CROSS-SECTION FOR DOUBLE HEAVY QUARKONIUM PRODUCTION IN HIGH ENERGY ELECTRON-POSITRON COLLISIONS

By

TEFERI MEKONNEN

SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE IN PHYSICS
AT
ADDIS ABABA UNIVERSITY
ADDIS ABABA, ETHIOPIA
JUNE 2010

ADDIS ABABA UNIVERSITY

DEPARTMENT OF

PHYSICS

Supervisor: ______________________________

Dr.Shashank Bhatnagar

Examiners: ______________________________

Dr. Fesseha Kassahun

____

Prof. Vadim Mal'nev

ADDIS ABABA UNIVERSITY

Date: **JUNE 2010**

Author: **TEFERI MEKONNEN**

Title: **Cross-section for Double Heavy Quarkonium Production in high energy electron-positron collisions**

Department: **Physics**

Degree: **M.Sc.** Convocation: **JUNE** Year: **2010**

Permission is herewith granted to Addis Ababa University to circulate and to have copied for non-commercial purposes, at its discretion, the above title upon the request of individuals or institutions.

____

Signature of Author

[0] E-mail:tefemekon@yahoo.com

Table of Contents

List of Figures

Abstract

In this thesis we first give a Lagrangian formulation of the gauge theories QED and QCD. We derive the Feynman rules for QCD . AS an application of the rule, we first present the calculation of cross section for the process q+$\bar{q} \rightarrow$ g+g as a mathematical preliminary. We then derive the 2-particle Bethe-Salpeter equation. Then we evaluate the cross section for the production of double heavy quarkonium production in high energy electron-positron collisions at the center of mass energy $\sqrt{s} \approx 10.6$ GeV. The cross section for the process of $e^+e^- \rightarrow J/\psi + \eta_c$ at the stated energy has been calculated by applying the two-particle Bethe-Salpeter wave equation. We have used the heavy quark limit and calculated the cross section for the lowest order diagram. The results are in broad agreement with the Belle's data available recently.

Acknowledgements

Above all, I would like to thank the almighty; God, for letting me accomplish this work. I am deeply indebted Dr.Shashank Bhatnagar, my advisor, for his useful suggestions and continuous support and friendly approach during this research. I accomplished this work successfully with his help and encouragement. His tireless follow up and his consistent support will be in my memory forever.

I have derived materials from many research journals and books, and am indebted to the authors of those publications and books. Finally I would like to thank my intimate friends who gave me technical support during the process of writing this paper.

Introduction

The study of elementary particles and their interaction has brought about a better understanding of our universe from sub-nuclear scales to cosmological scales. There are four fundamental interactions observed in nature: gravitational, electromagnetic, weak and strong nuclear forces. The electromagnetic interaction is mediated by the exchange of virtual photons. The weak nuclear interaction is responsible for beta decay and the exchanged particles are W$\pm$ and Z_0 . For the strong interaction between quarks, the exchanged particles are gluons.

The theories that describe electromagnetic, weak nuclear and strong nuclear interactions are Quantum Electrodynamics (QED), the theory of Glashow, Weinberg and Salam (GWS theory),and Quantum Chromodynamics (QCD), respectively. It can be seen that fundamental particles have been classified based on their response to these interactions as well as their structure. The two main groups are hadrons and leptons. Hadrons experience all four the fundamental forces and they are composite particles while leptons do not experience the strong force. The particles which mediate the different forces are generally called gauge bosons.

The main focus of this study is the derivation of the cross section of double heavy quarkonium production in high energy in electron-positron collision by using the two-particle Bethe-Salpeter wave equation. The word quarkonium refers to the quark anti-quark bound states. Quarkonium physics is still an interesting research topic while the first quarkonium state, J/ψ, has been discovered for about thirty years. Due to its approximately non-relativistic nature, the description of the heavy quark and anti-quark system is one of the simplest applications of QCD. The interplay of perturbative and non-perturbative Quantum Chromodynamics(QCD) happens in the quarkonium production and decays, which can therefore stand as probes in investigating the non-perturbative nature of QCD.

It is widely expected that the B factories would provide clearer information about quarkonium production. The B-factory experiments recently reported their measurements on the prompt charmonium production at e^+e^- colliders at $\sqrt{s} = 10.6$ GeV [1, 2, 3]. To one's surprise, both their inclusive and exclusive measurements have big discrepancies with theoretical calculations [4, 5, 7]. Among the puzzling features of the B-factory data, in particular, the total cross section of the exclusive

$$e^+ + e^- \rightarrow J/\psi + \eta_c \tag{0.0.1}$$

process is found to be about an order larger than theoretical predictions [7, 8, 9].

The remainder part of this paper is organized as follows. In chapter 1, the high light of quantum electrodynamics is given. Under this chapter both the global and local gauge invariance of the quantum electrodynamics(QED) Lagrangian(L_{QED}) is proved based on $U(1)$ gauge transformation. In chapter 2,a brief explanation of quantum chromodynamics(QCD) including Feynman rules for QCD and invariance of the QCD Lagrangian both under global as well as under local $SU(3)_c$ gauge transformation is given. At the end of this chapter the cross section for quark antiquark pair annihilation into two gluons i.e, for the process

$$q + \bar{q} \rightarrow gg \tag{0.0.2}$$

is calculated by applying the Feynman rules for QCD merely as a mathematical preliminary. In chapter 3, more attention is given to the derivation of Bethe-Salpeter wave equation for two bound particles. In this chapter we also derived the hadron-quark vertex function for quark antiquark bound states which plays a crucial role for the calculation of the cross section in the next chapter. In chapter 4,we calculated the cross section for double heavy quarkonium production in high energy in electron-positron collisions by applying two-particle Bethe-Salpeter wave equation. In the process of the calculation,most of the terms involved in the calculation are derived by following step by step reasonings and by using reasonable approximations. Conclusion and interpretation of our results are given in chapter 5. Finally in the Appendix,relativistic notations,four-vector formalism, Dirac algebra of gamma matrix and trace theorems,and so on are given.

Chapter 1

Quantum Electrodynamics(QED)

Quantum electrodynamics, or QED, [11] is a quantum theory of the interactions of charged particles with the electromagnetic field. It describes mathematically not only all interactions of light with matter but also those of charged particles with one another trough mediation of carrier gauge field. The development of QED theory was essential in the verification and development of quantum field theory and it allows physicists to predict how subatomic particles are created or destroyed. QED is the merger of special relativity quantum mechanics and theory of electromagnetic field. QED is a gauge-invariant theory because its predictions are not affected by variations in space or time. The practical value of QED theory is that it allows physicists to make calculations regarding the absorption and emission of light by atoms,electromagnetic moments, Lamb shift,scattering cross-section, etc. In addition, QED provides very accurate predictions regarding the interactions between photons and charged atomic particles such as electrons. Thus it predictes accurately almost all processes involving charged particles. As quarks, gluons, and other subatomic particles became known, QED became increasingly important in explaining the structure, properties and reactions of these particles. QED, also known as the quantum theory of light, eventually became one of the most precise, accurate, and well tested theories in science. QED predictions of the magnetic moments of some subatomic particles, offer results accurate to eleventh place of decimal.

1.1 Gauge Theory QED

Gauge theory is a field theory in which the fields and potentials are described by a symmetry group (the gauge group). A gauge theory arises when a particular symmetry is imposed on a theory even when the parameter labeling the symmetry is allowed to vary over space-time.

The earliest physical theory to have a gauge symmetry was classical electrodynamics, the theory of electricity and magnetism originated by James Clerk Maxwell. It turns out that electrodynamic theory is invariant under redefinition of the electrostatic potential, which corresponds eventually to the law of conservation of electric charge. The mathematics of gauge theories was not fully developed, however, until the early 20th century by the German mathematician Hermann Weyl. In quantum theory, for example, the phase of the wave functions describing a system can be changed by an arbitrary amount without altering the physical content or structure of the theory, provided that all the wave functions are changed in the same way everywhere in space. To gauge this symmetry the phase change is allowed to vary over space-time. In order to maintain this as a symmetry of the theory, it turns out that the electromagnetic force must be introduced. Thus quantum electrodynamics (QED) is an example of a gauge theory.

In a gauge theory there is a group of transformations of the field variables (gauge transformations) that leaves the basic physics of the quantum field unchanged. For example the Lagrangian,electric field($\vec{E}$),magnetic field($\vec{B}$),electromagnetic field strength tensor $F_{\mu\nu}$ and so on are invariant under gauge transformation, $A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x)$. This property, called gauge invariance, gives the theory a certain symmetry, which governs its equations. The classical theory of the electromagnetic field, proposed by Maxwell in 1864, is the prototype of gauge theories, though the concept of gauge transformation was not fully developed until the early 20th century by Weyl. In Maxwell's theory the basic field variables are the strengths of the electric and magnetic fields, which may be described in terms of auxiliary variables (e.g. the scalar and vector potentials). The gauge transformations in this theory consist of certain alterations in the values of those potentials that do not result in a change of the electric and magnetic fields. This gauge invariance is preserved in the quantum theory of electromagnetism called quantum electrodynamics, or QED. Modern work on gauge theories began with the attempt of Chen Ning Yang and Robert L. Mills (1954) to formulate a gauge theory of the strong interaction. The group of gauge transformations in this theory dealt with the isospin of strongly interacting particles. In the late 1960s Steven Weinberg,Sheldon Glashow, and Abdus Salam developed a gauge theory that treats electromagnetic and weak interactions in a unified manner. This theory, now commonly called the electroweak theory, has had notable success and is widely accepted. During the mid-1970s much work was done toward developing Quantum Chromodynamics (QCD), a gauge theory of strong interactions between quarks. For various theoretical reasons, the concept of gauge invariance seems fundamental, and many physicists believe that the final unification of the fundamental interactions (i.e., gravitational, electromagnetic, strong, and weak) will be achieved by a gauge theory. The Standard Model is also built out of a gauge theory.

1.1.1 U(1)Gauge Theory:

We start from the free Dirac Lagrangian,

$$\pounds_D = -\bar{\psi}(x)(\gamma_\mu \partial_\mu + m)\psi(x) \tag{1.1.1}$$

where $\bar{\psi}(x)$ is the adjoint fermion field $\bar{\psi}(x) = \psi^\dagger(x)\gamma_4$ .This free Dirac Lagrangian is invariant under a global U(1) gauge transformation. That is, the Lagrangian does not change under the transformation

$$\psi(x) \xrightarrow{U(1)} \psi'(x) = e^{(i\theta)}\psi(x) \tag{1.1.2}$$

where θ is an arbitrary real constant. This means that $\partial_\mu(i\theta)$=0. It can be easily checked that the the Lagrangian is invariant i.e,

$$\pounds'_D = \pounds_D. \tag{1.1.3}$$

Since the transformation is global, the kinetic energy term of the free Dirac Lagrangian is invariant as well since the derivative ∂_μ doesn't affect it.
However, the free Dirac Lagrangian is no-longer invariant, if the phase transformation depends on space-time coordinate,i.e under local gauge U(1) transformation where $\theta = \theta(x)$. Under local U(1) gauge transformation the spinor field transformed as

$$\psi(x) \equiv \psi'(x) = e^{i\theta(x)}\psi(x). \tag{1.1.4}$$

We can easily see that under this transformation, the Lagrangian transforms as,

$$\pounds'_D = -\bar{\psi}(x)(i\gamma_\mu\partial_\mu\theta(x))\psi(x) + \pounds_D \tag{1.1.5}$$

which is not invariant. To make the Lagrangian invariant under local U(1) gauge transformation some additional term which cancels $\partial_\mu\theta(x)$ should be added to the Lagrangian.

$$\pounds = -\bar{\psi}(x)(\gamma_\mu\partial_\mu + m)\psi(x) - ieA_\mu(x)\bar{\psi}(x)\gamma_\mu\psi(x) \tag{1.1.6}$$

This can be done by introducing a new spin 1 gauge field $A_\mu(x)$ which transforms as

$$A_\mu(x) \xrightarrow{U(1)} A'_\mu(x) = A_\mu(x) + i\frac{1}{e}\partial_\mu\theta(x) \tag{1.1.7}$$

and also by introducing the covariant derivative

$$D_\mu\psi(x) \equiv [\partial_\mu - ieA_\mu(x)]\psi(x). \tag{1.1.8}$$

Hence, the Lagrangian after transformation is

$$\begin{aligned}
&\pounds' = -\bar{\psi}'(x)(\gamma_\mu D'_\mu + m)\psi'(x) \\
&= -\bar{\psi}(x)e^{-i\theta(x)}[\gamma_\mu(\partial_\mu - ieA_\mu(x) - i\partial_\mu(x)\theta(x)) + m]e^{i\theta(x)}\psi(x) \\
&= -\bar{\psi}(x)(\gamma_\mu\partial_\mu\theta(x))\psi(x) - \bar{\psi}(x)(\gamma_\mu\partial_\mu + m)\psi(x) \\
&+iQ\partial_\mu\theta(x)\bar{\psi}(x)\gamma_\mu\psi(x) - ieA_\mu(x)\bar{\psi}(x)\gamma_\mu\psi(x) \\
&= -\bar{\psi}(x)(\gamma_\mu\partial_\mu + m)\psi(x) - ieA_\mu(x)\bar{\psi}(x)\gamma_\mu\psi(x) \\
&\Rightarrow \pounds' = \pounds_o - ieA_\mu(x)\bar{\psi}(x)\gamma_\mu\psi(x)
\end{aligned} \tag{1.1.9}$$

which is then invariant under local U(1) gauge transformation. In equation (1.1.9) the second term that is

$$\pounds_{int} = -ie\bar{\psi}(x)\gamma_\mu A_\mu(x)\psi(x) \tag{1.1.10}$$

represents the interaction Lagrangian. This is the Lagrangian for the interaction between the electromagnetic field and Dirac particles(matter), where e is the coupling constant that couples matter with electromagnetic field.

It can be checked that to make $\pounds_D$ locally invariant we just have to introduce a covariant derivative

$$D_\mu = \partial_\mu - ieA_\mu \tag{1.1.11}$$

where $\partial_\mu \to D_\mu$ in $\pounds_D$. This adjustment of the derivative operator is called the minimal coupling prescription. The term

$$\bar{\psi}'\gamma_\mu D'_\mu\psi' = \bar{\psi}\gamma_\mu D_\mu\psi \tag{1.1.12}$$

is invariant under a local U(1) transformation.

Thus the gauge principle generates the interaction between the Dirac field and the gauge field $A_\mu(x)$. Note that the corresponding electromagnetic charge $'e'$ is completely arbitrary.

If one wants $A_\mu(x)$ to be a true propagating field, one has to add a gauge-invariant kinetic term

$$\pounds_{em(kin)} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} \tag{1.1.13}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor. Finally,by introducing the gauge term, the invariant form of the QED Lagrangian becomes

$$\pounds_{QED} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} - \bar{\psi}(x)(\gamma_\mu\partial_\mu + m)\psi(x) - ieA_\mu(x)\bar{\psi}(x)\gamma_\mu\psi(x) - \frac{1}{2}(\partial_\mu A_\mu(x))^2 \tag{1.1.14}$$

where

$$\pounds_{kin} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} \tag{1.1.15}$$

is the electromagnetic(kinetic) Lagrangian.

$$\pounds_{Dirac} = -\bar{\psi}(x)(\gamma_\mu \partial_\mu + m)\psi(x) \tag{1.1.16}$$

is the free fermion Dirac Lagrangian and

$$\pounds_{int} = -ieA_\mu(x)\bar{\psi}(x)\gamma_\mu\psi(x) \tag{1.1.17}$$

is the interaction Lagrangian and $'e'$ signifies the coupling constant between electromagnetic field and matter. The total Lagrangian in equation (1.1.9) and gives rise to the well-known Maxwell equations. From a simple gauge-symmetry requirement, we have deduced the right QED Lagrangian, which leads to a very successful quantum field theory. The QED predictions have been tested to a very high accuracy, as exemplified by the electron and muon anomalous magnetic moments $a_l \equiv (g_l$ -2)/2, where $\mu_l \equiv g_l(e\hbar/2m_l)$[12]:

$$\begin{aligned} a_e &= (115965214.0 \pm 2.8) \times 10^{-11} \text{ (Theory)} \\ &= (115965219.3 \pm 1.0) \times 10^{-11} \text{ (Experiment)} \end{aligned} \tag{1.1.18}$$

$$\begin{aligned} a_\mu &= (1165919.2 \pm 1.9) \times 10^{-9} \text{ (Theory)} \\ &= (1165923.0 \pm 8.4) \times 10^{-9} \text{ (Experiment)} \end{aligned} \tag{1.1.19}$$

Thus we see that imposition of local gauge invariance on free Dirac Lagrangian leads automatically to introduction of a certain gauge field $A_\mu(x)$ which transforms as $A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu\Lambda(x)$ which would couple to free Dirac field in a mathematically self consistent manner where the charge $'e'$ now plays the role of coupling constant between field and matter.

Thus this provides an example of how fundamental interactions are generated by imposition of local gauge symmetry on a Lagrangian. We now extend this idea to quark interactions in the next chapter.

Chapter 2

Quantum Chromodynamics(QCD)

QCD is a non-Abelian quantum gauge field theory for the strong interaction which governs the structure and interactions of nucleons ,and is a generalization of quantum electrodynamics (QED) to non-abelian gauge group SU(3). In QCD, the carriers of the strong force are massless, electrically neutral but colored particles called gluons. The strength of the interaction is defined by the strong version of the fine structure constant,called the strong coupling constant α_s. However, the coupling is not a constant,and runs with momentum transfer Q^2. The analogous quantity to electric charge is the strong charge or color to strong interactions is what electric charge is to electromagnetic interactions. There are three different colors and their corresponding anticolors. The name is inspired by the fact that, in addition to the color-anticolor combination, the combination of all three colors also results in a color neutral object. In contrast to QED, where the carriers of the field are electrically neutral, gluons themselves carry color. The theory predicts that forces between particles are very weak at short distances, a phenomenon known as asymptotic freedom while increasing roughly linearly with distance at large separations. As a result, only colorless objects are thought to be observable in isolation. This theory postulates that quarks carry a color charge which comes in three varieties: red, blue, and green. The interaction between these color charges is responsible for binding quarks into colorless states such as hadrons, which are composed of three quarks each having a

distinct color. Such states are baryons. Mesons, which are composed of a quark and an antiquark,one carrying a certain color while the other carries the appropriate anticolor. The quanta of the chromodynamic force are called gluons. The QCD interaction is so strong that it is thought to be impossible to liberate individual quarks from a bound state. This hypothesis is called confinement.
Quantum chromodynamics(QCD) is a non-Abelian gauge theory with gauge group SU(3), where the gauge field is coupled to fermions known as quarks.

Quarks are fundamental constituents of hadronic bound state systems. Quarks are spin-$\frac{1}{2}$particles and they are thought to carry a fractional electronic charge of $\pm\frac{2}{3}$ or $\pm\frac{1}{3}$. In addition to electric charge, quarks are thought to carry color charge. The interaction of quarks via their color charge is thought to be so strong that quarks are permanently confined in mesons($q\bar{q}$) or baryons(qqq) or antibaryons($\bar{q}\bar{q}\bar{q}$). Quarks come in three families or generations. In the first generation there are the up(u) quark and down (d) quark;in the second generation are the charmed(c) and strange quark(s); in the third generation are the top(t) and bottom(b) quark. Each generation is similar to the preceding one except for being more massive. It is unknown if there are more families or generations of quarks beyond those given above. The quarks transform in the fundamental representation of the gauge group, which is three-dimensional, and the gauge charge carried by the quarks is called colour. The number of colours is Nc = 3, and a quark can have any of three colours, usually called red(R),green(G) and blue(B). Here when we say color it does not have any relation with the ordinary definition of the color that we already know. The gauge sector consists of gluons which are massless gauge bosons that transform to one of the eight generators $\lambda_1 ... \lambda_8$ of the $S(U)_3$ gauge group. There are therefore N_c^2-1=8 gluons each corresponding to the eight generators of the SU(3) gauge group.

Thus in the theory of Quantum Chromodynamics (QCD), the interactions between quarks are mediated by eight massless vector bosons called gluons. However, a number of complications effectively prevent the properties of hadrons to be predicted from QCD. First of all, the theory is nonlinear due to gluon self-interactions, and it describes systems that interact strongly enough so that perturbative methods are inapplicable at low energies. Only at the very highest energy scales, where quarks become asymptotically free and the coupling between them small, can the predictions of perturbative QCD be compared with experimental results. At low energies, the quarks interact strongly, are confined into hadronic bound states and acquire effective masses. These constituent quark masses are for the light u, d quarks of the order ∼300 MeV. At present, the only way to analyze QCD at a fundamental level is the method of "lattice QCD" simulations, where the properties of hadrons are probed by means of numerical Monte Carlo algorithms. Although much progress is being made in the development of more efficient algorithms and the inclusion of dynamical fermions (unquenched lattice QCD) into the simulations, the applicability of such methods is still limited by the huge demands on computing power. In such a situation, it is natural to attempt to understand the properties of hadrons by means of effective theories and phenomenological QCD-motivated models. The physical motivation of such an approach is that the fundamental degrees of freedom of QCD are quarks and gluons, whereas low-energy experiments observe hadrons, which at least at long range interact by Yukawa-type meson exchange. It is, therefore, a reasonable expectation that the low-energy properties of QCD can be described in terms of an effective theory.

2.1 Quantum Chromodynamics:SU_3 gauge theory

We denote a quark field of color α and flavour f by ψ_f^α [10, 12, 13, 18]. The free Lagrangian of the quantum chromodynamics can be written as

$$\pounds_o = -\sum_f \bar{\psi}_f^\alpha(\gamma_\mu \partial_\mu + m_f)\psi_f^\alpha. \tag{2.1.1}$$

where

$$\psi_f^\alpha = \begin{pmatrix} \psi_f^R \\ \psi_f^B \\ \psi_f^G \end{pmatrix} \tag{2.1.2}$$

the flavour index f = 1,2,...6 corresponds to the up(u), down(d), strange(s), charm(c), bottom(b) and top(t) quarks in order of increasing mass. And

$$\bar{\psi}_f^\alpha = (\psi_f^\alpha)^\dagger \gamma_4 \tag{2.1.3}$$

is the adjoint quark field. The quark masses vary quite drastically from the very light up and down quarks that have masses of the order of a few MeV, to the very heavy top quark, whose mass is around 175 GeV, i.e., almost 200 times heavier than the proton.

The free Dirac Lagrangian in equation (2.1.1) is invariant under an arbitrary global $SU(3)_c$ transformation of the quark field. The transformation of the field is

$$\psi_f^\alpha \longrightarrow (\psi_f^\alpha)' = U_\beta^\alpha \psi_f^\beta \tag{2.1.4}$$

Since, ψ_f^α is a 3×1 column matrix. U should be a 3×3 unitary matrix ,that is

$$U^\dagger U = UU^\dagger = 1\text{and det(U)} = 1 \tag{2.1.5}$$

Any unitary 3×3 matrix U can be expressed as

$$U = e^{iH} \tag{2.1.6}$$

where H is a 3×3 Hermitian matrix which can be expressed as a linear superposition of a unit matrix *1* and eight 3×3 Gellman-matrices $\lambda_1 ... \lambda_8$. Thus we can write it as

$$H = \mathbf{1}\Theta + \theta_a \lambda_a \tag{2.1.7}$$

where Θ and θ_a are arbitrary parameters.

Now consider an SU(3) global gauge transformation on $\psi(x)$ i.e

$$\psi_f^\alpha \longrightarrow (\psi_f^\alpha)' = U_\beta^\alpha \psi_f^\beta \text{ and}$$
$$U = exp(-ig_s\frac{\lambda^a}{2}\theta_a) \tag{2.1.8}$$

where $\lambda^a(a = 1, 2, ..., 8)$ represents the generators $SU(3)_c$ algebra, g_s is the strong coupling constant and θ_a are arbitrary parameters. The matrices λ^a are traceless and satisfy the commutation relation

$$[\lambda^a, \lambda^b] = 2if^{abc}\lambda^c \tag{2.1.9}$$

where f^{abc} is the structure constant of $SU(3)_c$, which are real and totally antisymmetric.

To make the Lagrangian invariant under local $SU(3)_c$ gauge transformation the derivative has to be replaced by the covariant derivative as

$$D_\mu\psi_f(x) \equiv [\partial_\mu - ig_s\frac{\lambda^a}{2}A_\mu(x)^a]\psi_f(x) \equiv [\partial_\mu - ig_sA_\mu(x)]\psi_f(x) \tag{2.1.10}$$

where

$$A_\mu(x) = \frac{\lambda^a}{2}A_\mu(x)^a. \tag{2.1.11}$$

The covariant derivative is transformed as

$$\begin{aligned}
&D_\mu\psi_f(x) \longrightarrow (D_\mu\psi_f(x))' = U(D_\mu\psi_f(x)) \\
&\Rightarrow [\partial_\mu - ig_sA_\mu(x)']U\psi_f(x) = U[\partial_\mu - ig_sA_\mu(x)]\psi_f(x) \\
&\Rightarrow \partial_\mu U - ig_sA_\mu(x)'U = -ig_sUA_\mu(x) \\
&\Rightarrow A_\mu(x)u' = UA_\mu(x)U^\dagger - \frac{i}{g_s}(\partial_\mu U)U^\dagger.
\end{aligned} \tag{2.1.12}$$

Here the non-abelianess of the SU(3) matrices makes the gauge transformation of the gluon fields more complicated than the U(1) gauge transformation in QED which is abelian. The corresponding field strengths in QCD is given by

$$G_{\mu\nu}(x) \equiv \frac{i}{g_s}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig_s[A_\mu, A_\nu] \equiv \frac{\lambda^a}{2}G^a_{\mu\nu}(x) \tag{2.1.13}$$

where

$$G^a_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu + g_s f^{abc} A^b_\mu A^c_\nu \tag{2.1.14}$$

showing the self coupling between the gauge(gluon) fields in contrast to QED where gauge fields do not self interact. Under local gauge transformation the field strength is transformed as

$$G_{\mu\nu} \xrightarrow{SU(3)_c} (G_{\mu\nu})' = UG_{\mu\nu}U^\dagger. \tag{2.1.15}$$

We can thus show the invariance of free gluon field Lagrangian, $\pounds_G = -\frac{1}{2}Tr(G_{\mu\nu}G_{\mu\nu})$. Its proof can be taken as follows

$$\begin{aligned} Tr(G'_{\mu\nu}G'_{\mu\nu}) = \quad & Tr(G'_{\mu\nu}U^\dagger UG'_{\mu\nu}U^{\dagger)} \\ & = Tr(G_{\mu\nu}G_{\mu\nu}U^\dagger U) \\ & = Tr(G_{\mu\nu}G_{\mu\nu}) \end{aligned} \tag{2.1.16}$$

Substituting for $G_{\mu\nu}$ from equation (2.1.13) into equation (2.1.16) we obtain

$$Tr(G_{\mu\nu}G_{\mu\nu}) = \frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu}Tr(\lambda^a\lambda^a). \tag{2.1.17}$$

But

$$Tr(\lambda^a\lambda^a) = 2. \tag{2.1.18}$$

Then by substituting from equation (2.1.18) into equation (2.1.17) we see that the trace is invariant. That is

$$Tr(G'_{\mu\nu}G'_{\mu\nu}) = Tr(G_{\mu\nu}G_{\mu\nu}) = \frac{1}{2}G^a_{\mu\nu}G^a_{\mu\nu}. \tag{2.1.19}$$

Thus the gauge field part of the the Lagrangian is invariant under local SU(3) transformation. By using the covariant derivative, the local invariance of the full QCD Lagrangian can be proved easily as follows.

$$\begin{aligned}
\pounds'_{QCD} = \quad & -\sum_f \bar{\psi}'(\gamma_\mu D'_\mu + m)\psi' - \frac{1}{2}Tr(G'_{\mu\nu}G'_{\mu\nu}) \\
& = -\sum_f \bar{\psi}U^\dagger U(\gamma_\mu D_\mu + m_f)U^\dagger U\psi - \frac{1}{2}Tr(G_{\mu\nu}G_{\mu\nu}) \\
& = -\sum_f \bar{\psi}(\gamma_\mu D_\mu + m_f)\psi - \frac{1}{2}Tr(G_{\mu\nu}G_{\mu\nu}).
\end{aligned} \tag{2.1.20}$$

Decomposing the Lagrangian into its different components it can be written as

$$\begin{aligned}
\pounds_{QCD} = \quad & -\sum_f \bar{\psi}(\gamma_\mu D_\mu + m_f)\psi - \frac{1}{2}Tr(\frac{\lambda^a}{2}G^a_{\mu\nu}\frac{\lambda^a}{2}G^a_{\mu\nu}) \\
& = -\sum_f \bar{\psi}(\gamma_\mu D_\mu + m_f)\psi - \frac{1}{8}G^a_{\mu\nu}G^a_{\mu\nu}Tr(\lambda^a\lambda^a) \\
& = -\sum_f \bar{\psi}(\gamma_\mu D_\mu + m_f)\psi - \frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu}.
\end{aligned} \tag{2.1.21}$$

But

$$G^a_{\mu\nu} = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + g_s f^{abc}A^b_\mu A^c_\nu. \tag{2.1.22}$$

Then inserting for $G^a_{\mu\nu}$ from equation (2.1.22) into equation (2.1.21) the Lagrangian becomes

$$\begin{aligned}
\pounds_{QCD} = \quad & -\sum_f \bar{\psi}(\gamma_\mu D_\mu + m_f)\psi - \frac{1}{4}[(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) + g_s A^b_\mu A^c_\nu f^{abc}] \\
& \times[(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) + g_s A^d_\mu A^e_\nu f^{ade}] \\
& = -\sum_f \bar{\psi}(\gamma_\mu D_\mu + m_f)\psi - \frac{1}{4}[(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + g_s A^b_\mu A^c_\nu f^{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) \\
& +g_s f^{ade}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)A^d_\mu A^e_\nu + g_s^2 f^{abc}f^{ade}A^a_\mu A^c_\nu A^d_\mu A^e_\nu]
\end{aligned} \tag{2.1.23}$$

Substituting for D_μ from equation (2.1.10) into equation (2.1.23) we obtain the gauge invariant Lagrangian of QCD as

$$\begin{aligned}
& \pounds_{QCD} = -\sum_f \bar{\psi}(\gamma_\mu \partial_\mu + m_f)\psi + ig_s\sum_f \bar{\psi}A^a_\mu \frac{\lambda^a}{2}\psi - \frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 \\
& -\frac{g_s}{4}A^b_\mu A^c_\nu f^{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) - \frac{g_s}{4}f^{ade}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)A^d_\mu A^e_\nu \\
& -\frac{g_s^2}{4}f^{abc}f^{ade}A^a_\mu A^c_\nu A^d_\mu A^e_\nu.
\end{aligned} \tag{2.1.24}$$

In equation (2.1.24) the first term that is

$$-\sum_f \bar{\psi}(x)(\gamma_\mu \partial_\mu + m_f)\psi(x) = \pounds_D \tag{2.1.25}$$

represents the Dirac Lagrangian.

The second term

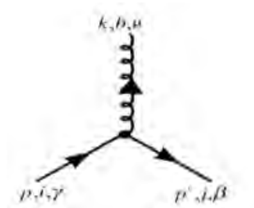


Figure 2.1: Diagram for the interaction Lagrangian

$$ig_s \sum_f \bar{\psi} A_\mu^a \frac{\lambda^a}{2} \psi = \pounds_{\text{int}} \tag{2.1.26}$$

represents the interaction Lagrangian.

The third term that is

$$-\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 \tag{2.1.27}$$

represents the free propagator.

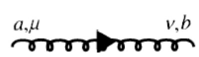


Figure 2.2: Free propagator for gluon

The fourth and fifth terms

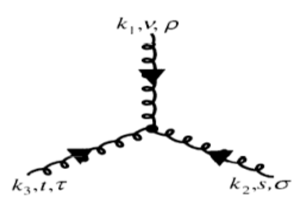


Figure 2.3: Self interaction among three gluons.

$$-\frac{g_s}{4} A_\mu^b A_\nu^c f^{abc}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a), -\frac{g_s}{4} f^{ade}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) A_\mu^d A_\nu^e. \tag{2.1.28}$$

represent the self interaction among three gluons.

Finally the last term that is

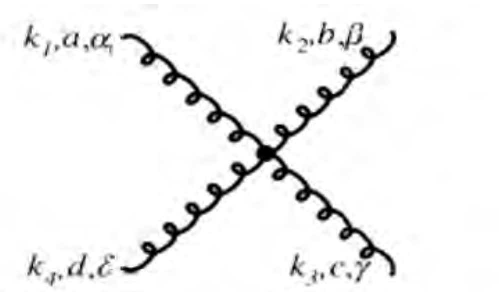


Figure 2.4: Diagram for self interaction among four gluons.

$$-\frac{g_s^2}{4} f^{abc} f^{ade} A_\mu^a A_\nu^c A_\mu^d A_\nu^e \tag{2.1.29}$$

represents the self interaction among four gluons. The strength of these interactions is given by the same coupling g_s which appears in the fermionic piece of the Lagrangian. The existence of self interactions among the gauge fields is a new feature that was not present in the QED case. It seems then reasonable to expect that these gauge self-interactions could explain properties like asymptotic freedom and confinement, which do not appear in QED. Now we try to derive the Feynman rules for QCD in the next section.

2.2 Feynman Rules for QCD

The mathematics of quantum field theory have been distilled into a series of operations called the Feynman rules. The rules can be thought of as prescriptions to describe all manner of processes in quantum field theory and are embodied in a pictorial form, the famous Feynman diagrams. The ultimate goal of Feynman diagrams is to compute observables like decay width,cross section,amplitudes,and etc,for various physical processes . In quantum theory experimental predictions can be made by calculating the probability amplitude that a process will occur. This remains true in quantum field theory, where we calculate amplitudes for particle interactions such as decays and scattering events. The primary tool used to do such calculations is known as the S matrix. Any given physical process can be considered as a transition from an initial state $|i> =|\alpha(t_0)>$to a final output state we denote by $|f> =|\alpha(t)>$. But here the main point is not , how the initial and final states are represented. The main point is on how the Feynman rules are derived. We first present here the derivation of Feynman rules for QCD.[13, 18]

1. External Lines:

(i).For an external quark with momentum p, spin s, and color c

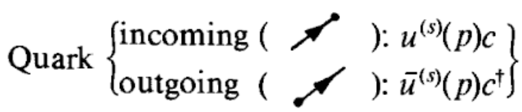


Figure 2.5: Diagram for incoming and outgoing quarks

(ii).For an external anti-quark Where c represents the color of the corresponding quark.

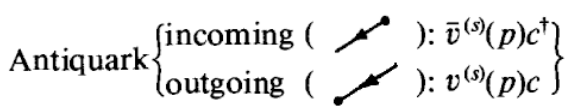


Figure 2.6: Diagram for incoming and outgoing antiquarks

(iii).For an external gluon of momentum p, polarization ϵ, and color a,by including the the color factor $\alpha\mu$ the Feynman rule can be given as

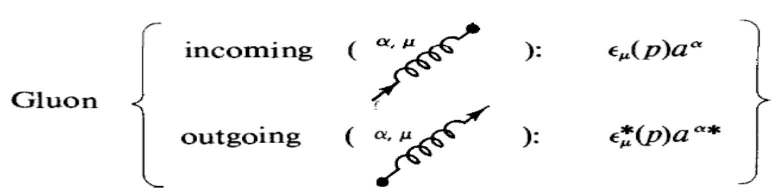


Figure 2.7: Diagram for incoming and outgoing gluons

2.Quark Propagators: The propagator of the the quark and anti quark in momentum representation can be derived by applying Dirac equation to Green's function of Dirac operator as,

$$(\gamma_\mu\partial_\mu + m)S_F(x) = -i\delta^{(4)}(x). \tag{2.2.1}$$

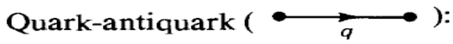


Figure 2.8: Free propagator of quarks

But the Greens function of Dirac operator can also be found by Fourier transformation as

$$S_F(x) = \int \frac{d^4q}{(2\pi)^4} e^{iq.x} S_F(q). \tag{2.2.2}$$

By operating on both sides by Dirac operator we obtain

$$\begin{aligned}
&(\gamma_\mu \partial_\mu + m) S_F(x) = \int \frac{d^4p}{(2\pi)^4} (i\not{q} + m) e^{iq.x} S_F(q) \\
&-i\delta^{(4)}(x) = (i\not{q} + m)\delta^{(4)}(x) S_F(q) \\
&\Rightarrow S_F(q) = \frac{-i}{(i\not{q} + m)} = \frac{-i(-i\not{q} + m)}{q^2 + m^2}.
\end{aligned} \tag{2.2.3}$$

Gluon propagators:

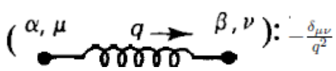


Figure 2.9: Diagram for free gluon propagator

The free gluon propagator can be derived from the free gauge Lagrangian in the QCD Lagrangian in equation (2.1.23) by applying Lagranges equations of motion as

$$\frac{\partial \pounds}{\partial A_\mu^a} - \partial_\nu \left(\frac{\partial \pounds}{\partial(\partial_\nu A_\mu^a)} \right) = 0. \tag{2.2.4}$$

Since in the free Lagrangian there is no A_μ^a the partial derivative with respect to A_μ^a vanishes. That is

$$\frac{\partial \pounds}{\partial A_\mu^a} = 0. \tag{2.2.5}$$

Therefore,

$$\partial_\nu \left(\frac{\partial \pounds}{\partial(\partial_\nu A_\mu^a)} \right) = 0. \tag{2.2.6}$$

Let the free Lagrangian $\pounds_{GF}$ be

$$\pounds_{GF} = [(\partial_\lambda A_\sigma^a)(\partial_\delta) - (\partial_\lambda A_\sigma^a)^2]. \tag{2.2.7}$$

Then equation (2.2.6) can be written as

$$
\begin{aligned}
\partial_\nu\left(\frac{\partial \pounds}{\partial(\partial_\nu A^a_\mu)}\right) = \quad & \partial_\nu[(\partial_\lambda A^a_\sigma) \\
& +(\partial_\sigma A^a_\lambda\delta_{\nu\lambda}\delta\mu\sigma) - 2(\partial_\lambda A^a_\sigma)\delta_{\nu\lambda}\delta_{\mu\sigma}] = 0 \\
& \Rightarrow \Box^2 - \partial_\mu(\partial_\nu A^a_\nu) = 0. \qquad (2.2.8)
\end{aligned}
$$

However, under Lorentz gauge condition

$$\partial_\nu A_\nu = 0. \qquad (2.2.9)$$

This implies that

$$\Box^2 A^2_\mu = 0. \qquad (2.2.10)$$

Taking the Fourier integral transform $A_\mu(x)$ we obtain

$$A_\mu(x) = \int \frac{d^4q}{(2\pi)^4} e^{iq.x} A_\mu(q). \qquad (2.2.11)$$

Substituting for $A_\mu(x)$ from equation (2.2.11) into equation (2.2.10) we will get

$$-q^2\delta_{\mu\nu}A_\mu(q) = 0. \qquad (2.2.12)$$

Now let

$$
\begin{aligned}
c_{\mu\nu} &= -q^2\delta_{\mu\nu} \\
(c^{-1})_{\mu\nu} &= Bq^2\delta_{\mu\nu} \qquad (2.2.13)
\end{aligned}
$$

such that

$$
\begin{aligned}
& c_{\mu\nu}(c^{-1})_{\nu\lambda} = \delta_{\mu\lambda} \\
& -q^2\delta_{\mu\nu}B\delta_{\nu\lambda} = \delta_{\mu\lambda} \\
& \Rightarrow B = \frac{-1}{q^4} \\
& \Rightarrow (c^{-1}) = \frac{-\delta_{\mu\nu}}{q^2} \equiv (D_F(q))_{\mu\nu}. \qquad (2.2.14)
\end{aligned}
$$

So this is the free gluon propagator.

3.Interaction Vertices:

i) The quark-gluon vertex function: The quark-gluon vertex function is the function at the gluon quark interaction point. That is $\frac{-ig_s}{2}\lambda_\alpha\gamma_\mu$.

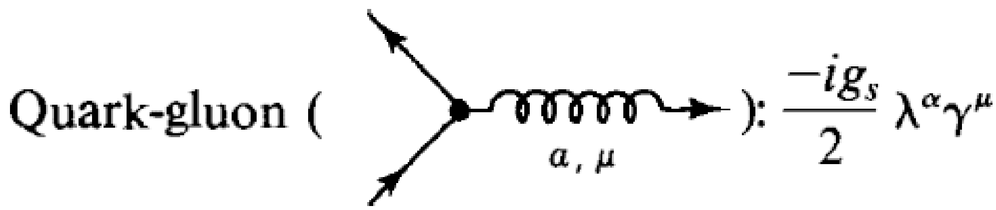


Figure 2.10: Quark-gluon vertex

ii) Three gluon vertex function:

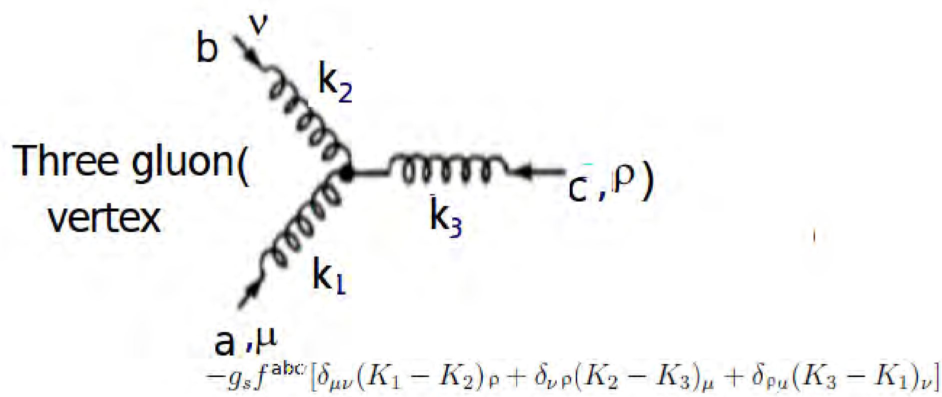


Figure 2.11: Diagram for three gluon interaction

The Feynman rule for the three gluon vertex can be obtained from the fourth and fifth terms in equation (2.1.23) as follows.

$$\begin{aligned} g_s f_{a'b'c'} A^{b'}_\mu A^{c'}_\nu(\partial_\mu A^{a'}_\nu) = \quad & g_s f_{abc} A^b_\mu A^c_\nu(\partial_\mu A^a_\nu) + g_s f_{bca} A^c_\mu A^a_\nu(\partial_\mu A^b_\nu) \\ & + g_s f_{cab} A^a_\mu A^b_\nu(\partial_\mu A^c_\nu) + g_s f_{acb} A^c_\mu A^b_\nu(\partial_\mu A^a_\nu) \\ & + g_s f_{cba} A^b_\mu A^a_\nu(\partial_\mu A^c_\nu) + g_s f_{bac} A^a_\mu A^c_\nu(\partial_\mu A^b_\nu) \qquad (2.2.15) \end{aligned}$$

To complete the above equation lets take those terms corresponding to a have momentum k_1 ,those with b have momentum k_2 and those with c have momentum k_3. And lets take

$$A_\mu(x) = \int \frac{d^4k}{(2\pi)^4} e^{ik.x} A_\mu(k). \qquad (2.2.16)$$

Then by differentiating we can write the above equation as

$$\begin{aligned} g_s f_{a'b'c'} A_\mu^{b'} A_\nu^{c'} (\partial_\mu A_\nu^{a'}) = \quad & g_s f_{abc} A_\mu^b A_\nu^c (ik_{1\mu} A_\nu^a) + g_s f_{bca} A_\mu^c A_\nu^a (ik_{2\mu} A_\nu^b) \\ & + g_s f_{cab} A_\mu^a A_\nu^b (ik_{3\mu} A_\nu^c) + g_s f_{acb} A_\mu^c A_\nu^b (ik_{1\mu} A_\nu^a) \\ & + g_s f_{cba} A_\mu^b A_\nu^a (ik_{3\mu} A_\nu^c) + g_s f_{bac} A_\mu^a A_\nu^c (ik_{2\mu} A_\nu^b). \quad (2.2.17) \end{aligned}$$

In equation (2.2.17) f_{abc} is completely antisymmetric. That is

$$\begin{aligned} f_{abc} &= f_{bca} = f_{cab} \\ f_{acb} &= f_{cba} = f_{bac} = -f_{abc}. \end{aligned} \quad (2.2.18)$$

So by using this antisymmetric relation and by taking the dot product of the terms with the same indices it is possible to write equation (2.2.17) as

$$\begin{aligned} & g_s f_{a'b'c'} A_\mu^{b'} A_\nu^{c'} (\partial_\mu A_\nu^{a'}) = i g_s f_{abc} \left[A^b.k_1)(A^c.A^a) + (A^c.k_2)(A^a.A^b) + (A^a.k_3)(A^b.A^c) \right] \\ & -i g_s f_{abc} \left[(A^c.k_1)(A^b.A^a) + (A^b.k_3)(A^a.A^c) + (A^a.k_2)(A^c.A^b) \right]. \end{aligned} \quad (2.2.19)$$

By simplifying the above equation we can write it in the form of

$$\begin{aligned} & g_s f_{a'b'c'} A_\mu^{b'} A_\nu^{c'} (\partial_\mu A_\nu^{a'}) = i g_s f_{abc} \left[A^b.(k_1 - k_3)(A^c.A^a) \right] \\ & + i g_s f_{abc} \left[A^c.(k_2 - k_1)(A^a.A^b) + A^a.(k_3 - k_2)(A^b.A^c) \right] \end{aligned} \quad (2.2.20)$$

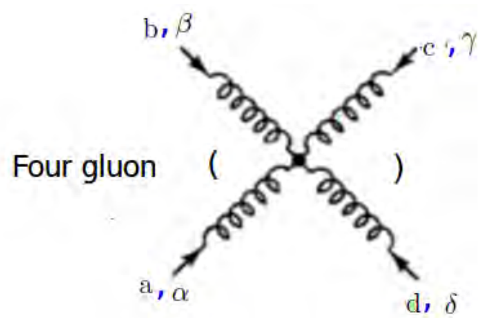
Finally by removing the electromagnetic fields we can write the final vertex function for three interacting gluons as

$$\begin{aligned} & g_s f_{a'b'c'} A_\mu^{b'} A_\nu^{c'} (\partial_\mu A_\nu^{a'}) = i g_s f_{abc} \left[(k_1 - k_3)_\nu \delta_{\mu\rho} \right] \\ & + i g_s f_{abc} \left[(k_2 - k_1)_\rho \delta_{\mu\nu} + (k_3 - k_2)_\mu \delta_{\nu\rho}) \right] \\ & = -i g_s f_{abc} \left[(k_3 - k_1)_\nu \delta_{\mu\rho} + (k_1 - k_2)_\rho \delta_{\mu\nu} + (k_2 - k_3)_\mu \delta_{\nu\rho}) \right]. \end{aligned} \quad (2.2.21)$$

As it is shown in the diagram the gluon momenta k_1,k_2 and k_3 are assumed to point into the vertex. If any of the momentum point outward from the vertex we have to change the sign of the momentum.

iii) Four gluon vertx:

To find all the terms which contribute to the vertex function for the four gluon interaction lets take the fifth term in equation (2.1.23) $f_{nab}f_{ncd} \rightarrow f_{na'b'}f_{nc'd'}$. All the



$$-g_s^2\,[f_{nab}f_{ncd}(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}) + f_{nac}f_{nbd}(\delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha\delta}\delta_{\gamma\beta}) + f_{nad}f_{nbc}(\delta_{\alpha\beta}\delta_{\delta\gamma} - \delta_{\delta\beta}\delta_{\alpha\gamma})]\}$$

Figure 2.12: Diagram for four gluon interaction.

terms which contribute to the vertex function can be found as,

$$\begin{aligned}
& f_{na'b'}f_{nc'd'}A^{a'}_{\mu}A^{b'}_{\nu}A^{c'}_{\mu}A^{d'}_{\nu} = \left[f_{nab}f_{ncd}A^{a}_{\mu}A^{b}_{\nu}A^{c}_{\mu}A^{d}_{\nu}\right] \\
& + \left[f_{nab}f_{ndc}A^{a}_{\mu}A^{b}_{\nu}A^{d}_{\mu}A^{c}_{\nu} + f_{nba}f_{ncd}A^{b}_{\mu}A^{a}_{\nu}A^{c}_{\mu}A^{d}_{\nu}\right] \\
& + \left[f_{nba}f_{ndc}A^{b}_{\mu}A^{a}_{\nu}A^{d}_{\mu}A^{c}_{\nu} + f_{ncb}f_{nad}A^{c}_{\mu}A^{b}_{\nu}A^{a}_{\mu}A^{d}_{\nu}\right] \\
& + \left[f_{nbc}f_{nad}A^{b}_{\mu}A^{c}_{\nu}A^{a}_{\mu}A^{d}_{\nu} + f_{ncb}f_{nda}A^{c}_{\mu}A^{b}_{\nu}A^{d}_{\mu}A^{a}_{\nu}\right] \\
& + \left[f_{nbc}f_{nda}A^{b}_{\mu}A^{c}_{\nu}A^{d}_{\mu}A^{a}_{\nu} + f_{nac}f_{nbd}A^{a}_{\mu}A^{c}_{\nu}A^{b}_{\mu}A^{d}_{\nu}\right] \\
& + \left[f_{nac}f_{ndb}A^{a}_{\mu}A^{c}_{\nu}A^{d}_{\mu}A^{b}_{\nu} + f_{nca}f_{nbd}A^{c}_{\mu}A^{a}_{\nu}A^{b}_{\mu}A^{d}_{\nu}\right] \\
& + \left[f_{nca}f_{ndb}A^{c}_{\mu}A^{a}_{\nu}A^{d}_{\mu}A^{b}_{\nu} + f_{nad}f_{nbc}A^{a}_{\mu}A^{d}_{\nu}A^{b}_{\mu}A^{c}_{\nu}\right] \\
& + \left[f_{nad}f_{ncb}A^{a}_{\mu}A^{d}_{\nu}A^{c}_{\mu}A^{b}_{\nu} + f_{nda}f_{nbc}A^{d}_{\mu}A^{a}_{\nu}A^{b}_{\mu}A^{c}_{\nu}\right] \\
& + \left[f_{nda}f_{ncb}A^{d}_{\mu}A^{a}_{\nu}A^{c}_{\mu}A^{b}_{\nu} + f_{nbd}f_{nac}A^{b}_{\mu}A^{d}_{\nu}A^{a}_{\mu}A^{c}_{\nu}\right] \\
& + \left[f_{ndb}f_{nac}A^{d}_{\mu}A^{b}_{\nu}A^{a}_{\mu}A^{c}_{\nu} + f_{nbd}f_{nca}A^{b}_{\mu}A^{d}_{\nu}A^{c}_{\mu}A^{a}_{\nu}\right] \\
& + \left[f_{ndb}f_{nca}A^{d}_{\mu}A^{b}_{\nu}A^{c}_{\mu}A^{a}_{\nu} + f_{ncd}f_{nab}A^{c}_{\mu}A^{d}_{\nu}A^{a}_{\mu}A^{b}_{\nu}\right] \\
& + \left[f_{ncd}f_{nba}A^{c}_{\mu}A^{d}_{\nu}A^{b}_{\mu}A^{a}_{\nu} + f_{ndc}f_{nab}A^{d}_{\mu}A^{c}_{\nu}A^{a}_{\mu}A^{b}_{\nu}\right] \\
& + f_{nc}f_{na}A^{d}_{\mu}A^{c}_{\nu}A^{b}_{\mu}A^{a}_{\nu}.
\end{aligned} \tag{2.2.22}$$

Using the antisymmetric relation that is

$$f_{nab} = -f_{nba} \ \ \text{and} f_{ncd} = -f_{ndc} \tag{2.2.23}$$

and collecting the same terms together we obtain,

$$\begin{aligned}
f_{na'b'}f_{nc'd'}A_\mu^{a'}A_\nu^{b'}A_\mu^{c'}A_\nu^{d'} = \quad & 2f_{nab}f_{ncd}\left(A_\mu^aA_\nu^bA_\mu^cA_\nu^d - A_\mu^bA_\nu^aA_\mu^cA_\nu^d\right) \\
& +2f_{nab}f_{ncd}\left(A_\mu^bA_\nu^aA_\mu^dA_\nu^c - A_\mu^aA_\nu^bA_\mu^dA_\nu^c\right) \\
& +2f_{nbc}f_{nnad}\left(A_\mu^bA_\nu^cA_\mu^aA_\nu^d - A_\mu^cA_\nu^bA_\mu^aA_\nu^d\right) \\
& +2f_{ncb}f_{nda}\left(A_\mu^cA_\nu^bA_\mu^dA_\nu^a - A_\mu^bA_\nu^cA_\mu^dA_\nu^a\right) \\
& +2f_{nac}f_{nbd}\left(A_\mu^cA_\nu^aA_\mu^dA_\nu^b - A_\mu^cA_\nu^aA_\mu^bA_\nu^d\right) \\
& +2f_{nac}f_{nbd}\left(A_\mu^aA_\nu^cA_\mu^bA_\nu^d - A_\mu^aA_\nu^cA_\mu^dA_\nu^b\right).
\end{aligned} \tag{2.2.24}$$

Taking the dot product of terms with the same indices it is possible to modify equation (2.2.24) as

$$\begin{aligned}
f_{na'b'}f_{nc'd'}A_\mu^{a'}A_\nu^{b'}A_\mu^{c'}A_\nu^{d'} = \quad & 2f_{nab}f_{ncd}\left[(A^a.A^c)(A^b.A^d) - (A^b.A^c)(A^a.A^d)\right] \\
& +2f_{nab}f_{ncd}\left[(A^b.A^d)(A^a.A^c) - (A^a.A^d)(A^b.A^c)\right] \\
& +2f_{nad}f_{nbc}\left[(A^b.A^a)(A^c.A^d) - (A^c.A^a)(A^b.A^d)\right] \\
& +2f_{nad}f_{nbc}\left[(A^c.A^d)(A^b.A^a) - (A^b.A^d)(A^c.A^a)\right] \\
& +2f_{nac}f_{nbd}\left[(A^c.A^d)(A^a.A^b) - (A^c.A^b)(A^a.A^d)\right] \\
& +2f_{nac}f_{nbd}\left[(A^a.A^b)(A^c.A^d) - (A^a.A)(A^b.A^c)\right]
\end{aligned} \tag{2.2.25}$$

By adding the same terms together we will get,

$$\begin{aligned}
f_{na'b'}f_{nc'd'}A_\mu^{a'}A_\nu^{b'}A_\mu^{c'}A_\nu^{d'} = \quad & 4f_{nab}f_{ncd}\left[(A^a.A^c)(A^b.A^d) - (A^b.A^c)(A^a.A^d)\right] \\
& +4f_{nad}f_{nbc}\left[(A^b.A^a)(A^c.A^d) - (A^c.A^a)(A^b.A^d)\right] \\
& +4f_{nac}f_{nbd}\left[(A^a.A^b)(A^c.A^d) - (A^a.A)(A^b.A^c)\right]
\end{aligned} \tag{2.2.26}$$

On inserting this result from equation (2.2.25) into equation (2.1.23) and dropping the gauge field terms, we get the contribution of the four gluon vertices i.e

$$-g_s^2 f_{na'b'} f_{nc'd'} A_\mu^{a'} A_\nu^{b'} A_\mu^{c'} A_\nu^{d'} = -g_s^2 \left[f_{nab} f_{ncd} (\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}) \right]$$
$$-g_s^2 \left[f_{nac} f_{nbd} (\delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha\delta}\delta_{\gamma\beta}) + f_{nad} f_{nbc} (\delta_{\alpha\beta}\delta_{\delta\gamma} - \delta_{\delta\beta}\delta_{\alpha\gamma}) \right]. \tag{2.2.27}$$

This is the vertex function for 4-gluon vertex.

4. Conservation of energy and momentum For each vertex ,we have to write a delta function of the form $(2\pi)^4\delta^4(k_1 + k_2 + k_3)$ where the $k's$ are the three four momenta coming into the vertex (if any arrow leads outward, we have to change the sign of k). This factor enforces the conservation of energy and momentum at each vertex.

5. Integrate over internal momenta

For each internal momentum q, we have to write a factor

$\frac{d^4q}{(2\pi)^2}$.

6. Cancel the delta function. The result includes a factor

$(2\pi)^4\delta^4(p_1 + p_2 + ... - p_n)$

corresponding to the overall energy -momentum conservation.

7. Antisymmetrization. Include a minus sign between diagrams that differ only in the interchange of two incoming (or outgoing quarks) or (antiquarks) or of an incoming quark with an outgoing antiquarks (or vice-versa)[18].

2.3 Quark-antiquark Pair annihilation into two gluons (q+$\bar{q}$ $\rightarrow$ g+g)

As an illustration of Feynman rules for QCD, we calculate the high energy process $q + \bar{q} \rightarrow gg$ merely as a preliminary exercise in QCD. In QED in pair annihilation of electron positron into photons there are only two diagrams which contribute to the calculation of scattering amplitude. This is because in QED there is no self interaction among photons. However, in pair annihilation of quark-antiquark into two gluons in QCD, there are three diagrams which contribute to the calculation of the scattering amplitude in lower order. This is due to the self interaction of gluons with each other. The diagrams which contribute to the amplitude can be given as shown in the following figure[18].

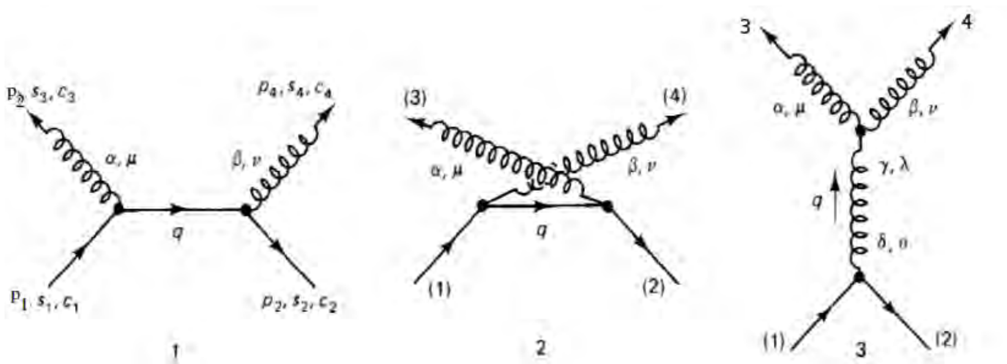


Figure 2.13: Quark-antiquark pair annihilation diagrams.

The covariant matrix element corresponding to the 1^{st} diagram is

$$M_1 = \bar{v}(p_2)c_2^\dagger \left(-ig_s\frac{\lambda_\beta}{2}\gamma_\nu\right)(\epsilon_4^* a_{4\beta}^*)\left(\frac{-i(-i\not{q}+m)}{q^2+m^2}\right) \times \left(-ig_s\frac{\lambda_\alpha}{2}\gamma_\mu\right)(\epsilon_{3\mu}^* a_{3\alpha}^*)\,u(p_1)c \tag{2.3.1}$$

where c represents the color and s represents the spin.

The internal momentum for the internal line(q) is given by

$$q = p_1 - p_3. \tag{2.3.2}$$

The four momentum p is also given by

$$p = (\vec{p}, iE). \tag{2.3.3}$$

Then

$$\begin{aligned} q^2+m^2 = \quad & (p_1-p_3)^2+m^2 = p_1^2+p_3^2-2p_1.p_3+m^2 \\ & = \vec{p}_1^2+\vec{p}_3^2-(E_1^2+E_3^2)-2p_1.p_3+m^2 \\ & = \vec{p}^2+m^2-E^2-2p_1.p_3, \ but\ \vec{p}^2+m^2=E^2 \\ & \Rightarrow q^2+m^2 = -2p_1.p_3. \end{aligned} \tag{2.3.4}$$

By leaving the symbol (*) on gluon polarization vector and color states and by substituting for q^2+m^2 from equation (2.3.4) and for q from equation (2.3.2) into equation (2.3.1) the amplitude M_1 can be expressed as

$$M_1 = \frac{ig_s^2}{8p_1.p_3}\bar{v}(p_2)\left(\not{\epsilon}_4(-i\not{p}_1 + i\not{p}_3 + m)\not{\epsilon}_3\right)u(p_1)a_{3\alpha}a_{4\beta}(c_2^\dagger\lambda_\beta\lambda_\alpha c_1). \tag{2.3.5}$$

Similarily by applying Feynman rules of QCD the amplitude for diagram 2 can be expressed as

$$\begin{aligned} M_2 = \quad & \bar{v}(p_2)c_2^\dagger\left(-ig_s\frac{\lambda_\alpha}{2}\gamma_\mu\right)(\epsilon_3^* a_{3\alpha}^*)\left(\frac{-i(-i\not{q}+m)}{q^2+m^2}\right) \\ & \times\left(-ig_s\frac{\lambda_\beta}{2}\gamma_\nu\right)\left(\epsilon_{4\nu}^* a_{4\beta}^*\right)u(p_1)c_1 \end{aligned} \tag{2.3.6}$$

But here for diagram 2,

$$q = p_1 - p_4. \tag{2.3.7}$$

By following the same procedure as for diagram 1 and making appropriate substitution the amplitude for diagram 2 can be written in a simplified form as

$$M_2 = \frac{ig_s^2}{8p_1.p_4}\bar{v}(p_2)\left(\not{\epsilon}_3(-i\not{p}_1 + i\not{p}_4 + m)\not{\epsilon}_4\right)u(p_1)a_{3\alpha}a_{4\beta}(c_2^\dagger\lambda_\alpha\lambda_\beta c_1). \tag{2.3.8}$$

Finally the amplitude for diagram 3 is given by

$$\begin{aligned} & M_3 = \bar{v}(p_2)c_2^\dagger\left(-ig_s\frac{\lambda_\delta}{2}\gamma_\sigma\right)u(p_1)c_1\left(\frac{-i\delta_{\sigma\lambda}\delta_{\delta\gamma}}{q^2}\right)(ig_s f_{\alpha\beta\gamma}) \\ & \times\{[\delta_{\mu\nu}(-p_3+p_4)_\lambda + \delta_{\nu\lambda}(-p_4-q)_\mu + \delta_{\lambda\mu}(q+p_3)_\nu](\epsilon_{3\mu}a_{3\alpha})(\epsilon_{4\nu}a_{4\beta})\} \end{aligned} \tag{2.3.9}$$

For this case q=p_3+p_4.Therefore,

$$q^2 = p_3^2 + p_4^2 + 2p_3.p_4 = 2p_3.p_4. \tag{2.3.10}$$

This result is due to the condition that the gluon is massless.

$$\begin{aligned} & M_3 = \bar{v}(p_2)c_2^\dagger\left(\frac{-ig_s^2}{4p_3.p_4}\lambda_\delta\gamma_\sigma\delta_{\sigma\lambda}\delta_{\delta\gamma}\right) \\ & \times\{[\delta_{\mu\nu}(-p_3+p_4)_\lambda + \delta_{\nu\lambda}(-2p_4-p_3)_\mu + \delta_{\lambda\mu}(2p_3+p_4)_\nu](\epsilon_3^\mu a_3^\alpha)u(p_1)c_1 f_{\alpha\beta\gamma}\epsilon_{4\nu}a_{4\beta}\lambda_\gamma)\} \\ & = \bar{v}(p_2)\left(\frac{-ig_s^2\gamma_\lambda}{4p_3.p_4}\right)\{\epsilon_3.\epsilon_4(-p_3+p_4)_\lambda + (-2p_4.\epsilon_3 - p_3.\epsilon_3)\epsilon_{4\lambda} + (2p_3\epsilon_4 + p_4.\epsilon_4)\epsilon_{3\lambda}])\} \\ & \times(a_{3\alpha}u(p_1)c_1 f_{\alpha\beta\gamma}\lambda_\gamma a_{4\beta}). \end{aligned} \tag{2.3.11}$$

Thus by using the orthonormality condition $\epsilon_3.p_3 = \epsilon_4.p_4 = 0$, we can drop two terms from equation (2.3.11) and then the amplitude for diagram 3 can be written in a simplified form as

$$
\begin{aligned}
M_3 &= \bar{v}(p_2)\left(\frac{-ig_s^2\gamma_\lambda}{4p_3.p_4}\right)\{[\epsilon_3.\epsilon_4(-p_3+p_4)_\lambda + (-2p_4.\epsilon_3)\epsilon_{4\lambda} + (2p_3\epsilon_4)\epsilon_{3\lambda}])\} \\
&\times (a_{3\alpha}u(p_1)c_1 f_{\alpha\beta\gamma}\lambda_\gamma a_{4\beta}) \\
&= \bar{v}(p_2)\left(\frac{-ig_s^2}{4p_3.p_4}\right)\left\{\epsilon_3.\epsilon_4(-\not{p}_3+\not{p}_4) + (-2p_4.\epsilon_3)\not{\epsilon}_4 + (2p_3.\epsilon_4)\not{\epsilon}_3)\right\} \\
&\times \left(u(p_1)a_{3\alpha}a_{4\beta}f_{\alpha\beta\gamma}c_2^\dagger\lambda_\gamma c_1\right). \qquad (2.3.12)
\end{aligned}
$$

So, this is the general term and it is a little bit more complicated to evaluate. To make things more managable let's assume the case in which the initial(incoming particles) are at rest [18]. In other words this means the space components of their momentum is zero. That is

$$p_1 = p_2 = (0, iE) \text{ and } p_3 = (\vec{P}, iE)\ ,\ p_4 = (-\vec{P}, iE). \qquad (2.3.13)$$

Then

$$p_1.p_3 = p_1.p_4 = -E^2 = -m^2 \text{ and } p_3.p_4 = -\vec{P}^2 - E^2 = -2E^2 = -2m^2 \qquad (2.3.14)$$

here m is the relativistic mass. Meanwhile in the Coulomb gauge,

$$p_3.\epsilon_4 = -p_4.\epsilon_4 = 0, \quad \text{and likewise } p_4.\epsilon_3 = 0. \qquad (2.3.15)$$

(N.B In the Coulomb gauge the time component of ϵ is zero.) Thus by using these assumptions we can write equation (2.3.12) as

$$M_3 = \left(\frac{ig_s^2}{8m^2}\right)\bar{v}(p_2)\left\{\epsilon_3.\epsilon_4(-\not{p}_3+\not{p}_4)\right\}\left(u(p_1)a_{3\alpha}a_{4\beta}f_{\alpha\beta\gamma}c_2^\dagger\lambda_\gamma c_1\right). \qquad (2.3.16)$$

By using the assumption that we made for the center of mass frame case we can write the amplitude M_1 given in equation (2.3.5) in a modified form as

$$M_1 = \frac{g_s^2}{8m^2}\bar{v}_{(p_2)}\left(\not{\epsilon}_4\not{p}_3\not{\epsilon}_3\right)u_{(p_1)}a_{3\alpha}a_{4\beta}(c_2^\dagger\lambda_\beta\lambda_\alpha c_1). \tag{2.3.17}$$

And in the same way we can write M_2 as

$$M_2 = \frac{g_s^2}{8m^2}\bar{v}(p_2)\left(\not{\epsilon}_3\not{p}_4\not{\epsilon}_4\right)u(p_1)a_{3\alpha}a_{4\beta}(c_2^\dagger\lambda_\alpha\lambda_\beta c_1). \tag{2.3.18}$$

Thus we can find the total amplitude by linear superposition of M_1, M_2 and M_3 as

$$\begin{aligned}
M_{total} &= M_1 + M_2 + M_3 = \frac{g_s^2}{8m^2}\bar{v}(p_2)\left(\not{\epsilon}_4\not{p}_3\not{\epsilon}_3\right)u(p_1)a_{3\alpha}a_{4\beta}(c_2^\dagger\lambda_\beta\lambda_\alpha c_1)\\
&+\frac{g_s^2}{8m^2}\bar{v}(p_2)\left(\not{\epsilon}_3\not{p}_4\not{\epsilon}_4\right)u(p_1)a_{3\alpha}a_{4\beta}(c_2^\dagger\lambda_\alpha\lambda_\beta c_1)\\
&+\left(\frac{ig_s^2}{8m^2}\right)\bar{v}(p_2)\left\{\epsilon_3.\epsilon_4(-\not{p}_3+\not{p}_4)\right\}\left(u(p_1)a_{3\alpha}a_{4\beta}f_{\alpha\beta\gamma}c_2^\dagger\lambda_\gamma c_1\right)\\
&= \frac{g_s^2}{8m^2}a_{3\alpha}a_{4\beta}\bar{v}(p_2)c_2^\dagger\left[\not{\epsilon}_4\not{p}_3\not{\epsilon}_3\lambda_\beta\lambda_\alpha + \not{\epsilon}_3\not{p}_4\not{\epsilon}_4\lambda_\alpha\lambda_\beta\right]c_1u(p_1)\\
&+i\frac{g_s^2}{8m^2}a_{3\alpha}a_{4\beta}\bar{v}(p_2)c_2^\dagger\left[\epsilon_3.\epsilon_4(-\not{p}_3+\not{p}_4)f_{\alpha\beta\gamma}\lambda_\gamma\right]c_1u(p_1).
\end{aligned} \tag{2.3.19}$$

By considering that the orientation of our coordinates in such a way that the z-axis lies along the $\vec{P}$ direction we get the relation

$$\begin{aligned}
&\not{p}_3 = m(\gamma_4+\gamma_3),\ \not{p}_4 = m(\gamma_4-\gamma_3)\\
&\text{and } \not{p}_4 - \not{p}_3 = -2m\gamma_3
\end{aligned} \tag{2.3.20}$$

where γ_3 is the component along the z-axis. By using the anticommutation relation for γ matrices we can write

$$\begin{aligned}
\not{\epsilon}_4\not{p}_3\not{\epsilon}_3 &= \not{\epsilon}_4(2\epsilon_3.p_3 - \not{\epsilon}_3\not{p}_3)\\
&= -\not{\epsilon}_4\not{\epsilon}_3\not{p}_3
\end{aligned} \tag{2.3.21}$$

and

$$\begin{aligned}
\not{\epsilon}_3\not{p}_4\not{\epsilon}_3 &= \not{\epsilon}_3(2\epsilon_4.p_4 - \not{\epsilon}_4\not{p}_4)\\
&= -\not{\epsilon}_3\not{\epsilon}_4\not{p}_4.
\end{aligned} \tag{2.3.22}$$

On substituting these results into equation (2.3.19) we obtain, M_{total}=$M_1 + M_2 + M_3$ which can be written as,

$$\begin{aligned} M_{total} = \quad & \frac{g_s^2}{8m^2} a_{3\alpha} a_{4\beta} \bar{v}(p_2) c_2^\dagger \left[-\not\epsilon_4 \not\epsilon_3 m(\gamma_4+\gamma_3)\lambda_\beta\lambda_\alpha - \not\epsilon_3\not\epsilon_4 m(\gamma_4-\gamma_3)\lambda_\alpha\lambda_\beta \right] c_1 u(p_1) \\ & +i\frac{g_s^2}{8m^2} a_{3\alpha} a_{4\beta} \bar{v}(p_2) c_2^\dagger \left[\epsilon_3.\epsilon_4(-2m\gamma_3) f_{\alpha\beta\gamma}\lambda_\gamma \right] c_1 u(p_1). \end{aligned} \tag{2.3.23}$$

By using the relation

$$\begin{aligned} 2if_{\alpha\beta\gamma}\lambda_\gamma &= [\lambda_\alpha, \lambda_\beta] \\ \not\epsilon_3\not\epsilon_4 &= \epsilon_3.\epsilon_4 + i(\epsilon_3 \times \epsilon_4).\Sigma \\ \not\epsilon_4\not\epsilon_3 &= \epsilon_3.\epsilon_4 - i(\epsilon_3 \times \epsilon_4).\Sigma \end{aligned} \tag{2.3.24}$$

and simplifying equation (2.3.23) can be written as

$$\begin{aligned} M_{total} = \quad & \frac{-g_s^2}{8m} a_{3\alpha} a_{4\beta} \bar{v}(p_2) c_2^\dagger \left(\epsilon_3.\epsilon_4 \{\lambda_\alpha, \lambda_\beta\} \gamma_4 + i(\epsilon_3\times\epsilon_4).\Sigma[\lambda_\alpha,\lambda_\beta]\gamma_4 \right) c_1 u(p_1) \\ & \frac{-g_s^2}{8m} a_{3\alpha} a_{4\beta} \bar{v}(p_2) c_2^\dagger \left(-\epsilon_3.\epsilon_4[\lambda_\alpha,\lambda_\beta]\gamma_3 - i(\epsilon_3\times\epsilon_4).\Sigma \{\lambda_\alpha, \lambda_\beta\} \gamma_3 \right) c_1 u(p_1) \\ & \frac{-g_s^2}{8m} a_{3\alpha} a_{4\beta} \bar{v}(p_2) c_2^\dagger \left(+\epsilon_3.\epsilon_4[\lambda_\alpha,\lambda_\beta]\gamma_3 \right) c_1 u(p_1). \end{aligned} \tag{2.3.25}$$

Simplifying the above equation we will get

$$\begin{aligned} M_{total} = & \frac{-g_s^2}{8m} a_{3\alpha} a_{4\beta} \bar{v}(p_2) c_2^\dagger \left(\epsilon_3.\epsilon_4 \{\lambda_\alpha, \lambda_\beta\} \gamma_4 \right) c_1 u(p_1) \\ & \frac{-g_s^2}{8m} a_{3\alpha} a_{4\beta} \bar{v}(p_2) c_2^\dagger i(\epsilon_3\times\epsilon_4).\Sigma \left([\lambda_\alpha,\lambda_\beta]\gamma_4 - \{\lambda_\alpha, \lambda_\beta\}\gamma_3 \right) c_1 u(p_1). \end{aligned} \tag{2.3.26}$$

Where

$$\Sigma = \begin{pmatrix} \sigma & o \\ 0 & \sigma \end{pmatrix} \tag{2.3.27}$$

and the curly brackets denote the anticommutator $\{A, B\}$=AB + BA.

Since the triplet states cannot go to two gluons, we can put the quarks in spin-0 (singlet). Then the amplitude becomes

$$M = \frac{(M_{\uparrow\downarrow} - M_{\downarrow\uparrow})}{\sqrt{2}}. \tag{2.3.28}$$

For $M_{\uparrow\downarrow}$,

$$\bar{v}(p_2)\gamma_4 u(p_1) = 0$$
$$\bar{v}(p_2)\Sigma\gamma_3 u(p_1) = 2mi\hat{z}. \tag{2.3.29}$$

Since

$$M_{\downarrow\uparrow} = -M_{\uparrow\downarrow} \tag{2.3.30}$$

we can write the amplitude obtained in equation (2.3.28) as

$$M = \frac{2M_{\uparrow\downarrow}}{\sqrt{2}}. \tag{2.3.31}$$

Substituting these results into equation (2.3.26) we obtain

$$\begin{aligned} M = \quad & \frac{-g_s^2}{2\sqrt{2}} a_{3\alpha} a_{4\beta} c_2^\dagger (\epsilon_3 \times \epsilon_4) \left\{\lambda_\alpha, \lambda_\beta\right\} c_1 \\ = & \frac{-\sqrt{2}g_s^2}{4} a_{3\alpha} a_{4\beta} (\epsilon_3 \times \epsilon_4)_z c_2^\dagger \left\{\lambda_\alpha, \lambda_\beta\right\} c_1. \end{aligned} \tag{2.3.32}$$

But

$$\epsilon_3 \times \epsilon_4 = i\hat{k}. \tag{2.3.33}$$

Substituting this result into equation (2.3.32) we obtain

$$M = \frac{-\sqrt{2}g_s^2}{4} a_{3\alpha} a_{4\beta} (i\hat{k}) c_2^\dagger \left\{\lambda_\alpha, \lambda_\beta\right\} c_1. \tag{2.3.34}$$

The color factor can be obtained as

$$f = \frac{1}{8} a_{3\alpha} a_{4\beta} c_2^\dagger \left\{\lambda_\alpha, \lambda_\beta\right\} c_1. \tag{2.3.35}$$

In particular, if the quarks occupy the color singlet state ,$\frac{1}{\sqrt{3}}(r\bar{r} + b\bar{b} + g\bar{g})$, then the color factor given in equation (2.3.34) becomes

$$\begin{aligned} f = \quad & \frac{1}{8}a_{3\alpha}a_{4\beta}(\frac{1}{\sqrt{3}}(1 \quad 0 \quad 0)\left\{\lambda_\alpha, \lambda_\beta\right\}\begin{pmatrix}1\\0\\0\end{pmatrix} \\ & +\frac{1}{8}a_{3\alpha}a_{4\beta}\frac{1}{\sqrt{3}}(0 \quad 1 \quad 0)\left\{\lambda_\alpha, \lambda_\beta\right\}\begin{pmatrix}0\\1\\0\end{pmatrix} \\ & +\frac{1}{8}a_{3\alpha}a_{4\beta}\frac{1}{\sqrt{3}}(0 \quad 0 \quad 1)\left\{\lambda_\alpha, \lambda_\beta\right\}\begin{pmatrix}0\\0\\1\end{pmatrix}) \\ & = \frac{1}{8\sqrt{3}}a_{3\alpha}a_{4\beta}Tr\left\{\lambda_\alpha, \lambda_\beta\right\}. \end{aligned} \tag{2.3.36}$$

But

$$Tr\left\{\lambda_\alpha, \lambda_\beta\right\} = 4\delta_{\alpha\beta}. \tag{2.3.37}$$

Substituting this result into equation (2.3.36) we obtain

$$f = \frac{1}{2\sqrt{3}}a_{3\alpha}a_{4\alpha}. \tag{2.3.38}$$

Now the singlet state for two gluons is given by

$$\begin{aligned} |singlet\rangle &= \frac{1}{\sqrt{8}}|n\rangle_1|n\rangle_2 \\ a_{3\alpha}a_{4\alpha} &= \frac{1}{\sqrt{8}}(8) = 2\sqrt{2}. \end{aligned} \tag{2.3.39}$$

Inserting this result from equation (2.3.39) into equation (2.3.38) we obtain

$$f = \sqrt{\frac{2}{3}}. \tag{2.3.40}$$

For the $q + \bar{q} \rightarrow g + g$ in the color singlet state configuration,with the quarks at rest the amplitude is

$$M = \frac{-4ig_s^2}{\sqrt{3}}. \tag{2.3.41}$$

Finally the square of the amplitude becomes

$$|M|^2 = \frac{16g_s^4}{3}. \tag{2.3.42}$$

Finally the differential cross section is calculated as

$$\int \frac{d\sigma}{d\Omega}\bigg|_{cm} = \frac{1}{(4\pi)^2} \frac{1}{E_{tot}^2} \frac{|p_{fin}|}{|p_{init}|} \frac{1}{4} |M|^2. \tag{2.3.43}$$

But

$$\begin{aligned} E_{tot} = & \quad E_1 + E_2 = 2m \\ & |P_{init}| = |P_{fin}| = 2m. \end{aligned} \tag{2.3.44}$$

Then the differential cross section will be

$$\int \frac{d\sigma}{d\Omega}\bigg|_{cm} = \frac{1}{(4\pi)^2} \frac{1}{(2m)^2} \frac{1}{4} |M|^2. \tag{2.3.45}$$

By substituting for $|M|^2$ from equation (2.3.42) into equation (2.3.45) we obtain,

$$\begin{aligned} \int \frac{d\sigma}{d\Omega}\bigg|_{cm} = & \quad \frac{1}{(4\pi)^2} \frac{1}{(2m)^2} \frac{4g_s^4}{3} \\ & = \frac{4}{3(2m)^2} \left(\frac{g_s^2}{4\pi} \right)^2. \end{aligned} \tag{2.3.46}$$

But the fine structure constant for strong interaction (QCD) is $\frac{g_s^2}{4\pi} = \alpha_s$. Hence the differential cross section becomes

$$\int \frac{d\sigma}{d\Omega}\bigg|_{cm} = \frac{4}{3} \left(\frac{\alpha_s}{2m} \right)^2. \tag{2.3.47}$$

Finally the total cross section can be evaluated as

$$\begin{aligned} \int d\sigma &= \frac{4}{3} \left(\frac{\alpha_s}{2m} \right)^2 \int_0^{4\pi} d\Omega \\ \Rightarrow \sigma &= \frac{16\pi}{3} \left(\frac{\alpha_s}{2m} \right)^2. \end{aligned} \tag{2.3.48}$$

So this is the total cross section for the pair annihilation of quark-antiquark into two gluons.

After this we proceed to the next chapter and derive the Bethe-Salpeter equation for two-particle interactions which we would use to calculate the cross section for double heavy quarkonium production in chapter 4.

Chapter 3

Bethe Salpeter Equation

The Bethe-Salpeter equation [11, 14] describes the bound states of a two-body (particles) quantum mechanical system in a relativistically covariant formalism. Examples of two-particle systems described by the Bethe-Salpeter equation are positronium, bound state of an electron-positron pair. The Bethe-Salpeter formalism is generally accepted to represent the appropriate framework for the description of bound states within relativistic quantum field theory. Within this formalism, a bound state is described by its Bethe-Salpeter amplitude, which is defined as the time-ordered product of the field operators of the bound-state constituents between the vacuum and the bound state.

In principle, this bound-state amplitude should be found as a solution of the homogeneous Bethe-Salpeter equation. However, apart from a very few special cases such as the famous Wick-Cutkosky model which describes the interaction of two scalar particles by exchange of a massless scalar particle the Bethe-Salpeter equation turns out to be practically not tractable. One of the main reasons for this fact is the appearance of time-like variables in the equation of motion. Consequently people usually consider some three dimensional reduction of the Bethe-Salpeter equation. The most popular among these three-dimensional reductions is based on the assumption that the interaction between the bound-state constituents is instantaneous in the center-of momentum frame of the bound state. The result of this is called the "Salpeter equation" or "instantaneous Bethe-

Salpeter equation". This equation may be formulated as eigenvalue problem for the mass M of the bound state. Its dynamical quantity is the "Salpeter amplitude," obtained from the Bethe-Salpeter amplitude by equating the time variables of the involved bound-state constituents.

To derive the Bethe-Salpeter equation first it is necessary to start from the one particle Dirac propagator then generalize to the two bound state.

3.1 Two Particle propagator

We know from propagator theory that one particle propagator in an external field can be expressed as an interaction series

$$K(x_1,x_2) = K_0(x_1,x_2) - e\int d^4x_3 K_0(x_1,x_3)\not{A}_{(X_3)}(K_0(x_3,x_2)$$
$$+(-e)^2\int d^4x_3 K_0(x_1,x_3)\not{A}_{(X_3)}K_0(x_3,x_2)\not{A}_{(X_4)}K(x_4,x_2) + ... \tag{3.1.1}$$

This is the propagator series eqn. for a one particle in an external field $A_\mu(x)$.

We now generalize the expression for the propagator obtained for one particle in an external field to two-particles in mutual interaction consider two particles a and b. First let's take the condition in which the two propagating particles are freely propagating simultaneously from space-time points x_3 and x_4 to space-time points x_1 and x_2. In this case their propagation can be sketched graphically as shown in the following figure. Where p_1 and p_2 are the initial momenta at space-time points x_3 and x_4 respectively and

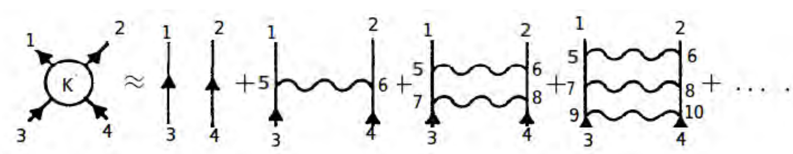


Figure 3.1: Ladder approximation diagram for two interacting particles[11].

p_1' and p_2' are the final momenta at space-time points x_1 and x_2 respectively. Here in this interaction there is an exchange of one quanta.

The free 2-particle propagator given in the figure can be written as the product of two single particle propagator as,

$$k_{0ab}(x_1, x_2; x_3, x_4) = k_{0a}(x_1, x_3)k_{0b}(x_2, x_4). \tag{3.1.2}$$

To find the form of the propagator for these two-interacting particles, let us consider the simplest case(Moller interaction, $e^- + e^- \longrightarrow e^- + e^-$).

Now let us consider the interaction between them be Coulomb interaction and the Coulomb potential is

$$V = \frac{e^2}{r_{ab}} \tag{3.1.3}$$

where $r_{ab} = r_a - r_b$ is the separation between them.

In eqn.(3.1.3) the Coulomb interaction potential is not in covariant form. To write it in a covariant form first $\frac{1}{r_{ab}}$ has to be transformed in to momentum space by Fourier transform. The Fourier transform of $\frac{1}{r_{ab}}$ is as follows

$$\frac{1}{|r|} \longrightarrow \frac{1}{|p|^2}. \tag{3.1.4}$$

Therefore, the Fourier transform of $\frac{1}{r_{ab}}$ can be written as

$$\frac{1}{|r_{ab}|} \longrightarrow \frac{1}{(p_1 - p_1')^2}. \tag{3.1.5}$$

Still in eqn.(3.1.5) the potential is not in its covariant form. Finally by using the property of the Dirac γ matrices it is possible to write this potential covariantly as

$$\begin{aligned} V &= \frac{e^2\gamma_\mu^a\gamma_\mu^b}{(p_1 - p_1')^2} \\ V &= -\gamma_\mu^a e^2(\frac{-1}{q^2})\gamma_\mu^b \end{aligned} \tag{3.1.6}$$

where q=$(p_1 - p_1')$. We then generate the interaction potential e$\gamma_\mu A_\mu$ obtained for the one-particle Dirac propagator to two Dirac particles as e$\gamma_\mu A_\mu \longrightarrow -\gamma_\mu^a e^2 D_F(x_5 - x_6)\gamma_\mu^b$ where

$$D_F(x_5 - x_6) = \frac{-1}{(2\pi)^4}\int d^4q \frac{e^{iq.(x_5-x_6)}}{q^2} \tag{3.1.7}$$

is the photon propagator. To remove the singularity in the propagator when $q^2 = 0$, it has to be modified as

$$D_F(x_5 - x_6) = \frac{-1}{(2\pi)^4} \int d^4q \frac{e^{iq.(x_5-x_6)}}{q^2 + i\varepsilon} \tag{3.1.8}$$

where ϵ is an infinitesimal real constant. By using this photon propagator the propagator for the two interacting particles propagating simultaneously can be written as

$$\begin{aligned} &K(x_1, x_2; x_3, x_4) = K_{0a}(x_1, x_3)K_{0b}(x_3, x_4) \\ &- \int d^4x_5 d^4x_6 K_0(x_1, x_2; x_5, x_6)\gamma_\mu^a e^2 D_F(x_5 - x_6)\gamma_\mu^b K(x_5, x_6; x_3, x_4). \end{aligned} \tag{3.1.9}$$

If we let $-\gamma_\mu^a e^2 D_F(x_5 - x_6)\gamma_\mu^b = I(x_5, x_6)$ then eqn.(3.1.9) becomes

$$\begin{aligned} &K(x_1, x_2; x_3, x_4) = K_0(x_1, x_2; x_3, x_4) \\ &+ \int d^4x_5 d^4x_6 K_0(x_1, x_2; x_5, x_6)I(x_5, x_6)K(x_5, x_6; x_3, x_4). \end{aligned} \tag{3.1.10}$$

Introducing another intermediate space-time point x_7 and x_8 between points x_5 ,x_3 and x_6 ,x_4 the propagator for the simultaneous propagation from x_5 to x_3 and x_6 to x_4 that is $K(x_5, x_6; x_3, x_4)$ can be written as

$$\begin{aligned} &K(x_5, x_6; x_3, x_4) = K_0(x_5, x_6; x_3, x_4) \\ &+ \int d^4x_7 d^4x_8 K_0(x_5, x_6; x_7, x_8)I(x_7, x_8)K(x_7, x_8; x_3, x_4). \end{aligned} \tag{3.1.11}$$

Substituting for $K(x_5, x_6; x_3, x_4)$ in eqn.(3.1.10) from eqn.(3.1.11) we will obtain

$$\begin{aligned} &K(x_1, x_2; x_3, x_4) = K_0(x_1, x_2; x_3, x_4) \\ &\int d^4x_5 d^4x_6 K_0(x_1, x_2; x_5, x_6)I(x_5, x_6)[K_0(x_5, x_6; x_3, x_4) \\ &+ \int d^4x_7 d^4x_8 K_0(x_5, x_6; x_7, x_8)I(x_7, x_8)K(x_7, x_8; x_3, x_4)] \end{aligned} \tag{3.1.12}$$

Thus we can write

$$
\begin{aligned}
&\Rightarrow K(x_1,x_2;x_3,x_4) = K_0(x_1,x_2;x_3,x_4) \\
&+ \int d^4x_5 d^4x_6 K_0(x_1,x_2;x_5,x_6) I(x_5,x_6)[K_0(x_5,x_6;x_3,x_4) K_0(x_5,x_6;x_3,x_4) \\
&+ \int d^4x_5 d^4x_6 d^4x_7 d^4x_8 K_0(x_1,x_2;x_5,x_6) I(x_5,x_6) \\
&\times [K_0(x_5,x_6;x_7,x_8) I(x_7,x_8) K(x_7,x_8;x_3,x_4)] + ... \qquad (3.1.13)
\end{aligned}
$$

This implies that if we continue introducing other intermediate space-time points and solve for the propagators iteratively we will get the above series expansion for the propagator. The expression obtained in eqn.(3.1.12) can be represented graphically by adding a one quantum exchange term graph to the initial graph as follows.

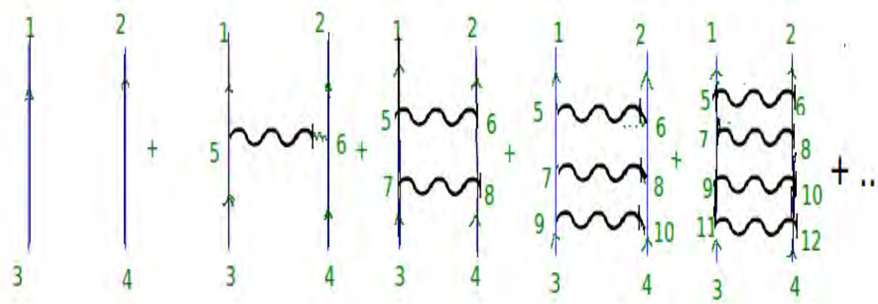


Figure 3.2: Graphical representation of the interaction kernel of two-particle bound system

3.2 Bethe-salpeter two-particle wave function

In non-relativistic quantum mechanics for a particle propagating from a space-time point $(\vec{r}', t')$ to a space-time $(\vec{r}, t)$, if the wave function[11, 15] at $(\vec{r}', t')$ is known, then the wave function at $(\vec{r}, t)$ is given by

$$\psi(\vec{r}, t) = \int d^3\vec{r}' K(\vec{r}, t; \vec{r}', t') \psi(\vec{r}', t'). \qquad (3.2.1)$$

The above condition can be applied for the two bound particles propagating simultaneously from space-time x_3, x_4 to space-time point x_1, x_2. That is if the wave funcion of the bound system at x_3, x_4 is known, then the wave function at x_1, x_2 can be given as

$$\psi(x_1, x_2) = \int d^3x_3 d^3x_4 K(x_1, x_2; x_3, x_4)\psi(x_3, x_4). \tag{3.2.2}$$

By substituting for $K(x_1, x_2; x_3, x_4)$ from eqn.(3.1.10) into eqn.(3.2.2) we obtain

$$\begin{aligned}
&\psi(x_1, x_2) = \int d^3x_3 d^3x_4 [K_0(x_1, x_2; x_3, x_4) \\
&+ \int d^4x_5 d^4x_6 K_0(x_1, x_2; x_5, x_6) I(x_5, x_6) K(x_5, x_6; x_3, x_4)]\psi(x_3, x_4) \\
&\Rightarrow \psi(x_1, x_2) = \psi_0(x_1, x_2) + \int d^3x_3 d^3x_4 d^4x_5 d^4x_6 K_0(x_1, x_2; x_5, x_6) \\
&\times I(x_5, x_6) K(x_5, x_6; x_3, x_4)\psi(x_3, x_4).
\end{aligned} \tag{3.2.3}$$

But

$$\int d^3x_3 d^3x_4 K(x_5, x_6; x_3, x_4)\psi_{(x_3,x_4)} = \psi_{(x_5,x_6)}. \tag{3.2.4}$$

On substituting from eqn.(3.2.4) into eqn.(3.2.3) we obtain

$$\psi(x_1, x_2) = \psi_{0(x_1,x_2)} + \int d^4x_5 d^4x_6 K_0(x_1, x_2; x_5, x_6) I(x_5, x_6)\psi(x_5, x_6). \tag{3.2.5}$$

For two particles forming a bound state after propagation $\psi_0(x_1, x_2)$=0. Therefore, under this condition eqn.(3.2.5) will be reduced to

$$\psi(x_1, x_2) = \int d^4x_5 d^4x_6 K_0(x_1, x_2; x_5, x_6) I(x_5, x_6)\psi(x_5, x_6). \tag{3.2.6}$$

Multiplying eqn. (3.2.6) from left by free Dirac operators on both sides we can write Bethe-Salpeter equation(BSE) in coordinate representation as,

$$\begin{aligned}
&(\gamma_\mu \partial_{1\mu} + m_1)(\gamma_\mu \partial_{2\mu} + m_2)\psi(x_1, x_2) = \\
&(\gamma_\mu \partial_{1\mu} + m_1)(\gamma_\mu \partial_{2\mu} + m_2) \int d^4x_5 d^4x_6 K_0(x_1, x_2; x_5, x_6) \\
&I(x_5, x_6)\psi(x_5, x_6).
\end{aligned} \tag{3.2.7}$$

Making use of

$$\begin{aligned}
(\gamma_\mu \partial_{1\mu} + m_1) K_0(x_1, x_5) &= -i\delta^{(4)}(x_1 - x_5) \\
(\gamma_\mu \partial_{2\mu} - m_2) K_0(x_2, x_6) &= -i\delta^{(4)}(x_2 - x_6).
\end{aligned} \tag{3.2.8}$$

and substituting from eqn. (3.2.8) into eqn. (3.2.7) we obtain

$$(\gamma_\mu \partial_{1\mu} + m_1)(\gamma_\mu \partial_{2\mu} + m_2)\psi(x_1, x_2) = \\ \int d^4x_5 d^4x_6 (-i\delta^{(4)}(x_1 - x_5))(-i\delta^{(4)}(x_2 - x_6)) I(x_5, x_6)\psi(x_5, x_6). \tag{3.2.9}$$

Thus eqn.3.2.9 becomes

$$(\gamma_\mu \partial_{1\mu} + m_1)(\gamma_\mu \partial_{2\mu} + m_2)\psi(x_1, x_2) = -I(x_1, x_2)\psi(x_1, x_2). \tag{3.2.10}$$

Let $-I(x_1, x_2) = K(x_1, x_2)$. Where $k(x_1, x_2)$ is an interaction kernel. Then the BSE can be written as

$$(\gamma_\mu \partial_{1\mu} + m_1)(\gamma_\mu \partial_{2\mu} + m_2)\psi(x_1, x_2) = K(x_1, x_2)\psi(x_1, x_2). \tag{3.2.11}$$

This is also another form of BSE in coordinate representation. Here this $K(x_5, x_6)$ is not the same as the propagator.

Now let us consider the hadron moves with 4-momentum P_μ. In the center-of-mass frame, the center of mass coordinate is given by

$$\bar{X} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}. \tag{3.2.12}$$

If we consider the condition $m_1 = m_2 = m$ then the center-of-mass coordinate becomes

$$\bar{X} = \frac{x_1 + x_2}{2} \tag{3.2.13}$$

and the relative separation between the two particles is also given by

$$x = x_1 - x_2. \tag{3.2.14}$$

From the above relation we can get

$$\partial_{1\mu} = \frac{\partial}{\partial x_1} = \frac{\partial \bar{X}}{\partial x_1}\frac{\partial}{\partial \bar{X}} + \frac{\partial x}{\partial x_1}\frac{\partial}{\partial x} = \frac{1}{2}\partial_{\bar{x}} + \partial_{x\mu} \\ \partial_{2\mu} = \frac{\partial}{\partial x_2} = \frac{\partial \bar{X}\mu}{\partial x_2}\frac{\partial}{\partial \bar{X}} + \frac{\partial x}{\partial x_2}\frac{\partial}{\partial x} = \frac{1}{2}\partial_{\bar{x}\mu} - \partial_{x\mu}. \tag{3.2.15}$$

Now by using $\bar{X}$ and x we can write $\psi(x_1, x_2)$ as follows. That is

$$\psi(x_1, x_2) = \psi(\bar{X})\psi(x). \tag{3.2.16}$$

Where

$$\psi(\bar{X}) = \frac{1}{\sqrt{V}} e^{iP.\bar{X}}. \tag{3.2.17}$$

Substituting for $\psi_{(x_1,x_2)}$ from equation (3.2.17) into equation (3.2.11) we will obtain

$$(\frac{1}{2}\partial_{\bar{x}} + \partial_{x\mu} + m)(\frac{1}{2}\partial_{\bar{x}\mu} - \partial_{x\mu} + m)e^{iP.\bar{X}}\psi(x) = k(x)e^{iP.\bar{X}}\psi(x). \tag{3.2.18}$$

Differentiating with respect to $\partial_{\bar{x}}$ equation (3.2.18) will reduce to

$$(i\frac{1}{2}\not{P} + \partial_{x\mu} + m)(i\frac{1}{2}\not{P} - \partial_{x\mu} + m)\psi(x) = k(x)\psi(x). \tag{3.2.19}$$

where $\not{P}$=$\gamma_\mu P_\mu$. Here in the above equation equation (3.2.19) $K(x)$ is taken as a function of x only. This is by assuming that the interaction between the particles depends on the relative separation between the two particles. Now by taking the Fourier transform of $\psi(x)$ and $K(x_0)$ as

$$\begin{aligned} \psi_{(x)} &= \frac{1}{(2\pi)^4}\int d^4q e^{iq.x}\psi_{(q)} \\ K(x) &= \frac{1}{(2\pi)^4}\int d^4k e^{ik.x}K(k). \end{aligned} \tag{3.2.20}$$

Then by substituting from equation (3.2.20) into equation (3.2.19) we obtain

$$\begin{aligned} &(i\frac{1}{2}\not{P} + \partial_{x\mu} + m)(i\frac{1}{2}\not{P} - \partial_{x\mu} + m)\frac{1}{(2\pi)^4}\int d^4q e^{iq.x}\psi_{(q)} \\ &= \frac{1}{(2\pi)^4}\int d^4q' e^{iq'.x}\psi_{(q')}\frac{1}{(2\pi)^4}\int d^4k' e^{ik'.x}k_{(k')} \\ &\Rightarrow (i\frac{1}{2}\not{P} + i\not{q} + m)(i\frac{1}{2}\not{P} - i\not{q} + m)e^{iq.x}\psi_{(q)} = \frac{1}{(2\pi)^4}\int d^4q' d^4k' e^{i(q'+k').x}\psi(q'). \end{aligned} \tag{3.2.21}$$

Let K'+q'=q for a given value of q'. Then $d^4k = d^4q$. Therefore, (3.2.21) becomes

$$\begin{aligned} &(i\not{p}_1 + m)(i\not{p}_2 + m)\int d^4q e^{iq.x}\psi_{(q)} = \frac{1}{(2\pi)^4}\int d^4q d^4q' e^{iq.x}k(q-q')\psi_{(q')} \\ &\Rightarrow (i\not{p}_1 + m)(i\not{p}_2 + m)\psi_{(q)} = \frac{1}{(2\pi)^4}\int d^4q' k(q-q')\psi_{(q')} \end{aligned} \tag{3.2.22}$$

where, $p_1=\frac{1}{2}$P+q and $p_2=\frac{1}{2}P$-q.

Finally by using the fermion propagator eqn. (3.2.22) can be writen as

$$\begin{aligned} S^{-1}_{F(p_1)} S^{-1}_{F(p_2)} \psi_{(q)} = & \int \frac{d^4 q'}{(2\pi)^4} k(q-q') \psi_{(q')} \\ & \Rightarrow \psi_{(q)} = \frac{-i}{(i\not{p}_1 + m)} \frac{-i}{(i\not{p}_2 + m)} \int \frac{d^4 q'}{(2\pi)^4} d^4 q' k(q,q') \psi_{(q')} \\ & \Rightarrow \psi_{(q)} = \left(\frac{-i(-i\not{p}_1 + m)}{(p_1^2 + m^2)} \right) \left(\frac{-i(-i\not{p}_2 + m)}{(p_2^2 + m^2)} \right) \\ & \times \int \frac{d^4 q'}{(2\pi)^4} k(q,q') \psi_{(q')} \end{aligned} \quad (3.2.23)$$

where $S^{-1}_{F(p_1)}$ is the Dirac field inverse propagator. This can also be written as

$$\psi_{(q)} = \frac{i(-i\not{p}_1 + m)}{\Delta_1} \frac{i(-i\not{p}_2 + m)}{\Delta_2} \int \frac{d^4 q'}{(2\pi)^4} k(q,q') \psi_{(q')} \quad (3.2.24)$$

where $\triangle_1$=$(p_1^2 + m^2)$ and$\triangle_2$ =$(p_2^2 + m^2)$are the inverse propagators for two fermion fields.

The four-momentum of the quark and anti-quark,p_1 and p_2 are related to the internal momentum q_μ of the hadron and the four-momentum P_μ of the hadron by the relation[20, 22]:

$$p_{1,2\mu} = \hat{m}_{1,2} P_\mu \pm q_\mu \quad (3.2.25)$$

where $\hat{m}_{1,2} = [1 \pm \frac{(m_1^2 - m_2^2)}{M^2}]/2$ is the Weightman-Garding(WG) definition of masses of the conistituent quarks of the hadron.

For spinless quarks the Bethe-Salpeter equation is given by

$$i(2\pi)^4 \ \triangle_1 \triangle_2 \ \phi(p,q) = \int d^4 q' K(q,q') \phi(p,q'). \quad (3.2.26)$$

Using the ansatz on bound state kernel which is assumed to depend on 3D variables $\hat{q}_\mu$ i.e

$$K(q,q') = K(\hat{q}, \hat{q'}) \quad (3.2.27)$$

where the internal momentum q_μ is split into its transverse and longitudinal components as

$$q_\mu = (\hat{q}_\mu, iM\sigma) \quad (3.2.28)$$

where

$$\hat{q}_\mu = q_\mu - \frac{q.P}{P^2}P_\mu \tag{3.2.29}$$

is the transverse component and

$$M\sigma = M\frac{q.P}{P^2} \tag{3.2.30}$$

is the longitudinal component of the internal momentum q_μ. When we say the transverse and longitudinal it is with respect to the total momentum of the hadron. As it can easily be verified from equation (3.2.29) the transverse component $\hat{q}_\mu$ is orthogonal to the total momentum of hadron P_μ.That means

$$\hat{q}_\mu.P_\mu = q_\mu.P_\mu - \frac{q.P}{P^2}P_\mu.P_\mu = q.P - q.P = 0. \tag{3.2.31}$$

By using the above relations, the 4-dimensional volume element in momentum space can be expressed as

$$d^4q = d^3\hat{q}Md\sigma. \tag{3.2.32}$$

Substituting for $d^4q = d^3\hat{q}Md\sigma$ from eqn. (3.2.32) into eqn. (3.2.26) we will get

$$i(2\pi)^4\ \triangle_1\triangle_2\ \phi(p,q) = \int d^3q' K(q,q')Md\sigma\phi(p,q'). \tag{3.2.33}$$

By introducing a 3-D wave function

$$\phi(\hat{q}) = \int_{-\infty}^{\infty} Md\sigma\phi(p,q) \tag{3.2.34}$$

eqn. (3.2.33) can be written as

$$i(2\pi)^4\ \triangle_1\triangle_2\ \phi(p,q) = \int d^3\hat{q}' K(q,q')\phi(\hat{q}'). \tag{3.2.35}$$

Integrating both sides of the eqn. (3.2.35) by $\int_{-\infty}^{\infty} d^3\hat{q}Md\sigma\phi(p,q)$ it can be reduced to

$$\begin{aligned}\int_{-\infty}^{\infty} Md\sigma\phi(p,q) = \quad & \frac{1}{(2\pi)^4 i}\int_{-\infty}^{\infty}\frac{Md\sigma}{\triangle_1\triangle_2}\int d^3\hat{q}' K(q,q')\phi\hat{q}' \\ & \Rightarrow \phi(q) = \frac{1}{D(\hat{q})(2\pi)^3}\int d^3\hat{q}' K(q,q')\phi(\hat{q}') \\ & \Rightarrow D(\hat{q})(2\pi)^3\phi(q) = \int d^3\hat{q}' K(q,q')\phi(\hat{q}'). \end{aligned} \tag{3.2.36}$$

By comparing eqn. 3.2.35 and eqn. 3.2.36 it is possible to obtain the BS vertex function as

$$\triangle_1\triangle_2\,\phi(p,q)=\frac{D(\hat{q})\phi(\hat{q})}{2\pi i}\equiv\Gamma(\hat{q}) \tag{3.2.37}$$

where $\Gamma(\hat{q})$ is the BS vertex function [21] under CIA and

$$\frac{1}{D(\hat{q})}=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{Md\sigma}{\triangle_1\triangle_2}. \tag{3.2.38}$$

The value of$\frac{1}{D(\hat{q})}=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{Md\sigma}{\triangle_1\triangle_2}$ can be evaluated by contour integration in the complex σ-plane.It can be carried out as follows.

$$\begin{aligned}\frac{1}{D(\hat{q})}= \quad & \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{Md\sigma}{\triangle_1\triangle_2}\\ &\Rightarrow=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{Md\sigma}{(m_1^2+p_1^2)(m_2^2+p_1^2)}.\end{aligned} \tag{3.2.39}$$

But interms of σ ,$\triangle_1$ *and* $\triangle_2$ can be expressed as [21] $\triangle_1=\omega_1^2-M^2(\sigma+\frac{1}{2})^2$ and $\triangle_2=\omega_2^2-M^2(\frac{1}{2}-\sigma)^2$. Then

$$\begin{aligned}\frac{1}{D(\hat{q})}= \quad & \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{Md\sigma}{(\omega_1^2-M^2(\sigma+\frac{1}{2})^2)(\omega_2^2-M^2(\frac{1}{2}-\sigma)^2)}\\ &\Rightarrow=\frac{1}{2\pi i M^3}\int_{-\infty}^{\infty}\frac{d\sigma}{(\frac{\omega_1^2}{M^2}-(\sigma+\frac{1}{2})^2)(\frac{\omega_2^2}{M^2}-(\frac{1}{2}-\sigma)^2)}.\end{aligned} \tag{3.2.40}$$

Let

$$\begin{aligned}I= \quad & \int_{-\infty}^{\infty}\frac{d\sigma}{(\frac{\omega_1^2}{M^2}-(\sigma+\frac{1}{2})^2)(\frac{\omega_2^2}{M^2}-(\frac{1}{2}-\sigma)^2)}\\ &=2\pi i\sum(\text{residues enclosed in the upper half-plane}).\end{aligned} \tag{3.2.41}$$

The poles of the integration are at $\sigma=-\frac{1}{2}\pm\frac{\omega_1}{M}$ and at $\sigma=\frac{1}{2}\pm\frac{\omega_2}{M}$. Since the position of the poles lie on the real axis, we have to shift the positions of the poles by introducing an infinitesimal real constant ϵ. Therefore,the position of the poles will be at $\sigma=-\frac{1}{2}\pm\frac{\omega_1}{M}\mp i\epsilon$ and at $\sigma=\frac{1}{2}\pm\frac{\omega_2}{M}\mp i\epsilon$. So the positions of the poles in the upper half plane are at $\sigma=-\frac{1}{2}-\frac{\omega_1}{M}+i\epsilon$ and $\sigma=\frac{1}{2}-\frac{\omega_2}{M}+i\epsilon$. Then the residue at $\sigma=-\frac{1}{2}-\frac{\omega_1}{M}+i\epsilon$ is

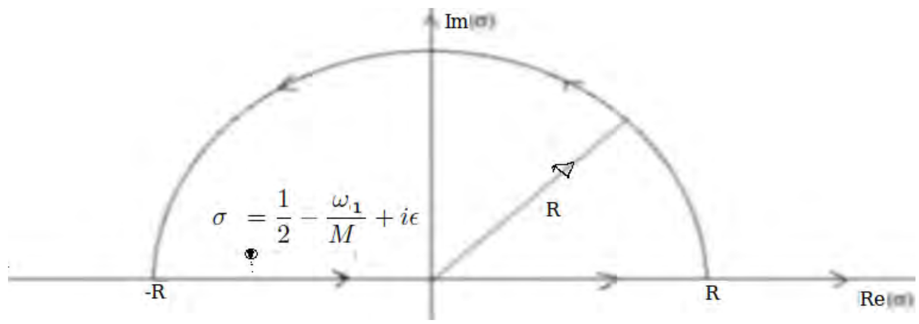


Figure 3.3: Figure for contour integration over complex σ plane.

$$
\begin{aligned}
a_{-1} = &\lim_{\sigma\to\frac{-1}{2}-\frac{\omega1}{M}-i\epsilon}\frac{(\sigma+\frac{1}{2}+\frac{\omega_1}{M}-i\epsilon)}{(\sigma+\frac{1}{2}+\frac{\omega_1}{M}-i\epsilon)(\sigma+\frac{1}{2}-\frac{\omega_1}{M}+i\epsilon)((\sigma-\frac{1}{2}+\frac{\omega_2}{M}+i\epsilon)(\sigma-\frac{1}{2}-\frac{\omega_2}{M}+i\epsilon)}\\
&\Rightarrow a_{-1} = \frac{-M^3}{2\omega_1(M+\omega_1-\omega_2)(M+\omega_1+\omega_2)}. \qquad (3.2.42)
\end{aligned}
$$

The other residue is at $\sigma = \frac{1}{2} - \frac{\omega_2}{M} + i\epsilon$. That is

$$
\begin{aligned}
a_{-1} = &\lim_{\sigma\to\frac{1}{2}-\frac{\omega2}{M}+i\epsilon}\frac{(\sigma-\frac{1}{2}+\frac{\omega_2}{M}-i\epsilon)}{(\sigma+\frac{1}{2}+\frac{\omega_1}{M}-i\epsilon)(\sigma+\frac{1}{2}-\frac{\omega_1}{M}+i\epsilon)((\sigma-\frac{1}{2}+\frac{\omega_2}{M}+i\epsilon)(\sigma-\frac{1}{2}-\frac{\omega_2}{M}+i\epsilon)}\\
&\Rightarrow a_{-1} = \frac{M^3}{2\omega_1(M+\omega_1-\omega_2)(-M+\omega_1+\omega_2)}. \qquad (3.2.43)
\end{aligned}
$$

Therefore, the sum of the residues becomes

$$
\begin{aligned}
\sum(\text{residues}) = &\frac{M^3}{(M+\omega_1-\omega_2)}\left(\frac{-1}{2\omega_1(M+\omega_1+\omega_2)}+\frac{-1}{2\omega_2(-M+\omega_1+\omega_2)}\right)\\
&\Rightarrow = M^3\left(\frac{\omega_1}{2\omega_1\omega_2[-M^2+(\omega_1+\omega_2)^2]}+\frac{\omega_1}{2\omega_1\omega_2[-M^2+(\omega_1+\omega_2)^2]}\right)\\
&\Rightarrow = \frac{-M^3}{(-M^2+(\omega_1+\omega_2)^2)}\left(\frac{1}{2\omega_1}+\frac{1}{2\omega_1}\right). \qquad (3.2.44)
\end{aligned}
$$

Finally the contour integration over the complex σ plane becomes

$$
I = \frac{2\pi i M^3}{(-M^2+(\omega_1+\omega_2)^2)}\left(\frac{1}{2\omega_1}+\frac{1}{2\omega_1}\right) \qquad (3.2.45)
$$

and

$$
\begin{aligned}
\frac{1}{D(\hat{q})} = &\frac{1}{2\pi i M^3}\frac{2\pi i M^3}{(-M^2+(\omega_1+\omega_2)^2)}\left(\frac{1}{2\omega_1}+\frac{1}{2\omega_1}\right)\\
&\Rightarrow D(\hat{q}) = \frac{(\omega_1+\omega_2)^2-M^2}{\frac{1}{2\omega_1}+\frac{1}{2\omega_1}}\\
&\Rightarrow D(\hat{q}) = \frac{D_0(\hat{q})}{\frac{1}{2\omega_1}+\frac{1}{2\omega_1}} \qquad (3.2.46)
\end{aligned}
$$

where $D_0(\hat{q}) = (\omega_1 + \omega_2)^2 - M^2$. But for the case $m_1 = m_2 = m$, $\omega_1 = \omega_2 = \omega$ eqn. 3.2.46 becomes

$$D(\hat{q}) = \omega \left(4\omega^2 - M^2\right). \tag{3.2.47}$$

Substituting this result into equation (3.2.37) we obtain the final expression for the hadron-quark vertex function for case scalar quarks constituting the hadron as,

$$\Gamma(\hat{q}) = \frac{D(\hat{q})}{2\pi i}\phi(\hat{q}). \tag{3.2.48}$$

We generalize this to fermionic quarks now. For the case of P-meson, the hadron-quark vertex function is taken as

$$\Gamma_P(\hat{q}) = \gamma_5 \frac{D(\hat{q})}{2\pi i}\phi(\hat{q}) \tag{3.2.49}$$

while for vector mesons the hadron-quark vertex function is taken as

$$\Gamma_v(\hat{q}) = i(\gamma.\epsilon)\frac{D(\hat{q})}{2\pi i}\phi(\hat{q}). \tag{3.2.50}$$

Hence in the next chapter applying this result and the two particle Bethe-Salpeter wave function we can find the cross section for the production of double heavy quarkonium production in high energy electron-positron collisions.

Chapter 4

Cross-section for double heavy quarkonium production in high energy e^-e^+ collisions

Before we proceed to the the calculation of the cross-section, it is better to have an idea of what cross section refers to. Cross-section is a quantity that provides the information regarding the interaction in the scattering process. That is, based on the available initial and final states of colliding particles, it tells the probability for a particular physical process to take place. Note that this does not refer to the geometry of the target particle, rather it refers to the effective area over which two or more particles interact such that they make a transition from an initial state to a final state.

Different processes of the production and decay of heavy mesons consisting of heavy b (bottom) and c(charm) quarks provide the means for revealing the role of the color and spin dependent quark forces. The aim of many present experiments consists in the increase of experimental accuracy that is important for the detailed comparison of different existing theoretical approaches to the heavy quark problem. The exclusive production of a pair of doubly heavy mesons with c-quarks in e^+e^- annihilation has attracted considerable attention in the last years. This is due to the fact that the cross section of the process

$e^+ + e^- \rightarrow J/\psi + \eta_C$ which was measured in the experiments on Babar and Belle detectors at the energy $\sqrt{s} = 10.6$ GeV has great dicrepancy with the theoretical predictions.

To reduce the discrepancy many efforts have been made. For example, as discussed in [8], corrections from pure electromagnetic interactions are introduced into the non-relativistic QCD (NRQCD) factorization formalism. The next-to-leading order contributions of strong interaction are taken into account in [24, 25]. In [7],the authors take into account corrections to the J/ψ leptonic decay width and the scale dependence of the leading-order prediction; etc.

Discussions given in [24, 25] suggest that to reduce the large discrepancy between the experimental results and the theoretical predictions based on NRQCD for the process $e^-e^+ \rightarrow J/\psi + \eta_c$ with the final state which is composed of two charmonia, large next-to-leading-order (NLO) corrections may appear (the 'NLO' contribution is about 1.8 to 2.1 times of the leading-order one). Including this large NLO contributions, their results are close to the lower bound set by the Babar and Belle collaborations for the double-charmonia production. The authors also indicate that including the relativistic corrections can further enhance the estimated value.

In this work, by properly including relativistic effects, we try to calculate the cross section for the process of double heavy quarkonium production $e^-e^+ \rightarrow J/\psi + \eta_c$ by applying the two-particle Bethe-Salpeter (BS) wave equation which we derived in the previous (chapter 3). The BS equation is in principle established in the frame work of relativistic quantum field theory. Therefore, it is supposed to include all relativistic effects. Moreover, inorder to solve this problem in practice, one needs to take some approximations such as the instantaneous approximation where part of the relativistic effects are lost. However,in many cases, such loss is not that much significant. Inorder to simplify the calculation we will use the heavy quark limit and the ultrarelativistic limit throughout the process of the calculation.

To calculate the cross section we will start by evaluating the scattering amplitude and then finally we calculate the total cross section for the process as follows.

4.1 Amplitude for double heavy quarkonium production

To find the cross-section for the above process, first we have to find the scattering amplitude by applying Feynman rules to the following diagrams[26].

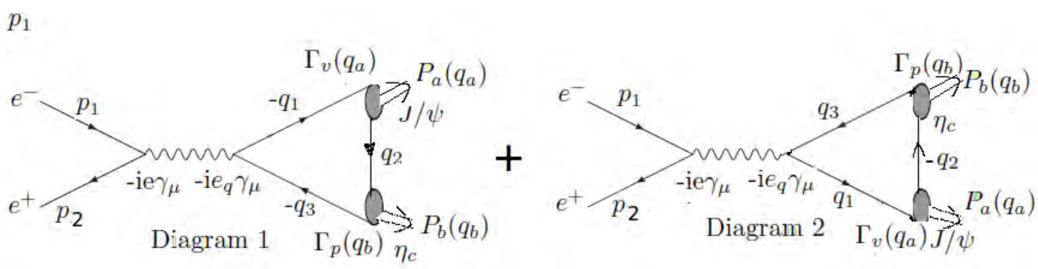


Figure 4.1: Diagram for vector and pseudoscalar meson production from e^+e^- pair annihilation.

$$\begin{aligned} M_1 = \quad & \frac{c_f[\bar{v}(p_2)(-ie\gamma_\mu)u(p_1)]}{s}\int\frac{d^4q_a}{(2\pi)^4}\frac{d^4q_b}{(2\pi)^4} \\ & \times Tr[(-ie_Q\gamma_\mu)SF(-q_3)\Gamma_p(\hat{q}_b)S_F(q_2)\Gamma_v(\hat{q}_a)S_F(-q_1)] \end{aligned} \tag{4.1.1}$$

where $c_f = \frac{N^2-1}{2N} = \frac{9-1}{6} = \frac{4}{3}$ is the color factor and $\sqrt{s}$ is the total energy.

Let

$$C = \frac{4}{3}\frac{[\bar{v}(p_2)(-ie\gamma_\mu)u(p_1)]}{s}. \tag{4.1.2}$$

Then equation (4.1.1) can be written as

$$\begin{aligned} M_1 = \quad & C\int\frac{d^4q_a}{(2\pi)^4}\frac{d^4q_b}{(2\pi)^4} \\ & \times Tr[(-ie_Q\gamma_\mu)S_F(-q_3)\Gamma_p(\hat{q}_b)S_F(q_2)\Gamma_v(\hat{q}_a)S_F(-q_1)]. \end{aligned} \tag{4.1.3}$$

The hadron-quark vertex function for p-meson (η_c) is given by

$$\Gamma_p(\hat{q}_b) = \frac{\gamma_5 N_p D_p(\hat{q}_b)\phi_p(\hat{q}_b)}{2\pi i} \tag{4.1.4}$$

and the hadron-quark vertex function for vector(J/ψ) vertex is given by

$$\Gamma_v(\hat{q_a}) = \frac{i\gamma.\varepsilon N_v D_v(\hat{q_a})\phi_p(\hat{q_a})}{2\pi i} \tag{4.1.5}$$

where N_p and N_v are the normalization constants for the respective mesons.

The propagators for the three quarks are

$$\begin{aligned} S_F(-q_1) &= -i\left(\frac{i\not{q}_1 + m_1}{q_1^2 + m_1^2}\right) \\ S_F(q_2) &= -i\left(\frac{-i\not{q}_2 + m_2}{q_2^2 + m_2^2}\right) \\ S_F(-q_3) &= -i\left(\frac{i\not{q}_3 + m_3}{q_3^2 + m_3^2}\right). \end{aligned} \tag{4.1.6}$$

In equation (4.1.1) $N_p(\hat{q_b}), N_v(\hat{q_a}), D_p(\hat{q_b}), D_v(\hat{q_a}), \phi_p(\hat{q_b}), \phi_v(\hat{q_a})$ can be taken out of the trace. By substituting the above results in equation (4.1.1) the trace term can be calculated as

$$\begin{aligned} Tr[(-ie_Q\gamma_\mu)S_F(-q_3)\mathit{\Gamma}_p(\hat{q_b})S_F(q_2)\mathit{\Gamma}_v(\hat{q_a})S_F(-q_1)] &= \frac{ie_Q N_p D_p(\hat{q_b})\phi_p(\hat{q_b})N_v D_v(\hat{q_a})\phi_p(\hat{q_a})}{(2\pi)^2} \\ \times Tr\left\{\gamma_\mu\left(\frac{i\not{q}_3 + m_3}{q_3^2 + m_3^2}\right)\gamma_5\left(\frac{-i\not{q}_2 + m_2}{q_2^2 + m_2^2}\right)(\gamma.\varepsilon)\left(\frac{i\not{q}_1 + m_1}{q_1^2 + m_1^2}\right)\right\} \end{aligned} \tag{4.1.7}$$

For the sake of mathematical simplicity we let

$$\frac{-e_Q N_p D_p(\hat{q_b})\phi_p(\hat{q_b})N_v D_v(\hat{q_a})\phi_p(\hat{q_a})}{(2\pi)^2} = C' \tag{4.1.8}$$

and thus equation (4.1.7) can be written as

$$\begin{aligned} & Tr[(-ie_Q\gamma_\mu)S_F(-q_3)\mathit{\Gamma}_p(\hat{q_b})S_F(q_2)\mathit{\Gamma}_v(\hat{q_a})S_F(-q_1)] = \\ & C'Tr\left\{i\gamma_\mu\left(\frac{i\not{q}_3 + m_3}{q_3^2 + m_3^2}\right)\gamma_5\left(\frac{-i\not{q}_2 + m_2}{q_2^2 + m_2^2}\right)(\gamma.\varepsilon)\left(\frac{i\not{q}_1 + m_1}{q_1^2 + m_1^2}\right)\right\} \\ & = \frac{C'}{\triangle_1\triangle_2\triangle_3}Tr\left\{i\gamma_\mu(i\not{q}_3 + m_3)\gamma_5(-i\not{q}_2 + m_2)(\gamma.\varepsilon)(i\not{q}_1 + m_1)\right\} \\ & = \frac{C'}{\triangle_1\triangle_2\triangle_3}[-Tr(\gamma_\mu\not{q}_3\gamma_5\not{q}_2\not{\varepsilon}\not{q}_1) + im_1 Tr(\gamma_\mu\not{q}_3\gamma_5\not{q}_2\not{\varepsilon}) - im_2 Tr(\gamma_\mu\not{q}_3\gamma_5\not{\varepsilon}\not{q}_1) \\ & -m_1 m_2 Tr(\gamma_\mu\not{q}_3\gamma_5\not{\varepsilon}) + im_3 Tr(\gamma_\mu\gamma_5\not{q}_2\not{\varepsilon}\not{q}_1) + m_1 m_3 Tr(\gamma_\mu\gamma_5\not{q}_2\not{\varepsilon}) \\ & -m_3 m_2 Tr(\gamma_\mu\gamma_5\not{\varepsilon}\not{q}_1) + im_1 m_2 m_3 Tr(\gamma_\mu\gamma_5\not{\varepsilon})] \end{aligned} \tag{4.1.9}$$

where,

$$\begin{aligned}\triangle_1 &= q_1^2 + m_1^2\\ \triangle_2 &= q_2^2 + m_2^2\\ \triangle_3 &= q_3^2 + m_3^2.\end{aligned} \tag{4.1.10}$$

Since $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$, in equation (4.1.9) only the trace over even numbered product of γ matrices survive. That means the trace over odd numbered product of γ matrices vanish. So, in equation (4.1.9) only the trace over three terms survive. Therefore,

$$\begin{aligned}&Tr[(-ie_Q\gamma_\mu)S_F(-q_3)\mathit{\Gamma}_p(\hat{q}_b)S_F(q_2)\mathit{\Gamma}_v(\hat{q}_a)S_F(-q_1)] = \left(\frac{C'}{\triangle_1\triangle_2\triangle_3}\right)\\ &\times\left\{im_1Tr(\gamma_\mu \not{q}_3\gamma_5\not{q}_2\not{\epsilon}) - im_2Tr(\gamma_\mu\not{q}_3\gamma_5\not{\epsilon}\not{q}_1) + im_3Tr(\gamma_\mu\gamma_5\not{q}_2\not{\epsilon}\not{q}_1)\right\}.\end{aligned} \tag{4.1.11}$$

Since only the c quark is present in both the mesons $m_1 = m_2 = m_3$=m. Then equation (4.1.11) can be written in a simplified form as

$$\begin{aligned}&Tr[(-ie_Q\gamma_\mu)S_F(-q_3)\mathit{\Gamma}_p(\hat{q}_b)S_F(q_2)\mathit{\Gamma}_v(\hat{q}_a)S_F(-q_1)]\\ &= \frac{imC'}{\triangle_1\triangle_2\triangle_3}\left\{Tr(\gamma_\mu\not{q}_3\gamma_5\not{q}_2\not{\epsilon} - Tr(\gamma_\mu\not{q}_3\gamma_5\not{\epsilon}\not{q}_1) + Tr(\gamma_\mu\gamma_5\not{q}_2\not{\epsilon}\not{q}_1)\right\}.\end{aligned} \tag{4.1.12}$$

But

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4. \tag{4.1.13}$$

Then by using the commutation relation $\gamma_5\gamma_\mu = -\gamma_\mu\gamma_5$ equation (4.1.12) can be written in the modified form as

$$\begin{aligned}&Tr[(-ie_Q\gamma_\mu)S_F(-q_3)\mathit{\Gamma}_p(\hat{q}_b)S_F(q_2)\mathit{\Gamma}_v(\hat{q}_a)S_F(-q_1)]\\ &= \frac{imC'}{\triangle_1\triangle_2\triangle_3}\left\{Tr(\gamma_5\gamma_\mu\not{q}_3\not{q}_2\not{\epsilon} - Tr(\gamma_5\gamma_\mu\not{q}_3\not{\epsilon}\not{q}_1) - Tr(\gamma_5\gamma_\mu\not{q}_2\not{\epsilon}\not{q}_1)\right\}.\end{aligned} \tag{4.1.14}$$

Now let a_μ=$(0,1,0,0)$ where 1 represents the μ component of a. Then the above equation can be written in terms of $\not{a}$ as

$$\begin{aligned}&Tr[(-ie_Q\gamma_\mu)S_F(-q_3)\mathit{\Gamma}_p(\hat{q}_b)S_F(q_2)\mathit{\Gamma}_v(\hat{q}_a)S_F(-q_1)]\\ &= \frac{imC'}{\triangle_1\triangle_2\triangle_3}\left\{Tr(\gamma_5\not{a}\not{q}_3\not{q}_2\not{\epsilon} - Tr(\gamma_5\not{a}\not{q}_3\not{\epsilon}\not{q}_1) - Tr(\gamma_5\not{a}\not{q}_2\not{\epsilon}\not{q}_1)\right\}.\end{aligned} \tag{4.1.15}$$

But

$$Tr(\gamma_5 \not{a}\not{b}\not{c}\not{d}) = 4\epsilon_{\alpha\beta\gamma\delta} a_\alpha b_\beta c_\gamma d_\delta. \tag{4.1.16}$$

where $\epsilon_{\alpha\beta\gamma\delta}$ is a completely antisymmetric tensor. That is

$$\epsilon_{\alpha\beta\gamma\delta} \begin{cases} = +1 \text{ if } \alpha,\beta,\gamma,\delta \text{ are even permutation of (1,2,3,4)} \\ = -1 \text{ if } \alpha,\beta,\gamma,\delta \text{ are odd permutation of (1,2,3,4) and} \\ = 0 \text{ if any two indices are equal.} \end{cases}$$

So by using the above relation the equation for the trace can be simplified and written as

$$\begin{aligned} & Tr[(-ie_Q\gamma_\mu)S_F(-q_3)\mathit{\Gamma}_p(\hat{q}_b)S_F(q_2)\mathit{\Gamma}_v(\hat{q}_a)S_F(-q_1)] \\ & = \frac{4imC'\varepsilon_\lambda}{\Delta_1\Delta_2\Delta_3}\left\{\epsilon_{\mu\gamma\beta\lambda}q_{3\gamma}q_{2\beta} - \epsilon_{\mu\gamma\lambda\alpha}q_{3\gamma}q_{1\alpha} - \epsilon_{\mu\beta\lambda\alpha}q_{2\beta}q_{1\alpha}\right\}. \end{aligned} \tag{4.1.17}$$

Inserting the values obtained for the trace part and by substituting for C' from equation (4.1.8) the amplitude can be written as

$$\begin{aligned} M_1 = & \frac{-4ime_QC\varepsilon_\lambda}{(2\pi)^2}\int \frac{d^3\hat{q}_a}{(2\pi)^4}M_a d\sigma_a \frac{d^3\hat{q}_b}{(2\pi)^4}M_b d\sigma_b N_p D_p(\hat{q}_b)\phi_p(\hat{q}_b)N_v D_v(\hat{q}_a)\phi_v(\hat{q}_a) \\ & \times\left(\frac{\epsilon_{\mu\gamma\beta\lambda}q_{3\gamma}q_{2\beta} - \epsilon_{\mu\gamma\lambda\alpha}q_{3\gamma}q_{1\alpha} - \epsilon_{\mu\beta\lambda\alpha}q_{2\beta}q_{1\alpha})}{[\omega_b^2 - M_b^2(\frac{1}{2}-\sigma_b)^2][\omega_a^2 - M_a^2(\frac{1}{2}-\sigma_a)^2][\omega_a^2 - M_a^2(\frac{1}{2}+\sigma_a)^2]}\right). \end{aligned} \tag{4.1.18}$$

To integrate over the complex σ plane let the other terms which are independent of σ be C''. That is

$$\begin{aligned} C'' = & \frac{-4ime_QC\varepsilon_\lambda}{(2\pi)^2}(\frac{1}{(2\pi)^8})(\epsilon_{\mu\gamma\beta\lambda}q_{3\gamma}q_{2\beta} - \epsilon_{\mu\gamma\lambda\alpha}q_{3\gamma}q_{1\alpha} - \epsilon_{\mu\beta\lambda\alpha}q_{2\beta}q_{1\alpha}) \\ & \times(N_pD_p(\hat{q}_b)\phi_p(\hat{q}_b)N_vD_v(\hat{q}_a)\phi_v(\hat{q}_a)). \end{aligned} \tag{4.1.19}$$

Therefore, eqn.(4.1.18) can be written as

$$M_1 = C''\int M_a d\sigma_a \int M_b d\sigma_b \left(\frac{1}{[\omega_b^2 - M_b^2(\frac{1}{2}-\sigma_b)^2][\omega_a^2 - M_a^2(\frac{1}{2}-\sigma_a)^2][\omega_a^2 - M_a^2(\frac{1}{2}+\sigma_a)^2]}\right). \tag{4.1.20}$$

As it can easily be seen from equation (4.1.20) the poles of the integration are at $\sigma_a = \frac{1}{2} \pm \frac{\omega_a}{M_a}$, $\sigma_a = \frac{-1}{2} \pm \frac{\omega_a}{M_a}$ and $\sigma_b = \frac{1}{2} \pm \frac{\omega_b}{M_b}$. By taking those terms which are independent of σ_b as constant, the integration over σ_b can be evaluated as

$$\int_{-\infty}^{\infty} \frac{M_b d\sigma_b}{\omega_b^2 - M_b^2(\frac{1}{2}-\sigma_b)^2} = \int_{-\infty}^{\infty} \frac{d\sigma_b}{M_b(\frac{\omega_b^2}{M_b^2} - (\frac{1}{2}-\sigma_b)^2)}. \tag{4.1.21}$$

Since the poles of the integration lie on the real axis of σ_b,it is possible to shift the position of the pole by introducing an infinitesimal imaginary increment $i\epsilon$ to the poles. Therefore,equation (4.1.21) can be written as

$$\int_{-\infty}^{\infty} \frac{M_b d\sigma_b}{\omega_b^2 - M_b^2(\frac{1}{2}-\sigma_b)^2} = \int_{-\infty}^{\infty} \frac{d\sigma_b}{M_b(\sigma_b + \frac{1}{2} + \frac{\omega_b}{M_b} - i\epsilon)(\sigma_b + \frac{1}{2} - \frac{\omega_b}{M_b} + i\epsilon)}$$
$$= I_b = 2\pi i \sum(\text{ residues in the upper half plane}). \tag{4.1.22}$$

The residue at this pole is

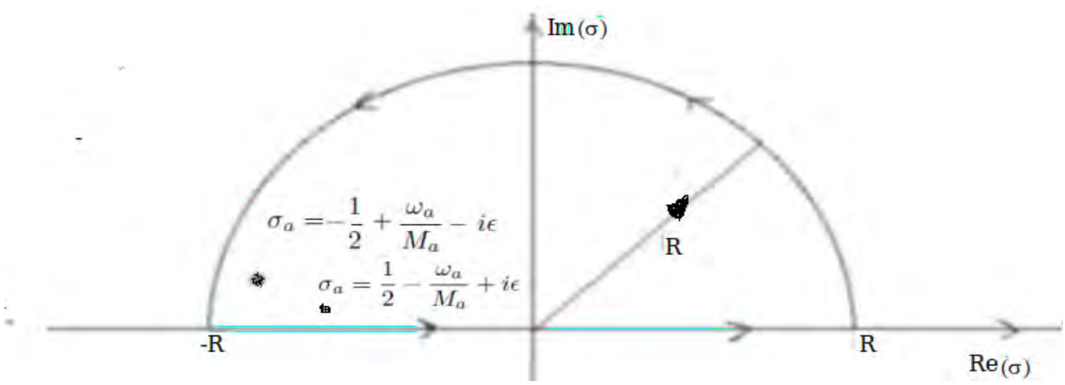


Figure 4.2: Digram for pole integration over σ_a complex plane

$$a_{-1} = \lim_{\sigma_b \to (\frac{1}{2} - \frac{\omega_b}{M_b} + i\epsilon)} \frac{(\sigma_b - \frac{1}{2} + \frac{\sigma_b}{M_b} - i\epsilon)}{M_b(\sigma_b - \frac{1}{2} - \frac{\omega_b}{M_b} + i\epsilon)(\sigma_b + \frac{1}{2} - \frac{\omega_b}{M_b} - i\epsilon)}$$
$$= -\frac{1}{2\omega_b}. \tag{4.1.23}$$

By substituting for the residue in equation (4.1.22) from equation (4.1.23) the pole integration over the complex σ_b becomes

$$I_b = -\frac{i\pi}{\omega_b}. \tag{4.1.24}$$

Similarily the pole integration over σ_a can be evaluated as

$$\begin{aligned} I_a = & \int_{-\infty}^{\infty} \frac{M_a d\sigma_a}{M_a^4[(\sigma_a - \frac{1}{2})^2 - \frac{\omega_a^2}{M_a^2}][(\sigma_a + \frac{1}{2})^2 - \frac{\omega_a^2}{M_a^2}]} \\ & = \frac{1}{M_a^3} \int_{-\infty}^{\infty} \frac{d\sigma_a}{[(\sigma_a - \frac{1}{2})^2 - \frac{\omega_a^2}{M_a^2}][(\sigma_a + \frac{1}{2})^2 - \frac{\omega_a^2}{M_a^2}]} \\ & = 2\pi i \sum (\text{ residues in the upper half plane}). \end{aligned} \tag{4.1.25}$$

For this pole integration the poles of the integration are at $\sigma_a = \frac{1}{2} \pm \frac{\omega_a}{M_a}$ and at $\sigma_a = -\frac{1}{2} \pm \frac{\omega_a}{M_a}$. The residue at $\sigma_a = \frac{1}{2} - \frac{\omega_a}{M_a}$ is

$$\begin{aligned} a_{-1} = & \lim_{\sigma_a \to \frac{1}{2} - \frac{\omega_a}{M_a} + i\epsilon} \frac{(\sigma_a - \frac{1}{2} + \frac{\omega_a}{M_a} - i\epsilon)}{(\sigma_a - \frac{1}{2} + \frac{\omega_a}{M_a} - i\epsilon)(\sigma_a - \frac{1}{2} - \frac{\omega_a}{M_a} + i\epsilon)(\sigma_a + \frac{1}{2} - \frac{\omega_a}{M_a} + i\epsilon)(\sigma_a + \frac{1}{2} + \frac{\omega_a}{M_a} - i\epsilon)} \\ & = \frac{1}{(-\frac{2\omega_a}{M_a})(1 - \frac{2\omega_a}{M_a})} \\ & = \frac{M_a^2}{2\omega_a(2\omega_a - M_a)}. \end{aligned} \tag{4.1.26}$$

Similarily the residue at $\sigma_a = -\frac{1}{2} - \frac{\omega_a}{M_a}$ is

$$\begin{aligned} a_{-1} = & \lim_{\sigma_a \to -\frac{1}{2} - \frac{\omega_a}{M_a} + i\epsilon} \frac{(\sigma_a + \frac{1}{2} + \frac{\omega_a}{M_a} - i\epsilon)}{(\sigma_a - \frac{1}{2} + \frac{\omega_a}{M_a} - i\epsilon)(\sigma_a - \frac{1}{2} - \frac{\omega_a}{M_a} + i\epsilon)(\sigma_a + \frac{1}{2} - \frac{\omega_a}{M_a} + i\epsilon)(\sigma_a + \frac{1}{2} + \frac{\omega_a}{M_a} - i\epsilon)} \\ & = \frac{1}{(-\frac{2\omega_a}{M_a})(1 + \frac{2\omega_a}{M_a})} \\ & = \frac{-M_a^2}{2\omega_a(2\omega_a + M_a)}. \end{aligned} \tag{4.1.27}$$

Finally the sum of the residues will be

$$\begin{aligned} \sum(\text{residues}) = & \frac{M_a^2}{2\omega_a}\left(\frac{1}{-M_a + 2\omega_a + \frac{-1}{M_a + 2\omega_a}}\right) \\ = & \frac{M_a^3}{\omega_a}\left(\frac{1}{4\omega_a^2 - M_a^2}\right). \end{aligned} \tag{4.1.28}$$

By substituting for the sum of the residues from equation (4.1.28) into equation (4.1.25) the pole integration over σ_a will be

$$I_a = \frac{2\pi i}{\omega_a}\left(\frac{1}{4\omega_a^2 - M_a^2}\right). \tag{4.1.29}$$

Hence the pole integration over σ_a and σ_b plane will be

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{M_b d\sigma_b M_a}{\Delta_1\Delta_2\Delta_3} = & \left(\frac{-\pi i}{\omega_b}\right)\left(\frac{2\pi i}{\omega_a}\right)\left(\frac{1}{4\omega_a^2 - M_a^2}\right)\\ & = \frac{2\pi^2}{\omega_a\omega_b}\left(\frac{1}{4\omega_a^2 - M_a^2}\right).\end{aligned} \tag{4.1.30}$$

Substituting for the result obtained for pole integration over σ_a and σ_b from eqn. 4.1.27 into eqn. 4.1.18 we get,

$$\begin{aligned}M_1 = & \frac{-4ime_Q C\varepsilon_\lambda}{(2\pi)^2}\left(\frac{2\pi^2}{\omega_a\omega_b}\right)\left(\frac{1}{4\omega_a^2 - M_a^2}\right)\int\frac{d^3\hat{q}_a}{(2\pi)^4}\frac{d^3\hat{q}_b}{(2\pi)^4}N_p D_p(\hat{q}_b)\phi_p(\hat{q}_b)N_v D_v(\hat{q}_a)\phi_v(\hat{q}_a)\\ & \times\left(\epsilon_{\mu\gamma\beta\lambda}q_{3\gamma}q_{2\beta} - \epsilon_{\mu\gamma\lambda\alpha}q_{3\gamma}q_{1\alpha} - \epsilon_{\mu\beta\lambda\alpha}q_{2\beta}q_{1\alpha}\right).\end{aligned} \tag{4.1.31}$$

Since the indices of the $q's$ in the above equation are independent the antisymmetric tensor ϵ can be taken out as a common factor by using the following rearrangement. That is by renaming $\alpha \to \beta$ in the second term and $\alpha \to \gamma$ in the third term. Therefore,

$$\begin{aligned}M_1 = & \frac{-4ime_Q C\varepsilon_\lambda}{(2\pi)^2}\left(\frac{2\pi^2}{\omega_a\omega_b}\right)\left(\frac{1}{4\omega_a^2 - M_a^2}\right)\int\frac{d^3\hat{q}_a}{(2\pi)^4}\frac{d^3\hat{q}_b}{(2\pi)^4}N_p D_p(\hat{q}_b)\phi_p(\hat{q}_b)N_v D_v(\hat{q}_a)\phi_v(\hat{q}_a)\\ & \times\left(\epsilon_{\mu\gamma\beta\lambda}q_{3\gamma}q_{2\beta} - \epsilon_{\mu\gamma\lambda\beta}q_{3\gamma}q_{1\beta} - \epsilon_{\mu\beta\lambda\gamma}q_{2\beta}q_{1\gamma}\right).\end{aligned} \tag{4.1.32}$$

Then by using the relation

$$\begin{aligned}\epsilon_{\mu\gamma\lambda\beta} &= -\epsilon_{\mu\gamma\beta\lambda}\\ \epsilon_{\mu\beta\lambda\gamma}q_{2\beta} &= \epsilon_{\mu\gamma\beta\lambda}q_{2\beta}\end{aligned} \tag{4.1.33}$$

and finally by taking the common factor equation (4.1.32) can be written as

$$\begin{aligned}M_1 = & \frac{-4ime_Q C\varepsilon_\lambda}{(2\pi)^2}\int\frac{d^3\hat{q}_a}{(2\pi)^4}\frac{d^3\hat{q}_b}{(2\pi)^4}\left(\frac{2\pi^2}{\omega_a\omega_b}\right)\left(\frac{1}{4\omega_a^2 - M_a^2}\right)N_p D_p(\hat{q}_b)\phi_p(\hat{q}_b)N_v D_v(\hat{q}_a)\phi_v(\hat{q}_a)\\ & \times\epsilon_{\mu\gamma\beta\lambda}\left(q_{3\gamma}q_{2\beta} + q_{3\gamma}q_{1\beta} - q_{2\beta}q_{1\gamma}\right).\end{aligned} \tag{4.1.34}$$

The Bethe-Salpeter normalizers N_p and N_v for η_c and J/ψ mesons respectively can be taken out of the integral operators and equation (4.1.34) can be written as,

$$\begin{aligned}M_1 = & \frac{-4ime_Q C\varepsilon_\lambda}{(2\pi)^2}N_p N_v\int\frac{d^3\hat{q}_a}{(2\pi)^4}\frac{d^3\hat{q}_b}{(2\pi)^4}\left(\frac{2\pi^2}{\omega_a\omega_b}\right)\left(\frac{1}{4\omega_a^2 - M_a^2}\right)D_p(\hat{q}_b)\phi_p(\hat{q}_b)D_v(\hat{q}_a)\phi_v(\hat{q}_a)\\ & \times\epsilon_{\mu\gamma\beta\lambda}\left(q_{3\gamma}q_{2\beta} + q_{3\gamma}q_{1\beta} - q_{2\beta}q_{1\gamma}\right).\end{aligned} \tag{4.1.35}$$

Since an integration is needed to obtain the amplitude and since the propagators of the quark depend on the relative momenta q_a and q_b one can expect that the calculation is very complicated. To simplify the calculation, we assume that the propagators of the quark and the gluon are independent of relative momenta q_a and q_b. This simplification is appropriate since the masses of heavy quarks are large compared with the relative momenta,which are of order $\alpha_s m_Q$. Then the momenta $q_i's$ (i=1,2,3)are large compared with the relative momenta q_a and q_b . One may expect that, in the heavy quark limit, the calculation without taking into account the relative momenta should be exact. So for the heavy quark approximation, that is for the condition the mass of the quarks is much larger than the internal relative momenta, we can take the relation

$$\begin{aligned} q_{1\mu} &= \frac{1}{2}P_{a\mu} + q_{a\mu} \simeq \frac{1}{2}P_{a\mu} \\ q_{2\mu} &= \frac{1}{2}P_{a\mu} - q_{a\mu} \simeq \frac{1}{2}P_{a\mu} \\ q_{3\mu} &= \frac{1}{2}P_{b\mu} + q_{b\mu} \simeq \frac{1}{2}P_{b\mu}. \end{aligned} \tag{4.1.36}$$

Substituting from equation (4.1.36) into equation (4.1.35) we obtain

$$\begin{aligned} M_1 = \quad & \frac{-4ime_QC\varepsilon_\lambda}{(2\pi)^2}N_pN_v\epsilon_{\mu\gamma\beta\lambda}\left(\frac{1}{4}P_{b\gamma}P_{a\beta} + \frac{1}{4}P_{b\gamma}P_{a\beta} - \frac{1}{4}P_{a\beta}P_{b\gamma}\right) \\ & \times\int \frac{d^3\hat{q}_a}{(2\pi)^4}\frac{d^3\hat{q}_b}{(2\pi)^4}\left(\frac{2\pi^2}{\omega_a\omega_b}\right)\left(\frac{1}{4\omega_a^2 - M_a^2}\right)D_p(\hat{q}_b)\phi_p(\hat{q}_b)D_v(\hat{q}_a)\phi_v(\hat{q}_a) \\ & \Rightarrow M_{fi} = \frac{-4ime_QC}{(2\pi)^2}N_pN_v\varepsilon_\lambda\epsilon_{\mu\gamma\beta\lambda}(\frac{1}{4}P_{b\gamma}P_{a\beta})\int\frac{d^3\hat{q}_a}{(2\pi)^4}\frac{d^3\hat{q}_b}{(2\pi)^4}\left(\frac{2\pi^2}{\omega_a\omega_b}\right)\left(\frac{1}{4\omega_a^2 - M_a^2}\right) \\ & \times D_p(\hat{q}_b)\phi_p(\hat{q}_b)D_v(\hat{q}_a)\phi_v(\hat{q}_a). \end{aligned} \tag{4.1.37}$$

Substituting for $D_p(\hat{q}_b)$ and $D_v(\hat{q}_a)$ from equation (3.2.47) into equation (4.1.37) we will get

$$\begin{aligned} M_1 = \quad & \frac{-4ime_QC\varepsilon_\lambda}{(2\pi)^2}N_pN_v\epsilon_{\mu\gamma\beta\lambda}(\frac{1}{4}P_{b\gamma}P_{a\beta})\int\frac{d^3\hat{q}_a}{(2\pi)^4}\frac{d^3\hat{q}_b}{(2\pi)^4}\left(\frac{2\pi^2}{\omega_a\omega_b}\right)\left(\frac{1}{4\omega_a^2 - M_a^2}\right) \\ & \times\omega_b\left(4\omega_b^2 - M_b^2\right)\phi_p(\hat{q}_b)\omega_a\left(4\omega_a^2 - M_a^2\right)\phi_v(\hat{q}_a) \\ & \Rightarrow M_1 = \frac{-4ime_QC}{(2\pi)^2}N_pN_v\varepsilon_\lambda\epsilon_{\mu\gamma\beta\lambda}(\frac{1}{4}P_{b\gamma}P_{a\beta}) \\ & \times\int\frac{d^3\hat{q}_a}{(2\pi)^4}\frac{d^3\hat{q}_b}{(2\pi)^4}\left(2\pi^2\right)\left(4\omega_b^2 - M_b^2\right)\phi_p(\hat{q}_b)\phi_v(\hat{q}_a). \end{aligned} \tag{4.1.38}$$

Furthermore, due to the heavy quarks in the mesons, we will make use of approximation

$$M_a \simeq M_b \simeq M \simeq 2m \tag{4.1.39}$$

in the calculation of the cross section in this paper. Then the amplitude will be

$$\begin{aligned} M_1 = \quad & \frac{-4ime_Q C\varepsilon_\lambda}{(2\pi)^2} N_p N_v \epsilon_{\mu\gamma\beta\lambda}(\frac{1}{4}P_{b\gamma}P_{a\beta}) \\ & \times \int \frac{d^3\hat{q}_a}{(2\pi)^4}\frac{d^3\hat{q}_b}{(2\pi)^4}\left(2\pi^2\right)\left(4\omega_b^2 - M^2\right)\phi_p(\hat{q}_b)\phi_v(\hat{q}_a). \end{aligned} \tag{4.1.40}$$

The ground state wave function[20] $\phi(\hat{q})$ has a gausian structure and is expressed as:

$$\phi(\hat{q}) \simeq e^{-\hat{q}^2/2\beta^2}. \tag{4.1.41}$$

The structure of β is given by

$$\begin{aligned} \beta^2 &= \left(\frac{1}{2}M\omega_{q\bar{q}}^2/\gamma^2\right)^{1/2} \\ \gamma^2 &= 1 + \frac{2\omega_{q\bar{q}}^2 C_0}{M\omega_0^2}. \end{aligned} \tag{4.1.42}$$

Where

$$\omega_{q\bar{q}}^2 = 2M\omega_0^2\alpha_s(M^2). \tag{4.1.43}$$

Inserting for $\phi_p(\hat{q}_b)$ and $\phi_v(\hat{q}_a)$ into equation (4.1.40) we will get

$$\begin{aligned} M_1 = \quad & \frac{-4ime_Q C\varepsilon_\lambda}{(2\pi)^2} N_p N_v \epsilon_{\mu\gamma\beta\lambda}(\frac{1}{4}P_{b\gamma}P_{a\beta}) \\ & \times \int \frac{d^3\hat{q}_a}{(2\pi)^4}\frac{d^3\hat{q}_b}{(2\pi)^4}\left(2\pi^2\right)\left(4\omega_b^2 - M^2\right)e^{-\hat{q}_b^2/2\beta^2}e^{-\hat{q}_a^2/2\beta^2} \\ & = \frac{-4ime_Q C}{(2\pi)^2} N_p N_v \varepsilon_\lambda \epsilon_{\mu\gamma\beta\lambda}(\frac{1}{4}P_{b\gamma}P_{a\beta}) \\ & \times \int \frac{d^3\hat{q}_a}{(2\pi)^4}\frac{d^3\hat{q}_b}{(2\pi)^4}\left(2\pi^2\right)\left(4(\hat{q}_b^2 + m^2) - 4m^2\right)e^{-\hat{q}_b^2/2\beta^2}e^{-\hat{q}_a^2/2\beta^2} \\ & \Rightarrow= \frac{-4ime_Q C}{(2\pi)^2} N_p N_v \varepsilon_\lambda \epsilon_{\mu\gamma\beta\lambda}(P_{b\gamma}P_{a\beta}) \\ & \times \int \frac{d^3\hat{q}_a}{(2\pi)^4}\frac{d^3\hat{q}_b}{(2\pi)^4}\left(2\pi^2\right)\left(\hat{q}_b^2\right)e^{-\hat{q}_b^2/2\beta^2}e^{-\hat{q}_a^2/2\beta^2}. \end{aligned} \tag{4.1.44}$$

Finally by substituting for C from equation 4.1.2 then the equation for the amplitude can be given by

$$
\begin{aligned}
M_1 = \quad & \frac{-4ime_Q}{(2\pi)^2}\frac{4}{3}\frac{[\bar{v}(p_2)(-ie\gamma_\mu)u(p_1)]}{s}N_pN_v\varepsilon_\lambda\epsilon_{\mu\gamma\beta\lambda}(\frac{1}{4}P_{b\gamma}P_{a\beta}) \\
& \times\int\frac{d^3\hat{q}_a}{(2\pi)^4}\frac{d^3\hat{q}_b}{(2\pi)^4}\left(2\pi^2\right)\left(4(\hat{q}_b^2+m^2)-4m^2\right)e^{-\hat{q}_b^2/2\beta^2}e^{-\hat{q}_a^2/2\beta^2} \\
& \Rightarrow = \frac{-16mee_Q}{3(2\pi)^2}\frac{[\bar{v}(p_2)(\gamma_\mu)u(p_1)]}{s}N_pN_v\varepsilon_\lambda\epsilon_{\mu\gamma\beta\lambda}(P_{b\gamma}P_{a\beta}) \\
& \times\int\frac{d^3\hat{q}_a}{(2\pi)^4}\frac{d^3\hat{q}_b}{(2\pi)^4}\left(2\pi^2\right)\left(\hat{q}_b^2\right)e^{-\hat{q}_b^2/2\beta^2}e^{-\hat{q}_a^2/2\beta^2}.
\end{aligned}
\tag{4.1.45}
$$

So the result obtained in equation (4.1.45) is the amplitude for diagram 1.

We now evaluate the amplitude for diagram 2 which can be evaluated in the same way as for diagram 1. To obtain the amplitude for diagram 2 it is enough to evaluate the trace part only. All the other terms take the same values they possess in diagram 1. Therefore the amplitude for diagram 2 is given by

$$
\begin{aligned}
M_2 = \quad & \frac{c_f[\bar{v}(p_2)(-ie\gamma_\mu)u(p1)]}{s}\int\frac{d^4q_a}{(2\pi)^4}\frac{d^4q_b}{(2\pi)^4} \\
& \times Tr[(-ie_Q\gamma_\mu)S_F(q_3)\mathit{\Gamma}_p(\hat{q}_b)S_F(-q_2)\mathit{\Gamma}_v(\hat{q}_a)S_F(q_1)].
\end{aligned}
\tag{4.1.46}
$$

Where the trace part can be expressed as,

$$
\begin{aligned}
& Tr[(-ie_Q\gamma_\mu)S_F(q_3)\mathit{\Gamma}_p(\hat{q}_b)S_F(-q_2)\mathit{\Gamma}_v(\hat{q}_a)S_F(q_1)] \\
& = \left(\frac{-ie_QN_pD_p(\hat{q}_b)\phi_p(\hat{q}_b)N_vD_v(\hat{q}_a)\phi_v(\hat{q}_a)}{(2\pi)^2\ \triangle_1\triangle_2\triangle_3}\right) \\
& \times Tr\left\{\gamma_\mu(-i\not{q}_3+m)\Gamma_5(\not{q}_2+m)\not{\epsilon}(-i\not{q}_1+m)\right\}.
\end{aligned}
\tag{4.1.47}
$$

Putting

$$
C' = \frac{-e_QN_pD_p(\hat{q}_b)\phi_p(\hat{q}_b)N_vD_v(\hat{q}_a)\phi_v(\hat{q}_a)}{(2\pi)^2}.
\tag{4.1.48}
$$

we get

$$
\begin{aligned}
& Tr[(-ie_Q\gamma_\mu)S_F(q_3)\mathit{\Gamma}_p(\hat{q}_b)S_F(-q_2)\mathit{\Gamma}_v(\hat{q}_a)S_F(q_1)] \\
& = \frac{iC'}{\triangle_1\triangle_2\triangle_3}Tr\left\{\gamma_\mu(-i\not{q}_3+m)\Gamma_5(\not{q}_2+m)\not{\epsilon}(-i\not{q}_1+m)\right\}.
\end{aligned}
\tag{4.1.49}
$$

Expanding the terms in equation (4.1.49) we obtain

$$Tr[(-ie_Q\gamma_\mu)S_F(q_3)\mathit{\Gamma}_p(\hat{q}_b)S_F(-q_2)\mathit{\Gamma}_v(\hat{q}_a)S_F(q_1)]$$
$$= \frac{iC'}{\triangle_1\triangle_2\triangle_3}Tr[-i\gamma_\mu \not{q}_3\Gamma_5\not{q}_2\not{\epsilon}\not{q}_1 + m\gamma_\mu\not{q}_3\Gamma_5\not{q}_2\not{\epsilon} - m\gamma_\mu\not{q}_3\Gamma_5\not{\epsilon}\not{q}_1 - im^2\gamma_\mu\not{q}_3\Gamma_5\not{\epsilon}$$
$$+m\gamma_\mu\Gamma_5\not{q}_2\not{\epsilon}\not{q}_1 + im^2\gamma_\mu\Gamma_5\not{q}_2\not{\epsilon} - im^2\gamma_\mu\Gamma_5\not{q}_2\not{\epsilon}\not{q}_1 + m^3\gamma_\mu\Gamma_5\not{\epsilon}]. \tag{4.1.50}$$

From the above trace terms only the trace of three terms will survive. Thus, we get

$$Tr[(-ie_Q\gamma_\mu)S_F(q_3)\mathit{\Gamma}_p(\hat{q}_b)S_F(-q_2)\mathit{\Gamma}_v(\hat{q}_a)S_F(q_1)]$$
$$= \frac{iC'}{\triangle_1\triangle_2\triangle_3}m\left(Tr(\gamma_\mu\not{q}_3\Gamma_5\not{q}_2\not{\epsilon}) - Tr(\gamma_\mu\not{q}_3\Gamma_5\not{\epsilon}\not{q}_1) + Tr(\gamma_\mu\Gamma_5\not{q}_2\not{\epsilon}\not{q}_1)\right). \tag{4.1.51}$$

By applying the relation

$$\gamma_\mu\mathit{\Gamma}_5 = -\mathit{\Gamma}_5\gamma_\mu. \tag{4.1.52}$$

we can write equation (4.1.51) as

$$Tr[(-ie_Q\gamma_\mu)S_F(q_3)\mathit{\Gamma}_p(\hat{q}_b)S_F(-q_2)\mathit{\Gamma}_v(\hat{q}_a)S_F(q_1)]$$
$$= \frac{iC'}{\triangle_1\triangle_2\triangle_3}m\left(Tr(\Gamma_5\gamma_\mu\not{q}_3\not{q}_2\not{\epsilon}) - Tr(\Gamma_5\gamma_\mu\not{q}_3\not{\epsilon}\not{q}_1) - Tr(\Gamma_5\gamma_\mu\not{q}_2\not{\epsilon}\not{q}_1)\right). \tag{4.1.53}$$

To write all terms in slashed notation,let

$$\gamma_\mu a_\mu = \not{a}, \text{where } a_\mu = (0,1,0,0)\text{ , and 1 is the } \mu \text{ component.}$$
$$\gamma_\alpha q_{1\alpha} = \not{q}_1$$
$$\gamma_\beta q_{2\beta} = \not{q}_2$$
$$\gamma_\gamma q_{3\alpha} = \not{q}_3$$
$$\gamma_\lambda\varepsilon_\lambda = \not{\epsilon}. \tag{4.1.54}$$

By using the relation given in equation (4.1.16) we can write equation (4.1.53) as

$$Tr[(-ie_Q\gamma_\mu)S_F(q_3)\mathit{\Gamma}_p(\hat{q}_b)S_F(-q_2)\mathit{\Gamma}_v(\hat{q}_a)S_F(q_1)]$$
$$= \frac{iC'}{\triangle_1\triangle_2\triangle_3}m\left(4\epsilon_{\mu\gamma\beta\lambda}q_{3\gamma}q_{2\beta}\epsilon_\lambda - 4\epsilon_{\mu\gamma\lambda\alpha}q_{3\gamma}\epsilon_\lambda q_{1\alpha} - 4\epsilon_{\mu\beta\lambda\alpha}q_{2\beta}\epsilon_\lambda q_{1\alpha}\right). \tag{4.1.55}$$

Using the antisymmetric relation for ϵ given in equation (4.1.33) equation (4.1.55) can simplified as

$$\begin{aligned}&Tr[(-ie_Q\gamma_\mu)S_F(q_3)\Gamma_p(\hat{q}_b)S_F(-q_2)\Gamma_v(\hat{q}_a)S_F(q_1)]\\&=\frac{4iC'm}{\triangle_1\triangle_2\triangle_3}\epsilon_{\mu\gamma\beta\lambda}\epsilon_\lambda\left(q_{3\gamma}q_{2\beta}+q_{3\gamma}q_{1\beta}-q_{2\beta}q_{1\gamma}\right).\end{aligned} \tag{4.1.56}$$

Inserting for q_1, q_2 and q_3 the approximate values given in equation (4.1.36) we obtain

$$\begin{aligned}&Tr[(-ie_Q\gamma_\mu)S_F(q_3)\Gamma_p(\hat{q}_b)S_F(-q_2)\Gamma_v(\hat{q}_a)S_F(q_1)]\\&=\frac{4iC'm}{\triangle_1\triangle_2\triangle_3}\epsilon_{\mu\gamma\beta\lambda}\epsilon_\lambda\left(\frac{1}{4}P_{b\gamma}P_{a\beta}+\frac{1}{4}P_{b\gamma}P_{a\beta}-\frac{1}{4}P_{a\beta}P_{b\gamma}\right).\end{aligned} \tag{4.1.57}$$

It is possible to write equation (4.1.57) in a simplified form as

$$\begin{aligned}&Tr[(-ie_Q\gamma_\mu)S_F(q_3)\Gamma_p(\hat{q}_b)S_F(-q_2)\Gamma_v(\hat{q}_a)S_F(q_1)]\\&=\frac{4iC'm}{\triangle_1\triangle_2\triangle_3}\epsilon_{\mu\gamma\beta\lambda}\epsilon_\lambda\left(\frac{1}{4}P_{b\gamma}P_{a\beta}\right).\end{aligned} \tag{4.1.58}$$

We get the result obtained in equation (4.1.58) which shows that the trace part for the amplitude of diagram 2 is the same as the trace part of the amplitude for diagram 1.

Since all the other remaining terms in the equation of the amplitude for diagram 2 are identical with their corresponding terms in the equation of the amplitude for diagram 1, we can easily conclude that the amplitude for both diagrams are equal. That is

$$\begin{aligned}M_2=M_1=\quad&\frac{-4ime_Q}{(2\pi)^2}\frac{4}{3}\frac{[\bar{v}(p_2)(-ie\gamma_\mu)u(p_1)]}{s}N_pN_v\varepsilon_\lambda\epsilon_{\mu\gamma\beta\lambda}(\frac{1}{4}P_{b\gamma}P_{a\beta})\\&\times\int\frac{d^3\hat{q}_a}{(2\pi)^4}\frac{d^3\hat{q}_b}{(2\pi)^4}\left(2\pi^2\right)\left(4(\hat{q}_b^2+m^2)-4m^2\right)e^{-\hat{q}_b^2/2\beta^2}e^{-\hat{q}_a^2/2\beta^2}\\&\Rightarrow=\frac{-16mee_Q}{3(2\pi)^2}\frac{[\bar{v}(p_2)(\gamma_\mu)u(p_1)]}{s}N_pN_v\varepsilon_\lambda\epsilon_{\mu\gamma\beta\lambda}(P_{b\gamma}P_{a\beta})\\&\times\int\frac{d^3\hat{q}_a}{(2\pi)^4}\frac{d^3\hat{q}_b}{(2\pi)^4}\left(2\pi^2\right)\left(\hat{q}_b^2\right)e^{-\hat{q}_b^2/2\beta^2}e^{-\hat{q}_a^2/2\beta^2}\end{aligned} \tag{4.1.59}$$

The total amplitude for the process $e^+e^- \to J/\psi + \eta_c$ can be obtained by a linear superposition of amplitudes M_1 and M_2 of the diagrams given in figure (4.1)as

$$\begin{aligned} M = \quad & M_1 + M_2 = 2M_1 \\ & \Rightarrow M = \frac{-32mee_Q}{3(2\pi)^2} \frac{[\bar{v}(p_2)(\gamma_\mu)u(p_1)]}{s} N_p N_v \varepsilon_\lambda \epsilon_{\mu\gamma\beta\lambda}(P_{b\gamma}P_{a\beta}) \\ & \times \int \frac{d^3\hat{q}_a}{(2\pi)^4} \frac{d^3\hat{q}_b}{(2\pi)^4} \left(2\pi^2\right) \left(\hat{q}_b^2\right) e^{-\hat{q}_b^2/2\beta^2} e^{-\hat{q}_a^2/2\beta^2}. \end{aligned} \tag{4.1.60}$$

4.2 The total cross section

The unpolarized total cross section is obtained by summing over various J/ψ spin states and averaging over those of the initial states of e^+e^- as [26],

$$\sigma = \frac{1}{32\pi} \frac{\sqrt{s - 16m^2}}{s^{3/2}} \int \frac{1}{4} \sum_{spins} |M_{total}|^2 d(cos\theta). \tag{4.2.1}$$

where the masses of the electron and positron are ignored in the calculation. This is because of the masses of the electron and positron are much smaller than the masses of the quarks. The explicit expression for the total amplitude can be obtained as follows.

$$\begin{aligned} \sum_{s_1,s_2,s_1',s_2'} |M_{total}|^2 = \quad & \sum_{s_1,s_2,s_1',s_2'} \left(\frac{-32mee_Q}{3(2\pi)^2}\right)^2 \left(\frac{\bar{v}(p_2)(\gamma_\mu)u(p_1)}{s}\right)^2 (\varepsilon_\lambda \epsilon_{\mu\gamma\beta\lambda} P_{b\gamma} P_{a\beta})^2 \\ & \times \left(N_p N_v \int \frac{d^3\hat{q}_a}{(2\pi)^4} \frac{d^3\hat{q}_b}{(2\pi)^4} \left(2\pi^2\right) \left(\hat{q}_b^2\right) e^{-\hat{q}_b^2/2\beta^2} e^{-\hat{q}_a^2/2\beta^2}\right)^2 \end{aligned} \tag{4.2.2}$$

where s_1, s_2, s_1', s_2' are the spin states.

To find the square of the amplitude we first obtain the square of each term one by one. For instance, the square of

$$\begin{aligned} & \sum_{s_1,s_2,s_1',s_2'} (\bar{v}(p_2)(\gamma_\mu)u(p_1))^2 = \sum_{s_1,s_2,s_1',s_2'} [\bar{v}(p_2)_\alpha(\gamma_\mu)_{\alpha\beta}u(p_1)_\beta]^\dagger [\bar{v}'(p_2)_\gamma(\gamma_\nu)_{\gamma\delta}u'(p_1)_\delta] \\ & = \sum_{s_1,s_2,s_1',s_2'} [\bar{u}(p_1)_\alpha(\bar{\gamma}_\mu)_{\alpha\beta}v(p_2)_\beta] [\bar{v}'(p_2)_\gamma(\gamma_\nu)_{\gamma\delta}u'(p_1)_\delta]. \end{aligned} \tag{4.2.3}$$

But

$$\sum_{s_2,s_2'} v(p_2)\bar{v}'(p_2) = \frac{-i\not{p}_2 - m_e}{2m_e}\delta_{s_2s_2'}, \text{ where } m_e \text{ is the mass of the positron.} \tag{4.2.4}$$

Substituting this result from equation (4.2.4) into equation (4.2.3) we obtain

$$\sum_{s_1,s_2,s_1',s_2'} (\bar{v}(p_2)(\gamma_\mu)u(p_1))^2 = \sum_{s_1,s_1'} \left[\bar{u}(p_1)_\alpha(\bar{\gamma}_\mu)_{\alpha\beta}\left(\frac{-i\not{p}_2 - m_e}{2m_e}\right)_{\alpha\gamma}(\gamma_\nu)_{\gamma\delta}u'(p_1)_\delta\right]$$
$$= \sum_{s_1,s_1'} Tr\left[\bar{u}(p_1)(\bar{\gamma}_\mu)\left(\frac{-i\not{p}_2 - m_e}{2m_e}\right)(\gamma_\nu)u'(p_1)\right]$$
$$= \sum_{s_1,s_1'} Tr\left[(\bar{\gamma}_\mu)\left(\frac{-i\not{p}_2 - m_e}{2m_e}\right)(\gamma_\nu)u'(p_1)\bar{u}(p_1)\right]. \tag{4.2.5}$$

But

$$\sum_{s_1,s_1'} u'(p_1)\bar{u}(p_1) = \left(\frac{-i\not{p}_1 + m_e}{2m_e}\right)\delta_{s_1s_1'},\ \text{where}m_e\text{is the mass of the electron.} \tag{4.2.6}$$

Then by substituting this result into equation (4.2.5) we get

$$(\bar{v}(p_2)(\gamma_\mu)u(p_1))^2 = Tr\left[\bar{\gamma}_\mu\left(\frac{-i\not{p}_2 - m_e}{2m_e}\right)\gamma_\nu\left(\frac{-i\not{p}_1 + m_e}{2m_e}\right)\right]$$

but $\bar{\gamma}_\mu = -\gamma_\mu$

$$Tr\left[-\gamma_\mu\left(\frac{-i\not{p}_2 - m_e}{2m_e}\right)\gamma_\nu\left(\frac{-i\not{p}_1 + m_e}{2m_e}\right)\right]$$
$$= \frac{1}{4m_e^2}\left[Tr(\gamma_\mu\not{p}_2\gamma_\nu\not{p}_1) + iTr(\gamma_\mu\not{p}_2\gamma_\nu) - im_eTr(\gamma_\mu\gamma_\nu\not{p}_1) + m_e^2Tr(\gamma_\mu\gamma_\nu)\right]. \tag{4.2.7}$$

Since the trace for odd numbered γ matrices is zero,from the above expression only two terms servive. Therefore equation (4.2.7) can be written as

$$(\bar{v}(p_2)(\gamma_\mu)u(p_1))^2 = \frac{1}{4m_e^2}\left[Tr(\gamma_\mu\not{p}_2\gamma_\nu\not{p}_1) + m_e^2Tr(\gamma_\mu\gamma_\nu)\right]. \tag{4.2.8}$$

To evaluate the trace of the γ matrices let$a_\mu = (0,1,0,0)$ where 1 is the μ component and $b_\nu = (0,0,1,0)$ where 1 is the ν component of b. Thus we can write equation (4.2.8) as,

$$(\bar{v}(p_2)(\gamma_\mu)u(p_1))^2 = \frac{1}{4m_e^2}\left[Tr(\not{a}\not{p}_2\not{b}\not{p}_1) + m_e^2Tr(\gamma_\mu\gamma_\nu)\right] \tag{4.2.9}$$

where $\not{a} = \gamma_\mu a_\mu$ and $\not{b} = \gamma_\nu b_\nu$. But

$$Tr(\not{a}\not{p}_2\not{b}\not{p}_1) = 4\left[(a_\mu p_{2\mu})(b_\nu p_{1\nu}) - (p_1 \dot{p}_2)\delta_{\mu\nu} + (a_\mu p_\mu)(b_\nu p_{1\upsilon})\right]$$

$$= 4\left[p_{2\mu} p_{1\nu}) - (p_1.p_2)\delta_{\mu\nu} + (p_{1\mu} p_{2\nu})\right]$$

$$\text{and } Tr(\gamma_\mu \gamma_\nu) = 4\delta_{\mu\nu}. \tag{4.2.10}$$

Substituting this result into equation (4.2.9) we obtain

$$(\bar{v}(p_2)(\gamma_\mu)u(p_1))^2 = \frac{1}{m_e^2}\left[p_{2\mu}p_{1\nu}) - (p_1.p_2)\delta_{\mu\nu} + (p_{1\mu}p_{2\nu}) + m_e^2\delta_{\mu\nu}\right]$$

$$= \frac{1}{m_e^2}\left[p_{2\mu}p_{1\nu}) + (p_{1\mu}p_{2\nu}) - (p_1.p_2 - m_e^2)\delta_{\mu\nu}\right]. \tag{4.2.11}$$

Since p_1 and p_2 are the 4-momenta of the incoming leptons,lets introduce a leptonic tensor $L_{\mu\nu}$, where

$$L_{\mu\nu} = \frac{1}{m_e^2}\left[p_{2\mu}p_{1\nu}) + (p_{1\mu}p_{2\nu}) - (p_1.p_2 - m_e^2)\delta_{\mu\nu}\right]. \tag{4.2.12}$$

The second term which should be evaluated explicitly is

$$(\epsilon_{\mu\gamma\beta\lambda}\epsilon_\lambda P_{a\beta}P_{b\gamma})^2 = \epsilon_{\mu\gamma\beta\lambda}\epsilon_{\nu\sigma\rho\tau}\epsilon_\lambda^*\epsilon_\tau P_{a\beta}P_{b\gamma}P_{a\rho}P_{b\sigma}. \tag{4.2.13}$$

But

$$\sum_{\lambda,\tau}\epsilon_\lambda^*\epsilon_\tau = \delta_{\lambda\tau}. \tag{4.2.14}$$

This shows that $\epsilon_\lambda^*\epsilon_\tau \neq 0$ only if $\lambda = \tau$. Then by using this relation equation (4.2.13) can be written as

$$(\epsilon_{\mu\gamma\beta\lambda}\epsilon_\lambda P_{a\beta}P_{b\gamma})^2 = \epsilon_{\mu\gamma\beta\lambda}\epsilon_{\nu\sigma\rho\tau}P_{a\beta}P_{b\gamma}P_{a\rho}P_{b\sigma}. \tag{4.2.15}$$

But the product of two antisymmetric tensors in four dimensions with one index in common is given by

$$\begin{aligned}\epsilon_{\lambda\mu\gamma\beta}\epsilon_{\lambda\nu\sigma\rho} = \quad & -\delta_{\mu\rho}\delta_{\gamma\sigma}\delta_{\beta\nu} + \delta_{\mu\sigma}\delta_{\gamma\rho}\delta_{\beta\nu} - \delta_{\mu\nu}\delta_{\gamma\rho}\delta_{\beta\sigma} \\ & -\delta_{\mu\sigma}\delta_{\gamma\nu}\delta_{\beta\sigma} + \delta_{\mu\nu}\delta_{\gamma\sigma}\delta_{\beta\rho}.\end{aligned} \tag{4.2.16}$$

By plugging this result into equation (4.2.13) we obtain

$$\begin{aligned}\epsilon_{\mu\gamma\beta\lambda}\epsilon_{\nu\sigma\rho\tau}P_{a\beta}P_{b\gamma}P_{a\rho}P_{b\sigma} = & [-\delta_{\mu\rho}\delta_{\gamma\sigma}\delta_{\beta\nu} + \delta_{\mu\sigma}\delta_{\gamma\rho}\delta_{\beta\nu} - \delta_{\mu\nu}\delta_{\gamma\rho}\delta_{\beta\sigma} - \delta_{\mu\sigma}\delta_{\gamma\nu}\delta_{\beta\sigma}]P_{a\beta}P_{b\gamma}P_{a\rho}P_{b\sigma} \\ & +[\delta_{\mu\nu}\delta_{\gamma\sigma}\delta_{\beta\rho}]P_{a\beta}P_{b\gamma}P_{a\rho}P_{b\sigma} \\ & = [-P_{a\mu}P_{a\nu}P_b^2 + P_{a\nu}P_{b\mu}(P_a.P_b)P_{a\mu} + P_{b\nu}(P_a.P_b) - \delta_{\mu\nu}(P_a.P_b)^2 \\ & +\delta_{\mu\nu}P_a^2P_b^2 - P_{b\mu}P_{b\nu}P_a^2]. \qquad (4.2.17)\end{aligned}$$

Since P_{a1} and P_{b2} are the hadronic 4-momenta of the outgoing mesons,lets derive hadronic tensor $H_{\mu\nu}$ as,

$$\begin{aligned}H_{\mu\nu} = & [-P_{a\mu}P_{a\nu}P_b^2 + P_{a\nu}P_{b\mu}(P_a.P_b)P_{a\mu} + P_{b\nu}(P_a.P_b) - \delta_{\mu\nu}(P_a.P_b)^2 \\ & +\delta_{\mu\nu}P_a^2P_b^2 - P_{b\mu}P_{b\nu}P_a^2]. \qquad (4.2.18)\end{aligned}$$

By taking the dot product of Hadronic tensor in equation (4.2.18) with the leptonic tensor in equation (4.2.12) and simplifying we get

$$\begin{aligned}L_{\mu\nu}H_{\mu\nu} = & \frac{1}{m_e^2}\left\{-2(p_1.P_a)(p_2.P_a)P_b^2 + 2[(p_1.P_a)(p_2.P_b) + (p_1.P_b)(p_2.P_a)](P_a.P_b)\right\} \\ & +\frac{1}{m_e^2}\left\{-2(p_1.P_b)(p_2.P_b)P_a^2 - 2m_e^2[(P_a.P_b)^2 - P_a^2P_b^2]\right\} \qquad (4.2.19)\end{aligned}$$

where m_e is the mass of the electron.

By choosing the center of mass frame we use the following kinematics of collision process:

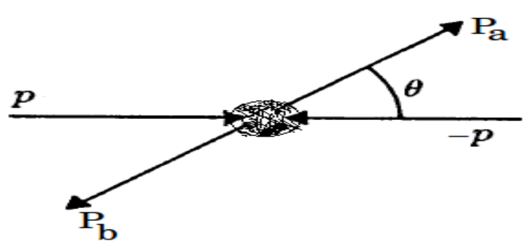


Figure 4.3: Two-body scattering in the center of mass frame.

$$\begin{aligned}p_1 = (\vec{p}_1, iE_1) = (\vec{p}, iE),\ p_1 = (\vec{p}_2, iE_1) = (-\vec{p}, iE) \\ P_a = (\vec{P}_a, iE_a) = (\vec{P}', iE'),\ P_b = (\vec{P}_b, iE_b) = (-\vec{P}', iE') \qquad (4.2.20)\end{aligned}$$

and we can also take the following relation by considering the angle between p_1 and P_a be θ.

$$p_1.p_2 = -(\vec{p}^2 + E^2), \text{ but } \vec{p}^2 = E^2 - m_0^2. \tag{4.2.21}$$

For ultrarelativistic collision,that is for $E^2 >> m_0^2$ we can approximate $\vec{p}^2 \simeq E^2$. Then using this relation equation (4.2.21) can be written as

$$p_1.p_2 = -(\vec{p}^2 + E^2) \simeq -2E^2. \tag{4.2.22}$$

Similarily,the results for dot products of various momenta can be expressed as,

$$\begin{aligned} p_1.P_a &= -EE' + E|\vec{P'}|cos\theta \; p_1.P_b = -EE' - E|\vec{P'}|cos\theta \\ p_2.P_a &= -EE' - E|\vec{P'}|cos\theta, \; p_2.P_b = -EE' + E|\vec{P'}|cos\theta \\ P_a.P_b &= -2E'^2 + M^2, \; P_a^2 = -M^2, \; P_b^2 = -M^2. \end{aligned} \tag{4.2.23}$$

Inserting these results into equation (4.2.19) and simplifying we obtain

$$L_{\mu\nu}H_{\mu\nu} = \frac{1}{m_e^2}\left\{8M^2E^2E'^2(1 + cos^2\theta) - 8E'^4E^2(1 + cos^2\theta) + 8M^2E'^2m_e^2 - 8E'^4m_e^2\right\}. \tag{4.2.24}$$

Finally, since $E^2 >> m_e^2$, we can write equation (4.2.24) as,

$$L_{\mu\nu}H_{\mu\nu} = \frac{1}{m_e^2}\left\{8M^2E^2E'^2(1 + cos^2\theta) - 8E^2E'^4(1 + cos^2\theta)\right\}. \tag{4.2.25}$$

Inserting for$L_{\mu\nu}H_{\mu\nu}$ from equation (4.2.25) into equation (4.2.2) we obtain

$$\begin{aligned} \sum_{s_1,s_2,s_1',s_2'} |M_{total}|^2 = & \frac{1}{m_e^2}\left(\frac{-32mee_Q}{3(2\pi)^2 s}\right)^2 \left\{8M^2E^2E'^2(1 + cos^2\theta) - 8E^2E'^4(1 + cos^2\theta)\right\} \\ & \times \left(N_pN_v \int \frac{d^3\hat{q}_a}{(2\pi)^4}\frac{d^3\hat{q}_b}{(2\pi)^4}\left(2\pi^2\right)\left(\hat{q}_b^2\right) e^{-\hat{q}_b^2/2\beta^2} e^{-\hat{q}_a^2/2\beta^2}\right)^2 \\ & = \frac{1}{m_e^2}\left(\frac{2^{10}m^2e^2e_Q^2}{3^2(2\pi)^4 s^2}\right)\left\{2M^2E^2E'^2(1 + cos^2\theta) - 2E^2E'^4(1 + cos^2\theta)\right\} \\ & \times \left(N_pN_v \int \frac{d^3\hat{q}_a}{(2\pi)^3}\frac{d^3\hat{q}_b}{(2\pi)^3}\left(\hat{q}_b^2\right) e^{-\hat{q}_b^2/2\beta^2} e^{-\hat{q}_a^2/2\beta^2}\right)^2. \end{aligned} \tag{4.2.26}$$

Now let

$$\psi_a = N_v \int \frac{d^3\hat{q}_a}{(2\pi)^3} e^{-\hat{q}_a^2/2\beta^2}$$
$$\psi_b = N_p \int \frac{d^3\hat{q}_b}{(2\pi)^3} \hat{q}_b^2 e^{-\hat{q}_b^2/2\beta^2}. \tag{4.2.27}$$

On substituting from equation (4.2.27) into equation (4.2.26)we obtain

$$\sum_{s_1,s_2,s_1',s_2'} |M_{total}|^2 = \frac{E^2}{m_e^2}\left(\frac{2^{10}m^2e^2e_Q^2}{3^2(2\pi)^4 s^2}\right)\left\{2M^2E'^2(1+cos^2\theta) - 2E'^4(1+cos^2\theta)\right\}\psi_a^2\psi_b^2. \tag{4.2.28}$$

Plugging for $\sum_{s_1,s_2,s_1',s_2'} |M_{total}|^2$ from equation (4.2.28) into equation (4.2.1) the total cross section can be written as

$$\begin{aligned}
\sigma &= \frac{E^2}{m_e^2}\frac{1}{32\pi}\frac{\sqrt{s-16m^2}}{s^{3/2}}\int_1^{-1}\frac{1}{4}\left(\frac{2^{10}m^2e^2e_Q^2}{3^2(2\pi)^4s^2}\right)\\
&\times\left\{2M^2E'^2(1+cos^2\theta) - 2E'^4(1+cos^2\theta)\right\}\psi_a^2\psi_b^2 d(cos\theta)\\
&= \frac{E^2}{m_e^2}\frac{1}{64\pi}\frac{\sqrt{s-16m^2}}{s^{7/2}}\int_1^{-1}\left(\frac{2^{10}m^2e^2e_Q^2}{3^2(2\pi)^4}\right)\\
&\left\{M^2E'^2(1+cos^2\theta) - E'^4(1+cos^2\theta)\right\}\psi_a^2\psi_b^2 d(cos\theta).
\end{aligned} \tag{4.2.29}$$

To integrate over cosθ, let y=$cos\theta$. For θ from 0 to π, the corresponding limit for y will be from 1 to -1. Then

$$\begin{aligned}
\sigma &= \frac{1}{64\pi}\frac{\sqrt{s-16m^2}}{s^{7/2}}\int_1^{-1}\left(\frac{2^{10}m^2e^2e_Q^2}{3^2(2\pi)^4}\right)\\
&\left[\frac{E^2}{m_e^2}\left(M^2E'^2(1+y^2) - E'^4(1+y^2)\right)\right]\psi_a^2\psi_b^2 dy\\
&= \frac{1}{64\pi}\frac{\sqrt{s-16m^2}}{s^{7/2}}\left(\frac{2^{10}m^2e^2e_Q^2}{3^2(2\pi)^4}\right)\\
&\left[\frac{E^2}{m_e^2}\left(M^2E'^2(y\big|_1^{-1} + \frac{y^3}{3}\big|_1^{-1}) - E'^4(y\big|_1^{-1} + \frac{y^3}{3}\big|_1^{-1})\right)\right]\psi_a^2\psi_b^2\\
&= \frac{1}{64\pi}\frac{\sqrt{s-16m^2}}{s^{7/2}}\left(\frac{2^{10}m^2e^2e_Q^2}{3^2(2\pi)^4}\right)\\
&\left[\frac{E^2}{m_e^2}\left(M^2E'^2(-2+\frac{-2}{3}) - E'^4(-2+\frac{-2}{3})\right)\right]\psi_a^2\psi_b^2.
\end{aligned} \tag{4.2.30}$$

So the final result for the total cross section will be

$$\sigma = \frac{1}{32\pi}\frac{\sqrt{s-16m^2}}{s^{7/2}}\left(\frac{2^{14}m^2e^2e_Q^2}{3^3(2\pi)^4}\right)\left[\frac{E^2}{m_e^2}\left(E'^2(E'^2-M^2)\right)\right].\psi_a^2\psi_b^2 \tag{4.2.31}$$

Since we are dealing with charm quarks with charge $e_q = \frac{+2}{3}e$, thus by substituting this value into equation (4.2.31) we obtain

$$\sigma = \frac{1}{32\pi}\frac{\sqrt{s-16m^2}}{s^{7/2}}\left(\frac{2^{14}m^2e^2(\frac{4}{9})}{3^3(2\pi)^4}\right)\left[\frac{E^2}{m_e^2}\left(E'^2(E'^2-M^2)\right)\right]\psi_a^2\psi_b^2. \tag{4.2.32}$$

By using the fine structure constant in natural units ($\hbar$=c=1),

$$\alpha_{em} = \frac{e^2}{4\pi} \tag{4.2.33}$$

we can also write the result obtained in equation (4.2.32) in a more compact form as

$$\sigma = \frac{1}{32\pi}\frac{\sqrt{s-16m^2}}{s^{7/2}}\left(\frac{2^{14}m^2\alpha_{em}^2}{3^5\pi^2}\right)\left[\frac{E^2}{m_e^2}\left(E'^2(E'^2-M^2)\right)\right]\psi_a^2\psi_b^2. \tag{4.2.34}$$

Substituting for M =2m into equation (4.2.34) we get the total cross section

$$\sigma = \frac{1}{32\pi}\frac{\sqrt{s-16m^2}}{s^{7/2}}\left(\frac{2^{16}m^2\alpha_{em}^2}{3^5\pi^2}\right)\left[\frac{E^2}{m_e^2}\left(E'^4-4m^2E'^2\right)\right]\psi_a^2\psi_b^2. \tag{4.2.35}$$

This is the analytical expression for total cross section for the double heavy quarkonium production in high energy electron-positron collisions. Where m=mass of the quarks, m_e= mass of the electron, E=energy of one of the incoming(initial state) particles (electron or positron), E' =energy of one of the outgoing(final state) particles(J/ψ or η_c) and $\sqrt{s}$ = total energy of either the incoming or outgoing particles in the center of mass frame. In equation (4.2.35) ψ_a and ψ_b can be calculated numerically by inserting appropriate values of the input parameters in the expressions of ψ_a and ψ_b.

We take the appropriate values of the input parameters calibrated earlier to mass spectra of q$\bar{q}$ mesons in the calculation, m=1.5GeV, s=10.6GeV, N_p=0.0168, N_v=0.0121 ,$\beta_{J/\psi}$=0.501 ,β_{η_c}=0.4388 ,we get the numerical value of the total cross section for the process $e^+e^- \to J/\psi + \eta_c$ =36.22fb.Our result is in broad agreement with the Belle's data for the same process $\sigma[e^+e^- \to J/\psi + \eta_c] \geq 25.6 \pm 2.8 \pm 3.4 fb$ which has only recently been available. This numerical result is obtained without incorporating the gluonic exhange in the diagram.

Chapter 5

CONCLUSION AND SUMMARY

In this thesis we first give a Lagrangian formulation of the gauge theories QED and QCD. We derive the Feynman rules for QCD . AS an application of the rule, we first present the calculation of cross section for the process q+$\bar{q}$ $\rightarrow$ g+g as a mathematical preliminary. We then derive the 2-particle Bethe-Salpeter equation. Then we study the cross section for the double heavy quarkonium production in high energy electron-positron collisions through the process of $e^+e^- \rightarrow J/\psi + \eta_c$ at the center of mass energy $\sqrt{s}$=10.6GeV in the framework of the Bethe-Salpeter wave equation. The amplitude and the cross section for the above process has been worked out rigorousily and the formula for them have been explicitly obtained. Then by taking the appropriate values of the input parameters calibrated earlier to mass spectra of q$\bar{q}$ mesons in the calculation and by making use of the heavy quark limit, we find the numerical value of the total cross section for the process $e^+e^- \rightarrow J/\psi + \eta_c$ =36.22fb. We compare this value with Belle's data for cross section for the same process $\sigma[e^+e^- \rightarrow J/\psi + \eta_c] \geq 25.6 \pm 2.8 \pm 3.4 fb$. Our result is in broad agreement with the Belle's data which has only recently been available. The small difference in the numerical results may be due to the fact that we have considered the lowest order process. The incorporation of gluonic exchange in the diagram would make the results more closer to the data.

For considering relativistic bound states, the Bethe-Salpeter equation which is firmly rooted in field theory is a powerful tool to evaluate the scattering amplitude to get the total cross section for the double heavy quarkonium production in high energy electron-positron collisions. Since the equation is based on the relativistic quantum field theory,and incorporates the results of summation over infinite ladders of one quantum exchanges between the interacting particles, it is expected that almost all relativistic effects are incorporated in the calculation. Furthermore, the covariant instantaneous approximation ensures that the vertex functions $\Gamma_p(\hat{q})$ and $\Gamma_v(\hat{q})$ used have a wide range of applicabiliy for a vast number of physical processes.

Appendices

Appendix A

A

A.1 Appendix A

A.1.1 The Units and relativistic notations

We used the units which are mostly used in most particle physics and quantum field theory books.

$$c = \hbar = 1 \tag{A.1.1}$$

In this system,

$$[Length] = [time] = [energy]^{-1} = [mass]^{-1}. \tag{A.1.2}$$

$$x_\mu = (\vec{x}, x_4)\text{where } x_4\text{=it,}\vec{x}\text{=}\vec{x}_i\text{,i=1,2,3.} \tag{A.1.3}$$

Here the indices i=1,2,3 represent the three spatial coordinates.

The covariant derivative ∂_μ is given by

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = (\vec{\nabla}, \frac{\partial}{i\partial t}). \tag{A.1.4}$$

In this representation the four momentum is given by

$$p = (\vec{p}, iE). \tag{A.1.5}$$

For a massive particle in the representation for μ=1 to 4 we have the relation

$$p^2 = p_\mu p_\mu = -E^2 + \vec{p}^2 = -m^2 \tag{A.1.6}$$

In this representation

$$p_\mu = p^\mu. \tag{A.1.7}$$

A.1.2 The Dirac gamma matrices

$$\gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix} \tag{A.1.8}$$

where σ_i represent the Pauli matrices and I represents a 2×2 identity matrix.

$$\gamma_i^2 = \gamma_4^2 = I \tag{A.1.9}$$

The anticommutation relation for gamma matrices is

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu \tag{A.1.10}$$

and

$$\gamma_\mu\gamma_\nu = 2\delta_{\mu\nu}. \tag{A.1.11}$$

The fifth Dirac gamma matrix which is derived from the other gamma matrices is given by

$$\Gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4. \tag{A.1.12}$$

$$\Gamma_5^2 = I \tag{A.1.13}$$

A.1.3 Trace theorems

Theorem 1. *Trace of odd product of γ matrices is zero.*

Proof: By using the anti commutation relation of the γ matrices and the property of the trace that is Tr(AB)=Tr(BA), for odd n the trace can be obtained as'

$$\begin{aligned} Tr(\not{a}_1\not{a}_2...\not{a}_n) = \quad & Tr(\not{a}_1\not{a}_2...\not{a}_n\Gamma_5\Gamma_5) \\ & = (-1)^n Tr(\Gamma_5\not{a}_1\not{a}_2...\not{a}_n)\Gamma_5) \\ & = (-1)^n Tr(\Gamma_5\Gamma_5\not{a}_1\not{a}_2...\not{a}_n) \end{aligned} \tag{A.1.14}$$

But $\Gamma_5\Gamma_5 = 1$. Then for odd n it becomes.

$$Tr(\not{a}_1\not{a}_2...\not{a}_n) = -Tr(\not{a}_1\not{a}_2...\not{a}_n)$$
$$\Rightarrow Tr(\not{a}_1\not{a}_2...\not{a}_n) = 0 \tag{A.1.15}$$

Theorem 2. *For any arbitrary four-vectors a and b,*
$Tr(\not{a}\not{b})$=*4a.b.*

Proof:

$$Tr(\not{a}\not{b}) = \frac{1}{2}(\not{a}\not{b} + \not{b}\not{a})$$
$$= \frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu)a_\mu b_\nu$$
$$= a_\mu b_\nu \delta_{\mu\nu} Tr(I) = 4a.b. \tag{A.1.16}$$

Theorem 3. $Tr(\Gamma_5\not{a}\not{b})$=*0*

Proof: To prove it we have to show that $Tr(\Gamma_5\gamma_\mu\gamma_\nu)$=0.

$$Tr(\Gamma_5\gamma_\mu\gamma_\nu) = Tr(\Gamma_5\gamma_\mu\gamma_\nu\gamma_\beta^{-1}\gamma_\beta)$$
$$= Tr(\gamma_\beta\Gamma_5\gamma_\mu\gamma_\nu\gamma_\beta^{-1})$$
$$= (-1)^3 Tr(\Gamma_5\gamma_\mu\gamma_\nu\gamma_\beta\gamma_\beta^{-1})$$
$$= -Tr(\Gamma_5\gamma_\mu\gamma_\nu)$$
$$\Rightarrow Tr(\Gamma_5\gamma_\mu\gamma_\nu) = 0 \tag{A.1.17}$$

Theorem 4. *For any arbitrary four-vectors a_1, a_2, a_3 and a_4,*
$Tr(\not{a}_1\not{a}_2\not{a}_3\not{a}_4)$=*4[*$(a_1.a_2)(a_3.a_4)$-$(a_1.a_4)(a_2.a_3)$ +$(a_1.a_3)(a_2.a_4)$*]*

Proof: This proof can be done by using the cyclic permutation of the trace and by using the relation, $\not{a}_1\not{a}_2$=-$\not{a}_2\not{a}_1$ +$2a_1.a_2$ and the relation given in theorem (2). Therefore,by applying the relations we will reach at a more simplified form,

$$Tr(\not{a}_1\not{a}_2\not{a}_3\not{a}_4) = (a_1.a_2)Tr(\not{a}_3\not{a}_4)$$
$$-(a_1.a_3)Tr(\not{a}_2\not{a}_4) + (a_1.a_4)Tr(\not{a}_2\not{a}_3)$$
$$= 4[(a_1.a_2)(a_3.a_4) - (a_1.a_4)(a_2.a_3) + (a_1.a_3)(a_2.a_4)] \tag{A.1.18}$$

Appendix B

B

B.1 Appendix B

B.1.1 The SU(3) algebra

The eight generators of the algebra are usually written down in terms of the following matrices, known as the Gell-Mann matrices,

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{B.1.1}$$

Bibliography

[1] B. Aubert et al. [BABAR Collaboration], Phys. Rev. Lett. 87 (2001) 162002 [arXiv:hep-ex/0106044].

[2] K. Abe et al. [BELLE Collaboration], Phys. Rev. Lett. 88 (2002) 052001 [arXiv:hep-ex/0110012].

[3] K. Abe et al. [Belle Collaboration], Phys. Rev. Lett. 89, (2002) 142001.

[4] Peter Cho, Adam K. Leibovich, Phys. Rev. D 54 (1996) 6690 [arXiv:hep-ph/9606229].

[5] S. Baek, P. Ko, J. Lee and H. S. Song, J. Korean Phys. Soc. 33 (1998) 97 [arXiv:hep-ph/9804455].

[6] F. Yuan, C.-F. Qiao, and K.T. Chao, Phys. Rev. D 56, (1997) 321.

[7] K. Hagiwara, E. Kou, and C.F. Qiao, to appear in PLB, hep-ph/0305102.

[8] E. Braaten and J. Lee, Phys. Rev. D67 (2003) 054007.

[9] K.Y. Liu, Z.G. He, and K.T. Chao, Phys. Lett. B557 (2003) 45.

[10] An Introduction to Quantum Field Theory by Miceal E. Peskin and Daniel V. Schroeder 1995

[11] W.Greiner.Joachim,Reinhardt.Quantum Electrodynamics,(3ed.,Springer,2003)

[12] Dr. Moliner 50, E-46100 ,arXiv:hep-ph/9505231v1 4 May 1995

[13] W.Greiner.S.Schramm.E.Stein,Quantum chromodyamics,second edition,2002

[14] Wolfgang Lucha,KhinMaung Maung,Franz Schöbert, Physical review D volume 63,056002.

[15] Modern Quantum Mechanics Revised Edition by J.J Sakurai 1994

[16] T. Kinoshita (editor), Quantum Electrodynamics, Advanced Series on Directions in High Energy Physics, Vol. 7 (World Scientific, Singapore, 1990);

[17] David McMahon,Quantum Field Theory Demystified,2008

[18] David griffiths,Introduction to elementary particles,John Weily and Sons,Inc. 1987

[19] Advanced Quantum Mechanics,J.J. Sakurai 1967, Chicago,Illinois.

[20] S.Bhatnagar,Physics of Quarks:New Research(Horizons in World Physics Vol.265),Nova science Pubilishers,W.Y USA(2009)(ISBN:978-1-60456-802-8)

[21] S.Bhatnagar,S.Y.Li,J.Phys.G32(2006)

[22] S.Bhatnagar,S-Y. Li,Intl.J.Mod.Phys.E18(2009)

[23] T. Kinoshita, Phys. Rev. D47 (1993) 5013.

[24] Y. Zhang, Y. Gao and K. Chao, Phys. Rev. Lett. 96,092001 (2006).

[25] B. Gong and J. Wang, arXiv:0712.4220[hep-ph].

[26] Xin-Heng Guo , Hong-Wei Ke , Xue-Qian Li , and Xing-Hua Wu,arXiv:0804.08949V1 [hep-ph] 7 Apr 2008

[27] Kaoru *Hagiwara*[1], Emi *Kou*[2], and Cong-Feng *Qiao*[3,4],arXiv:hep-ph/0305102v1 9 May,2003.

[28] J.P. Ma,arXiv:hep-ph/0405111V2,6,Jul,2004

Declaration

This thesis is my original work, has not been presented for a degree in any other University and that all the sources of material used for the thesis have been dully acknowledged.

Name: Teferi Mekonnen

Signature:

E-mail:tefemekon@yahoo.com

Place and time of submission: Addis Ababa University, June 2010

This thesis has been submitted for examination with my approval as University advisor.

Name: Dr. Shashank Bhatnagar

Signature: