NONLINEAR TRANSPORTATION PROBLEMS

By

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1. Introduction

Network models and integer programs are well known variety of decision problems. A very useful and widespread area of application is the management and efficient use of scarce resources to increase productivity. These applications include operational problems such as the distributions of goods, production scheduling and machine sequencing. And planning problems such as capital budgeting facility allocation, portfolio selection, and design problems such as telecommunication and transportation network design.[20]

The transportation problem which is one of network integer programming problems is a problem that deals with distributing any commodity from any group of 'sources' to any group of destinations or 'sinks' in the most cost effective way with a given 'supply' and 'demand' constraints.

Depending on the nature of the cost function, the transportation problem can be categorized into linear and nonlinear transportation problem.

In the linear transportation problem(ordinary transportation problem) the cost per unit commodity shipped from a given source to a given destination is constant, regardless of the amount shipped. Also it is always supposed that the mileage (distance) from every source to every destination is fixed. To solve such transportation problem we have the streamlined simplex algorithm which is very efficient[10]. However, in actuality we can see at least two cases that the transportation problem fails to be linear.

First, the cost per unit commodity transported may not be fixed for volume discounts sometimes are available for large shipments. This would make the cost function either piecewise linear or just separable concave function,. In this case the problem may be formulated as piecewise linear or concave programming problem with linear constraints [10]

Second, in special conditions such as transporting emergency materials when natural calamity occurs or transporting military supplies during war time, where carrying network may be destroyed , mileage from some sources to some destination are no longer definite. So the choice of different measures of distance leads to nonlinear (quadratic, convex,..)objective function [2].

In both the above cases solving the transportation problem is not as simple as that of the linear one.

In this work, solution procedures to the generalized transportation problem taking nonlinear cost function are investigated. In particular, the nonlinear transportation problem considered in this paper is stated as follow; We are given

1. A set of n sources of commodity with known supply capacity and a set of m destinations with known demands.
2. The function of transportation cost, nonlinear,and differentiable for a unit of product from each source to each destination.

   We are required to find the amount of product to be supplied from each source (may be market) to meet the demand of each destination in such a way as to minimize the total transportation cost

Our approach to solve this problem is applying the existing general nonlinear programming algorithms to it making a suitable modifications in order to use the special structure of the problem.

The paper is organized as follows: In the first chapter, definitions, basic concepts and theorems necessary for the understanding of the consequent chapters are explained and in the second chapter, the linear transportation problem.

In the third, where the main objective of the paper lies, we discuss different algorithms to the nonlinear transportation problem.

Finally, in the fourth chapter,variations to the problem and some practical matters associated with solving the problem are discussed.

2. Literature review

The transportation problem was formalized by the French mathematician Gaspard Monge in 1781. Major advances were made in the field during World War II by the Soviet/Russian mathematician and economist Leonid Kantorovich. Consequently, the problem as it is now stated is sometimes known as the Monge-Kantorovich transportation problem.[37]

Many scientific disciplines have contributed toward analyzing problems associated with the transportation problem, including operation research, economics, engineering, Geographic Information Science and geography. It is explored extensively in the mathematical programming and engineering literatures.

Sometimes referred to as the facility location and allocation problem, the transportation optimization problem can be modeled as a large-scale mixed integer linear programming problem.[11]

The problem with the production capacity of each source fixed and with constant unit transportation cost was originally formulated by Hitchcock [13] in 1941 and was subsequently dealt with independently by Koopmans [16] during second World War. Analytical solutions to this problem have been given by several authors.Stringer and Haley [29] have developed a method of solution using a mechanical analogue.

May be the first algorithm to find an optimal solution for the uncapacitated transportation problem was that of Efroymson and Ray [5].They assumed that each of the unit production cost functions has a fixed charge form. But they remark that their branch-and - bound method can be extended to the case in which each of these functions is concave and consists of several linear segment.And each unit transportation cost function is linear.

Spielberg [27] has considered essentially the same problem,but his algorithm included some features that added speed computation time.The method can also accommodate such side conditions as budget constraints on plant expense and mutually exclusive alternatives, Khumawala [15] has reported good computational results for this class of problems.

Algorithms for the capacitated case have been presented by Davis and Ray [4], Ellwein [6], Gray [12]Marks,[17], and S$\acute{A}$[23]. In all of these the cost functions are assumed to be linear and the production cost is linear where ever the production is and zero where not. Ellwein's technique allows the easy incorporation of configuration constraints that restrict the allowable combinations of open plants and generalization of the production.

Vidale [30] has given a graphical solution based on successive approximations when the production costs vary with the volume of resource produced.

J.Frank Sharp.et.al [9] developed an algorithm for reaching an optimal solution to the production- transportation problem for the convex case. The

algorithm utilizes the decomposition approach it iterates between a linear programming transportation problem which allocates previously set plant production quantities to various markets and a routine which optimally sets plant production quantities to equate total marginal production costs, including a shadow price representing a relative location cost determined from the transportation problem.
Williams [31] applied the decomposition principle of Dantzing and Wolf to the solution of the Hitchcock transportation problem and to several generalizations of it.In this generalizations,the case in which the costs are piecewise linear convex functions is included. He decomposed the problem and reduced to a strictly linear program.In addition he argued that the two problems are the same by a theorem that he called the reduction theorem.

The algorithm given by him, to solve the problem, is a variation of the simplex method with "generalized pricing operation". It ignores the integer solution property of the transportation problem so that some problems of not strictly transportation type, and for which the integer solution property may not hold be solved.

Shetty [25] in 1959 also formulated an algorithm to solve transportation problems taking nonlinear costs. he considered the case when a convex production cost is included at each supply center besides the linear transportation cost.

As Feldman et al.[1] assessed, the concavity of the cost curve brought about by economies of scale leads to multiple-optima, and thus problems like these are not susceptible to conventional mathematical techniques. The power of the simplex method in solving linear programs is based on the general theorem which states that the number of variables - including slack variables, whose values are positive in an optimal solution, is at most equal to the number of constraints in the problem. For this reason, nearsighted computational techniques are used to examine the corners of the feasible region (basic solution). Unfortunately, these myopic computational and optimality testing techniques can be employed only when the problem involves a convex feasible region and increasing marginal cost (the case on which we are also will be focusing).

Some of the approaches used to solve the concave transportation problem are presented as follows.

The branch and bound algorithm approach is based on using a convex approximation to the concave cost functions. It is equivalent to the solution of a finite sequence of transportation problems. The algorithm was developed as a particular case of the simplified algorithm for minimizing separable concave functions over linear polyhedral as Falk and Soland [7]. Piece-wise linear over approximation is also the other approach to solve the nonlinear concave transportation problem [18].

Richard M. Soland (1971)[22], presented a branch and bound algorithm to solve concave separable transportation problem which he called it the "Simplified algorithm" in comparison with similar algorithm given by Falk and himself in 1969 [22][7]
The algorithm reduces the problem to a sequence of linear transportation problem with the same constraint set as the original problem.

A.C. Caputo. et. al. [3] presented a methodology for optimally planning long-haul road transport activities through proper aggregation of customer orders in separate full-truckload or less-than-truckload shipments in order to minimize total transportation costs. They have demonstrated that evolutionary computation techniques may be effective in tactical planning of transportation activities. The model shows that substantial savings on overall transportation cost may be achieved adopting the methodology in a real life scenario.

Chapter 1

Preliminaries

1.1 Polyhedral sets

Definition 1.1.1 *A set $\boldsymbol{P}$ in an n dimensional normed vector space $\boldsymbol{E}^n$ is called polyhedral set if it is the intersection of a finite number of closed- half spaces, i.e $\boldsymbol{P} = \{x : p_i^t x \leq \alpha_i \quad , i = 1 \cdots m\}$, where $\boldsymbol{p}_i$ is a non zero vector in $\boldsymbol{E}^n$ and α_i is scalar.*

The following are well known facts:

- A polyhedral set is a closed convex set.
- A polyhedral set can be represented by a finite number of inequalities and/ or equations.

We consider the polyhedral set $\mathbf{P} = \{x : Ax = b, x \geq 0\}$, where A is an mXn matrix and b is an m - vector, assume also that the rank of A is m. If not, assuming that $Ax = b$ is consistent, we can leave aside any redundant equations.

1.1.1 Extreme Points.

Definition 1.1.2 *Let P be non empty convex set in $\boldsymbol{E}^n$. A vector $x \in P$ is called an extreme point of P if $x = \lambda x_1 + (1 - \lambda)x_2$ with x_1 and x_2 elements* *of P and $\lambda \in (0,1)$ then $x = x_1 = x_2$.*

The following are basic theorems concerning extreme points; and for their proofs one can refer to [2].

Theorem 1.1.3 *Let $P = \{x : Ax = b, \quad x \geq 0\}$, where A is mXn matrix of rank m, and b is an m vector. A point $\boldsymbol{x}$ is an extreme point of P if and only if A can be decomposed in to [B,N] such that*

$$\boldsymbol{x} = \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}$$

where B is an mXm invertible matrix satisfying $B^{-1}b \geq 0$. Any such solution is called a basic feasible solution (BFS) for P.

Corollary 1.1.4 *The number of extreme points of P is finite.*

Theorem 1.1.5 *(Existence of extreme points)*
Let $P = \{x : Ax = b, \quad x \geq 0\}$ be non empty; where A is an mXn matrix of rank m and b is an m - vector. Then P has at least one extreme point.

1.1.2 Extreme Direction

Definition 1.1.6 *Let P be a non empty polyhedral set in $\boldsymbol{E}^n$. A non zero vector d in $\boldsymbol{E}^n$ is called direction or a recession direction of P if $x + \lambda d \in P$ for each $x \in P$ and for all $\lambda \geq 0$.*

It follows that, d is a direction of P if and only if

$$Ad = 0$$

$$d \geq 0$$

Theorem 1.1.7 ***Characterizations of Extreme Directions***

Let $\boldsymbol{P} = \{x : Ax = b, \quad x \geq 0\} \neq \emptyset$, where A is an mXn matrix of rank m, and b is an m vector. A vector $\bar{d}$ is an extreme direction of $\boldsymbol{P}$ if and only if A can be decomposed into [B,N] such that $B^{-1}a_j \leq 0$ for some column a_j of N, and $\bar{d}$ is a positive multiple of $d = \begin{pmatrix} -B^{-1}a_j \\ e_j \end{pmatrix}$, where e_j is an n-m vector of zeros except for in position j which is 1.

Theorem 1.1.8 ***Representation theorem***

Let $\boldsymbol{P} = \{x : Ax = b, \quad x \geq 0\} \neq \emptyset$. Let $x_1 \cdots x_k$ be the extreme points of $\boldsymbol{P}$ and $d_1, d_2, \cdots, d_l$ be the extreme direction of $\boldsymbol{P}$. Then, $x \in \boldsymbol{P}$ if and only if x can be written as:

$$x = \sum_{j=1}^{k} \lambda_j x_j + \sum_{j=1}^{l} \mu_i d_j$$

$$\sum_{j=1}^{k} \lambda_j = 1$$

$$\lambda_j \geq 0$$

$$\mu_j \geq 0$$

Corollary 1.1.9 ***Existence of extreme directions***

Let $\boldsymbol{P} = \{x : Ax = b, \quad x \geq 0\}$ where A is an mXn matrix with rank m. Then, $\boldsymbol{P}$ has at least one extreme direction if and only if it is unbounded.

1.2 The Karush -Kuhn - Tucker (KKT) optimality condition for nonlinear programming problem

Given the nonlinear programming problem:

$$\text{(NPP)} \qquad \min \quad f(x)$$

$$s.t \quad g_i(x) \leq 0 \qquad i = 1, \cdots, k$$

$$h_j(x) = 0 \qquad j = 1 \cdots, l$$

1.2.1 KKT Necessary optimality conditions

Theorem 1.2.1 *Given the objective function $f : I\!R^n \to I\!R$ and the constraint functions are $g_i : I\!R^n \to I\!R$ and $h_j : I\!R^n \to I\!R$ and $I = \{i : g_i(x^*) = 0\}$. In addition, suppose they are continuously differentiable at a feasible point x^* and $\nabla g_i(x^*)$ for $i \in I$ and $\nabla h_j(x^*)$ for $j = 1, \cdots, l$ be linearly independent. If x^* is minimizer of the problem (NPP), then there exist scalars $\overline{\lambda}_i \quad i = 1, \cdots, k$ and $\overline{\mu}_j \quad j = 1, \cdots, l$, called Lagrange multipliers, such that*

$$\nabla f(x^*) + \sum_{i=1}^{k} \overline{\lambda}_i \nabla g_i(x^*) + \sum_{j=1}^{l} \overline{\mu}_j \nabla h_j x^* = 0$$

$$\overline{\lambda}_i g_i(x^*) = 0$$

$$\overline{\lambda}_i \geq 0 \quad , \quad \overline{\mu}_j \in I\!R$$

Proof: Refer any standard nonlinear programming text book ⧫

1.2.2 KKT Sufficient Optimality conditions for convex NPP

Further, if f and each g_i are convex, each h_j is affine, then the above necessary optimality conditions will be also sufficient.

Justification

Let x be any feasible point different form x^*. From the first KKT condition we get

$$\nabla f(x^*)(x-x^*) = -\left(\sum_{i=1}^{k}\overline{\lambda}_i \nabla g_i^t(x^*)(x-x^*) + \sum_{j=1}^{l}\overline{\mu}_j \nabla h_j(x^*)(x-x^*)\right)$$

Since each $g_i(x)$ is convex, $\overline{\lambda}_i \geq 0$ and $\nabla h_j(x^*)(x-x^*) = 0 \quad \forall j$, we also have

$$\sum_{i=1}^{k}\overline{\lambda}_i \nabla g_i^t(x^*)(x-x^*) \leq \sum_{i=1}^{k}\overline{\lambda}_i [g_i(x) - g_i(x^*)]$$

$$\Rightarrow \nabla f(x^*)(x-x^*) \geq -\sum \overline{\lambda}_i g_i(x) \geq 0$$

From convexity of $f(x)$,therefore, we obtain

$$f(x) - f(x^*) \geq 0$$

$$\Leftrightarrow f(x^*) \leq f(x) \quad \text{for any feasible} \quad x. \blacklozenge$$

Chapter 2

The Linear Transportation Problem

The linear transportation problem is concerned with distributing any commodity from any group of supply centers, called *sources*, to any group of receiving centers, called *destinations* in such a way as to minimize the total distribution cost, where the cost per commodity is constant regardless of the amount transported.

By letting z to be the total distribution cost and x_{ij} the number of units to be distributed from source i (s_i) to destination j (d_j) the linear programming formulation of this problem becomes:

$$\min z = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij}$$

$$\begin{aligned} \text{s.t} \quad \sum_{j=1}^{m} x_{ij} &= s_i \qquad \text{for} \quad i = 1, 2, \cdots n \\ \sum_{i=1}^{n} x_{ij} &= d_j \qquad \text{for} \quad j = 1, 2, \cdots m \\ x_{ij} &\geq 0 \qquad \forall i, j \end{aligned}$$

The following is the structure of the constraint coefficient matrix where entries with zero left empty.

$$
A = \begin{pmatrix}
1 & & \cdot & \cdot & \cdot & & 1 & & & & & & & & & & & & & & & \\
& & & & & & 1 & & \cdot & \cdot & \cdot & & & & 1 & & & & & & & \\
& & & & & & & & & & & & & & & 1 & & \cdot & \cdot & \cdot & 1 & \\
\cdot & & & \cdot & & & & \cdot & & & & & & & & \cdot & & & & & & \\
\cdot & & & \cdot & & & & \cdot & & & & & & & & \cdot & & & & & & \\
\cdot & & & \cdot & & & & \cdot & & & & & & & & \cdot & & & & & & \\
1 & & & & 1 & & & & 1 & & \cdot & \cdot & \cdot & & & & 1 & & & & & \\
& 1 & & & & 1 & & & & 1 & & & & & & & & 1 & & & & \\
& & \cdot & & & & \cdot & & & & \cdot & & & & & & & & \cdot & & & \\
& & & \cdot & & & & \cdot & & & & \cdot & & & & & & & & \cdot & & \\
& & & & \cdot & & & & \cdot & & & & \cdot & & & & & & & & \cdot & \\
& & & & & 1 & & & & 1 & & & & 1 & & & & & & & & 1
\end{pmatrix}
$$

The transportation problem possesses two special and important properties due to the structure of the above coefficient matrix.

1. *Integer solution property*

   For a transportation problem with every s_i and d_j integer values, all the basic variables in every basic feasible solution also have integer values. **Idea of the proof**: Follows from uni modularity of the constraint coefficient matrix i.e each k by k sub matrix of $\mathbf{A}$ has determinant equals to 1. ⧫

2. *Feasible solution property*

   A necessary and sufficient condition for a transportation problem to have any feasible solution is that :

   $$\sum_{i=1}^{n} s_i = \sum_{j=1}^{m} d_j \quad \text{(In this case we say the rim requirement is satisfied)}$$

Proof :

$$\text{Let} \quad T = \sum_{i=1}^{n} s_i = \sum_{j=1}^{m} d_j,$$

$$\text{Take} \quad x_{ij} = \frac{s_i d_j}{T} \quad \text{(is feasible).}\blacklozenge$$

Definition 2.0.2 *:A transportation problem in which the rim requirement is satisfied is called* `balanced` *transportation problem otherwise it is called* `unbalanced`*.*

Theorem 2.0.3 *In a balanced transportation problem having n origins and m destinations (where $m, n \geq 0$) the exact number of basic variables is m +n-1.*

proof: trivial.

Loops in the transportation tableau.

In the transportation tableau, an ordered set of four or more cells is said to form a loop:

1. If and only if two consecutive cells in the ordered set lie either in the same row or column; and
2. If the first and the last cells of the set lie either in the same row or column.

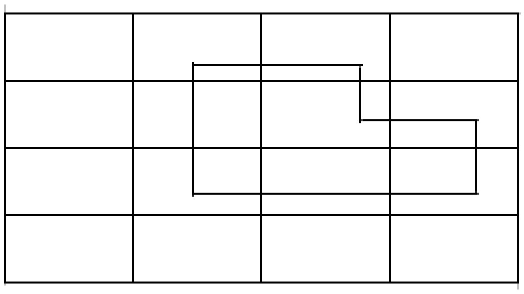

Figure 2.1: Loop in the transportation tableau

Definition 2.0.4 *A set S of cells of a transportation table is said to contain a loop if the cells of S or any subset of S can be ordered so as to form a loop.*

Theorem 2.0.5 *A set X of column vectors of the coefficient matrix of transportation problem is linearly dependent if and only if their corresponding cells in the transportation table contain loop.*

2.1 Solution Procedure to the linear transportation problem

Since the transportation problem is special type of linear programming problem it can be solved by the simplex method. Alternatively, because the transportation problem is also a network flow problem, the *labelling* method can be used to solve the problem. However, by exploiting the special structure of the coefficient matrix some tremendous computational shortcuts can be taken by using primal methods as the stream lined *simplex method*, a primal dual transportation simplex algorithm,. In this case a specially designed tableau called *transportation tableau* is used.

In general, primal methods can be applied to any distributional model that can be represented by a standard transportation tableau. This assures not only widespread solutional potential but also permits greater computational efficiency.

		Destination					
		1	2	...	n	Supply	u_i
Source	1	c_{11}	c_{12}	...	c_{1n}	s_1	
	2	c_{21}	c_{22}	...	c_{2n}	s_2	
	⋮	...	...	...	...	⋮	
	m	c_{m1}	c_{m2}	...	c_{mn}	s_m	
Demand		d_1	d_2	...	d_n	Z=	
	v_j						

Figure 2.2: Transportation tableau

Solution Procedure

***Part I*: Determination of an initial basic feasible solution.**

Here are the methods we use to find the initial basic feasible solution to the transportation problem.

Alternative Criteria for selecting the initial basic variables

1. **Northwest corner rule:**

   Begin with $x_{11} = min\{s_1, d_1\}$. Thereafter if x_{ij} was the last basic variable selected, then next select $x_{i,j+1}$ if source i has any supply remaining. Otherwise, next select $x_{j+1,j}$.

2. **Vogel's approximation method:**

   For each row and column remaining under consideration, calculate its *difference*, the difference between the smallest and next-to-the smallest unit cost c_{ij} still remaining in that row or column. In that row or column having the largest difference, select the variable having the smallest remaining unit cost.

3. **Russell's approximation method:**

   For each i determine its r_i ,the largest unit cost c_{ij} still remaining in that row. And for each j, determine its n_j , the largest unit cost c_{ij} still remaining in that column. For each variable x_{ij}, calculate $\triangle_{ij} = c_{ij} - r_i - n_j$. Select the variable having the largest (in absolute value) negative value of $\triangle_{ij}$.

4. **Row minima (cost minima) method:**

   Starting from the first row, select the smallest cost say c_{ij} allocate $x_{ij} = min\{s_i, d_j\}$ then eliminate the row or the column in which the minimum occurs until all the initial basic variables are identified.

***Part II*: Optimality condition (Improving the basic feasible solution)**

- **Determination of $c_{ij} - z_{ij}$**

  Out of $m+n$ constraint equations of the transportation problem, only $m+n-1$ of them are linearly independent. Hence there are $m+n$ dual variables to the problem of which one variable can be assigned arbitrarily and all variables are unrestricted in sign.

  Let $w = (u_1, u_2, \cdots, u_n, v_1, \cdots, v_m) = (u, v)$ be the dual variables. The dual constraints are given by $u_i + v_j \leq c_{ij}$ with no restriction in sign on u_i and v_j .

  Now let B^{-1} the inverse of the basis of the primal problem at the optimal stage and c_B is the cost vector associated with B; then the dual solution is given by:

  $$w = c_B^T B^{-1}$$

  If a_j is a column coefficient vector corresponding to the non basic variables then

  $$c_j - z_j = c_{ij} - c_B^T B^{-1} a_j$$

  Therefore $c_{ij} - z_{ij}$ corresponding to non basic cells in a transportation problem are given by

  $$c_{ij} - z_{ij} = c_{ij} - c_B^T {-1} b^{-1} a_{ij}$$

  $$= c_{ij} - w a_{ij}$$

  $$= c_{ij} - (u, v) a_{ij}$$

But

$$a_{ij} = e_i + e_{n+j}$$

$$\Rightarrow c_{ij} - z_{ij} = c_{ij} - u_i - v_j$$

Now from the complementary slackness condition and since all the basic variables are non negative, we obtain

$$c_{ij} - z_{ij} = c_{ij} - (u_i + v_j) = 0 \qquad \text{for the basic cell;}$$

$$\text{and} \quad c_{ij} - (u_i + v_j) \geq 0 \qquad \text{for the non-basic cell.}$$

- **Determination of the entering cell and entering variable**

The entering vector is selected in the following manner. Let

$$c_{rl} - u_r - v_l = \min\{c_{ij} - u_i - v_j : c_{ij} - u_i - v_j < 0\}$$

i.e., if the minimum net evaluation occurs at cell (r, k),then (r, k) will be the entering cell and x_{rl}, the entering variable. If the minimum is not unique, select any one corresponding to the minimum value of the net evaluation.

- **Determination of the departing cell and variable**

The departing cell, the cell which leaves the basis for the next iteration, can be identified geometrically from the transportation tableau as follows.

The number of basic cells including the entering one is $m+n$ so that the set of column vectors corresponding to this value is linearly dependent.

Therefore by *Theorem 2.0.5,* it is always possible to construct a unique loop. Now as we increase the value of x_{rl}, there will be a chain reaction in the loop; i.e to keep feasibility there will be some basic variables to be increased and some to be subtracted from. We call the basic cells corresponding to the ones to be increased *recipient* cells. and to the ones to be decreased from *donor* cells.

Now allocate $\theta, = \min\{x_{Bkl}\}$, where $\min\{x_{Bl}\}$ is allocation in the donor cell (k, l), to the cell (r,l).

Then adjust the basic variables in the ordered set of cells containing the simple loop by adding and subtracting , alternatively from the corresponding quantities such that all the the constraints are satisfied.All basic variables in the cells not in the loop remain unchanged.

By doing so the first basic variable that falls to zero i.e θ will be the departing variable and the corresponding cell will be the departing cell. Ties are broken arbitrarily.

If the ordered set of cells containing the simple loop are (r,l),(r,t),(t,f),(f,s) etc. the new variables corresponding to the cells are $\theta, x_{tf} - \theta, x_{tr} + \theta, x_{rs} - \theta$,etc. where $x_{pt}, \cdots,$ are the basic variables in the current iteration.

The transportation simplex algorithm

Initialization

- Constructing initial BF solution
- There are m+n-1 basic variables.
- The procedure selects the m + n - 1 basic variables one at a time.

Optimality test:

- A BF solution is optimal if and only if $c_{ij} - u_i - v_j \geq 0$ for every (i, j) such that $x_{ij}$is non basic.
- Since $c_{ij} - u_i - v_j$ is required to be zero if x_{ij} is a basic variable,u_i and v_j satisfy the set of equations $c_{ij} = u_i + v_j$.

Iteration

- Step 1: Determine the entering basic variable by selecting the one having the larger negative value of $c_{ij} - u_i - v_j$.
- Step 2: Determine the leaving basic variable, the first variable to be decreased to zero while increasing the entering basic variable from zero.
- Step 3: Identify the resulting BF by adding the value of the leaving basic variable to the allocation for each recipient cell and subtracting this same amount from the allocation for each donor cell.

Chapter 3

Solution procedures to the Nonlinear Transportation Problem (NTP)

In this section, we consider a transportation problem with nonlinear cost function. We try to find different solution procedures depending on the nature of the objective function.

Before going to the different special cases, let's formulate the KKT condition and general algorithm for the problem.

Given a differentiable function $C : \mathbb{R}^{nm} \to \mathbb{R}$

We consider a nonlinear transportation problem (NTP),

$$\begin{aligned} \min \quad & C(\mathbf{x}) \\ s.t \quad & A\mathbf{x} = b \\ & \mathbf{x} \geq 0 \end{aligned}$$

where

$$
\mathbf{x} = \begin{pmatrix} x_{11} \\ \vdots \\ x_{ij} \\ \vdots \\ x_{nm} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} s_1 \\ \vdots \\ s_n \\ d_1 \\ \vdots \\ d_m \end{pmatrix}, \quad A = \begin{pmatrix}
1 & 1 & 1 & 1 & & & & & & & & & & & \\
 & & & & 1 & 1 & 1 & 1 & & & & & & & \\
 & & & & & & & & 1 & 1 & 1 & 1 & . & . & . \\
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 & 1 & & & & 1 & & & & 1 & & & & & \\
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 & & & 1 & & & & 1 & & & & 1 & . & . & . \\
 & & & . & & & & . & & & & . & . & & \\
 & & & . & & & & . & & & & . & & . & \\
 & & & . & & & & . & & & & . & & & .
\end{pmatrix}
$$

The KKT Optimality Condition for the NTP

The transportation table is given as:-

$\frac{\partial C(\overline{x})}{\partial x_{11}}$	$\cdots$	$\cdots$	$\cdots$	$\frac{\partial C(\overline{x})}{\partial x_{1m}}$	s_1	u_1
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\frac{\partial C(\overline{x})}{\partial x_{ij}}$	$\cdots$	$\cdots$	s_i	u_i
$\frac{\partial C(\overline{x})}{\partial x_{n1}}$	$\cdots$	$\cdots$	$\cdots$	$\frac{\partial C(\overline{x})}{\partial x_{nm}}$	s_n	u_n
d_1	$\cdots$	$\cdots$	$\cdots$	d_m		
v_1	$\cdots$	v_j	$\cdots$	v_m		

where $\overline{x}$ is the current basic solution.

The Lagrange function for the NTP is formulated as

$$z(\mathbf{x}, \lambda, w) = C(\mathbf{x}) + w(b - A\mathbf{x}) - \lambda\mathbf{x}$$

where λ and w are Lagrange multipliers and

$$\lambda \in \mathbb{R}_+^{nm} \bigcup \{0\}$$

$$w \in \mathbb{R}^{n+m}$$

The optimal point $\overline{x}$ should satisfy the KKT conditions;

$$\begin{aligned}\nabla z = \nabla C(\overline{x}) - w^T A - \lambda = 0\\ \lambda\overline{x} = 0\\ \lambda \geq 0\\ \overline{x} \geq 0\end{aligned}$$

Specifically for each cell (i, j) we have

$$\frac{\partial z}{\partial x_{ij}} = \frac{\partial C(\overline{x})}{\partial x_{ij}} - (u, v)(e_i, e_{n+j}) - \lambda_{ij} = 0 \tag{3.1}$$

$$\lambda_{ij} x_{ij} = 0$$

$$x_{ij} \geq 0$$

$$\lambda_{ij} \geq 0$$

where $k = 1 \cdots nm$ and $w = (u, v) = (u_1, u_2, \cdots, u_n, v_1, \cdots, v_m)$, $e_k \in \mathbb{R}^{m+n}$ is a vector of zeros except at position k which is 1.

From the conditions (3.1) and $\lambda_k \geq 0$, we get,

$$\frac{\partial z}{\partial x_{ij}} = \frac{\partial C(\overline{x})}{\partial x_{ij}} - (u_i + v_j) \geq 0 \tag{3.2}$$

$$x_{ij} \frac{\partial z}{\partial x_{ij}} = x_{ij}\left(\frac{\partial C(\overline{x})}{\partial x_{ij}} - (u_i + v_j)\right) = 0 \tag{3.3}$$

$$x_{ij} \geq 0$$

General Solution Procedure for the NTP

- **Initialization**

Find an initial basic feasible solution $\overline{x}$

- **Iteration**

Step 1 If $\overline{x}$ is KKT point, stop. Otherwise go to the next step.

Step 2 Find the new feasible solution that improves the cost function and go to step 1.

3.1 Transportation Problem with Concave Cost Functions

For large shipments, volume discount may be available sometimes. In this case the cost function of the transportation problem generally takes concave structure for it is separable and the marginal cost (cost per unit commodity shipped) decreases with increase of the amount of shipment; and increasing, because of the total cost increase per addition of unit commodity shipped. The discount

1. May be either directly related to the unit commodity.

2. Or have the same rate for some amount.

Case 1: If the discount is directly related to the unit commodity the resulting cost function will be continues and have continues first partial derivatives. The graph of $c_{ij}(x_{ij})$ looks like,

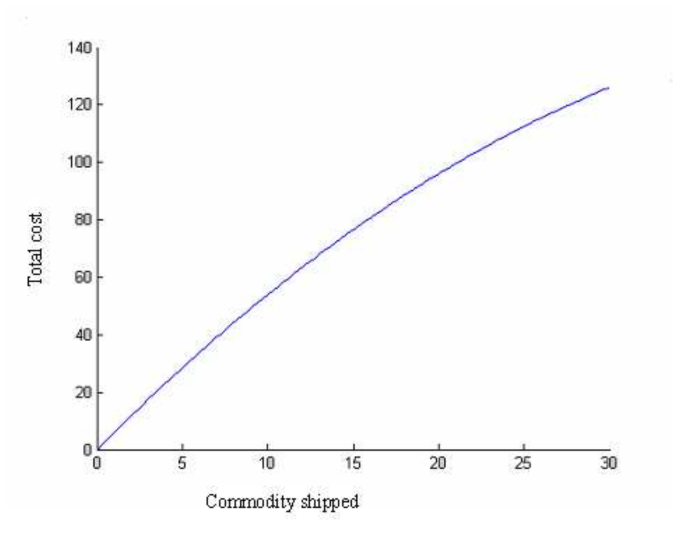

Figure 3.1: Transportation problem with continuous volume discount

Nonlinear programming formulation of such a problem is given by

$$\min \sum_{i=1}^{n} \sum_{j=1}^{m} C_{ij}(x_{ij})$$

$$\text{s.t} \quad \sum_{j=1}^{m} x_{ij} = s_i$$

$$\sum_{i=1}^{n} x_{ij} = d_j$$

$$i = 1, 2, \cdots, n \qquad j = 1, 2, \cdots, m$$

where

$$C_{ij} : \mathbb{R} \to \mathbb{R}$$

Now before going to look for an optimal solution let's state an important theorem:

Theorem 3.1.1 *Let f be concave and continues function and* $\mathbf{P}$ *be a non empty compact polyhedral set. The optimal solution to the problem* $\min f(x), x \in \mathbf{P}$ *exists and can be found at an* ***extreme*** *point of* $\mathbf{P}$.

Proof

Let $E = \{x_1, x_2, \cdots, x_k, \cdots, x_n\}$ be the set of extreme points of P, and $x_k \in E$ such that $f(x_k) = \min\{f(x_i) : \quad i = 1, \cdots, n\}$. Now since P is compact and f is continuous, f attains its minimum in P, call it $\overline{x}$,

If $\overline{x}$ is extreme point, we are done. Otherwise, we have that

$$\overline{x} = \sum_{i=1}^{n} \lambda_i x_i, \qquad \sum_{i-1}^{n} \lambda_i = 1, \qquad \lambda_i > 0$$

where $x_1, x_2, \cdots, x_n$ are extreme points of $\mathbf{P}$.

Then by concavity of f it follows that,

$$f(\overline{x}) = f(\sum_{i=1}^{n} \lambda_i x_i) \geq \sum_{i=1}^{n} \lambda_i f(x_i) \geq f(x_k) \sum_{i=1}^{n} \lambda_i$$

$$\Rightarrow f(\overline{x}) \geq f(x_k) \qquad \text{(Since for each} \quad i = 1, \cdots, n; f(x_k) \leq f(x_i) \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i = 1)$$

Since $\overline{x}$ is minimizer, in addition we have,

$$f(\overline{x}) \leq f(x_k)$$

The above two relations imply

$$f(\overline{x}) = f(x_k)$$

This completes the proof. ♦

Solution Procedure

Because of the above theorem, it suffices to consider only the extreme points to find the minimum; the following is the procedure
After we find the initial basic feasible solution (they are n+m-1 in number), let $\overline{x}$ be the basic solution we have in the current iteration.

Next let's decompose our $\overline{x}$ to $(\overline{x}_B, \overline{x}_N)$ where $\overline{x}_B$ and $\overline{x}_N$ are the basic and nonbasic variables respectively. Since $\overline{x}_B > 0$, the complementary slackness condition given in equation (3.3) gives as $m + n - 1$ equations;

$$\frac{\partial z}{\partial x_{Bij}} = \frac{\partial C(\overline{x})}{\partial x_{Bij}} - (u_i + v_j) = 0. \tag{3.4}$$

From the above relation we can determine the values of u_i and v_j by assigning one of $u_i's$ the value zero for we have m+n variables ,u_i and v_j.
Then we calculate $\frac{\partial z}{\partial x_{ij}}$ for the non basic variables x_{ij}. Since all x_{ij} are zero at the extreme, the complementary slackness condition is satisfied. Therefore if equation (3.2) is satisfied for all non basic variables x_{ij}, $\overline{x}$ is a KKT point. Otherwise, if

$$\frac{\partial z}{\partial x_{ij}} - (u_i + v_j) < 0,$$

We will move to look for better basic solution such that all the constraints (feasibility conditions) are satisfied.We do this by using the same procedure as the transportation simplex algorithm.

3.1.1 The Transportation Concave Simplex Algorithm (TCS)

Initialization

Find the initial basic feasible solution using some rule like west corner rule.

Iteration

Step 1: Determine the values of u_i and v_j from the equation,

$$\frac{\partial C(\overline{x})}{\partial x_{Bij}} - (u_i + v_j) = 0,$$

where x_{Bij} are the basic variables.

Step 2: If

$$\frac{\partial C(\overline{x})}{\partial x_{ij}} - (u_i + v_j) \geq 0,$$

for all x_{ij} - non basic, stop, $\overline{x}$ is KKT point. Otherwise go to step 3.

Step 3: Calculate

$$\frac{\partial z}{\partial x_{rl}} = \min\{\frac{\partial C(\overline{x})}{\partial x_{ij}} - u_i - v_j\}$$

x_{rl} will enter the basis. Allocate $x_{rl} = \theta$ where θ is found as in the linear transportation case.

Adjust the allocations so that the constraints are satisfied.

Determine the leaving variable say x_{Brk}, where x_{Brk} is the basic variable comes to zero first while making the adjustment. Then find the new basic variables and go to step 1.

Finite Convergence of the Algorithm

The feasible set of our problem is a non empty polyhedral set. And by definition, a polyhedral set P is a set bounded with a finite number of hyperplanes from which it follows that it possesses finite number of extreme points.
In each step of the algorithm, we jump from one extreme point to another looking for a better feasible solution implying that the algorithm will terminate after a finite iteration. In addition since for all i and j, $0 \leq x_{ij} \leq \max\{s_i, d_j\}$, P is bounded that guarantees the existence of minimum value.

Example The following example illustrates the algorithm.

Consider the transportation problem

$$\min \sum_{i=1}^{3} \sum_{j=1}^{4} C_{ij}(x_{ij})$$

$$\text{s.t} \quad x_{11} + x_{12} + x_{13} + x_{14} = 15$$

$$x_{21} + x_{22} + x_{23} + x_{24} = 25$$

$$x_{31} + x_{32} + x_{33} + x_{34} = 10$$

$$x_{11} + x_{21} + x_{31} = 20$$

$$x_{12} + x_{22} + x_{32} = 10$$

$$x_{13} + x_{23} + x_{33} = 8$$

$$x_{14} + x_{24} + x_{34} = 12$$

where

$$C_{11}(x_{11}) = 15x_{11} - 0.02x_{11}^2 \qquad C_{13}(x_{13}) = 4x_{13} - 0.04x_{13}^2$$

$$C_{21}(x_{21}) = 7x_{21} - 0.01x_{21}^2 \qquad C_{23}(x_{23}) = 8x_{23} - 0.03x_{23}^2$$

$$C_{31}(x_{31}) = x_{31} - 0.005x_{31}^2 \qquad C_{33}(x_{33}) = 5x_{33} - 0.015x_{33}^2$$

$$C_{12}(x_{12}) = 10x_{12} - 0.01x_{12}^2 \qquad C_{14}(x_{14}) = 20x_{14} - 0.07x_{14}^2$$

$$C_{22}(x_{22}) = 6x_{22} - 0.04x_{22}^2 \qquad C_{24}(x_{24}) = 3x_{24} - 0.02x_{24}^2$$

$$C_{32}(x_{32}) = 9x_{32} - 0.03x_{32}^2 \qquad C_{11}(x_{34}) = 2x_{34} - 0.01x_{34}^2$$

Using the West Corner rule we get the initial basic solution;

$$\overline{x} = (x_{B11}, x_{12}, x_{13}, x_{14}, x_{B21}, x_{B22}, x_{B23}, x_{B24}, x_{31}, x_{32}, x_{33}, x_{B34})$$

$$= (15, 0, 0, 0, 5, 10, 8, 2, 0, 0, 0, 10)$$

The partial derivatives at $\overline{x}$ are given as:

$$\frac{\partial f(\overline{x})}{\partial x_{11}} = 14.4 \quad \frac{\partial f(\overline{x})}{\partial x_{21}} = 6.9 \quad \frac{\partial f(\overline{x})}{\partial x_{31}} = 1$$

$$\frac{\partial f(\overline{x})}{\partial x_{12}} = 10 \quad \frac{\partial f(\overline{x})}{\partial x_{22}} = 5.2 \quad \frac{\partial f(\overline{x})}{\partial x_{32}} = 9$$

$$\frac{\partial f(\overline{x})}{\partial x_{13}} = 4 \quad \frac{\partial f(\overline{x})}{\partial x_{23}} = 7.52 \quad \frac{\partial f(\overline{x})}{\partial x_{33}} = 5$$

$$\frac{\partial f(\overline{x})}{\partial x_{14}} = 20 \quad \frac{\partial f(\overline{x})}{\partial x_{24}} = 2.92 \quad \frac{\partial f(\overline{x})}{\partial x_{34}} = 1.8$$

Letting $u_1 = 0$, from the equations;

$$\frac{\partial z}{x_{Bij}} = \frac{\partial f(\overline{x})}{\partial x_{Bij}} - u_i - v_j = 0$$

we have found

$$u_2 = -7.5 \qquad v_1 = 14.4 \qquad v_3 = 15.02$$

$$u_3 = -6.62 \qquad v_2 = 12.7 \qquad v_4 = 8.42$$

Then the reduced costs for the non basic variables is

$$\frac{\partial z}{\partial x_{12}} = \frac{\partial f(\overline{x})}{\partial x_{12}} - u_1 - v_2 = -2.3 \qquad \frac{\partial z}{\partial x_{13}} = \frac{\partial f(\overline{x})}{\partial x_{13}} - u_1 - v_3 = -11.02$$

$$\frac{\partial z}{\partial x_{14}} = \frac{\partial f(\overline{x})}{\partial x_{14}} - u_1 - v_4 = 11.58 \qquad \frac{\partial z}{\partial x_{31}} = \frac{\partial f(\overline{x})}{\partial x_{31}} - u_3 - v_1 = -6.78$$

$$\frac{\partial z}{\partial x_{32}} = \frac{\partial f(\overline{x})}{\partial x_{32}} - u_3 - v_2 = 2.92 \qquad \frac{\partial z}{\partial x_{33}} = \frac{\partial f(\overline{x})}{\partial x_{33}} - u_3 - v_3 = -3.4$$

$$\frac{\partial z}{x_{13}} = \min\left\{\frac{\partial z}{x_{ij}}\right\} = -11.02$$

Therefore x_{13} should enter the basis; after adjusting the values x_{23} left the basis.

Continuing in the same manner, after three iterations (excluding the first) the reduced costs for the non basic ones at a basic feasible point

$$\overline{x}^4 = (x_{11}, x_{B12}, x_{B13}, x_{14}, x_{B21}, x_{B22}, x_{23}, x_{B24}, x_{B31}, x_{32}, x_{33}, x_{34})$$

$$= (0, 7, 8, 0, 10, 3, 0, 12, 10, 0, 0, 0)$$

will be;

$$\frac{\partial z}{\partial x_{11}} = \frac{\partial f(\overline{x}^4)}{\partial x_{11}} - u_1 - v_1 = 4.1 \qquad \frac{\partial z}{\partial x_{14}} = \frac{\partial f(\overline{x}^4)}{\partial x_{14}} - u_1 - v_4 = 13.38$$

$$\frac{\partial z}{\partial x_{23}} = \frac{\partial f(\overline{x}^4)}{\partial x_{23}} - u_2 - v_3 = 8.74 \qquad \frac{\partial z}{\partial x_{32}} = \frac{\partial f(\overline{x}^4)}{\partial x_{32}} - u_3 - v_2 = 9.14$$

$$\frac{\partial z}{\partial x_{33}} = \frac{\partial f(\overline{x}^4)}{\partial x_{33}} - u_3 - v_3 = 11.64 \qquad \frac{\partial z}{\partial x_{34}} = \frac{\partial f(\overline{x}^4)}{\partial x_{34}} - u_3 - v_4 = 5.38$$

All are non negative, implying that $\overline{x}^4$ is a KKT point. In fact optimal solution to our problem.

Case 2: In the case when the volume discount is fixed for some amount of commodity, rather than varying with unit amount shipped, the transportation cost function will be piecewise linear concave yet increasing.
The graph is like;

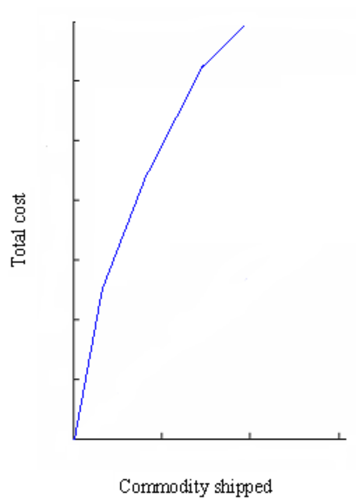

Figure 3.2: Transportation problem with piecewise linear concave cost

To avoid complication, assuming that to each combination of source and destination, the interval in which the the marginal cost(cost per unit commodity) changes is the same, the cost of shipping x_{ij} units from source i to destination j is given by $C_{ij}(x_{ij})$,then the nonlinear programming formulation of the problem is given by

$$\min \sum_{i=1}^{n} \sum_{j=1}^{m} C_{ij}(x_{ij})$$

$$\text{s.t} \quad \sum_{j=1}^{m} x_{ij} = s_i$$

$$\sum_{i=1}^{n} x_{ij} = d_j$$

$$i = 1, 2, \cdots, n \qquad j = 1, 2, \cdots, m$$

where,

$$C_{ij}(x_{ij}) = \begin{cases} C_{ij}^0(x_{ij}), & 0 \leq x_{ij} \leq a_1 \\ C_{ij}^1(x_{ij}), & a_1 \leq x_{ij} \leq a_2 \\ \vdots & \\ C_{ij}^l(x_{ij}), & a_l \leq x_{ij} \leq a_{l+1} \\ \vdots & \\ C_{ij}^{k-1}(x_{ij}), & a_{k-1} \leq x_{ij} \leq a_k \\ C_{ij}^k(x_{ij}), & a_k \leq x_{ij} \leq b = \max\{s_i, d_j\} \end{cases}$$

And

1. $\{0, a_1, \cdots, a_l \cdots, a_{k-1}, a_k, b\}$ is the partition of the interval $[0, b]$ in to $k + 1$ sub intervals

2. Each C_{ij}^l is linear in the sub interval $[a_l, a_{l+1}]$

To solve this problem, as we can see from the structure of the cost function, it's impossible to directly apply the algorithm of the previous section for non differentiability of the total cost function hinders as to do so.
But, since the function ,also, has a simple structure and differentiability fails at discrete points, it can be easily approximated using differentiable functions like Chebshev, trigonometric or Legendre polynomials.
We choose to approximate it by the so called *shifted Legendre* polynomials. These set of Legendre polynomials say $\{p_0, p_1, \cdots, p_r\}$ is orthogonal in [0,1] with respect to weight function $w(x) = 1$, where the inner product on $C[0, 1]$ is defined by

$$< f, g >= \int_0^1 f(x)g(x)dx, \quad \text{for all} \quad f, g \in C[0, 1],$$

where $C[0,1]$ is the space of continuous functions on $[0,1]$

The first four of them are,

$$p_0(x) = 1$$

$$p_1(x) = 2x - 1$$

$$p_2(x) = 6x^2 - 6x + 1$$

$$p_3(x) = 20x^3 - 30x^2 + 12x - 1$$

and the others can be obtained from

$$p_r(x) = \frac{1}{2^r!}\frac{d^r}{dx^2}\left[(x^2-1)^r\right]$$

Then, the space spanned by $\{p_0, p_1, \cdots, p_r\}$ is a subspace of $C[0.1]$. Hence, given a ny $f(x) \in C[0,1]$, we can find a unique least square approximation of f in the subspace. Note that every element of the subspace spanned $\{p_0.p_1, \cdots, p_r\}$ is at least twice differentiable.

The least square approximation of any function $f(x)$ with r of these polynomials in $[0,1]$ is given by,

$$\overline{f}(x) = a_0p_0(x) + a_1p_1(x) + \cdots + a_ip_i(x)\cdots + a_rp_r(x)$$

where

$$a_i = \frac{\int_0^1 p_if(x)dx}{\int_0^1 [p_i(x)]^2dx}, \quad i = 0, 1, \cdots, r$$

To approximate our functions $C_{ij}(x_{ij})$,in the same manner,we define a one to one correspondence between [0,b] to [0,1] by

$$g : [0,b] \to [0,1]$$

$$g(x_{ij}) = \frac{1}{b}x_{ij}$$

That is, we substitute x_{ij} by $\frac{1}{b}x_{ij}$ so that it's domain will be [0,1] then we have,

$$C_{ij}(x_{ij}) \longrightarrow \hat{C}_{ij}(x_{ij}) = C_{ij}(\frac{1}{b}x_{ij}) = \begin{cases} C_{ij}^0(\frac{1}{b}x_{ij}), & 0 \leq x_{ij} \leq \frac{a_1}{b} \\ C_{ij}^1(\frac{1}{b}x_{ij}), & \frac{a_1}{b} \leq x_{ij} \leq \frac{a_2}{b} \\ \vdots \\ C_{ij}^k(\frac{1}{b}x_{ij}), & \frac{a_k}{b} \leq x_{ij} \leq 1 \end{cases}$$

Now, after approximating $\hat{C}_{ij}x_{ij}$ by the shifted Legendre polynomials on [0,1],assume we have found it's best approximation $\tilde{C}_{ij}(x_{ij})$.

Then, substituting back the x_{ij} in $\tilde{C_{ij}}$ by bx_{ij} gives us the approximation to $C_{ij}(x_{ij})$ over [0,b]. Therefore the best approximation of $C_{ij}(x_{ij})$ over [0,b] will be

$$\overline{C}_{ij}(x_{ij}) = \tilde{C}_{ij}(bx_{ij}),$$

which has continuous derivatives.

Consequently, we solve the problem

$$\min \sum_{i=1}^{n}\sum_{j=1}^{m}\overline{C}(x_{ij}) = \sum_{l=0}^{2}\sum_{i=1}^{n}\sum_{j=1}^{m} a_l p_l(x_{ij})$$

$$\text{s.t} \quad \sum_{j=1}^{m} x_{ij} = \frac{s_i}{b}$$

$$\sum_{i=1}^{n} x_{ij} = \frac{d_j}{b}$$

$$i = 1, 2, \cdots, n \qquad j = 1, 2, \cdots, m$$

using exactly the same procedure as the previous case.

3.2 Convex Transportation Problem

This case may arise when the objective function is composed of not only the unit transportation cost but also of production cost related to each commodity[25]. Or in the case when the distance from each source to each destination is not fixed.

The problem can be formulated as

$$\min C(\mathbf{x})$$

$$\text{s.t} \quad A\mathbf{x} = b$$

$$\mathbf{x} \geq 0$$

where C($\mathbf{x}$) is convex, continuous and has continuous first order partial derivatives .

The Convex Simplex solution procedure for Transportation Problem

In the case when the cost function is convex, the minimum point may not be attained necessarily at an extreme; it may be found before reaching a boundary of the feasible set.

What precisely happens is that there may be non basic variable with positive allocation while non of the basis is driven to zero.

To solve this problem, we use the idea of the convex simplex algorithm of Zangwill [32] which was originally designed to take care of convex and pseudo convex problem with linear constraints. Actually the original procedure is used to look for a local optimal solution for any other linearly constrained programming problem. We use the special structure of transportation problem in the procedure so as to make it efficient for our particular problem.

The method reduces to the ordinary transportation simplex algorithm whenever the objective is linear, to the method of Beal when it is quadratic and to the above concave simplex procedure when the function is concave.

We partition the variable $\mathbf{x} = (x_{11}, \cdots, x_{nm})$ to $(\mathbf{x_B}, \mathbf{x_N})$. where $\mathbf{x_B}$ is $n+m-1$ component vector of basic variables and $\mathbf{x_N}$ is $nm-(n+m-1))$ component vector of nonbasic variables, corresponding to the $(n+m-1)X(n+m-1)$ basic sub matrix and $(n+m-1)X(nm-(n+m-1))$ non basic sub matrix of A.

Suppose we have the initial basic feasible solution $\overline{x}^0$.

In the procedure what we do is to find a mechanism in which non optimal basic solution $\overline{x}$ at a given iteration is improved until it satisfies the KKT conditions which are also sufficient conditions for convex transportation problem, i. e, until for each cell we have;

$$x_{ij}(\frac{\partial f(\overline{x})}{\partial x_{ij}} - (u_i + v_j)) = 0$$

and

$$\frac{\partial f(\overline{x})}{\partial x_{ij}} - (u_i + v_j) \geq 0$$

Since we have each basic variable $x_{Bij} > 0$, the above complementary slackness condition implies that for each basic cell, we must have

$$\frac{\partial f(\overline{x})}{\partial x_{Bij}} - (u_i + v_j) = 0$$

x_{Bij} - basic variable.

Since we have $n+m-1$ of such equations, by letting $u_1 = 0$ we obtain all the values of u_i and v_j as we have done exactly for the concave and linear cases.

Now for a non basic cell,at a feasible iterate point $\overline{x}$; we may have:

$$\text{I.} \quad \frac{\partial f(\overline{x})}{\partial x_{ij}} - (u_i + v_j) > 0 \quad , \quad x_{ij}(\frac{\partial f(\overline{x})}{\partial x_{ij}} - (u_i + v_j)) > 0,$$

$$\text{II.} \quad \frac{\partial f(\overline{x})}{\partial x_{ij}} - (u_i + v_j) < 0 \quad , \quad x_{ij}(\frac{\partial f(\overline{x})}{\partial x_{ij}} - (u_i + v_j)) < 0,$$

$$\text{III.} \quad \frac{\partial f(\overline{x})}{\partial x_{ij}} - (u_i + v_j) < 0 \quad , \quad x_{ij}(\frac{\partial f(\overline{x})}{\partial x_{ij}} - (u_i + v_j)) = 0$$

Or for all non basic x_{ij}, we may have;

$$\text{IV.} \quad \frac{\partial f(\overline{x})}{\partial x_{ij}} - (u_i + v_j) \geq 0 \quad , \quad x_{ij}(\frac{\partial f(\overline{x})}{\partial x_{ij}} - (u_i + v_j)) = 0$$

From the KKT conditions given earlier, the last case occurs when $\overline{x}$ is optimal.

But if the solution $\overline{x}$ falls on either of the other three, it must be improved as follows.

Let $IJ = \{ij : x_{ij} \quad \text{is non basic variable}\}$ and suppose that we are in the k^{th} iteration.

We first begin by computing;

$$\frac{\partial z}{\partial x_{rl}} = \min \left\{ \frac{\partial f(\overline{x}^k)}{\partial x_{ij}} - u_i - v_j \right\} \qquad ij \in IJ$$

$$x_{st}(\frac{\partial z}{\partial x_{st}}) = \max \left\{ x_{ij} \left(\frac{\partial f(\overline{x}^k)}{\partial x_{ij}} - u_i - v_j \right) \right\} \qquad ij \in IJ$$

Here we don't want to improve (decrease) a positive - valued non basic variable x_{ij} unless its partial derivative is positive. Therefore we only focus on positive values of the product $\frac{\partial z}{\partial x_{ij}} x_{ij}$.

Now the variables to be adjusted is selected as;

Case 1 If
$$\frac{\partial z}{\partial x_{rl}} \geq 0 \quad \text{and} \quad x_{st}(\frac{\partial z}{\partial x_{st}}) > 0,$$

Decrease x_{st} by the value θ using the transportation table as in the linear and concave cases.

Let $\mathbf{y}^k = (y^k_{11}, y^k_{12}, \cdots, y^k_{nm})$ be the value of $\overline{x}^k = (\overline{x}^k_{11}, \cdots, \overline{x}^k_{nm})$ after making the necessary adjustment by adding and subtracting θ in the loop containing x_{st}so that all the constraints are satisfied.

By doing so, either x_{st} itself or a basic variable say x_{Bst} will be driven to zero.

Now $\mathbf{y}^k$ may not be the next iterate point; since the function is convex, a better point could be found before reaching $\mathbf{y}^k$ to check this, we solve problem;

$$f(\overline{x}^{k+1}) = \min\{f(\lambda\overline{x}^k + (1-\lambda)\mathbf{y}^k) : \quad 0 \leq \lambda \ \leq 1\} \qquad (3.5)$$

and get $\overline{x}^{k+1} = \overline{\lambda}\overline{x}^k + (1-\overline{\lambda})\mathbf{y}^k$ where $\overline{\lambda}$ is the optimal solution of equation 3.5.

Before the next iteration,
If $\overline{x}^{k+1} = \mathbf{y}^k$ and if a basic variable became zero during the adjustment made, we change the basis.
If $\overline{x}^{k+1} \neq \mathbf{y}^k$ or if $\overline{x}^{k+1} = \mathbf{y}^k$ and x_{st} is driven to zero, we don't change the basis by substituting the leaving basic variable by x_{st}.

case 2 If
$$\frac{\partial z}{\partial x_{rl}} < 0 \quad \text{and} \quad x_{st}(\frac{\partial z}{\partial x_{st}}) \leq 0$$

In this case the value of x_{rl} should be increased by θ and then we find $\mathbf{y}^k$, where θ and $\mathbf{y}^k$ are defined as in the case 1.

Note that: as we increase the value of x_{rl} one of the basic variables, say, x_{Bt} will be driven to zero, and this is the exit criteria of the linear and concave transportation simplex algorithm and $\mathbf{y}^k$ would have been the next iterate point of the procedure.
But now after solving for $\overline{x}^{k+1}$ from 3.5, before going to the next iteration, we will have the following possibilities.

If $\overline{x}^{k+1} = \mathbf{y}^k$, we change the former basis. substitute x_{Bt} by x_{rl}
If $\overline{x}^{k+1} \neq \mathbf{y}^k$, we do not change the basis.
All the basic variables outside of the loop will remain unchanged.

case 3 If

$$\frac{\partial z}{\partial x_{rl}} < 0 \quad \text{and} \quad x_{st}(\frac{\partial z}{\partial x_{st}}) > 0$$

In this case either we decrease x_{st} as in the case 1 or increase x_{rl} according to case 2.

3.2.1 The Transportation Convex Simplex Algorithm

Now we write the formal algorithm* for solving the convex transportation problem.

Initialization

Find the initial basic feasible solution.

Iteration

Step 1: Determine all u_i and v_j from

$$\frac{\partial f(\overline{x}^k)}{\partial x_{Bij}} - u_i - v_j = 0$$

for each basic cell.

Step 2: For each non basic cell, calculate;

$$\frac{\partial z}{\partial x_{rl}} = \min\left\{\frac{\partial f(\overline{x}^k)}{\partial x_{ij}} - u_i - v_j\right\}$$

$$x_{st}(\frac{\partial z}{\partial x_{st}}) = \max\left\{x_{ij}\left(\frac{\partial f(\overline{x}^k)}{\partial x_{ij}} - u_i - v_j\right)\right\}$$

If

$$\frac{\partial z}{\partial x_{rl}} \geq 0 \quad \text{and} \quad x_{st}(\frac{\partial z}{\partial x_{st}}) = 0$$

Stop. Otherwise go to step 3.

Step 3: Determine the non basic variable to change.

Decrease x_{st} according to case 1 if

$$\frac{\partial z}{\partial x_{rl}} \geq 0 \quad \text{and} \quad x_{st}(\frac{\partial z}{\partial x_{st}}) > 0$$

Increase x_{rl} according to case 2 if

$$\frac{\partial z}{\partial x_{rk}} < 0 \quad \text{and} \quad x_{st}(\frac{\partial z}{\partial x_{st}}) \leq 0$$

Either increase x_{rl} or decrease x_{st} if

$$\frac{\partial z}{\partial x_{rk}} < 0 \quad \text{and} \quad x_{st}(\frac{\partial z}{\partial x_{st}}) > 0$$

Step 4 : Find the values of $\mathbf{y}^k$, by means of θ, and $\overline{x}^{k+1}$, from 3.5

If $\mathbf{y}^k = \overline{x}^{k+1}$ and a basic variable is driven to zero, change the basis.

Otherwise do not change the basis.

$$\overline{x}^k = \overline{x}^{k+1}$$

go to step 1.

---

* Detailed convergence proof of the algorithm is given on [32]

An Example
This example is used by Zangwill [32]to illustrate the Convex Simplex Algorithm.

$$\min z = f(x) = x_{11} + 2x_{12} + x_{12}^2 + x_{13}^2 + x_{21}^2 + 3x_{22} + 2x_{23}^2 + e^{x_{11}x_{21}}$$

$$\begin{aligned}
s.t \quad x_{11} + x_{12} + x_{13} \qquad\qquad &= 3 = s_1 \\
x_{21} + x_{22} + x_{23} &= 2 = s_2 \\
x_{11} + \qquad\qquad x_{21} \qquad &= 1 = d_1 \\
x_{12} + \qquad x_{22} &= 2 = d_2 \\
x_{13} \qquad + x_{23} &= 2 = d_3 \\
x_{ij} &\geq 0
\end{aligned}$$

Using the west corner rule, we found

$$\overline{x}^1 = (x_{B11}, x_{B12}, x_{13}, x_{21}, x_{B22}, x_{B23},) = (1, 2, 0, 0, 0, 2)$$

as initial basic feasible solution.

Then

$$\frac{\partial f(\overline{x}^1)}{\partial x_{11}} = 1, \quad \frac{\partial f(\overline{x}^1)}{\partial x_{12}} = 2, \quad \frac{\partial f(\overline{x}^1)}{\partial x_{13}} = 0,$$
$$\frac{\partial f(\overline{x}^1)}{\partial x_{21}} = 1, \quad \frac{\partial f(\overline{x}^1)}{\partial x_{22}} = 3, \quad \frac{\partial f(\overline{x}^1)}{\partial x_{23}} = 8$$

The reduced costs ,

$$\frac{\partial z}{\partial x_{ij}} = \frac{\partial f(\overline{x}^1)}{\partial x_{ij}} - u_i - v_j$$

are calculated and we get the values

$$\frac{\partial z}{\partial x_{11}} = 0, \quad \frac{\partial z}{\partial x_{12}} = 0, \quad \frac{\partial z}{\partial x_{13}} = -7$$

$$\frac{\partial z}{\partial x_{21}} = -1, \quad \frac{\partial z}{\partial x_{22}} = 0, \quad \frac{\partial z}{\partial x_{23}} = 0$$

$$\frac{\partial z}{\partial x_{13}} = \min\{\frac{\partial f(\overline{x}^1)}{\partial x_{ij}} - u_i - v_j\} = -7 < 0$$

$$x_{st}\frac{\partial z}{\partial x_{st}} = \min\{x_{ij}(\frac{\partial f(\overline{x}^1)}{\partial x_{ij}} - u_i - v_j)\} = 0$$

Therefore x_{13} must be increased by θ where

$$\theta = min\{x_{12}, x_{23}\} = 2$$

there by making the value of $\mathbf{y}^1$=(1,0,2,0,2,0)
Now to find the value of $\overline{x}^2$ we solve

$$f(\overline{x}^2) = \min\{f(\lambda\overline{x}^1 + (1-\lambda)\mathbf{y^1}), \quad 0 \leq \lambda \leq 1\}$$

This gives us

$$\lambda = \frac{5}{12} \quad \text{and} \quad \overline{x}^2 = (x_{B11}, x_{B12}, x_{13}, x_{21}, x_{B22}, x_{B23},) = (1, \frac{5}{6}, \frac{7}{6}, 0, \frac{7}{6}, \frac{5}{6}).$$

Basis are not changed Calculating the reduced costs we get,

$$\frac{\partial z}{\partial x_{11}} = 0, \quad \frac{\partial z}{\partial x_{12}} = 0, \quad \frac{\partial z}{\partial x_{13}} = 0$$

$$\frac{\partial z}{\partial x_{21}} = -1, \quad \frac{\partial z}{\partial x_{22}} = 0, \quad \frac{\partial z}{\partial x_{23}} = 0$$

$$\frac{\partial z}{\partial x_{21}} = \min\{\frac{\partial f(\overline{x}^1)}{\partial x_{ij}} - u_i - v_j\} = -1 < 0$$

$$x_{st}\frac{\partial z}{\partial x_{st}} = \min\{x_{ij}(\frac{\partial f(\overline{x}^1)}{\partial x_{ij}} - u_i - v_j)\} = 0$$

Therefore we increase x_{21} by θ calculated as before. Then we get

$$\mathbf{y}^2 = (0, \frac{11}{6}, \frac{7}{6}, 1, \frac{1}{6}, \frac{5}{6})$$

And using equation (3.5) we obtain $\overline{x}^3 = \mathbf{y}^2$.
This time x_{21} becomes basic while x_{11} leaves the basis.
The reduced costs are

$$\frac{\partial z}{\partial x_{11}} = 1, \quad \frac{\partial z}{\partial x_{12}} = 0, \quad \frac{\partial z}{\partial x_{13}} = 0$$

$$\frac{\partial z}{\partial x_{21}} = 0, \quad \frac{\partial z}{\partial x_{22}} = 0, \quad \frac{\partial z}{\partial x_{23}} = 0$$

And all

$$x_{ij}\frac{\partial z}{\partial \mathbf{x_{ij}}} = \{x_{ij}(\frac{\partial f(\overline{x}^1)}{\partial x_{ij}} - u_i - v_j)\} = 0$$

Therefore

$$\mathbf{x_3} = (0, \frac{11}{6}, \frac{7}{6}, 1, \frac{1}{6}, \frac{5}{6})$$

is optimal solution.

Chapter 4

Some practical matters in the Transportation Problem

4.1 The Unbalanced Transportation Problem

In the standard transportation problem,the feasible solution property requires that the system be in balance.But some times this condition is not met if the problem has physical significance.
A transportation problem in which the total supply of commodities is not equal to the total demand is called *unbalanced transportation problem*.

In this section we will present solution procedure to the unbalanced transportation problem. In fact in the procedure, the problem will be transformed to the usual *balanced* transportation problem by introducing a fictitious "source" called *dummy source* or "demand" called *dummy destination*to take up the slack to convert the inequalities to equations.
As we can see from the definition, the imbalance occurs whenever;

1. The total supply exceeds the total demand and i.e.

$$\sum_{i=1}^{n} s_i > \sum_{j=1}^{m} d_j$$

Or

2. The total supply is less than the total demand.

$$\sum_{i=1}^{n} s_i < \sum_{j=1}^{m} d_j$$

Solution to the Unbalanced transportation problem

If a transportation problem has a total supply that is strictly less than total demand the problem has no feasible solution. There is no doubt that in such a case one or more of the demand will be left unmet. Generally in such situations a penalty cost is often associated with unmet demand and as one can guess this time the total penalty cost is desired to be minimum

If total supply exceeds total demand, we can balance the problem by adding dummy demand point. Since shipments to the dummy demand point are not real, they are assigned a cost of zero.
We present the procedure with the following illustration.

4.1.1 A model and Example to the Unbalanced Transportation Problem

An Example

A certain company produces a textile machine and installs for different textile factories.
The company has been working under some contracts to deliver a considerable number of machines in the near future,and the production of the machines must now be scheduled for the next 4 months.
To meet the contracted dates for delivery, the company must supply machines for installation in the quantities mentioned in table 6.4 below.

Table 6.4

Month	Scheduled Installation	Maximum of Production
1	20	35
2	10	30
3	25	30
4	15	20
5	30	15

Thus the cumulative number of machines produced by the end of 1^{st},2^{nd},3^{rd} 4^{th} and 5^{th} months must be at least $20, 30, 75, 125$ respectively.
The facilities that will be available for producing the machines vary according to other production, maintenance, etc. during this period, resulting in variations in production costs. Because of this variations, it may be necessary to produce some of the machines a month or more before they are scheduled for installation, and this possibility is being considered. While considering this possibility, the problem is that such machines must be stored until the scheduled installation at some storage cost.
The production manager wants a schedule for the number of machines to be produced in each of the five months in order that the total cost (production, stor-

age)will be minimum.

Formulation of the problem
This problem can be modeled as a nonlinear programming problem that doesn't fit the transportation problem. but our interest is to formulate this problem as a transportation problem that requires much less effort to solve.

Here we describe the problem in terms of sources and destinations and then identify the corresponding x_{ij}, c_{ij} s_i, and d_j.

Because the commodities being distributed are machines, each of which is to be scheduled for production in a particular month and then installed in a particular (may be different) month,

Source $i=$ Production of machine in month i, $(i = 1, 2, 3, 4, 5)$

Destination $j=$ Installation of machine in month j

$x_{ij}=$ Number of machines produced in month i for installation in month j

$c_{ij}(x_{ij})=$ Cost associated with each unit of x_{ij}
$=$ cost per unit machine for production and any storage if $i \leq j$
$=?$ if $i > j$

$s_i=$?

$d_j=$ Number of scheduled installation in month j

The corresponding incomplete transportation table is given below

	Destination					
Source	1	2	3	4	5	s_i
1	$c(x_{11})$	$c(x_{12})$	$c(x_{13})$	$c(x_{14})$	$c(x_{15})$	?
2	?	$c(x_{22})$	$c(x_{23})$	$c(x_{24})$	$c(x_{25})$	?
3	?	?	$c(x_{33})$	$c(x_{34})$	$c(x_{35})$	?
4	?	?	?	c_{44}	$c(x_{45})$	?
5	?	?	?	?	$c(x_{55})$	?
d_j	20	10	25	15	30	

Now we are left with identifying the missing cost and supplies.

Since it is impossible to produce machines in one month for installation in an earlier month, x_{ij} must be zero if $i > j$. Therefore there is no cost associated with such x_{ij}. However, in order to have a well-defined transportation problem, we need to assign some value for the unidentified costs.Here we can use a big M, large number to these costs so that the corresponding x_{ij} be forced to be zero in the final solution.

Even though the "supplies", the amounts produced in the respective months, are not fixed, they do exist in the form of upper bounds on the amount that can be supplied for the maximum production of each month is given,as,

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} \leq 35$$
$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} \leq 30$$
$$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} \leq 30$$
$$x_{41} + x_{42} + x_{43} + x_{44} + x_{45} \leq 20$$
$$x_{51} + x_{52} + x_{53} + x_{54} + x_{55} \leq 15$$

To convert these inequalities to equations, we add slack variables. In this case the slack variables are allocations to a dummy destination.
Now the demand for the dummy destination ,6', (with associated cost zero), is

$$(35 + 30 + 30 + 20 + 15) - (20 + 10 + 25 + 15 + 30) = 130 - 100 = 30,$$

resulting in the final complete transportation table.

	Destination						
Source	1	2	3	4	5	6'	s_i
1	$c(x_{11})$	$c(x_{12})$	$c(x_{13})$	$c(x_{14})$	$c(x_{15})$	0	35
2	M	$c(x_{22})$	$c(x_{23})$	$c(x_{24})$	$c(x_{25})$	0	30
3	M	M	$c(x_{33})$	$c(x_{34})$	$c(x_{35})$	0	30
4	M	M	M	c_{44}	$c(x_{45})$	0	20
5	M	M	M	M	$c(x_{55})$	0	15
d_j	20	10	25	15	30	30	

4.2 The Transshipment Problem

A transportation problem allows only shipments that go directly from supply points to demand points. Sometimes there may also be points (called transshipment points) through which goods can be transshipped on their journey from a supply point to a demand point. This extension of the transportation problem to include the routing decisions is referred to as *The transshipment problem.*

There is a simple way to formulate the problem as a standard transportation problem so that the TP algorithms can be applied to solve the problem.

What we exactly do is to consider all the sources, junctions, and destinations as potential sources and potential destinations then if there are n sources, m destinations, and k transshipment junctions, we will finally have a transportation problem of $m + n + k$ sources and $m + n + k$ destinations.

Each supply point will have a supply equal to its original supply, and each demand point will have a demand to its original demand. Let $s=$ total available supply. Then each transshipment point will have a supply equal to (points original supply)$+s$ and a demand equal to (points original demand)$+s$. This ensures that any transshipment point that is a net supplier will have a net outflow equal to points original supply and a net demander will have a net inflow equal to points original demand. Although we dont know how much will be shipped through each transshipment point, we can be sure that the total amount will not exceeds.
We must consider the following steps before we come to solve the problem using transportation problem techniques.

Step1 . If necessary, add a dummy demand point (with a supply of 0 and a demand equal to the problems excess supply) to balance the problem. Shipments to the dummy and from a point to itself will be zero.

Step2 . Construct a transportation tableau Here note that A row in the tableau will be needed for each supply point and transshipment point, and a column will be needed for each demand point and transshipment point.

4.3 Degeneracy in the Transportation Problem

Definition 4.3.1 *In a standard transportation problem with n sources of supply and m demand destinations, the test of optimality of any feasible solution requires allocations in $m + n - 1$ independent cells. If the number of allocations is short of the required number, then the solution is said to be degenerate[24].*

Alternatively, in any subsequent iteration, if two or more basic variables tie for being the leaving basic variable simultaneously, as the entering variable is increased, all the tied basic variables reach zero.

When nonempty cells are fewer than necessary, it is impossible to obtain a unique set of u_i and v_j and in some cases, to form a closed loop circuites. Thus the problem never converges on an optimal solution but continues to cycle through the same non optimal solution over and over.

To resolve degeneracy i.e. to overcome this problem of endless cycling, some techniques are available

Arbitrary tie breaking rule
In the current iteration, where the degenerate solution appear, if the degenerate cell is to be increased, we can just treat it as a nonempty or basic cell and continue.

If one of the degenerate cells is to be reduced from,we can shift the zero to a cell that is to be increased keeping it in the same row or column. As long as the zero is kept in the same row or column, it doesn't change the shipping schedule or violet the constraints.[21]

The ϵ perturbation technique

This technique consists of

- Before starting to solve the problem, perturbing the set of supplies at each source $\{s_1, s_2, \cdots, s_n\}$ by an infinitesimal positive amount ϵ, then we will have a new set of supplies $\{s_1 + \epsilon, \cdots, s_n + \epsilon\}$. And we substitute the set of demands $\{d_1, \cdots, d_m\}$ by $\{d_1, d_2, \cdots, d_{m-1}, d_m + n\epsilon\}$ so that

  $$\sum_{i=1}^{n} s_i = \sum_{j=1}^{m} d_j \quad \text{be satisfied.}$$

  This ensures that degeneracy will never occur and that the iterative procedure to solve the problem can be carried out to reach optimal solution in a finite steps.

- On the current iteration where degenerate solution appear, allocating the infinitesimally small ϵ to one of the degenerate variable, once this is done, the optimality is tested, if necessary, and the solution is improved in the normal way until optimality is reached.

But with this method there will be a number of iterations that in which there is no improvement in the value of the cost function resulting in shifting ϵ from one independent cell to an other. Even if this number , generally, is less than that in the case of arbitrary tie breaking rule , it is still quite many if ,specially, the order of degeneracy is of high order.

A Shaffat and S.K Goyal [24] suggested a method of locating the independent cell where to allocate ϵ so that improvement of the solution or recognition of its optimality is ensured there by avoiding unnecessary iterations when the order of degeneracy is one.
Next we present the procedure with a bit modification for the concave transportation problem (It was originally for the linear TP) starting with a degenerate solution.

Procedure for Determining The cell for the allocation of ϵ

Step 1: Determine a set of dual variables u_i and v_j $(i = 1, 2, \cdots, n, j = 1, 2, \cdots, m)$ starting with $u_1 = 0$.Using the relation, $u_i + v_j = \frac{\partial f}{\partial x_{Bij}}$ for all x_{Bij} with allocation. At some stage, since our solution is degenerate, no further u_i or v_j values can be obtained. So we assign a value y to one of the undetermined u_i or v_j and determine all the remaining.

Step2: Determine $\frac{\partial z}{x_{ij}} = \frac{\partial f}{x_{ij}} - u_i - v_j$. Here for all allocated cells, $\frac{\partial z}{\partial x_{ij}} = 0$
And for all other (i, j), it's either constant or an expression of the form $a_{ij} - y$ or $b_{ij} + y$. (a_{ij} and b_{ij} are constants)

Step 3: Allocate the ϵ to any of the cells for which a_{ij} or b_{ij} is minimum.
Determine y by using the property that the reduced cost for the cell to which ϵ has been allocated is zero. Then $\frac{\partial z}{\partial x_{ij}}$ for any cell with $a_{ij} - y$ or $b_{ij} - y$ can be determined by substituting the value of y obtained.

Step 4: Finally compute the reduced cost for all cells with allocation zero and check for optimality. If the current solution is optimal, stop. And substitute ϵ by 0. If not proceed the iteration.

Summery and Conclusion

Some times there may be many different ways to model a particular problem, choosing the best one minimizes the complexity of the problem and time to solve.

Since, as we have said earlier, any programming problem with constraint matrix structure the same as the transportation problem is regarded as a transportation problem regardless of it's physical meaning and because of it's simple structure, modeling problems as transportation problems requires much less effort to solve than that of modeling it differently.

In this thesis the nonlinear transportation problem is considered as nonlinear programming problem and algorithms to solve this particular problem are given. The first algorithm is similar to that of the transportation simplex algorithm except the nonlinearity assumption. The second algorithm is dependent on the simplex algorithm of Zangwill [32] that we modified it to use the special property of the coefficient matrix of the transportation problem so that we may take shortcuts to make problem solving simple. However the algorithms are not been compered to other previous algorithms. Therefore in the future further work must be done :

1. To measure the efficiency of the algorithm.
2. To check how near the solution of the approximated problem of the piecewise linear transportation problem is to the optimal solution of the original problem. And finally
3. to implement the algorithm to the real world problem.

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Declaration

This thesis is my original work and has not been presented for a degree in any other university and that all sources of information used for this thesis have been fully acknowledged.

Kidist Terefe - - - - - -

Signature

This thesis is submitted for Examination with my approval as a university advisor.

Advisor Signature

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