GRADUATE SEMINAR REPORT
ON
STABILITY, CONTROLLABILTY AND OBSERVABILITY IN CONTROL THEORY

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Preface

This seminar report introduces important concepts in classical and modern control theories. Classical control theory is based on Laplace transforms and one of its objectives is to design-in terms of the transfer function-a system which satisfies certain assigned specifications; and modern control theory is not only applicable to linear autonomous systems but also to time-varying & is useful when dealing with non-linear systems-in contrast to the classical control theory. The approach is based on the concepts of state.

The seminar paper consists of three chapters .The first chapter provides some basic propositions of control theory which are very important for the study of stability, controllability and Observability. The second chapter discusses the concept of stability, the necessary and sufficient conditions for stability, and introduces some criteria of system stability. The third chapter deals with system controllability and observability, the necessary and sufficient conditions for controllability and observability of dynamic systems.

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1. SOME BASIC PRELIMINARIES OF CONTROL SYSTEM

In this chapter we will give some basic definitions, notions, properties and examples of control system which are important for the next chapters of stability, controllability and observability, that is, the main topic of this seminar paper. In this seminar paper a **system** refers dynamical system.

1.1. Basic Notions

The fundamental notion of control theory in the notion of a system.

A system is a combination of elements intended to act together to accomplish an objective.

Example 1.1:

A car's engine is a system whose elements are the carburetor, the ignition, the crank shaft etc. On a higher level, the car itself can be considered as a system with the engine as an element.

Definition 1.1:

A system becomes **dynamical** when one or more aspects of system change with time. The effects which originate outside the system and act on it directly and are not changed by changes in the system are known as **input** of the system. That is, the input of the system is the driving force applied to the system. The **output** is the response to the input to be obtained.

1.2. Review of Laplace Transforms and its Inverse

A very important instrument of control theory in the Laplace Transform.

Definition 1.2:

Given a function f (t) that satisfies the condition

$$\int_0^{\infty} |f(t)e^{-\alpha t}|\, dt < \infty \qquad (1.1)$$

for some finite real α,then the Laplace transform of f (t) is defined as

$$F(s) = \int_0^{\infty} f(t)e^{-st}\, dt \qquad (1.2)$$

or using the linear operator $\mathcal{L}$ we have

$$F(s) = \mathcal{L}[f(t)] \, , \tag{1.3}$$

where $s \in \mathbb{C}$,i.e. $s = \alpha + i\omega$.

1.2.1. Some Important Properties of the Laplace Transform

We have the following rules for Laplace transform:

1. $\mathcal{L}[kf(t)] = k\,\mathcal{L}[f(t)] = k\,F(s)$ (1.4)

2. $\mathcal{L}[f_1(t) \pm f_2(t)] = \mathcal{L}[f_1(t)] \pm \mathcal{L}[f_2(t)]$

   $= F_1(s) \pm F_2(s)$. (1.5)

3. $\mathcal{L}\left[\frac{df(t)}{dt}\right] = s\,F(s) - f(0)$. (1.6)

4. $\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots - f^{(n-1)}(0)$. (1.7)

5. $\mathcal{L}\left[\ \int_0^t f(\tau)d\tau\ \right] = \frac{F(s)}{s}$. (1.8)

6. $\mathcal{L}\left[\int_0^{t_1}\int_0^{t_2} \dots \int_0^{t_n} f(\tau)d\tau dt_1 dt_2 \dots dt_{n-2} dt_{n-1}\right] = \frac{F(s)}{s^n}$. (1.9)

7. $\mathcal{L}[f(t-\alpha)] = e^{-\alpha s} F(s)$. (1.10)

8. $F_1(s)F_2(s) = \mathcal{L}[\int_0^t f_1(\tau) f_2(t-\tau)d\tau]$

   $= \mathcal{L}[\int_0^t f_2(\tau) f_1(t-\tau)d\tau]$

   $= \mathcal{L}[f_1(t) * f_2(t)]$, (1.11)

where * denotes complex convolution.

We will use in many cases the unit step, and the unit impulse function given in the following definitions.

Definition 1.3:

Let **u** be a function defined by

$$u(t) := \begin{cases} 0, for\ t < 0 \\ k, for\ t \geq 0. \end{cases}$$

Then **u** is called a step function, and if k=1, then u is called a unit step function.

Example 1.2:

Find the Laplace transform of the unit step function.

By equation (1.2), we have

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty u(t)\,e^{-st}\,dt = \int_0^\infty e^{-st}\,dt = \frac{1}{s}\ .$$

Let $f(t) := \begin{cases} \frac{1}{c}, 0 \leq t \leq c \\ 0, \quad t > c. \end{cases}$

f can be expressed in terms of the step function (see example1.2) as

$f(t) = \frac{1}{c}u(t) - \frac{1}{c}u(t - c)$, that is, a step function beginning at t=0 minus a step function beginning at t = c. This leads to the definition of the unit impulse function or the **Dirac delta function** denoted by $\delta(t)$.

Definition 1.3:

δ is said to be the unit impulse function or Dirac delta function if

$$\delta(t) := \lim_{c\to 0} \frac{u(t)-u(t-c)}{c}\ .$$

Example 1.3:

Find the Laplace transform of the unit impulse function.

Using the results of (example 1.2) and (equation 1.10), we have

$$\mathcal{L}\,[\delta(t)] = \lim_{c\to 0}\frac{1}{sc}(1 - e^{-cs}).$$

Using l'Hospitals rule we get

$$\mathcal{L}\,[\delta(t)] = \lim_{c\to 0}\frac{\frac{d}{dc}[1-e^{-cs}]}{\frac{d}{dc}[sc]} = 1.$$

1.2.2. Inverse Laplace Transformation

Inverse Laplace transformation is a mathematical method by which any function F(s) is changed from complex variable to time variable function f (t).The metod is represented by

$$\mathcal{L}^{-1}\,[F(s)] = f\,(t) \tag{1.12}$$

where F(s) is the Laplace transform of f(t) and $\mathcal{L}^{-1}$ [F(s)] is the inverse Laplace transform of F(s). In practice the inverse Laplace transforms are not obtained through mathematical operations but are directly obtained from the tables of Laplace transforms.

We apply the partial fraction method for the functions in which inverse Laplace transforms cannot be directly obtained from the standard Laplace transform table.

Partial Fraction Method:

In this method the function for which the inverse Laplace transforms is to be determined is divided in to partial fractions.

Heaveside's Partial Fraction Method

Let the function F(s) be given by:

$$F(s) = \frac{a_m s^m + a_{m-1}s^{m-1} + \cdots + a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}. \tag{1.13}$$

Let D(s):= the denominator of F(s) given in equation(1.13).
Then after factorizing D(s),we see that D(s) has different roots.

A) D(s) has real roots.

The function F(s) of this type is

$$F(s) = \frac{(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_k)\ldots(s+p_n)}. \tag{1.14}$$

The partial fractions of this type of function are given by:

$$F(s) = \frac{A_1}{(s+p_1)} + \frac{A_2}{(s+p_2)} + \ldots + \frac{A_k}{(s+p_k)} \ldots + \frac{A_n}{(s+p_n)} \tag{1.15}$$

where $A_1, A_2, A_3, \ldots, A_n$ are coefficients of partial fractions.

Hence the coefficient of partial fraction for $(s+p_k)$ is obtained by

$$A_k = [(s+p_k)F(s)]|_{s=-p_k}. \tag{1.16}$$

Example 1.4:

By the inverse Laplace transform of the function

$$F(s) = \frac{10s+60}{s^4+7s^3+14s^2+8s}$$

we get the function f. We want to find f.

Now $F(s) = \frac{10s+60}{s^4+7s^3+14s^2+8s} = \frac{10(s+6)}{s(s+1)(s+2)(s+4)}$.

By (1.15) we have

$$F(s) = \frac{A_1}{s} + \frac{A_2}{s+1} + \frac{A_3}{s+2} + \frac{A_4}{s+4}.$$

And applying (1.16), we get

$$A_1 = \frac{15}{2}, A_2 = \frac{-50}{3}, A_3 = 10, \text{ and } A_4 = \frac{-5}{6}.$$

Substitution yields

$$F(s) = \frac{\frac{15}{2}}{s} + \frac{\frac{-50}{3}}{s+1} + \frac{10}{s+2} + \frac{\frac{-5}{6}}{s+4}.$$

By inverse Laplace transformation we get

$$f(t) = \frac{15}{2} - \frac{50}{3}e^{-t} + 10e^{-2t} - \frac{5}{6}e^{-4t}.$$

B) D(s) has real roots of multiple order r.

The function F(s) of this type is

$$F(s) = \frac{(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)^{r_1}(s+p_2)^{r_2}\ldots(s+p_n)^{r_n}}. \tag{1.17}$$

Partial fractions of this type are given by

$$F(s) = \frac{A_{r_1}}{(s+p_1)^{r_1}} + \frac{A_{r_1-1}}{(s+p_1)^{r_1-1}} + \frac{A_{r_1-2}}{(s+p_1)^{r_2-2}} + \ldots + \frac{A_1}{(s+p_1)}$$

$$+ \frac{B_{r_2}}{(s+p_2)^{r_2}} + \frac{B_{r_2-1}}{(s+p_2)^{r_2-1}} + \frac{B_{r_2-2}}{(s+p_2)^{r_2-2}} + \ldots + \frac{B_1}{(s+p_2)}$$

$$+ \frac{C_{r_n}}{(s+p_n)^{r_n}} + \frac{C_{r_n-1}}{(s+p_n)^{r_n-1}} + \frac{C_{r_n-2}}{(s+p_n)^{r_n-2}} + \ldots + \frac{C_1}{(s+p_n)} \quad . \tag{1.18}$$

Thus the value of any coefficient C_k for roots s=$-p_k$ is given by

$$C_k = [\frac{1}{(r_n-k)!}\frac{d^{(r_n-k)}}{ds^{(r_n-k)}}[(s+p_n)^{r_n}]]\Big|_{s=-p_k} , \tag{1.19}$$

where $\frac{d^{(r_n-k)}}{ds^{(r_n-k)}}$ is the $(r_n - k)th$ derivative of $(s + p_n)^{r_n}$.

Example 1.5:

By the inverse Laplace transform of the function

$$F(s) = \frac{10}{(s+3)^2(s+4)^3}$$

We get the function f.

$$F(s) = \frac{10}{(s+3)^2(s+4)^3} .$$

By (1.18) we have

$$F(s) = \frac{A_2}{(s+3)^2} + \frac{A_1}{s+3} + \frac{B_3}{(s+4)^3} + \frac{B_2}{(s+4)^2} + \frac{B_1}{s+4},$$

and applying (1.19) we get

$$A_1 = -30, A_2 = 10, B_3 = 10, B_2 = 20, B_1 = 30.$$

This implies

$$F(s) = \frac{10}{(s+3)^2} - \frac{30}{s+3} + \frac{10}{(s+4)^3} + \frac{20}{(s+4)^2} + \frac{30}{s+4}.$$

Hence , we have

$$f(t) = 30(e^{-3t} - e^{-4t}) + 10t(e^{-3t} + 2e^{-4t}) + 10t^2e^{-4t} .$$

C) D(s) possesses complex conjugate roots.

The function of this type is

$$F(s) = \frac{(s+z_1)(s+z_2)...(s+z_m)}{(s+\alpha_1+i\omega_1)(s+\alpha_1-i\omega_1)...(s+\alpha_n+i\omega_n)(s+\alpha_n-i\omega_n)} \quad . \tag{1.20}$$

Hence partial fractions of this type are given by:

$$F(s) = \frac{A_1}{(s+\alpha_1+i\omega_1)} + \frac{A_1^*}{(s+\alpha_1-i\omega_1)} + \ldots + \frac{A_n}{(s+\alpha_n+i\omega_n)} + \frac{A_n^*}{(s+\alpha_n-i\omega_n)} \quad . \tag{1.21}$$

Here the A_k and A_k^* are obtained by

$$A_k = [[s + (\alpha_k + i\omega_k)]F(s)]|_{s=-(\alpha_k+i\omega_k)}$$

and

$$A_k^* = [[s + (\alpha_k - i\omega_k)]F(s)]|_{s=-(\alpha_k-i\omega_k)} \quad . \tag{1.22}$$

Example 1.6:

By the inverse Laplace transform of the function

$$F(s) = \frac{5}{s^2+6s+10}$$

we get the function f. We want to find f.

By (1.21) we have

$$F(s) = \frac{A}{(s+3+i)} + \frac{A^*}{(s+3-i)},$$

and applying (1.22) we get

$A = \frac{5}{-2i}$, *and* $A^* = \frac{5}{2i}$.

Hence using simplification we get

$f(t) = 5e^{-3t} sint.$

D) F(s) possesses imaginary roots.

The function of this type is

$$F(s) = \frac{(s+z_1)(s+z_2)...(s+z_m)}{(s^2+p_1^2)(s^2+p_2^2)...(s^2+p_n^2)} . \tag{1.23}$$

The partial fraction coefficients of this class of functions are given by

$$F(s) = \frac{A}{s+ip_k} + \frac{A^*}{s-ip_k} .$$

Hence A_k and A_k^* are obtained by

$$A_k = [(s+ip_k)F(s)]|_{s=-ip_k} \quad ,$$
$$A_k^* = [(s-ip_k)F(s)]|_{s=+ip_k} . \tag{1.24}$$

Example 1.7:

By the inverse Laplace transform of the function

$$F(s) = \frac{3}{s^2+4}$$

we get the function f.

$$F(s) = \frac{3}{s^2+4} = \frac{3}{(s+2i)(s-2i)} = \frac{A}{s+2i} + \frac{A^*}{s-2i} .$$

By (1.24) we have

$A = \frac{3}{-4i}$ and $A^* = \frac{3}{4i}$.

Thus by simplifying we get

$f(t) = \frac{3}{2} sint$.

1.2.3. Applications of Laplace Transform to Differential Equations

By taking the Laplace transform of a differential equation it is transformed in to an algebraic equation in the variable **s** .This equation is rearranged so that all the terms involving the dependent variable are on one side and the output response is obtained by taking the inverse Laplace transform as illustrated by the following example.

Example 1.8:

We want to solve the initial value problem

$D^2y - 6Dy + 9y = e^{-2t}$, with initial conditions y(0) = 1, $y'(0) = 2$.

Taking Laplace transform throughout the equation and using equation (1.7), we get

$$[s^2Y(s) - sy(0) - y'(0)] - 6[sY(s) - y(0)] + 9Y(s) = \frac{1}{s+2} ,$$

that is ,

$$[s^2 - 6s + 9]Y(s) - s + 4 = \frac{1}{s+2} ,$$

where Y(s) = $\mathcal{L}[y(t)]$.

Since y (0) =1 and $y'(0)$ =2, we have

$$Y(s) = \frac{1}{(s+2)(s-3)^2} + \frac{s-4}{(s-3)^2} = \frac{s^2-2s-7}{(s+2)(s-3)^2} = \frac{k_1}{s+2} + \frac{k_2}{s-3} + \frac{k_3}{(s-3)^2} .$$

Solving for $k_1, k_2, and\ k_3$ (by 1.16) we get

$$k_1 = \frac{1}{25}, k_2 = \frac{24}{25}, \quad k_3 = \frac{-4}{5} .$$

Hence $Y(s) = \frac{s^2-2s-7}{(s+2)(s-3)^2} = \frac{\frac{1}{25}}{s+2} + \frac{\frac{24}{25}}{s-3} + \frac{\frac{-4}{5}}{(s-3)^2}$.

Thus

$$y(t) = \frac{1}{25}e^{-2t} + \frac{24}{25}e^{3t} + \frac{-4}{5}te^{3t}.$$

1.3. Transfer Functions

We can often describe the dynamic behavior of a system by a linear n^{th} order differential equation. The n^{th} order system having a single input and a single output may have an associated differential equation:

$$b_0D^ny + b_1D^{n-1}y + \cdots + b_ny = a_0D^mu + a_1D^{m-1}u + \cdots + a_mu \quad (1.25)$$

where u(t) is the input and y(t) is the output. Once the input and the initial conditions $y(0) = 0, y'(0) = 0, \dots, y^n(0) = 0$ are given, the output response y (t) is found by solving equation (1.25). The Laplace transform of equation (1.25) becomes:

$$[b_0s^n + b_1s^{n-1} + \cdots + b_n]Y(s) = [a_0s^m + a_1s^{m-1} + \cdots + a_m]U(s) \,.$$

Definition 1.4:

The function G given by

$$G(s) := \frac{Y(s)}{U(s)} = \frac{a_0s^m + a_1s^{m-1} + \cdots + a_m}{b_0s^n + b_1s^{n-1} + \cdots + b_n} \;, (m<n) \quad (1.26)$$

is called transfer function for the initial value problem (1.25)

where Y(s) is the Laplace transform of y, U(s) is the Laplace transform of u.

The equation $b_0s^n + b_1s^{n-1} + \cdots + b_n = 0$ is called the **characteristic equation.**

The transfer function G(s) can be rewritten as

$$Y(s) = G(s)\, U(s). \quad (1.27)$$

This shows that G(s) is always a factor of Y(s), whatever u (t) is applied.

Example 1.9:

Let u = δ. Then if the input to the system is u (t) = $\delta(t)$, then Y(s) = G(s), since in this case U(s) = 1.

Taking the inverse Laplace transform, we obtain y (t) = g (t), where

g (t) = $\mathcal{L}^{-1}[G(s)]$ which is called the impulse response or the **weighting function** for the system.

Example 1.10:

Find the transfer function of the system described by the initial value problem:

$D^2y + 5Dy + 6y = u$, $y(0) = 0$, $y'(0) = 0$.

Applying Laplace transform to both sides, we obtain

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{(s+2)(s+3)}.$$

1.4. State Space Analysis of Control Systems

In the previous section we have discussed the characterization of a system by its transfer function. The basis of transfer function has the following properties:
1. The initial conditions in system equation are considered as zero.
2. Applicable only to single- input- single- output (SISO) systems.
3. Applicable only to linear time- invariant system equations.
4. The analysis of system output for a specified input is obtained.

In view of these properties a more general mathematical representation of a control system is required which covers limitations of transfer function approach.

The modern approach known as state space analysis is preferred to transfer function approach.

The state space mathematical model of a control system takes into account the following points.

1. The initial conditions pertaining to the system are taken in to consideration.
2. The mathematical model is in the form of first order differential equations of linear time invariant or variant systems.
3. The analysis is carried out in time domain.
4. The mathematical model covers both single- input- single- output (SISO) and multiple-input-multiple-output (MIMO) systems.
5. The concept of state space model forms the basis for analysis of advance control systems.
6. The state space model gives complete description of the system.

1.4.1. The State Space Approach

We will use in the following notations:

State variable: The variables which determine the state of a dynamic system. It is not necessary that the state variables be physically measurable and observable quantities.

State: The state of a system is the smallest set of state variables such that the knowledge of variables at $t=t_0$ (initial conditions) together with the inputs completely determine the behavior of the system for any time $t > t_0$.

State vector: If n state variables are necessary to determine the behavior of a given system, the variables are considered as n components of a vector.

State space: The n dimensional state variables are elements of n dimensional space called state space.

State equations: The system equations written in the form of first order differential equations in time domain are known as state equations. The state model for a given differential equation is determined below:

The nth order time invariant differential equation with constant coefficients is represented by a set of n-first order differential equations in terms of state variables $x_1, x_2, \dots x_n$ as follows

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \cdots b_{1m}u_m$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \cdots b_{2m}u_m$$

$$\vdots$$

$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \cdots b_{nm}u_m \,. \tag{1.28}$$

Equation (1.28) can be written in compact form as

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \tag{1.29}$$

where $\boldsymbol{x} := [x_1, x_2, \dots, x_n]^T$

is denoted as state vector and

$$\boldsymbol{u} := [u_1, u_2, \dots, u_m]^T$$

is denoted as input vector. Furthermore,

$$\boldsymbol{A} := \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ & \cdots & \\ a_{n1} & & a_{nn} \end{bmatrix}$$

and

$$B := \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ b_{21} & \dots & b_{2m} \\ \vdots & \dots & \vdots \\ b_{n1} & & b_{nm} \end{bmatrix},$$

are constant matrices.

For a time varying system A and B are functions of time.

Output equation: The dynamic state of a system is represented by state variables. The output variables are denoted as $y_1, y_2, \dots y_p$ which are combinations of state variables. Therefore, the output equation is

$$y = Cx \tag{1.30}$$

where $C = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ c_{21} & \dots & c_{2n} \\ \vdots & \dots & \vdots \\ c_{p1} & & c_{pn} \end{bmatrix}$

is constant.

In some cases, the output is a function of x and u, that is,

$$y = Cx + Du \tag{1.31}$$

where $D = \begin{bmatrix} d_{11} & \cdots & d_{1m} \\ d_{21} & \dots & d_{2m} \\ \vdots & \dots & \vdots \\ d_{p1} & & c_{pm} \end{bmatrix}$

is constant .

For a single-input-single-output system the state and output equations are given by

$$\begin{aligned} \dot{x} &= Ax + Bu , \\ y &= Cx , \end{aligned} \tag{1.32}$$

where B is n×1 matrix, u is a scalar and C is 1×n matrix.
For the case where the output of a single-input-single-output system is a combination of state vector as well as input, the state equation and output equations are respectively

$$\begin{aligned} \dot{x} &= Ax + Bu , \\ y &= Cx + Du , \end{aligned} \tag{1.33 .a}$$

where the terms D and u are scalars .

The general state model of a multiple-input-multiple-output system based on state and output equation is given by:

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu},$$
$$\boldsymbol{y} = \boldsymbol{Cx} + \boldsymbol{Du}. \quad (1.33.\text{b}).$$

1.4.2. The Companion and Diagonal State Space Form

Let us consider a system characterized by an n[th] order differential equation. The system equation has the form

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1}\dot{y} + a_n y = u, \quad (1.34)$$

It is assumed that $y(0) = 0, \dot{y}(0) = 0, \dots, y^{(n-1)}(0) = 0$.

Let $x_1 := y,\ x_2 := \dot{y}, \dots,\ x_n := y^{(n-1)}$, then we can write equation (1.34) as a system of n simultaneous differential equations, each of order 1, i.e.,

$$\dot{x}_1 = x_2 = \dot{y},$$

$$\dot{x}_2 = x_3 = \ddot{y},$$

$$\vdots$$

$$\dot{x}_{n-1} = x_n = y^{(n-1)},$$

$$\dot{x}_n = y^{(n)} = -a_n x_1 - a_{n-1}x_2 - \cdots - a_1 x_n + u.$$

This can be written as a matrix differential equation as follows

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix}}_{=\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}}_{=A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}}_{=x} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}}_{=B} u. \quad (1.35)$$

$$y = \underbrace{[1\ \ 0\ \ 0 \dots 0]}_{=C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}. \quad (1.36).$$

From equations (1.35) and (1.36) we have equations (1.32). Here the matrix A is said to be in **companion form**.

Consider any non-singular matrix P of order n×n.

Let $\quad x = P\,z$. (1.37)

Then z is also a state vector and equation (1.32) can be written as:

$P\dot{z} = APz + Bu$,
$y = CP\,z$.
or as

$\dot{z} = A_1 z + B_1 u$,
$y = C_1 z$, (1.38)

where $A_1 = P^{-1}AP, B_1 = P^{-1}B$ and $C_1 = CP$.

The transformation defined by equation (1.37) is called a state transformation, and the matrices A and A_1 are similar.

We are particularly interested in the transformation when A_1 is diagonal denoted by **Λ** and A is in the companion form. It is assumed that the matrix A has distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Corresponding to the eigenvalue λ_i there is eigenvector x_i such that

$$Ax_i = \lambda_i x_i \,, (x_i \neq 0). \quad (1.39)$$

Define the matrix V^T whose columns are the eigenvectors $x_1, x_2, \dots, x_n$, that is, $V = [x_1 \; x_2 \; \dots \; x_n]$. V is called the **modal** matrix, and it is non-singular and can be used as the transformation matrix P above. We can write the n equations defined by equation (1.39) as

$$AV = V\Lambda \quad (1.40)$$

where $\Lambda = \begin{bmatrix} \lambda_1 & 0 \dots & 0 \\ 0 & \lambda_2 \dots & 0 \\ 0 & 0 \dots & \lambda_n \end{bmatrix}$ =diag $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$.

From equation (1.40), we obtain

$$\Lambda = V^{-1}A\,V\,. \quad (1.41)$$

The matrix A has the companion form. The characteristic equation is

$|\lambda I - A| = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0$.

By solving this equation we obtain the eigenvalue $\lambda_1, \lambda_2, \dots, \lambda_n$.

Consider one of the eigenvalues λ and the corresponding eigenvectors

$x := [x_1, x_2, \dots, x_n]^T$

Since A is in companion form the equation A x = λ x, $(x \neq 0)$ corresponds to the system of equations

$$
\begin{aligned}
x_2 &= \lambda x_1, \\
x_3 &= \lambda x_2, \\
&\vdots \\
x_n &= \lambda x_{n-1}.
\end{aligned}
$$

Setting x_1=1, we obtain

$$x = [1, \lambda, \lambda^2, \dots, \lambda^{n-1}]^T$$

Hence the modal matrix in this case takes the form

$$V = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}. \tag{1.42}$$

In this form the matrix V is called a Vandermonde matrix, and is therefore non-singular, if $\lambda_i \neq \lambda_j \; for \; i \neq j.$

Example 1.10 :

Let $\dddot{y} - 2\ddot{y} + \dot{y} - 2y = u.$

Setting $x_1 = y,\ x_2 = \dot{y},\ and\ \dot{x}_2 = \ddot{y}$,

we get the companion forms of state equations of the system as follows

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= 2x_1 - x_2 + 2x_3 + u,
\end{aligned}
$$

or in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u,$$

$$y = [\,1 \quad 0 \quad 0\,]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Now we want to find the diagonal form.
The eigenvalues of λ are { 2, i, -i }

From equation (1.42) the modal matrix is

$$V = \begin{bmatrix} 1 & 1 & 1 \\ 2 & i & -i \\ 4 & -1 & -1 \end{bmatrix}$$

and its inverse is

$$V^{-1} = \frac{1}{20}\begin{bmatrix} 4 & 0 & 4 \\ 8+4i & -10i & -2+4i \\ 8-4i & 10i & -2-4i \end{bmatrix}.$$

The transformation is defined by equation (1.37), that is

$$x = V\,z \text{ or } z = V^{-1}x.$$

The original choice for x is $x = [y, \dot{y}, \ddot{y}]^T$. Then

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 4 & 0 & 4 \\ 8+4i & -10i & -2+4i \\ 8-4i & 10i & -2-4i \end{bmatrix}\begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \end{bmatrix}, \text{ i..e.,}$$

$$z_1 = \frac{1}{5}y + \frac{1}{5}\ddot{y}$$
$$z_2 = \frac{1}{5}(2+i)y - \frac{1}{2}i\dot{y} + \frac{1}{10}(-1+2i)\ddot{y}$$
$$z_3 = \frac{1}{5}(2-i)y + \frac{1}{2}i\dot{y} - \frac{1}{10}(1-2i)\ddot{y}.$$

The state equations are now in the form

$$\dot{z} = A_1\,z + B_1 u,$$

$$y = C_1\,z,$$

where $A_1 = V^{-1}AV = diag\{2, i, -i\}$,

$$B_1 = V^{-1}B = \frac{1}{10}\begin{bmatrix} 2 \\ -1+2i \\ -1-2i \end{bmatrix},$$

$$C_1 = CV = [1\ \ 1\ \ 1]\,.$$

1.4.3. The Transfer Function from the State Equation

Taking the Laplace transform of equations (1.32) we obtain

s X(s) = A X(s) +B U(s)

Y(s) = C X(s).

From the first of these equations, we obtain

X(s) = $[sI - A]^{-1}BU(s)$, if the inverse of $(\,sI - A)$ exists

so that

$$G(s) = \frac{Y(s)}{U(s)} = CX(s) = C[sI - A]^{-1}B\ . \qquad (1.43)$$

Example 1.11:

Let $$\dot{x} = \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} x + \begin{bmatrix} -4 \\ 3 \end{bmatrix} u\,,$$

y = [2 3] x .

Then

$$G(s) = [2\ \ 3]\begin{bmatrix} s-2 & 3 \\ 1 & s-4 \end{bmatrix}^{-1}\begin{bmatrix} -4 \\ 3 \end{bmatrix} = \frac{s+8}{(s-1)(s-5)}$$

1.4.4. Matrix function

Definition 1.5:

Since $e^z = \sum_0^\infty \frac{1}{n!} z^n \quad , z \in \mathbb{C}.$

we define $e^A = \sum_0^\infty \frac{1}{n!} A^n = A^0 + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$

$$= I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

where $A^0 = I$ for every square matrix A.

From Linear Algebra we know that if A has distinct eigenvalues, $\lambda_1, \dots, \lambda_n$,then there exists a non-singular matrix P such that

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \dots & 0 \\ 0 & \lambda_2 \dots & 0 \\ 0 & 0 \dots & \lambda_n \end{bmatrix} = \text{diag}\ \{\lambda_1, \lambda_2, \dots, \lambda_n\} = \mathbf{\Lambda}\ .$$

Then we have

$$A = P\Lambda P^{-1}$$

so that

$$A^2 = (P\Lambda P^{-1})(P\Lambda P^{-1}) = P\Lambda(P^{-1}P)\Lambda P^{-1} = P\Lambda^2 P^{-1}$$

Generally we get

$$A^r = P\Lambda^r P^{-1}\ .$$

If we consider a matrix polynomial, $f(A) = A^2 - 2A + I$, we can write it as

$$f(A) = P\Lambda^2 P^{-1} - 2\, P\Lambda P^{-1} + PIP^{-1}$$

$$= P[\Lambda^2 - 2\,\Lambda + \mathrm{I}]\, P^{-1}$$

$$= Pf(\Lambda)P^{-1} = P\text{diag}\ \{ f(\lambda_1), f(\lambda_2), \dots, f(\lambda_n)\}P^{-1}\ .$$

Hence

$$f(\Lambda) = \Lambda^2 - 2\,\Lambda + \mathrm{I}$$

$$= \begin{bmatrix} \lambda_1^2 & 0 \dots & 0 \\ 0 & \lambda_2^2 \dots & 0 \\ 0 & 0 \dots & \lambda_n^2 \end{bmatrix} -2 \begin{bmatrix} \lambda_1 & 0 \dots & 0 \\ 0 & \lambda_2 \dots & 0 \\ 0 & 0 \dots & \lambda_n \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1^2 - 2\lambda_1 + I & 0 \ldots & 0 \\ 0 & \lambda_2^2 - 2\lambda_2 + I \ldots & 0 \\ 0 & 0 \ldots & \lambda_n^2 - 2\lambda_n + I \end{bmatrix}$$

$= \text{diag}\{ f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n)\}.$

In general for a polynomial of degree n, taking

$f(A) = e^{At}$, we obtain

$$e^{At} = P \text{ diag } \{e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t}\} P^{-1} . \tag{1.44}$$

Example 1.12:

Let $A = \begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix}$.

To find e^{At} we have the eigenvalues of A which are $\lambda = -1$ and $\lambda = -2$.

$P = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$.

Using equation (1.44), we obtain

$$e^{At} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & 2e^{-t} - 2e^{-2t} \\ e^{-2t} - e^{-t} & 2e^{-2t} - e^{-t} \end{bmatrix}.$$

The **derivative** and the **integra**l of a matrix A (t) whose elements are functions of t are given in the following

Definition 1.6:

Let A (t) = $[a_{ij}(t)]$, $t \in [0, \infty)$. Then

1. $\frac{d}{dt}A(t) = \dot{A}(t) = \left[\frac{d}{dt}(a_{ij})\right]$,
2. $\int A(t)dt = [\int a_{ij}(t)dt]$.

If α *and* β are constants, and A and B are matrices, then

i. $\frac{d}{dt}(\alpha A + \beta B) = \alpha \dot{A} + \beta \dot{B}$.

ii. $\int_a^b (\alpha A + \beta B)dt = \alpha \int_a^b A dt + \beta \int_a^b B dt$.

iii. $\frac{d}{dt}(AB) = A\dot{B} + \dot{A}B$.

iv. If A is a constant matrix, then $\frac{d}{dt}(e^{At}) = A\, e^{At}$.

1.4.5. Solution of the State- Equation

By a solution to the state equation (1.29) we mean finding x at any time t, given u(t), $\forall t$, and the value of x at some specified time t_0 ,i.e., given $x(t_0) = x_0$.Thus we have

$$\dot{x} - \boldsymbol{Ax} = \boldsymbol{Bu}.$$

Multiplying both sides by the matrix e^{-At} , we obtain

$$e^{-At}(\dot{x} - Ax) = e^{-At}(Bu),$$

or

$$\frac{d}{dt}[e^{-At}x] = e^{-At}(Bu).$$

Integrating both sides we get

$$e^{-At}x(t) - x(0) = \int_0^t e^{-A\tau}Bu(\tau)d\tau\,, \tau \in [0,t]$$

so that

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\,. \quad (1.45)$$

The term $e^{At}x(0)$, dependent only on x_0 is called **Complementary function** which is the solution of the equation $\dot{\boldsymbol{x}} = \boldsymbol{Ax}$ (when forcing function or input is zero).

The term $\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$, dependent only on the forcing function u, is called the **particular integral.**

In the solution to the unforced system equation $x = e^{At}x(0)$, the eigenvalues $\lambda_1, \dots, \lambda_n$ are called **poles** and $e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_n t}$ are called **modes** of the system.

Example 1.13:

A system is characterized by the state equation

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 2\\ -1 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u.$$

If the forcing function u (t) = 1 for t ≥ 0 and x (0) = $[1, -1]^T$, the state x of the system at time t is obtained as follows

Using the value of e^{At}(see example1.12) we have that

$$e^{At}x(0) = \begin{bmatrix}2e^{-t} - e^{-2t} & 2e^{-t} - 2e^{-2t}\\ e^{-2t} - e^{-t} & 2e^{-2t} - e^{-t}\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$= \begin{bmatrix}e^{-2t}\\ -e^{-2t}\end{bmatrix}$$

$$= e^{-2t}\begin{bmatrix}1\\ -1\end{bmatrix}.$$

Here the initial condition x (0) is such that it eliminates the mode e^{-t} from the unforced system solution.

The second term of equation (1.45) is

$$\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = \int_0^t e^{A(t-\tau)}\begin{bmatrix}0\\ 1\end{bmatrix}d\tau$$

$$= \int_0^t \begin{bmatrix}2e^{-(t-\tau)} - 2e^{-2(t-\tau)}\\ 2e^{-2(t-\tau)} - e^{-(t-\tau)}\end{bmatrix}d\tau$$

$$= \begin{bmatrix}1 - 2e^{-t} + e^{-2t}\\ e^{-t} - e^{-2t}\end{bmatrix}.$$

Therefore, the state of the system at time t is

$$x(t) = \begin{bmatrix}e^{-2t}\\ -e^{-2t}\end{bmatrix} + \begin{bmatrix}1 - 2e^{-t} + e^{-2t}\\ e^{-t} - e^{-2t}\end{bmatrix}$$

$$= \begin{bmatrix}1 - 2e^{-t} + 2e^{-2t}\\ e^{-t} - 2e^{-2t}\end{bmatrix}.$$

The matrix e^{At} in the solution equation (1.45) is called the **state transition matrix**

or **the fundamental matrix** and denoted by $\Phi(t)$ so that

$$\Phi(t) := e^{At}. \quad (1.46)$$

For the unforced system (when u (t) = 0) the solution equation (1.45) becomes

$$x(t) = \Phi(t)\, x(0)$$

so that $\Phi(t)$ transforms the system from its state x(0) at some initial time t=0 to the state x(t) at some subsequent time t.

Since

$$e^{At}e^{-(At)} = I, \text{ it follows that } [e^{At}]^{-1} = e^{-At}\ .$$

Hence $\Phi^{-1}(t) = e^{-At} = \Phi(-t)$.

Also $\Phi(t)\ \Phi(-\tau) = e^{At}e^{-A\tau} = e^{A(t-\tau)} = \Phi(t-\tau)$.

With this notation equation (1.45) becomes

$$x(t) = \Phi(t)\, x(0) + \int_0^t \Phi(t-\tau)\ Bu(\tau)d\tau,\ \tau \in [0,t]. \qquad (1.47)$$

1.4.6. Solution of the State-Equation by Laplace Transforms

Since the state equation is in a vector form we must first define the Laplace transform of a vector

Let $x(t) = \ [x_1, x_2, \ldots, x_n]^T$.

Then $\mathcal{L}[x(t)] = \begin{bmatrix} \mathcal{L}[x_1(t)] \\ \mathcal{L}[x_2(t)] \\ \vdots \\ \mathcal{L}[x_n(t)] \end{bmatrix} = \begin{bmatrix} X_1(s) \\ X_2(s) \\ \vdots \\ X_n(s) \end{bmatrix} = X(s).$

From this and by equation (1.6) we can get

$$\mathcal{L}[\dot{x}(t)] = \begin{bmatrix} \mathcal{L}[\dot{x}_1(t)] \\ \mathcal{L}[\dot{x}_2(t)] \\ \vdots \\ \mathcal{L}[\dot{x}_n(t)] \end{bmatrix} = \begin{bmatrix} sX_1(s) - x_1(0) \\ sX_2(s) - x_2(0) \\ \vdots \\ sX_n(s) - x_n(0) \end{bmatrix} = sX(s) - x(0).$$

Applying the Laplace transform on the state equation (1.29) we obtain

$$sX(s) - x(0) = A\,X(s) + BU(s)\,,$$

where $U(s) = \ \mathcal{L}[u(t)]$

or

$$[sI - A]\,X(s) \ = \ x(0) + BU(s).$$

The matrix $[sI - A]$ is non-singular, so that the above equation can be solved giving

$$X(s) = [sI - A]^{-1}x(0) + [sI - A]^{-1}\, BU(s) \tag{1.48}$$

and the solution $x(t)$ is found by taking the inverse Laplace transform of equation(1.48).

Definition 1.7:

Let A be as in equation (1.29). Then

$[sI - A]^{-1}$ is called the **re-solvent** matrix of the system.

Comparing equations (1.47) and (1.48) we find that

$$\Phi(t) = \mathcal{L}^{-1}\{[sI - A]^{-1}\}.$$

Example 1.14:

Using Laplace transform, the state x (t) of the system described in Example1.13 can be evaluated as

$$[sI - A] = \begin{bmatrix} s & -2 \\ 1 & s+3 \end{bmatrix},$$

so that

$$[sI - A]^{-1} = \frac{1}{s(s+3)+2}\begin{bmatrix} s+3 & 2 \\ -1 & s \end{bmatrix} = \begin{bmatrix} \frac{s+3}{(s+1)(S+2)} & \frac{2}{(s+1)(S+2)} \\ \frac{-1}{(s+1)(S+2)} & \frac{s}{(s+1)(S+2)} \end{bmatrix}.$$

To evaluate the inverse transform, we must express each element of the matrix as a partial fraction, that is, as

$$[sI - A]^{-1} = \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{2}{s+1} - \frac{2}{s+2} \\ \frac{-1}{s+1} + \frac{1}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

So that $\mathcal{L}^{-1}\{[sI - A]^{-1}\} = \begin{bmatrix} 2e^{-t} - e^{-2t} & 2e^{-t} - 2e^{-2t} \\ e^{-2t} - e^{-t} & 2e^{-2t} - e^{-t} \end{bmatrix} = \Phi(t).$

Hence the complementary function is as in Example1.13.For the particular integral, we note that since

$$\mathcal{L}[u(t)] = \frac{1}{s},$$

$$[sI - A]^{-1}\, BU(s) = \frac{1}{(s+1)(s+2)}\frac{1}{s}\begin{bmatrix} s+3 & 2 \\ -1 & s \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{2}{s(s+1)(s+2)} \\ \dfrac{1}{(s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} \frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2} \\ \frac{1}{s+1} - \frac{1}{s+2} \end{bmatrix}.$$

Taking the inverse Laplace transform, we obtain

$$\begin{bmatrix} 1 - 2e^{-t} + e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

which is the particular integral part of the solution obtained in Example1.13.

1.5. Linear Independence and the Rank of a Matrix

Let $x_1, x_2, x_3 \dots x_n$ be vectors and $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ be scalars. We consider the equation

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = 0 \tag{1.49}$$

If at least one $\lambda_i \neq 0$ in the equation(1.49), the vectors $x_1, x_2, x_3 \dots x_n$ are said to be linearly dependent and the equation is said to have a non-trivial solution. If, on the other hand, the above equation is satisfied only when$\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$, the vectors are said to be linearly independent, and this solution is said to be trivial.

Let A be a matrix of order m×n, i.e.,

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \cdots & \vdots \\ a_{m1} & & a_{mn} \end{bmatrix}.$$

We can write this matrix as

A = [a_1 a_2 a_3 , … , a_n]

where $a_i = [\,a_{1i}, x_{2i}, \dots, a_{mi}\,]^T$, i ∈ { 1,2,…,n}.

The rank of the matrix A can be defined as the number of linearly independent vectors in the set $\{a_1, a_2, a_3, \cdots, a_n\}$,that is the number of linearly independent columns of the matrix , or the common dimension of the row space and column space of a matrix A is called the rank of A and is denoted by r(A) or rank(A).

A square matrix A of order n×n , is non-singular, if and only if it is linearly independent and the rank of A is n. Also r (A) < n, if and only if A is singular.

1.6. Definite and Semi- definite Matrices

Definition 1.8:

Let A be an n×n symmetric matrix.
A is said to be positive definite if $x^T Ax > 0$ for all non-zero x and is said to be a positive semi-definite if $x^T Ax \geq 0$.

Similarly, if $x^T Ax < 0$ for all non-zero x, then A is called negative definite; and if $x^T Ax \leq 0$ for all x, then A is called negative semi-definite.

1.7. Quadratic Form

Definition 1.9:

A quadratic form is a scalar function v(x) of variables

$[x_1 , x_2 , x_3 , \dots , x_n]^T$ defined by

$$v(x) := x^T Px = [x_1 \ \ x_2 \dots x_n] \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n1} & \cdots & P_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= P_{11}x_1^2 + P_{12}x_1x_2 + P_{13}x_1x_3 + \cdots + P_{1n}x_1x_n +$$

$$+ P_{21}x_2x_1 + P_{22}x_2^2 + P_{31}x_3x_1 + \cdots + P_{2n}x_2x_n +$$

$$+ \cdots + P_{n1}x_nx_1 + P_{n2}x_nx_2 + \cdots + P_{nn}x_n^2$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} P_{ij}x_ix_j$$

The matrix P is of order n×n and is symmetric, that is

$P_{ij} = P_{ji}$ (all i,j; i≠ j) so that $P^T = P$

Example 1.15:

Let n=2, and $v(x) = x_1^2 + 2x_2^2 - 4x_1x_2$.

Then we can write this in a quadratic form as

$$v(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

In problems involving more than two or three variables, the determination of whether or not a quadratic form is a positive-definite can be difficult to deal with or overcome, for such a case there is a procedure known as **Sylvester's criterion**, which is very helpful.

Sylvester's criterion: states that the necessary and sufficient condition for a quadratic form $v(x) = x^T P x$ to be positive-definite is that all the successive principal minors of the symmetric matrix P are positive, that is,

$$|P_{11}| > 0\,, \begin{vmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{vmatrix} > 0\,, \begin{vmatrix} P_{11} & P_{12} & P_{13} \\ P_{12} & P_{22} & P_{23} \\ P_{13} & P_{23} & P_{33} \end{vmatrix} > 0, \ldots, \begin{vmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n1} & \cdots & P_{nn} \end{vmatrix} > 0.$$

Example 1.16:

Let v(x) = $4x_1^2 + 3x_2^2 + x_3^2 + 4x_1x_2 - 2x_1x_3 - 2x_2x_3$. To show this is a positive definite we use sylvester's criterion.

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12} & P_{22} & P_{23} \\ P_{13} & P_{23} & P_{33} \end{bmatrix} = \begin{bmatrix} 4 & 2 & -1 \\ 2 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

Since 4>0, $\begin{vmatrix} 4 & 2 \\ 2 & 3 \end{vmatrix} = 8 > 0$ and $\begin{vmatrix} 4 & 2 & -1 \\ 2 & 3 & -1 \\ -1 & -1 & 1 \end{vmatrix} = 5 > 0.$

Thus sylvester's criterion assures us that v(x) is a positive definite.

CHAPTER 2

2. STABILITY

2.1. The concept of stability

Stability is possibly the most important consideration when designing a control system.

The meaning of system stability is that a finite duration of disturbance causes a response (output) of finite duration, after which the system resumes a steady state conditions, that is, if any oscillations setup in a system in consequence to application of an input are damped out with respect to time, the system is said to be **stable.**

When a system is said to be unstable, we mean that its response readily diverges from an initial value and the output either changes un directionally or oscillates with ever-increasing amplitude.

Stability of a control system is a very important characteristic of the transient performance of a system.

A linear time-invariant system is stable if the following two conditions are satisfied:

(I) for a bounded input , the output should be bounded ,

(II) in the absence of system input, the output should be zero whatever may be the initial conditions.

Here a system is said to be **asymptotically stable** if its weighting function (impulse response) decays to zero as t tends to infinity.

Some methods of testing for system stability use the **transfer function** as the starting point; others make use of the **state-space equations**. It must be remembered that the state-space equations are themselves a representation of the differential equations characterizing the system.

Now let us consider a single-input, single-output (SISO) system with transfer function given by equation (1.29). From which, we obtain equation (1.30).

Applying the inverse Laplace transform on (1.30), the output is given by

$$y(t) = \mathcal{L}^{-1}[\ G(s)\, U(s)], \qquad (2.1)$$

assuming that the initial conditions to be zero. Using the rule of complex multiplication (1.11), we obtain

$$y(t) = \int_0^\infty g(\lambda)\, u(t-\lambda) d\lambda\,, \quad t \in [0,\infty). \tag{2.2}$$

where $g(\lambda)$ is the impulse response of the system. From equation (2.2) we have the absolute value given by

$$|y(t)| = \left|\int_0^\infty g(\lambda)\, u(t-\lambda) d\lambda\right|\,, \quad t \in [0,\infty). \tag{2.3}$$

Since the absolute value of an integral is less than the integral of the absolute value of the integrand, we have

$$\begin{aligned} |y(t)| &\le \left|\int_0^\infty g(\lambda)\, u(t-\lambda) d\lambda\right| \\ &\le \int_0^\infty |g(\lambda)|\, |u(t-\lambda)| d\lambda\,, \quad t \in [0,\infty). \end{aligned} \tag{2.4}$$

By bounded input we mean

$$|u(t)| \le M_1 < \infty$$

and the output is bounded implies

$$|y(t)| \le M_2 < \infty\,.$$

Let $\int_0^\infty |g(\lambda)|\, |u(t-\lambda)| d\lambda = \bar{M}_2 > M_2\,.$

Then from equation (2.4) we have for bounded input, the bounded output condition becomes

$$|y(t)| \le \int_0^\infty |g(\lambda)|\, |u(t-\lambda)| d\lambda = \bar{M}_2\,. \tag{2.5}$$

2.2. Stability Analysis of Control System

Characteristic Equation Root Location in Relation to Stability

The transfer function of a dynamic system is given by equation (1.29).The characteristic equation of the system and the output time response is obtained by the s-plane location of the roots of the characteristic equation.

Example 2.1:

For a unit step input, the output time response of the following examples justify the above statement.

A) Let

$$\frac{Y(s)}{U(s)} = \frac{2}{s^2+2s+2}.$$

Then

$$\frac{Y(s)}{U(s)} = \frac{2}{s^2+2s+2}. \quad = \frac{2}{(s-(-1+i))(s-(-1-i))} \;,$$

by partial fraction we get

$$= \frac{K_1}{s-(-1-i)} + \frac{K_2}{s-(-1+i)}$$ .Solving for K_1 *and* K_2 we have

$K_1 = i$, $K_2 = -i$ .Using the table of Laplace transform we obtain

$y(t) = ie^{-(1+i)t} - ie^{-(1-i)t}$, $t \in [0,\infty)$ and simplification yields

$y(t) = 2e^{-t}sint$, $t \in [0,\infty)$.This is the output time response for a unit step input.

Using Mathematica, see [4], (take for example $t = [0,15.0064]$) the corresponding output time response for a unit step input is shown in Fig 2.1.

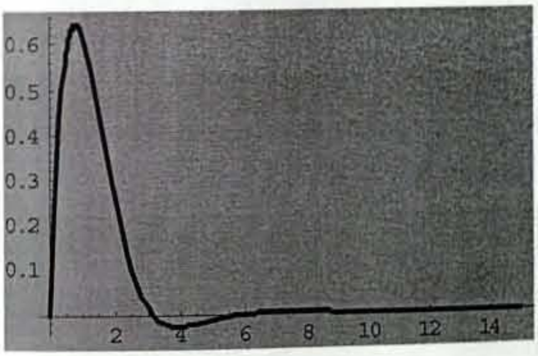

Fig 2.1.The system is asymptotically stable.

From this graph we observe that for the output time response of a system to decay with time and settle to a steady- state value, it is necessary that real part of all the roots of the characteristic equation be of negative sign and the system is said to be asymptotically stable.

B) Let $\frac{Y(s)}{U(s)} = \frac{2}{s^2-2s+2}$ .In the same procedure we obtain

$y(t) = -ie^{(1+i)t} + ie^{(1-i)t}$, $t \in [0,\infty)$ and simplification yields

$y(t) = 2e^{t}sint, t \in [0,\infty)$.

Hence, using Mathematica, (take $t = [0,15.433]$) the corresponding output time response for a unit step input is shown in Fig 2.2.

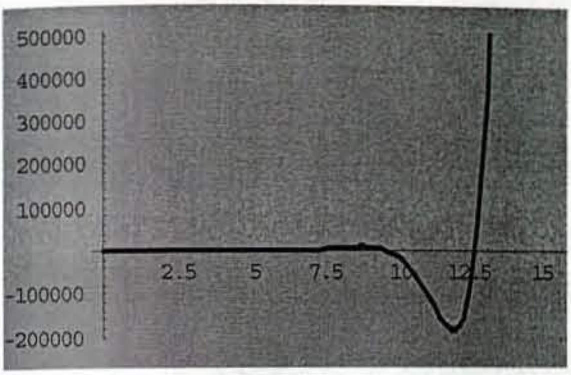

Fig 2.2.The system is unstable.

From Fig 2.2.we observe that for the output time response to increase exponentially with respect to time, at least one of the roots of characteristic equation is having positive real part. This growing output time response leads the system to instability. i.e., it makes the feedback control system unstable.

C) Let

$\frac{Y(s)}{U(s)} = \frac{1}{s^2+1}$ .Using the same way, we have

$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \sin t\ , t \in [0, \infty).$

For $t = [0,35]$, the corresponding output time response for a unit step input is shown in Fig 2.3.

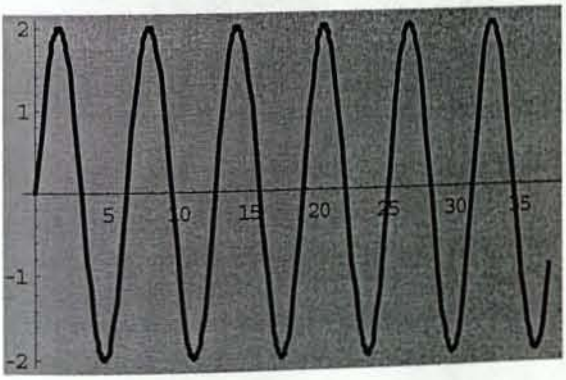

Fig 2.3.The system is marginally stable.

From Fig 2.3.we observe that in case when the real part of complex conjugate root is nil, the output time response shows sustained oscillations and the system is said to be marginally stable. If the characteristic equation is of higher order, then it is not convenient to determine its roots.

In general, the impulse response g(t) $=\mathcal{L}^{-1}[G(s)]$ has a nature dependent on location of the roots of the characteristic equation. The roots of the characteristic equation may be both real and complex and may have multiplicity of different order. Some of the response terms contributed by various types of roots are shown below.

Now let us consider a dynamic system described by some of the transfer functions G(s). Here G(s),Y(s) and U(s) are expressed as in section(1.3),K is constant.

1. $G(s) = \frac{Y(s)}{U(s)} = \frac{K}{S}$. The system has only one root at s=0. Notice that the impulse response for U(s)=1.Then the output response of the system is given by

$y(t) = \mathcal{L}^{-1}\left[K\frac{1}{s}\right] = K, t = [0,\infty)$ which is marginally stable.

By using mathematica, (take $t = [0,6], K = 1$) the response graph is shown in Fig 2.4.

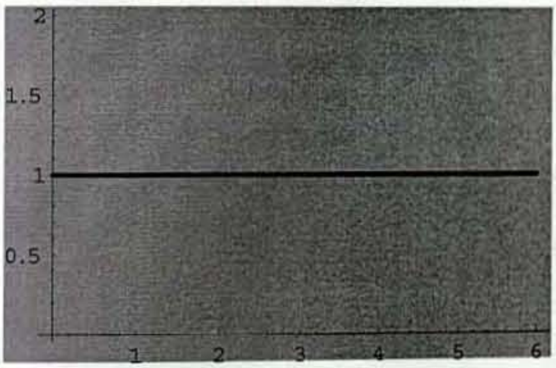

Fig 2.4. The system is marginally stable.

2. $G(s) = \frac{Y(s)}{U(s)} = \frac{K}{s^r}$ .The system has roots of multiplicity r at the origin. Using partial fraction and inverse Laplace transform, see section (1.2.2), the output response of the system is given by $y(t) = K_1 + K_2t + K_3t^2 + \cdots + K_rt^{r-1}, t = [0,\infty)$ which is unstable.
   By using mathematica, ($t = [0,6]$,$K_1 = 0.5, K_2 = 4, K_3 = 0.1, K_4 = 1$) the response graph is shown in Fig 2.5.

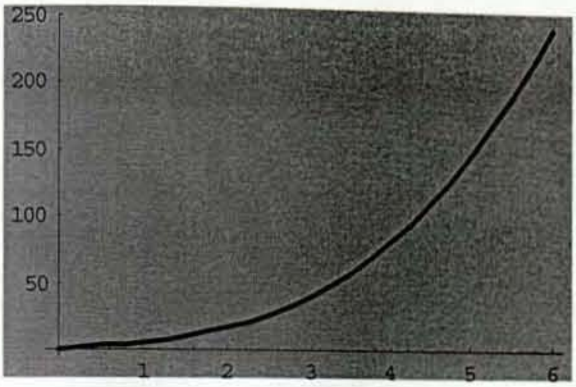

Fig 2.5. The system is unstable.

3a) G(s) = $\frac{Y(s)}{U(s)}=\frac{K}{s-\alpha}$, $(\alpha < 0)$. The system has only one root at s=α, i.e., real and negative root. By similar procedure we obtain the output response of the system:

$y(t) = Ke^{\alpha t}$, $t = [0, \infty)$ which is asymptotically stable.
By mathematica,(take $t = [0,6], K = 2, \alpha = -0.7$), the response graph is shown in Fig 2.6.

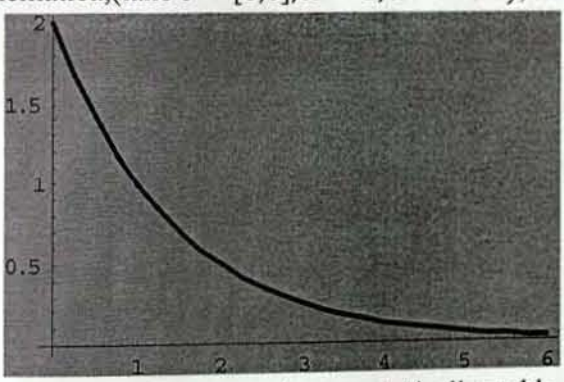

Fig 2.6.The system is asymptotically stable.

3b) G(s) = $\frac{Y(s)}{U(s)}=\frac{K}{s-\alpha}$, $(\alpha > 0)$. The system has only one root at s=α, i.e., real and positive root. By similar procedure we obtain the output response of the system, that is

$y(t) = Ke^{\alpha t}$, $t = [0, \infty)$ which is asymptotically stable. By mathematica, (take $t = [0,6]$, $K = 2, \alpha = 0.7$), the response graph is shown in Fig 2.7.

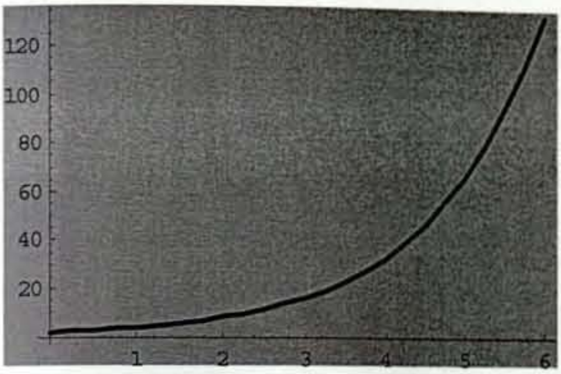

Fig 2.7.The system is unstable.

4a) $G(s) = \frac{Y(s)}{U(s)} = \frac{k}{(s-\alpha)^r}$, $(\alpha > 0)$ .The system has roots of multiplicity r at s=α . In a similar way we have the response function given by

$y(t) = [\,K_1 + K_2 t + K_3 t^2 + \cdots + K_r t^{r-1}]e^{\alpha t}, t = [0, \infty)$ which is unstable. The response graph for ($t = [0,6]$, $\alpha = 0.049, K_1 = 0.5, K_2 = 4, K_3 0.1,\ \ K_4 = 1$) is shown in Fig 2.8.

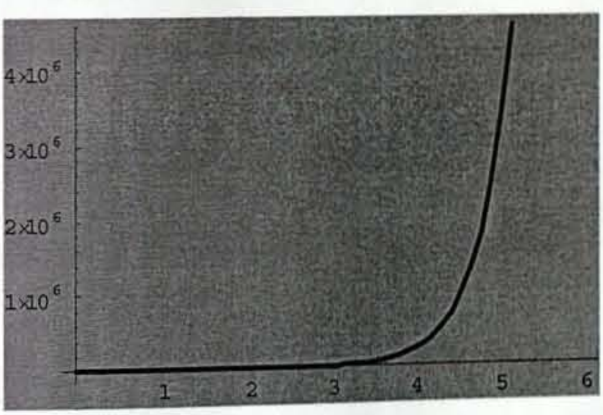

Fig 2.8. The system is unstable.

4b) $G(s) = \frac{Y(s)}{U(s)} = \frac{k}{(s-\alpha)^r}$, $(\alpha < 0)$ .The system has roots of multiplicity r at s=α.In a similar way we have the response function given by:

$y(t) = [\,K_1 + K_2 t + K_3 t^2 + \cdots + K_r t^{r-1}]e^{\alpha t}, t = [0, \infty)$ which is asymptotically stable. The response graph for ($t = [0,6], K_1 = 0.5, K_2 = 4,$

$K_3 = 0.1,\ \ K_4 = 1, \alpha = -2$) is shown in Fig 2.9.

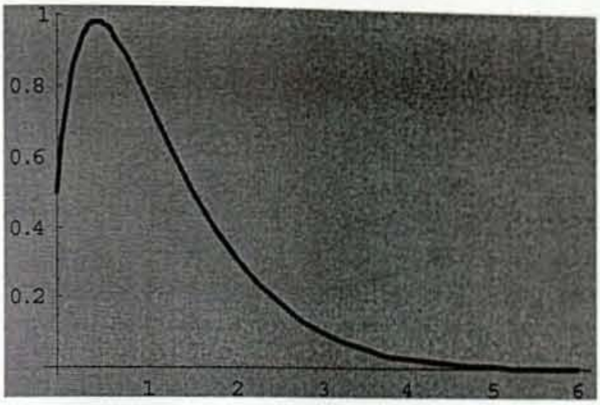

Fig 2.9. The system is asymptotically stable.

5a) $G(s) = \frac{Y(s)}{U(s)} = \frac{K_1}{(s-\alpha+i\omega)(s-\alpha-i\omega)}$, $(\alpha < 0)$. The system has a single complex conjugate root pair $s = \alpha \pm i\omega$,with negative real part. Applying the above procedure, the output response is given by

$y(t) = Ke^{\alpha t}(sin\omega t + \beta), t = [0, \infty)$ which is asymptotically stable, where $K = \frac{K_1}{\omega}$. The response graph for ($t = [0,35], K = 1, \alpha = -0.049, \omega = 1.005, \beta = 2\pi$) is as shown in Fig 2.10.

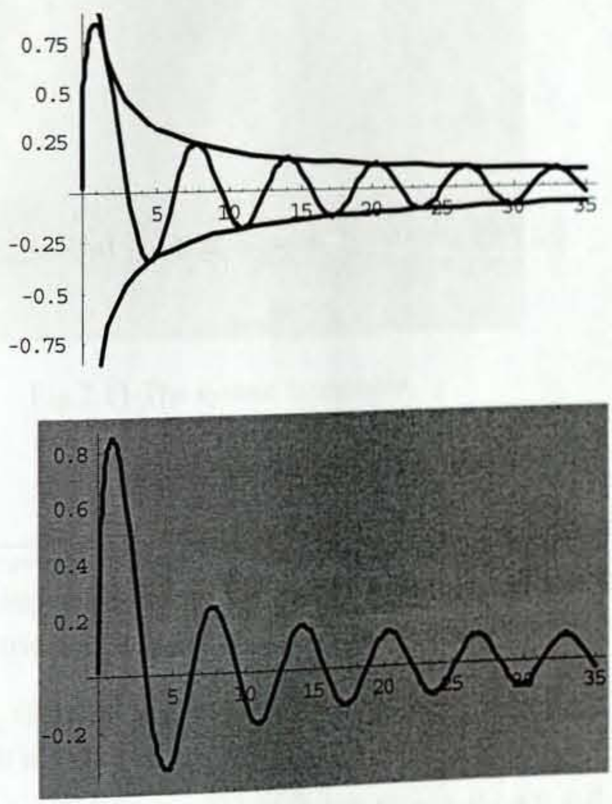

Fig 2.10. The system is asymptotically stable.

5b) $G(s) = \frac{Y(s)}{U(s)} = \frac{K_1}{(s-\alpha+i\omega)(s-\alpha-i\omega)}$, $(\alpha > 0)$.The system has a single complex conjugate root pair $s = \alpha \pm i\omega$,with positive real part. Applying the above procedure, the output response is given by

$y(t) = Ke^{\alpha t}(sin\omega t + \beta), t = [0, \infty)$ which is unstable, where $K=\frac{K_1}{\omega}$.

The response graph for ($t = [0,45], K = 1, \alpha = 0.049, \omega = 1.005, \beta = 2\pi$) is as shown in Fig 2.11.

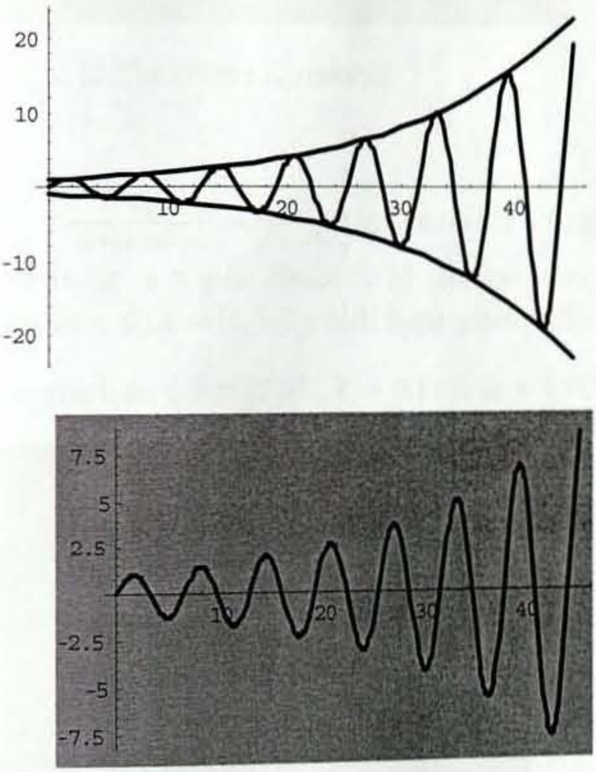

Fig 2.11.The system is unstable.

6.$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{[(s-\alpha-i\omega)(s-\alpha+i\omega)]^r}$, $(\alpha > 0) = \frac{K}{[(s-\alpha)^2+\omega^2]^r}$. The system has complex conjugate root pairs of multiplicity r at $s = \alpha \pm i\omega$, with positive real part. In a similar way we can obtain the output response, i.e.,

$$y(t) = [K_1 \sin(\omega t + \beta_1) + K_2 t \sin(\omega t + \beta_2) + \cdots + K_r t^{r-1} \sin(\omega t + \beta_r)]e^{\alpha t}$$

$t = [0, \infty)$ which is unstable.

The response graph for ($t = [0,20]$, $K_1 = 0.2, K_2 = 1, K_3 = 0.4, \beta_1 = 2\pi, \beta_2 = 4\pi, \beta_3 = 3\pi$,

$\alpha = 0.049$, $\omega = 2$) is as shown in Fig 2.12.

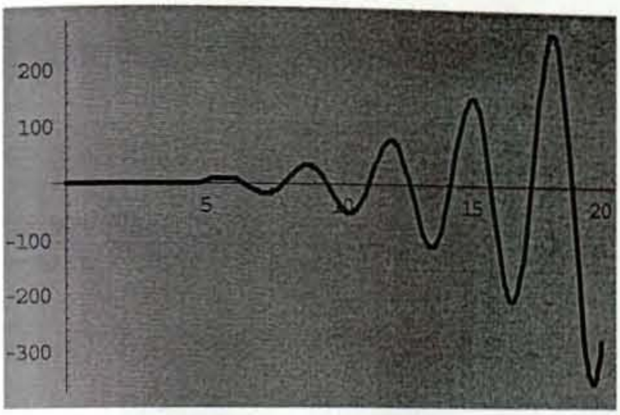

Fig 2.12.The system is unstable.

7.G(s) $= \frac{Y(s)}{U(s)} = \frac{K_1}{(s+i\omega)(s-i\omega)} = \frac{K_1}{s^2+\omega^2}$. The system has a single complex conjugate root pair on the imaginary axis at $s = \pm i\omega$. Hence by similar way we get the output response as $y(t) = Ksin(\omega t + \beta), t = [0, \infty)$ which is marginally stable.

The response graph for ($t = [0,37], K = 0.145,\ \omega = 1.005,\ \beta = 2\pi$) is as shown in Fig 2.13.

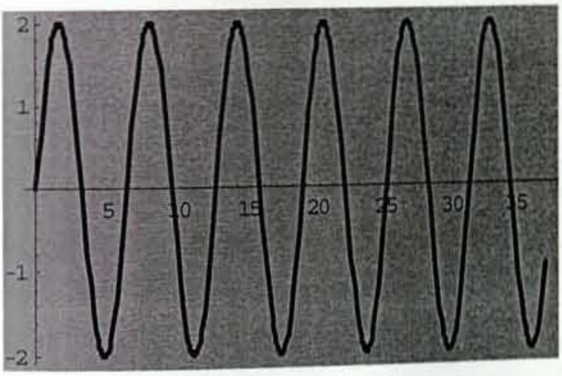

Fig 2.13.The system is marginally stable.

8.G(s) $= \frac{Y(s)}{U(s)} = \frac{K}{[(s-i\omega)(s+i\omega)]^r} = \frac{K}{[s^2+\omega^2]^r}$ .The system has complex conjugate root pairs of multiplicity r on the imaginary axis at s = $\pm i\omega$. As discussed above , the output response of the system is given by:

$$y(t) = [K_1 \sin(\omega t + \beta_1) + K_2 t \sin(\omega t + \beta_2) + \cdots + K_r t^{r-1} \sin(\omega t + \beta_r)],$$
$$t = [0, \infty) \text{ which is unstable.}$$

The response graph for ($t = [0,10]$, $K_1 = 0.2, K_2 = 1, K_3 = 0.4, \beta_1 = 2\pi, \beta_2 = 4\pi, \beta_3 = 3\pi$, $\omega = 3.8$) is as shown in Fig 2.14.

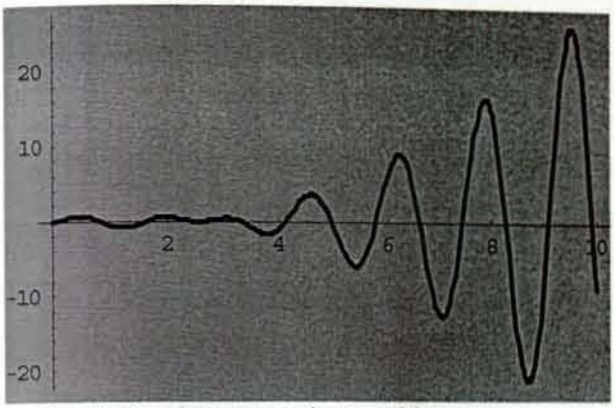

Fig 2.14.The system is unstable

From the above response discussion we have the following conclusions:

If all the roots of the given characteristic equation have negative real parts then the system is stable.

If any of the roots of the given characteristic equation has a repeated root on the imaginary axis or a root having positive real part of this system, then the system is unstable.

2.3. Necessary Conditions for Stability

The stability of a system can be obtained from the coefficients of its characteristic equation. From equation(1.29) the characteristic equation of a control system can be written in the form of a polynomial in s as given below :

$$b_0 s^n + b_1 s^{n-1} + \cdots + b_n = 0 \,. \tag{2.6}$$

If **all the coefficients of the characteristic equation (2.6) should be positive and no term should be missing in the characteristic equation**, then these conditions are necessary for a system to be stable. However, if characteristic equation contains only odd or even powers of s, then this indicates that the roots have no real parts and have only imaginary parts and output response shows sustained oscillations. This makes the system marginally stable. If some of the coefficients are zero or negative it can be concluded that the system is not stable.

Example 2.2:

A) The system described by the characteristic equation $2s^4+s^2+3s+2=0$ is unstable.

B) The system described by the characteristic equation $3s^4-2s^3+5s^2+s+16=0$ is unstable.

C) The system described by the characteristic equation $s^2+2=0$ is marginally stable.

If **all** the coefficients of the characteristic equation are positive or negative there is a possibility of stability of the system to exist and let us now proceed to examine the sufficient conditions of stability.

2.4. Routh Stability Criterion

The criterion does not depend on finding the roots of the characteristic equation and also does not distinguish between real and complex roots but on inspecting the coefficients of the characteristic equation and determining under what conditions it is possible for a root to be in the right-half of the s-plane and hence for the system to be unstable.

Let the characteristic polynomial (the denominator of the transfer function after cancellation of common factor with numerator) be given by equation (2.6).

Then the Routh's criterion requires the construction of the following array **s** of coefficients and values dependent on the coefficients.

$$
\begin{array}{c|cccc}
s^n & a_0 & a_2 & a_4 & a_6 \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 \cdots \\
s^{n-2} & b_1 & b_3 & b_5 & b_7 \cdots \\
s^{n-3} & c_1 & c_3 & c_5 & c_7 \cdots \\
\vdots & & \cdots & & \vdots \\
s^0 & & a_n & &
\end{array},
$$

where the first two rows are obtained from the coefficients of the characteristic equation (2.6). Having written down the first two rows the subsequent rows are easily constructed using the rule defined by the relations:

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$
$$b_3 = \frac{a_1 a_4 - a_0 a_5}{a_1},$$

$$\vdots$$

$$c_1 = \frac{b_1 a_3 - a_1 b_3}{b_1},$$

$$c_3 = \frac{b_1 a_5 - a_1 b_5}{b_1},$$

$$\vdots,$$

up to a_n.

Remark: The proof of Routh's criterion is based on Sturm's theorem which is concerned with sequence of real polynomials, is complicated and of great algebraic interest, so that we shall here state and illustrate without proof.

While preparing the Routh array for a given polynomial some of the elements may not exist. In calculating the entries in the line that follows these elements are considered to be zero. The procedure is illustrated in the example that follow.

Example 2.3:

Consider the system described by

$$\frac{Y(s)}{U(s)} = \frac{s+3}{s^3+5s^2+8s+4}.$$

The characteristic equation of the system is

$$s^3 + 5s^2 + 8s + 4 = 0.$$

Applying the Routh stability criterion, we have

$$\begin{array}{c|cc} s^3 & 1 & 8 \\ s^2 & 5 & 4 \\ s^1 & \frac{36}{5} & 0 \\ s^0 & 4 & \end{array}$$

Since each entry of the first column of Routh array are positive, the system is stable.

Generalization of Criterion of Routh

(a).The roots of the characteristic equations or **its modified equation** all have negative real parts if and only if the elements of the first column of the Routh array are positive.

(b). The number of roots with positive real parts is equal to the number of changes in sign of the elements of the first column. From this the necessary and sufficient conditions for stability follows.

2.5. Necessary and Sufficient Conditions for Stability

For a system to be stable, it is necessary and sufficient that each entry of the first column of Routh array of its characteristic equation (if $a_0 > 0$) or **its modified equation** be positive. If this condition is not satisfied, the system is unstable and the number of sign changes of the terms of the first column of the Routh array corresponds to the number of roots of the characteristic equation in the right half of the s-plane.

Routh's Test-Difficulties and Remedies

There are certain difficulties encountered while applying the Routh stability criterion. The difficulties encountered are generally of the following type.

Difficulty 1:

When the first term in any row of the Routh array is zero while rest of the row has at least one non-zero term. Due to this zero term, the terms in the next row become infinite and Routh's test breaks down. The remedies used to overcome this difficulty are as follows:

Remedy 1:

Put $s = \frac{1}{z}$ in the original characteristic equation and apply the Routh's test on the modified equation in terms of **z.** The number of **z** –roots with positive real parts are the same as the number of **s** –roots with positive real parts. This method works in most but not all cases. The following example illustrates this method:

Example 2.4:

Let us consider the characteristic equation

$s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$. Then the Routh array is:

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 3 \\ s^4 & 1 & 2 & 5 \\ s^3 & 0 & -2 & \\ s^2 & \infty & . & \end{array}$$

Putting $s = \frac{1}{z}$ in the characteristic equation, we obtain

$$\left(\frac{1}{z}\right)^5 + \left(\frac{1}{z}\right)^4 + 2\left(\frac{1}{z}\right)^3 + 2\left(\frac{1}{z}\right)^2 + 3\left(\frac{1}{z}\right) + 5.$$

This implies

$5z^5 + 3z^4 + 2z^3 + 2z^2 + z + 1 = 0.$

The Routh array for this equation is:

$$\begin{array}{c|ccc} z^5 & 5 & 2 & 1 \\ z^4 & 3 & 2 & 1 \\ z^3 & \frac{-4}{3} & \frac{-2}{3} & 0 \\ z^2 & \frac{1}{2} & 1 & \\ z^1 & 2 & 0 & \\ z^0 & 1 & & \end{array}$$

There are two changes of sign in the first column of the Routh array, and so there are two z-roots in the right half z-plane. Therefore, the number of s-roots in the right half s-plane is also two.

Difficulty 2:

When all the elements in any one row of the Routh array are zero. When such a condition happens, it indicates that there are symmetrically located roots in the s-plane. The polynomial whose coefficients are the elements of the row of zeros in the Routh array is called **Auxiliary poly-**

nomial, denoted by **A(s).** This polynomial gives the number and location of root pairs of the characteristic equation which are symmetrically located in the s-plane. The order of the auxiliary polynomial is always even.

Remedy 2:

Due to a zero array, the Routh's test breaks down. This situation is overcome by replacing the row of zeros in the Routh array by a row of coefficients of the polynomial generated by taking the first derivative of the auxiliary polynomial A(s). The following example illustrates the procedure.

Example 2.5:

Let now consider the characteristic equation

$s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$. The Routh array is:

s^6	1	8	20	16	
s^5	2	12	16	0	
s^5	1	6	8	0	
s^4	2	12	16	0	
s^4	1	6	8	0	← Auxiliary polynomial, A(s)
s^3	0	0			← Row of zeros.

The Routh's breaks down, since the terms in the s^3-row are all zero. The auxiliary polynomial A(s) is given by: A(s) = $s^4 + 6s^2 + 8$. The derivative of the polynomial A(s) with respect to s is $\frac{d}{ds}[A(s)] = 4s^3 + 12s$. Then the zeros in the s^3-row are replaced by 4&12. Thus the Routh array becomes:

s^6	1	8	20	16
s^5	1	6	8	0
s^4	1	6	8	0
s^3	1	3	0	
s^2	3	8		
s^1	$\frac{1}{3}$	0		
s^0	8			

There is no change of sign in the first column of the new array. By solving $s^4 + 6s^2 + 8 = 0$,we have $s = \pm i\sqrt{2}$ *and* $s = \pm 2i$. These two pairs of roots are also the roots of the original characteristic equation.

As there is no change in sign in the new array, there are no roots having positive real part. Hence the system considered here is stable.

Example 2.6:

Let now consider the dynamical system given by

$$\frac{dy^{(3)}}{dt^3} + 4\frac{dy^{(2)}}{dt^2} + 5\frac{dy}{dt} + 2y = 2u.$$

Assuming all the initial conditions are zero. To see whether this system is stable, first we transform the system to transfer equation as discussed in section (1.3).Hence we have

$$\frac{Y(s)}{U(s)} = \frac{2}{s^3+4s^2+5s+2} = \frac{2}{s+2} - \frac{2}{s+1} + \frac{2}{(s+1)^2} .$$

Now the Routh array is

s^3	1	5	0
s^2	4	2	0
s^1	$\frac{9}{2}$	0	0
s^0	2	0	

By the Routh criterion, the system is asymptotically stable. By the inverse Laplace transform the output response of this system is

y (t) = $2e^{-2t} - 2e^{-t}$ +$2te^{-t}, t \in [0, \infty)$.

The response graph (for $t \in [0,3]$)is shown in Fig 2.14.

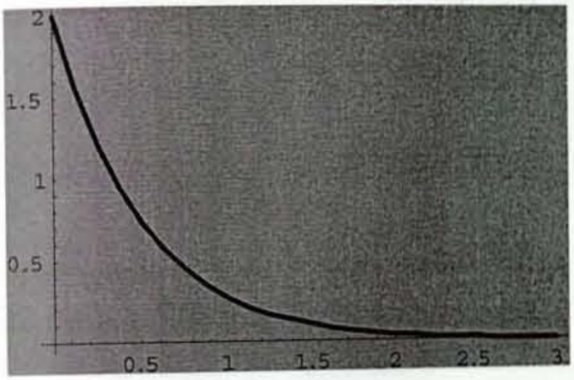

Fig 2.14. Asymptotically stable.

2.7. Liapunov Criterion for Stability

Introduction

Liapunov's method is the most general method we know for system stability analysis .In the previous section we considered the system transfer function whereas in this section we are examining the state equation. For this we use the so called Liapunov's function

$v(x) := (x_1, x_2)$ which has the following properties:

1) v(x) and its partial derivatives , $\frac{\partial v}{\partial x_i}$, i =1, 2,⋯, n are continuous.

2) v(x) > 0, for x≠ 0 (in the nbd. i.e., neighborhood of x = 0),and v(0) = 0 .

3) $\dot{v}(x) < 0$, for $x \neq 0$ (in the nbd of x = 0),and $\dot{v}(0) = 0$.

The stability of a system depends on the dynamics of the system, that is, on the unforced state equation $\dot{x} = Ax$.

The matrix A is said to be **stable**, if the corresponding system defined by equation $\dot{x} = Ax$ is stable. To determine whether A is stable we seek a symmetric matrix P so that

$v(x) = x^T P x$ is a positive definite function and

By the Routh criterion, the system is asymptotically stable. By the inverse Laplace transform the output response of this system is

$y(t) = 2e^{-2t} - 2e^{-t} + 2te^{-t}, t \in [0, \infty)$.

The response graph (for $t \in [0,3]$)is shown in Fig 2.14.

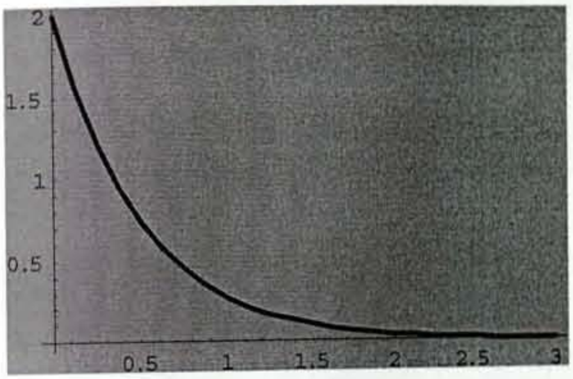

Fig 2.14. Asymptotically stable.

2.7. Liapunov Criterion for Stability

Introduction

Liapunov's method is the most general method we know for system stability analysis .In the previous section we considered the system transfer function whereas in this section we are examining the state equation. For this we use the so called Liapunov's function

$v(x) := (x_1, x_2)$ which has the following properties:

1) v(x) and its partial derivatives , $\frac{\partial v}{\partial x_i}$, i =1, 2,⋯, n are continuous.

2) v(x) > 0, for x$\neq$ 0 (in the nbd. i.e., neighborhood of x = 0),and v(0) = 0 .

3) $\dot{v}(x) < 0$, for $x \neq 0$ (in the nbd of x = 0),and $\dot{v}(0) = 0$.

The stability of a system depends on the dynamics of the system, that is, on the unforced state equation $\dot{x} = Ax$.

The matrix A is said to be **stable**, if the corresponding system defined by equation $\dot{x} = Ax$ is stable. To determine whether A is stable we seek a symmetric matrix P so that

$v(x) = x^T Px$ is a positive definite function and

$$\dot{v}(x) = \dot{x}^T P x + x^T P \dot{x},$$

$$= (Ax)^T P x + x^T P A x,$$

$$= x^T [A^T P + PA] x, \quad (2.7)$$

which is negative-definite.

We have $[A^T P + PA]^T = [P^T A + A^T P^T] = [PA + A^T \text{ P}]$ (since P is symmetric).

Hence $\dot{v}(x)$ is a quadratic form(see section 1.8)

Let $-Q = A^T P + PA.$ (2.8)

For $\dot{v}(x)$ to be negative-definite, $x^T Q x$ must be Positive-definite.

Given a positive-definite matrix Q, the matrix A is **stable** if the solution to the (matrix) equation (2.8) results in a positive-definite matrix P. The solution to equation (2.8) can be very tedious. To make the calculations as simple as possible we choose Q to be simple, that is, $Q = I$

Example 2.7:

For the system described by

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -5\, x_1 - 2\, x_2\,,$$

we can use Liapunvo's direct method to test whether the systems is asymptotically stable.

$A = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix}$ (in companion), and by equation (2.8) we get

$$\begin{bmatrix} 0 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

which implies

$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 17 & 1 \\ 1 & 3 \end{bmatrix}$, which, by Sylvester's criterion, is a positive-definite. Hence the system is stable.

Example 2.9:

Let now consider the system described by the equation

$\dddot{y} + 4\ddot{y} + 5\dot{y} + 2y = 0$ To find the state equation:

Define $x_1 := y$, $\dot{x}_1 := x_2 = \dot{y}, \dot{x}_2 := x_3 = \ddot{y}$, $\dot{x}_3 = \dddot{y} = -4x_3 - 5x_2 - 2x_1$.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{bmatrix}, \text{(in companion)}.$$

Now by equation (2.8) we obtain P as follows

$$\begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & -5 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12} & P_{22} & P_{23} \\ P_{13} & P_{23} & P_{33} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12} & P_{22} & P_{23} \\ P_{13} & P_{23} & P_{33} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Simplification yields

$P = \frac{1}{36} \begin{bmatrix} 73 & 52 & 9 \\ 52 & 87 & 14 \\ 9 & 14 & 8 \end{bmatrix}$, which is a positive definite by Sylvester's criterion, and hence the system is stable.

CHAPTER 3

3. CONTROLLABILITY and OBSERVABILITY

3.1. Introduction

The concepts of controllability and observability are central to modern control theory. Controllability and observability are concerned with the following fundamental questions:

(1) Can a control function u (t) be found which will transform the initial state x_0 of a system to some desired final state x_f in finite time? If the answer to this question is yes, the system is **controllable**.

(2) Can the state of the system be determined by measuring the system output over a finite time interval? If the answer is yes, then the system is **observable.**

3.2. CONTROLLABILITY

We consider a system described by the state equations (1.32)

where A is n x n, B is n x m and C is r x n matrix with the transformation (1.32) .

We can transform equation (1.35) in to the form of equation (1.38), where $A_1 = P^{-1}AP$, $B_1 = P^{-1}B$ and $C_1 = CP$.

Assuming that A has distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ we can choose P so that A_1 is a diagonal matrix, i.e., A_1= diag $\{\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n\}$.

Let n = m = r = 2, the first of the equations (1.38) has the form

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix} = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} + \begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

which is written as

$$\begin{aligned}\dot{z}_1 &= \lambda_1 z_1 + b_1^T u\\ \dot{z}_2 &= \lambda_2 z_2 + b_2^T u\end{aligned} \qquad (3.1)$$

where b_1^T and b_2^T are the row vectors of the matrix B_1.

The output equation is

$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$, which can be written as

$$y_1 = c_{11}z_1 + c_{12}z_2$$

$$y_2 = c_{21}z_2 + c_{22}z_2 \quad \text{or}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix} z_1 + \begin{bmatrix} c_{12} \\ c_{22} \end{bmatrix} z_2$$

so that $y = c_1z_1 + c_2z_2$ (3.2)

where c_1 and c_2 are the column vectors of C_1 .So, in general, equation (3.1) can be written in the form

$$\dot{z}_i = \lambda_i z_i + b_i^T u \quad \text{(i=1, 2, } \cdots\text{, n)}$$

$$y = \sum_{i=1}^{n} c_i \, z_i \tag{3.3}$$

It is seen from equation (3.3) that if b_i^T , the i^{th} row of B_1, has all zero components, then

$\dot{z}_i = \lambda_i z_i + 0$ (i=1, 2, ⋯, n)

and the input u (t) has no influence on the i^{th} mode of the system. The mode is said to be **uncontrollable,** and a system having one or more such modes is uncontrollable. Otherwise, where all the modes are controllable the system is said to be **completely state controllable or just controllable.**

The system equation (3.3) is controllable if and only if a control function u (t) can be found to transform the initial state

$$z_0 = [z_{01}, z_{02}, \dots, z_{0n}]^T \quad \text{to a specified state}$$

$$z_1 = [z_{11}, z_{12}, \dots, z_{1n}]^T$$

It follows that for complete state controllability each of the components of z must be transformed by an appropriate choice of u (t) from

$$z_{0y} \text{ to } z_{1y} \; (y = 1,2,..,n)$$

If the i^{th} mode is uncontrollable, this can not be achieved for the i^{th} component of z. It follows that the system is uncontrollable.

Example 3.1:

The system having the state-space representation

$$\dot{x} = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} x + \begin{bmatrix} 4 \\ 6 \end{bmatrix} u$$

$y = [1 \quad -2]x$ has the characteristic equation

$|\lambda I - A| = \lambda^2 - 3\lambda + 2 = 0$. From this we have $\lambda = 1$ and $\lambda = 2$.

The corresponding eigenvectors are

$x_1 = [1 \quad 1]^T$ and $x_2 = [2 \quad 3]^T$ so that the modal matrix is

$$P = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \text{ and } P^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}.$$

Using the transformation x=P z equation(1.38), the state equation becomes

$$\dot{z} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} z + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$$

$$y = [-1 \quad -4]z.$$

This equation shows that the first mode is uncontrollable, that is, the system is not influenced by the input u_1 and so the system is uncontrollable.

Example 3.2:

The system having the state-space representation

$$\dot{x} = \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} x + \begin{bmatrix} -4 \\ 3 \end{bmatrix} u$$

$y = [2 \quad 3]x$ has the characteristic equation

$|\lambda I - A| = \lambda^2 - 6\lambda + 5 = 0$. From this we have $\lambda = 1$ and $\lambda = 5$.

In similar way we obtain the corresponding eigenvectors so that the modal matrix is

$$P = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \text{ and } P^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix}.$$

Using the transformation x=P z, the state equation now becomes

$$\dot{z} = \frac{1}{4}\begin{bmatrix} -18 & 10 \\ -10 & 10 \end{bmatrix} z + (-\tfrac{1}{4})\begin{bmatrix} 1 \\ 15 \end{bmatrix} u$$

$$y = [11 \quad -1]z.$$

Since each of the modals is different from zero the system is controllable.

On the basis of the above result, we now **state** and **prove** an extremely useful criterion for determining whether a system is controllable.

The Controllability criterion

The dynamical system $\dot{x} = Ax + Bu$

$y = Cx$ (where A, B, and C are as defined in equations (1.32))
is said to be controllable if and only if the

$$\text{rank }[B, AB, A^2B, \ldots, A^{n-1}B] = n. \quad (3.4)$$

Proof: To simplify the notation and the mathematical manipulation, we consider a SISO system, so that in equations (1.32) B is a one column matrix, a column vector b , and C is a row matrix, a row vector c_1^T .The result holds for the more general case when the system is multivariable.

Equations (1.32) and (1.38) are then written as

$$\dot{x} = Ax + bu$$

$$y = C^T x \quad (3.5)$$

and

$$\dot{z} = A_1 z + b_1 u$$

$$y = c^T z \quad (3.6)$$

Hence the necessary condition for the system defined by equation (3.5) to be controllable is that the components of the vector

$b_1 = [\beta_1 \quad \beta_2 \quad \cdots \quad \beta_n]^T$ in equation (3.6) are all non-zero.

In equation (3.6) the matrix

A_1= diag $\{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n\}$, where the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are distinct and not equal to zero. Hence we have for the well-known Vandermonde matrix

$$\begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

has linearly independent columns, so that it is non-singular.

It follows that the necessary condition for the system to be controllable is the (partition matrix

$$Q_1 = [b_1 \vdots A_1 b_1 \vdots \cdots \vdots A_1^{n-1} b_1] = \begin{bmatrix} \beta_1 & \lambda_1\beta_1 & \cdots & \lambda_1^{n-1}\beta_1 \\ \beta_2 & \lambda_2\beta_2 & \cdots & \lambda_2^{n-1}\beta_2 \\ \vdots & \vdots & \cdots & \vdots \\ \beta_n & \lambda_n\beta_n & \cdots & \lambda_n^{n-1}\beta_n \end{bmatrix} \quad (3.7)$$

is non-singular. This is also a sufficient condition for controllability.

Since $A_1 = P^{-1}AP$ and $b_1 = P^{-1}b$, we have

$A_1 b_1 = P^{-1}Ab,\ A_1^2 b_1 = P^{-1}A^2 b,\ \ldots, A_1^{n-1} b_1 = P^{-1}A^{n-1}b$, so that

$$Q_1 = P^{-1}[b \vdots Ab \vdots \cdots \vdots A^{n-1}b] = P^{-1}Q, \quad (3.8)$$

where Q = $[b \vdots Ab \vdots \cdots \vdots A^{n-1}b]$ or $PQ_1 = Q$.

Since Q_1 (for a controllable system) and P^{-1} are both non-singular, Q is non-singular. As Q is non-singular, its n columns are linearly independent. So that the rank of the matrix Q written as r(Q) is n. The matrix Q of equation (3.8) is called the system controllability matrix.

It follows that if the rank of the matrix Q in equation (3.8) is n (Q is non-singular) the system is controllable. If the rank is less than n (Q is singular), the system is uncontrollable. For a multivariable system Q = $[B \vdots AB \vdots \cdots \vdots A^{m-1}B]$.

Example 3.3:

a) Using the controllability criterion verify that the system examined in example (3.1) is uncontrollable.

For the system in example(3.1) we have

$$b = \begin{bmatrix} 4 \\ 6 \end{bmatrix},\ Ab = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix},$$

so that

$$r(Q) = r[b, Ab] = r[\begin{bmatrix} 4 & 8 \\ 6 & 12 \end{bmatrix}] = 1.$$

Since the rank of Q is less than 2, the system is uncontrollable.

Example 3.3:

b) Using the controllability criterion verify that the system examined in example (3.2) is controllable.

For the system in example(3.2) we have

$$b = \begin{bmatrix} -4 \\ 3 \end{bmatrix}, \quad Ab = \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -17 \\ 16 \end{bmatrix},$$

so that

$$r(Q) = r[b, Ab] = r\begin{bmatrix} -4 & -17 \\ 3 & 16 \end{bmatrix} = 2.$$

Since the rank of Q is equal to 2, the system is controllable.

Example 3.4:

Determine whether the system, governed by the equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

is controllable.

$$r(Q) = r[B, AB] = r\left[\begin{array}{cc|cc} 3 & 6 & -6 & -12 \\ -1 & -2 & 2 & 4 \end{array}\right].$$

It is obvious that the rank of this matrix is 1. The system is therefore uncontrollable.

3.3. OBSERVABILITY

Using the transform x = P z, as in the previous section, we end up with the system state equations in the form of equation (1.38), i.e.,

$$\dot{z} = A_1 z + B_1 u$$

$$y = C_1 z$$

If a column of the matrix C_1 is zero, the corresponding mode of the system will not appear in the output y. In this case the system is unobservable, since we can not determine the state variable corresponding to the row of zeros in C_1 from y. Otherwise, where all the modes are observable, the system is said to be **observable.**

Example 3.5:

Determine whether the system having the state equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$ is observable.

The characteristic equation for this system is

$|\lambda I - A| = \lambda^2 - 1 = 0. \mathit{This\ implies}\ \lambda = -1\ \mathit{and}\ \lambda = 1.$ The corresponding eigenvectors are $[1 \quad 1]^T$ *and* $[2 \quad 3]^T$ and the modal matrix is

$P = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ so that $P^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$.

The transformation x=P z, transforms the state-equations in to

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

This result shows that the system is unobservable since the output y is not influenced by the state variable z_2.

The above example illustrates the importance of the observability concept. In this case we have a non-stable system, whose instability is not observed in the output measurement.

Example 3.6:

Determine whether the system having the state equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$ is observable.

The characteristic equation for this system is

$|\lambda I - A| = \lambda^2 + \lambda - 12 = 0. This\ implies\ \lambda = 3\ and\ \lambda = -4$. The corresponding eigenvectors are $[1 \quad -1]^T$ *and* $[8 \quad -1]^T$ and the modal matrix is

$$P = \begin{bmatrix} 1 & 8 \\ -1 & -1 \end{bmatrix} \text{ so that } P^{-1} = -\frac{1}{7}\begin{bmatrix} -1 & -8 \\ 1 & 1 \end{bmatrix}.$$

The transformation x=P z, transforms the state-equations in to

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} u$$

$$y = [-9 \quad -23]\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

This result shows that the system is observable since the output y is influenced by the state variables z_1 *and* z_2.

We now **state** and **prove** a criterion for observability in a similar manner to that of the controllability criterion.

The Observability Criterion

The dynamical system $\dot{x} = Ax + Bu$

$y = Cx$ (where A, B, and C are as defined in equation (1.32))

is said to be observable if and only if

$$\text{the rank } \begin{bmatrix} C^T \\ C^T A \\ \vdots \\ C^T A^{n-1} \end{bmatrix} = n. \tag{3.9}$$

Proof: Again for simplicity we consider a SISO system, but the result holds for the more general multivariable system. Hence the necessary conditions for systems defined by equation (3.5) to be observable is that the components of the vector

$c_1^T := [\gamma_1 \quad \gamma_2 \quad \cdots \quad \gamma_n]$ in equation (3.5) are all non-zero.

In equation (3.5) the matrix

A_1= diag $\{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n\}$, where the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are distinct and not equal to zero. Considering the vandrmonde matrix we have that

$$R_1 = \begin{bmatrix} c_1^T \\ c_1^T A_1 \\ \vdots \\ c_1^T A_1^{n-1} \end{bmatrix} = \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_n \\ \gamma_1\lambda_1 & \gamma_2\lambda_2 & \cdots & \gamma_n\lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_1\lambda_1^{n-1} & \gamma_2\lambda_2^{n-1} & \cdots & \gamma_n\lambda_n^{n-1} \end{bmatrix} \quad (3.10)$$

It follows that the necessary condition for the system to be observable is the matrix given in equation (3.10) is non-singular. This is also a sufficient condition for observability.

Since $A_1 = P^{-1}AP$ and $c_1^T = C^T P$, we have

$c_1^T A_1 = C^T AP, c_1^T A_1^2 = C^T A^2 P, \ldots, c_1^T A_1^{n-1} = C^T A^{n-1} P$, so that

$$R_1 = \begin{bmatrix} C^T \\ C^T A \\ \vdots \\ C^T A^{n-1} \end{bmatrix} P = RP \text{ where } R = \begin{bmatrix} C^T \\ C^T A \\ \vdots \\ C^T A^{n-1} \end{bmatrix} \quad (3.11).$$

Since R_1 (for an observable system) and P are non-singular, R must also be non-singular. As R is non-singular, its n columns are linearly independent, so that the rank of the matrix R written as rank (R) or r (R) is n. The matrix R of equation (3.11) is called the system observability matrix.

It follows that if the rank of the matrix R in equation (3.11) is n (R is non-singular), the system is observable. If the rank is less than n (R is singular), the system is unobservable. For a multivariable system, the matrix R becomes the (partitioned) matrix in which the vector c^T is replaced by the matrix C defined in equation (1.32).

Example 3.7:

a) Using the observability criterion, verify that the system examined in example (3.5) is unobservable. For this system

$C^T = [3 \quad -2]$, $A = \begin{bmatrix} -5 & 4 \\ -6 & 5 \end{bmatrix}$. $Hence$ $R = \begin{bmatrix} c^T \\ c^T A \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -3 & 2 \end{bmatrix}$, since the columns of R are linearly dependent r (R) < 2, that is, r (R) = 1. It follows that the system is unobservable.

Example 3.7:

b) Using the observability criterion, verify that the system examined in example (3.6) is observable. For this system

$C^T = [-2 \quad 7],\ \ A = \begin{bmatrix} 4 & 8 \\ -1 & -5 \end{bmatrix}.\ Hence\ R = \begin{bmatrix} c^T \\ c^T A \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ -15 & -51 \end{bmatrix}$, since the columns of R are linearly independent r (R). It follows that the system is observable.

Table of Laplace Transform

No.	$f(t)$	$F(s) = \mathcal{L}[f(t)]$
1	t^n	$\frac{n!}{s^{n+1}}$
2	ω	$\frac{\omega}{s}$
3	t	$\frac{1}{s^2}$
4	$e^{\omega t}$	$\frac{1}{s-\omega}$, where $Re\, s > Re\, \omega$
5	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$, where $Re\, s > -Im\, \omega$
6	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$, where $Re\, s > -Im\, \omega$
7	$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$
8	$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$
9	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
10	$t \sin \omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$
11	$t \cos \omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
12	$te^{-\omega t}$	$\frac{1}{(s+\omega)^2}$ where $\omega > 0$
13	$t^n e^{-\omega t}$	$\frac{n!}{(s+\omega)^{n+1}}$ where $\omega > 0$
14	$e^{\omega t} f(t)$	$F(s-\omega)$
15	$e^{-\omega t} sin\beta t$	$\frac{\beta}{(s+\omega)^2+\beta^2}$ where $\omega > 0$
16	$e^{-\omega t} sinh\beta t$	$\frac{\beta}{(s+\omega)^2-\beta^2}$ where $\omega > 0$
17	$e^{-\omega t} cos\beta t$	$\frac{s+\omega}{(s+\omega)^2+\beta^2}$ where $\omega > 0$

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