = 0$ a.e.in Ω and $a_0(x) \geq 0$.

Generally speaking, though, we are unable to demonstrate that the bilinear form B is coercive; rather, we can state that it is weakly coercive, meaning that $\lambda_0 \in \mathbb{R}$ exists such that:

$$\widetilde{\mathrm{B}}(u, v) = B(u, v) + \lambda_0(u, v)_0 \equiv B(u, v) + \lambda_0 \int_\Omega uv\, dx$$

From the inequality:

$$|ab| \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2,\ \forall\varepsilon > 0$$

we get

$$\left|\int_{\Omega} (\mathbf{b}-\mathbf{c})u \cdot \nabla u d\mathbf{x}\right| \leq (\beta+\gamma)\int_{\Omega} |u \cdot \nabla u| d\mathbf{x} \leq \varepsilon \parallel \nabla u \parallel_0^2 + \frac{(\beta+\gamma)^2}{4\varepsilon} \parallel u \parallel_0^2$$

Therefore

$$\widetilde{\mathrm{B}}(u,u) \geq \alpha \parallel \nabla u \parallel_0^2 + \lambda_0 \parallel u \parallel_0^2 - \varepsilon \parallel \nabla u \parallel_0^2 - \left(\frac{(\beta+\gamma)^2}{4\varepsilon}+\gamma\right) \parallel u \parallel_0^2. \qquad (3.15)$$

If we choose $\varepsilon = \alpha/2$ and $\lambda_0 = (\beta+\gamma)^2/4\varepsilon + \gamma$, we obtain

$$\widetilde{\mathrm{B}}(u,u) \geq \frac{\alpha}{2} \parallel \nabla u \parallel_0^2$$

which shows the coercivity of $\widetilde{\mathrm{B}}$. Introduce now the Hilbert triplet

$$V = H_0^1(\Omega), H = L^2(\Omega), V^* = H^{-1}(\Omega)$$

Since Ω is a bounded, Lipschitz domain, $H_0^1(\Omega)$ is dense and compactly embedded in $L^2(\Omega)$. Finally, we define the adjoint bilinear form of B by

$$B^*(u,v) \equiv \int_{\Omega} \{(\mathbf{A}^{\top}\nabla u + \mathbf{c}u)\cdot\nabla v - \mathbf{b}v\cdot\nabla u + a_0 uv\}\, d\mathbf{x} = B(v,u)$$

associated with the formal adjoint of $\mathcal{E}$

$$\mathcal{E}^* u = -\mathrm{div}(\mathbf{A}^{\top}\nabla u + \mathbf{c}u) - \mathbf{b}\cdot\nabla u + a_0 u$$

The conclusions are:

1) The subspaces $\mathcal{N}_B$ and $\mathcal{N}_{B^*}$ of the solutions of the homogeneous problems

$$B(u,v) = 0, \forall v \in H_0^1(\Omega)$$

and

$$B^*(w,v) = 0,\ \forall v \in H_0^1(\Omega)$$

share the same dimension $d, 0 \leq d < \infty$.

2) The problem

$$B(u, v) = Fv,\ \forall v \in H_0^1(\Omega)$$

has a solution if and only if $\mathrm{Fw} = 0$ for every $w \in \mathcal{N}_{B^*}$.

Theorem 3.3. Let Ω be a bounded, Lipschitz domain, $f \in L^2(\Omega) and$ $\boldsymbol{f} \in L^2(\Omega; R^n)$.Then, we have:

a) Either ε is an isomorphism between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ and therefore problem (3.9) has a unique weak solution, with

$$\| \nabla u \|_0 \leq C(n, a, K, \beta, \gamma)\{\| f \|_0 + \| \mathbf{f} \|_0\}$$

or the homogeneous and the adjoint homogeneous problems

$$\begin{cases} \mathcal{E}u & = 0 & \text{in } \Omega \\ u & = 0 & \text{on } \partial\Omega, \end{cases} \text{and} \begin{cases} \mathcal{E}^* w = 0 & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

have each d-linearly independent solutions.

b) Moreover, problem (3.9) has a solution if and only if

$$\int_\Omega \{fw - \mathbf{f} \cdot \nabla w\} d\mathbf{x} = 0 \tag{3.16}$$

for every solution w of the adjoint homogeneous problem

Theorem 3.3 implies that if we can show the uniqueness of the solution of problem (3.9), then automatically we infer both the existence and the stability estimate.

3.3.2 Neumann problem

Let Ω be a bounded, Lipschitz domain. The Neumann condition for an operator in the divergence form (3.6) assigns on $\partial\Omega$ the flux. This flux is composed by two terms: $A\nabla u \cdot v\ (-divA\nabla u)$, and $-bu \cdot v\ (div(bu))$ are the diffusion and transport terms, respectively. We set

$$\partial_v^\varepsilon u \equiv (\boldsymbol{A}\nabla u - \boldsymbol{b}u) \cdot v$$

We call $\partial_v^\varepsilon u$ conormal derivative of u. Thus, the correct Neumann problem is:

$$\begin{cases} \mathcal{E}u = f & \text{in } \Omega \\ \partial_\nu^{\mathcal{E}} u = g & \text{on } \partial\Omega. \end{cases} \tag{3.17}$$

with f ∈ $L^2(\Omega)$ and g ∈ $L^2(\partial\Omega)$. The variational formulation of problem (3.17)

may be obtained by the usual integration by parts technique. It is enough to note, that, multiplying the differential equation $\varepsilon u = f$ by a test function $v \in H^1(\Omega)$ and using the Neumann condition, we get, formally:

$$\int_\Omega \{(\boldsymbol{A}\nabla u - \boldsymbol{b}u)\nabla v + (\boldsymbol{c} \cdot \nabla u)v + a_0\, uv\} d\boldsymbol{x} = \int_\Omega f v d\boldsymbol{x} + \int_{\partial\Omega} gv\, d\sigma$$

Introducing the bilinear form

$$B(u,v) = \int_\Omega \{(\mathbf{A}\nabla u - \mathbf{b}u)\nabla v + (\mathbf{c} \cdot \nabla u)v + a_0 uv\} d\mathbf{x} \tag{3.18}$$

and the linear functional

$$Fv = \int_\Omega f v d\boldsymbol{x} + \int_{\partial\Omega} gv\, d\sigma$$

The weak formulation, when all the data are smooth
Find out $u \in H^1(\Omega)$ such that

$$B(u,v) = Fv, \forall v \in H^1(\Omega). \tag{3.19}$$

If the size of $b - c$ *is* so small enough, problem (3.19) is well-posed, as the following proposition shows.

Proposition 3.3. Assume that $f \in L^2(\Omega)$ *and* $g \in L^2(\partial\Omega)$. *If* $a_0(\mathrm{x}) \geq C_0 > 0$ a.e. in Ω and

$$\alpha_0 \equiv \min\{\alpha - (\beta + \gamma)/2, c_0 - (\beta + \gamma)/2\} > 0, \tag{3.20}$$

then, problem (3.19) has a unique solution. Moreover, the following stability estimate holds:

$$\| u \|_{1,2} \leq \frac{1}{\alpha_0}\left\{\| f \|_0 + \bar{C}(n,\Omega) \| g \|_{L^2(\partial\Omega)}\right\}$$

Proof . Since,

$$|B(u,v)| \leq (M + \beta + \gamma + \gamma_0) \| u \|_{1.2} \| v \|_{1.2}$$

B is continuous in $H^1(\Omega)$. Moreover,

$$\mathrm{B}(u,u) \geq \alpha \int_{\Omega} |\nabla u|^2 d\mathbf{x} - \left| \int_{\Omega} [(\mathbf{b}-\mathbf{c}) \cdot \nabla u] u d\mathbf{x} \right| + \int_{\Omega} a_0 u^2$$

From Schwarz's inequality and the inequality $2ab \leq a^2 + b^2$, we obtain

$$\left| \int_{\Omega} [(\mathbf{b}-\mathbf{c}) \cdot \nabla u] u d\mathbf{x} \right| \leq (\beta+\gamma) \parallel \nabla u \parallel_0 \parallel u \parallel_0 \leq \frac{(\beta+\gamma)}{2} \parallel u \parallel_{1,2}^2.$$

Thus, if (3.20) holds, we get $B\ (u,u)\ \geq\ \alpha_0 ||u||_{1,2}^2$ and therefore B is coercive.

Finally, using the trace inequality, we check that $F\ \in\ H^1(\Omega)^*$, with

$$\parallel v \parallel_{L^2(\partial\Omega)} \leq \bar{C}(n,\Omega) \parallel v \parallel_{1,2}$$

$$\parallel F \parallel_{H^1(\Omega)^*} \leq \parallel f \parallel_0 + \bar{C}(n,\Omega) \parallel g \parallel_{L^2(\partial\Omega)}$$

Theorem 3.4. Let Ω be a bounded, Lipschitz domain. Assume that (3.7) and (3.8) hold. Then, if $f\ \in\ L^2(\Omega)$ and $g\ \in\ L^2(\partial\Omega)$:

a) Either the non homogeneous problem (3.17) has a unique solution $u \in H^1(\Omega)$ and

$$\parallel u \parallel_{1,2} \leq C(n,\alpha,M,\beta,\gamma,\gamma_0)\{\parallel f \parallel_0 + \parallel g \parallel_{L^2(\partial\Omega)}\}$$

Or

The homogeneous and the adjoint homogeneous problems

$$\begin{cases} \mathcal{E}u &= 0 \quad \text{in } \Omega \\ (\mathbf{A}\nabla u - \mathbf{b}u) \cdot \boldsymbol{\nu} &= 0 \quad \text{on } \partial\Omega \end{cases} \text{and} \begin{cases} \mathcal{E}^* w &= 0 \quad \text{in } \Omega \\ (\mathbf{A}^\top \nabla w + \mathbf{c}w) \cdot \boldsymbol{\nu} &= 0 \quad \text{on } \partial\Omega \end{cases}$$

have d-linearly independent solutions.

b) Moreover, equation (3.19) has a solution if and only if

$$Fw = \int_{\Omega} fw d\mathbf{x} + \int_{\partial\Omega} gw d\sigma = 0 \qquad (3.21)$$

for every solution w of the adjoint homogeneous problem

Remark 3.2. uniqueness implies existence

Note that if $b = c = 0$ and $a_0 = 0$, then the solutions of the adjoint homogeneous problems are the constant functions i.e. dimension, $d = 1$ and the compatibility condition of (3.21) reduce to the equation.

$$\int_{\Omega} f d\mathbf{x} + \int_{\partial\Omega} g d\sigma = 0.$$

Remark 3.3. In the right hand side of (3.17) there is no term of the form div **f**, Consider, the problem $-\Delta u = divf$, $\partial_{\nu} u = 0$. A weak formulation would be after integration by parts:

$$\int_{\Omega} \nabla u \cdot \nabla v d\mathbf{x} = \int_{\partial\Omega} (\mathbf{f} \cdot \nu) v d\sigma - \int_{\Omega} \mathbf{f} \cdot \nabla v d\mathbf{x} \forall v \in H^1(\Omega). \quad (3.22)$$

However, even if **f** is smooth (3.22) is equivalent to $div(\nabla u + \boldsymbol{f}) = 0$ in the sense of distributions, but with

$$(\nabla u + \mathbf{f}) \cdot \nu = 0 \text{ on } \partial\Omega$$

3.3.3 Robin (mixed problems)

Robin problem.

Consider

$$\begin{cases} \mathcal{E}u = f \text{ in } \Omega \\ \partial_{\nu}^{\mathcal{E}} u + hu = g \text{ on } \partial\Omega. \end{cases} \quad (3.23)$$

The variational formulation of (3.23) is obtained by replacing the bilinear form B in problem (3.19) by

$$\tilde{B}(u, v) = B(u, v) + \int_{\partial\Omega} huv d\sigma$$

If $0 \leq h(x) \leq h_0$a.e. on $\partial\Omega$,

In robin problem the bilinear form is weakly coercive analogous to Neumann problem

Mixed (Dirichlet-Neumann) problem. Let Γ_D be a non empty relatively open subset of $\partial\Omega$ and $\Gamma_N = \partial\Omega \setminus \Gamma_D$. Consider the mixed problem

$$\begin{cases} \mathcal{E}u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \partial_{\nu}^{\mathcal{E}} ug & \text{on } \Gamma_N \end{cases}$$

We select $H^1_{0,\Gamma_D}(\Omega)$ as the correct functional setting is with the norm $\| u \|_{H^1_{0,\Gamma_D}(\Omega)} = \| \nabla u \|_0$. the linear functional,

$$\boldsymbol{Fv} = \int_{\Omega} f v d\mathbf{x} + \int_{\Gamma_N} g v d\sigma,$$

The variational formulation is

Find out $u \in H^1_{0,\Gamma_D}(\Omega)$ such that

$$\mathrm{B}\,(\mathrm{u},\mathrm{v}) = \mathrm{F}\,\mathrm{v}, \forall \mathrm{v} \in H^1_{0,\Gamma_D}(\Omega). \tag{3.24}$$

Proposition 3.4. Assume that hypotheses (3.7) and (3.8) hold and that $f \in L^2(\Omega), g \in L^2(\Gamma_N)$. If b and c have Lipschitz components and

$$(\mathbf{b}-\mathbf{c})\cdot\nu \leq 0 \text{ a.e. on } \Gamma_N, \ \frac{1}{2}\operatorname{div}(\mathbf{b}-\mathbf{c}) + a_0 \geq 0, \ \text{ a.e. in } \Omega$$

then problem (3.24) has a unique solution $u \in H^1_{0,\Gamma_D}(\Omega)$. Moreover, the following stability estimate holds:

$$\| u \|_1 \leq \frac{1}{\alpha}\{\| f \|_0 + \bar{C} \| g \|_{L^2(\Gamma_N)}\}$$

Remark 3.4 If $u = g_0$ *on* Γ_D, i.e. if the Dirichlet data are nonhomogeneous, set $w = u - \tilde{g}_0$, *where* $\tilde{g}_0 \in H^1(\Omega)$ is an extension of g_0. Then $W \in H^1_{0,\Gamma_D}(\Omega)$ and solves

$$B(w,v) = B(\tilde{g}_0, v) + \int_\Omega fv d\mathbf{x} + \int_{\Gamma_N} gv d\sigma \ \forall v \in H^1_{0,\Gamma_D}(\Omega).$$

In general, the bilinear form B for the mixed problem is also weakly coercive
Note that the compatibility conditions (3.24) take the form

$$Fw = \int_\Omega fw d\mathbf{x} + \int_{\Gamma_N} gw d\sigma = 0$$

for every solution w of the adjoint problem

$$\begin{cases} \mathcal{E}^* w & = 0 & \text{in } \Omega \\ w & = 0 & \text{on } \Gamma_D \\ (\mathbf{A}^\top \nabla w + \mathbf{c} w)\cdot\nu & = 0 & \text{on } \Gamma_N. \end{cases}$$

3.4 Weak Maximum Principles

Due to the setting of the sobolev functions, we need to introduce a notion of "positivity or negativity on $\partial\Omega$" which is stronger than the a.e. sense on the boundary.

Let $u \in H^1(\Omega)$ and let Ω be a bounded, Lipschitz domain. We say that $u \geq 0$, if there exists a sequence $\{v_k\}_{k\geq 1} \subset C^1(\bar{\Omega})$, $C^1(\bar{\Omega})$ *is dense in* H^1 such that $v_k \to u$ *in* $H^1(\Omega)$ and $v_k \geq 0$. The trace of u gets its non-negativity from the sequence $\{v_k\}_{k\geq 1}$. It is clear that if $u \geq 0$ on $\partial\Omega$, the trace is non-negative a.e. on $\partial\Omega$, but the reverse implication is not true. Since $v_k \geq 0$ is equivalent to saying that the negative part $v_k^- = max\{-v_k, 0\}$ has zero trace on $\partial\Omega$. it is also shown that $u \geq 0$ *on* $\partial\Omega$ if and only if $u^- \in H^1_0(\Omega)$. Similarly, $u \leq 0$ *on* $\partial\Omega$ if and only if $u^+ \in H^1_0(\Omega)$.

Other inequalities follow the same order, for example, $u \leq v\ on\ \partial\Omega$ if $u - v \leq 0\ on\ \partial\Omega$. Thus, we define:

$$\sup_{\partial\Omega} u = \inf\{k \in \mathbb{R}: u \leq k \text{ on } \partial\Omega\},\ \inf_{\partial\Omega} u = \sup\{k \in \mathbb{R}: u \geq k \text{ on } \partial\Omega\}$$

which coincide with the usual greatest lower bound and lowest upper bound when $u \in C(\partial\Omega)$.

Consider the equation

$$B(u,v) = \int_{\Omega} \{(\mathbf{A}\nabla u - \mathbf{b}u)\nabla v + \mathbf{c}v \cdot \nabla u + a_0 uv\} d\mathbf{x} = 0 \qquad (3.25)$$

for every $v \in H_0^1(\Omega)$.

Theorem 3.5. (Weak maximum principle). Assume that $u \in H^1(\Omega)$ satisfies (3.25) and that (3.7) and (3.8) hold. Moreover, let b Lipschitz and

$$\text{divb } + a_0 \geq 0 \text{ a.e. in } \Omega. \qquad (3.26)$$

Then

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ \text{ and } \inf_{\Omega} \geq \inf_{\partial\Omega} u^- \qquad (3,27)$$

Proof. For $b = c = 0$, and $a_0 \geq 0$ a.e. in Ω. We have:
if $u \in H^1(\Omega)$ then its positive and negative part, $u^+ = max\ \{u, 0\}$ and $u^- = max\ \{-u, 0\}$, belong to $H^1\ (\Omega)$ as well.

$$\int_{\Omega} \mathbf{A}\nabla u \cdot \nabla v d\mathbf{x} = -\int_{\Omega} a_0 uv d\mathbf{x},\ \forall v \in H_0^1(\Omega)$$

Let

$$l = \sup_{\partial\Omega} u^+$$

Assume that $l < \infty$, otherwise there is nothing to be proved. Select as a test function $v = \max\{u - l, 0\} \geq 0$, which belongs to $H_0^1(\Omega)$.

Now, observe that in the set $\{u > l\}$, where $v > 0$ a.e., we have $\nabla v = \nabla u$ so that, using the uniform ellipticity condition and (3.27), we obtain

$$\alpha \int_{\{u>l\}} |\nabla v|^2 d\mathbf{x} \leq \int_{\Omega} \mathbf{A}\nabla u \cdot \nabla v d\mathbf{x} = -\int_{\{u>l\}} a_0 u(u-l) d\mathbf{x} \leq 0.$$

Thus, either $|\{u > l\}| = 0$ or $\nabla v = 0$. In any case, since $v \in H_0^1(\Omega)$, the inference $v = 0$, when $u < 1$

The second inequality (3.27) may be proved in similar way .

Remark 3.5 Note that Theorem 3.5 implies that if $u \leq 0 \; or \; u \geq 0 \; on \; \partial\Omega$, then $u \leq 0$ or $u \geq 0 \; in \; \Omega$. In particular, if $u = 0 \; on \; \partial\Omega \; then \; u = 0 \; in \; \Omega$
Also, it is not possible to substitute $sup_{\partial\Omega} \, u^+$ by $sup_{\partial\Omega}$ u or$inf_{\partial\Omega} \, u^-$ with $inf_{\partial\Omega}$ u in (3.27). A counterexample in dimension one is shown in figure 3.1. The solution of $-u'' + u = 0 \; in \; (0,1), u\,(0) = u\,(1) = -1$, has a negative maximum which is greater than -1

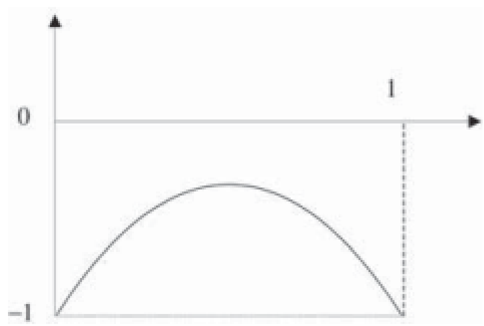

Fig. 3.1. The solution of $-u'' + u = 0 \; in \; (0,1), u\,(0) = u\,(1) = -1$

Corollary 3.1. Under the hypotheses of Theorem 3.5, the Dirichlet problem

$$\begin{cases} \mathcal{E}u = f + \text{div}\boldsymbol{f} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique solution $u \in H_0^1(\Omega)$ and

$$\| \nabla u \|_0 \leq C(n, \alpha, K, \beta, \gamma)\{\| f \|_0 + \| \mathbf{f} \|_0\}$$

A similar maximum principle holds for Robin (mixed conditions), yielding uniqueness and therefore well-posedness, for the corresponding problems.
Suppose for instance that $u \in H^1(\Omega)$ satisfies the equation

$$B(u, v) = 0, \forall v \in H^1_{0,\Gamma_D}(\Omega). \qquad (3.28)$$

Then u is a solution of a mixed problem with f = g = 0.

Theorem 3.6. Let $\Gamma_D \subset \partial\Omega$, open, , $\Gamma_D \neq \emptyset$. Assume that $u \in H^1(\Omega)$ satisfies (3.28) and that (3.7) and (3.8) hold. Moreover, let b be Lipschitz and

$$\text{b}.\nu \leq 0 \text{ a.e. on } \Gamma_N, \; \text{divb} + a_0 \geq 0 \text{ a.e. in } \Omega.$$

Then

$$\sup_{\Omega} u \leq \sup_{\Gamma_D} u^{+} \text{ and } \inf \geq \inf_{\Gamma_D} u^{-}$$

3.5 Regularity

Consider the poisson problem

$$\begin{cases} -\Delta u + u = \boldsymbol{F} \; in \; \Omega \\ u = o \; on \; \partial\Omega \end{cases}$$

Where $F \in H^{-1}(\Omega)$

The establishment of the optimal regularity of a weak solution in relation to the degree of smoothness of the data. Under this conditions, the Lax-Milgram theorem yields a solution $u \in H_O^1(\Omega)$ and doesn't bring much more smoothness. Again from sobolev inequalities a solution $u \in L^p(\Omega)$ with $p = \frac{2n}{n-2}$ for n ≥ 3, or $u \in L^2(\Omega)$, with $2 \leq p < \infty \, if \; n = 2$. doesn't increase the smoothness of u

Reversing our point of view and starting from a function in $H_O^1(\Omega)$ and applying to it second order operator "two orders of differentiability are lost", the loss of one order drives from $H_O^1(\Omega)$ in to $L^2(\Omega)$ while a further loss leads to $H^{-1}(\Omega)$, '-1' shows lack of one order of differentiability.

Nevertheless, consider the case in which $u \in H^1(R^n)$ is a solution of the equation

$$-\Delta u + u = f \text{ in } \mathbb{R}^n \tag{3.29}$$

$If \, f \in L^2(R^n)$ the optimal regularity of u is: we start from $u \in H^1(R^n)$, but applying the second order operator $-\Delta + I$, where I denotes the identity operator, we find $f \in L^2(R)$. This shows that the starting function should really be in $H^2(R^n)$ rather than $H^1(R^n)$. Indeed this is true and can be easily proved using the Fourier transform. Since

$$\widehat{\partial_{x_i} u}(\boldsymbol{\xi}) = i\xi_i \hat{u}(\boldsymbol{\xi}), \; \widehat{\partial_{x_i x_j}} u(\boldsymbol{\xi}) = -\xi_i \xi_j \hat{u}(\boldsymbol{\xi})$$

we have

$$-\widehat{\Delta \mathrm{u}}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^2 \hat{u}(\boldsymbol{\xi})$$

and equation (3.29) becomes

$$(1 + |\boldsymbol{\xi}|^2)\hat{u}(\boldsymbol{\xi}) = \hat{f}(\boldsymbol{\xi})$$

Whence

$$\hat{u}(\xi) = \frac{\hat{\mathrm{f}}(\xi)}{1 + |\xi|^2}. \tag{3.30}$$

From (3.30) we confirm that every second order derivative of u belongs to $L^2(R^n)$. This comes from the following facts:

•The formula:

$$\| \hat{v} \|^2_{L^2(\mathbb{R}^n)} = (2\pi)^n \| v \|^2_{L^2(\mathbb{R}^n)}$$

- the elementary inequality

$$2|\xi_i \xi_j| < 1 + |\xi|^2, \ \forall i, j = 1, \dots, n$$

- the simple computation

$$\int_{\mathbb{R}^n} |\partial_{xixj} u(\mathbf{x})|^2 d\mathbf{x} \quad = \int_{\mathbb{R}^n} \xi_i^2 \xi_j^2 |\hat{u}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} = \int_{\mathbb{R}^n} \frac{\xi_i^2 \xi_j^2}{(1 + |\boldsymbol{\xi}|^2)^2} |\hat{f}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi}$$
$$< \frac{1}{4} \int_{\mathbb{R}^n} |\hat{f}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} = \frac{(2\pi)^n}{4} \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 d\mathbf{x}$$

Thus, $u \in H^2(\mathbb{R}^n)$ and moreover, we have obtained the estimate

$$\| u \|_{H^2(\mathbb{R}^n)} \leq C \| f \|_{L^2(\mathbb{R}^n)}$$

If f $\in H^1$ (R^n), i.e. if f has first partials in L^2 (R^n), a similar computation yields $u \in H^3$ (R^n) from this iteration we conclude that, for every $m \geq 0$,

if $f \in H^m(\mathbb{R}^n)$ then $u \in H^{m+2}(\mathbb{R}^n)$.

The inference takes us if m becomes sufficiently large, u is a classical solution. In fact, if $u \in H^{m+2}$ (R^n) then

$$u \in C^k(\mathbb{R}^n) \text{ for } k < m + 2 - \frac{n}{2},$$

and therefore it is enough that $m > \frac{n}{2}$ to have u at least in $C^2(\mathbb{R}^n)$. An immediate Consequence is:

$$\text{if } f \in C^\infty(\mathbb{R}^n) \text{ then } u \in C^\infty(\mathbb{R}^n).$$

We extend this kind of results to the solutions of Dirichlet, Neumann and Robin problems and to uniformly elliptic operators in divergence form.

There are two kinds of regularity results, interior regularity and global (up to the boundary) regularity, respectively.

we only state the main results.

In all the theorems below u is a weak solution of

$$\mathcal{E}u = f \text{ in } \Omega.$$

• **Interior regularity**

Theorem 3.7. (H^2-interior regularity). Let the coefficients a_{ij} be Lipschitz in Ω. Then $u \in H^2_{loc}(\Omega)$ and if $\Omega' \subset\subset \Omega$,

$$\| u \|_{H^2(\Omega')} \leq C\{\| f \|_{L^2(\Omega)} + \| u \|_{L^2(\Omega)}\} \tag{3.31}$$

Thus, u is a strong solution in Ω. The constant C depends on all the parameters $\alpha, \beta, \gamma, \gamma_0, M$ and also on the distance of Ω' from $\partial\Omega$ and the Lipschitz constant of a_{ij} and b_j, $i, j = 1, \ldots, n$.

If we increase the regularity of the coefficients, the smoothness of u increases according to the following theorem:

Theorem 3.8 (Higher interior regularity). Let a_{ij}, $b_j \in C^{m+1}(\Omega)$ and c_j, $a_0 \in C^m(\Omega)$, $m \geq 1, i, j = 1, \ldots, n$. Then $u \in H^{m+2}_{loc}(\Omega)$ and if $\Omega' \subset\subset \Omega$,

$$\| u \|_{H^{m+2}(\Omega')} \leq C\{\| f \|_{H^m(\Omega)} + \| u \|_{L^2(\Omega)}\}$$

As a consequence, if $a_{ij}, b_j, c_j, a_0, f \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$ as well.

- **Global regularity**

Consider first H^2-regularity. If $u \in H^2(\Omega)$, its trace on $\partial\Omega$ belongs to $H^{3/2}(\partial\Omega)$ so that a Dirichlet data g_D has to be taken in this space. On the other hand, the trace of the normal derivative belongs to $H^{1/2}(\partial\Omega)$ and hence we have to assign a Neumann or a Robin data g_N in this space. Also, the domain has to be smooth enough, say C^2, in order to define the traces of u and $\partial_\nu u$.

Thus, assume that u is a solution of $\mathcal{E}u = f$ in Ω, with one of the following boundary conditions:

$$u = g_D \in H^{3/2}(\partial\Omega)$$

or

$$\partial_\nu^{\mathcal{E}} + hu = g_N \in H^{1/2}(\partial\Omega)$$

With

$$0 \leq h(\boldsymbol{\sigma}) \leq h_0 \text{ a.e. on } \partial\Omega$$

Theorem 3.9 Let Ω be a bounded, C^2−domain. Assume that a_{ij}, b_j, $i, j = 1, \ldots, n$, are Lipschitz in Ω and $f \in L^2(\Omega)$. Then $u \in H^2(\Omega)$ and

$$\| u \|_{H^2(\Omega)} \leq C \left\{ \| u \|_0 + \| f \|_0 + \| g_D \|_{H^{3/2}(\partial\Omega)} \right\} \quad \text{(Dirichlet)},$$
$$\| u \|_{H^2(\Omega)} \leq C \left\{ \| u \|_0 + \| f \|_0 + \| g_R \|_{H^{1/2}(\partial\Omega)} \right\} \quad \text{(Neumann/Robin)}.$$

If we increase the regularity of the domain, the coefficients and the data, the smoothness of u increases accordingly to the following theorem.

Theorem 3.10. Let Ω be a bounded $C^{m+2}-$domain. Assume that $a_{ij}, b_j \in C^{m+1}(\overline{\Omega})$, C_J, $a_0 \in C^m(\overline{\Omega})$, $i, j = 1, \ldots, n, f \in H^m(\Omega)$. If $g_D \in H^{m+\frac{3}{2}}(\partial\Omega)$ or $g_N \in H^{m+\frac{1}{2}}(\partial\Omega)$ and $h \in C^{m+1}(\partial\Omega)$, then $u \in H^{m+2}(\Omega)$ and moreover,

$$\| u \|_{H^{m+2}(\Omega)} \leq C \left\{ \| u \|_0 + \| f \|_{H^m(\Omega)} + \| g_D \|_{H^{m+3/2}(\partial\Omega)} \right\} \text{ (Dirichlet)},$$
$$\| u \|_{H^{m+2}(\Omega)} \leq C \left\{ \| u \|_0 + \| f \|_{H^m(\Omega)} + \| g_R \|_{H^{m+1/2}(\partial\Omega)} \right\} \text{ (Neumann, Robin)}.$$

In particular, if Ω is a C^∞-domain, all the coefficients are in $C^\infty(\bar{\Omega})$ and the boundary data are in $C^\infty(\partial\Omega)$, then $u \in C^\infty(\bar{\Omega})$.

- A particular case. Let Ω be a $C^2 -$ domain and $f \in L^2(\Omega)$. The Lax-Milgram

Theorem imply that the solution of the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases}$$

belongs to $H^2(\Omega) \cap H_0^1(\Omega)$ and that

$$\| u \|_{H^2(\Omega)} \leq C \| f \|_0 = C \| \Delta u \|_0 . \tag{3.32}$$

Since we clearly have

$$\| \Delta u \|_0 \leq \| u \|_{H^2(\Omega)}$$

Corollary 3.2. If $u \in H^2(\Omega) \cap H_0^1(\Omega)$, then

$$\| \Delta u \|_0 \leq \| u \|_{H^2(\Omega)} \leq C_b \| \Delta u \|_0$$

In other words, $\| \Delta u \|_0$ and $\| u \|_{H^2(\Omega)}$ are equivalent norms in $H^2(\Omega) \cap H_0^1(\Omega)$.
Let us see the application of Corollary 3.2 to an equilibrium problem for a bent plate.

Example 3.2. Consider the plane sector:

$$S_\alpha = \left\{(r,\theta): 0 < r < 1, -\frac{\alpha}{2} < \theta < \frac{\alpha}{2}\right\} (0 < \alpha < 2\pi)$$

$$u(r,\theta) = r^{\frac{\pi}{\alpha}} \cos\frac{\pi}{\alpha}\theta$$

Is harmonic function in S_α, since it is the real part of $f(z) = z^{\frac{\pi}{\alpha}}$, which is holomorphic in S_α. Furthermore,

$$u\left(r, -\frac{\alpha}{2}\right) = u\left(r, \frac{\alpha}{2}\right) = 0, 0 \le r \le 1 \qquad (3.33)$$

And

$$u(1,\theta) = \cos\frac{\pi}{\alpha}\theta \,, 0 \le \theta \le \alpha. \qquad (3.34).$$

Focusing on a neighborhood of the origin. If $\alpha = \pi, S_\alpha$ is a semicircle and

$$u(r,\theta) = \mathrm{Re}\, z = x \in C^\infty(\bar{S}_\alpha)$$

Suppose $\alpha \neq \pi$. Since

$$|\nabla u|^2 = u_r^2 + \frac{1}{r^2} u_\theta^2 = \frac{\pi^2}{\alpha^2} r^{2\left(\frac{\pi}{\alpha}-1\right)}$$

we have

$$\int_{S_\alpha} |\nabla u|^2 dx_1 dx_2 = \frac{\pi^2}{\alpha} \int_0^1 r^{2\frac{\pi}{\alpha}-1} dr = \frac{\pi}{2}$$

so that $u \in H^1(S_\alpha)$ and is the unique weak solution of $\Delta u = 0 \; in \; S_\alpha$,with the boundary conditions (3.33), (3.34). It is easy to check that for every $i, j = 1, 2$, whence

$$\int_{S_o} \left|\partial_{xixj} u\right|^2 dx_1 dx_2 \simeq \int_0^1 r^2 \frac{\pi}{\alpha} - 3dr$$

This integral is convergent only for

$$2\frac{\pi}{\alpha} - 3 > -1$$

The conclusion is that $u \in H^2(S_\alpha)$ if and only if $\alpha \leq \pi$, i.e. if the sector is convex. If $\alpha > \pi, u \notin H^2(S_\alpha)$.

Conclusion: in a neighborhood of a non convex angle, we expect a low degree of regularity of the solution (less than H^2).

Example 3.3. As a second example, the function $u\,(r, \theta) = r^{\frac{1}{2}} \sin \frac{\theta}{2}$ is a weak solution in the half circle

$$S_\pi = \{(r, \theta): 0 < r < 1, 0 < \theta < \pi\}$$

of the mixed problem

$$\begin{cases} \Delta u = 0 & \text{in } S_\pi \\ u(1,\theta) = \sin \dfrac{\theta}{2} & 0 < \theta < \pi \\ u(r,0) = 0 \text{ and } \partial_{x_2} u(r,\pi) = 0 & 0 \leq r < 1. \end{cases}$$

Namely,

$$|\nabla u|^2 = \frac{1}{4r}$$

so that

$$\int_{S_n} |\nabla u|^2 dx_1 dx_2 = \frac{\pi}{4}$$

whence $u \in H^1(S_\pi)$. Moreover,

$$\partial_{x_2} u = u_r \sin\theta + \frac{1}{r} u_\theta \cos\theta = \frac{1}{2\sqrt{r}} \cos\frac{\theta}{2}$$

hence

$$\partial_{x_2} u(r, \pi) = 0$$

However, along the half-line $\theta = \pi/2$, for example, we have

$$\left|\partial_{xixj} u\right| \sim r^{-\frac{3}{2}}, r \sim 0$$

so that

$$\int_{S_\alpha} \left|\partial_{x_{ix_j}} u\right|^2 dx_1 dx_2 \sim \int_0^1 r^{-2} dr = \infty$$

and therefore $u \notin H^2(S_\pi)$.

Thus, the solution has a low order of regularity near the origin, even though the boundary of S_π is flat there. Note that the origin separates the Dirichlet and Neumann regions (see fig 3.3)

Conclusion: In general, the optimal regularity of the solution of a mixed problem is less than H^1 near the boundary between the Dirichlet and Neumann regions.

3.6 Equilibrium of a plate

Let us investigate the vertical deflection $u = u\,(x, y)$ resulting from the application of a normal load on a thin, bent plate. It can be shown that if $\Omega \subset R^2$ is the transversal section of the plate, then u is determined by the fourth order equation.

$$\Delta\Delta u = \Delta^2 u = \frac{q}{D} \equiv f \quad \text{in } \Omega$$

wherein q is the loading density and D denotes the elastic properties of the material. Bi-harmonic functions are the solutions to the Variational Formulation of Elliptic Problems Δ^2 u= 0; biharmonic or bi-laplacian is the operator Δ^2. In two dimensions, the explicit expression of Δ^2 is given by

$$\Delta^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\,\partial y^2} + \frac{\partial^4}{\partial y^4}$$

If the plate is rigidly fixed along its boundary (clamped plate), then u and its normal derivative must vanish on $\partial\Omega$.

Consider the problem:

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega \\ u = \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.35}$$

to obtain a formulation that is variational. Select the set of functions in $C^2(\Omega)$compactly supported in Ω, or $C_0^1(\Omega)$as the space of test functions. Multiply by function v $\in C_0^2(\Omega)$ biharmonic equation and integrate over Ω:

$$\int_\Omega \Delta^2 uv d\mathbf{x} = \int_\Omega fv d\mathbf{x}. \tag{3.36}$$

Integrating by parts twice and using the conditions $v = \partial_\nu v = 0$ on $\partial\Omega$, we get:

$$\int_\Omega \Delta^2 uv d\mathbf{x} = \int_\Omega (\operatorname{div}\nabla\Delta u)v d\mathbf{x} = \int_{\partial\Omega} \partial_\nu(\Delta u)v d\sigma - \int_\Omega \nabla\Delta u \cdot \nabla v d\mathbf{x}$$
$$= -\int_{\partial\Omega} \Delta u\,\partial_\nu v d\sigma + \int_\Omega \Delta u \Delta v d\mathbf{x} = \int_\Omega \Delta u \Delta v d\mathbf{x}$$

Thus, (3.35) becomes

$$\int_\Omega \Delta u \Delta v d\mathbf{x} = \int_\Omega fv d\mathbf{x}. \tag{3.37}$$

Now we extend the space of test functions by taking the closure of C_0^2 (Ω) in $H^2(\Omega)$, which is H_0^2 (Ω). Note that the space of functions u such that u and $\partial_\nu u$ have zero trace on $\partial\Omega$. Thus H_0^2 $(\Omega) \subset H_0^1$ $(\Omega) \cap H^2$ (Ω), and from Corollary 3.2 we know that in this space we may choose $||u||_2 = ||\Delta u||_0$ as a norm.

The variational formulation is:
Figure out $u \in H_0^2$ (Ω) such that

$$\int_\Omega \Delta u \Delta v d\mathbf{x} = \int_\Omega f v d\mathbf{x}, \forall v \in H_0^2(\Omega). \tag{3.38}$$

The following result holds:
Proposition 3.5. If $f \in L^2(\Omega)$, there exists a unique solution $u \in H_0^1$ (Ω) of (3.38). Moreover,

$$\| \Delta u \|_0 \leq C_b \| f \|_0.$$

Proof. Note that the bilinear form

$$B(u,v) = \int_\Omega \Delta u \cdot \Delta v d\mathbf{x}$$

coincides with the inner product in $H_0^2(\Omega)$. On the other hand, setting,

$$Lv = \int_\Omega f v d\mathbf{x}$$

We have

$$|L(v)| = \int_\Omega |fv| d\mathbf{x} \leq \| f \|_0 \| v \|_0 \leq C_b \| f \|_0 \| \Delta v \|_0$$

Let u be the solution of problem (3.38). Setting $w = \Delta u$, we have $\Delta w = f$ with $f \in L^2(\Omega)$. Thus, Corollary 3.2 implies $w \in H^2(\Omega)$ which, in turn, yields $u \in H^4$ (Ω)

3.7 Lower and upper Barriers for Semi linear Equations

To solve non-linear boundary value problems the iteration schemes are formulated using the weak maximum principle
consider the problem:

$$\begin{cases} -\Delta u = f(u) \text{ in } \Omega \\ \ \ u = g \text{ on } \partial\Omega. \end{cases} \tag{3.39}$$

Assume that Ω is a smooth domain and that $f \in C^1$ (R), $g \in H^{\frac{1}{2}}(\partial\Omega)$ are also smooth

A weak solution of problem (3.39) is a function $u \in H^1(\Omega)$ such that $u = g \; on \; \partial\Omega$ and

$$\int_\Omega \nabla u \cdot \nabla v d\mathbf{x} = \int_\Omega f(u) v d\mathbf{x}, \;\; \forall v \in H_0^1(\Omega). \tag{3.40}$$

Introducing the weak *sub* and *super* solutions. We say that $u_* \in H^1(\Omega)$ is a weak sub solution of problem (3.39) if $u_* \leq g$ on $\partial\Omega$ and

$$\int_\Omega \nabla u_* \cdot \nabla v d\mathbf{x} \leq \int_\Omega f(u_*) v d\mathbf{x}, \;\; \forall v \in H_0^1(\Omega), v \geq 0 \text{ a.e. in } \Omega$$

Similarly, we say that $u^* \in H^1(\Omega)$ is a weak super solution of problem (3.39) if $u^* \geq g$ on $\partial\Omega$ and

$$\int_\Omega \nabla u^* \cdot \nabla v d\mathbf{x} \geq \int_\Omega f(u^*) v d\mathbf{x} \quad \forall v \in H_0^1(\Omega), v \geq 0 \text{ a.e. in } \Omega$$

Theorem 3.11 Assume that g is bounded on $\partial\Omega$ and that there exist a weak subsolution $u_*$and a weak super solution $u^*$of problem (3.39) such that:

$$\boldsymbol{a} \leq u_* \leq g \leq u^* \leq b \qquad\qquad a, b \in \mathbb{R}.$$

Then, there exists a solution u of problem (3.39) such that

$$u_* \leq u \leq u^*$$

Proof. Let $M = \max_{[a,b]} |f'|$. Then the function $F(s) = f(s) + Ms$ is nondecreasing. Write Poisson's equation in the form

$$-\Delta u + Mu = F(u)$$

To define the sequence recursively we apply the linear theory as the sequence $\{u_k\}_{k\geq 1}$ of functions: let u_1 be the solution of

$$\begin{cases} -\Delta u_1 + M u_1 = F(u_*) & \text{in } \Omega \\ \qquad u_1 = g & \text{on } \partial\Omega. \end{cases}$$

Given u_k, let u_{k+1} be the solution of

$$\begin{cases} -\Delta u_{k+1} + M u_{k+1} = F(u_k) & \text{in } \Omega \\ \qquad u_k = g & \text{on } \partial\Omega. \end{cases} \tag{3.41}$$

We claim that u_k is non decreasing and trapped between $u_*$and u^* :

$$u_* \leq u_k \leq u_{k+1} \leq u^* \text{ a.e. in } \Omega.$$

Based on the claim, we deduce that u_k converges a.e in Ω to some bounded function u, as $k \to +\infty$. Since $F(a) \le F(u_k) \le F(b)$, by the Dominated Convergence Theorem we infer that

$$\int_\Omega F(u_k) v d\mathbf{x} \to \int_\Omega F(u) v d\mathbf{x} \text{ as } k \to +\infty$$

for every $v \in H_0^1(\Omega)$. Now it is enough to show that there is a subsequence $\{u_{kj}\}$ which converges weakly in $H^1(\Omega)$ to u, in order to pass to the limit in the equation

$$\int_\Omega \left(\nabla u_{k_j+1} \cdot \nabla v + M u_{k_j+1} v\right) d\mathbf{x} = \int_\Omega F\left(u_{k_j}\right) v d\mathbf{x} \; \forall v \in H_0^1(\Omega)$$

and obtain (3.39)

We now prove the claim. Let us check that $u_* \le u_1$ a.e in Ω. Set $h_0 = u_* - u_1$. Then $\sup_{\partial\Omega} h_0^+ = 0$ and

$$\int_\Omega (\nabla h_0 \cdot \nabla v + M h_0 v) d\mathbf{x} \le 0, \; \forall v \in H_0^1(\Omega), v \ge 0 \text{ a.e. in } \Omega.$$

since $u_* \le u_1 \Rightarrow u_* - u_1 \le 0$ we deduce $u_0 \le 0$. Similarly, we infer that

$u_1 \le u^*$.Now assume inductively that

$$u_* \le u_{k-1} \le u_k \le u^* \text{ a.e. in } \Omega.$$

We prove that $u_* \le u_k \le u_{k+1} \le u^*$ a.e. in Ω. Let $w_k = u_k - u_{k+1}$. We have $w_k = 0$ on $\partial\Omega$ and

$$\int_\Omega (\nabla w_k \cdot \nabla v + M w_k v) d\mathbf{x} = \int_\Omega [F(u_{k-1}) - F(u_k)] v d\mathbf{x} \; \forall v \in H_0^1(\Omega)$$

Since F is nondecreasing, we deduce that $F(u_{k-1}) - F(u_k) \le 0$ a.e. in Ω so that

$$\int_\Omega (\nabla w_k \cdot \nabla v + M w_k v) d\mathbf{x} \quad \le 0 \; \forall v \in H_0^1(\Omega), v \ge 0 \text{ a.e. in } \Omega.$$

The above proof yields $w_k \le 0$ a.e. in Ω.

Similarly, we infer that $u_* \leq u_k$and $u_{k+1} \leq u^*$. To complete the proof we have to show that $u_k \to u$, weakly in $H^1(\Omega)$. This follows from the estimate for the non homogeneous Dirichlet problem (3.41):

$$\begin{aligned} \|u_k\|_{1,2} &\leq C(n, M, \Omega)\left\{\|F(u_{k-1})\|_0 + \| g \|_{H^{1/2}(\partial\Omega)}\right\} \\ &\leq C_1(n, M, \Omega)\left\{F(b) + \| g \|_{H^{1/2}(\partial\Omega)}\right\} \end{aligned}$$

Since $\{u_k\}$ is bounded in $H^1(\Omega)$, there exists a subsequence weakly convergent to u.

The functions u_* and u^* in the above theorem are called lower and upper barrier, respectively. Consider the following example to show non-uniqueness

The stationary fisher equation:

$$\begin{cases} -\Delta u = u(1-u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Clearly, $u_* \equiv 0$ is a solution. If we assume that the domain Ω is smooth and that the first Dirichlet eigenvalue for the Laplace operator is $\lambda_1 < 1$, we can show that there exists a solution which is positive in Ω. In fact, $u^* \equiv 1$ is an upper barrier. We now exhibit a positive lower barrier. Let w_1 be the nonnegative normalized

Eigenfunction corresponding to λ_1
we know that $w_1 > 0$ inside Ω and from elliptic regularity, $w_1$is smooth up to $\partial\Omega$. Let $u_* = \partial w_1$. We claim that, if σ is positive and small enough, u∗ is a lower barrier. Indeed, since $-\Delta w_1 = \lambda_1 w_1$, we have,

$$-\Delta u_* - u_*(1-u_*) = \sigma w_1(\lambda_1 - 1 + \sigma w_1) \qquad (3.42)$$

If m $= max_{\overline{\Omega}} w_1$ *and* $\sigma < \lambda_1/m$, then the right hand side of (3.42) is negative and u_* is a lower barrier. we infer the existence of a solution u such that $w_1 \leq u \leq 1$. □

The uniqueness of the solution of problem (3.39) is guaranteed if, for instance, f is non increasing:

$$f'(s) \leq 0, s \in \mathbb{R}$$

Then, if $u_1$and $u_2$are two solutions of (3.39), we have $w = u_1 - u_2 \in H_0^1(\Omega)$ and we can write

$$-\Delta w = f(u_1) - f(u_2) = c(\mathbf{x})w$$

where $c(\mathbf{x}) = f'(\bar{u}(\mathbf{x}))$, for a suitable $\bar{u}$ between u_1 and u_2. Since $c \leq 0$ we conclude from the maximum principle that $w \equiv 0$ or $u_1 = u_2$.

Summary

The thesis, titled "Elliptic Problems and the Variational Form," investigates the mathematical structure and solutions of elliptic boundary value problems through variational (or weak) formulations. It begins by deriving weak formulations for various boundary conditions, including Dirichlet, Neumann, Robin, and mixed conditions, particularly in relation to the Poisson equation. The work addresses the well-posedness of these problems, ensuring the existence, uniqueness, and stability of solutions by employing tools such as the Lax-Milgram theorem.

Key points covered include: The thesis underscores the significance of variational formulations in solving boundary value problems while preserving originality. It also introduces essential tools from functional analysis and Hilbert (Sobolev) spaces to construct an appropriate variational framework. A comprehensive examination of the Poisson equation is provided, exploring various boundary conditions and their respective variational formulations.

Further, the it highlights the advantages of weak formulations, which are more adaptable and conducive to numerical methods like the Galerkin method.

The Lax-Milgram theorem plays a pivotal role in establishing the existence, uniqueness, and appropriateness of solutions for weak formulations.

In addition , the thesis also applies eigenvalue problems for the Laplace operator, demonstrating how eigenfunctions can be utilized in the separation of variables method and in analyzing the asymptotic stability of steady-state solutions.

Finally, this thesis concludes by emphasizing the broad applicability of variational formulation theory beyond linear problems, extending to non-linear and semi-linear partial differential equations. It also highlights how the theory aids in finding weak subsolutions or supersolutions (lower and upper barriers) essential for complex boundary value problems.

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