

**A DEGENERATE THREE-LEVEL LASER
COUPLED
TO A SQUEEZED VACUUM RESERVOIR**

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Abstract

Employing the master equation for a three-level laser driven by coherent light and coupled to a squeezed vacuum reservoir, we obtain stochastic differential equations associated with the normal ordering. Using the solutions of the stochastic differential equations, we calculate the quadrature variance, the squeezing spectrum, the mean photon number, and the variance of the photon number. It turns out that the degree of squeezing increases with the linear gain coefficient or the squeeze parameter. It is also found that the driving coherent light decreases the mean photon number.

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1. INTRODUCTION

In quantum optics, the annihilation and creation operator describing a single-mode radiation can be decomposed into two component operators, referred to as quadrature operators. For a single-mode radiation in any state, the product of the fluctuations in the two quadratures satisfies the uncertainty relation [1]. In a squeezed state the quantum noise in one quadrature is below the vacuum level at the expense of enhanced fluctuations in the conjugate quadrature, with the product of the variance in the two quadratures satisfying the uncertainty relation. Squeezing like photon antibunching or sub-Poissonian photon statistics is a nonclassical feature of light [2]. Parametric oscillation and second harmonic generation are typical processes leading to the production of squeezed light modes [2, 3, 4]. It is also worth mentioning that squeezed light has potential applications in the detection of weak signals, noiseless communication, and precision measurement [2, 4, 5, 6].

The squeezing and statistical properties of the light produced by a degenerate three-level laser coupled to a vacuum reservoir have been investigated by several authors when either the atoms are initially prepared in a coherent superposition of the top and bottom levels [7, 8, 9], or when these levels are coupled by a strong coherent light [10, 11, 12]. Moreover, the squeezing and statistical properties of the light produced by a degenerate three-level laser coupled to a squeezed vacuum reservoir and in which the atoms injected into the cavity are initially prepared in a coherent superposition of the top and the bottom levels have been investigated recently [14]. It is found that a three-level laser generates under certain condition squeezed light [2, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16].

The main objective of this thesis is to analyze the squeezing and statistical properties of the light produced by a three-level laser coupled to a squeezed vacuum reservoir via a single-port mirror and in which the top and bottom levels of the three-level atoms injected into the cavity are coupled by a strong coherent light. We carry out the analysis of this quantum optical system using the pertinent stochastic differential equations for the cavity mode variables, associated with the normal ordering. The solution of the resulting equations are then used to calculate the quadrature variance, the squeezing spectrum, the mean photon number, and the variance of the photon number.

2. STOCHASTIC DIFFERENTIAL EQUATIONS

A three-level laser consists of a cavity into which three-level atoms in a cascade configuration are injected at a constant rate r_a and removed from the cavity after a certain time τ . We represent the top, middle, and bottom levels by $|a\rangle$, $|b\rangle$, and $|c\rangle$, respectively. In addition, we assume the cavity mode to be at resonance with the two transitions $|a\rangle \rightarrow |b\rangle$ and $|b\rangle \rightarrow |c\rangle$, with direct transition between levels $|a\rangle$ and $|c\rangle$ to be dipole forbidden [2]. We consider the case in which the atoms are initially prepared in a superposition of the top and bottom levels and in which these levels are coupled by a strong coherent light. The Hamiltonian describing the coupling between levels $|a\rangle$ and $|c\rangle$ by the coherent light can be expressed at resonance as

$$\hat{H}' = ig'(\hat{b}^\dagger|c\rangle\langle a| - \hat{b}|a\rangle\langle c|), \quad (2.1)$$

where g' is the coupling constant and \hat{b} is the annihilation operator for the coherent light. Assuming the coherent light to be strong, we replace \hat{b} by β , which is taken to be real, positive, and constant. In view of this, we can write the Hamiltonian as

$$\hat{H}' = \frac{i\Omega}{2}(|c\rangle\langle a| - |a\rangle\langle c|), \quad (2.2a)$$

where

$$\Omega = 2g'\beta \quad (2.2b)$$

is called the Rabi frequency. On the other hand, the Hamiltonian describing the interaction between a three-level atom and the cavity mode can be written at resonance as

$$\hat{H}'' = ig[\hat{a}^\dagger(|b\rangle\langle a| + |c\rangle\langle b|) - \hat{a}(|a\rangle\langle b| + |b\rangle\langle c|)], \quad (2.3)$$

where \hat{a} is the annihilation operator for the cavity mode. On account of Eqs. (2.2a) and (2.3), the Hamiltonian describing the interaction of a three-level atom with the coherent light and the cavity mode is

$$\hat{H} = \frac{i\Omega}{2}(|c\rangle\langle a| - |a\rangle\langle c|) + ig[\hat{a}^\dagger(|b\rangle\langle a| + |c\rangle\langle b|) - \hat{a}(|a\rangle\langle b| + |b\rangle\langle c|)]. \quad (2.4)$$

We take the initial state of a three-level atom to be

$$|\psi(0)\rangle = C_a(0)|a\rangle + C_c(0)|c\rangle \quad (2.5)$$

and hence the initial density operator for a single atom has the form

$$\hat{\rho}_A = \rho_{aa}^{(0)}|a\rangle\langle a| + \rho_{ac}^{(0)}|a\rangle\langle c| + \rho_{ca}^{(0)}|c\rangle\langle a| + \rho_{cc}^{(0)}|c\rangle\langle c|, \quad (2.6a)$$

where

$$\rho_{aa}^{(0)} = |C_a|^2, \quad (2.6b)$$

$$\rho_{ac}^{(0)} = C_a C_c^*, \quad (2.6c)$$

$$\rho_{ca}^{(0)} = C_c C_a^*, \quad (2.6d)$$

and

$$\rho_{cc}^{(0)} = |C_c|^2. \quad (2.6e)$$

With the assumption that $\rho_{ac}^{(0)} = \rho_{ca}^{(0)}$, the equation of evolution of the density operator for the cavity mode, commonly known as the master equation, is expressible as [10]

$$\begin{aligned} \frac{d}{dt}\hat{\rho} = & p(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\rho}) \\ & + q(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) \\ & + u(\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^{\dagger 2}) + v(\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger - \hat{a}^{\dagger 2}\hat{\rho}) \\ & + u^*(\hat{a}\hat{\rho}\hat{a} - \hat{a}^2\hat{\rho}) + v^*(\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^2), \end{aligned} \quad (2.7)$$

where

$$p = \frac{A}{2B} \left[\rho_{aa}^{(0)} \left(1 + \frac{\Omega^2}{4\gamma^2} \right) + \rho_{cc}^{(0)} \left(\frac{3\Omega^2}{4\gamma^2} \right) - \rho_{ac}^{(0)} \left(\frac{3\Omega}{2\gamma} \right) + \frac{\kappa B}{A} N \right], \quad (2.8a)$$

$$q = \frac{A}{2B} \left[\rho_{aa}^{(0)} \left(\frac{3\Omega^2}{4\gamma^2} \right) + \rho_{cc}^{(0)} \left(1 + \frac{\Omega^2}{4\gamma^2} \right) + \rho_{ac}^{(0)} \left(\frac{3\Omega}{2\gamma} \right) + \frac{\kappa B}{A} (N + 1) \right], \quad (2.8b)$$

$$u = \frac{A}{2B} \left[-\rho_{aa}^{(0)} \left(\frac{\Omega}{2\gamma} \right) \left(1 - \frac{\Omega^2}{2\gamma^2} \right) + \rho_{cc}^{(0)} \left(\frac{\Omega}{\gamma} \right) \left(1 + \frac{\Omega^2}{4\gamma^2} \right) - \rho_{ac}^{(0)} \left(1 - \frac{\Omega^2}{2\gamma^2} \right) - \frac{\kappa B}{A} M \right], \quad (2.8c)$$

$$v = \frac{A}{2B} \left[-\rho_{aa}^{(0)} \left(\frac{\Omega}{\gamma} \right) \left(1 + \frac{\Omega^2}{4\gamma^2} \right) + \rho_{cc}^{(0)} \left(\frac{\Omega}{2\gamma} \right) \left(1 - \frac{\Omega^2}{2\gamma^2} \right) - \rho_{ac}^{(0)} \left(1 - \frac{\Omega^2}{2\gamma^2} \right) - \frac{\kappa B}{A} M \right], \quad (2.8d)$$

in which

$$A = \frac{2g^2 r_a}{\gamma^2} \quad (2.8e)$$

is called the linear gain coefficient,

$$N = \sinh^2 r, \quad (2.8f)$$

and

$$M = \cosh r \sinh r \quad (2.8g)$$

are reservoir parameters. It is very important to note that the presence of the quadratic terms \hat{a}^2 and $\hat{a}^{\dagger 2}$ in the master equation is a signature that the system under consideration may generate squeezed light.

The expectation value of an operator \hat{A} evolves in time in the Schrödinger picture according to

$$\frac{d}{dt} \langle \hat{A} \rangle = Tr \left(\frac{d\hat{\rho}}{dt} \hat{A} \right). \quad (2.9)$$

With the aid of (2.7) and (2.9), and employing the cyclic property of the trace operation along with the relations

$$[\hat{a}, f(\hat{a}, \hat{a}^\dagger)] = \frac{\partial}{\partial \hat{a}^\dagger} f(\hat{a}, \hat{a}^\dagger), \quad (2.10a)$$

$$[\hat{a}^\dagger, f(\hat{a}, \hat{a}^\dagger)] = -\frac{\partial}{\partial \hat{a}} f(\hat{a}, \hat{a}^\dagger), \quad (2.10b)$$

it can easily be verified that

$$\frac{d}{dt} \langle \hat{a} \rangle = (p - q) \langle \hat{a} \rangle + (u - v) \langle \hat{a}^\dagger \rangle, \quad (2.11)$$

$$\frac{d}{dt} \langle \hat{a}^2 \rangle = 2[(p - q) \langle \hat{a}^2 \rangle + (u - v) \langle \hat{a}^\dagger \hat{a} \rangle - v], \quad (2.12)$$

$$\frac{d}{dt} \langle \hat{a}^\dagger \hat{a} \rangle = 2(p - q) \langle \hat{a}^\dagger \hat{a} \rangle + (u - v) [\langle \hat{a}^{\dagger 2} \rangle + \langle \hat{a}^2 \rangle] + 2p. \quad (2.13)$$

It proves to be convenient to work with c-number variables than with operators. In view of this, we wish to convert the operator equations into c-number equation associated with the normal ordering. We note Eqs. (2.12), (2.13), are already in the normal order. Hence one can write

$$\frac{d}{dt} \langle \alpha(t) \rangle = -C \langle \alpha(t) \rangle + D \langle \alpha^*(t) \rangle, \quad (2.14)$$

$$\frac{d}{dt} \langle \alpha^2(t) \rangle = 2[-C \langle \alpha^2(t) \rangle + D \langle \alpha^*(t) \alpha(t) \rangle - v], \quad (2.15)$$

$$\frac{d}{dt} \langle \alpha^*(t) \alpha(t) \rangle = -2C \langle \alpha^*(t) \alpha(t) \rangle + D [\langle \alpha^2(t) \rangle + \langle \alpha^{*2}(t) \rangle] + 2p, \quad (2.16)$$

where

$$C = q - p, \quad (2.17a)$$

and

$$D = u - v. \quad (2.17b)$$

Here α and α^* are the c-number variables corresponding to the operators \hat{a} and \hat{a}^\dagger , respectively.

On the basis of (2.14), one can write the stochastic differential equation

$$\frac{d}{dt}\alpha(t) = -C\alpha(t) + D\alpha^*(t) + f(t), \quad (2.18)$$

where $f(t)$ is a noise force whose properties remain to be determined. We notice that

$$\frac{d}{dt}\langle\alpha(t)\rangle = -C\langle\alpha(t)\rangle + D\langle\alpha^*(t)\rangle + \langle f(t)\rangle. \quad (2.19)$$

On comparing Eqs. (2.14) and (2.19), we see that

$$\langle f(t)\rangle = 0. \quad (2.20)$$

We note that

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = 2\langle\alpha(t)\dot{\alpha}(t)\rangle. \quad (2.21)$$

On combining (2.21) with (2.18), there follows

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = 2[-C\langle\alpha^2(t)\rangle + D\langle\alpha^*(t)\alpha(t)\rangle + \langle\alpha(t)f(t)\rangle]. \quad (2.22)$$

Comparison of Eqs. (2.15) and (2.22) shows that

$$\langle\alpha(t)f(t)\rangle = -v. \quad (2.23)$$

We also note that

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = \langle\alpha^*(t)\dot{\alpha}(t)\rangle + \langle\alpha(t)\dot{\alpha}^*(t)\rangle \quad (2.24)$$

Employing Eq. (2.18) and its complex conjugate, we arrive at

$$\begin{aligned} \frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle &= -2C\langle\alpha^*(t)\alpha(t)\rangle + D[\langle\alpha^2(t)\rangle + \langle\alpha^{*2}(t)\rangle] \\ &\quad + \langle\alpha(t)f^*(t)\rangle + \langle\alpha^*(t)f(t)\rangle. \end{aligned} \quad (2.25)$$

Comparison of this with Eq. (2.16) indicates that

$$\langle\alpha(t)f^*(t)\rangle + \langle\alpha^*(t)f(t)\rangle = 2p. \quad (2.26)$$

Furthermore, a formal solution of (2.18) can be written as

$$\alpha(t) = \alpha(0)e^{-ct} + \int_0^t e^{-c(t-t')} [D\alpha^*(t') + f(t')]dt'. \quad (2.27)$$

Multiplying (2.27) by $f(t)$ and taking the expectation value of both sides, we get

$$\langle \alpha(t)f(t) \rangle = \langle \alpha(0)f(t) \rangle e^{-ct} + \int_0^t e^{-c(t-t')} [D\langle \alpha^*(t')f(t) \rangle + \langle f(t')f(t) \rangle]dt'. \quad (2.28)$$

With the assumption that the noise force at a time t and a cavity mode variable at an earlier time t' are not correlated, we can set

$$\langle \alpha(t')f(t) \rangle = \langle \alpha(t') \rangle \langle f(t) \rangle. \quad (2.29)$$

Thus on account of (2.20), we have

$$\langle \alpha(t')f(t) \rangle = 0. \quad (2.30)$$

Hence in view of (2.23) and (2.30), Eq.(2.28) reduces to

$$\int_0^t e^{-c(t-t')} \langle f(t')f(t) \rangle dt' = -v. \quad (2.31)$$

On the basis of this result, one can write

$$\langle f(t')f(t) \rangle = -2v\delta(t-t'). \quad (2.32)$$

Moreover, one easily obtains

$$\begin{aligned} \langle \alpha(t)f^*(t) \rangle + \langle \alpha^*(t)f(t) \rangle &= \langle \alpha(0)f^*(t) \rangle e^{-ct} + \langle \alpha^*(0)f(t) \rangle e^{-ct} \\ &+ \int_0^t e^{-c(t-t')} [D\langle \alpha^*(t')f^*(t) \rangle + \langle f(t')f^*(t) \rangle]dt' \\ &+ \int_0^t e^{-c(t-t')} [D\langle \alpha(t')f(t) \rangle + \langle f^*(t')f(t) \rangle]dt'. \end{aligned} \quad (2.33)$$

Applying (2.30) and (2.26), we get

$$\int_0^t e^{-c(t-t')} [\langle f(t')f^*(t) \rangle + \langle f^*(t')f(t) \rangle]dt' = 2p. \quad (2.34)$$

On the basis of this result, one can write

$$\langle f(t)f^*(t') \rangle = 2p\delta(t-t'). \quad (2.35)$$

It is worth mentioning that Eqs. (2.20), (2.32) and (2.35) describe the correlation properties of the noise force $f(t)$ associated with the normal ordering.

Now introducing a new variable defined by

$$\alpha_{\pm}(t) = \alpha^*(t) \pm \alpha(t), \quad (2.36)$$

we easily obtain

$$\frac{d}{dt}\alpha_{\pm}(t) = -\lambda_{\mp}\alpha_{\pm}(t) + f^*(t) \pm f(t), \quad (2.37a)$$

where

$$\lambda_{\mp}(t) = C \mp D. \quad (2.37b)$$

The solution of Eq. (2.37a) can be expressed as

$$\alpha_{\pm}(t) = \alpha_{\pm}(0)e^{-\lambda_{\mp}(t)} + \int_0^t e^{-\lambda_{\mp}(t-t')} [f^*(t') \pm f(t')] dt'. \quad (2.38)$$

It then follows that

$$\alpha(t) = A(t)\alpha(0) + B(t)\alpha^*(0) + F(t), \quad (2.39)$$

where

$$A(t) = \frac{1}{2} (e^{-\lambda_- t} + e^{-\lambda_+ t}), \quad (2.40a)$$

$$B(t) = \frac{1}{2} (e^{-\lambda_- t} - e^{-\lambda_+ t}), \quad (2.40b)$$

and

$$F(t) = F_+(t) + F_-(t), \quad (2.40c)$$

with

$$F_{\pm}(t) = \frac{1}{2} \int_0^t e^{-\lambda_{\mp}(t-t')} [f(t') \pm f^*(t')] dt'. \quad (2.40d)$$

3. QUADRATURE FLUCTUATIONS

In this section we wish to calculate the quadrature variance and squeezing spectrum for the cavity mode under consideration.

3.1 Quadrature Variance

Here we seek to calculate the variance of the quadrature operators

$$\hat{a}_+ = \hat{a}^\dagger + \hat{a}, \quad (3.1a)$$

and

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}). \quad (3.1b)$$

With the aid of (3.1a) and (3.1b), one can express the quadrature variance in the normal order as

$$\Delta a_{\pm}^2 = 1 \pm \langle (\hat{a}^{\dagger 2} + \hat{a}^2 \pm 2\hat{a}^\dagger \hat{a}) \rangle \mp \langle (\hat{a}^\dagger \pm \hat{a}) \rangle^2 \quad (3.2)$$

and the c-number expression corresponding to (3.2) is

$$\Delta a_{\pm}^2 = 1 \pm \langle \alpha_{\pm}^2(t) \rangle \mp \langle \alpha_{\pm}(t) \rangle^2, \quad (3.3)$$

where $\alpha_{\pm}(t)$ is defined by Eq. (2.36). We consider here the case for which the cavity mode is initially in a vacuum state. Hence on account of (2.20) and (2.38), we see that

$$\langle \alpha_{\pm}(t) \rangle = 0. \quad (3.4)$$

In view of this result, Eq.(3.3) reduces to

$$\Delta a_{\pm}^2 = 1 \pm \langle \alpha_{\pm}^2(t) \rangle. \quad (3.5)$$

Moreover, employing Eq. (2.37a), one easily gets

$$\frac{d}{dt}\langle\alpha_{\pm}^2(t)\rangle = 2[-\lambda_{\mp}\langle\alpha_{\pm}^2(t)\rangle + \langle\alpha_{\pm}(t)f^*(t)\rangle \pm \langle\alpha_{\pm}(t)f(t)\rangle]. \quad (3.6)$$

Multiplying Eq. (2.36) by $f(t)$ and taking into account Eq. (2.23) and Eq. (2.26), we have

$$\langle\alpha_{\pm}(t)f(t)\rangle = p \mp v, \quad (3.7a)$$

similarly, one readily finds

$$\langle\alpha_{\pm}(t)f^*(t)\rangle = -v \pm p. \quad (3.7b)$$

Therefore, in view of this result, Eq. (3.6) can be rewritten as

$$\frac{d}{dt}\langle\alpha_{\pm}^2(t)\rangle = 2[-\lambda_{\mp}\langle\alpha_{\pm}^2(t)\rangle - 2v \pm 2p]. \quad (3.8)$$

With the cavity mode initially in a vacuum state, the solution of this equation has the form

$$\langle\alpha_{\pm}^2(t)\rangle = \left(\frac{-2v \pm 2p}{\lambda_{\mp}}\right)(1 - e^{-2\lambda_{\mp}t}). \quad (3.9)$$

Now combination of Eqs. (3.5) and (3.9) yields

$$\Delta a_{\pm}^2 = 1 \pm \left(\frac{-2v \pm 2p}{\lambda_{\mp}}\right)(1 - e^{-2\lambda_{\mp}t}) \quad (3.10)$$

and at steady state this equation reduces to

$$\Delta a_{\pm}^2 = \frac{\lambda_{\mp} + 2p \mp 2v}{\lambda_{\mp}}. \quad (3.11)$$

Upon substituting Eq. (2.37b) along with Eqs. (2.17a) and (2.17b) into Eq. (3.11), one arrives at the result

$$\Delta a_{\pm}^2 = \frac{(q \mp u) + (p \mp v)}{(q \mp u) - (p \mp v)}, \quad (3.12)$$

where

$$(q \mp u) = \frac{A}{2B} \left[\frac{\kappa B}{A} (1 + N \pm M) + \rho_{aa}^{(0)} \left(\frac{3\Omega^2}{4\gamma^2} \pm \frac{\Omega}{2\gamma} \left(1 - \frac{\Omega^2}{2\gamma^2} \right) \right) \right. \\ \left. + \rho_{cc}^{(0)} \left(\left(1 + \frac{\Omega^2}{4\gamma^2} \right) \mp \frac{\Omega}{\gamma} \left(1 + \frac{\Omega^2}{4\gamma^2} \right) \right) + \rho_{ac}^{(0)} \left(\frac{3\Omega}{2\gamma} \pm \left(1 - \frac{\Omega^2}{2\gamma^2} \right) \right) \right], \quad (3.13a)$$

and

$$(p \mp v) = \frac{A}{2B} \left[\frac{\kappa B}{A} (N \pm M) + \rho_{aa}^{(0)} \left(\left(1 + \frac{\Omega^2}{4\gamma^2} \right) \pm \frac{\Omega}{\gamma} \left(1 + \frac{\Omega^2}{4\gamma^2} \right) \right) \right. \\ \left. + \rho_{cc}^{(0)} \left(\frac{3\Omega^2}{4\gamma^2} \mp \frac{\Omega}{2\gamma} \left(1 - \frac{\Omega^2}{2\gamma^2} \right) \right) + \rho_{ac}^{(0)} \left(-\frac{3\Omega}{2\gamma} \pm \left(1 - \frac{\Omega^2}{2\gamma^2} \right) \right) \right]. \quad (3.13b)$$

It proves to be more convenient to introduce a new parameter defined by

$$\rho_{aa}^{(0)} = \frac{1 - \eta}{2}, \quad (3.14a)$$

so that in view of the fact that

$$\rho_{aa}^{(0)} + \rho_{cc}^{(0)} = 1, \quad (3.14b)$$

we have

$$\rho_{cc}^{(0)} = \frac{1 + \eta}{2}. \quad (3.14c)$$

Using the relations

$$\rho_{ac}^{(0)} = |\rho_{ac}^{(0)}| e^{i\theta}, \quad (3.15a)$$

$$\rho_{ca}^{(0)} = |\rho_{ac}^{(0)}| e^{-i\theta}, \quad (3.15b)$$

and the assumption that $\rho_{ac}^{(0)} = \rho_{ca}^{(0)}$, one easily obtains

$$|\rho_{ac}^{(0)}|^2 = \rho_{ac}^{(0)} \rho_{ca}^{(0)}. \quad (3.16a)$$

Now on account of Eqs. (2.6b), (2.6c), (2.6d) and (2.6e), we arrive at

$$\rho_{ac}^{(0)} \rho_{ca}^{(0)} = \rho_{aa}^{(0)} \rho_{cc}^{(0)}, \quad (3.16b)$$

so that in view of Eq. (3.16a) and (3.16b), we have

$$|\rho_{ac}^{(0)}|^2 = \rho_{aa}^{(0)} \rho_{cc}^{(0)}. \quad (3.17a)$$

Hence with the aid of Eq. (3.14a) and (3.14c) Eq. (3.17a) takes the form

$$\rho_{ac}^{(0)} = \frac{1}{2}(1 - \eta^2)^{1/2}. \quad (3.17b)$$

Now on combining Eqs. (3.13a), (3.13b), (3.14a), (3.14c), and (3.17b), there follows

$$(q \mp u) + (p \mp v) = \left[\frac{\kappa}{2}(1 + 2N \pm 2M) + \frac{A(\beta(2\beta \mp 3\eta) \mp (1 - \eta^2)^{\frac{1}{2}}(\beta^2 - 2) + 2)}{2(1 + \beta^2)(1 + \frac{\beta^2}{4})} \right] \quad (3.18a)$$

and

$$(q \mp u) - (p \mp v) = \left[\frac{\kappa}{2} + \frac{A(\eta(2 - \beta^2) \mp \beta(1 + \beta^2) + 3\beta(1 - \eta^2)^{\frac{1}{2}})}{2(1 + \beta^2)(1 + \frac{\beta^2}{4})} \right], \quad (3.18b)$$

where

$$B = (1 + \frac{\Omega^2}{\gamma^2})(1 + \frac{\Omega^2}{4\gamma^2}) \quad (3.19a)$$

and

$$\beta = \frac{\Omega}{\gamma}. \quad (3.19b)$$

In view of (3.18), the variance of the quadrature operators takes the form

$$\Delta a_{\pm}^2 = \frac{2\kappa(1 + \beta^2)(1 + \frac{\beta^2}{4})e^{\pm 2r} + A[\beta(2\beta \mp 3\eta) \mp (1 - \eta^2)^{\frac{1}{2}}(\beta^2 - 2) + 2]}{2\kappa(1 + \beta^2)(1 + \frac{\beta^2}{4}) + A[\eta(2 - \beta^2) \mp \beta(1 + \beta^2) + 3\beta(1 - \eta^2)^{\frac{1}{2}}]}, \quad (3.20)$$

in which we have made use of Eqs. (2.8f) and (2.8g).

We next seek to consider some special cases of interest. We then note that for $\beta = 0$, expression (3.20) reduces to

$$\Delta a_+^2 = \frac{\kappa e^{2r} + A[1 + (1 - \eta^2)^{\frac{1}{2}}]}{(\eta A + \kappa)}, \quad (3.21)$$

and

$$\Delta a_-^2 = \frac{\kappa e^{-2r} + A[1 - (1 - \eta^2)^{\frac{1}{2}}]}{(\eta A + \kappa)}. \quad (3.22)$$

This represents the quadrature variance in the absence of the driving light.

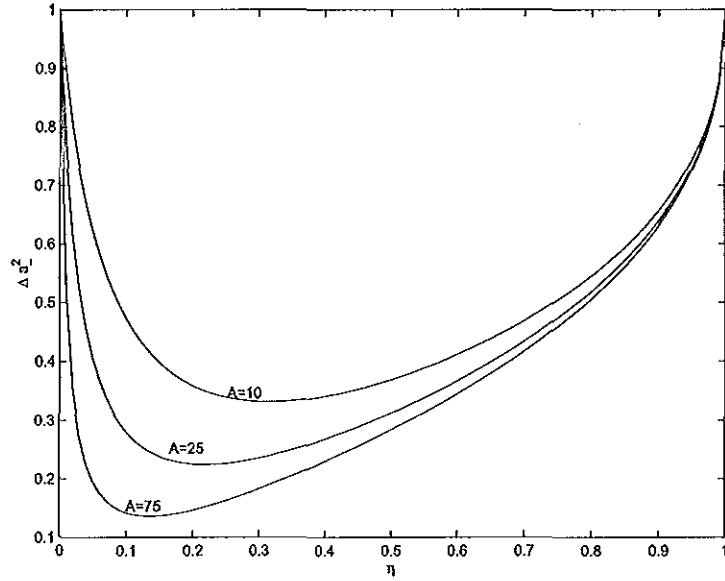


Fig. 3.1 Plots of the quadrature variance Δa^2 versus η for $r = 0$, $\kappa = 0.8$, $\beta = 0$, and for different values of the linear gain coefficient .

Fig. (3.1) or (3.2) shows that the cavity mode is in a squeezed state for all values of η between zero and one and the degree of squeezing increases with the linear gain coefficient A or the squeeze parameter r .

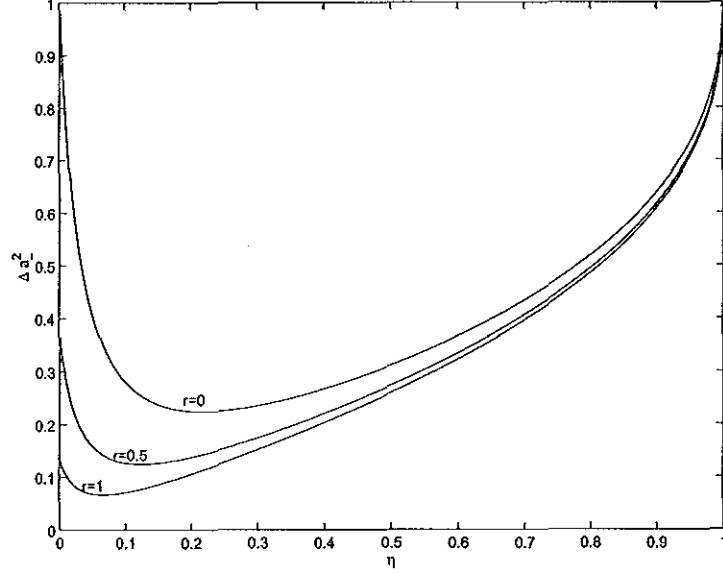


Fig. 3.2 Plots of the quadrature variance Δa_{\pm}^2 versus η for $A = 25$, $\kappa = 0.8$, $\beta = 0$ and for different values of the squeeze parameter .

Moreover, we want to consider the case when all atoms are initially in the upper level and when the upper and lower levels are coupled by a strong coherent light. Thus upon setting $\eta = -1$ in Eq. (3.20), we have

$$\Delta a_{+}^2 = \frac{2\kappa(1 + \beta^2)(1 + \frac{\beta^2}{4})e^{2r} + A[\beta(2\beta + 3) + 2]}{2\kappa(1 + \beta^2)(1 + \frac{\beta^2}{4}) + A[(\beta^2 - 2) - \beta(1 + \beta^2)]} \quad (3.23a)$$

and

$$\Delta a_{-}^2 = \frac{2\kappa(1 + \beta^2)(1 + \frac{\beta^2}{4})e^{-2r} + A[\beta(2\beta - 3) + 2]}{2\kappa(1 + \beta^2)(1 + \frac{\beta^2}{4}) + A[(\beta^2 - 2) + \beta(1 + \beta^2)]}. \quad (3.23b)$$

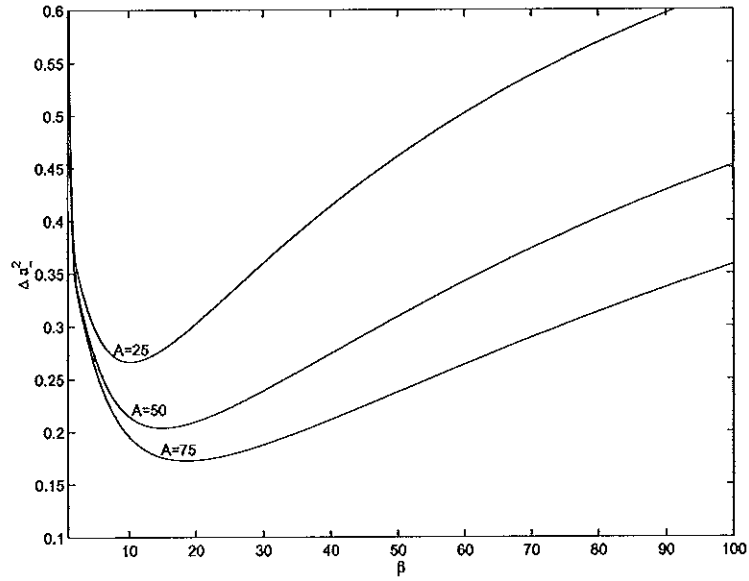


Fig. 3.3 Plots of the quadrature variance Δa^2 versus β for $r = 0$, $\kappa = 0.8$, $\eta = -1$, and for different values of the linear gain coefficient .

From Fig. (3.3) we note that a relatively better squeezing can be achieved for large values of the linear gain coefficient A . And Fig. (3.4) indicates that the degree of squeezing increases with the squeeze parameter r .

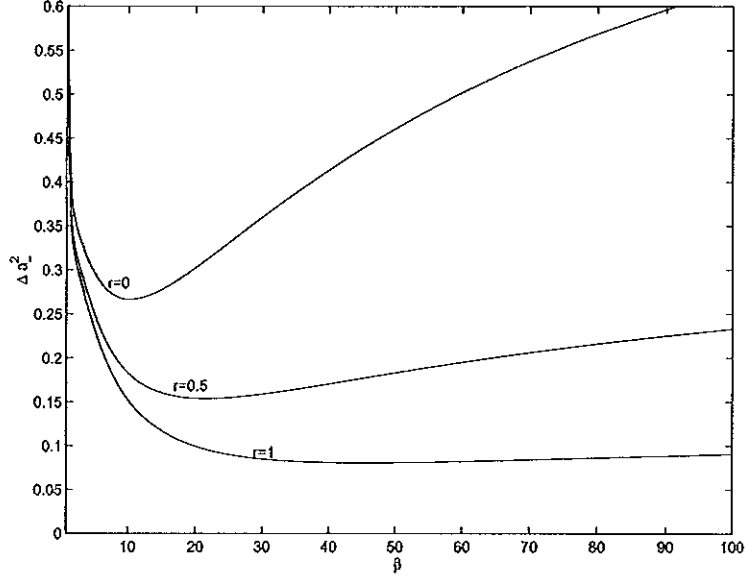


Fig. 3.4 Plots of the quadrature variance Δa_{\pm}^2 versus β for $A = 25$, $\kappa = 0.8$, $\eta = -1$, and for different values of the squeeze parameter .

Furthermore, we want to consider the case when half of the atoms are initially in the upper level while the remaining half are in the lower level and when the upper and the lower levels are coupled by a strong coherent light. Thus upon setting $\eta = 0$ in Eq. (3.20), we have

$$\Delta a_{+}^2 = \frac{2\kappa(1 + \beta^2)(1 + \frac{\beta^2}{4})e^{2r} + A\beta^2}{2\kappa(1 + \beta^2)(1 + \frac{\beta^2}{4}) + A[\beta(2 - \beta^2)]} \quad (3.24a)$$

and

$$\Delta a_{-}^2 = \frac{2\kappa(1 + \beta^2)(1 + \frac{\beta^2}{4})e^{-2r} + 3A\beta^2}{2\kappa(1 + \beta^2)(1 + \frac{\beta^2}{4}) + A[\beta(4 + \beta^2)]}. \quad (3.24b)$$

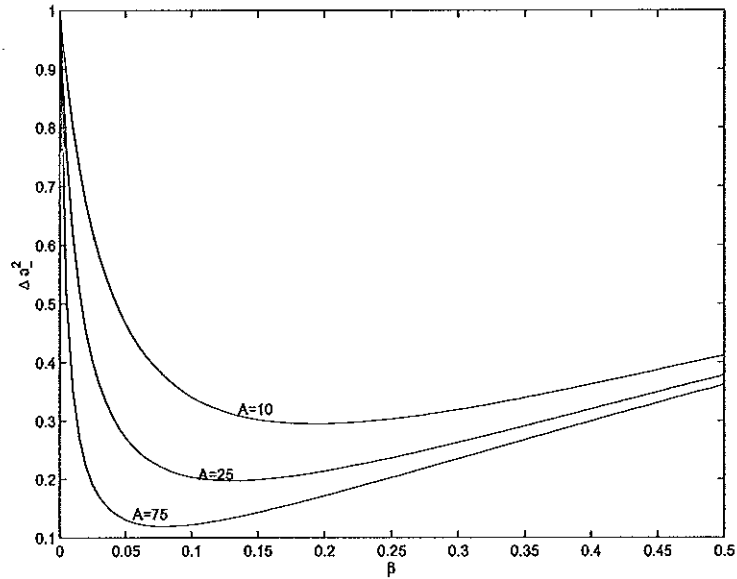


Fig. 3.5 Plots of the quadrature variance Δa_+^2 versus β for $r = 0$, $\kappa = 0.8$, $\eta = 0$, and for different values of the linear gain coefficient .

Fig. (3.5) or (3.6) indicates that the degree of squeezing increases with the linear gain coefficient A or the squeeze parameter r . We also note that a relatively better squeezing can be achieved for very small values of β .

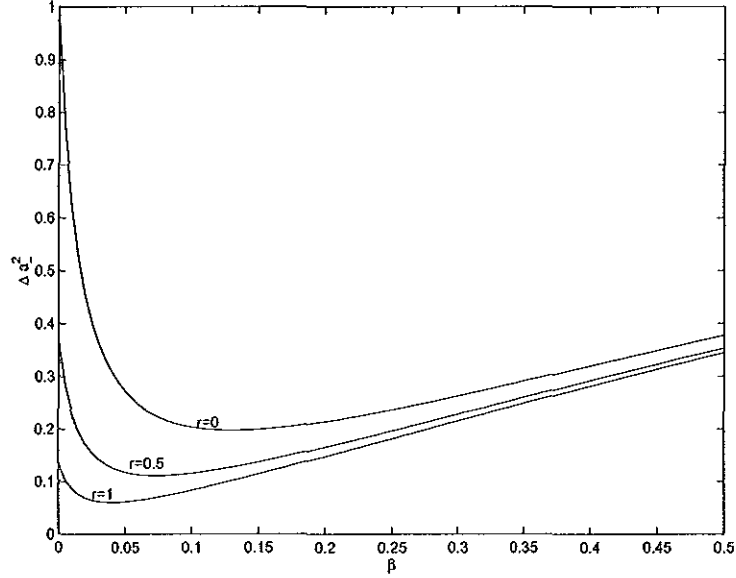


Fig. 3.6 Plots of the quadrature variance Δa_{\pm}^2 versus β for $A = 25$, $\kappa = 0.8$, $\eta = 0$, and for different values of the squeeze parameter .

3.2 Squeezing Spectrum

The squeezing spectrum of the output radiation is defined by [2].

$$S_{\pm}^{out}(\omega) = 2Re \int_0^{\infty} \langle \hat{a}_{\pm}^{out}(t), \hat{a}_{\pm}^{out}(t + \tau) \rangle_{ss} e^{i\omega\tau} d\tau, \quad (3.25)$$

where *ss* stands for steady state and

$$\hat{a}_{+}^{out}(t) = \hat{a}_{out}^{\dagger}(t) + \hat{a}_{out}(t), \quad (3.26a)$$

$$\hat{a}_{-}^{out}(t) = i(\hat{a}_{out}^{\dagger}(t) - \hat{a}_{out}(t)). \quad (3.26b)$$

Making use of Eqs. (3.26a) and (3.26b) along with

$$\langle \hat{A}, \hat{B} \rangle = \langle \hat{A}\hat{B} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle, \quad (3.27)$$

one finds

$$\begin{aligned}
\langle \hat{a}_{\pm}^{out}(t), \hat{a}_{\pm}^{out}(t + \tau) \rangle &= \langle \hat{a}_{out}^{\dagger}(t) \hat{a}_{out}(t + \tau) \rangle + \langle \hat{a}_{out}(t) \hat{a}_{out}^{\dagger}(t + \tau) \rangle \\
&\pm \langle \hat{a}_{out}^{\dagger}(t) \hat{a}_{out}^{\dagger}(t + \tau) \rangle \pm \langle \hat{a}_{out}(t) \hat{a}_{out}(t + \tau) \rangle \\
&\mp \langle \hat{a}_{out}^{\dagger}(t) \pm \hat{a}_{out}(t) \rangle \langle \hat{a}_{out}^{\dagger}(t + \tau) \pm \hat{a}_{out}(t + \tau) \rangle.
\end{aligned} \tag{3.28}$$

With the aid of

$$[\hat{a}(t), \hat{a}^{\dagger}(t + \tau)] = \delta(\tau), \tag{3.29}$$

one can put Eq. (3.28) in the normal order as

$$\begin{aligned}
\langle \hat{a}_{\pm}^{out}(t), \hat{a}_{\pm}^{out}(t + \tau) \rangle &= \delta(\tau) + \langle \hat{a}_{out}^{\dagger}(t) \hat{a}_{out}(t + \tau) \rangle + \langle \hat{a}_{out}^{\dagger}(t + \tau) \hat{a}_{out}(t) \rangle \\
&\pm \langle \hat{a}_{out}^{\dagger}(t) \hat{a}_{out}^{\dagger}(t + \tau) \rangle + \langle \hat{a}_{out}(t) \hat{a}_{out}(t + \tau) \rangle \\
&\mp \langle \hat{a}_{out}^{\dagger}(t) \pm \hat{a}_{out}(t) \rangle \langle \hat{a}_{out}^{\dagger}(t + \tau) \pm \hat{a}_{out}(t + \tau) \rangle.
\end{aligned} \tag{3.30}$$

The c-number equation corresponding to Eq. (3.30) is

$$\begin{aligned}
\langle \hat{a}_{\pm}^{out}(t), \hat{a}_{\pm}^{out}(t + \tau) \rangle &= \delta(\tau) \pm \langle \alpha_{out}^*(t) \alpha_{out}^*(t + \tau) \rangle + \langle \alpha_{out}^*(t) \alpha_{out}(t + \tau) \rangle \\
&+ \langle \alpha_{out}^*(t + \tau) \alpha_{out}(t) \rangle \pm \langle \alpha_{out}(t) \alpha_{out}(t + \tau) \rangle \\
&\mp \langle \alpha_{out}^*(t) \pm \alpha_{out}(t) \rangle \langle \alpha_{out}^*(t + \tau) \pm \alpha_{out}(t + \tau) \rangle.
\end{aligned} \tag{3.31}$$

With the aid of (3.27) this can be rewritten as

$$\langle \hat{a}_{\pm}^{out}(t), \hat{a}_{\pm}^{out}(t + \tau) \rangle = \delta(\tau) \pm \langle \alpha_{\pm}^{out}(t), \alpha_{\pm}^{out}(t + \tau) \rangle, \tag{3.32}$$

where

$$\alpha_{\pm}^{out}(t) = \alpha_{out}^*(t) \pm \alpha_{out}(t). \tag{3.33}$$

Substitution of (3.32) into Eq. (3.25) yields

$$S_{\pm}^{out}(\omega) = 2Re \int_0^{\infty} \delta(\tau) e^{i\omega\tau} d\tau \pm 2Re \int_0^{\infty} \langle \alpha_{\pm}^{out}(t), \alpha_{\pm}^{out}(t + \tau) \rangle_{ss} e^{i\omega\tau} d\tau. \quad (3.34)$$

It proves to be more convenient to rewrite the first integral in (3.34) as

$$2Re \int_0^{\infty} \delta(\tau) e^{i\omega\tau} d\tau = \int_0^{\infty} \delta(\tau) e^{i\omega\tau} d\tau + \int_0^{\infty} \delta(\tau) e^{-i\omega\tau} d\tau. \quad (3.35)$$

Upon replacing τ by $-\tau$ in the second integral, one readily finds

$$2Re \int_0^{\infty} \delta(\tau) e^{i\omega\tau} d\tau = \int_{-\infty}^{\infty} \delta(\tau) e^{i\omega\tau} d\tau = 1. \quad (3.36)$$

On account of this result, Eq. (3.34) takes the form

$$S_{\pm}^{out}(\omega) = 1 \pm 2Re \int_0^{\infty} \langle \alpha_{\pm}^{out}(t), \alpha_{\pm}^{out}(t + \tau) \rangle_{ss} e^{i\omega\tau} d\tau. \quad (3.37)$$

For a cavity mode coupled to a squeezed vacuum reservoir, the output and intracavity variables are related by

$$\alpha_{\pm}^{out}(t) = \sqrt{\kappa} \alpha_{\pm}(t) - \alpha_{in\pm}(t), \quad (3.38)$$

where

$$\alpha_{\pm}^{out}(t) = \alpha_{out}^*(t) \pm \alpha_{out}(t), \quad (3.39a)$$

and

$$\alpha_{in\pm}(t) = \frac{1}{\sqrt{\kappa}} (f_R^*(t) \pm f_R(t)), \quad (3.39b)$$

with $f_R(t)$ being the noise force associated with the reservoir. Therefore on account of Eqs. (3.38) and (3.4) the squeezing spectrum can be put in the form

$$\begin{aligned} S_{\pm}^{out}(\omega) &= 1 \pm 2\kappa Re \int_0^{\infty} \langle \alpha_{\pm}(t) \alpha_{\pm}(t + \tau) \rangle_{ss} e^{i\omega\tau} d\tau \\ &\mp 2\sqrt{\kappa} Re \int_0^{\infty} \langle \alpha_{\pm}(t) \alpha_{in\pm}(t + \tau) \rangle_{ss} e^{i\omega\tau} d\tau \\ &\mp 2\sqrt{\kappa} Re \int_0^{\infty} \langle \alpha_{in\pm}(t) \alpha_{\pm}(t + \tau) \rangle_{ss} e^{i\omega\tau} d\tau \\ &\pm 2Re \int_0^{\infty} \langle \alpha_{in\pm}(t) \alpha_{in\pm}(t + \tau) \rangle_{ss} e^{i\omega\tau} d\tau. \end{aligned} \quad (3.40)$$

Furthermore, the solution of Eq. (2.37a) can be written as

$$\alpha_{\pm}(t + \tau) = \alpha_{\pm}(t) e^{-\lambda_{\mp}\tau} + e^{-\lambda_{\mp}\tau} \int_0^{\tau} e^{\lambda_{\mp}\tau'} [f^*(t + \tau') \pm f(t + \tau')] d\tau'. \quad (3.41)$$

Multiplying Eq. (3.41) by $\alpha_{\pm}(t)$ and taking the expectation value of both sides, we get

$$\begin{aligned} \langle \alpha_{\pm}(t) \alpha_{\pm}(t + \tau) \rangle &= \langle \alpha_{\pm}^2(t) \rangle e^{-\lambda_{\mp}\tau} \\ &+ e^{-\lambda_{\mp}\tau} \int_0^{\tau} e^{\lambda_{\mp}\tau'} [\langle \alpha_{\pm}(t) f^*(t + \tau') \rangle \pm \langle \alpha_{\pm}(t) f(t + \tau') \rangle] d\tau'. \end{aligned} \quad (3.42)$$

On account of the fact that the reservoir noise force at time t and a cavity mode variable at an earlier time are uncorrelated, one can write

$$\langle \alpha_{\pm}(t) f^*(t + \tau') \rangle = \langle \alpha_{\pm}(t) \rangle \langle f^*(t + \tau') \rangle = 0, \quad (3.43a)$$

$$\langle \alpha_{\pm}(t) f(t + \tau') \rangle = \langle \alpha_{\pm}(t) \rangle \langle f(t + \tau') \rangle = 0, \quad (3.43b)$$

where we have made use of (3.4). In view of Eqs. (3.43), Eq. (3.42) takes the form

$$\langle \alpha_{\pm}(t) \alpha_{\pm}(t + \tau) \rangle = \langle \alpha_{\pm}^2(t) \rangle e^{-\lambda_{\mp}\tau}. \quad (3.44a)$$

With the aid of Eq. (3.9), Eq. (3.44a) becomes

$$\langle \alpha_{\pm}(t) \alpha_{\pm}(t + \tau) \rangle_{ss} = \left(\frac{-2v \pm 2p}{\lambda_{\mp}} \right) e^{-\lambda_{\mp}\tau}. \quad (3.44b)$$

Furthermore, multiplying Eq.(3.39b) by $\alpha_{\pm}(t)$ and taking the expectation value of both sides, we have

$$\langle \alpha_{\pm}(t) \alpha_{in\pm}(t + \tau) \rangle = \frac{1}{\sqrt{\kappa}} (\langle \alpha_{\pm}(t) f_R^*(t + \tau) \rangle \pm \langle \alpha_{\pm}(t) f_R(t + \tau) \rangle), \quad (3.45)$$

in view of the fact that the reservoir force at a time t and cavity mode at an earlier time are uncorrelated, one can write

$$\langle \alpha_{\pm}(t) f_R^*(t + \tau) \rangle = \langle \alpha_{\pm}(t) \rangle \langle f_R^*(t + \tau) \rangle, \quad (3.46a)$$

$$\langle \alpha_{\pm}(t) f_R(t + \tau) \rangle = \langle \alpha_{\pm}(t) \rangle \langle f_R(t + \tau) \rangle, \quad (3.46b)$$

so that with the aid of Eqs. (3.4), (3.46), Eq. (3.45) becomes

$$\langle \alpha_{\pm}(t) \alpha_{in\pm}(t + \tau) \rangle = 0, \quad (3.47)$$

Moreover, multiplying Eq. (3.41) by $\alpha_{in\pm}(t)$ and taking the expectation value of both sides, we find

$$\begin{aligned} \langle \alpha_{in\pm}(t)\alpha_{\pm}(t+\tau) \rangle &= \langle \alpha_{in\pm}(t)\alpha_{\pm}(t) \rangle e^{-\lambda\mp\tau} \\ &+ e^{-\lambda\mp\tau} \int_0^{\tau} e^{\lambda\mp\tau'} [\langle \alpha_{in\pm}(t)f^*(t+\tau') \rangle \pm \langle \alpha_{in\pm}(t)f(t+\tau') \rangle] d\tau', \end{aligned} \quad (3.48)$$

so that with the help of (2.38) and (3.39b), we have

$$\begin{aligned} \langle \alpha_{in\pm}(t)\alpha_{\pm}(t+\tau) \rangle &= \frac{e^{-\lambda\mp\tau}}{\sqrt{\kappa}} \langle f_R^*(t)\alpha_{\pm}(0) \rangle e^{-\lambda\mp t} \pm \frac{e^{-\lambda\mp\tau}}{\sqrt{\kappa}} \langle f_R(t)\alpha_{\pm}(0) \rangle \\ &+ \frac{e^{-\lambda\mp\tau}}{\sqrt{\kappa}} \int_0^t e^{-\lambda\mp(t-t')} [\langle f_R^*(t)f^*(t') \rangle \pm \langle f_R^*(t)f(t') \rangle] dt' \\ &+ \frac{e^{-\lambda\mp\tau}}{\sqrt{\kappa}} \int_0^{\tau} e^{\lambda\mp\tau'} [\langle f_R^*(t)f^*(t+\tau') \rangle \pm \langle f_R(t)f^*(t+\tau') \rangle \\ &\quad \pm \langle f_R^*(t)f(t+\tau') \rangle + \langle f_R(t)f(t+\tau') \rangle] d\tau'. \end{aligned} \quad (3.49)$$

Assuming the noise force at time t does not affect the cavity mode variable at an earlier times and taking into account the fact that

$$\langle f_R(t) \rangle = 0, \quad (3.50)$$

we see that

$$\langle \alpha_{\pm}(0)f_R(t) \rangle = \langle \alpha_{\pm}(0) \rangle \langle f_R(t) \rangle = 0. \quad (3.51)$$

On account of this result, Eq. (3.49) reduces to

$$\begin{aligned} \langle \alpha_{in\pm}(t)\alpha_{\pm}(t+\tau) \rangle &= \frac{e^{-\lambda\mp\tau}}{\sqrt{\kappa}} \int_0^t e^{-\lambda\mp(t-t')} [\langle f_R^*(t)f^*(t') \rangle \pm \langle f_R^*(t)f(t') \rangle] dt' \\ &+ \frac{e^{-\lambda\mp\tau}}{\sqrt{\kappa}} \int_0^{\tau} e^{\lambda\mp\tau'} [\langle f_R^*(t)f^*(t+\tau') \rangle \pm \langle f_R(t)f^*(t+\tau') \rangle \\ &\quad \pm \langle f_R^*(t)f(t+\tau') \rangle + \langle f_R(t)f(t+\tau') \rangle] d\tau'. \end{aligned} \quad (3.52)$$

The noise force is a sum of the system and reservoir noise forces:

$$f(t) = f_R(t) + f_S(t), \quad (3.53)$$

where $f_R(t)$ and $f_S(t)$ are the reservoir and system noise forces. With the aid of this, (3.52) can be rewritten as

$$\begin{aligned}
\langle \alpha_{in\pm}(t)\alpha_{\pm}(t+\tau) \rangle &= \frac{e^{-\lambda\mp\tau}}{\sqrt{\kappa}} \int_0^t e^{-\lambda\mp(t-t')} [\langle f_R^*(t)f_R^*(t') \rangle + \langle f_R^*(t)f_S^*(t') \rangle \\
&\quad \pm \langle f_R^*(t)f_R(t') \rangle \pm \langle f_R^*(t)f_S(t') \rangle] dt' \\
&+ \frac{e^{-\lambda\mp\tau}}{\sqrt{\kappa}} \int_0^\tau e^{\lambda\mp\tau'} [\langle f_R^*(t)f_R^*(t+\tau') \rangle + \langle f_R^*(t)f_S^*(t+\tau') \rangle \\
&\quad \pm \langle f_R(t)f_R^*(t+\tau') \rangle \pm \langle f_R(t)f_S^*(t+\tau') \rangle \\
&\quad \pm \langle f_R^*(t)f_R(t+\tau') \rangle \pm \langle f_R^*(t)f_S(t+\tau') \rangle \\
&\quad + \langle f_R(t)f_R(t+\tau') \rangle + \langle f_R(t)f_S(t+\tau') \rangle] d\tau'. \tag{3.54}
\end{aligned}$$

With the aid of (3.53), Eq. (2.32) can be rewritten as

$$\langle f_R(t)f_R(t') \rangle + \langle f_R(t)f_S(t') \rangle + \langle f_S(t)f_R(t') \rangle + \langle f_S(t)f_S(t') \rangle = -2v\delta(t-t'), \tag{3.55a}$$

in which

$$\begin{aligned}
v &= \frac{A}{2B} \left[-\rho_{aa}^{(0)} \left(\frac{\Omega}{\gamma} \right) \left(1 + \frac{\Omega^2}{4\gamma^2} \right) + \rho_{cc}^{(0)} \left(\frac{\Omega}{2\gamma} \right) \left(1 - \frac{\Omega^2}{2\gamma^2} \right) \right. \\
&\quad \left. - \rho_{ac}^{(0)} \left(1 - \frac{\Omega^2}{2\gamma^2} \right) \right] - \frac{\kappa}{2} M, \tag{3.550b}
\end{aligned}$$

Since the reservoir and the system noise forces are uncorrelated, we see that

$$\langle f_R(t)f_S(t') \rangle = \langle f_R(t) \rangle \langle f_S(t') \rangle = 0, \tag{3.56a}$$

$$\langle f_S(t)f_R(t') \rangle = \langle f_S(t) \rangle \langle f_R(t') \rangle = 0. \tag{3.56b}$$

Thus Eq. (3.55a) reduces to

$$\langle f_R(t)f_R(t') \rangle + \langle f_S(t)f_S(t') \rangle = -2v\delta(t-t'), \tag{3.57a}$$

from which follows

$$\langle f_R(t)f_R(t') \rangle = \kappa M \delta(t-t'), \tag{3.57b}$$

$$\begin{aligned} \langle f_S(t)f_S(t') \rangle = \frac{A}{2B} \left[-\rho_{aa}^{(0)} \left(\frac{\Omega}{\gamma} \right) \left(1 + \frac{\Omega^2}{4\gamma^2} \right) + \rho_{cc}^{(0)} \left(\frac{\Omega}{2\gamma} \right) \left(1 - \frac{\Omega^2}{2\gamma^2} \right) \right. \\ \left. - \rho_{ac}^{(0)} \left(1 - \frac{\Omega^2}{2\gamma^2} \right) \right] \delta(t-t'). \end{aligned} \quad (3.57c)$$

Moreover employing (3.53), Eq. (2.35) can be rewritten as

$$\langle f_R(t')f_R^*(t) \rangle + \langle f_R(t')f_S^*(t) \rangle + \langle f_S(t')f_R^*(t) \rangle + \langle f_S(t')f_S^*(t) \rangle = 2p\delta(t-t'), \quad (3.58a)$$

in which

$$p = \frac{A}{2B} \left[\rho_{aa}^{(0)} \left(1 + \frac{\Omega^2}{4\gamma^2} \right) + \rho_{cc}^{(0)} \left(\frac{3\Omega^2}{4\gamma^2} \right) - \rho_{ac}^{(0)} \left(\frac{3\Omega}{2\gamma} \right) \right] + \frac{\kappa}{2}N. \quad (3.58b)$$

Using the fact that

$$\langle f_R(t')f_S^*(t) \rangle = \langle f_S(t')f_R^*(t) \rangle = 0, \quad (3.59a)$$

we observe that

$$\langle f_R(t')f_R^*(t) \rangle = \kappa N \delta(t-t'), \quad (3.59b)$$

$$\langle f_S(t')f_S^*(t) \rangle = \frac{A}{2B} \left[\rho_{aa}^{(0)} \left(1 + \frac{\Omega^2}{4\gamma^2} \right) + \rho_{cc}^{(0)} \left(\frac{3\Omega^2}{4\gamma^2} \right) - \rho_{ac}^{(0)} \left(\frac{3\Omega}{2\gamma} \right) \right] \delta(t-t'). \quad (3.59c)$$

On account of Eqs. (3.56), (3.57b), (3.59a), and (3.59b), Eq. (3.54) takes the form

$$\langle \alpha_{in\pm}(t)\alpha_{\pm}(t+\tau) \rangle = 2\sqrt{\kappa}(M \pm N)e^{-\lambda \mp \tau}. \quad (3.60)$$

Furthermore, on account of (3.39b), one can write

$$\langle \alpha_{in\pm}(t)\alpha_{in\pm}(t+\tau) \rangle = \frac{1}{\kappa} [\langle f_R^*(t)f_R^*(t+\tau) \rangle \pm \langle f_R^*(t)f_R(t+\tau) \rangle \pm \langle f_R(t)f_R^*(t+\tau) \rangle + \langle f_R(t)f_R(t+\tau) \rangle], \quad (3.61)$$

so that using Eqs. (3.57b) and (3.59b), one gets

$$\langle \alpha_{in\pm}(t)\alpha_{in\pm}(t+\tau) \rangle = 2(M \pm N)\delta(\tau). \quad (3.62)$$

Now with the aid of (3.44b), (3.47), (3.60), and (3.62), the squeezing spectrum can be put in the form

$$\begin{aligned}
S_{\pm}^{out}(\omega) &= 1 \pm 2\kappa Re \int_0^{\infty} \left(\frac{-2v \pm 2p}{\lambda_{\mp}} \right) e^{-\lambda_{\mp}\tau + i\omega\tau} d\tau \\
&\mp 2\sqrt{\kappa} Re \int_0^{\infty} 2\sqrt{\kappa}(M \pm N) e^{-\lambda_{\mp}\tau + i\omega\tau} d\tau \\
&\pm 2Re \int_0^{\infty} 2(M \pm N)\delta(\tau) e^{i\omega\tau} d\tau.
\end{aligned} \tag{3.63}$$

Upon performing the integration and taking into account Eqs. (3.14a), (3.14c) and (3.17b), we get

$$S_{\pm}^{out}(\omega) = 1 + 4\kappa(p \mp v) \mp \frac{4\kappa(M \pm N)\lambda_{\mp}}{\omega^2 + \lambda_{\mp}^2} + 2N \pm 2M, \tag{3.64}$$

where

$$\begin{aligned}
(p \mp v) &= \frac{\kappa}{2}(M \pm N) + \frac{A}{4B} \left[(1 + \beta^2)(1 \pm \frac{\beta}{2}) + \eta(\frac{\beta}{2}(\beta \mp 3) - 1) \right] \\
&\quad + \frac{A\sqrt{1-\eta^2}}{4B} \left[-\frac{\beta}{2}(3 \pm \beta) \pm 1 \right],
\end{aligned} \tag{3.65a}$$

$$B = (1 + \beta^2)(1 + \frac{\beta^2}{4}), \tag{3.65b}$$

and

$$\lambda_{\mp} = \frac{\kappa}{2} + \frac{A}{4B} \left[\eta(2 - \beta^2) \mp \beta(1 + \beta^2) + \sqrt{1 - \eta^2}(3\beta) \right]. \tag{3.65c}$$

Setting $\eta = 0$ and taking into account Eqs. (2.8f) and (2.8g), we have

$$S_{+}^{out} = e^{2r} + \frac{4\kappa(p - v) - 2\kappa(e^{2r} - 1)\lambda_{-}}{\omega^2 + \lambda_{-}^2}, \tag{3.66a}$$

and

$$S_{-}^{out} = e^{-2r} + \frac{4\kappa(p + v) + 2\kappa(1 - e^{-2r})\lambda_{+}}{\omega^2 + \lambda_{+}^2}, \tag{3.66b}$$

where

$$p - v = \frac{\kappa}{4}(e^{2r} - 1) + \frac{A(\beta^3 + \beta^2 - 2\beta + 4)}{(1 + \beta^2)(2 + \frac{\beta^2}{2})}, \tag{3.67a}$$

$$p + v = \frac{\kappa}{4}(e^{-2r} - 1) - \frac{A(\beta^3 - 3\beta^2 + 4\beta)}{(1 + \beta^2)(2 + \frac{\beta^2}{2})}, \tag{3.67b}$$

$$\lambda_- = \frac{\kappa}{2} \left(1 + \frac{A\beta(2 - \beta^2)}{2\kappa(1 + \beta^2)(1 + \frac{\beta^2}{4})} \right), \quad (3.67c)$$

$$\lambda_+ = \frac{\kappa}{2} \left(1 + \frac{2A\beta}{\kappa(1 + \beta^2)} \right). \quad (3.67d)$$

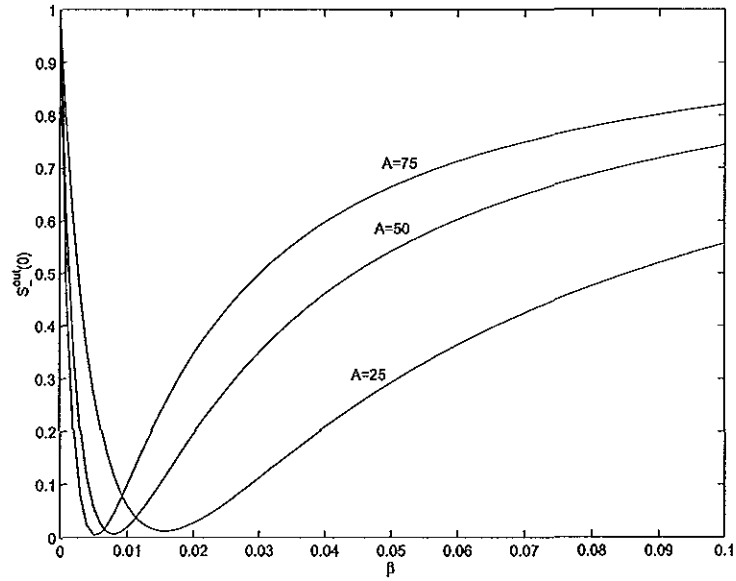


Fig. 3.7 Plots of squeezing spectrum versus β for $r = 0$, $\kappa = 0.8$, $\eta = 0$, $\omega = 0$ and for different values of the linear gain coefficient .

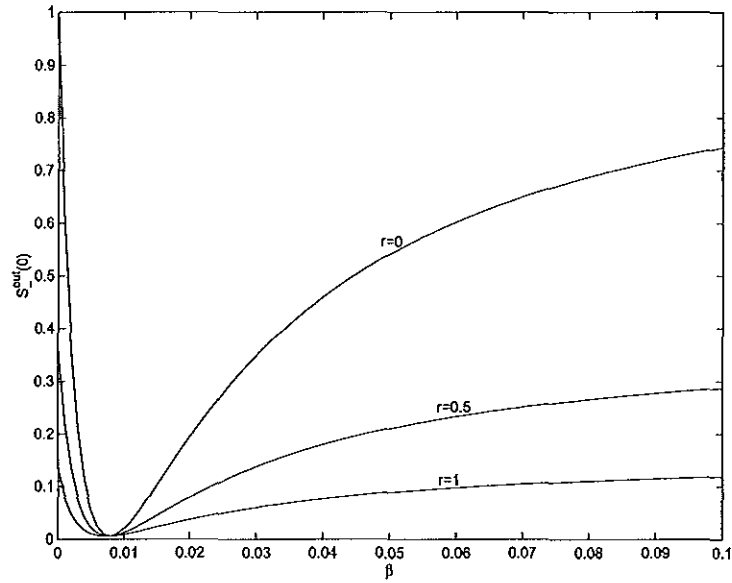


Fig. 3.8 Plots of squeezing spectrum versus β for $A = 50$, $\kappa = 0.8$, $\omega = 0$, $\eta = 0$ and for different values of the squeeze parameter .

From Fig. (3.7) or (3.8) we note that the degree of squeezing of the output mode increases with linear gain coefficient A or the squeeze parameter r . In addition, both figures show that there is almost perfect squeezing for small value of β .

4. PHOTON STATISTICS

In this section we wish to calculate the mean and the variance of the photon number for the cavity mode under consideration.

4.1 The Mean Photon Number

The mean photon number can be expressed in terms of c-number variable associated with the normal ordering as

$$\langle \hat{n} \rangle = \langle \alpha^*(t)\alpha(t) \rangle. \quad (4.1)$$

Making use of Eq. (2.39) along with its complex conjugate, we obtain

$$\begin{aligned} \langle \alpha^*(t)\alpha(t) \rangle &= A^2(t)\langle \alpha^*(0)\alpha(0) \rangle + B^2(t)\langle \alpha(0)\alpha(0)^* \rangle \\ &\quad + A(t)B(t)[\langle \alpha^2(0) \rangle + \langle \alpha^{*2}(0) \rangle] \\ &\quad + A(t)[\langle \alpha^*(0)F(t) \rangle + \langle F^*(t)\alpha(0) \rangle] \\ &\quad + B(t)[\langle \alpha(0)F(t) \rangle + \langle F^*(t)\alpha^*(0) \rangle] \\ &\quad + \langle F^*(t)F(t) \rangle, \end{aligned} \quad (4.2)$$

where $A(t)$, $B(t)$, and $F(t)$ are given by Eq. (2.40). Employing Eq. (2.30) together with the fact that the cavity mode is initially in a vacuum state, one obtains

$$\langle \alpha^*(t)\alpha(t) \rangle = \langle F^*(t)F(t) \rangle. \quad (4.3)$$

On account of Eqs. (2.40c) and (2.40d), we can write

$$\begin{aligned}
\langle F^*(t)F(t) \rangle &= \frac{1}{4}e^{-2\lambda_-t} \int_0^t e^{\lambda_-(t'+t'')} [\langle f^*(t'')f(t') \rangle + \langle f^*(t'')f^*(t') \rangle \\
&\quad + \langle f(t'')f(t') \rangle + \langle f(t'')f^*(t') \rangle] dt' dt'' \\
&+ \frac{1}{4}e^{-(\lambda_-+\lambda_+)t} \int_0^t e^{\lambda_-(t'+t'')} [\langle f^*(t'')f(t') \rangle - \langle f^*(t'')f^*(t') \rangle \\
&\quad + \langle f(t'')f(t') \rangle - \langle f(t'')f^*(t') \rangle] dt' dt'' \\
&+ \frac{1}{4}e^{-(\lambda_-+\lambda_+)t} \int_0^t e^{\lambda_-(t'+t'')} [\langle f^*(t'')f(t') \rangle + \langle f^*(t'')f^*(t') \rangle \\
&\quad - \langle f(t'')f(t') \rangle - \langle f(t'')f^*(t') \rangle] dt' dt'' \\
&+ \frac{1}{4}e^{-2\lambda_+t} \int_0^t e^{\lambda_+(t'+t'')} [\langle f^*(t'')f(t') \rangle - \langle f^*(t'')f^*(t') \rangle \\
&\quad - \langle f(t'')f(t') \rangle + \langle f(t'')f^*(t') \rangle] dt' dt''. \tag{4.4}
\end{aligned}$$

Applying Eqs. (2.32) and (2.35), Eq. (4.4) can be put in the form

$$\begin{aligned}
\langle F^*(t)F(t) \rangle &= (p-v)e^{-2\lambda_-t} \int_0^t e^{\lambda_-(t'+t'')} \delta(t''-t') dt' dt'' \\
&\quad + (p-v)e^{-2\lambda_+t} \int_0^t e^{\lambda_+(t'+t'')} \delta(t''-t') dt' dt''. \tag{4.5}
\end{aligned}$$

On performing the integration using the property of Dirac delta function

$$\int_0^t f(t'') \delta(t''-t') dt'' = f(t'), \tag{4.6}$$

one readily finds

$$\langle F^*(t)F(t) \rangle = \left[\frac{p-v}{2\lambda_-} \right] (1 - e^{-2\lambda_-t}) + \left[\frac{p+v}{2\lambda_+} \right] (1 - e^{-2\lambda_+t}). \tag{4.7}$$

With the aid of Eqs. (4.3), (2.37b), (2.17a) and (2.17b), the mean photon number of the cavity mode at steady state turns out to be

$$\bar{n}_{ss} = \frac{p(q-p) - v(u-v)}{(p-q)^2 - (u-v)^2}. \tag{4.8}$$

4.2 The Variance of the Photon Number

The variance of the photon number for the cavity mode is given in the normal order by

$$\Delta n^2(t) = \langle \hat{a}^{\dagger 2}(t) \hat{a}^2(t) \rangle + \langle \hat{a}^\dagger(t) \hat{a}(t) \rangle - \langle \hat{a}^\dagger(t) \hat{a}(t) \rangle^2. \quad (4.9)$$

The c-number equation corresponding to (4.9) is

$$\Delta n^2(t) = \langle \alpha^*(t) \alpha(t) \rangle + \langle \alpha^{*2}(t) \alpha^2(t) \rangle - \langle \alpha^*(t) \alpha(t) \rangle^2. \quad (4.10)$$

Since the stochastic differential equation for $\alpha(t)$ has the form given by Eq. (2.18) and noise operator $f(t)$ has the correlation properties described by Eqs. (2.20), (2.32) and (2.35), α is a Gaussian variable. Therefore one can write [2]

$$\langle \alpha^{*2}(t) \alpha^2(t) \rangle = \langle \alpha^{*2}(t) \rangle \langle \alpha^2(t) \rangle + 2 \langle \alpha^*(t) \alpha(t) \rangle^2, \quad (4.11)$$

so that Eq. (4.10) can be expressed as

$$\Delta n^2(t) = \langle \alpha^{*2}(t) \rangle \langle \alpha^2(t) \rangle + \langle \alpha^*(t) \alpha(t) \rangle^2 + \langle \alpha^*(t) \alpha(t) \rangle. \quad (4.12)$$

With the aid of Eqs. (2.39) and (2.40), one readily obtains

$$\langle \alpha^2(t) \rangle = \left[\frac{p-v}{2\lambda_-} \right] (1 - e^{-2\lambda_- t}) - \left[\frac{p+v}{2\lambda_+} \right] (1 - e^{-2\lambda_+ t}). \quad (4.13)$$

And at steady state

$$\langle \alpha^2(t) \rangle_{ss} = \langle \alpha^{*2}(t) \rangle_{ss} = \frac{p(u-v) - v(q-p)}{(q-p)^2 - (u-v)^2}. \quad (4.14)$$

Hence on account of Eqs. (4.12) together with (4.8) and (4.14), the variance of the photon number for the cavity mode, at steady state, can be expressed as

$$\Delta n_{ss}^2 = \bar{n}_{ss}^2 + \bar{n}_{ss} + \left[\bar{n}_{ss} - \frac{(p+v)}{(q+u) - (p+v)} \right]. \quad (4.15)$$

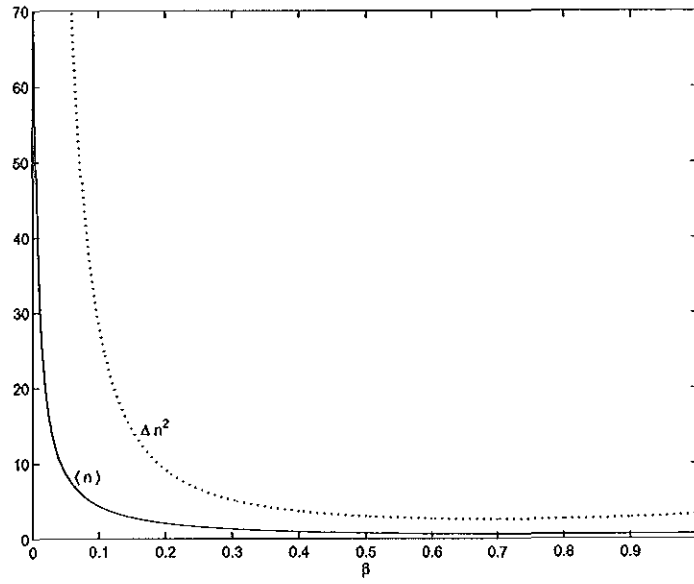


Fig. 3.9 Plots of mean photon number and variance of the photon number versus β for $A = 75$, $\kappa = 0.8$, $r = 0$.

Fig. (3.9) shows that the mean photon number decreases as β increases. We also note that the photon statistics of the cavity mode under consideration is super-Poissonian.

5. CONCLUSION

We have considered a three-level laser coupled to a squeezed vacuum reservoir and in which the top and bottom levels are coupled by a strong coherent light. Employing the master equation, for the cavity mode under consideration, we have obtained stochastic differential equations. Applying the solutions of these equations, we have calculated the quadrature variance and squeezing spectrum. From the plots of the quadrature variance versus η , we have seen that the cavity mode under consideration is in a squeezed state for all values of η between zero and one and the degree of squeezing increases with the linear gain coefficient or the the squeeze parameter. We have also seen from the plots of the quadrature variance versus β , when all the atoms are initially in the upper level or when half of the atoms are initially in the upper level with the remaining half being in the lower level and when these levels are coupled by a strong coherent light, that the cavity mode is in a squeezed state for small values of β and the degree of squeezing increases with the linear gain coefficient or the squeeze parameter.

Moreover, the plots of the squeezing spectrum versus β for $\eta = 0$ show that there is almost perfect squeezing for large values of the linear gain coefficient or the squeeze parameter.

In addition, applying the solutions of the stochastic differential equations, we have calculated the mean photon number and the variance of the photon number for the cavity mode. We see that the photon statistics of the cavity mode is super-Poissonian. Fig. (3.9) shows that the mean photon number decreases as β increases.

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