On Polynomial Functions on a Variety

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Legesse Abebe

Addis Ababa, Ethiopia

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Declaration

I declare that this project has been composed by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship or any other similar title to me.

Legesse Abebe

Signature: __________

Permission

This is to certify that this project is compiled by Legesse Abebe in the Department of Mathematics, Addis Ababa University, under my supervision. I hereby also confirm that the project can be submitted for evaluation by examiners and eventual defense.

Dr. Tilahun Abebaw

Signature: __________

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Abstract

Let K be a field and given a polynomials in $K[x_1, x_2, . . . , x_n]$, we can define an affine varieties in K^n and ideals in a polynomials ring $K[x_1, x_2, . . . , x_n]$. This project considers the polynomial functions on a variety. The algebraic properties of polynomial functions on a variety yield many insights in to the geometric properties of the variety. The collection of polynomial functions from the variety V to the field K (or the coordinate ring $K[V]$) has the sum and product operations constructed using the sum and product operations in $K[x_1, x_2, . . . , x_n]$. The construction of the coordinate ring $K[V]$ is a special case of the quotient ring $K[x_1, x_2, ..., x_n]/I$. In particular, we relate the quotient ring $K[x_1, x_2, ..., x_n]/\boldsymbol{I}(V)$ to the ring $K[V]$ of polynomial functions on V. And the relation between two isomorphic varieties and two coordinate rings of an affine varieties are considered.

List of Mathematical Symbols

K : Field

$\mathbb{Q}$: The set of rational numbers

$\mathbb{R}$: The set of real numbers

$\mathbb{C}$: The set of complex numbers

Introduction

Algebraic geometry is the study of polynomial equations in one or more variables. We will discuss polynomials in n variables $x_1, x_2, \ldots, x_n$ with coefficients in an arbitrary field K. The solutions of a system of polynomial equations form a geometric object called a variety over the set of rational numbers, real numbers, complex numbers and so on; the corresponding algebraic object called an ideal in the polynomial ring $K[x_1, x_2, \ldots, x_n]$. There are close relationships between ideals and varieties, integral domain of coordinate rings and irreducible varieties which reveal the intimate link between algebra and geometry. Varieties can be classed in to affine varieties, quasi− affine varieties, projective varieties and quasi− projective varieties, but in this project, we will consider only affine varieties. After having defined affine varieties, then this project, focus on polynomial functions on a variety.

This project contains two parts. In the first part, we will introduce about polynomials and affine space, affine varieties, parameterizations of affine varieties and ideals. In the second part, we will study polynomial mappings between varieties, quotients of polynomial rings and coordinate ring of an affine variety.

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CHAPTER ONE

1. Preliminaries

In this chapter, to link algebra and geometry, we will study polynomials in n variables $x_1, x_2, . . . , x_n$ with coefficients in an arbitrary field K. The geometry we are interested in concerns affine varieties, which are curves and surfaces (and higher dimensional objects) defined by polynomial equations. To understand affine varieties, we will need to study ideals in the polynomial ring $K[x_1, x_2, . . . , x_n]$. The basic geometric objective of this chapter is an affine variety and the algebraic objective is ideal. We can also identify those ideals that consist of all polynomials which vanish on some variety V. Finally, we will discuss the relations between irreducible varieties and prime ideals.

Definition 1.1. A commutative ring consists of a set R and two binary operations ‘‘.’’ and ‘‘+’’ defined on R for which the following conditions are satisfied :

i) Associative. That is $(a.b).c = a.(b.c)$ and $(a+b)+c = a+(b+c)$ for all $a, b, c \in R$

ii) Commutative. That is $a.b = b.a$ and $a + b = b + a$ for all $a, b \in R$

iii) Distributive. That is $a.(b + c) = a.b + a.c$ for all $a, b, c \in R$

iv) Identities. There are $0, 1 \in R$ such that $a + 0 = 0 + a = a$ and $a.1 = 1.a = a$ for all $a \in R$

v) Additive inverses. That is given $a \in R$ there is $b \in R$ such that $a + b = 0$

A field is a commutative ring with identity $1 \neq 0$ in which every non-zero element has a multiplicative inverse. That is given $a \in R, a \neq 0$ there exists $c \in R$ such that $a.c = 1$.

Definition 1.2. Let R, S be commutative rings. Then a mapping $\phi : R \longrightarrow S$ is said to be **ring isomorphism** if

a) ϕ preserves sums : $\phi(r + r') = \phi(r) + \phi(r')$ for all $r, r' \in R$

b) ϕ preserves products : $\phi(r.r') = \phi(r).\phi(r')$ for all $r, r' \in R$

c) ϕ is one-to-one and onto.

1.1 Polynomials and Affine Space

We can now define polynomials in n variables $x_1 , x_2 , \ . \ . \ . \ , x_n$ with coefficients in arbitrary field K. We start by defining monomials.

Definition 1.1.1. A monomial in $x_1 , x_2 , \ . \ . \ . \ , x_n$ is a product of the form $x_1{}^{\alpha_1}.x_2{}^{\alpha_2}...x_n{}^{\alpha_n}$, where all of the exponents $\alpha_1 , \alpha_2 , \ . \ . \ . \ , \alpha_n$ are nonnegative integers. The total degree of this monomial is the sum of $\alpha_1 , \alpha_2 , \ . \ . \ . \ , \alpha_n$.

The notations for monomials are as follows:

Let α = $(\alpha_1 , \alpha_2 , \ . \ . \ . \ , \alpha_n)$ be an $n -$tuples of nonnegative integers.

Then we set $x^\alpha = x_1{}^{\alpha_1}.x_2{}^{\alpha_2}...x_n{}^{\alpha_n}$ and if α = (0 , 0 , . . . , 0), then x^α =1.

We also let |α| = $\alpha_1 + \alpha_2 + \ . \ . \ . + \alpha_n$, the total degree of the monomial x^α.

Definition 1.1.2. A polynomial f in $x_1 , x_2 , \ ... \ , x_n$ with coefficients in K is a finite linear combination (with coefficients in K) of monomials. We can write a polynomial f in the form f = $\sum_\alpha a_\alpha x^\alpha$, $a_\alpha \in K$, where the sum is over a finite number of n-tuples α = $(\alpha_1 , \alpha_2 , \ . \ . \ . \ , \alpha_n)$. The set of all polynomials in $x_1 , x_2 , \ . \ . \ . \ , x_n$ with coefficients in K are denoted by $K[x_1 , x_2 , \ . \ . \ . \ , x_n]$. Then under addition and multiplication $K[x_1 , x_2 , \ . \ . \ . \ , x_n]$ is clearly commutative ring, but not field, because $\frac{1}{x_1}$ is not polynomial.

Definition 1.1.3. Let f = $\sum_\alpha a_\alpha x^\alpha$ be polynomial in $K[x_1 , x_2 , \ . \ . \ . \ , x_n]$.

i) We call a_α the coefficient of the monomial x^α.
ii) If $a_\alpha \neq 0$, then we call $a_\alpha x^\alpha$ a term of f.
iii) The total degree of f, denoted deg(f) is the maximum |α| such that the coefficient a_α is non-zero.

Example 1. Take f = $x^2 + x^4y^2z^3 - \frac{2}{3} y^2z^2x$ a polynomial in $\mathbb{Q}[x, y, z]$, where $\mathbb{Q}$ is rational numbers. Then f has three terms and total degree 9.

Definition 1.1.4. Given a field K and a positive integer n, we define the $n-$ dimensional **affine space** over K to be the set

$K^n = \{(a_1, a_2, \ . \ . \ . \ , a_n) \ : \ a_1, a_2, \ . \ . \ . \ , a_n \in \ K\}$.

Example 2. Consider K is equal to real numbers. Then we call $K^1 = K$ the affine line and K^2 the affine plane.

Let us see how polynomials are related to affine space. The key idea is that a polynomial $\mathrm{f} = \sum_\alpha a_\alpha x^\alpha \ \in K[x_1, x_2, \ . \ . \ . \ , x_n]$ gives a function $\mathrm{f} : \ K^n \rightarrow K$ defined as : given $(a_1, a_2, \ . \ . \ . \ , a_n) \in \ K^n$, then $f(a_1, a_2, \ . \ . \ . \ , a_n) \in \ K$.

This dual nature of polynomials has some unexpected consequences.

Example 3. If $\mathrm{f} = 0$ has two potential meanings:

$a)$ f is the zero polynomial which means that all the coefficients a_α are zero .

$b)$ f is the zero function which means that $f(a_1, a_2, \ . \ . \ . \ , a_n) = 0$.

for all $(a_1, a_2, \ . \ . \ . \ , a_n) \in K^n$.

But the two statements are ***not*** equivalent. To verify these two statements are not equivalent. Let take $\mathbb{F}_2 = \{0, 1\}$, since field consists at least 0 and 1 implies $\mathbb{F}_2$ is a field. Now consider the polynomial $\mathrm{f} = x^2 - x = x(x-1) \ \in \mathbb{F}_2[x]$. Then f vanishes at 0 and 1. We have found a non- zero polynomial which gives the zero function on the affine space ${\mathbb{F}_2}^1$. However, as long as K is infinite there is no problem.

Proposition 1.1.1. Let K be an infinite field and let $f \in K[x_1, x_2, \ . \ . \ . \ , x_n]$. Then $\mathrm{f} = 0$ in $K[x_1, x_2, \ . \ . \ . \ , x_n]$ if and only if $\mathrm{f} : K^n \longrightarrow K$ is the zero function.

1.2 Affine Varieties

Definition 1.2.1. Let K be a field and let $f_1, f_2, . \ . \ . , f_s$ be a polynomials in $K[x_1, x_2, \ . \ . \ . \ , x_n]$. Then we set

$\mathbf{V}(f_1, f_2, . \ . \ . , f_s) = \{(a_1, a_2, . \ . \ . , a_n) \in \ K^n : f_i \ (a_1, a_2, . \ . \ . , a_n) = 0$ for $1 \leq \ i \ \leq s\}$.

We call $\mathbf{V}(f_1, f_2, . \ . \ . , f_s)$ the **affine variety** defined by $f_1, f_2, . \ . \ . , f_s$. Thus, an affine variety $\mathbf{V}(f_1, f_2, . \ . \ . , f_s) \subseteq K^n$ is the set of all solutions of the system of equations

$f_1(x_1, x_2, \ldots, x_n) = f_2(x_1, x_2, \ldots, x_n) = \ldots = f_s(x_1, x_2, \ldots, x_n) = 0.$

Note: we use the letters $V, W,$ etc. to denote affine varieties.

Examples

1. Prove that a single point $(a_1, a_2, \ldots, a_n) \in K^n$ is affine variety.

   Proof: Consider the set $V = \{(a_1, a_2, \ldots, a_n)$, then for V to be an affine variety, there must a collection of polynomials $f_1, f_2, \ldots, f_s \in K[x_1, x_2, \ldots, x_n]$ such that f_i vanishes at $(a_1, a_2, \ldots, a_n)$. Let $f_i = (x_i - a_i)$ so we found a set of polynomials that vanish at V for $i = 1,2,\ldots,n$.

   Hence, a single point is an affine variety.

2. $V = \mathbf{V}((x-1)^2 + (y-2)^2 - 4)$ is a variety on $\mathbb{R}^2$.

3. The graphs of polynomial functions are affine varieties.

   i.e. the graph of $y = f(x)$ is $\mathbf{V}(y - f(x))$.

4. A curve $\mathbb{R}^3$ is the twisted cubic which is the affine variety $\mathbf{V}(y - x^2, z - x^3)$.

Lemma 1.2.1. If $V, W \subseteq K^n$ are affine varieties, then so are $V \cap W$ and $V \cup W$.

Proof: Suppose that $V = \mathbf{V}(f_1, f_2, \ldots, f_s)$ and $W = \mathbf{V}(g_1, g_2, \ldots, g_t)$.

Then i) $V \cap W = \mathbf{V}(f_1, f_2, \ldots, f_s, g_1, g_2, \ldots, g_t)$.

ii) $V \cup W = \mathbf{V}(f_i g_j : 1 \leq i \leq s, 1 \leq j \leq t)$.

Therefore, $V \cap W$ and $V \cup W$ are affine varieties.

Example 5. Consider the union of the (x, y) plane and the $z-$axis in affine 3-space. Then, we have $\mathbf{V}(z) \cup \mathbf{V}(x, y) = \mathbf{V}(xz, yz)$.

Example 6. For the intersection case take

$\mathbf{V}(y - x^2) \cap \mathbf{V}(z - x^3) = \mathbf{V}(y - x^2, z - x^3)$.

1.3 Parameterizations of Affine Varieties

In this section, we will discuss the problem of describing the points of an affine variety **V**$(f_1, f_2, \ldots, f_s)$. This reduces to asking whether there is a way to ''write down'' the solutions of the system of polynomial equations $f_1 = f_2 = \ldots = f_s = 0$**.** When there are finitely many solutions, the goal is simply to list them all. But what do we do when infinitely many? This is question leads to the notation of parameterizing an affine variety**.**

Examples

1. Let the field be $\mathbb{R}$ and consider the system of equations

$$\begin{aligned} x+y+z&=1 \\ x+2y-z&=3 \end{aligned} \tag{1.1}$$

Geometrically equation (1.1) represents the line $\mathbb{R}^3$ which is the intersection of the planes $x + y + z = 1$ and $x + 2y - z = 3$. It follows that there are infinitely many solutions.

To describe the solutions, we use row operations on equation (1.1) to obtain the equation

$$\begin{aligned} x+3z&=-1 \\ y-2z&=2 \end{aligned}$$

Let $z = t$, where t is arbitrary, this implies that all solution of equation (1.1) are given by

$$\begin{aligned} x &= -1 - 3t \\ y &= 2 + 2t \\ z &= t \end{aligned} \tag{1.2}$$

as t varies over $\mathbb{R}$. We call t is a parameter and equation (1.2) is the parameterization of the solutions of equation (1.1).

2. Let us look the unit circle

$$x^2 + y^2 = 1 \tag{1.3}$$

A common way to parameterize the circle is using trigonometric functions

$$\begin{aligned} x &= \sin(t) \\ y &= \cos(t) \end{aligned}$$

In algebraic way the parameterization of this circle

$$x = \frac{1-t^2}{1+t^2}$$
$$y = \frac{2t}{1+t^2} \qquad (1.4)$$

To see this notice that each nonvertical line through $(-1\ ,0)$ will intersect the circle in a unique point $(x\ ,y)$

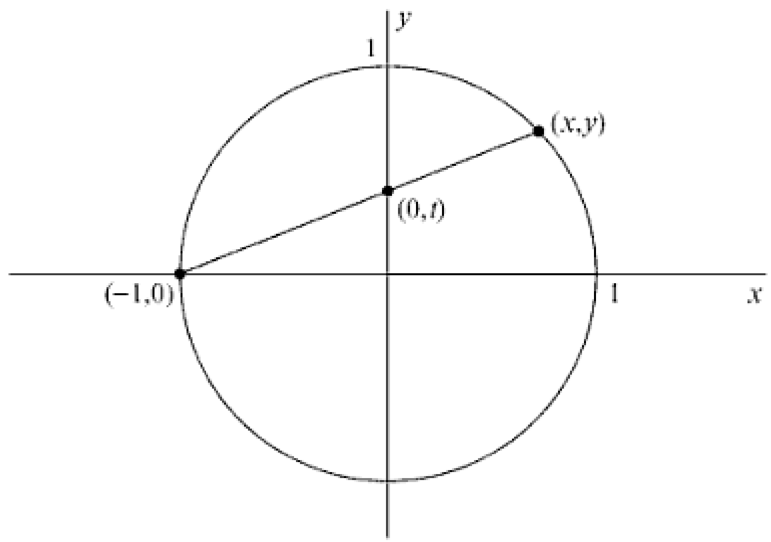

Figure 1

Each nonvertical line also meets the y-axis at $(0\ ,t)$ .This gives geometric parameterization of the circle give t draw the line connecting $(-1\ ,0)$ to $(0\ ,t)$ and let $(x\ ,y)$ be the point where the line meets $x^2 + y^2 = 1$.

To find the explicit formulas for x and y in terms of t. Consider the slope of the line in the above picture .We can compute the slope in two ways, using either the points $(-1\ ,0)$ and $(0\ ,t)$ or $(-1\ ,0)$ and (x, y). Then we have

$$\frac{t-0}{0-(-1)} = \frac{y-0}{x-(-1)} \text{ which implies } t = \frac{y}{x+1}$$

Thus, $y = t(x+1)$. If we substitute this in to $x^2 + y^2 = 1$, we get

$x^2 + t^2(x+1)^2 = 1$

This gives the quadratic equation

$$(1 + t^2)x^2 + 2t^2x + (t^2 - 1) = 0 \qquad (1.5)$$

This equation gives the x-coordinates of where the line meets the circle and it is quadratic, since there are two intersections. One of the points -1, so that $x + 1$ is a factor of (1.5). To find the other factor, we can rewrite as

$(x + 1)((1 + t^2)x - (1 - t^2)) = 0$ which implies $(1 + t^2)x = 1 - t^2$

Hence, $x = \frac{1-t^2}{1+t^2}$.

Furthermore, $y = t(x + 1)$ then by substituting $x = \frac{1-t^2}{1+t^2}$,

we can get $y = \frac{2t}{1+t^2}$.

Hence, equation (1.4) is the parameterization of $x^2 + y^2 = 1$.

But this parameterization does not describe the whole circle since $x = \frac{1-t^2}{1+t^2}$ can never equal to -1 , the point $(-1, 0)$ is not covered.

1.4 Ideals and Radical Ideals

Definition 1.4.1. A subset $I \subseteq K[x_1, x_2, \ldots, x_n]$ is an **ideal** if it satisfies

i) $0 \in I$.

ii) If $f, g \in I$ then $f + g \in I$.

iii) If $f \in I$ and $h \in K[x_1, x_2, \ldots, x_n]$, then $f h \in I$.

The goal of this section is introduce ideals relate to affine varieties.

Definition 1.4.2. Let $f_1, f_2, \ldots, f_s$ be a polynomials in $K[x_1, x_2, \ldots, x_n$. Then we set $< f_1, f_2, \ldots, f_s > = \{\sum_{i=1}^{s} h_i f_i \ : \ h_1, h_2, \ldots, h_s \in K[x_1, x_2, \ldots, x_n]$.

So, one can easily show that $< f_1, f_2, \ldots, f_s >$ is an ideal of $K[x_1, x_2, \ldots, x_n]$.

Proposition 1.4.1. If $f_1, f_2, \ldots, f_s$ and $g_1, g_2, \ldots, g_t$ are bases of the same ideal in $K[x_1, x_2, \ldots, x_n]$, that is, $< f_1, f_2, \ldots, f_s > = < g_1, g_2, \ldots, g_t >$, then $\mathbf{V}(f_1, f_2, \ldots, f_s) = \mathbf{V}(g_1, g_2, \ldots, g_t)$.

Proof: Suppose $< f_1, f_2, \ldots, f_s > = < g_1, g_2, \ldots, g_t >$.

We claim that $\mathbf{V}(f_1, f_2, \ldots, f_s) = \mathbf{V}(g_1, g_2, \ldots, g_t)$.

Let $(a_1, a_2, \ldots, a_n) \in \mathbf{V}(g_1, g_2, \ldots, g_t)$ which implies

$g_i(a_1, a_2, \ldots, a_n) = 0$ for all $1 \leq i \leq t$.

Since $f_1, f_2, \ldots, f_s \in < g_1, g_2, \ldots, g_t >$ by assumption, $f_i = \sum_{j=1}^{t} h_j\, g_j$ for $h_j \in K[x_1, x_2, \ldots, x_n]$ and $i = 1,2,\ldots,s$, then $f_i(a_1, a_2, \ldots, a_n) = 0$ for all $1 \leq i \leq s$.

Thus, $(a_1, a_2, \ldots, a_n) \in \mathbf{V}(f_1, f_2, \ldots, f_s)$.

Hence, $\mathbf{V}(g_1, g_2, \ldots, g_t) \subseteq \mathbf{V}(f_1, f_2, \ldots, f_s)$.

Similar, $\mathbf{V}(f_1, f_2, \ldots, f_s) \subseteq \mathbf{V}(g_1, g_2, \ldots, g_t)$.

Therefore, $\mathbf{V}(f_1, f_2, \ldots, f_s) = \mathbf{V}(g_1, g_2, \ldots, g_t)$.

Example 1. Consider the variety $\mathbf{V}(2x^2 + 3y^2 - 11, x^2 - y^2 - 3)$ in $\mathbb{Q}[x, y]$. Then prove that $\mathbf{V}(2x^2 + 3y^2 - 11, x^2 - y^2 - 3) = \mathbf{V}(x^2 - 4, y^2 - 1)$.

It is enough to show $< 2x^2 + 3y^2 - 11, x^2 - y^2 - 3 > = < x^2 - 4, y^2 - 1 >$ by the above Proposition.

Since $5(x^2 - 4) = (2x^2 + 3y^2 - 11) + 3(x^2 - y^2 - 3)$.

Then $x^2 - 4 = \frac{1}{5}(2x^2 + 3y^2 - 11) + \frac{3}{5}(x^2 - y^2 - 3)$.

Which implies $x^2 - 4 \in < 2x^2 + 3y^2 - 11, x^2 - y^2 - 3 >$.

Since $5(y^2 - 1) = (2x^2 + 3y^2 - 11) - 2(x^2 - y^2 - 3)$.

Then $y^2 - 1 = \frac{1}{5}(2x^2 + 3y^2 - 11) - \frac{2}{5}(x^2 - y^2 - 3)$.

Which implies $y^2 - 1 \in < 2x^2 + 3y^2 - 11, x^2 - y^2 - 3 >$.

Hence, $< x^2 - 4, y^2 - 1 > \subseteq < 2x^2 + 3y^2 - 11, x^2 - y^2 - 3 >$.

Since $2(x^2 - 4) + 3(y^2 - 1) = 2x^2 + 3y^2 - 11 \in < x^2 - 4, y^2 - 1 >$ and

$(x^2 - 4) - (y^2 - 1) = x^2 - y^2 - 3 \in < x^2 - 4, y^2 - 1 >$.

Thus, $< 2x^2 + 3y^2 - 11, x^2 - y^2 - 3 > \subseteq < x^2 - 4, y^2 - 1 >$.

Therefore, $\mathbf{V}(2x^2 + 3y^2 - 11, x^2 - y^2 - 3) = \mathbf{V}(x^2 - 4, y^2 - 1) = \{(\pm 2, \pm 1)\}$.

Definition 1.4.3. Let $V \subseteq K^n$ be an affine variety. Then we set

$\mathbf{I}(V) = \{f \in K[x_1, x_2, \dots, x_n] : f(a_1, a_2, \dots, a_n) = 0$ for all $(a_1, a_2, \dots, a_n) \in V\}$.

Lemma 1.4.2. If $V \subseteq K^n$ is an affine variety, then $\mathbf{I}(V) \subseteq K[x_1, x_2, \dots, x_n]$ is an ideal. We call $\mathbf{I}(V)$ is the **ideal** of V.

Proof: i) $0 \in \mathbf{I}(V)$ since the zero polynomial vanishes on all of the K^n, and so, in particularly it vanishes on V.

Let $f, g \in \mathbf{I}(V)$ and $h \in K[x_1, x_2, \dots, x_n]$.

ii) Suppose $(a_1, a_2, \dots, a_n)$ be an arbitrary points of V.

Then $(f+g)(a_1, a_2, \dots, a_n) = f(a_1, a_2, \dots, a_n) + g(a_1, a_2, \dots, a_n) = 0$

Hence, $f + g \in \mathbf{I}(V)$.

iii) $(f\,h)(a_1, a_2, \dots, a_n) = f(a_1, a_2, \dots, a_n)\, h(a_1, a_2, \dots, a_n) = 0$

Hence, $fh \in \mathbf{I}(V)$.

Therefore, $\mathbf{I}(V)$ is an ***ideal*** of V.

Example 2. For the ideal of variety .Consider the variety $\{(0, 0)\}$ consisting of the origin in K^2. Then its ideal $\mathbf{I}(\{(0, 0)\})$ consists of all polynomials that vanish at the origin.

We claim that $\mathbf{I}(\{(0, 0)\}) = \langle x, y \rangle$.

Proof: Since for any polynomial of the form $A(x, y)x + B(x, y)y$ vanishes at the origin.

Thus, $\langle x, y \rangle \subseteq \mathbf{I}(\{(0, 0)\})$.

For the other way suppose that $f = \sum_{i,j} a_{ij} x^i y^j$ vanishes at the origin.

Then $a_{00} = f(0, 0) = 0$.

Consequently $f = a_{00} + \sum_{i,j \neq 0} a_{ij} x^i y^j$

$$= 0 + \Big(\sum_{\substack{i,j \\ i>0}} a_{ij} x^{i-1} y^j\Big) x + \Big(\sum_{j>0} a_{0j} y^{j-1}\Big) y \in \langle x, y \rangle.$$

Thus, $\mathbf{I}(\{(0, 0)\}) \subseteq \langle x, y \rangle$.

Therefore, $\mathbf{I}(\{(0, 0)\}) = \langle x, y \rangle$.

Example 3. Consider the case when V is all of the K^n. Then $\mathbf{I}(K^n)$ consists of polynomials that vanish everywhere and hence, by Proposition 1.1.1, we have $\mathbf{I}(K^n) = \{0\}$ when K is infinite. For the finite case see Example three of section1.1.

Definition 1.4.4. Let $I \subseteq K[x_1 , x_2 , \ldots , x_n]$ be an ideal. We will denote by $\mathbf{V}(I)$ the set $\mathbf{V}(I) = \{(a_1 , a_2 , . . . , a_n) \in K^n : f(a_1 , a_2 , . . . , a_n) = 0$ for all $f \in I\}$

Example 4: Consider $I = < x^2 > \subseteq \mathbb{R}[x]$. Then its affine variety is $V = \mathbf{V}(I) = \{0\}$.

Definition 1.4.5. An ideal I is ***radical*** if $f^m \in I$ for some integer $m \geq 1$ implies $f \in I$.

Remark: $\mathbf{I}(V)$ is radical ideal.

Proof: If $f \in \mathbf{I}(V)$, then $f^m \in \mathbf{I}(V)$.

Let $f^m \in \mathbf{I}(V)$, then $f^m (a_1 , a_2 , . . . , a_n) = 0$ for all elements $(a_1 , a_2 , . . . , a_n)$ of V, but $f^m(a_1 , a_2 , . . . , a_n) = (f(a_1 , a_2 , . . . , a_n))^m$ which implies $f \in \mathbf{I}(V)$ since K^n is integral domain.

Hence, $\mathbf{I}(V)$ is radical ideal.

Definition 1.4.6. Let $I \subseteq K[x_1 , x_2 , \ldots , x_n]$ be an ideal .The **radical** of I, denoted by $\sqrt{I}$ is the set $\{ f : f^m \in$ I for some integer $m \geq 1\}$.

Note that we always have $I \subseteq \sqrt{I}$ since $f \in I$ implies $f^1 \in I$ and hence, $f \in \sqrt{I}$.

Proposition 1.4.3. Let V and W be affine varieties in K^n. Then

i) $V \subseteq W$ if and only if $\mathbf{I}(V) \supseteq \mathbf{I}(W)$.

ii) $V = W$ if and only if $\mathbf{I}(V) = \mathbf{I}(W)$.

Proof: i) Suppose that $V \subseteq W$.

We need to show $\mathbf{I}(V) \supseteq \mathbf{I}(W)$.

Since $V \subseteq W,$ then any polynomial vanishes on W also must be vanish on V and hence, $\mathbf{I}(V) \supseteq \mathbf{I}(W)$.

To the other case suppose $\mathbf{I}(V) \supseteq \mathbf{I}(W)$.

We need to show $V \subseteq W$.

We know that W is the variety defined by some polynomials

$f_1 , f_2 , . . . , f_s \in K[x_1 , x_2 , ... , x_n]$, then $f_1 , f_2 , . . . , f_s \in \mathbf{I}(W) \subseteq \mathbf{I}(V)$.

Thus, $f_1 , f_2 , . . . , f_s \in \mathbf{I}(V)$.

Hence, $f_1 , f_2 , . . . , f_s$ vanishes on V.

Since W consists of all common zeros of the $f_1 , f_2 , . . . , f_s$

Hence, $V \subseteq W$

ii) The prove of this followed from (i).

1.5 Irreducible Varieties and Prime Ideals

Definition 1.5.1. An affine variety $V \subseteq K^n$ is **irreducible** if V is written in the form $V = V_1 \cup V_2$, where V_1 and V_2 are affine varieties, then either $V = V_1$ or $V = V_2$.

Definition 1.5.2. An ideal $I \subseteq K[x_1 , x_2 , . . . , x_n]$ is **prime** if $f, g \in K[x_1 , x_2 , . . . , x_n]$ and $fg \in I$, then either $f \in I$ or $g \in I$.

Proposition 1.5.1. Let $V \subseteq K^n$ be affine variety. Then V is irreducible if and only if $\mathbf{I}(V)$ is prime ideal.

Proof: Suppose V be irreducible.

We need to show that $\mathbf{I}(V)$ is prime ideal.

Let $fg \in \mathbf{I}(V)$. Set $V_1 = V \cap \mathbf{V}(f)$ and $V_2 = V \cap \mathbf{V}(g)$. Since the intersection of two affine varieties is also variety then V_1 and V_2 are affine varieties.

We claim that i) $V = V_1 \cup V_2$.

ii) either $f \in \mathbf{I}(V)$ or $g \in \mathbf{I}(V)$.

To prove the first claim

Since $V_1 = V \cap \mathbf{V}(f) \subseteq V$ and $V_2 = V \cap \mathbf{V}(g) \subseteq$ V, then $V_1 \cup V_2 \subseteq V$.

Now $fg \in \mathbf{I}(V)$ and let $(a_1 , a_2 , . . . , a_n) = x \in V$, then $(fg)(x) = 0$.

Which implies $x \in \mathbf{V}(fg)$, so that $x \in V \cap \mathbf{V}(f\,g)$

Since $\mathbf{V}(f\,g) = \mathbf{V}(f) \cup \mathbf{V}(g)$ by Lemma 1.2.1, then $x \in V \cap (\mathbf{V}(f) \cup \mathbf{V}(g))$.

Which implies $x \in (V \cap \mathbf{V}(f)) \cup (V \cap \mathbf{V}(g))$, then $x \in V_1 \cup V_2$.

Thus, $V \subseteq V_1 \cup V_2$.

Hence, $V = V_1 \cup V_2$.

Since V is irreducible either $V = V_1$ or $V = V_2$.

With out loss of generality say $V = V_2 = V \cap \mathbf{V}(g)$ which implies g vanish on V.

Hence, $g \in \mathbf{I}(V)$.

Therefore, $\mathbf{I}(V)$ is prime ideal.

Assume that $\mathbf{I}(V)$ be prime ideal and let $V = V_1 \cup V_2$

Suppose that $V \neq V_1$. We claim that $\mathbf{I}(V) = \mathbf{I}(V_2)$

$\mathbf{I}(V) \subseteq \mathbf{I}(V_2)$ since $V_2 \subseteq V$.

For the opposite inclusion $\mathbf{I}(V) \subsetneq \mathbf{I}(V_1)$ since $V_1 \subsetneq V$.

Thus, we can pick $f \in \mathbf{I}(V_1) - \mathbf{I}(V)$. Now take any $g \in \mathbf{I}(V_2)$, since $V = V_1 \cup V_2$ which implies fg vanishes on V and hence, $fg \in \mathbf{I}(V)$.

But $\mathbf{I}(V)$ is prime and $f \notin \mathbf{I}(V)$. Thus, $g \in \mathbf{I}(V)$ and then $\mathbf{I}(V_2) \subseteq \mathbf{I}(V)$.

Hence, $\mathbf{I}(V) = \mathbf{I}(V_2)$

Then by the Proposition 1.4.3 of (ii) $V = V_2$

Therefore, V is irreducible.

CHAPTER TWO

2. Polynomial Functions on a Variety

2.1 Introduction

One of the unifying themes of modern mathematics is that in order to understand any class of mathematical objects, one should also study mapping between those objects and especially the mapping which preserve some property of interest. For instance, in linear algebra after studying vector spaces, we also studied the properties of linear mapping between vector spaces (mapping that preserve the vector space operations of sum and scalar product).

In this chapter, we will consider mapping between varieties, serve as an introduction to (and motivation for) the idea of a quotient ring and coordinate ring of an affine variety. Next to that given a variety V, we would like to relate the quotient ring $K[x_1, x_2, \dots, x_n]/\boldsymbol{I}(V)$ to the ring $K[V]$ of polynomial functions on V, a close relation between ideals in the quotient $K[x_1, x_2, \dots, x_n]/I$ and ideals in $K[x_1, x_2, \dots, x_n]$ and one reasonable answer to the question: what should it mean for two affine varieties to be " isomorphic"? Another important thing is that the relation between two isomorphic varieties and two coordinate ring of an affine varieties.

2.2 Polynomial Mappings

Definition 2.2.1. Let $V \subseteq K^m$ and $W \subseteq K^n$ be varieties.

A function $\phi : V \longrightarrow W$ is said to be a **polynomial mapping** if there exist polynomials $f_1, f_2, . . ., f_n \in K[x_1, x_2, \dots, x_m]$ such that

$\phi(a_1, a_2, . . ., a_m) = (f_1(a_1, a_2, . . ., a_m), . . . , f_n(a_1, a_2, . . ., a_m))$ for all $(a_1, a_2, . . ., a_m) \in V$. We say that the n-tuple of polynomials $(f_1, f_2, . . ., f_n) \in (K[x_1, x_2, \dots, x_m])^n$ represents ϕ.

The map ϕ is a polynomial mapping from $V \subseteq K^m$ to $W \subseteq K^n$ represented by $(f_1, f_2, \ldots, f_n)$ means that $(f_1(a_1, a_2, \ldots, a_m), \ldots, f_n(a_1, a_2, \ldots, a_m))$ must satisfy the defining equations of W for all $(a_1, a_2, \ldots, a_m) \in V$.

Example 1. Consider $V = \mathbf{V}(y - x^2, z - x^3) \subseteq K^3$ (twisted cubic) and $W = \mathbf{V}(y^3 - z^2) \subseteq K^2$.
Then the projection $\pi_1 : K^3 \longrightarrow K^2$ represented by (y, z) gives a polynomial mapping $\pi_1 : V \longrightarrow W$. Since every point in $\pi_1(V) = \{(x^2, x^3) : x \in K\}$ satisfies the defining equation of W.

In the case $W = K$, ϕ becomes a scalar polynomial function defined on the variety V. One reason to consider polynomial functions from V to K is that a general polynomial mapping $\phi : V \longrightarrow K^n$ is constructed by using any n polynomial functions $\phi_i : V \longrightarrow K$ as the components. Hence, if we understand functions $\phi : V \longrightarrow K$, we understand how to construct all mapping $\phi : V \longrightarrow K^n$ as well.

Example 2. The i^{th} coordinate map $\sigma_i : K^n \longrightarrow K$ defined by $\sigma_i(x_1, x_2, \ldots, x_n) = x_i$ is polynomial mapping for all $i = 1, \ldots, n$.

Example 3. Let $V = \mathbf{V}(y^2 - x^2 - x^3)$ be an affine variety in K^2. Then function $\phi : K \longrightarrow V$ defined by $\phi(t) = (t^2 - 1, t(t^2 - 1))$ is polynomial mapping.

To begin the study of polynomial functions for $V \subseteq K^m$, Definition 2.2.1 says that a mapping $\phi : V \longrightarrow K$ is a polynomial function if there exists a polynomial $f \in K[x_1, x_2, \ldots, x_m]$ representing ϕ. However, the cases where a representative is uniquely determined are very rare.

Example 4. Consider the variety $V = \mathbf{V}(y - x^2) \subseteq \mathbb{R}^2$. The polynomial $f = x^3 + y^3$ represents a polynomial function from V to $\mathbb{R}$. However, $g = x^3 + y^3 + y - x^2$, $h = x^3 + y^3 + x^4 y - x^6$ and $F = x^3 + y^3 + A(x, y)(y - x^2)$ for any $A(x, y)$ define the same polynomial function on V. Indeed, since $\mathbf{I}(V)$ is the set of polynomials which are zero at every point of V, adding any element of $\mathbf{I}(V)$ to f does not change the values of the polynomial at the points of V.

Proposition 2.2.1. Let $V \subseteq K^m$ be an affine variety. Then

i) f and $g \in K[x_1, x_2, \dots, x_m]$ represent the same polynomial function on V if and only if $f - g \in \mathbf{I}(V)$.

ii) $(f_1, f_2, \dots, f_n)$ and $(g_1, g_2, \dots, g_n)$ represent the same polynomial mapping from V to K^n if and only if $f_i - g_i \in \mathbf{I}(V)$ for each i $1 \leq i \leq n$.

Proof: Assume that f and g represent the same polynomial function on V. Then at every $p = (a_1, a_2, \dots, a_m) \in V$, $f(p) - g(p) = 0$.
Thus, $f - g \in \mathbf{I}(V)$ by definition.
Assume that $f - g = h \in \mathbf{I}(V)$. Then $p = (a_1, a_2, \dots, a_m) \in V$,
$f(p) - g(p) = h(p) = 0$.
Thus, $f(p) = g(p)$ for any $p \in V$.
Hence, f and g represent the same polynomial function on V.
ii) follows directly from (i).

There is a way of dealing in describing polynomial functions on a variety:
"In rough terms, we can "lump together" all the polynomials $f \in K[x_1, x_2, \dots, x_m]$ that represent the same function on a variety V and think of collection as a "new object" in its own right. We can then take the collection of polynomials as our description of the function on V."

Note: The collection of polynomial functions $\phi : V \longrightarrow K$ are denoted by $K[V]$

Since K is field, we can define a sum and product function for any pair of functions $\psi, \phi : V \longrightarrow K$ by adding and multiplying images.
For each $p \in$ V, $(\phi + \psi)(p) = \phi(p) + \psi(p)$,
$(\phi.\psi)(p) = \phi(p).\psi(p)$.
Furthermore, if we pick specific representatives $f, g \in K[x_1, x_2, \dots, x_m]$ for ϕ, ψ respectively, then, by definition, the polynomial sum $f + g$ represents $\phi + \psi$ and the

polynomial product $f \, . \, g$ represents $\phi.\psi$. It follows $that$ $\phi + \psi$ and $\phi.\psi$ are polynomial functions on V.

Thus, we see that $K[V]$ has sum and product operations constructed using the sum and product operations in $K[x_1 \, , \, x_2 \, , \ldots , x_m]$. All of the usual properties of sum and products of polynomials also hold for functions in $K[V]$. Thus, $K[V]$ is commutative ring.

Proposition 2.2.2. Let $V \subseteq K^n$ be an affine variety. The following statements are equivalent:

i) V is irreducible.

ii) $\mathbf{I}(V)$ is a prime ideal.

iii) $K[V]$ is an integral domain.

Proof: The proof of (i) $\Leftrightarrow$ (ii) is in Proposition 1.5.1.

(i) $\Rightarrow$ (iii) Suppose V be irreducible.

We need to show that $K[V]$ is an integral domain.

Suppose not! Then there must be polynomials f , $g \in K[x_1 , x_2 , \ldots , x_n]$ such that neither f nor g vanishes identically on V, but their product vanishes on V. Since $V \cap \mathbf{V}(f)$ and $V \cap \mathbf{V}(g)$ are varieties and their union is also variety, by Proposition 1.5.1, we have $V = (V \cap \mathbf{V}(f)) \cup (V \cap \mathbf{V}(g))$. This implies either $V = V \cap \mathbf{V}(f)$ or $V = V \cap \mathbf{V}(g)$ since V be irreducible. Say $V = V \cap \mathbf{V}(g)$, then g vanishes on V, this is contradiction to that g is not vanishes on V.

Therefore, $K[V]$ is an integral domain.

(iii) $\Rightarrow$ (i) Suppose that $K[V]$ be an integral domain.

We need to show that V is irreducible.

Suppose V is reducible. Then we can write $V = V_1 \cup V_2$, where V_1 and V_2 are nonempty proper subvarieties of V.

Let $f_1 \in K[x_1 , x_2 , \ldots , x_n]$ be polynomial that vanishes on V_1 , but not identically zero on V_2 and $f_2 \in K[x_1 , x_2 , \ldots , x_n]$ be polynomial that vanishes on V_2, but not identically zero on V_1. Such polynomials must exist since V_1 and V_2 are varieties and neither is contained in the other. Then neither f_1 nor f_2 represents the zero function in $K[V]$. However, the product $f_1 . f_2$ vanishes at all points of $V = V_1 \cup V_2$, and hence, the product function is zero function in $K[V]$. This is contradiction to our hypothesis that $K[V]$ was an integral domain.

Hence, V is irreducible.

Example 1: The affine variety $V = \mathbf{V}(x^2 - y^2)$ in $\mathbb{C}^2$ is the union of lines $x = y$ and $x = -y$ is reducible. Consider the two functions $f = x - y$ and $h = x + y$ on the variety V. Neither of these functions is identically zero at $(1, -1)$ and $(1,1)$, but their product vanishes on V. Hence, $\mathbb{C}[V]$ is not an integral domain.

2.3 Quotients of Polynomial Rings

The construction of $K[V]$ is a special case of what is called the quotient of $K[x_1, x_2, \dots, x_n]$ modulo an ideal I. Forming the quotient will indicate the sort "lump together" of polynomials when describing the elements $\phi \in K[V]$. The quotient construction is a fundamental tool in commutative algebra and algebraic geometry, if we pursue this project further, the acquaintance we make with quotient ring here will be valuable. To begin, we introduce some new terminology.

Definition 2.3.1. Let $I \subseteq K[x_1, x_2, \dots, x_n]$ be an ideal and $f,\ g \in K[x_1, x_2, \dots, x_n]$. We say f and g are **congruent modulo *I*** if $f - g \in I$. Denoted by $f \equiv g (\text{mod } I)$.

Example 1. If $I = \langle x^2 - y^2\ ,\ x + y^3 + 1 \rangle \subseteq K[x, y]$, then $f = x^4 - y^4 + x$ and $g = x + x^5 + x^4 y^3 + x^4$ are congruent modulo I . Since

$f - g = x^4 - y^4 + x - x + x^5 + x^4 y^3 + x^4$

$= x^4 - y^4 + x^5 + x^4 y^3 + x^4$

$= (x^2 + y^2)(x^2 - y^2) - x^4 (x + y^3 + 1) \in I$

Thus, $f - g \in I$.

Therefore, $f \equiv g (\text{mod } I)$.

Remark: Let $I \subseteq K[x_1, x_2, \ldots, x_n]$ be an ideal. Then congruent modulo I is an equivalence relation on $K[x_1, x_2, \ldots, x_n]$.

For any $f \in K[x_1, x_2, \ldots, x_n]$, the equivalence class of f is the set
$[f] = \{g \in K[x_1, x_2, \ldots, x_n] : g \equiv f(\text{mod } I)\}$
In the special case that $I = \mathbf{I}(V)$ is the ideal of the variety V, $f \equiv g(\text{mod } \mathbf{I}(V))$ if and only if f and g define the same polynomial function on V by Proposition 2.2.1.

Definition 2.3.2. The **quotient** of $K[x_1, x_2, \ldots, x_n]$ modulo I, written $K[x_1, x_2, \ldots, x_n]/I$ is the set of equivalence classes for congruence modulo I:

$K[x_1, x_2, \ldots, x_n]/I = \{[f]: f \in K[x_1, x_2, \ldots, x_n]\}$.

Example 2. Consider $K = \mathbb{R}$, $n = 1$ and $I = < x^2 - 2 >$. By division algorithm, every $f \in \mathbb{R}[x]$ can be written as $f = g(x^2 - 2) + r$, where $r = ax + b$ for some $a, b \in \mathbb{R}$. Then $f - r = g(x^2 - 2) \in I$ which implies $f \equiv r(\text{mod } I)$. Thus every element of $\mathbb{R}[x]$ belongs to one of the equivalence classes $[ax + b]$ and $\mathbb{R}[x]/I = \{[ax + b] : a, b \in \mathbb{R}\}$.

Because of $K[x_1, x_2, \ldots, x_n]$ is ring, given any two classes $[f], [g] \in K[x_1, x_2, \ldots, x_n]/I$, we can define the sum and product operations on classes by using the corresponding operations on elements of $K[x_1, x_2, \ldots, x_n]$. That is

$$[f] + [g] = [f + g] \qquad (\text{sum in } K[x_1, x_2, \ldots, x_n]) \qquad (2.1)$$

$$[f].[g] = [f.g] \qquad (\text{product in } K[x_1, x_2, \ldots, x_n]) \qquad (2.2)$$

Proposition 2.3.1. Let I be an ideal of $K[x_1, x_2, \ldots, x_n]$. Then $K[x_1, x_2, \ldots, x_n]/I$ is a commutative ring with the operations defined above.

Proof: Suppose $[f'] = [f]$ and $[g'] = [g]$.
From these, we have $f' \in [f]$ and $g' \in [g]$, then $f' = f + a$ and $g' = g + b$, where $a, b \in I$.
Now $f' + g' = (f + a) + (g + b)$
$= (f + g) + (a + b)$.
Since I is an ideal, we have $a + b \in I$

Thus, $f' + g' \equiv (f + g) (\text{mod } I)$.

Hence, $[f' + g'] = [f + g]$. In other words the sum is well-defined.

For the second case using similar manner,

$$f' . g' = (f + a) . (g + b)$$
$$= f . g + a g + f b + a . b$$

Since I is an ideal and $a, b \in I$, we have $a g + f b + a . b \in I$.

Thus, $f' . g' \equiv (f . g) (\text{ mod } I)$.

Hence, $[f' . g'] = [f . g]$.

Therefore, the two operations are well-defined.

Since the sum and product in $K[x_1, x_2, ..., x_n] / I$ are defined in terms of the corresponding operations in $K[x_1, x_2, ..., x_n]$, then $K[x_1, x_2, ..., x_n]/I$ is a commutative ring.

Example 3. Consider the sum and product operations in $\mathbb{R}[x]/< x^2 - 2 >$.

From Example 2 the classes $[ax + b]$ where $a, b \in \mathbb{R}$ form a complete list of elements of $\mathbb{R}[x]/< x^2 - 2 >$.

The sum operation defined by $[ax + b] + [cx + d] = [(a + c)x + (b + d)]$.

The product operation is also given $[ax + b] . [cx + d] = [ac\, x^2 + adx + bcx + bd]$

$$= [acx^2 + (ad + bc)x + bd]$$

By dividing the quadratic polynomial $acx^2 + (ad + bc)x + bd$ with $x^2 - 2$ and using the remainder as our representative of the class of the product, then

$[ax + b] . [cx + d] = [(ad + bc)x + (bd + 2ac)]$

Theorem 2.3.2. Let $V \subseteq K^n$ be an affine variety. Then $K[V] \cong K[x_1, x_2, ..., x_n] / \boldsymbol{I}(V)$.

Proof: Let $\pi : K[x_1, x_2, ..., x_n] / \boldsymbol{I}(V) \longrightarrow K[V]$ be the mapping defined by $\pi([f]) = \phi$, where ϕ is the polynomial function represented by f.

i) Let ϕ and ψ are the polynomial functions represented by f and g respectively.

Suppose $[f] = [g]$. Then $f - g \in \mathbf{I}(V)$. This implies $f(p) - g(p) = 0$ for all p in V and then $\phi(p) - \psi(p) = 0$ for all p in V. Which implies $\phi = \psi$.

Thus, $\pi([f]) = \pi([g])$.

Hence, π is well-defined.

ii) Since every element of $K[V]$ is represented by some polynomial and hence, π is onto.

iii) To see π is one-to-one

Suppose $\pi([f]) = \pi([g])$. Then $\phi = \psi$. Which implies $f = g$ and then $f(p) - g(p) = 0$ for all p in V. This implies $f - g \in \mathbf{I}(V)$ and hence, $f \equiv g \pmod{\mathbf{I}(V)}$.

Thus, $[f] = [g]$ in $K[x_1, x_2, \dots, x_n]/\boldsymbol{I}(V)$.

Hence, π is one-to-one.

iv) To study sums and products

Let $[f], [g] \in K[x_1, x_2, \dots, x_n]/\boldsymbol{I}(V)$. Then $\pi([f] + [g]) = \pi([f + g])$ by the definition of sum in the quotient ring. If f represents the polynomial function ϕ and g represents the polynomial function ψ then $f + g$ represents $\phi + \psi$

Hence, $\pi([f] + [g]) = \pi([f + g]) = \phi + \psi = \pi([f]) + \pi([g])$

Thus, π preserves the sums.

Hence, π is an isomorphism.

Therefore, $K[V] \cong K[x_1, x_2, \dots, x_n]/\boldsymbol{I}(V)$.

Remark: From the above theorem 2.3.2, we get a ring $K[V] \cong K[x_1, x_2, \dots, x_n]/\boldsymbol{I}(V)$. A natural question to ask what happens if we replace $\mathbf{I}(V)$ by some other ideal I which defines V, could it be true that all the quotient rings $K[x_1, x_2, \dots, x_n]/I$ are isomorphic to $K[V]$? Answer is No.

Proof: Let $V = \{(0,0)\}$ and $\mathbf{I}(V) = \mathbf{I}(\{(0,0)\}) = < x, y >$

Thus, by the theorem 2.3.2, we have $K[V] \cong K[x, y]/\boldsymbol{I}(V)$

Our first claim is that the quotient ring $K[x, y]/\boldsymbol{I}(V)$ is isomorphic to the field K. The easiest way to see this is to note that a polynomial function on the one -point set $\{(0,0)\}$ can be represented by a constant since the function will have only one function value.

Consider a mapping $\Phi : K[x\,,y]/\boldsymbol{I}(V) \longrightarrow K$ by setting $\Phi([f]) = f(0\,,0)$ (the constant term of the polynomial).

We need to show Φ ring isomorphism.

Let $[f], [g] \in K[x\,,y]/\boldsymbol{I}(V)$. Then

$\Phi([f]\ +\ [g]) = \Phi([f\ +\ g]) = (f + g)(0\,,0) = f(0\,,0) + g(0\,,0) = \Phi([f]) + \Phi([g])$

Thus, $\Phi([f] + [g]) = \Phi([f]) + \Phi([g])$. Similarly,

$\Phi([f]\,.\,[g]) = \Phi([f\,.\,g]) = (f\,.\,g)(0\,,0) = f(0\,,0)\,.\,g(0\,,0) = \Phi([f])\,.\,\Phi([g])$

Thus, $\Phi([f]\,.\,[g]) = \Phi([f])\,.\,\Phi([g])$.

Hence, Φ ring homomorphism.

Suppose $\Phi([f]) = \Phi([g])$. Then $f(0\,,0) = g(0\,,0)$. This implies $f - g \in \mathbf{I}(V)$ and then $[f] = [g]$.

Thus, Φ is one-to-one.

Let $C \in K$, then $f(x\,,y) = C$ and $[f] \in K[x\,,y]/\boldsymbol{I}(V)$ such that $\Phi([f]) = f(0\,,0) = C$.

Thus, Φ is onto.

Therefore, Φ ring isomorphism.

Let $I = \langle x^3 + y^2\ ,\ 3y^4 \rangle \subseteq K[x\,,y]$, then $\mathbf{V}(I) = \{(0\,,0)\} = V$

We ask whether $K[x\,,y]/I$ is also isomorphic to K, the answer is no .

For instance, consider the class $[y] \in K[x\,,y]/I$, but $y \notin I$. In a ring $K[x\,,y]/I$, this shows that $[y] \neq [0]$. But we also have $[y]^4 = [y^4] = [0]$ since $y^4 \in I$.

Thus, there is an element of $K[x\,,y]/I$ which is not zero itself, but whose fourth power is zero. In a field, this is impossible. Thus, $K[x\,,y]/I$ is not field

Hence, $K[x\,,y]/\boldsymbol{I}(V)$ and $K[x\,,y]/I$ cannot be isomorphic since one is field and the other is not.

Proposition 2.3.3. Let I be an ideal in $K[x_1\,,x_2\,,\dots,x_n]$. The ideals in the quotient ring $K[x_1\,,x_2\,,\dots,x_n]/I$ are in one-to-one correspondence with the ideals of ring $K[x_1\,,x_2\,,\dots,x_n]$ containing I.

Proof: First we give a way to produce an ideal in $K[x_1, x_2, \dots, x_n]/I$ corresponding to each J containing I in $K[x_1, x_2, \dots, x_n]$

Suppose J be an ideal in $K[x_1, x_2, \dots, x_n]$ containing I.

Let $J/I = \{[j] \in K[x_1, x_2, \dots, x_n]/I : j \in J\}$.

We claim that J/I is an ideal in $K[x_1, x_2, \dots, x_n]/I$.

i) Since $0 \in J$, then $[0] \in J/I$.

ii) Let $[j], [k] \in J/I$. Then $[j] + [k] = [j + k]$ by definition of sum in $K[x_1, x_2, \dots, x_n]/I$. Since $j, k \in J$, we have $j + k \in J$.
Hence, $[j] + [k] \in J/I$.

iii) $[j] \in J/I$ and $[r] \in K[x_1, x_2, \dots, x_n]/I$, then $[r].[j] = [r.j]$ by the definition of product in $K[x_1, x_2, \dots, x_n]/I$. But $r.j \in J$ since J is an ideal in $K[x_1, x_2, \dots, x_n]$.
Hence, $[r].[j] \in J/I$.
Therefore, J/I is an ideal in $K[x_1, x_2, \dots, x_n]/I$.

Suppose $\hat{J} \subseteq K[x_1, x_2, \dots, x_n]/I$ is an ideal.

We need to show how to produce an ideal $J \subseteq K[x_1, x_2, \dots, x_n]$ which is containing I.

Let $J = \{j \in K[x_1, x_2, \dots, x_n] : [j] \in \hat{J}\}$.

Then we have $I \subseteq J$ since $[i] = [0] \in \hat{J}$ for any $i \in I$.

i) $0 \in I \subseteq J$, then $0 \in J$.

ii) Let $j, k \in J$, then $[j], [k] \in \hat{J}$ implies that $[j] + [k] = [j + k] \in \hat{J}$.
Thus, $j + k \in J$.

iii) Let $j \in J$ and $r \in K[x_1, x_2, \dots, x_n]$, then $[j] \in \hat{J}$, so $[r].[j] = [r.j] \in \hat{J}$.
Thus, $r.j \in J$.

Therefore, J is an ideal in $K[x_1, x_2, \dots, x_n]$.

We have, thus shown that there are correspondences between the two collections of ideals: $\{J : I \subseteq J \subseteq K[x_1, x_2, \dots, x_n]\}$ and $\{\hat{J} \subseteq K[x_1, x_2, \dots, x_n]/I\}$.

Therefore, the ideals of $K[x_1, \dots, x_n]/I$ are of the form J/I, where J is an ideal an of $K[x_1, x_2, \dots, x_n]$ which contains I.

Example 4: Consider the ideal $I = < x^2 - 4x + 3 >$ in $\mathbb{R}[x]$. Since every ideal in $\mathbb{R}[x]$ is generated by a single polynomial, then $\mathbb{R}[x]$ is principal ideal domain. The ideals containing I are precisely the ideals generated by polynomials that divide $x^2 - 4x + 3$. Hence, the quotient ring $\mathbb{R}[x]/I$ has exactly four ideals in this case:

ideals in $\mathbb{R}[x]/I$	ideals in $\mathbb{R}[x]$ containing I
$\{[0]\}$	I
$<[x-1]>$	$< x-1 >$
$< [x-3] >$	$< x-3 >$
$\mathbb{R}[x]/I$	$\mathbb{R}[x]$

As in another Example earlier in this section, we can compute in $\mathbb{R}[x]/I$ by computing remainders with respect to $x^2 - 4x + 3$. (See Example 2.)

2.4 The Coordinate Ring of an Affine Variety

In this section, we will apply the algebraic tools developed in section 2.3 to study the ring $K[V]$ of polynomial functions on an affine variety $V \subseteq K^n$, using the isomorphism $K[V] \cong K[x_1, x_2, \ldots, x_n]/\boldsymbol{I}(V)$ from section 2.3. Thus, given a polynomial $f \in K[x_1, x_2, \ldots, x_n]$, we let $[f]$ denote the polynomial function in $K[V]$ represented by f.

In particular, each variable x_i gives a polynomial function $[x_i] : V \longrightarrow K$ whose value at a point $p \in V$ is i^{th} coordinate of p. We call $[x_i] \in K[V]$ the i^{th} coordinate function on V.

Definition 2.4.1. $K[V]$ is called the **coordinate ring** of an affine variety $V \subseteq K^n$.

Definition 2.4.2. Let $V \subseteq K^n$ be affine variety.

i) For any ideal $J = < \phi_1, \phi_2, \ldots, \phi_s > \subseteq K[V]$, we define

$\boldsymbol{V}_V(J) = \{(a_1, a_2, \ldots, a_n) \in V : \phi(a_1, a_2, \ldots, a_n) = 0$ for all $\phi \in J\}$.

We call $\boldsymbol{V}_V(J)$ is a subvariety of V.

ii) For each subset $W \subseteq V$, we define

$\boldsymbol{I}_V(W) = \{ \phi \in K[V] : \phi(a_1, a_2, \ldots, a_n) = 0$ for all $(a_1, a_2, \ldots, a_n) \in W\}$.

Example 1. Let $V = \mathbf{V}(z - x^2 - y^2) \subseteq \mathbb{R}^3$. If we take $J = < [x] > \in \mathbb{R}[V]$ then $W = \boldsymbol{V}_V(J) = \{(0, y, y^2) \ : \ y \in \mathbb{R}\} \subseteq V$ is subvariety of V. This is the same as $\mathbf{V}(z - x^2 - y^2, x)$ in $\mathbb{R}^3$.

Notation: Given a fixed variety V, we can use $\boldsymbol{I}_V$ and $\boldsymbol{V}_V$ to relate subvarieties of V to ideals in $K[V]$.

Proposition 2.4.1. Let $V \subseteq K^n$ be affine variety.

i) For each ideal $J \subseteq K[V]$, $W = \boldsymbol{V}_V(J)$ is affine variety in K^n contained in V.

ii) For each subset $W \subseteq V$, $\boldsymbol{I}_V(W)$ is an ideal of $K[V]$.

iii) If $J \subseteq K[V]$ is an ideal, then $J \subseteq \sqrt{J} \subseteq \boldsymbol{I}_V((\boldsymbol{V}_V(J))$.

iv) If $W \subseteq V$ is subvariety, then $W = \boldsymbol{V}_V((\boldsymbol{I}_V(W))$.

Proof: i) Use the one-to-one correspondence of Proposition 2.3.3 between the ideals of $K[V]$ and the ideals in $K[x_1, x_2, \dots, x_n]$ containing $\mathbf{I}(V)$.

Let $\hat{J} = \{ f \in K[x_1, x_2, \dots, x_n] : [f] \in J \} \subseteq K[x_1, x_2, \dots, x_n]$ be the ideal corresponding to $J \subseteq K[V]$. Then $\mathbf{V}(\hat{J}) \subseteq V$, since $\mathbf{I}(V) \subseteq \hat{J}$ by Proposition 1.4.3. But we also have $\mathbf{V}(\hat{J}) = \boldsymbol{V}_V(J)$ by definition since the elements of $\hat{J}$ represent the functions in J on V.

Then W is considered as a subset K^n.

Thus, W is an affine variety.

ii. a) Since zero function vanishes everywhere, then $0 \in \boldsymbol{I}_V(W)$

b) Let $\phi, \psi \in \boldsymbol{I}_V(W)$ such that $\phi(a_1, \dots, a_n) = 0$ and $\psi(a_1, a_2, \dots, a_n) = 0$ for all $(a_1, a_2, \dots, a_n) \in W$. Then $\phi(a_1, a_2, \dots, a_n) + \psi(a_1, a_2, \dots, a_n) = 0$.

This implies $(\phi + \psi)(a_1, a_2, \dots, a_n) = 0$ and hence, $\phi + \psi \in \boldsymbol{I}_V(W$.

c) Let $\psi \in \boldsymbol{I}_V(W)$. Then $\psi(a_1, a_2, \dots, a_n) = 0$ for all $(a_1, \dots, a_n) \in W$ and $\pi \in K[V]$. Which implies $\pi. \psi(a_1, a_2, \dots, a_n) = 0$.

Hence, $\pi. \psi \in \boldsymbol{I}_V(W)$.

Therefore, $\boldsymbol{I}_V(W)$ is an ideal in $K[V]$.

iii) If $\phi \in J$, then $\phi^1 \in J$. Thus, $\phi \in \sqrt{J}$ and hence, $J \subseteq \sqrt{J}$

To show $\sqrt{J} \subseteq \boldsymbol{I}_V((\boldsymbol{V}_V(J))$

Let $\phi \in \sqrt{J}$ then by definition of radical of J which implies $\phi^m \in J$ for some $m \geq 1$.

Hence, ϕ^m vanishes on $\boldsymbol{V}_V(J)$, which implies that ϕ vanishes on $\boldsymbol{V}_V(J)$.

Thus, $\phi \in \boldsymbol{I}_V((\boldsymbol{V}_V(J))$

Therefore, $J \subseteq \sqrt{J} \subseteq \boldsymbol{I}_V((\boldsymbol{V}_V(J))$.

iv) Let $(a_1 , a_2, \dots , a_n) \in W$. Then $\phi\, (a_1 , a_2, \dots , a_n) = 0$ for all $\phi \in \boldsymbol{I}_V(W)$.

Which implies $(a_1 , a_2, \dots , a_n) \in \boldsymbol{V}_V((\boldsymbol{I}_V(W))$ and hence, $W \subseteq \boldsymbol{V}_V((\boldsymbol{I}_V(W))$.

Let $(b_1 , b_2, \dots , b_n) \in \boldsymbol{V}_V((\boldsymbol{I}_V(W))$. Then $\phi\, (b_1 , b_2, \dots , b_n) = 0$ for all $\phi \in \boldsymbol{I}_V(W)$.

This implies ϕ is vanishes on W and then $(b_1 , b_2, \dots , b_n) \in W$.

Hence, $\boldsymbol{V}_V((\boldsymbol{I}_V(W)) \subseteq W$.

Therefore, $W = \boldsymbol{V}_V((\boldsymbol{I}_V(W))$.

Proposition 2.4.2. An ideal $J \subseteq K[V]$ is radical if and only if the corresponding ideal $\hat{J} = \{ f \in K[x_1 , x_2 , \dots , x_n] : [f] \in J \} \subseteq K[x_1 , x_2 , \dots , x_n]$ is radical.

Proof: Suppose J is a radical.

Let $f \in K[x_1 , x_2 , \dots , x_n]$ satisfy $f^m \in \hat{J}$ for some $m \geq 1$. Then $[f^m] = [f]^m \in J$.

Since J is radical ideal which implies $[f] \in J$. Thus, $f \in \hat{J}$ and hence, $\hat{J}$ is radical ideal.

Conversely, if $\hat{J}$ is also radical and $[f]^m \in J$, then $[f^m] \in J$, so $f^m \in \hat{J}$. Since $\hat{J}$ is also radical ideal which implies $f \in \hat{J}$ **.** Thus, $[f] \in J$ and hence, J is radical.

Definition 2.4.3. Let $V \subseteq K^m$ and $W \subseteq K^n$ be affine varieties. We say that V and W are isomorphic if there exist polynomial mapping $\alpha : V \longrightarrow W$ and $\beta : W \longrightarrow V$ such that $\alpha \circ \beta = id_W$ and $\beta \circ \alpha = id_V$.

Note: For any variety V, we write id_V for the identity mapping from V to itself. This is always a polynomial mapping.

Example 2: Consider the following surfaces in $\mathbb{R}^3$

$Q_1 = \mathbf{V}(x^2 - xy - y^2 + z^2) = \mathbf{V}(f_1)$

$Q_2 = \mathbf{V}(x^2 - y^2 - z^2 - z) = \mathbf{V}(f_2)$

We wish to examine the intersection curve $C = \mathbf{V}(f_1, f_2)$ of the surfaces.

But in this case C is not easy to visualize. To make our task easier, we observe that

$C = \mathbf{V}(f_1, f_2) = \mathbf{V}(f_1, f_1 + bf_2)$ where $b \in \mathbb{R}$, $b \neq 0$ then

$\mathbf{V}(f_1, f_1 + bf_2) \subseteq \mathbf{V}(f_1 + bf_2)$

Let $F = \mathbf{V}(f_1 + bf_2)$ and set $b = -1$. Then $F = \mathbf{V}(f_1 - f_2) = \mathbf{V}(z - xy)$

The surface F is isomorphic as a variety to $\mathbb{R}^2$, by considering the following polynomial mappings

$\alpha : \mathbb{R}^2 \longrightarrow F \qquad (x, y) \mapsto (x, y, xy)$

$\pi : F \longrightarrow \mathbb{R}^2 \qquad (x, y, z) \mapsto (x, y)$

These mappings satisfy $\alpha \circ \pi = id_F$ and $\pi \circ \alpha = id_{\mathbb{R}^2}$

To visualize C, we can project to the curve $\pi(C) \subseteq \mathbb{R}^2$

The equation for $\pi(C)$ is $x^2 y^2 + x^2 - xy - y^2 = 0$ obtained by substituting z = xy in either f_1 or f_2 . Each point (a, b) on $\pi(C)$ corresponds to exactly one point (a, b, ab) on C.

Proposition 2.4.3. Let V and W be varieties (possibly in different affine space).

i) Let $\alpha : V \longrightarrow W$ be a polynomial mapping. Then for every polynomial function $\phi : W \longrightarrow K$, the composition $\phi \circ \alpha : V \longrightarrow K$ is also polynomial function. Furthermore, the map $\alpha^* : K[W] \longrightarrow K[V]$ defined by $\alpha^*(\phi) = \phi \circ \alpha$ is ring homomorphism which is the identity on the constant functions $K \subseteq K[W]$.

ii) Conversely, let f : $K[W] \longrightarrow K[V]$ be ring homomorphism which is the identity on constants. Then there is a unique polynomial mapping $\alpha : V \longrightarrow W$ such that f = α^*

Note that α^* "goes in the opposite direction" from α since α^* maps functions on W to functions on V. For this reason we call α^* the **pullback mapping** on functions.

Proof: i) Suppose that $V \subseteq K^m$ has coordinates $x_1, \ldots, x_m$ and $W \subseteq K^n$ has coordinates $y_1, \ldots, y_n$. Then $\phi : W \longrightarrow K$ can be represented by a polynomial f$(y_1, \ldots, y_n)$ and $\alpha : V \longrightarrow W$ can be represented by an n-tuple of polynomials:

$\alpha(x_1, \ldots, x_m) = (a_1(x_1, \ldots, x_m), \ldots, a_n(x_1, \ldots, x_m))$. We can compute

$\phi \circ \alpha$ by substituting $\alpha\,(x_1 , \; . \; . \; . \; , x_m)$ in to ϕ. Thus,

$(\phi \circ \alpha)\,(x_1 , \; . \; . \; . \; , x_m) = f(a_1\,(x_1 , \; . \; . \; . \; , x_m) , \; . \; . \; . \; , a_n\,(x_1 , \; . \; . \; . \; , x_m))$ which is a polynomial in $x_1 , \; . \; . \; . \; , x_m$.

Hence, $\phi \circ \alpha$ is a polynomial function on V.

It follows that we can define $\alpha^* : \; K[W] \longrightarrow \; K[V]$ by the formula $\alpha^*\,(\phi) = \phi \circ \alpha$.

We claim that α^* is ring homomorphism.

Let ψ be another element of $K[W]$ represented by a polynomial $g(y_1 , \; . \; . \; . \; , y_n)$. Then

$$\begin{aligned}(\alpha^*\,(\phi + \psi))\,(x_1 , \; . \; . \; . \; , x_m) &= f(a_1\,(x_1 , \; . \; . \; . \; , x_m) , \; . \; . \; . \; , \; a_n\,(x_1 , \; . \; . \; . \; , x_m)) \\ &\quad + \; g(a_1\,(x_1 , \; . \; . \; . \; , x_m) , \; . \; . \; . \; , \; a_n\,(x_1 , \; . \; . \; . \; , x_m)) \\ &= (\phi \circ \alpha)\,(x_1 , \; . \; . \; . \; , x_m) + (\psi \circ \alpha)\,(x_1 , \; . \; . \; . \; , x_m) \\ &= (\alpha^*\,(\phi))\,(x_1 , \; . \; . \; . \; , x_m) + (\alpha^*\,(\psi))(x_1 , \; . \; . \; . \; , x_m)\end{aligned}$$

And similarly for product case as follows

$$\begin{aligned}(\alpha^*\,(\phi \,.\, \psi))\,(x_1 , \; . \; . \; . \; , x_m) &= f(a_1\,(x_1 , \; . \; . \; . \; , x_m) , \; . \; . \; . \; , \; a_n\,(x_1 , \; . \; . \; . \; , x_m)) \\ &\quad . \; g(a_1\,(x_1 , \; . \; . \; . \; , x_m) , \; . \; . \; . \; , \; a_n\,(x_1 , \; . \; . \; . \; , x_m)) \\ &= (\phi \circ \alpha)\,(x_1 , \; . \; . \; . \; , x_m) \,.\, (\psi \circ \alpha)\,(x_1 , \; . \; . \; . \; , x_m) \\ &= (\alpha^*\,(\phi))\,(x_1 , \; . \; . \; . \; , x_m).(\alpha^*\,(\psi))(x_1 , \; . \; . \; . \; , x_m)\end{aligned}$$

Thus, $\alpha^*\,(\phi + \psi) = \alpha^*\,(\phi) + \alpha^*\,(\psi)$ and $\alpha^*\,(\phi .\, \psi) = \alpha^*\,(\phi) \,.\, \alpha^*\,(\psi)$

Hence, α^* is ring homomorphism.

Finally, consider $[b] \in K[W]$ for some $b \in K$. Then $[b]$ is constant function on W with value b and it follows that $\alpha^*\,([b]) = [b] \circ \alpha$ is constant on V , again with value b.

Thus, $\alpha^*\,([b]) = [b]$.

ii) Let $f : K[W] \longrightarrow \; K[V]$ be ring homomorphism which is the identity on constants.

We need to show f comes from a polynomial mapping $\alpha : V \longrightarrow W$.

Since $W \subseteq K^n$ has coordinates $y_1 , \; . \; . \; . \; , y_n$, we get the coordinate $[y_i] \in K[W]$.

Then $f\,([y_i]) \in K[V]$ and since $V \subseteq K^m$ has coordinates $x_1 , \; . \; . \; . \; , x_m$, we can write $f\,([y_i]) = [a_i\,(x_1 , \; . \; . \; . \; , x_m)] \in K[V]$ for some polynomial $a_i \in K[x_1 , \ldots , x_m]$. Then consider the polynomial mapping $\alpha = (a_1\,(x_1 , \; . \; . \; . \; , x_m) , \; . \; . \; . \; , \; a_n\,(x_1 , \; . \; . \; . \; , x_m))$.

We claim that α maps V to W and $f = \alpha^*$.

Given any polynomial $F \in K[y_1, \ldots, y_n]$, we first show that $[F \circ \alpha] = f([F])$ in $K[V]$. To prove this, note that

$[F \circ \alpha] = [F(a_1, \ldots, a_n)] = F([a_1], \ldots, [a_n]) = F(f([y_1]), \ldots, f([y_n]))$ where the second equality follows from the definitions of sum and product in $K[V]$ and the third follows from $[a_i] = f([y_i])$. But $[F] = [F(y_1, \ldots, y_n)]$ is k-linear combination of products of the $[y_i]$, so that $F(f([y_1]), \ldots, f([y_n])) = f([F(y_1, \ldots, y_n)]) = f([F])$ since f is ring homomorphism which is the identity on constants K.

Hence, $[F \circ \alpha] = f([F])$.

We can now prove that α maps V to W.

Given a point $(c_1, \ldots, c_n) \in V$, we must show that $\alpha(c_1, \ldots, c_n) \in W$.

If $F \in \mathbf{I}(W)$ then $[F] = 0$ in $K[W]$ and since f is ring homomorphism, we have $f([F]) = 0$ in $K[V]$ which this implies that $[F \circ \alpha]$ is the zero function on V.

In particular, $[F \circ \alpha](c_1, \ldots, c_n) = F(\alpha(c_1, \ldots, c_n)) = 0$ since F was an arbitrary element of $\mathbf{I}(W)$ and hence, $\alpha(c_1, \ldots, c_n) \in W$.

Once we know α maps V to W, $[F \circ \alpha] = f([F])$ implies that $[F] \circ \alpha = f([F])$ for any $[F] \in K[W]$. Since $\alpha^*([F]) = [F] \circ \alpha$ and hence, $f = \alpha^*$.

It remains to show that α is uniquely determined.

Suppose we have $\beta : V \longrightarrow W$ such that $f = \beta^*$.

If β is represented by

$\beta(x_1, \ldots, x_m) = (b_1(x_1, \ldots, x_m)), \ldots, b_n(x_1, \ldots, x_m))$, then

$\beta^*([y_i]) = [y_i] \circ \beta = [b_i(x_1, \ldots, x_m)]$.

Similarly $\alpha^*([y_i]) = [y_i] \circ \alpha = [a_i(x_1, \ldots, x_m)]$.

Since $\alpha^* = f = \beta^*$ we have $[a_i] = [b_i]$ for all i. Then a_i and b_i give the same polynomial function on V. Hence, $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_n)$ define the same mapping on V. Thus, $\alpha = \beta$.

Therefore, α is uniquely determined.

Theorem 2.4.4. Two affine varieties $V \subseteq K^m$ and $W \subseteq K^n$ are isomorphic if and only if there is an isomorphism $K[V] \cong K[W]$ of coordinate rings which is the identity on constant functions.

Proof: Suppose that V and W are isomorphic varieties, then there exist polynomial mappings $\alpha : V \longrightarrow W$ and $\beta : W \longrightarrow V$ such that $\alpha \circ \beta = id_W$ and $\beta \circ \alpha = id_V$.

Define $\alpha^* : K[W] \longrightarrow K[V]$ via $\alpha^* (\phi) = \phi \circ \alpha$.

$\beta^* : K[V] \longrightarrow K[W]$ via $\beta^* (\psi) = \psi \circ \beta$.

We need to show that $\alpha^* \circ \beta^* = id_{K[V]}$ and $\beta^* \circ \alpha^* = id_{K[W]}$.

Now $(\alpha \circ \beta)^*(\phi) = {id^*}_W (\phi) = \phi \circ id_W = \phi$ for all $\phi \in K[W]$.

However, we also have $(\alpha \circ \beta)^*(\phi) = \phi \circ (\alpha \circ \beta) = (\phi \circ \alpha) \circ \beta$

$$= \alpha^* (\phi) \circ \beta$$

$$= \beta^*(\alpha^* (\phi))$$

$$= (\beta^* \circ \alpha^*) (\phi)$$

Thus, $(\alpha \circ \beta)^*(\phi) = (\beta^* \circ \alpha^*) (\phi)$.

Hence, $(\alpha \circ \beta)^* = \beta^* \circ \alpha^* = id_{K[W]}$ as a mapping from $K[W]$ to itself.

Similarly, $(\beta \circ \alpha)^*(\psi) = {id^*}_V(\psi) = \psi \circ id_V = \psi$ for all $\psi \in K[V]$ and $(\beta \circ \alpha)^*(\psi) = (\alpha^* \circ \beta^*) (\psi)$.

Thus, $(\beta \circ \alpha)^* = \alpha^* \circ \beta^* = id_{K[V]}$ as a mapping from $K[V]$ to itself.

Hence, $K[V] \cong K[W]$.

Also by the Proposition 2.4.3 the isomorphism is the identity on constants.

For the converse, we must show that if we have a ring isomorphism f : $K[W] \longrightarrow K[V]$ which is the identity on K, then f and f^{-1} come from inverse polynomial mapping between V and W . By part (ii) of Proposition 2.4.3, we know that f = α^* for some $\alpha : V \longrightarrow W$ and $f^{-1} = \beta^*$ for $\beta : W \longrightarrow V$.

We need to show α and β are inverse mappings.

First consider the composite map $\alpha \circ \beta : W \longrightarrow W$. This is a polynomial map and since $(\alpha \circ \beta)^*(\phi) = (\beta^* \circ \alpha^*) (\phi)$ for any $\phi \in K[W]$, we have

$(\alpha \circ \beta)^*(\phi) = \beta^*(\alpha^* (\phi)) = f^{-1}(f(\phi)) = \phi$

Since the identity map $id_W : W \longrightarrow W$ is polynomial map on W and we saw above that $id^*_W(\phi) = \phi$ for all $\phi \in K[W]$. Since $(\alpha \circ \beta)^*(\phi) = \beta^*(\alpha^*(\phi)) = f^{-1}(f(\phi)) = \phi$, we conclude that $(\alpha \circ \beta)^* = id^*_W$ and then $\alpha \circ \beta = id_W$ follows from the uniqueness statement of part (ii) of Proposition 2.4.3 .

Similarly $(\beta \circ \alpha)^*(\psi) = \alpha^*(\beta^*(\psi)) = f(f^{-1}(\psi)) = \psi$.

Since the identity map $id_V : V \longrightarrow V$ is polynomial map on V, then $id^*_V(\psi) = \psi$.

Thus, $\beta \circ \alpha = id_V$.

Therefore, V and W are isomorphic.

Example 3: Consider the curve $V = \mathbf{V}(y^5 - x^2)$ in $\mathbb{R}^2$

We claim that V is not isomorphic to $\mathbb{R}$ as a variety.

Suppose $\alpha : \mathbb{R} \longrightarrow V$ be isomorphism, then the pullback mapping

$\alpha^* : \mathbb{R}[V] \longrightarrow \mathbb{R}[u]$ would be a ring isomorphism given by $\alpha * ([x]) = c(u)$

$$\alpha * ([y]) = d(u)$$

, where $c(u), d(u) \in \mathbb{R}[u]$ are polynomials. Since $y^5 - x^2$ represents the zero function on V, we must have

$$\begin{aligned}\alpha^*([y^5 - x^2]) &= \alpha^*([y^5]) - \alpha^*([x^2]) \\ &= \alpha^*([y])^5 - \alpha^*([x])^2 \\ &= (\alpha^*([y]))^5 - (\alpha^*([x]))^2 \\ &= (d(u))^5 - (c(u))^2 \\ &= 0 \text{ in } \mathbb{R}[u]\end{aligned}$$

Assume $c(0) = d(0) = 0$, then $\alpha(0) = (0, 0) \in V$. But let us examine the possible polynomial solutions

$c(u) = c_1u + c_2u^2 + c_3u^3 + \ldots$

$d(u) = d_1u + d_2u^2 + d_3u^3 + \ldots$

of the equation $(d(u))^5 = (c(u))^2$ since $(d(u))^5$ contains no power of u lower than u^5 and u^2 contains no power of u lower than u^2. However,

$(c(u))^2 = (c_1u + c_2u^2 + c_3u^3 + \ldots)^2$

$= (c_1 u + c_2 u^2 + c_3 u^3 + \ . \ . \ . \ ) (c_1 u + c_2 u^2 + c_3 u^3 + \ . \ . \ . \ )$

$= {c_1}^2 u^2 + \ 2\, c_1 c_2\, u^3 + ({c_2}^2 + 2 c_1 c_3)\, u^4 + - - - .$

The coefficients of u^2 and u^4 must be zero, which implies $c_1 = c_2 = 0$,

thus the smallest power of u that can be appears in c^2 is u^6 and the smallest power of $(\mathrm{d(u)})^5 = (d_1 u \ + d_2 u^2 + d_3 u^3 + \ . \ . \ . \ )^5$ is $d_1 u^5$ but $(d(u))^5 = (c(u))^2$ which implies that $d_1 = 0$. It follows that u can not be in the image of $\alpha *$ since the image of $\alpha *$ consists of polynomials in $c(u)$ and $d(u)$. This contradicts that $\alpha *$ is a ring isomorphism on to $\mathbb{R}[u]$.

Therefore, V is not isomorphic to $\mathbb{R}$.

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