Graduate Seminar Report
On
Interpolation by Spline functions

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1. Introduction

Many scientific and engineering phenomena being measured undergo a translation from one physical domain to another. Data obtained from these measurements are better represented by a set of piecewise continues curves rather than by a single curve. One of the difficulties with polynomial interpolation is that in some cases the oscillatory nature of high-degree polynomials can induce large fluctuations over the entire range when approximating a set of data points. One way of solving this problem is to divide the interval into a set of subintervals and construct a lower-degree approximating polynomial on each sub-interval. This type of approximation is called piecewise interpolation.

Piecewise polynomial functions, especially spline functions, have become increasingly popular. Most of the interest has centered on cubic spline functions because of the case of their applications to a variety of fields such as the solution of boundary value problems for differential equations and the method of finite element for the Numerical solution of partial differential equations. Now it becomes interesting to discuss on several types of piecewise polynomials for interpolating a given set of data points. The simplest of these is piecewise linear interpolation and the most popular one is cubic spline interpolation.

In interpolation by spline functions, the term "spline" is used to refer to a wide class of functions that are used in applications requiring data interpolation and/or smoothing. Splines may be used for interpolation and smoothing of either one-dimensional or multi-dimensional data.

On the surface you might wonder that approximating the given function by degree-three polynomial functions is better than that of degree seven-polynomial functions. Spline interpolation of degree three gives us a good approximation. Thus spline interpolation of degree three is commonly used for approximating a given curve or data points.

2. Polynomial Interpolation Theory

Given the data $x_0 < x_1 < x_2 < ... < x_n$ and $f(x_i) = y_i$ for all i=0,1,2,…,n.

Where x_i are assumed to be distinct, we want to identify the problem of finding a polynomial $P(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$ that interpolates the given data. Applying the condition that $P(x_i) = f(x_i)$ to the given tabulated points x_i gives the system.

$$
\begin{aligned}
&a_0 + a_1 x_0 + a_2 x_0^2 + ... + a_n x_0^n = y_0 \\
&a_0 + a_1 x_1 + a_2 x_1^2 + ... + a_n x_1^n = y_1 \\
&\quad . \\
&\quad . \\
&\quad . \\
&a_0 + a_1 x_n + a_2 x_n^2 + ... + a_n x_n^n = y_n
\end{aligned}
\qquad ... (1)
$$

This is a system of (n+1) linear equations in (n+1) unknowns $a_0, a_1, a_2, ..., a_n$. In matrix form, the system is

Xa =Y where X=$\lfloor x_i^j \rfloor$, i, j= 0,1,2,…,n …(2)

a= $(a_0, a_1, a_2, ..., a_n)^T$ and Y=$(y_0, y_1, y_2, …, y_n)^T$

The matrix X is known as the Vandermonde matrix. Thus, solving the system (1) is equivalent to solving the polynomial interpolation problem.

Theorem: Given n+1 distinct points $x_0 < x_1 < x_2 < ... < x_n$ and n+1 real values $y_0, y_1, y_2, ..., y_n$ there is a unique polynomial P of degree $\le n$ that interpolates the table 1

x_0	x_1	x_2	…	x_n
$y_0 = f(x_0)$	y_1	y_2	…	y_n

Table 1:

Proof: It can be shown that the determinant of the matrix in (2) is

$Det(X) = \prod_{0 \le j < i \le n} (x_i - x_j)$ and is non-zero, to show the polynomial obtained is unique, we assume that there is another polynomial $P^*(x)$ which also satisfies $P^*(x_i) = f(x_i)$, i= 0,1,2,…,n

Consider the polynomial Q(x)=P(x)-P^*(x). Since P(x) and P^*(x) are both polynomial of degree less than or equal to n, Q(x) is also a polynomial of degree less than or equal to n satisfying the conditions $Q(x_i)=P(x_i)-P^*(x_i)=0$, i=0,1,2,…,n

There fore Q(x) is a polynomial of degree $\le n$ which has n+1 distinct roots $x_0, x_1, x_2, \ldots, x_n$. This implies that $Q(x) \equiv 0$, because a polynomial Q(x) of degree n has exactly n roots, real or complex. There fore, P(x)=P^*(x)

Thus there is a unique solution for the a_i's, that is, there is a unique interpolating polynomial of degree n.

Spline Interpolation

The word "Spline" actually refers to a thin strip of wood or metal. At one time curves were designed for ships and planes by mounting actual strips of wood or metal so that they went through the desired data points but were otherwise free to move. For reasons of physics, such curves are approximately piecewise cubic with continuous second derivatives, if they are suitably parameterized.

In interpolation by spline functions, the term "spline" is used to refer to a wide class of functions that are used in applications requiring data interpolation and smoothing.

Let f be a real-valued function on some interval $[a,b]$ and let the set of data points in table below be given.

For simplicity, assume that a=$x_0 < x_1 < x_2 < \ldots < x_n$=b

Table 2

x	a=x_1	x_2	…	x_n=b
Y=f(x)	y_1	y_2	…	y_n

Definition:

A function S is called spline of degree k if it satisfies the following conditions.

1, S is defined in the interval $[a,b]$

2, $S^{(r)}$ is continuous on (a, b) for $0 \le r \le k-1$

3, S is a polynomial of degree $\le$k on each subinterval$[x_i, x_{i+1}]$, i=1, 2, 3… n-1

4, $S(x_i) = f(x_i)$ where f is a function defined on [a, b] and to be approximated by S(x).

Observe that in contrast to polynomial interpolation, the degree of the splines does not increase with the number of points. Here the degree is fixed and one uses more polynomial instead.

3.1. Linear Spline Interpolating Function

A simple and a familiar example of a piecewise polynomial interpolation is piecewise linear interpolation, which consists of connecting a set of data points in table (2) by a series of straight lines as shown in figure 1.

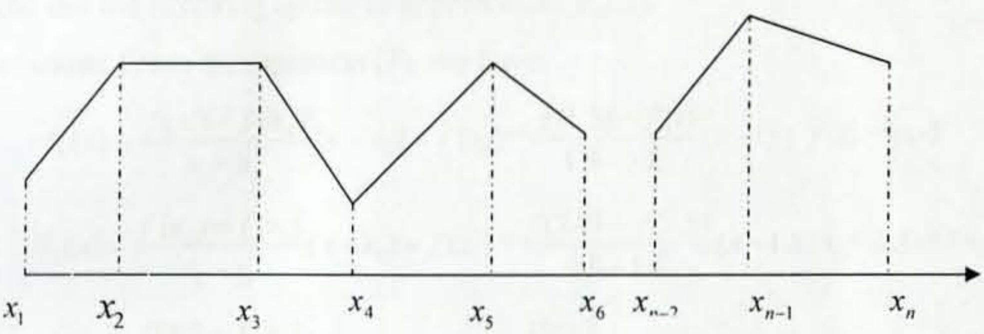


Figure 1: Piecewise linear interpolation.

This procedure can be described as follows:

Let f(x) be a real-valued function defined on some interval$[a,b]$. We wish to construct a piecewise linear polynomial function S(x), which interpolates f(x) at the data points given by a=$x_0 < x_1 < x_2 < ... < x_n$=b.

Using the formula of the equation of the line, it is easy to see that the function S(x) is defined by

$$S_i(x) = f(x_i) + \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}(x - x_i), i = 1,2,3,...,n-1 \qquad ... (3)$$

$$= f(x_i) + f[x_{i+1}, x_i](x - x_i) \text{ on each subinterval } [x_i, x_{i+1}] \quad ....(4)$$

Outside the interval $[a,b]$, S(x) is usually defined by

$$S(x) = \begin{cases} S_1(x), if\ x < a \\ S_{n-1}(x), if\ x > b \end{cases} \qquad ... (5)$$

The points $x_2, x_3, ..., x_{n-1}$, where S(x) changes from one polynomial to another, are called the break points or knots. Because S(x) is continuous on $[a,b]$,

it is called a spline of degree one.

Example1: Find a first degree interpolating polynomial function for the data points

x	1	1.5	2	2.5	3
f(x)	1	3	7	10	15

And use the resulting spline to approximate f(2.2).

Solution: From the equation (3), we have

$$S_1(x) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + f(x_0) = \frac{f(1.5) - f(1)}{1.5 - 1.0}(x - 1) + f(1) = 4x-3$$

$$S_2(x) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) + f(x_1) = \frac{f(2.0) - f(1.5)}{2.0 - 1.5}(x - 1.5) + f(1.5) = 8x-9$$

$$S_3(x) = \frac{f(x_3) - f(x_2)}{x_3 - x_2}(x - x_2) + f(x_2) = \frac{10 - 7}{0.5}(x - 2.0) + 7 = 6x - 5$$

$$S_4(x) = \frac{f(x_4) - f(x_3)}{x_4 - x_3}(x - x_3) + f(x_3) = \frac{f(3.0) - f(2.5)}{3.0 - 2.5}(x - 2.5) + f(2.5) = 10x-15$$

Hence $S(x) = \begin{cases} 4x - 3, if\ x \in [1, 1.5] \\ 8x - 9, if\ x \in [1.5, 2] \\ 6x - 5, if\ x \in [2, 2.5] \\ 10x - 15, if\ x \in [2.5, 3] \end{cases}$ and the value x=2.2 lies in $[2, 2.5]$ and so f(2.2)

$\approx S_3(2.2) = 6(2.2) - 5 = 8.2$

A Question that one may ask is about the goodness of fit when we interpolate a function by a first-degree spline. The answer is found in the following theorem.

Theorem (First degree spline accuracy)

Suppose f is twice differentiable and continuous on the interval $[a,b]$. If P(x) is a first-degree spline interpolating f through the knots $a = x_1 < x_2 < ... < x_n = b$, then, $|f(x) - P(x)| \le \frac{1}{8}Mh^2, a \le x \le b$, where $h = \max_i(x_{i+1} - x_i)$ and M denotes the maximum of $|f''(x)|$ on (a , b).

Proof: From error of Lagrange interpolation we have that

$f(x) - P(x) = \frac{1}{n!} f^{(n)}(\xi)\prod_{i=1}^{n}(x - x_i)$, where n is the number of nodes.

On the interval $[a,b]$, we have n=2 so $f(x) - P(x) = \frac{1}{2} f''(\xi)(x-a)(x-b)$, for some ξ on (a,b). Since $|f''(\xi)| \le M$ on (a, b), and $\max_{x\in[a,b]} |x-a||x-b| = \frac{(b-a)^2}{4}$, since the maximum value occurs at the midpoint. It follows that $|f(x) - P(x)| \le \frac{1}{2} M \frac{(b-a)^2}{4} = \frac{1}{8}Mh^2$

From this theorem one can understood that if the only thing we know is that the second derivative of our function is bounded, then we are guaranteed that the maximum interpolation error we make decreases to zero as $h \to 0$.

With polynomial interpolation, however, using for 10 data points, we had an error estimate in terms of the 10^{th} derivative.

Example 2: Assuming that we know M, find the smallest value of n to force the error bound for a first-degree spline to be less than a given tolerance ε for n equally spaced knots.

Solution: we have that $\left|f''(x)\right| \le M$, so $\frac{1}{8}Mh^2 \le \varepsilon$, now solving for h to get $h \le \sqrt{\frac{8\varepsilon}{M}}$, since $h = \frac{(b-a)}{n-1}$, it follows that $(b-a)\sqrt{\frac{M}{8\varepsilon}} \le n-1$ now solving for n to get $n = 1 + \left[(b-a)\sqrt{\frac{M}{8\varepsilon}}\right]$, where $[x]$ is the so called ceiling function. i.e. $[x]$ = the smallest integer $\ge x$

Question1: Determine whether the function $f(x) = \begin{cases} 2x-1, & if\ x \in [-1,1] \\ -x+2, & if\ x \in [1,2] \\ 5x, & if\ x \in [2,3] \end{cases}$ is first degree spline function or not.

3.2 Higher-Degree Spline Interpolation

Higher-degree spline interpolations are those spline interpolations whose degrees are two and more. These are used whenever smoothness and exactly good shape is needed in approximating the function.

If we want the approximating spline to have a continuous m^{th}-derivative, a spline of degree at least (m+1) is selected. Consider a situation in which knots $x_1 < x_2 < ... < x_n$ have been prescribed. Suppose that a piecewise polynomial of degree m is to be defined, with it species joined at the knots in such a way that the resulting spline S has m-continuous derivatives. At a typical interior knots we have the following circumstances.

To the left of t, S(x)=p(x) and to the right of t, S(x)= q(x), where p and q are m^{th}-degree polynomials.

The continuity of the m^{th}-derivatives $S^{(m)}$ implies the continuity of the lower order derivatives $S^{m-1}, S^{m-2}, S^{m-3}, ..., S^1, S$

So for any k $\lim_{x \to t^-} S^{(k)}(x) = \lim_{x \to t^+} S^{(k)}(x) \qquad (0 \le k \le m) \quad ... (1)$

From which we conclude that $\lim_{x \to t^-} p^{(k)}(x) = \lim_{x \to t^+} q^{(k)}(x) \qquad (0 \le k \le m) \quad ... (2)$

Here:- $\lim x \to t^+$ means that the limit is taken over x-values that converges to t from above to t. i.e. (x-t) is positive for all x-values.

Similarly $\lim x \to t^-$ means that the limit is taken over x-values that converge to t from below t. i.e. (x-t) is negative for all x values. Since p and q are polynomials their derivatives of all orders are continuous. So equation (2) is the same as $p^{(k)}(t) = q^{(k)}(t)$ $(0 \le k \le m)$.

This condition forces p and q to be the same polynomials. Since by Taylor theorem $p(x) = \sum_{k=0}^{m} \frac{1}{k!} p^{(k)}(t)(x-t)^k = \sum_{k=0}^{m} \frac{1}{k!} q^{(k)}(t)(x-t)^k = q(x)$

This argument can be applied at each of the knots $x_2 < x_3 < ... < x_{n-1}$ and we see that S is simply one polynomials through out the entire interval from x_1 to x_n. Thus we need a piecewise polynomial of degree m+1 with m continuous derivatives in order to have a spline function i.e. just a single polynomial through out the entire interval. [Recall that we already know that ordinary polynomials do not serve well in curve fitting.]

In general higher-degree splines are used whenever smoothness and shape is needed in the approximating the function.

3.2.1 Quadratic spline interpolation

The first-degree splines are although useful in certain applications, they suffer one obvious imperfection. The first-degree splines are not smooth i.e. at each knot the slope of the spline can change abruptly from one value to another. Technically, the failure of smoothness is in the pronounced discontinuity of the first –derivative. But these quadratic splines have continuous first-derivatives at the knots. Although Quadratic splines do not ensure equal second derivatives at the knots, they serve nicely to demonstrate the general procedure for developing higher order splines. The objectives in Quadratic splines are to derive a second degree polynomial for each interval between the data points. The polynomial for each interval can be represented generally as:-

$$f_i(x) = a_i x^2 + b_i x + c_i \qquad \dots (1)$$

In general, a quadratic spline satisfies the following properties:

I, $S(x_i) = f(x_i)$ i= 0, 1,2,…,n

II, On each subinterval $[x_{i-1}, x_i]$, $1 \le i \le n$, S(x) is a second-degree polynomial, except in the first or the last interval.

III, S(x) and $S^{/}(x)$ are continuous on (a, b)

We denote $S^{//}(x_i) = M_i$, if the second derivative exists. On each subinterval $[x_{i-1}, x_i]$, we approximate f(x) by a second degree polynomial as $S(x) = S_i(x) = a_i x^2 + b_i x + c_i$,

i= 1,2,3,…,n

There are 3n unknowns to be determined which are a_i, b_i, c_i , i= 1, 2, 3,…, n. Since S(x) is continuous at the internal nodes $x_1, x_2, x_3, ..., x_{n-1}$ we obtain the equations

On $[x_{i-1}, x_i] : S_i(x_i) = f(x_i) = a_i x^2 + b_i x + c_i$ … (2)

On $[x_i, x_{i+1}] : S_{i+1}(x_i) = f(x_i) = a_{i+1} x^2 + b_{i+1} x + c_{i+1}$ i= 1, 2, 3,…, n-1 … (3)

From this set we have 2n-2 equations. Since $S^{/}(x)$ is also continuous at the internal nodes, we obtain the equations from continuity at x_i,

$S_i^{/}(x_i) = S_{i+1}^{/}(x_i)$ or $2a_i x_i + b_i = 2a_{i+1} x_i + b_{i+1}$ i= 1,2,3,…,n-1 … (4)

From this set we have n-1 equations. At the end points x_0, x_n interpolators conditions give the equations

$f(x_0) = a_1 x_0^2 + b_1 x_0 + c_1$ and $f(x_n) = a_n x_n^2 + b_n x_n + c_n$ … (5)

Now, we have a total of (2n-2) + (n-1)+2=3n-1 equations to determine the 3n unknowns. We need one more equation to determine the polynomials uniquely. This extra condition can be provided in a number of ways. We may prescribe $M_0 = f^{//}(x_0) = S_1^{//}(x_0)$

This gives $f^{//}(x_0) = 2a_1 = p$ or $a_1 = \frac{p}{2}$ usually the value p=0 is chosen. In this case we get $a_1 = 0$. Hence in the first subinterval $[x_0, x_1]$ we are using a linear approximation, which is the first two points are joined by a straight line. Now the system of (3n) x (3n) linear algebraic equations are solved for a_i, b_i, c_i i= 1, 2, 3,…, n

However by rearranging the equations in a proper order, it is possible to solve 3 x 3 equations for each set of unknowns a_i, b_i, c_i, i= 1, 2, 3,…, n

Let us illustrate this procedure through an example

Example: Suppose that we have three subintervals $[x_0,x_1]$, $[x_1,x_2]$, $[x_2,x_3]$. Then, from equation (2),(3),(4) and (5) we have the equations

$a_1x_1^2+b_1x_1+c_1=f(x_1)$, $a_2x_1^2+b_2x_1+c_2=f(x_1)$

$a_2x_2^2+b_2x_2+c_2=f(x_2)$, $a_3x_2^2+b_3x_2+c_3=f(x_2)$

And $2a_1x_1+b_1=2a_2x_1+b_2$, $2a_2x_2+b_2=2a_3x_2+b_3$ and also $a_1x_0^2+b_1x_0+c_1=f(x_0)$ and

$a_3x_3^2+b_3x_3+c_3=f(x_3)$

Let us choose $M_0=f''(x_0)=0$ as the extra condition. This gives $a_1=0$.

Using the equations above we write them in the following order.

$b_1x_0+c_1=f(x_0)$ and $b_1x_1+c_1=f(x_1)$ since $a_1=0$ ...(6)

$$\begin{cases} a_2x_1^2+b_2x_1+c_2=f(x_1) \\ a_2x_2^2+b_2x_2+c_2=f(x_2) \\ 2a_1x_1+b_1=2a_2x_1+b_2 \end{cases} \quad \text{... (7)}$$

$$\text{And} \begin{cases} a_3x_2^2+b_3x_2+c_3=f(x_2) \\ 2a_2x_2+b_2=2a_3x_2+b_3 \\ a_3x_3^2+b_3x_3+c_3=f(x_3) \end{cases} \quad \text{... (8)}$$

The system of equation (6) are solved for b_1, c_1. Using these solutions, the systems of equations in (7) are solved. Finally the systems of equations in (8) are solved in the forward direction.

If $M_3=f''(x_3)=0$ is prescribed, then we rearrange the equations so that the solution is obtained in the backward direction, that is, we solve for b_3, c_3 first, then for a_2, b_2, c_2 etc.

Example: Given the data

x	0	1	2	3
f(x)	1	2	33	244

Fit Quadratic splines with $M(0)=f''(0)=0$. Hence find an estimate f(2.5)

Solution: we write the quadratic spline approximation as

$$S(x)=\begin{cases} S_1(x)=a_1x^2+b_1x+c_1, & 0\le x\le 1 \\ S_2(x)=a_2x^2+b_2x+c_2, & 1\le x\le 2 \\ S_3(x)=a_3x^3+b_3x+c_3, & 2\le x\le 3 \end{cases}$$

Since $M(0)=f''(0)=0$, we get $a_1=0$. Substituting

$x_0=0, x_1=1, x_2=2, x_3=3, f(x_0)=1, f(x_1)=2, f(x_2)=33, f(x_3)=244$ in equation (6), (7) and

(8) we obtain $\left.\begin{aligned} b_1(0)+c_1 &= 1 \\ b_1+c_1 &= 2 \end{aligned}\right\}$

$$\left.\begin{aligned} a_2+b_2+c_2 &= 2 \\ 4a_2+2b_2+c_2 &= 33 \\ 2a_2+b_2 &= 2a_1+b_1 \end{aligned}\right\}$$

$$\left.\begin{aligned} 4a_3+2b_3+c_3 &= 33 \\ 9a_3+3b_3+c_3 &= 244 \\ 4a_3+b_3 &= 4a_2+b_2 \end{aligned}\right\}$$

Solving the first system, we get, $c_1=1$, $b_1=1$. The second system become

$$\left.\begin{aligned} a_2+b_2+c_2 &= 2 \\ 4a_2+2b_2+c_2 &= 33 \\ 2a_2+b_2 &= 1 \end{aligned}\right\}$$

The solution of this system becomes $a_2=30$, $b_2=-59$, $c_2=31$

And the third system becomes $\left.\begin{aligned} 4a_3+2b_3+c_3 &= 33 \\ 9a_3+3b_3+c_3 &= 244 \\ 4a_3+b_3 &= 61 \end{aligned}\right\}$ and the solution of this system are

$a_3=150, b_3=-539, c_3=511$. Therefore the quadratic splines in the corresponding intervals can be written as

$$\begin{aligned} S_1(x) &= x+1 &&, \quad 0\le x\le 1 \\ S_2(x) &= 30x^2-59x+31 &&, \quad 1\le x\le 2 \\ S_3(x) &= 150x^2-539x+511 &&, \quad 2\le x\le 3 \end{aligned}$$

Hence an estimate at 2.5 is f(2.5)=101

Drawback of Quadratic Spline functions

Quadratic spline interpolations have a lot of application in mathematics. Even though it has drawbacks such as

I, A straight line connect the first two or the last two points.

II, The spline for the last interval or the first interval may swing high in the above cases.

3.2.2 CUBIC SPLINE INTERPOLATION

THEORY OF CUBIC SPLINE INTERPOLATION

Real world numerical data is usually difficult to analysis. Any function which would effectively correlate the data would be difficult to obtain and highly unwieldy (bulky, unmanageable).

To this end, the idea of the cubic spline was developed. Using this process, a series of unique cubic-polynomials are fitted between each of the data points, with the stipulation that the curve obtained be continuous and appear smooth. These cubic splines are then be used to determine rates of change and cumulative change over an interval. The fundamental idea behind cubic spline interpolation is based on the engineer's tool used to draw smooth curves through a number of points. These splines consist of weights attached to a flat surface at the points to be connected. A flexible strip is then bent across each of these weights, resulting in a pleasingly smooth curve. The Mathematical spline is similar in principle. The points in this case, are numerical data. The weights are n^{th}-coefficients on the cubic polynomials used to interpolate the data. These coefficients bend the line so that it passes through each of the data points without any erratic behavior or breaks in continuity.

The graph of cubic spline functions are shown on figure 2 below.

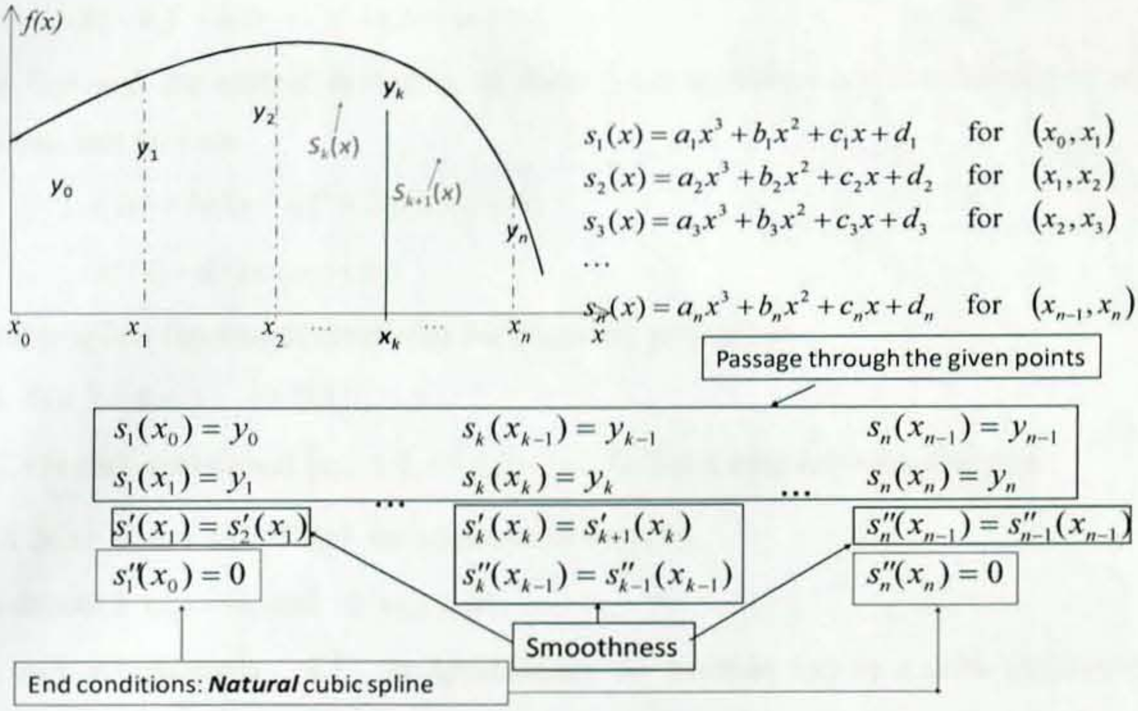


Figure 2: Cubic splines on $[x_0, x_n]$

3.2.2.1 Method 1

Interpolation by a cubic spline function: - using system of equations

Process: - The essential idea is to fit a piecewise function of the form:-

$$S(x) = \begin{cases} s_1(x), \text{ if } & x \in [x_1, x_2] \\ s_2(x), \text{ if } & x \in [x_2, x_3] \\ s_3(x), \text{ if } & x \in [x_3, x_4] \\ \quad . & \quad . \\ \quad . & \quad . \\ \quad . & \quad . \\ s_{n-1}(x), \text{ if } & x \in [x_{n-1}, x_n] \end{cases} \qquad \text{for each } i = 1,2,\ldots,n-1 \quad \ldots(1)$$

Where s_i is a third degree polynomial of the form

$$s_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i \qquad \dots (2)$$

The first and the second derivative of these (n-1) equations are fundamental to this process, and they are

$$s_i^{/}(x) = 3a_i(x - x_i)^2 + 2b_i(x - x_i) + c_i \qquad \dots (3)$$

$$s_i^{//}(x) = 6a_i(x - x_i) + 2b_i \qquad \dots (4)$$

A cubic spline function S(x) satisfies the following properties:-

i, $S(x_i) = f(x_i)$ $\quad i = 0,1,2,\dots,n$

ii, On each subinterval $[x_{i-1}, x_i]$, i = 1,2,…,n , S(x) is a third degree polynomial.

iii ,S(x) , $S^{/}(x)$ and $S^{//}(x)$ are continuous on (a , b).

We denote $S^{/}(x_i) = m_i$ and $S^{//}(x_i) = M_i$

On each subinterval $[x_{i-1}, x_i]$, we approximate the function f(x) by a cubic polynomial function as $S(x) = a_i x^3 + b_i x^2 + c_i x + d_i$ where i = 1,2,3,…,n $\qquad \dots (5)$

We have 4n unknowns i.e a_i, b_i, c_i and d_i, i= 1, 2, 3,…,n to be determined to determine the cubic interpolating polynomial function. Since S(x), $S^{/}(x)$ and $S^{//}(x)$ are continuous functions we have the following results.

I, From the continuity of S(x)

On $[x_{i-1}, x_i]$: $S_i(x_i) = f(x_i) = a_i x_i^3 + b_i x_i^2 + c_i x_i + d_i$

On $[x_i, x_{i+1}]$: $S_{i+1}(x_i) = f(x_i) = a_{i+1}x_i^3 + b_{i+1}x_i^2 + c_{i+1}x_i + d_{i+1}$, for i= 1,2,3,…,n-1

II, From the continuity of $S^{/}(x)$ at x_i. We have $S^{/}(x_i) = P_i^{/}(x_i) = P_{i+1}^{/}(x_i)$ which implies that $3a_i x_i^2 + 2b_i x_i + c_i = 3a_{i+1}x_i^2 + 2b_{i+1}x_i + c_{i+1}$, for i= 1,2,3,…,n-1

III, From the continuity of $S^{//}(x)$ at x_i, we have that $S^{//}(x_i) = S_i^{//}(x_i) = S_{i+1}^{//}(x_i)$ which implies that $6a_i x_i + 2b_i = 6a_{i+1}x_i + 2b_{i+1}$, for i= 1,2,3,…,n-1

IV, At the end points x_0 and x_n we have the interpolator conditions.

$$S(x_0) = S_1(x_0) = f(x_0) = a_1 x_0^3 + b_1 x_0^2 + c_1 x_0 + d_1$$

$$S(x_n) = S_n(x_n) = f(x_n) = a_n x_n^3 + b_n x_n^2 + c_n x_n + d_n$$

Now here we have 2(n-1) equations (conditions) from (I) and 2(n-1) equations from (II) and (III). Finally we have two more equations from (IV) that is a total of 4n-2 equations. Since we have 4n unknowns, to solve this system of equation (to determine the cubic spline function uniquely), we need two more equations.

In most cases, we prescribe $S''(x)$ at the two end points, that is $S''(x_0) = M_0 = p$ and $S''(x_n) = M_n = q$.

The end conditions, $M_0 = 0$, $M_n = 0$, lead to a natural spline.

However, we can use the conditions as $p \neq 0$ or/and $q \neq 0$. If the above two conditions are imposed, then we have 4n equations in 4n unknowns. These equations can be written in matrix form and solution can be obtained.

Example 1: Obtain the cubic spline approximation for the function defined by the data.

x	0	1	2	3
f(x)	1	2	33	244

With M(0)=0 and M(3)=0

Hence find an estimate of f(2.5)

Solution: We write the polynomial approximations as the following

$$S(x) = \begin{cases} S_1(x) = a_1 x^3 + b_1 x^2 + c_1 x + d_1 & ,0 \le x \le 1 \\ S_2(x) = a_2 x^3 + b_2 x^2 + c_2 x + d_2 & ,1 \le x \le 2 \\ S_3(x) = a_3 x^3 + b_3 x^2 + c_3 x + d_3 & 2 \le x \le 3 \end{cases}$$

From the continuity of S(x) we have the following equations

$$S(x_1) = f(x_1) = a_1 x_1^3 + b_1 x_1^2 + c_1 x_1 + d_1 = a_1 + b_1 + c_1 + d_1$$

$$S_2(x_1) = f(x_1) = a_2 x_1^3 + b_2 x_1^2 + c_2 x_1 + d_2 = a_2 + b_2 + c_2 + d_2$$

$$S_2(x_2) = f(x_2) = a_2 x_2^3 + b_2 x_2^2 + c_2 x_2 + d_2 = 8a_2 + 4b_2 + 2c_2 + d_2$$

$$S_3(x_2) = f(x_2) = a_3 x_2^3 + b_3 x_2^2 + c_3 x_2 + d_3 = 8a_3 + 4b_3 + 2c_3 + d_3$$

And from the continuity of $S'(x)$ we have the following equations

$$S_1'(x_1) = 3a_1 x_1^2 + 2b_1 x_1 + c_1 = S_2'(x_1) = 3a_2 x_1^2 + 2b_2 x_1 + c_2$$

$\Rightarrow\ 3a_1 + 2b_1 + c_1 = 3a_2 + 2b_2 + c_2$ and

$$S_2'(x_2) = 3a_2x_2^2 + 2b_2x_2 + c_2 = S_3'(x_2) = 3a_3x_2^2 + 2b_3x_2 + c_3$$

$\Rightarrow\ 12a_2 + 4b_2 + c_2 = 12a_3 + 4b_3 + c_3$

Continuity of $S''(x)$ gives the following equations.

$$S_1^{"}(x_1) = 6a_1x_1 + 2b_1 = S_2''(x_1) = 6a_2x_1 + 2b_2$$

$\Rightarrow$

$$6a_1 + 2b = 6a_2 + 2b_2$$

$$S_2''(x_2) = 6a_2x_2 + 2b_2 = S_3''(x) = 6a_3x_2 + 2b_3$$

$\Rightarrow\ 6a_2 + 2b_2 = 6a_3 + 2b_3$

At the end points, we have the interpolator conditions

$$S(x_0) = f(x_0) = a_1x_0^3 + b_1x_0^2 + c_1x_0 + d_1 = d_1$$

$$S_3(x_3) = f(x_3) = a_3x_3^3 + b_3x_3^2 + c_3x_3 + d_3 = 27a_3 + 9b_3 + 3c_3 + d_3$$

Form the given conditions, M (0) = 0, M(3) = 0 we get $b_1 = 0$ *and* $9a_3 + b_3 = 0$.

Substituting the values $b_1 = 0$ and $b_3 = -9a_3$ and $d_1 = 1$, we obtain the system of equations

$$\begin{cases} a_1 + c_1 = 1 \\ a_2 + b_2 + c_2 + d_2 = 2 \\ 8a_2 + 4b_2 + 2c_2 + 2c_2 + d_2 = 33 \\ -28a_3 + 2c_3 + d_3 = 33 \\ 3a_1 + c_1 - 3a_2 - 2b_2 - c_2 = 0 \\ 12a_2 + 4b_2 + c_2 + 24a_3 - c_3 = 0 \\ 3a_1 - 3a_2 - b_2 = 0 \\ 6a2 + b2 + 3a3 = 0 \\ -54a_3 + 3c_3 + d_3 = 244 \end{cases}$$

The solution of this system is a_1=-4, b_1= 0, c_1= 5, d_1= 1, a_2 = 50, b_2 = -162, c_2 = 167,

d_2 = -53, a_3= -46, b_3= 414, c_3= -985, d_3= 715

Hence, the interpolating cubic spline function is

$$S(x) = \begin{cases} S_1(x) & = -4x^3 + 5x + 1 & ,0 \le x \le 1 \\ S_2(x) & = 50x^3 - 162x^2 + 167x - 53 & ,1 \le x \le 2 \\ S_3(x) & = -46x^3 + 414x^2 - 985x + 715 & ,2 \le x \le 3 \end{cases}$$

An estimate of f(x) at x=2.5 is

$$f(2.5) \approx S_3(2.5) = -46(2.5)^3 + 414(2.5)^2 - 985(2.5) + 715 = 121.25$$

3.2.2.2 Method 2:

Interpolation by Cubic-Spline functions directly without using systems of Equations

We construct the cubic spline function as the following ways.

Since S(x) is to be a piecewise cubic polynomial, $S''(x)$ is a linear function of x in the interval $x_{i-1} \le x \le x_i$ and hence can be written as

$$S''(x) = \frac{(x_i - x)}{(x_i - x_{i-1})} S''(x_{i-1}) + \frac{(x - x_{i-1})}{(x_i - x_{i-1})} S''(x_i) \qquad \dots (6)$$

By using the Lagrange interpolating polynomial on $[x_{i-1}, x_i]$ for i= 1,2,3,…,n

Now integrating (**6**) two times with respect to x, we get

$$S(x) = \frac{(x_i - x)^3}{6h_i} M_{i-1} + \frac{(x - x_{i-1})^3}{6h_i} M_i + c_1 x + c_2 \qquad \dots (7)$$

where $M_i = S''(x_i)$ and c_1 and c_2 are arbitrary constants to be determined by using the conditions $S(x_{i-1}) = f(x_{i-1})$ and $S(x_i) = f(x_i)$,

We have $f(x_{i-1}) = \frac{1}{6h_i}(x_i - x_{i-1})^3 M_{i-1} + c_1 x_{i-1} + c_2$

$$= \frac{1}{6} h_i^2 M_{i-1} + c_1 x_{i-1} + c_2 \qquad \dots (8)$$

And $f(x_i) = \frac{1}{6h_i}(x_i - x_{i-1})^3 M_i + c_1 x_i + c_2$

$$= \frac{1}{6} h_i^2 M_i + c_1 x_i + c_2 \qquad \dots (9)$$

Subtracting equation (8) from (9) we get $c_1(x_i - x_{i-1}) = (f(x_i) - f(x_{i-1}) - \frac{1}{6}(M_i - M_{i-1})h_i^2$

$$\Rightarrow \; c_1 = \frac{1}{h_i}(f(x_i) - f(x_{i-1}) - \frac{1}{6}(M_i - M_{i-1})h_i$$

Solving for c_2 we obtain $c_2 = \frac{1}{h_i}(x_i f(x_{i-1}) - x_{i-1} f(x_i)) - \frac{1}{6}(x_i M_{i-1} - x_{i-1} M_i)h_i$

Substituting the expression for c_1 and c_2 in (7), we obtain

$$S(x)=\frac{(x_i-x)^3}{6h_i}M_{i-1}+\frac{(x-x_{i-1})^3}{6h_i}M_i+\frac{x}{h_i}(f(x_i)-f(x_{i-1}))-\frac{x}{h_i}(M_i-M_{i-1})h_i+$$

$$\frac{1}{h_i}(x_if(x_{i-1})-x_{i-1}f(x_i))-\frac{1}{6}(x_iM_{i-1}-x_{i-1}M_i)h_i$$

$$S(x)=\frac{1}{6h_i}\left[(x_i-x)\{(x_i-x)^2-h_i^2\}\right]M_{i-1}+\frac{1}{6h_i}\left[(x-x_{i-1})\{(x-x_{i-1})^2-h_i^2\}\right]M_i+$$

$$\frac{1}{h_i}(x_i-x)f(x_{i-1})+\frac{1}{h_i}(x-x_{i-1})f(x_i),\quad \text{Where } x_{i-1}\le x\le x_i \qquad \ldots (10)$$

Setting, i= i+1, we get

$$S'(x)=-\frac{(x_{i+1}-x)^2}{2h_{i+1}}M_i+\frac{(x-x_i)^2}{2h_{i+1}}M_{i+1}-\frac{1}{6}(M_{i+1}-M_i)h_{i+1}+\frac{f(x_{i+1})-f(x_i)}{h_{i+1}} \qquad \ldots(11)$$

$$\text{where } x_i\le x\le x_{i+1}$$

Now, we require that the derivatives $S'(x)$ be continues at $x=x_i\pm\varepsilon$ as $\varepsilon\to 0$.

Letting $S'(x_i-\varepsilon)=S'(x_i+\varepsilon)$ as $\varepsilon\to 0$, we get

$$\frac{h_i}{6}M_{i-1}+\frac{h_i}{3}M_i+\frac{1}{h_i}(f(x_i)-f(x_{i-1}))=-\frac{h_{i+1}}{3}M_i-\frac{h_{i+1}}{6}M_{i+1}+\frac{1}{h_{i+1}}(f(x_{i+1})-f(x_i))$$

This may be written as

$$\frac{h_i}{6}M_{i-1}+\frac{h_i+h_{i+1}}{3}M_i+\frac{h_{i+1}}{6}M_{i+1}=\frac{1}{h_{i+1}}(f(x_{i+1})-f(x_i))-\frac{1}{h_i}(f(x_i)-f(x_{i-1})) \quad \ldots (12)$$

$$i=1, 2, 3,\ldots, n-1$$

this gives a system of n-1 linear equations in n+1 unknowns i.e. $M_0, M_1, M_2, \ldots, M_n$

The two additional conditions may be taken in one of the following forms.

i, $M_0=M_n=0$ (natural spline)

ii, $M_0=M_n, M_1=M_{n+1}, f(x_0)=f(x_n), f(x_1)=f(x_{n+1}), h_1=h_{n+1}$

(A spline satisfying this condition is called a periodic spline).

iii, For a non-periodic spline , we use the conditions $S'(a) = f'(a) = f'(x_0)$ and $S'(b) = f'(b) = f'(x_n)$,Using (11), we get

$$2M_0 + M_1 = \frac{6}{h_1}(\frac{f(x_1) - f(x_0)}{h_1} - f'(x_0)) \text{ and}$$

$$M_{n-1} + 2M_n = \frac{6}{h_n}(f'(x_n) - \frac{f(x_n) - f(x_{n-1})}{h_n}) \qquad \dots (13)$$

For equispaced knots $h_i = h$ for all i, equation (10) and (12) reduce to

$$S(x) = \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\right] + \frac{1}{h}(x_i - x)(f(x_{i-1}) - \frac{h^2}{6}M_{i-1}) +$$

$$\frac{1}{h}(x - x_{i-1})(f(x_i) - \frac{h^2}{6}M_i) \text{ and} \qquad \dots (14)$$

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(f(x_{i+1}) - 2f(x_i) + f(x_{i-1})) \qquad \dots (15)$$

This method gives the values of $M_i = f''(x_i)$ i= 1,2,3,…,n-1. The solution obtained for M_i, i= 1,2,3,…,n-1 are substituted in (10) or (14) to find the cubic spline interpolation. It may be noted that in this method also, we need to solve only an (n-1)x(n-1) tridiagonal system of equations for finding M_i

Example: Obtain the cubic spline approximation for the function defined on example 1.

Solution: Since the points are equispaced with h=1, we obtain from (15) $M_{i-1} + 4M_i + M_{i+1} = 6(f(x_{i+1}) - 2f(x_i) + f(x_{i-1}))$ i=1,2

Therefore $M_0 + 4M_1 + M_2 = 6(f(x_2) - 2f(x_1) + f(x_0))$

$M_1 + 4M_2 + M_3 = 6(f(x_3) - 2f(x_2) + f(x_1))$ using M_0=0 and M_3=0 and the given functional values, we get $4M_1 + M_2 = 6(33 - 4 + 1) = 180$

$$M_1 + 4M_2 = 6(244 - 66 + 2) = 1080$$

Which gives $M_1 = -24$ and $M_2 = 276$. Thus using (14) the cubic spline in the corresponding intervals are obtained as follows:

On $[0,1]$: $S(x) = \frac{1}{6}\left[(1-x)^3 M_0 + (x-0)^3 M_1\right] + (1-x)(f(x_0) - \frac{1}{6}M_0) + (x-0)(f(x_1) - \frac{1}{6}M_1)$

$$=\frac{1}{6}\left[x^3(-24)\right]+(1-x)+x\left[2-\frac{1}{6}(-24)\right]=-4x^3+5x+1$$

On $[1,2]$: $S(x)=\frac{1}{6}\left[(2-x)^3M_1+(x-1)^3M_2\right]+(2-x)(f(x_1)-\frac{1}{6}M_1)+(x-1)(f(x_2)-\frac{1}{6}M_2)$

$$=\frac{1}{6}\left[(2-x)^3(-24)+(x-1)^3(276)\right]+(2-x)(2-\frac{1}{6}(-24))+(x-1)(33-\frac{1}{6}(276))$$

$$=50x^3-162x^2+167x-53$$

On $[2,3]$: $S(x)=\frac{1}{6}\left[(3-x)^3M_2+(x-2)^3M_3\right]+(3-x)(f(x_2)-\frac{1}{6}M_2)+(x-2)(f(x_3)-\frac{1}{6}M_3)$

$$=\frac{1}{6}\left[(27-27x+9x^2-x^3)(276)\right]+(3-x)(33-\frac{1}{6}(276))+(x-2)(244)$$

$$=-46x^3+414x^2-985x+715$$

Hence, the interpolating cubic spline function is

$$S(x)=\begin{cases} S_1(x) & =-4x^3+5x+1 & ,0\le x\le 1\\ S_2(x) & =50x^3-162x^2+167x-53 & ,1\le x\le 2\\ S_3(x) & =-46x^3+414x^2-985x+715 & ,2\le x\le 3\end{cases}$$ which is the same as

the spline function we get on example 1

3.2.2.3 Method 3:

Cubic Spline Interpolation using Hermite Interpolation

In this method, we use piecewise cubic Hermite interpolation and determine $f'(x_i)$ for i= 1,2,3,…,n-1 or $f'(x_i)$ for i= 0,1,2,…,n.

Consider the piecewise cubic Hermite interpolation polynomial given by

$$P_3(x) = \sum_{i=0}^{n} N_i(x) f(x_i) + \sum H_i(x) f'(x_i) \qquad \ldots(16)$$

In order to satisfy the condition of continuity of the second derivative of S(x) at x=x_i, , we differentiate the equation (16) twice at $x_i \pm \varepsilon, \varepsilon>0$, i= 1,2,3,…,n and we get

$$S''(x_i+\varepsilon) = N_i''(x_i+\varepsilon) f(x_i) + H_i''(x_i+\varepsilon) f'(x_i) +$$

$$N_{i+1}''(x_i+\varepsilon) f(x_{i+1}) + H_{i+1}''(x_i+\varepsilon) f'(x_{i+1}) \qquad \ldots(17)$$

$$S''(x_i-\varepsilon) = N_{i-1}''(x_i-\varepsilon) f(x_{i-1}) + H_{i-1}''(x_i-\varepsilon) f'(x_{i-1}) +$$

$$N_{i+1}''(x_i-\varepsilon) f(x_i) + H_i''(x_i-\varepsilon) f'(x_i) \qquad \ldots(18)$$

Denote $h_i = x_i - x_{i-1}$ and $h_{i+1} = x_{i+1} - x_i$

From

$$N_i(x) = \begin{cases} 0, & x \le x_{i-1} \\ \dfrac{(x-x_{i-1})^2}{(x_i-x_{i-1})^2}\left[1+\dfrac{2(x-x_i)}{x_{i-1}-x_i}\right] & ,x_{i-1} \le x \le x_i \\ \dfrac{(x-x_{i+1})^2}{(x_{i+1}-x_i)^2}\left[1+\dfrac{2(x-x_i)}{x_{i+1}-x_i}\right] & ,x_i \le x \le x_{i+1} \\ 0 & ,x \ge x_{i+1} \end{cases} \qquad \ldots(19)$$

And $$H_i(x) = \begin{cases} 0, & x \le x_{i-1} \\ \dfrac{(x-x_{i-1})^2}{(x_i-x_{i-1})^2}(x-x_i) & ,x_{i-1} \le x \le x_i \\ \dfrac{(x-x_{i+1})^2}{(x_{i+1}-x_i)^2}(x-x_i) & ,x_i \le x \le x_{i+1} \\ 0 & ,x \ge x_{i+1} \end{cases} \qquad \ldots(20)$$

We get

$$N_i^{//}(x_i+\varepsilon)=\frac{1}{h_{i+1}^2}\left[2+\frac{8(x_i+\varepsilon-x_{i+1})+4(x_i+\varepsilon-x_i)}{h_{i+1}}\right]=\frac{1}{h_{i+1}^2}\left[2+\frac{12\varepsilon-8h_{i+1}}{h_{i+1}}\right]$$

And $H_i^{//}(x_i+\varepsilon)=\frac{1}{h_{i+1}^2}\left[4(x_i+\varepsilon-x_{i+1})+2(x_i+\varepsilon-x_i)\right]=\frac{1}{h_{i+1}^2}\left[6\varepsilon-4h_{i+1}\right]$

Setting i=i+1 in (19) and (20) we get $N_{i+}^{//}(x_i+\varepsilon)=\frac{1}{h_{i+1}^2}\left[2+\frac{8(x_i+\varepsilon-x_i)+4(x_i+\varepsilon-x_{i+1})}{-h_{i+1}}\right]=$

$$\frac{1}{h_{i+1}^2}\left[2-\frac{12\varepsilon-4h_{i+1}}{h_{i+1}}\right]$$

$$H_{i+1}^{//}(x_i+\varepsilon)=\frac{1}{h_{i+1}^2}\left[4(x_i+\varepsilon-x_i)+2(x_i+\varepsilon-x_{i+1})\right]=\frac{1}{h_{i+1}^2}\left[6\varepsilon-2h_{i+1}\right]$$

Taking the limit as $\varepsilon\to 0$, we get from (17)

$\lim_{\varepsilon\to 0} S^{//}(x_i+\varepsilon)=$

$$\frac{1}{h_{i+1}^2}\lim_{\varepsilon\to 0}\left[\left\{\frac{12\varepsilon}{h_{i+1}}-6\right\}f(x_i)\{6\varepsilon-4h_{i+1}\}f^{/}(x_i)+\left\{\frac{-12\varepsilon}{h_{i+1}}+6\right\}f(x_{i+1})+\{6\varepsilon-2h_{i+1}\}f^{/}(x_{i+1})\right]$$

$$=\frac{6}{h_{i+1}^2}(f(x_{i+1})-f(x_i))-\frac{4}{h_{i+1}}f^{/}(x_i)-\frac{2}{h_{i+1}}f^{/}(x_{i+1}) \qquad \dots (21)$$

Similarly we have $N_i^{//}(x_i-\varepsilon)=\frac{1}{h_i^2}\left[2+\frac{8(x_i-\varepsilon-x_{i-1})+4(x_i-\varepsilon-x_i)}{-h_i}\right]$

$$=\frac{1}{h_i^2}\left[2+\frac{6\varepsilon-4h_i}{h_i}\right]$$

$$H_i^{//}(x_i-\varepsilon)=\frac{1}{h_i^2}\left[4(x_i-\varepsilon-x_{i-1})+2(x_i-\varepsilon-x_i)\right] = \frac{1}{h_i^2}\left[-6\varepsilon+4h_i\right]$$

Setting i=i-1 in (19) and (20) we get

$$N_{i-1}^{//}(x_i-\varepsilon)=\frac{1}{h_i^2}\left[2+\frac{8(x_i-\varepsilon-x_i)+4(x_i-\varepsilon-x_{i-1})}{h_i}\right]=\frac{1}{h_i^2}\left[2-\frac{12\varepsilon-4h_i}{h_i}\right]$$

$$H_{i-1}^{//}(x_i-\varepsilon)=\frac{1}{h_i^2}\left[4(x_i-\varepsilon-x_i)+2(x_i-\varepsilon-x_{i-1})\right]=\frac{1}{h_i^2}\left[-6\varepsilon+2h_i\right]$$

Taking a limit as $\varepsilon\to 0$, we get from (18)

$$\lim_{\varepsilon\to 0} S''(x_i-\varepsilon)=\frac{6}{h_i^2}(f(x_{i-1})-f(x_i))+\frac{2}{h_i}f'(x_{i-1})+\frac{4}{h_i}f'(x_i) \quad \dots (22)$$

Equating the right hand sides of (21) and (22), we obtain

$$\frac{1}{h_i}f'(x_{i-1})+(\frac{2}{h_i}+\frac{2}{h_{i+1}})f'(x_i)+\frac{1}{h_{i+1}}f'(x_{i+1})=-3(\frac{f(x_{i-1})-f(x_i)}{h_i^2})+3(\frac{f(x_{i+1})-f(x_i)}{h_{i+1}^2})$$

$$i=1,2,3,\dots,n-1 \quad \dots(23)$$

There are n-1 equations in n+1 unknowns $f'(x_0)$, $f'(x_1)$,…, $f'(x_n)$.

If $f''(x_0)$ and $f''(x_n)$ are prescribed, then from (21) and (22) for i=0 and i=n respectively,

we obtain $$\frac{2}{h_i}f'(x_0)+\frac{1}{h_1}f'(x_1)=3(\frac{f(x_1)-f(x_0)}{h_1^2})-\frac{1}{2}f''(x_0) \quad \dots (24)$$

$$\frac{1}{h_n}f'(x_{n-1})+\frac{2}{h_n}f'(x_n)=3(\frac{f(x_n)-f(x_{n-1})}{h_n^2})+\frac{1}{2}f''(x_n) \quad \dots (25)$$

The derivatives $f'(x_i)$, i=1,2,3,…,n are determined by solving the equations (23),(24) and (25). If $f'(x_0)$ and $f'(x_n)$ are specified, then we determine $f'(x_1)$, $f'(x_2)$,…, $f'(x_{n-1})$ from the equation (23).

For equispaced points, equation (23) to (25) become, respectively,

$$f'(x_{i-1})+4f'(x_i)+f'(x_{i+1})=3(\frac{f(x_{i+1})-f(x_{i-1})}{h}), \quad i=1,2,3,\dots,n-1 \quad \dots(26)$$

$$2f'(x_0)+f'(x_1)=3(\frac{f(x_1)-f(x_0)}{h})-\frac{h}{2}f''(x_0) \quad \dots (27)$$

$$f'(x_{n-1})+2f'(x_n)=3(\frac{f(x_n)-f(x_{n-1})}{h})+\frac{h}{2}f''(x_n)\text{ , where } x_i-x_{i-1}=h \quad \dots (28)$$

The above procedure gives the values of $f'(x_i)$ i= 0,1,2,…,n.

Substituting $f(x_i)$ and $f'(x_i)$, i=0,1,2,…,n in the piecewise cubic Hermite polynomial

$$P_{i,3}(x)=A_{i-1}(x)f(x_{i-1})+A_i(x)f(x_i)+B_{i-1}(x)f'(x_{i-1})+B_i(x)f'(x_i) \quad \dots (29)$$

We obtain the required cubic spline interpolation. It may be noted that we need to solve only an (n-1)x(n-1)or an (n+1)x(n+1) tridiagonal system of equations

for the solution of $f'(x_i)$.

Example: Obtain the cubic spline approximation for the function defined on example 1;

Solution: Since M(0)=0 and M(3)=0 are prescribed, we use equation (26), (27) and (28).

Denote $f'(x_i)=m_i$ then we have the following equations

$2m_0 + m_1 = 3(f(x_1) - f(x_0)) = 3(2-1) = 3$

$m_0 + 4m_1 + m_2 = 3(f(x_2) - f(x_0)) = 3(33-1) = 96$

$m_1 + 4m_2 + m_3 = 3(f(x_3) - f(x_1)) = 3(244-2) = 726$

$m_2 + 2m_3 = 3(f(x_3) - f(x_2)) = 3(244-33) = 633$

In matrix form, we write

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 96 \\ 726 \\ 633 \end{pmatrix}$$

The solution of the system is $m_0 = 5, m_1 = -7, m_2 = 119, m_3 = 257$

Let us now write the piecewise cubic Hermite interpolation polynomial

on $[0,1]$, we have from equation (29)

$P_3(x) = A_0(x)f(x_0) + A_1(x)f(x_1) + B_0(x)f'(x_0) + B_1(x)f'(x_1)$

$A_0(x) = (x-1)^2\left[1 + \frac{2(-x)}{-1}\right] = (1+2x)(x-1)^2$, $A\ (x) = x^2\left[1 + \frac{2(x-1)}{-1}\right] = (3-2x)x^2$

$B_0(x) = (x-1)^2 x$ and $B_1(x) = (x-1)x^2$, therefore

$P_3(x) = (1+2x)(x^2 - 2x + 1) + (3-2x)x^2(2) + x(x-1)^2(5) + (x-1)x^2(-7)$

$\quad = 4x^3 + 5x + 1$

Which the same polynomial is as obtained in example 1 or 2.

Since the approximating polynomials are unique, we get the same polynomials as obtained earlier in the intervals $[1,2]$ *and* $[2,3]$

Hence f(2.5)=121.25

4. Four Types of Splines

4.1. Clamped Spline

There exist a unique cubic spline with the derivative boundary $S'(x_0) = m_0$ and $S'(x_n) = m_n$ conditions. Where $m_0 = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$ and $m_n = \dfrac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$

The clamped spline involves slope at the ends. This spline can be visualized as the curve obtained when "a flexible elastic rod" is forced to pass through the points and the rod is clamped at each end with a fixed slope. This spline would be useful to a draftsman for drawing a "smooth" curve through several points.

4.2. Natural Spline

This first spline includes the stipulation that the second derivative be equal to zero at the end points.

$$M_1 = M_n = 0 \qquad \dots (29)$$

This results in the spline extending as a line outside the end points. This matrix for determining the M_1 through M_n values can be adapted accordingly.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & . & . & . & 0 & 0 & 0 & 0 \\ . & . & . & . & & & & . & . & . & . \\ . & . & . & . & & & & . & . & . & .. \\ . & . & . & . & & & & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & . & . & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ M_2 \\ M_3 \\ . \\ . \\ . \\ M_{n-2} \\ M_{n-1} \\ 0 \end{pmatrix} = \frac{6}{h^2} \begin{pmatrix} y_1 - 2y_2 + y_3 \\ y_2 - 2y_3 + y_4 \\ y_3 - 2y_4 + y_5 \\ . \\ . \\ . \\ y_{n-4} - 2y_{n-3} + y_{n-2} \\ y_{n-3} - 2y_{n-2} + y_{n-1} \\ y_{n-2} - 2y_{n-1} + y_n \end{pmatrix} \qquad \dots (30)$$

For reasons of convenience, the first and last columns of this matrix can be eliminated, as they correspond to the M_1 and M_n values, which are both 0.

$$\begin{pmatrix} 0 & 1 & 4 & 1 & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & & & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & & & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} M_2 \\ M_3 \\ \cdot \\ \cdot \\ \cdot \\ M_{n-2} \\ M_{n-1} \end{pmatrix} = \frac{6}{h^2} \begin{pmatrix} y_1 - 2y_2 + y_3 \\ y_2 - 2y_3 + y_4 \\ y_3 - 2y_4 + y_5 \\ \cdot \\ \cdot \\ \cdot \\ y_{n-4} - 2y_{n-3} + y_{n-2} \\ y_{n-3} - 2y_{n-2} + y_{n-1} \\ y_{n-2} - 2y_{n-1} + y_n \end{pmatrix} \quad \ldots (31)$$

This results in an (n-2) by (n-2) matrix, which will determine the remaining solutions for M_2 through M_{n-1}. The spline is now unique.

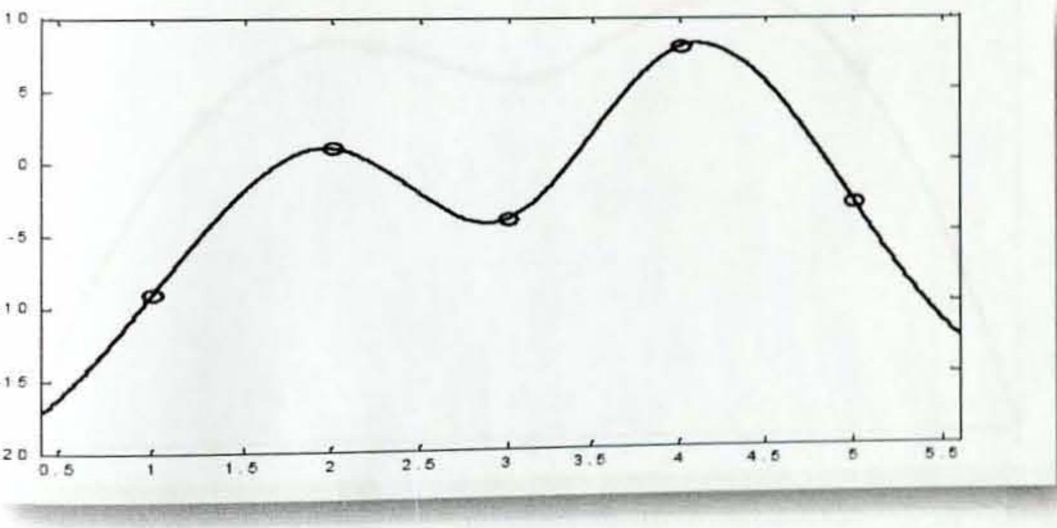


Figure 3: Natural interpolating curve

4. 3. Parabolic Run out Spline

The parabolic spline imposes the condition that the second derivative at the end points, M_1 and M_n, be equal to M_2 and M_{n-1} respectively.

$$M_1 = M_2 \text{ and } M_{n-1} = M_n \quad \ldots (32)$$

The result of this condition is the curve becomes a parabolic curve at the end point. This type of cubic spline is useful for periodic and exponential data.

The matrix equation for this type of spline is

$$\begin{pmatrix} 5 & 1 & 0 & . & . & . & 0 & 0 & 0 \\ 1 & 4 & 1 & . & . & . & 0 & 0 & 0 \\ 0 & 1 & 4 & & & & 0 & 0 & 0 \\ . & . & . & & & & . & . & .. \\ . & . & . & & & & . & . & . \\ 0 & 0 & 0 & . & . & . & 4 & 1 & 0 \\ 0 & 0 & 0 & . & . & . & 1 & 4 & 1 \\ 0 & 0 & 0 & & & & 0 & 1 & 5 \end{pmatrix} \begin{pmatrix} M_2 \\ M_3 \\ . \\ . \\ . \\ M_{n-2} \\ M_{n-1} \end{pmatrix} = \frac{6}{h^2} \begin{pmatrix} y_1 - 2y_2 + y_3 \\ y_2 - 2y_3 + y_4 \\ y_3 - 2y_4 + y_5 \\ . \\ . \\ . \\ y_{n-4} - 2y_{n-3} + y_{n-2} \\ y_{n-3} - 2y_{n-2} + y_{n-1} \\ y_{n-2} - 2y_{n-1} + y_n \end{pmatrix} \quad \dots (33)$$

We can now determine the values for M_2 through M_{n-1}, with the values for M_1 and M_n already determined.

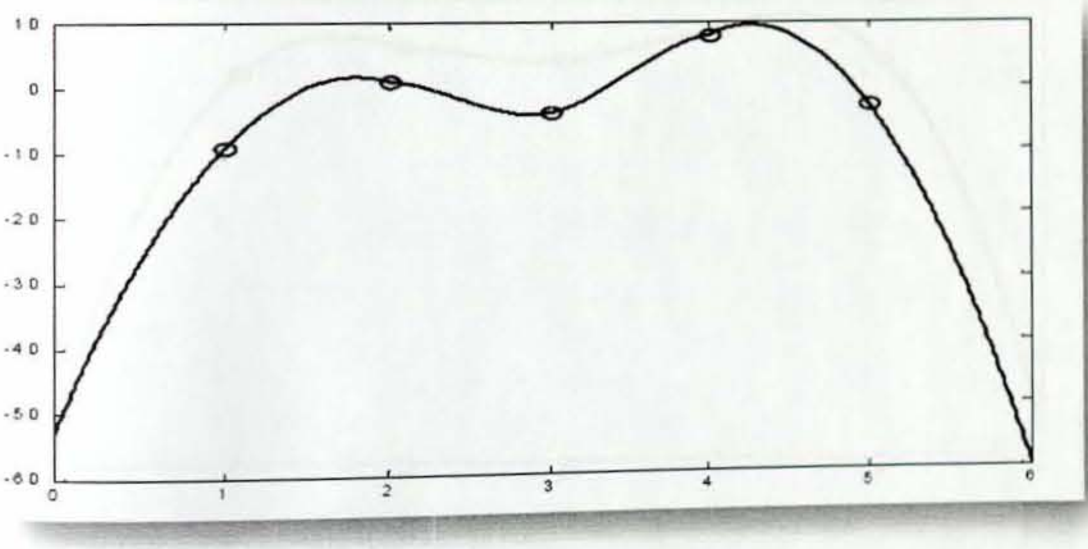


Figure 4: Parabolic Run out curve

4.4. Cubic Run out Spline

This last type of spline has the most extreme end-point behavior. It assigns $M_1 = 2M_2 - M_3$ and $M_n = 2M_{n-1} - M_{n-2}$. This causes the curve to degrade to a single cubic curve over the last two intervals, rather than two separate functions.

The matrix equation for this type is

$$\begin{pmatrix} 6 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 1 & 4 & 1 & \cdot & \cdot & \cdot & & 0 & 0 & 0 \\ 0 & 1 & 4 & & & & & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & & & & & \cdot & \cdot & \cdot\cdot \\ \cdot & \cdot & \cdot & & & & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & & 4 & 1 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & & 1 & 4 & 1 \\ 0 & 0 & 0 & & & & & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} M_2 \\ M_3 \\ \cdot \\ \cdot \\ \cdot \\ M_{n-2} \\ M_{n-1} \end{pmatrix} = \frac{6}{h^2} \begin{pmatrix} y_1 - 2y_2 + y_3 \\ y_2 - 2y_3 + y_4 \\ y_3 - 2y_4 + y_5 \\ \cdot \\ \cdot \\ \cdot \\ y_{n-4} - 2y_{n-3} + y_{n-2} \\ y_{n-3} - 2y_{n-2} + y_{n-1} \\ y_{n-2} - 2y_{n-1} + y_n \end{pmatrix} \quad \ldots (34)$$

The graph of cubic run out spline is shown as

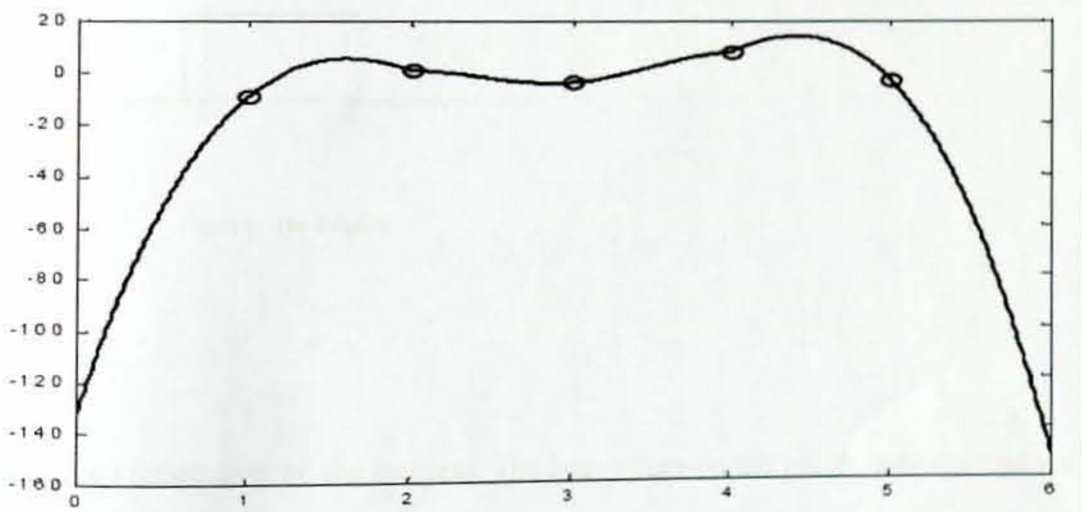


Figure 5: Cubic Run out curve

5. B-splines

This section is devoted to a system of spline functions from which all other spline functions can be obtained by forming linear combinations. These splines provide basis for certain knots are known, the B-splines are easily generated by recurrence relations. The B-splines are distinguished by their elegant theory and their model behavior in numerical calculations.

We begin with a system of knots, on the real line. For practical purposes, only a finite set of knots is ever needed, but for the theoretical development it is much easier to suppose that the knots from an infinite set extending to $+\infty$ on the right and to $-\infty$ on the left. $\ldots < x_{-2} < x_{-1} < x_0 < x_1 <_2 < \ldots$ This knot sequence is assumed to be fixed throughout this section, and all of our splines will be based on it.

5.1. B-splines of Degree 0

The B-splines of degree 0 are denoted by B_i^0 and have the appearance shown in figure 1.

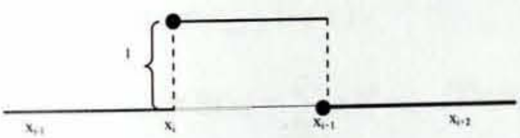


Figure 6: The B-Spline

The index i range over all the integers. The heavy dots on the graph indicate that we define $B_i^0(x_i) = 1$ and $B_i^0(x_{i+1}) = 0$. The formal definition is

$$B_i^0(x) = \begin{cases} 1 & ,if \quad x_i \le x < x_{i+1} \\ 0 & , \quad otherwise \end{cases}$$

These B-splines form an infinite sequence $\{B_i^0 : i \in I\}$.(Here I denote the set of all integers. Positive, negative, or 0)

We observe some of their salient properties.

1, the support of B_i^0, defined as the set of x where $B_i^0(x) \neq 0$, is the interval $[x_i, x_{i+1})$.

2, $B_i^0(x) \ge 0$ for all i and all x.

3, B_i^0 is continuous from the right on the entire real line.

4, $\sum_{i=-\infty}^{\infty} B_i^0(x) = 1$ for all x.

The last equation is verified by selecting any $x \in R$ and then determining the knot interval in which x lies, say $x_j \le x \le x_{j+1}$; then $\sum_{i=-\infty}^{\infty} B_i^0(x) = B_j^0(x) = 1$.

A final remark about the splines B_i^0 is that they do form a basis for all splines of degree 0 based on the given knot sequence, provided that we standardize such splines to be continuous from the right. To verify this assertion, suppose that S is such a spline function. Then it is piecewise constant and is defined by a set of rules of the form $S(x) = c_i$ if $x_i \le x \le x_{i+1}$, $(i \in I)$.

It is apparent that S(x) = $\sum_{i=-\infty}^{\infty} c_i B_i^0(x)$. (Thus, we have a basis. Each vector in the space has a unique representation as an infinite series $\sum_{i=-\infty}^{\infty} c_i B_i^0(x)$).

The function B_i^0 is the starting point for a recursive definition of all the higher degree B-splines.

The basic recurrence relation is $B_i^k(x) = (\frac{x - x_i}{x_{i+k} - x_i}) B_i^{k-1}(x) + (\frac{x_{i+k+1} - x}{x_{i+k+1} - x_{i+1}}) B_{i+1}^{k-1}(x), k \ge 1$ ...(1)

All the properties of the higher-order B-splines will follow from this recursive definition.

By introducing some special linear functions. $V_i^k(x) = \frac{x - x_i}{x_{i+k} - x_i}$ ... (2)

We can write the recurrence relation in the following more elegant form:

$B_i^k = V_i^k B_i^{k-1} + (1 - V_{i+1}^k) B_{i+1}^k$ ... (3)

Since B_i^0 is a piecewise polynomial of degree 0, and since V_i^k is linear, B_i^1 is a piecewise polynomial of degree≤ 1. The same reasoning shows that, in general, B_i^k will be a piecewise polynomial of degree$\le k$.

5.2. B-splines of Degree 1

With the aid of Equation (1), we can give an explicit formula for $B_i^1(x)$ as follows:

$$B_i^1(x) = \left(\frac{x - x_i}{x_{i+1} - x_i}\right)B_i^0(x) + \left(\frac{x_{i+2} - x}{x_{i+2} - x_{i+1}}\right)B_{i+1}^0(x)$$

$$= \begin{cases} 0 & , x > x_i \; or \; x \geq x_{i+2} \\ \dfrac{x - x_i}{x_{i+1} - x_i} & , x_i \leq x \leq x_{i+1} \\ \dfrac{x_{i+2} - x}{x_{i+2} - x_{i+1}} & , x_{i+1} \leq x < x_{i+2} \end{cases}$$

The graph of $B_i^1(x)$ is shown on figure 7:

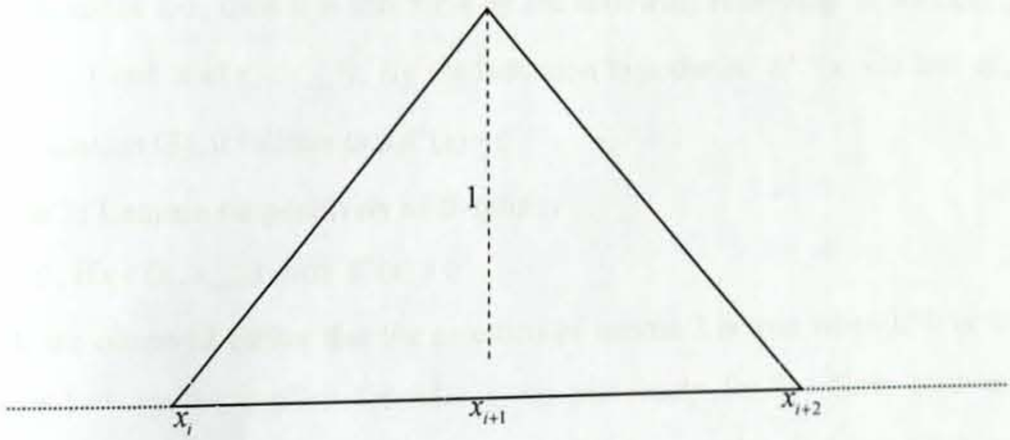


Figure 7: The B-spline B_i^1

Again, some observations can be made about the function B_i^1:

1, The support of B_i^1 is (x_i, x_{i+2})

2, $B_i^1 \geq 0$ for all i all x

3, B_i^1 is continuous and differentiable at every point except x_i, x_{i+1} and x_{i+2}.

4, $\sum_{i=-\infty}^{\infty} B_i^1(x) = 1$ for all x

To verify the last equation, take any $x \in R$. Since x_j converges to $+\infty$ when i increases, and it converges to $-\infty$ when i decreases, we can find an index j such that $x_j \leq x \leq x_{j+1}$ .Then $B_i^1(x) = 0$ for all i with the possible exception of i=j or i=j-1. Hence for this x,

$$\sum_{i=-\infty}^{\infty} B_i^1(x) = B_{j-1}^1(x) + B_j^1(x) = \frac{x_{j+1} - x}{x_{j+1} - x_j} + \frac{x - x_j}{x_{j+1} - x_j} = 1$$

Properties of B-splines

Now, in a sequence of Lammas, we shall develop the important properties of the family B_i^k $(i \in I, k \in N \text{U} 0)$.

Lemma 1: Lemma on support of B-splines

If $k \geq 1$ and $x \notin (x_i, x_{i+k+1})$, then $B_i^k(x) = 0$.

Proof: We have already observed that this is true for k=1, but not for k=0. If it is true for a certain index k-1, then it is true for k by the following reasoning: If $x \notin (x_i, x_{i+k+1})$ then $x \notin (x_i, x_{i+k})$ and $x \notin (x_{i+1}, x_{i+k+1})$. By the induction hypothesis, $B_i^{k-1}(x) = 0$ and $B_{i+1}^{k-1}(x) = 0$. From Equation (3), it follows that $B_i^k(x) = 0$.

Lemma 2: Lemma on positively of B-splines

Let $k \geq 0$, if $x \in (x_i, x_{i+k+1})$ then $B_i^k(x) > 0$

Proof: we observed earlier that the assertion of lemma 2 is true when k=0 or k=1. (It is true for k=1 by the explicit formulas given previously for B_i^1.) Now assume that the assertion is true for an index k-1, with $k \geq 2$. This assertion and lemma 1 imply that $B_i^{k-1}(x) \geq 0$ for all x and for all i. Let $x_i < x < x_{i+k+1}$. Then the linear factors on the right side of Equation (1) are positive. By the induction hypothesis, $B_i^{k-1}(x) > 0$ in (x_i, x_{i+k}), and $B_{i+1}^{k-1}(x) > 0$ in (x_{i+1}, x_{i+k+1}). Since $k \geq 2$, these two intervals overlap, and by Equation (1) we see that $B_i^k(x) > 0$

Since we expect to use the B-splines B_i^k as a basis for all splines of degree k, we shall be interested in linear combinations of the form $\sum_{i=-\infty}^{\infty} c_i B_i^k(x)$.

Lemma 3: Lemma on Recurrence Relation of B-splines

For all $k \geq 0$, we have $\sum_{i=-\infty}^{\infty} c_i B_i^k = \sum_{i=-\infty}^{\infty} \left[c_i V_i^k + c_{i-1}(1 - V_i^k) \right] B_i^{k-1}$

Proof: We use Equation (3) and elementary series manipulations as follows:

$$\sum_{i=-\infty}^{\infty} c_i B_i^k = \sum_{i=-\infty}^{\infty}\left[c_i V_i^k B_i^{k-1} + c_{i-1}(1-V_i^k)B_{i+1}^{k-1}\right]$$

$$= \sum_{i=-\infty}^{\infty}\left[c_i V_i^k B_i^{k-1}\right] + \sum_{i=-\infty}^{\infty}\left[c_{i-1}(1-V_i^k)B_{i+1}^{k-1}\right]$$

The graph of B-Splines of Degree 0, 1 and 2

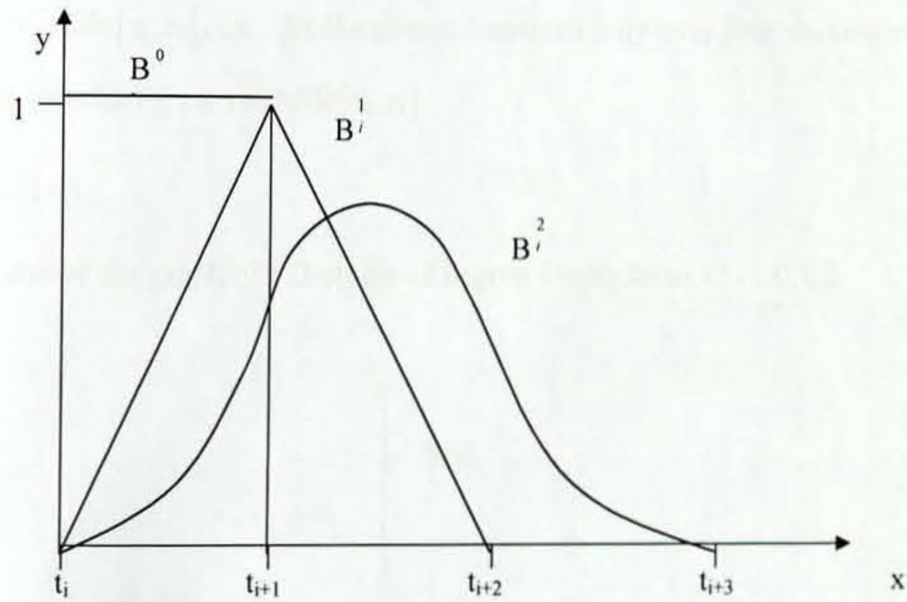


Figure 8: B-splines of degree 0,1and 2

5.3. Cubic B- spline (B-spline of degree 3)

The B-splines can be of any other applications, B-splines of degree 2 or 3 are generally found to be sufficient. The cubic B-spline resembles the ordinary cubic spline (a separate cubic is derived for each interval). A cubic B-spline (or B-spline of order four),denoted by $B_i^3(x)$, is a cubic spline with knots $x_{i-4}, x_{i-3}, x_{i-2}, x_{i-1}$ and x_i which is zero everywhere except in the range $x_{i-4} < x < x_i$ .In such a case, $B_i^3(x)$ is said to have a support $[x_{i-4}, x_i]$. It may be noted that a B-spline need not necessarily pass through any or all of the data points.

Let the set of data points be $(x_i, f(x_i))$ i= 1,2,3,…,m and $a \le x \le b$. Let S(x) be the cubic spline with knots $x_1, x_2, x_3, ..., x_p$ where $a < x_1 < x_2 < x_3 < ... < x_p < b$ Then the cubic spline $B_i^3(x)$ with knots x_1, x_2, x_3, x_4 and x_5 must satisfy the following properties.

1, On each interval, the B-spline must be a polynomial of degree 3 or less

2, The B-spline and its first-two derivatives must be continuous over the entire curve.

3, $B_i^3(x) > 0$ inside $[x_1, x_5]$, i.e., the B-spline is non-zero only over four successive intervals,

4, $B_i^3(x)$ is identically zero outside $[x_1, x_5]$.

5, $\sum_{i=-\infty}^{\infty} B_i^3(x) = 1$

Figure 9: shows the graph of a B-spline of degree 3 with knots -2,-1,0,1,2

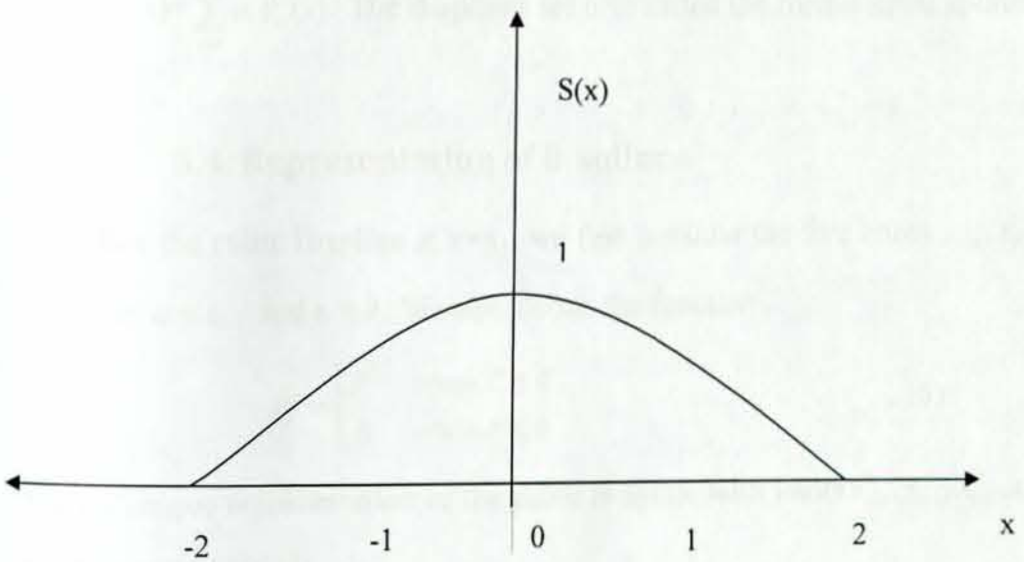


Figure 9: a B-spline of degree 3.

In figure 3, S(x) has the following properties

I, S(-2)=S(2)=0 and S(0)=1

II, S′ (-2)=S′(2)=0 and …(4)

III, $S''(-2)=S''(2)=0$

Suppose now we have p knots, i.e., $x_1,x_2,x_3,...,x_p$. To compute the cubic B-splines at x_1 and x_p, we require eight additional knots. To obtain the full set of B-splines, we then introduce eight additional knots, i.e., $x_{-3},x_{-2},x_{-1},x_{0,}x_{p+1},x_{p+2},x_{p+3}$ and x_{p+4}. These are chosen such that

$$\left.\begin{array}{l} x_{-3}<x_{-2}<x_{-1}<x_0=a \\ b=x_{p+1}<x_{p+2}<x_{p+3}<x_{p+4} \end{array}\right\} \qquad \dots (5)$$

Where $[a,b]$ is the given range such that $a<x_1$ and $x_p<b$.

We have now p+4 cubic B-splines in the range $a\le x\le b$ and then the cubic spline S(x) can be represented as a linear combination of the (p+4) cubic B-splines in the unique form S(x)= $\sum_{i=1}^{p+4}\alpha_i B_i^3(x)$. The B-splines are also called the fundamental splines.

5.4. Representation of B-splines

To define the cubic B-spline at x=x$_i$, we first consider the five knots $x_{i-4},x_{i-3},x_{i-2},x_{i-1}$ and x_i where $a<x_{i-4}$ and $x_i<b$. We also define the function

$$P_+^3=\begin{cases} P^3 & when, P\ge 0 \\ 0 & when, P\le 0 \end{cases} \qquad \dots (6)$$

Then a unique representation of the cubic B-spline with knots $x_{i-4},x_{i-3},x_{i-2},x_{i-1},x_i$ is given by (Goreville [1968])

$$S(x)=B_i^3(x)=\sum_{j=0}^{3}\alpha_j x^j+\sum_{m=i-4}^{i}\beta_m(x-x_m)_+^3 \qquad \dots (7)$$

Unfortunately, the representation of the cubic spline is given by (7) is computationally inefficient because of loss of accuracy through cancellation. Another representation of the B-spline, a traditional one, is through divided differences. The divided difference of fourth order of the function $(x_p-x)_+^3$ with respect to the knots $x_{i-4},x_{i-3},x_{i-2},x_{i-1}$ and x_i as arguments is denoted by [$x_{i-4},x_{i-3},x_{i-2},x_{i-1},x_i$]. We then have

$$B_i^3(x) = [x_{i-4}, x_{i-3}, x_{i-2}, x_{i-1}, x_i]$$

$$= \frac{(x_{i-4} - x)_+^3}{(x_{i-4} - x_{i-3})(x_{i-4} - x_{i-2})(x_{i-4} - x_{i-1})(x_{i-4} - x_i)} +$$

$$\frac{(x_{i-3} - x)_+^3}{(x_{i-3} - x_{i-4})(x_{i-3} - x_{i-2})(x_{i-3} - x_{i-1})(x_{i-3} - x_i)} + \quad \ldots (8)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots +$$

$$\frac{(x_i - x)_+^3}{(x_i - x_{i-4})(x_i - x_{i-3})(x_i - x_{i-2})(x_i - x_{i-1})}$$

Setting $\Pi_{4,i}(x) = (x - x_{i-4})(x - x_{i-3})(x - x_{i-2})(x - x_{i-1})(x - x_i)$ ... (9)

Equation (8) can be expressed in the more compact form

$$B_i^3(x) = \sum_{m=i-4}^{i} \frac{(x_m - x)_+^3}{\Pi_{4,i}'(x_m)} \quad \ldots (10)$$

More generally, a B-spline of order n (degree n-1) is defined by

$$B_i^{n-1}(x) = [x_{i-n}, x_{i-n+1}, \ldots, x_i] = \sum_{m=i-n}^{i} \frac{(x_m - x)_+^{n-1}}{\Pi_{n,i}'(x_m)} \quad \ldots (11)$$

Where $\Pi_{n,i}(x) = (x - x_{i-n})(x - x_{i-n+1}), \ldots, (x - x_i)$ ... (12)

Recalling that $[x_{i-4}, x_{i-3}, x_{i-2}, x_{i-1}, x_i] = \frac{[x_{i-3}, x_{i-2}, x_{i-1}, x_i] - [x_{i-4}, x_{i-3}, x_{i-2}, x_{i-1}]}{x_i - x_{i-4}}$ ...(13)

We obtain the relation $B_i^3(x) = \frac{B_i^2(x) - B_{i-1}^2(x)}{x_i - x_{i-4}}$ ... (14)

Which is a recurrence relation. Similarly, for B-splines $B_i^n(x)$ of order n+1 (degree n),

we obtain the relation $B_i^n(x) = \frac{B_i^{n-1}(x) - B_{i-1}^{n-1}(x)}{x_i - x_{i-n+1}}$ ...(15)

For a recursive computation of the B-splines $B_i^n(x)$

Example: Using the relation (7), determine the cubic B-spline S(x) with support [0, 4] on the knots 0,1,2,3,4. Show further that such a representation will be unique if S(1) is specified.

Solution: Since S(x) is a cubic B-spline over [0,4],we have

$S(0) = S'(0) = S''(0) = S(4) = S'(4) = S''(4) = 0$ ... i

Also $S'(2) = 0$, ...ii

$S(1) = S(3)$ by symmetry ...iii

On [0,1], Let S(x) be given by S(x)=$\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$ ...iv

Since S(0)=0, we obtain $\alpha_0 = 0$ also $S'(0) = S''(0) = 0$,gives $\alpha_1 = \alpha_2 = 0$

Accordingly, (iv) becomes $S(x) = \alpha_3 x^3$ ...(v)

Which is cubic B-spline on [0, 1] satisfying the conditions $S'(0) = 0 = S''(0)$

For definiteness, Let S(1)=S(3)= β_0 ...(vi)

Then (v) gives $\beta_0 = S(1) = \alpha_3$, so that $\alpha_3 = \beta_0$ ...(vii)

And (v) is written as $S(x) = \beta_0 x^3$ ...(viii)

Let the cubic B-spline on [0, 2] be written as $S(x) = \beta_0 x^3 + \beta_1 (x-1)_+^3$ ...(ix)

Where β_1 is to be determined. Using the condition $S'(2) = 0$, we obtain $0 = 12\beta_0 + 3\beta_1$ so that $\beta_1 = -4\beta_0$ ...(x)

Hence (ix) becomes $S(x) = \beta_0 x^3 - 4\beta_0 (x-1)_+^3$ ....(xi)

Which represents the cubic B-spline valid in the interval [0,2].

On the interval [0,3] Let the cubic B-spline be written as

$S(x) = \beta_0 x^3 - 4\beta_0 (x-1)_+^3 + \beta_1 (x-2)_+^3$ ...(xii)

Since S(3)= β_0 ,we obtain $\beta_0 = \beta_0(27) - 4\beta(8) + \beta_1$ so that $\beta_1 = 6\beta_0$

Then (xiii) becomes S(x)= $\beta_0[x^3 - 4(x-1)_+^3 + 6(x-2)_+^3]$ ...(xiii)

Finally, on the interval [0,4], Let the cubic B-spline be represented by

$S(x) = \beta_0[x^3 - 4(x-1)_+^3 + (x-2)_+^3] + \beta_2 (x-3)_+^3$ ... (xiv)

Since S(4)=0, we obtain 0= S(4)= $\beta_0[64 - 108 + 48] + \beta_2$ so that $\beta_2 = -4\beta_0$

Hence, the cubic B-spline with support [0,4] may be written as

$S(x) = \beta_0[x^3 - 4(x-1)_+^3 + 6(x-2)_+^3 - 4(x-3)_+^3]$ ...(xv)

If the value $\beta_0 = S(1)$ is specified, then the representation given by (xv) is unique. Further it is easily verified that $S'(4) = S''(4) = 0$

6. Comparison of Polynomial interpolation and Spline interpolation

Using Lagrange and cubic spline interpolation to interpolate the given data points we get the following results shown by the graphs.

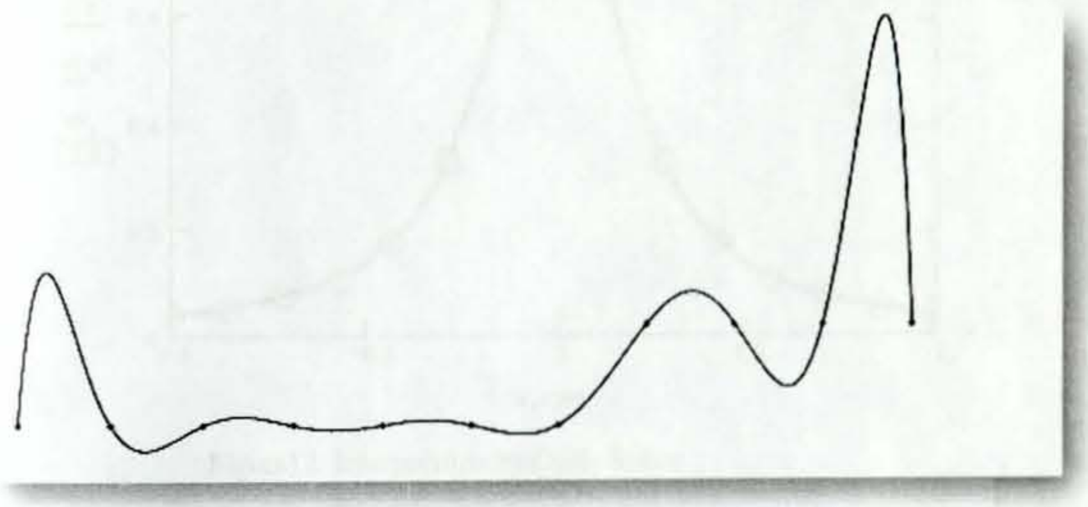


Figure10: General interpolating polynomial of the data points

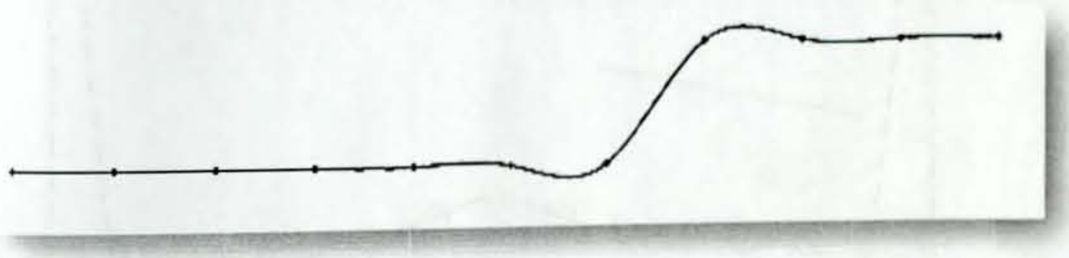


Figure11: Spline interpolation of the same data points

The spline curves were constructed by using a different cubic polynomial curve between each two data point. In other words, it is a piecewise cubic curve, made of pieces of different cubic curves glued together. Thus S(x) is so smooth that it has a second derivative everywhere and this derivative is continuous.

Example: Comparison on Spline and General Interpolation by using the Runge,s function $y(x)=\frac{1}{1+25x^2}$ by using 15 nodes in the interval [-1,1]

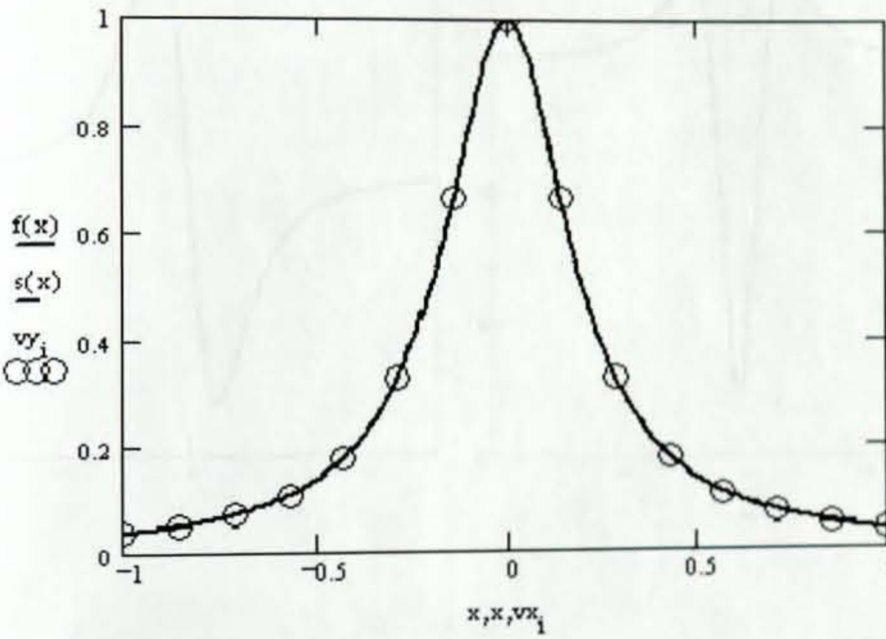


Figure12: Interpolation by Cubic Spline

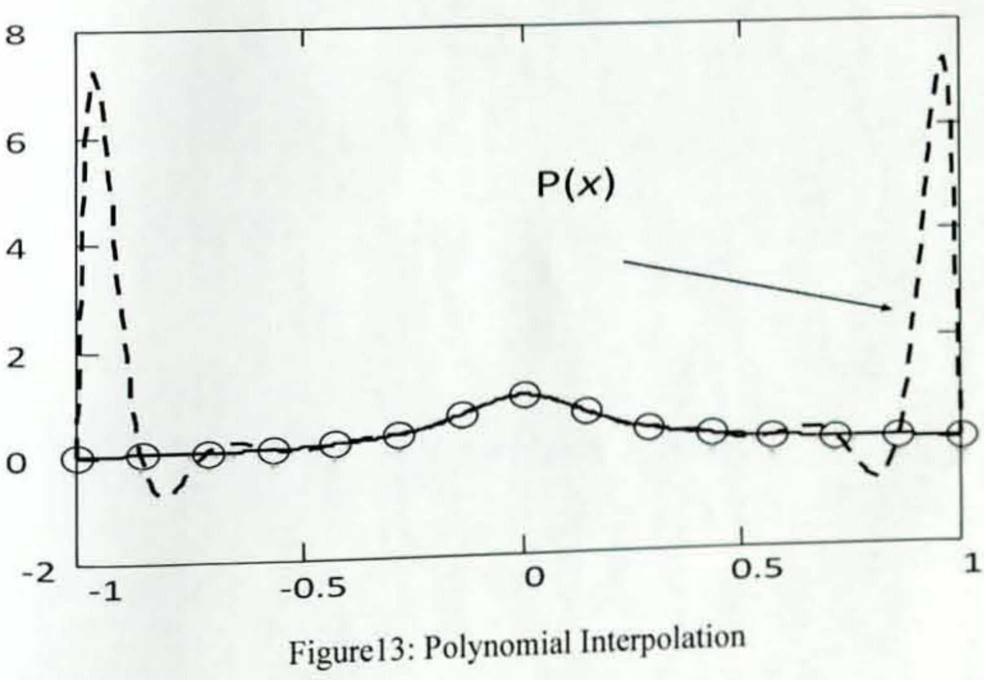
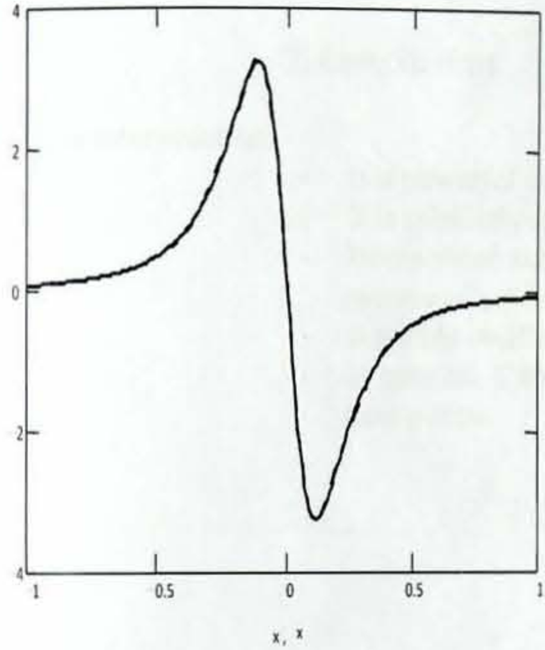
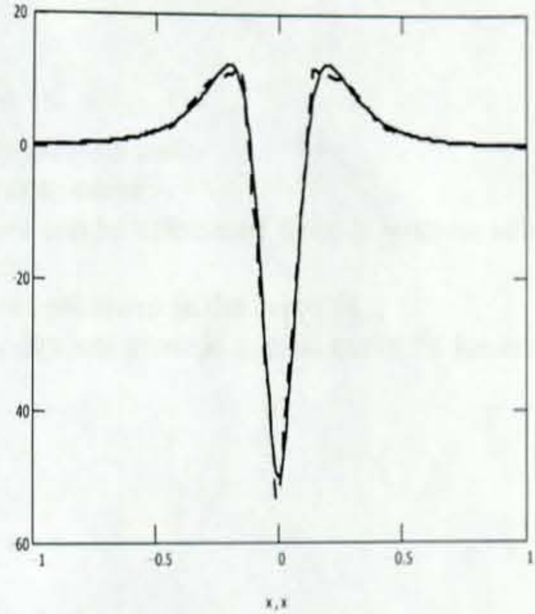


Figure13: Polynomial Interpolation

Derivatives of cubic spline:- the following graphs shows S'(x) and S''(x) respectively





7. Conclusion

Spline interpolation:-

- Is a powerful data analysis tool.
- It is relatively smooth curve.
- Interpolated values can be calculated directly without solving a system of equations.
- It avoids oscillation problems in the curve fit.
- In general, Cubic-Splines provide a good curve fit for arbitrary data points.

8. Error in the cubic spline and it's Derivatives

An estimation of error in the cubic spline and it's derivatives will be useful in practical applications.

The natural cubic spline yields a good approximation of a smooth function together with several derivatives, which is testified by the following Theorem:

Theorem: If $y \in c^2[a,b] a = x_0 < x_1 < ... < x_n = b$, and if $s(x)$ is the natural cubic spline for which $s(x_i) = y_i \; i = 0,1,2,...,n$ then $\max\limits_{x_0 \le x \le x_n} |y(x) - s(x)| \le \frac{Mh^2}{2}$ where $h = x_{i+1} - x_i \; i = 0,1,2,...,n$ and $M = \max |y''(x)|, \; x_0 \le x \le x_n$

It is clear that as the interval length h becomes smaller the better approximation the spline gives. This is in contrast to the known peculiarities of Lagrange interpolation. The error in the spline derivatives can be obtained by using the operator notation.

To find the error in the first derivatives, we start with the relation $m_{i-1} + 4m_i + m_{i+1} = \frac{3}{h}(y_{i+1} - y_{i-1})$ (from Hermite Cubic Spline)

That is $s'(x_{i-1}) + 4s'(x_i) + s'(x_{i+1}) = \frac{3}{h}(y_{i+1} - y_{i-1})$.

Using the operator notation, the above equation can be written as

$$(E^{-1} + 4 + E)s'(x_i) = \frac{3}{h}(E - E^{-1})y_i \qquad (1)$$

Since $E = e^{hD}$, where $D = d/dx$, then equation (1) becomes

$$(e^{-hD} + 4 + e^{hD})s'(x_i) = \frac{3}{h}(e^{hD} - e^{-hD})y_i \qquad (2)$$

Now, $e^{hD} = 1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + ...$ and $e^{-hD} = 1 - hD + \frac{h^2D^2}{2!} - \frac{h^3D^3}{3!} + ...$

Hence $e^{-hD} + e^{hD} = 2(1 + \frac{h^2D^2}{2!} + \frac{h^4D^4}{24} + \frac{h^6D^6}{720} + ...)$ and $e^{hD} - e^{-hD} = 2(hD + \frac{h^3D^3}{6} + \frac{h^5D^5}{120} + ...)$

Using the above expansions in (2), we obtain

$$[2(1 + \frac{h^2D^2}{2!} + \frac{h^4D^4}{24} + \frac{h^6D^6}{720} + ...) + 4]s'(x_i) = \frac{3}{h} 2(hD + \frac{h^3D^3}{6} + \frac{h^5D^5}{120} + ...)y_i =$$

$$6(D + \frac{h^2D^3}{6} + \frac{h^4D^5}{120} + ...)y_i$$

The above equation simplifies to

$s'(x) = (D - \frac{1}{180}h^4D^5 + ...)y_i$ Hence $s'(x_i) = y_i' - \frac{1}{180}h^4 y_i^v + O(h^6)$

In similar manner we can drive the relations

$$y''(x_i) = s''(x_i) + \frac{1}{12}h^2 y^{iv}(x_i) + O(h^4)$$

$$y'''(x_i) = \frac{1}{2}[s'''(x_i+) + s'''(x_i-)] + O(h^2)$$

$$y^{iv}(x_i) = \frac{1}{2}[s'''(x_i+) - s'''(x_i-)] + O(h^4)$$

9. References

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