ADDIS ABABA UNIVERSITY
COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES
DEPARTMENT OF MATHEMATICS

EXPLICIT FINITE DIFFERENCE SCHEME FOR 2D PARABOLIC PARTIAL DIFFERENTIAL EQUATION

A Project Report Submitted to the Department of Mathematics of Addis Ababa University in Partial Fulfillment of the Requirements of the Master of Science Degree in Mathematics

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Abstract

In this project report, explicit finite difference scheme for 2D parabolic partial differential equations is considered. This method is used to solve the partial derivatives in the partial differential equations at each gride point that are derived from neighbouring values by using Taylors theorem. The forward-time centered-space(FTCS)and explicit schemes are developed. The MATLAB implementation allows to experiment with the stability limit of the forward-time centered-space(FTCS).

Key words:- finite difference method, explicit Scheme, 2D parabolic differential equation, grid point

Notations

2D: Two dimensional.

R**:** Rectangular region.

$\parallel U \parallel_2$**:** Two dimensional norm of U.

FDM: Finite difference method.

PDEs: Partial differential equations.

$(i, j) = (i\Delta x, j\Delta y)$**:** Grid point in R

u=$u(x, y, t)$**:** The function approximated at a grid point(i, j).

Δt**:** The local distance between adjacent time step.

Δx **and** Δy**:** The local distance between adjacent point in space.

$\delta^2 u_k = u_{k+1} - 2u_k + u_{k-1}$**:** Second order center difference operator.

$O(\Delta x)^n$ **and** $O(\Delta y)^n$**:** Discretization error or Truncation error

$\mathcal{F}(u)$**:** The two dimensional discrete Fourier transform.

Introduction

Partial differential equations (PDEs)are equations involving unknown functions of two or more variables some of its partial derivatives.PDEs has importance in applied mathematics, physical problems, chemical and biological phenomena, and more recently PDEs are also useful in economics and financial forecasting. These problems are defined by the combination of time and space. So they are forever changing in complex way that can't be solved exactly. Therefore numerical solution of PDEs leads to some of the most important, and computationally intensive, task in all of numerical analysis and this described the finite difference method (FDM) which is numerical technique for the solution of PDEs.

The FDM consists of replacing each derivatives by a difference quotient in the classic formulation.It is simple to code and economic to compute. In a sense,a finite difference formulations offers a more direct approach to the numerical solution of PDEs than dose a method based on other formulations.It consists in approximating the differential operator by replacing the derivatives in the equation using differential quotients.

In this project, We will introduce the explicit FDM to solve the 2D parabolic PDE using grid points. The basic idea in the FDM is approximation of derivatives by difference quotients thereby replacing the partial derivatives involved in a given equation by difference quotients at the grid points and consequent reduction of a given differential equation into a system of algebraic equation.

We recall that the parabolic PDE involves several independent variable that requires some sort of regularity of the solution, u on a rectangular region R.

We need to choose a step size corresponding to each independent variable Δx and Δy to cover the rectangular region R with a grid. Consider rectangular region R such that;

$$R = [1, 0] \times [0, 1]$$

.

It is convenient to let $\Delta x = \Delta y$ (i.e the computation and analysis are more palatable) to denote the unknown function, $u = u(x, y, t)$ at the grid point (i, j) and at the nth time level by $u^n_{i,j}$.

Consequently the finite difference formulation of the 2D parabolic PDE;

$$
\begin{aligned}
u_t &= \nu(u_{xx} + u_{yy}) + F(x,y,t), (x,t) \in R, t > 0 \\
u(x,y,t) &= g(x,y,t) on \partial R, t > 0 \\
u(x,y,t) &= f(x,y)(x,y) \in \bar{R}
\end{aligned}
$$

Where R, ∂R and $\bar{R}$ are regions in a plane.

Lets us begin with difference approximation u_t , u_{xx} and u_{yy} at the grid point by
$(i\Delta x, j\Delta y, (n+1)\Delta t)$
by

$$
\begin{aligned}
u_t &= \frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} + O(\Delta t) \\
u_{xx} &= \frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} - u_{i-1,j}^{n}}{\Delta x^2} + O(\Delta x)^2 \\
u_{yy} &= \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} - u_{,ij-1}^{n}}{\Delta y^2} + O(\Delta y)^2
\end{aligned}
$$

respectively. This leads to the difference equation

$$
\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \nu(\frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} - u_{i-1,j}^{n}}{\Delta x^2} + O(\Delta x)^2 + \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} - u_{i,j-1}^{n}}{\Delta y^2} + O(\Delta y)^2) + F(x,y,t)
$$

further simplification yields

$$
u_{i,j}^{n+1} = u_{i,j}^{n} + (\gamma_x \delta_x^2 + \gamma_y \delta_y^2) u_{j,k}^{n} + \Delta t F_{i,j}^{n}
$$

This is the explicit scheme (forward time center space) of 2D parabolic PDE. We begin the following chapter by considering the model equation and then setting the outline how to discritezing using Taylor's Theorem to derive finite difference approximation to the first and second order derivatives of a function.

After the basic concepts have been introduced, we begin our discussion of finite difference schemes. The important concept of convergence, consistency and stability are presented.

Chapter 1

Preliminaries

1.1 Overview of Parabolic Partial Differential Equation

Parabolic Partial Differential Equations are large important class of PDE. To define the simplest kind of parabolic partial differential equation , consider a real-valued function $u(x, y)$ of two independent real variables, x and y. A second-order linear partial differential equation of u takes form:

$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} + D(x,y)u_x + E(x,y)u_y + F(x,y) = 0 \tag{1.1}$$

This PDE is parabolic if the coefficient satisfying the condition $B^2 - 4AC = 0$. Usually x represent one-dimensional position and y represent time,and the partial differential equation is solved subject to prescribed initial and boundary conditions.

The basic example of a parabolic equation is the one-dimensional heat equation,

$$u_t = \alpha u_{xx} \tag{1.2}$$

where $u(x, t)$ is the temperature at time t and at a position x along a thin rod, and α a positive constant (the thermal diffusivity).

In this project we will address the 2D parabolic partial differential equation by considering the model 2D parabolic equation given below:

$$u_t = \nu(u_{xx} + u_{yy}) + F(x, y, t); (x, t) \in R, t > 0 \quad (1.3)$$

$$u(x, y, t) = g(x, y, t); on\partial R, t > 0 \quad (1.4)$$

$$u(x, y, t) = f(x, y); (x, y) \in \bar{R} \quad (1.5)$$

Where R, ∂R and $\bar{R}$ are regions in a plane.

1.2 Finite Difference Method

The FDM proceeds by replacing the derivative in the differential equation with finite difference approximations.This gives a large but finite algebraic system of equations to be solved in place of the differential equation,something that can be done on a computer.The drawback of the finite difference method is accuracy and flexibility.

The finite difference approach is one of the premier mathematical tools employed to solve PDEs (i.e obtaining numerical solutions to PDEs). The most common finite difference methods for the solution of partial differential equation are

- explicit,
- implicit, and
- Crank Nicolson.

These are closely related but differ in stability, accuracy and execution speed. In the formulation of a partial differential equation problem, there are three components to be considered; these are:

- the partial differential equation,
- the region of space-time on which the partial differential equation is required to be satisfied, and
- the boundary conditions to be met.

The application of finite difference method to a particular differential equation problem includes the following steps:

1. Construction of a discrete finite-difference model of the problem:

- coverage of the computational domain by a space-time grid,
- approximation to derivatives, function, initial
  and/or boundary condition all at the grid point, and
- construction of a system of the finite difference equation.

2. Analysis of finite-difference model:

   - consistency and order of the approximation,
   - stability,and
   - convergence.

3. Numerical computations

1.2.1 Analysis of the Finite Difference Model

Every finite difference approximation of a PDE should satisfying consistency, stability and convergence in order to be valid solution of the problem.

- Consistency: A finite difference schemes for differential equation is consistent, if the difference between the partial differential equation and the finite difference equation vanishes as the interval (Δx) and the time step size (Δt) approaches zero. Consistency deals with how well the finite difference equation approximate the partial differential equation.

- Stability: It means that errors at one stage of the calculation do not cause increasingly large error as the computations are continued.

- Convergence:It means that the solution the finite difference equation approaches the true solution of the PDE as both the grid interval and time step size are reduced (as Δx and Δt tends to zero).

  The necessary sufficient condition for convergence are consistency and stability.

These three factors that characterize a numerical scheme are linked together by Lax equivalence theorem which states that a given well-posed linear initial value problem and a consistent finite difference scheme, stability is the necessary and sufficient condition for convergence.

In general, a problem is said to be well-posed, if:

- solution to the problem exists,
- the solution is unique when it exists, and
- the solution depends continuously on the problem data.

1.2.2 Grid(Mesh)

The mesh (grid) is the set of location where the discrete solution is computed. These points are called nodes, and if one were to draw the line between adjacent nodes in the domain the resulting image would resembled a net or mush. Two parameters of the mesh are Δx the local distance between adjacent points in space, and Δt the local distance between adjacent time steps.

The domain is partitioned in space and in time and approximations of the solution are computed at space or time points. In addition, there are practically useful sachems that can fail to yield a solution for bad combination of Δx and Δt.
Consider a computational domain in the two-dimensional space of variables (t, x). Cover this space by a grid of discrete points (t_i, x_j) given by

- $t_i = t_0 + i\Delta t$
- $x_i = x_0 + j\Delta x$,
- where $i = 0, 1, 2, ...$ and $j = 0, 1, 2, ...$

Here, Δx is usually called grid spacing, and Δt is called time step since t usually represent time.
A spacial grid that is the most appropriate for the problem under consideration should be chosen. In many application, the regular (uniform) grid with the grid spacing Δx is a natural and reasonable choice.

1.2.3 Truncation Error

In the limit as the mesh spacing (Δx and Δt) and time space (Δt) go to zero, the numerical solution obtain with any useful scheme will approach the true solution to the original differential equation. However, the rate at which the numerical solution approaches the true solution varies with the scheme. The error between the numerical solution and the exact solution is determined by the error that is committed by going form a differential operator to a difference operator. This error is called local discretization error or truncation error.The term truncation error reflects the fact that a finite part of a Taylor series is used in the approximation.

1.3 Driving Finite Difference Approximation

The finite difference approximation work by replacing over which the independent variables in the PDEs are defined by finite grid (also called mesh) of point at which the independent variable is approximated. The partial derivatives in the PDEs at each grid point are approximated from neighbouring value by using Taylors theorem.

1.3.1 Taylor Series

The Taylor series expansion of the function $u(x)$ about the point $x = x_0$ is given by the formula

$$u(x) = \sum_{n=0}^{\infty} \frac{u^n(x_0}{n!}(x - x_0)^n \tag{1.6}$$

where

$$\begin{aligned} u^n(x_0) &= \frac{d^n u}{dx^n} at x = x_0 \\ u^0 &= u(x_0) \end{aligned}$$

If we let $x = x_0 + \Delta x$, then $x - x_0 = \Delta x$, and the series can be written as

$$\begin{aligned} u(x_0 + \Delta x) &= \sum_{n=0}^{\infty} \frac{u^n(x_0)}{n!}.\Delta x^n \qquad (1.7) \\ &= u(x_0) + \frac{u^{'}(x_0)}{1!}.\Delta x + \frac{u^{''}(x_0)}{2!} + O(\Delta x)^3, \qquad (1.8) \end{aligned}$$

where the expression $O(\Delta x)^3$ represents the remaining terms of series and indicates that the leading term is of order Δx^3. Because Δx is small quantity, we can write $0 < \Delta x < 1$, and $\Delta x > \Delta x^2 > \Delta x^3 > \Delta x^4 > ....$ Therefore, the remaining of the series represented by $O(\Delta x)^3$ provides the order of the error incurred in neglecting this part of the series expansion when calculating $u(x_0 + \Delta x)$.

From the Taylor series expansion as shown above we can obtain an expression for the derivative $u^{'}(x_0)$ as

$$\begin{aligned} u^{'}(x_0) &= \frac{u(x_0 + \Delta x) - u(x_0)}{\Delta x} - \frac{u^{''}(x_0)}{2!}.\Delta x - O(\Delta x)^2 \qquad (1.9) \\ &= \frac{u(x_0 + \Delta x) - u(x_0)}{\Delta x} - O(\Delta x) \qquad (1.10) \end{aligned}$$

In practical application of finite difference, we will replace the first- order derivative $\frac{du}{dx}$ at $x = x_0$, with the expression $\frac{u(x_0+\Delta x)-u(x_0)}{\Delta x}$, selecting an appropriate value for Δx, and indicating that error introduced in the calculation is of order Δx, (i.e error = $O(\Delta x)$).

1.3.2 Taylor's Series Applied to the FDM

The finite difference approximation to partial derivatives are approximated based on Taylors series expansions of a function of one or more variables.
Let u(x,y) has n continuous derivatives of one variable over the interval (a,b). Then for $a < x$, $x + \Delta x < b$,

$$u(x_0+\Delta x) = u(x_0)+\frac{\Delta x}{1!}u_x(x_0)+\frac{(\Delta x)^2}{2!}u_{xx}(x_0)+...+(\Delta x)^{n-1}\frac{u_{n-1}(x_0)}{(n-1)!}+O(\Delta x)^n, \quad (1.11)$$

where,

- $u_x = \frac{du}{dx}$, $u_{xx} = \frac{d2u}{dx^2}$, $u_{xxx} = \frac{d^3u}{dx^3}$, ...
- $u_x(x_0)$ is the derivative of u with respect to x evaluated at $x = x_0$.
- $O(\Delta x)^n$ is discretization error or truncation error.

The usual interpretation of Taylor Series says that if we know the value of u and the values of its derivatives at point x_0 then we can write down the equation (1.11) for its value at a point $x_0 + \Delta x$. This expression contains an unknown quantity which is written in as $O(\Delta x)^n$ in equation (1.11)(i.e truncate the right hand side of equation (1.11) we get an approximation is $O(\Delta x^n)$).

In the finite difference method we know the u values at the grid points and we want to replace partial derivatives in the PDEs we are solving by approximations at this grid points. We do this by interpreting equation (1.11) in another way. In the FDM both x_0 and $x_0 + \Delta x$ are grid points and $u(x_0)$ and $u(x_0 + \Delta x)$ are known. This allows to rearrange equation.

1.3.3 Finite Difference Approximation to the Derivative of a Function

In this section, we will discuss three different types of first order difference approximation and a second order difference approximation for the derivative of a function. These are:

First Order Forward Difference

The first type of first order difference approximation for the derivative a function can be derived from Taylor's Series (equation (1.11)) given above as follow,

$$u(x_0 + \Delta x) = u(x_0) + \Delta x u_x(x_0) + O(\Delta x)^2 \quad (1.12)$$

then $u_x(x_0)$ form equation (1.12) solved as

$$u_x(x_0) = \frac{u(x_0+\Delta x)-u(x_0)}{\Delta x} + \frac{O(\Delta x)^2}{\Delta x} \quad (1.13)$$

$$= \frac{u(x_0+\Delta x)-u(x_0)}{\Delta x} + O(\Delta x) \quad (1.14)$$

Equation (1.14) is called a first order finite difference approximation to with approximation error (i.e $O(\Delta x)$) which depends on the first power of Δx. This approximation is called a forward finite difference approximation since we start at x_0 and step forward to the point $x_0+\Delta x$, Δx is called the step size ($\Delta x > 0$).

First Order Backward Difference

The second type of first order finite difference approximation for the derivative of a function is obtained by substituting $-\Delta x$ instead of Δx in equation (1.12).It give as

$$u(x_0-\Delta x) = u(x_0) - \Delta x u_x(x_0) + O(\Delta x)^2 \quad (1.15)$$

Then simplifying $u_x(x_0)$ from equation (1.15)

$$u_x(x_0) = \frac{u(x_0)-u(x_0-\Delta x)}{\Delta x} + \frac{O(\Delta x)^2}{\Delta x} \quad (1.16)$$

$$= \frac{u(x_0)-u(x_0-\Delta x)}{\Delta x} + O(\Delta x) \quad (1.17)$$

Equation (1.17) is called the backward difference formula because it involves the values of u at x_0 and $x_0-\Delta x$. The order of magnitude of the truncation error for the backward difference approximation is the same as that of the forward difference approximation.

First Order Central Difference

The third type of first order difference approximation for the derivative of a function can be determine by using the Tayler series expansion for $u(x_0+\Delta x)$ and $u(x_0-\Delta x)$ are:

$$u(x_0+\Delta x) = u(x_0) + \Delta x u_x(x_0) + \Delta x^2 \frac{u_{xx}(x_0)}{2!} + \Delta x^3 \frac{u_{xxx}(x_0)}{3!} + ... \quad (1.18)$$

$$u(x_0-\Delta x) = u(x_0) - \Delta x u_x(x_0) + \Delta x^2 \frac{u_{xx}(x_0)}{2!} - \Delta x^3 \frac{u_{xxx}(x_0)}{3!} + ... \quad (1.19)$$

Subtracting equation (1.19) from equation (1.18), then it would be

$$u(x_0+\Delta x) - u(x_0-\Delta x) = 2\Delta x u_x(x_0) + O(\Delta x)^3$$

Solve $u_x(x_0)$ it gives as

$$\begin{aligned} u_x(x_0) &= \frac{u(x_0+\Delta x)-u(x_0-\Delta x)}{2\Delta x}+\frac{O(\Delta x^3)}{\Delta x} && (1.20)\\ &= \frac{u(x_0+\Delta x)-u(x_0-\Delta x)}{2\Delta x}+O(\Delta x^2) && (1.21) \end{aligned}$$

This is known as first order central difference approximation. To get good approximation to the continuous problem small Δx is chosen. When $0 < \Delta x << 1$; the truncation error for the central difference approximation goes to zero much faster than the truncation error in equation (1.14)and (1.17). There is a complication with equation (1.21) because it does not include the value for $u(x_0)$. This may cause problems when the central difference approximation is included in an approximation to the differential equation.

Second Order Center Difference

The higher order finite difference approximation for the derivative of a function is obtain with the additional manipulations of the Taylor series expansion about $u(x_0)$. To determine it add equation (1.14) and (1.17)

$$u(x_0+\Delta x)+u(x_0-\Delta x)=2u(x_0)+\Delta x^2 u_{xx}(x_0)+2\Delta x^4\frac{u_4(x_0)}{4!}+... \quad (1.22)$$

then solving $u_{xx}(x_0)$;

$$\begin{aligned} u_{xx}(x_0) &= \frac{u(x_0+\Delta x)-2u(x_0)+u(x_0-\Delta x)}{\Delta x^2}+\Delta x^2\frac{u_4(x_0)}{12}\\ &= \frac{u(x_0+\Delta x)-2u(x_0)+u(x_0-\Delta x)}{\Delta x^2}+O(\Delta x)^2 \end{aligned}$$

Thus

$$u_{xx}(x_0)=\frac{u(x_0+\Delta x)-2u(x_0)+u(x_0-\Delta x)}{\Delta x^2}+O(\Delta x)^2 \quad (1.23)$$

And this is known as the central difference approximation for the second order derivative of a function.

1.3.4 Outline of the Method

When solving the 2D parabolic PDE , our approach will be cover the rectangular regions $\bar{R}$ with a grid and approximate our problem by a problem defined on that grid (the grid will be two dimensional). However, when we solve PDE numerically, infinitely many points is replaced by a finite set of grid points, then the PDE is replaced by algebraic equations and the algebraic equations are solved at each grid points. In short, we have to follow the following steps:

1. Descretize the domain. During this operation,the infinite many points replaced by a discrete domain (a finite set of points or grid)

2. Descretized the PDE.During this operation,all the partial derivatives which appearing the PDE is approximated by finite differences. During this process, the PDE is replaced by algebraic equations

3. Descretized the boundary conditions and the initial condition similar to step 2.

4. By solving the algebraic equation obtained in steps 2 and 3 at the points we obtain the solution at grid points.

Chapter 2

Explicit Scheme for 2D Parabolic PDE

2.1 Descretizing the 2D Parabolic PDE

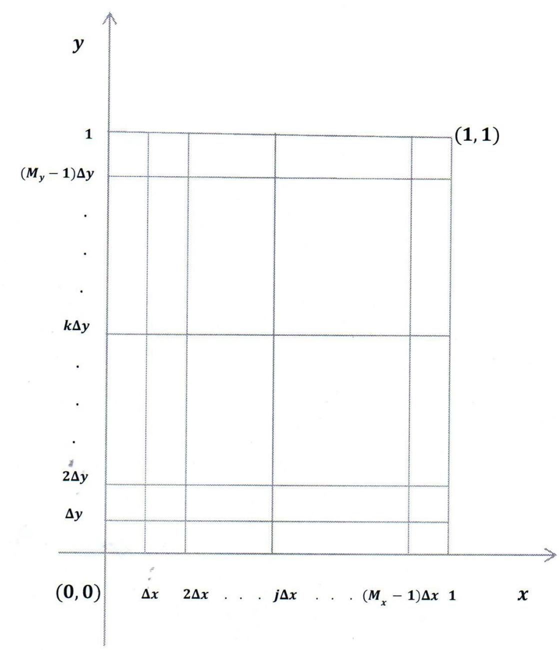
We begin our discussion of numerical method for the model 2D parabolic PDE by getting started solving the problem. (Note that if numerical values for the solution are desired, one can generally produced them faster by the numerical method than by solving the problem analytically and evaluating the solution).

To solve the problem our method is to reduce the model 2D parabolic PDE to a discrete problem we are able to solve. We begin by descretizing the spatial domain the rectangular region R by placing a grid, with spacing $\Delta x = \frac{1}{M_x}$ and $\Delta y = \frac{1}{M_y}$ where M_x and M_y are positive integers. Then domain of the model 2D parabolic PDE descretized by a grid as shown in the figure-1 of in the following pages.

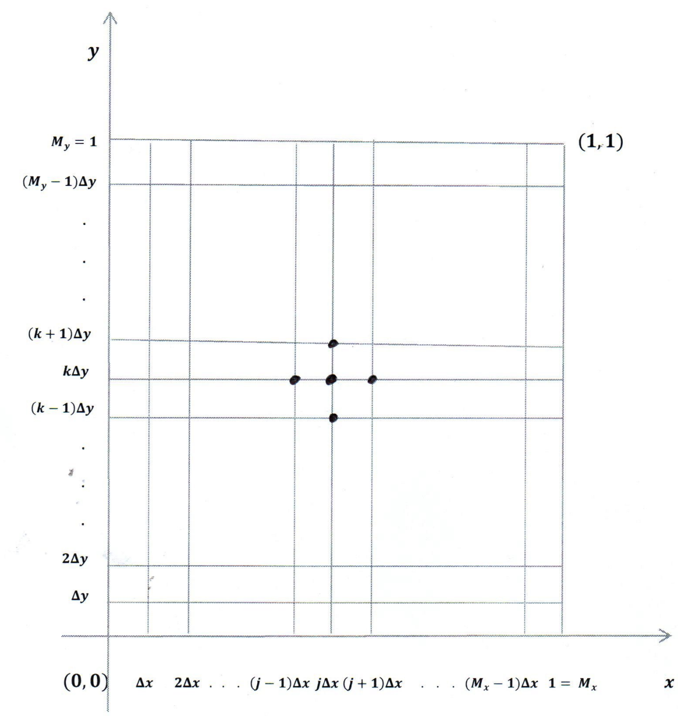
We must have some relatively consistent notation for which we denote points and defined on such grids. We shall use the convention of using an order pair (i, j) to denote the point $(i\Delta x, j\Delta y)$ in R, where $i = 0, 1, 2, ..., M_x$ and $j = 0, 1, 2, ..., M_y$ illustrated in figure-2 of in the following pages.

Like wise we descretized the time domain similarly by placing a temporal axis with grid spacing Δt. The resulting grid in the time-spacing domain is illustrated in figure-3 of in the following pages.

Consider rectangles $R = [0, 1] \times [0, 1]$. To cover R with a grid we must choose a Δx and Δy . Often it is convenient to let $\Delta x = \Delta y$. A function $u = u(x, y, t)$ approximated at the *nth* time level will be denoted by $u^n_{i,j}$. Thus we can approx-



Figer – 1: **Two dimensional grid on the region $[0, 1] \times [0, 1]$.**



$Figer - 2$: **Stencil for approximating $u_{xx} + u_{yy}$ on a two dimensional grid.**

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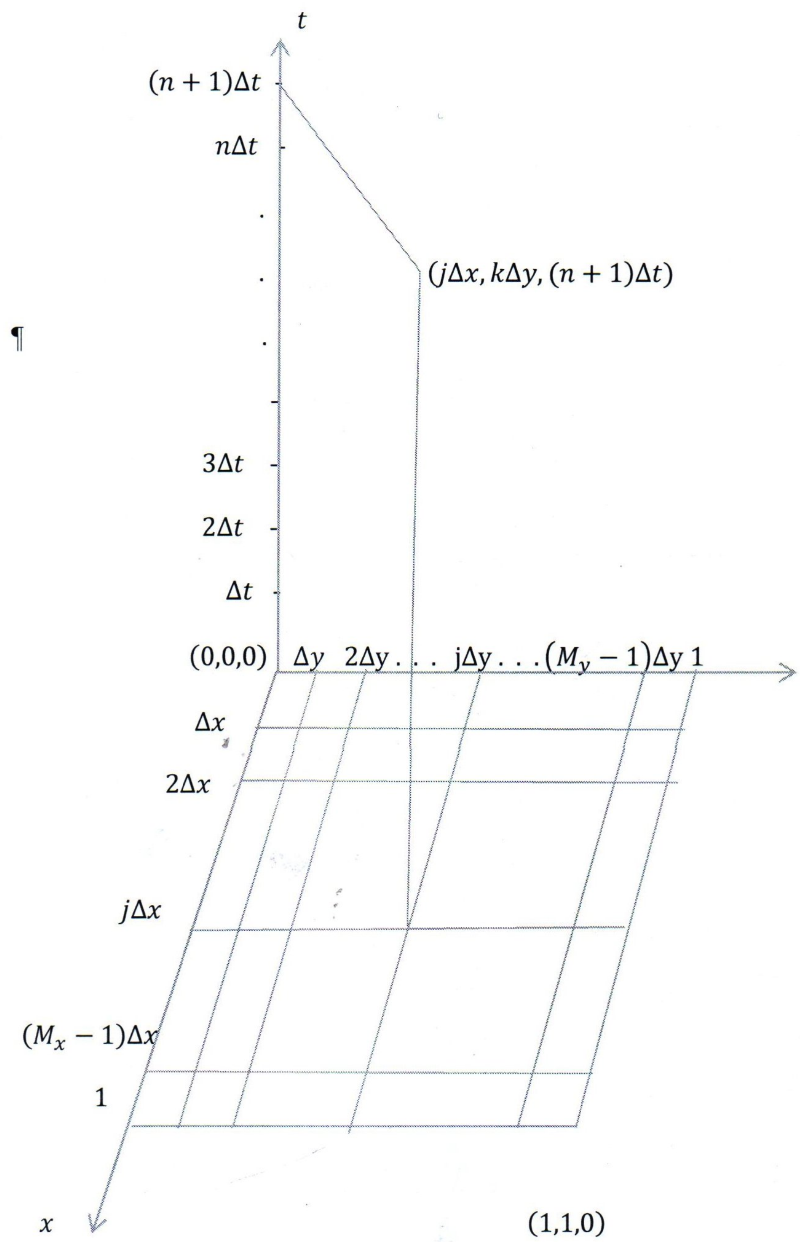


Figer-3: Grid point for the explicit Scheme.

imate u_t , u_{xx} and U_{yy} at the grid point $(i\Delta x, j\Delta y, (n+1)\Delta t)$ thus the model 2D parabolic PDE (1.3) by using the finite difference approximation (1.13) and (1.23):

$$u_t = \frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} + O(\Delta t)$$

,

$$u_{xx} = \frac{u_{i+1,j}^n - 2u_{i,j}^n - u_{i-1,j}^n}{\Delta x^2} + O(\Delta x)^2$$

and

$$u_{yy} = \frac{u_{i,j+1}^n - 2u_{i,j}^n - u_{i,j-1}^n}{\Delta y^2} + O(\Delta y)^2$$

thus we can approximate the partial differential equation (1.3) by substituting the above equations:

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} = \nu(\frac{u_{i+1,j}^n - 2u_{i,j}^n - u_{i-1,j}^n}{\Delta x^2} + O(\Delta x)^2 + \frac{u_{i,j+1}^n - 2u_{i,j}^n - u_{i,j-1}^n}{\Delta y^2} + O(\Delta y)^2) + F(x,y,t)$$

if $\Delta t \to 0$, then $O(\Delta x)^2 \to 0$ and $O(\Delta y)^2 \to 0$.So the above equation becomes:

$$\frac{u_{i,j,}^{n+1} - u_{i,j}^n}{\Delta t} = \nu(\frac{u_{i+1,j}^n - 2u_{i,j}^n - u_{i-1,j}^n}{\Delta x^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n - u_{i,j-1}^n}{\Delta y^2}) + F(x,y,t) \tag{2.1}$$

Let be $\delta_x^2 u_{i,j}^n = u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n$,$\delta_y^2 u_{i,j}^n = u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n$

and $F_{i,j}^n = F(i\Delta x, j\Delta y, n\Delta t)$.Then equation (2.1) becomes

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} = \frac{\nu}{\Delta x^2}\delta_x^2 u_{i,j}^n + \frac{\nu}{\Delta y^2}\delta_y^2 u_{i,j}^n + F_{i,j}^n \tag{2.2}$$

solving $u_{j,k}^{n+1}$ from equation (2.2)

$$u_{j,k}^{n+1} = u_{j,k} + \frac{\nu\Delta t}{\Delta x^2}\delta_x^2 u_{j,k}^n + \frac{\nu\Delta t}{\Delta y^2}\delta_y^2 u_{jk}^n + \Delta t F_{j,k}^n \tag{2.3}$$

Again let be $\gamma_x = \frac{\nu \Delta t}{\Delta x^2}$ and $\gamma_y = \frac{\nu \Delta t}{\Delta y^2}$, then equation (2.3) will be

$$u_{i,j}^{n+1} = u_{i,j}^n + (\gamma_x \delta_x^2 + \gamma_y \delta_y^2) u_{i,j}^n + \Delta t F_{i,j}^n \tag{2.4}$$

and this is the explicit scheme (forward time center space) for approximating partial differential equation (1.3).

The Dirichlet-Boundary condition (1.4) with $R = [0,1] \times [0,1]$ are approximated as

$$u_{0,j}^n = g(0, j\Delta y, n\Delta t); j = 0, 1, 2, ..., M_y, n \geq 0 \tag{2.5}$$

$$u_{M_x,j}^n = g(1, j\Delta y, n\Delta t); j = 0, 1, 2, ..., M_y, n \geq 0 \tag{2.6}$$

$$u_{i,0}^n = g(i\Delta x, 0, n\Delta t); i = 0, 1, 2, ..., M_x, n \geq 0 \tag{2.7}$$

$$u_{i,M_y}^n = g(i\Delta x, 1, n\Delta t); i = 0, 1, 2, ..., M_x, n \geq 0 \tag{2.8}$$

2.2 Descretizing Neumann Boundary Conditions

In the above problem we have solved the Dirichlet-Boundary condition. Next we consider what changes are necessary for Neumann boundary condition. Consider the following problem to treat Neumann boundary conditions for 2D parabolic PDE:

$$u_t = \nu(u_{xx} + u_{yy}), (x, y) \in (0, 1) \times (0, 1), t > 0 \tag{2.9}$$

$$u(0, y, t) = u(x, 0, t) = u(x, 1, t) = 0, x \in [0, 1], t \geq 0 \tag{2.10}$$

$$u_x(1, y, t) = -\pi e^{-5\nu\pi^2 t} \sin 2\pi y, y \in [0, 1], t \geq 0 \tag{2.11}$$

$$u(x, y, 0) = \sin \pi x \sin 2\pi y, (x, y) \in [0, 1] \times [0, 1] \tag{2.12}$$

as we have seen when $x = 1$, the Neumann boundary condition is

$$u_x(1, y, t) = -\pi e^{-5\nu\pi^2 t} \sin 2\pi y. \tag{2.13}$$

If we use a first order difference scheme to approximate the derivative in equation (2.13), we get

$$\frac{u^{n+1}_{M_x,k} - u^{n+1}_{M_x-1}, k}{\Delta x} = -\pi e^{-5\nu\pi^2(n+1)\Delta t} \sin 2\pi k\Delta y, k = 0, 1, 2, ..., M_y \tag{2.14}$$

or

$$u^{n+1}_{M_x,k} = u^{n+1}_{M_x-1,k} - \pi\Delta x e^{-5\pi\nu(n+1)\Delta t} sin 2\pi k\Delta y, k = 0, 1, 2, ..., M_y. \tag{2.15}$$

Then we can use difference scheme (2.4) along with the boundary conditions (2.5), (2.7), (2.8) and (2.15) to solve initial-boundary-value problem (2.9) to (2.12).

If we instead use the second order approximation of the Neumann boundary condition, we get

$$u^{n+1}_{M_x+1,k} = u^{n}_{M_x-1,k} - \pi\Delta x e^{-5\pi\nu(n+1)\Delta t} sin 2\pi k\Delta y; k = 0, 1, 2, ..., M_y. \tag{2.16}$$

as an approximation to Neumann boundary condition (2.13). Equation (2.4) be satisfied at $j = M_x$,

$$u^{n+1}_{M_x,k} = u^{n}_{M_x-1,k} + \nu\Delta t(\frac{1}{\Delta x^2}\delta^2_x + \frac{1}{\Delta y^2}\delta^2_y)u^{n}_{M_x,k}, k = 0, 1, 2, ..., M_y. \tag{2.17}$$

If we use equation (2.17) to eliminate the $u^{n}_{M_x+1,k}$ term from equation (2.16), thus

$$u^{n+1}_{M_x,k} = u^{n}_{M_x,k} + 2\gamma_x(u^{n}_{M_x-1,k} - u^{n}_{M_x,k}) + \gamma_y\delta^2_y u^{n}_{M_x,k} - 2\pi\nu\frac{\Delta t}{\Delta x}e^{-5\nu\pi^2 n\Delta t} sin 2\pi k\Delta y. \tag{2.18}$$

$k = 0, 1, 2, ..., M_y - 1$

Then, the difference scheme (2.4) along with the boundary conditions (2.5), (2.7), (2.8) and (2.18) gives us the second order accurate scheme for solving problem (2.9) to (2.12).

Chapter 3

Consistency, Stability and Convergence

Consistency, stability and convergence are the three fundamental properties that every finite difference approximation of a partial differential equation should satisfy in order be valid solution.
Consistency implies that the finite difference equation is a good approximation of the partial differential equation; stability implies that the solution of the difference is not too sensitive to small perturbations in the initial data; and convergence implies that the solution of the difference equation approaches the solution of the PDE as the computational mesh is refined.
These properties are often difficult to verify for realistic problems, but they can be explained and illustrated quite easily using difference schemes.

3.1 Consistency

A discrete approximation to a PDE is said to be consistent; if in the limit of the step size(s) going to zero, and the original PDE system is recovered (i.e the truncation error approaches zero).The local truncation error is the difference between the solution of the PDE and its finite difference approximation if the local truncation error tends to zero as the step size goes to zero. In these case the method is said to be consistent.

Definition 3.1.1. *Consider a differential equation* $Pu = 0$ *and a finite difference approximation of it* $P_{\triangle}U_i^j = 0$*. Let* $v(x, t)$ *be any smooth function, then the local descretization or local truncation error is:*

$$\tau_i^j = Pv(ih_x, jh_t) - P_{\triangle}v(ih_x, jh_t) \tag{3.1}$$

Remark 3.1.1. *P is a differential operator,* $Pu = v_t - \nu(v_{xx} + v_{yy}) - F(x, y, t)$ *is for the model two dimensional parabolic equation(1.3). Similarly,* $P_{\triangle}$ *is a finite* *difference operator; thus, the finite difference operator for the two dimensional operator is*

$$P_{\Delta}v = \frac{v_{i,j}^{n+1} - v_{i,j}^{n}}{\Delta t} - \nu\left(\frac{v_{i+1,j} - 2v_{i,j,} - v_{i-1,j}}{\Delta x^2} + \frac{v_{i,j+1} - 2v_{i,j} - v_{i,j-1}}{\Delta y^2}\right) - F(x,y,t)$$

Remark 3.1.2. *The function v is often regarded as the exact solution u of the partial differential equation. This is convenient but not necessary; thus, v may be any smooth function. This gives us a way of defining the local descretization error even when the solution of the partial differential has singularities.*

The local descretization error of the finite difference approximation for the 2D parabolic PDE is:

$$\tau_{i,j}^{n} = \frac{v_{i,j}^{n+1} - v_{i,j}^{n}}{\Delta t} - \nu\left(\frac{v_{i+1,j} - 2v_{i,j} - v_{i-1,j}}{\Delta x^2} + \frac{v_{i,j+1} - 2v_{i,j} - v_{i,j-1}}{\Delta y^2}\right) - F(x,y,t) \tag{3.2}$$

Using the Taylor Theorem

$$\tau_{j,k}^{n} = -\frac{\Delta t}{2}(v_{tt})_{i,j}^{n} + \frac{\nu \Delta x^2}{12}(v_{xxxx} + v_{yyyy})_{i,j}^{n} \tag{3.3}$$

The local error is often used in place of the local descretization error.

Definition 3.1.2. *The local error is the difference between solution of the partial differential equation and its finite difference approximation*
$u_{j,k}^{n+1} - U_{j,k}^{n+1}$ *assuming that no error were committed prior to time level* $j+1$.*The finite difference approximation of the 2D parabolic PDE is*

$$\frac{U_{i,j}^{n+1} - U_{j,k}^{n}}{\Delta t} - \nu\left(\frac{U_{i+1,j}^{n} - 2U_{i,j}^{n} + U_{i-1,j}^{n}}{\Delta x^2} - \left(\frac{U_{i,j+1}^{n} - 2U_{i,j}^{n} + U_{i,j-1}^{n}}{\Delta y^2}\right) - F(x,y,t) = 0$$

According to definition (3.1)we commit no errors prior to time level $n+1$; *thus* $U_{i,j}^{n} = u_{i,j}^{n}$ *and*

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} - \nu\left(\frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{\Delta x^2} - \left(\frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{\Delta y^2}\right) - F(x,y,t) = 0$$

Using (3.2) with v replaced by u

$$\tau_{i,j}^{n} = \frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} - \nu\left(\frac{u_{i+1,j} - 2u_{i,j} - u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} - u_{i,j-1}}{\Delta y^2}\right) - F(x,y,t)$$

Combining the previous two relations, we obtain

$$u_{i,j}^{n+1} - U_{i,j}^{n+1} = -\Delta t \tau_{i,j}^{n} \tag{3.4}$$

Thus, the local error is the negative of the product of the local descretization error and the time step. This relationship between local and local discretization error generally holds for explicit finite difference schemes.

Definition 3.1.3. *A finite difference scheme $P\Delta U_{j,k}^{n} = 0$ is consistent with a partial differential equation $Pu = 0$ if the local descretization error tends to zero as $\Delta t \to 0$, $\Delta x \to 0$, and $\Delta y \to 0$.*

Equation (3.3) is the local descretization error for the equation (1.3)

$$\tau_{j,k}^{n} = -\frac{\Delta t}{2}(v_{tt})_{j,k}^{n} + \frac{\nu \Delta x^2}{12}(v_{xxxx} + v_{yyyy})_{j,k}^{n}$$

The left hand side of this equation will tends to zero as $\Delta t \to 0$, $\Delta x \to 0$, and $\Delta y \to 0$. This means that the numerical scheme (2.4) tends to the exact equation at point (x_i,y_j) and time level t_n and therefore this approximation is consistent.

3.2 Stability

One of the central question that arise by numerical treatment of the problem is stability of the numerical scheme. An algorithm for solving a PDE is said to be stable, if the numerical solution at a fixed time remains bounded as the step size goes to zero, so the perturbations in the form of rounding error does not increase in time.

Definition 3.2.1. *A finite difference scheme for homogenous initial value problem is stable if there exists a constant c independent of the mesh spacing and initial data such that*
$\parallel U^n \parallel \leq c \parallel U^0 \parallel$ *as* $n \to \infty$, $\Delta x \to 0$, $\Delta t \to 0$ *and* $n\Delta t \leq T$.

Definition 3.2.2. *If we define the error to be the difference between the computed solutions and the exact solution of the discrete approximation, then the scheme is stable if the error remains uniformly bounded for successive iterations.*

For two dimensional difference scheme the most difficult step to obtain convergence via the Lax Theorem that we discuss the next section is proving stability. For this reason, we now proceed to study stability for two dimensional scheme.

As a first step, let us consider the difference scheme (2.4) as a scheme for an initial-value problem. We must realised that the stability analysis provided by considering the difference scheme(2.4) for an initial-value problem will at least provided us the necessary condition for convergence for the initial-boundary-value problems.

3.2.1 Stability of Initial-value Scheme

For proving stability of two dimensional difference scheme, we shall use the discrete Fourier transform as a tool. When we consider difference scheme having spatial dimension, we must consider the two dimensional discrete Fourier transform. For a sequence $u_{j,k}, -\infty < i, j < \infty$, we define

$$\mathcal{F}(u)(\xi,\eta) = \hat{u}(\xi,\eta) = \frac{1}{2\pi}\sum_{j,k=-\infty}^{\infty} e^{-ij\xi-ik\eta}u_{j,k} \tag{3.5}$$

It is very important to understand the following remarks:

Remark 3.2.1. *Parsevals's Identity holds for the two dimensional discrete Fourier transform, i.e* $\| u \|_2 = \| \hat{u} \|_2$*, where the norms are the two dimensional* l_2 *and* $L_2([-\pi,\pi]\times[-\pi,\pi])$ *norms, respectively.*

Remark 3.2.2. $\mathcal{F}(S_{x\pm}u) = e^{\pm i\xi}\mathcal{F}(U)$ *and* $\mathcal{F}(S_{y\pm}u) = e^{\pm i\eta}\mathcal{F}(U)$*, where* $S_{x\pm}$ *and* $S_{y\pm}$ *are the obvious two dimensional extension of the shift operators.*

3.2.2 The Stability of the Difference Scheme 2D Parabolic Differential Equation

To analyze the stability of the difference scheme (2.4) as a scheme for solving an initial-value problem, we take the discrete Fourier transform of the difference scheme (2.4) to get

$$\hat{u}^{n+1}(\xi,\eta) = \hat{u}^n(\xi,\eta)+\gamma_x(e^{-i\xi}\hat{u}^n(\xi,\eta)-2\hat{u}^n(\xi,\eta)+e^{i\xi}\hat{u}(\xi,\eta))+\gamma_y(e^{-i\eta}\hat{u}^n(\xi,\eta)-2\hat{u}^n(\xi,\eta)+e^{i\eta}\hat{u}(\xi,\eta)) \tag{3.6}$$

$$\hat{u}^{n+1}(\xi,\eta) = [1+\gamma_x(e^{-i\xi}-2+e^{i\xi})]\hat{u}^n(\xi,\eta) \tag{3.7}$$

As before we have an expression

$$\hat{u}^{n+1} = \rho(\xi,\eta)\hat{u}^n \tag{3.8}$$

where $\rho(\xi,\eta)$ is $[1+\gamma_x(e^{-i\xi}-2+e^{i\xi})]$ and we have not written the (ξ,η) in the $\hat{u}$ function and we will do the same for ρ. In this case the ρ term is the multiplier between steps n and $n+1$. Applying equation (3.8) times yields

$$\hat{u}^{n+1} = \rho^{n+1}\hat{u}^0 \tag{3.9}$$

if

$$\mid \rho(\xi,\eta) \mid \leq 1 + C\Delta t, \tag{3.10}$$

then

$$\mid \rho^{n+1} \mid \leq e^{C(n+1)\Delta t}$$

and

$$\| \hat{u}^{n+1} \|_2 \leq e^{C(n+1)\Delta t} \| \hat{u}^0 \|_2 \,. \tag{3.11}$$

Then equation (1.3),along with Parseval's Identity, yields

$$\| u^{n+1} \|_{2,\Delta x,\Delta y} \leq e^{C(n+1)\Delta t} \| u^n \|_{2,\Delta x,\Delta y} \tag{3.12}$$

Thus we see that all we have to do to obtain stability is to show that the symbol ρ satisfies (3.10). We fined where the function $\mid \rho \mid$ has its maximum values by finding the maximum and minimum values of ρ. It is clear that ρ can be written as

$$\rho = 1 + 2\gamma_x(\cos\xi - 1) + 2\gamma_y(\cos\eta - 1) = 1 - 4\gamma_x \sin^2\frac{\xi}{2} - 4\gamma_y \sin^2\frac{\eta}{2} \tag{3.13}$$

Differentiating equation (3.13) with respect to ξ and η, setting these derivatives equal to zero and solving for ξ and η, we fined the critical points for this function occur at all combinations of $\xi = -\pi, 0, \pi$ and $\eta = -\pi, 0, \pi$.

It is easy to see that the maximum of $\rho = 1$ occurs at $(\xi,\eta) = (0,0)$ and the minimum of $\rho = 1 - 4\gamma_x - 4\gamma_y$ occures at $(\xi,\eta) = (\pi,\pi)$.

Then requiring that ρ satisfies $\rho \geq -1$ yields the stability condition

$$\gamma_x + \gamma_y \leq \frac{1}{2} \tag{3.14}$$

Hence, we see that difference scheme (2.4) is conditional stable.

3.3 Convergence

Convergence is the most basic property that a scheme must have in order to be valid.

Definition 3.3.1. *Convergence means that the solution to a finite difference equation approaches the true solution of the partial differential equation as both grid interval and time step sizes are reduced.*

Even if it is fundamental property, it is difficult to verify directly. But Lax Equivalent Theorem help us to show convergency of a finite difference scheme from its consistency and stability properties.

Theorem 3.3.1. *The Lax-Richmyer Equivalent Theorem. A consistent finite difference scheme for a partial differential equation for which the initial value problem is well posed is convergent if only if it is stable.*

Here well-posedness of initial value problem means:

$$\parallel u(t,x) \parallel_2 \leq C_T \parallel u(0,x) \parallel_2, t\epsilon[0,T]$$

.

Proof: We will prove that a consistent and stable difference is convergent. A general formula for the evolution of the finite difference solution is the following:

$$U^{i+1} = AU^i + b^i \tag{3.15}$$

where A is the evolution matrix, and b is a vector containing forcing terms and the effects of boundary conditions. The vector U^{i+1} holds the vector of solution values at time $i+1$.The truncation error at a specific time level can be obtained by applying the above matrix operation to the vector of exact solution values:

$$u^{i+1} = Au^i + b^i + \tau^i \Delta t \tag{3.16}$$

Where τ is the vector of truncation error at time level i. Subtracting equation (3.16) from equation (3.15), then we get an equation for the error, namely:

$$e^{i+1} = Ae^i + \tau^i \Delta t. \tag{3.17}$$

Where $e^{i+1} = U^{i+1} - u^{i+1}$ is the total error at time $t_{i+1} = (i+1)\Delta t$. Equation (3.17) shows that the total error at time level $i+1$ is made up of two parts. The first one is the evolution of the error inherited from the previous time level, the first term on the right hand side of equation (3.17), and the second part is the truncation error committed at present time level. Since, this expression applies to a generic time level, the same expression holds for e^i :

$$e^i = Ae^{i-1} + \tau^{i-1}\Delta t. \tag{3.18}$$

Where we have assumed that the matrix A does not change with time to simplify the discussion (this tantamount to assuming constant coefficients for the partial differential equation). By repeated application of this argument we get:

$$\begin{aligned} e^{i+1} &= A^2e^{i-1} + (A\tau^{i-1} + \tau^i)\Delta t \\ &= A^3e^{i-2} + (A^2\tau^{i-2} + A\tau^{i-1} + \tau^i)\Delta t \\ ... &= A^{i+1}e^0 + (A^i\tau^0 + A^{i-1}\tau + ... + A\tau^{i-1} + \tau^i)\Delta t \end{aligned}$$

This shows that the error growth depends on the truncation error at all time levels, and on the discretization through the matrix A. We can use the triangle inequality to get a bounded on the norm of the error. Thus,

$$\parallel e^{i+1} \parallel \leq \parallel A^{i+1} \parallel \parallel e^0 \parallel + (\parallel A^{i+1} \parallel \parallel \tau^0 \parallel + \parallel A^i \parallel \parallel \tau^1 \parallel + ... + \parallel A \parallel \parallel \tau^{i-1} \parallel + \parallel \tau^i \parallel)\Delta t \tag{3.19}$$

In order to make further progress we assume that the norm of the truncation error at any time is bounded by a constant ϵ such that

$$\epsilon = \max_{0 \leq m \leq i+1} (\parallel \tau^m \parallel) \tag{3.20}$$

The right hand side inequality (3.19) can be bounded by

$$\parallel e^i \parallel \leq \parallel A^i \parallel \parallel e^0 \parallel + (\sum_{m=0}^{i} \parallel A^m \parallel) \tag{3.21}$$

The initial error and the subsequent truncation error are thus modulated by the evolution of matrices A^m. In order to prevent the unbounded growth of the error norm as $i \to \infty$, we need to put a limit on the norm of these matrices. This is in effect the stability property needed for convergence:

$$\parallel A^m \parallel \leq C = \max_{0 \leq m \leq i} (\parallel A^m \parallel)$$

Where C is a constant independent of $i, \Delta t$ and Δx. The sum in bracket can be bounded by the factor iC;the final expression becomes:

$$\parallel e^i \parallel \leq C(\parallel e^0 \parallel) + t_i\epsilon$$

Where $t_i = i\Delta t$ is the final integration time. When $\Delta x \to 0$, the initial error e^i can be made as small as desired. Furthermore, by consistency, the truncation error $\epsilon \to 0$ when $\Delta t, \Delta x \to 0$. The global error is hence guaranteed to go to zero as the computational grid is refined, and the scheme is convergent.

Remark 3.3.1. *The Lax Equivalence Theorem expresses relationship among consistency, stability and convergency.*

Therefor , the explicit scheme of 2D parabolic partial differential equation i.e finite difference scheme (2.4) is convergent by Lax Equivalent Theorem.

Chapter 4

Implementation

Matlab implementation of the explicit finite scheme of 2D parabolic partial differential equation is presented and demonstrated in this section.

4.1 Test Problem

Consider the model problem:

$$u_t = \nu(u_{xx} + u_{yy}) + F(x, y, t), (x, t) \in R, t > 0 \tag{4.1}$$
$$u(x, y, t) = g(x, y, t) on \partial R, t > 0 \tag{4.2}$$
$$u(x, y, t) = f(x, y)(x, y) \in \bar{R} \tag{4.3}$$

with the boundary condition equation (4.2)

$$u^n_{0,k} = g(0, k\Delta y, \Delta t), k = 0, 1, 2, ..., M_y, n \geq 0 \tag{4.4}$$

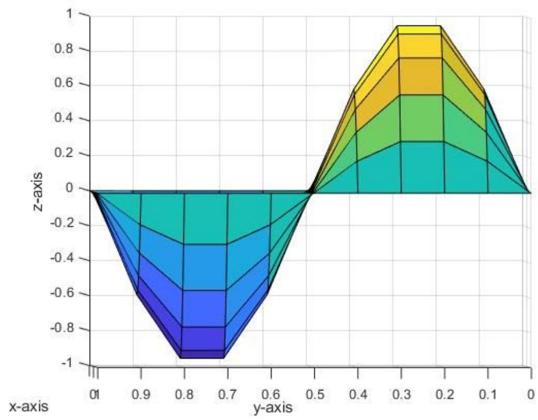
$$u^n_{M_x,k} = g(1, k\Delta y, n\Delta t), k = 0, 1, 2, ..., M_y, n \geq 0 \tag{4.5}$$

$$u^n_{j,0} = g(j\Delta x, 0, n\Delta t), j = 0, 1, 2, ..., M_x, n \geq 0 \tag{4.6}$$

$$u^n_{j,M_y} = g(j\Delta x, 1, n\Delta t), j = 0, 1, 2, ..., M_x, n \geq 0 \tag{4.7}$$

Use the explicit scheme (2.4) to simulate the following example.

Examplrs 4.1.1. *Test the result of explicit finite difference scheme* (2.4) *by using the difference scheme* $(4.4) - (4.7)$ *to approximate the solution of initial-value problem* $(4.1) - (4.3)$ *where* $R = (0, 1) \times (0, 1)$,$F(x, t) = sintsin2\pi xsin4\pi y$, $g = 0$, $f = 0$, *and* $\mu = 0.5$. *Use* $M_x = M_y = 20$ *and* $\Delta t = 0.0005$. *Give the result at time* $t = 1.5$, $t = 4.5$ *and* $t = 6.25$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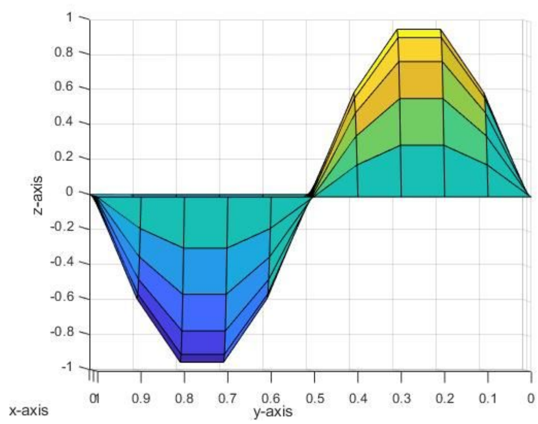
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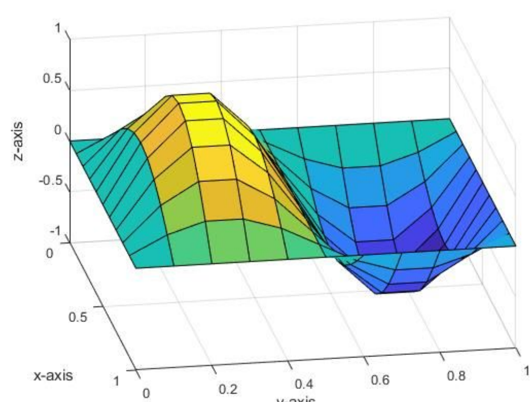
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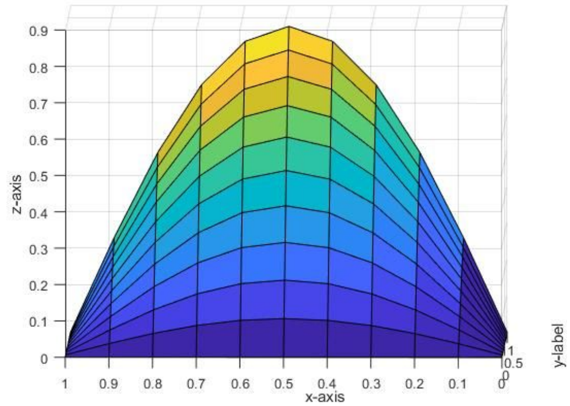
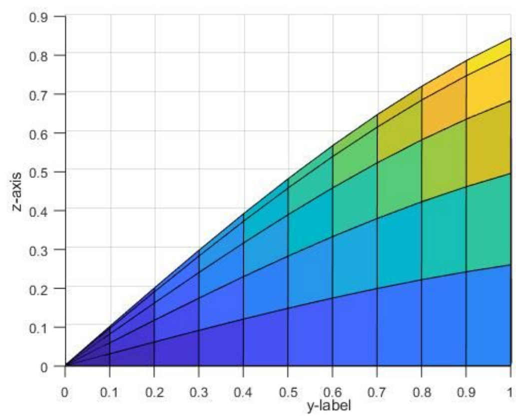
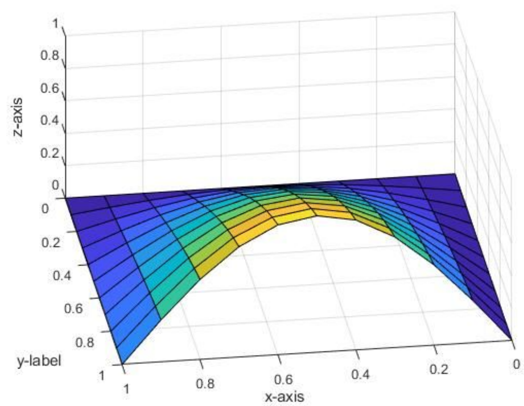




When we running the equation with the above parameter :

Examplrs 4.1.2. *Test the result of explicit finite difference scheme* (2.4) *by using the difference scheme* $(4.4)-(4.7)$ *to approximate the solution of initial-value problem* $(4.1)-(4.3)$ *where* $R = (0,1) \times (0,1)$*,* $F = 0$*,* $g(0, y, t) = sintsin\pi y$*,* $g(x, 0, t)sintsin\pi x$*,* $g(1, y, t) = g(x, 1, t) = 0$*,*$f = 0$ *and* $\mu = 1.0$ *by using* $M_x = M_y = 20$ *and* $\delta t = 0.001$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0.2
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x-axis

Conclusion

Using the forward time center spacing (FTCS) approximation of 2D parabolic partial differential equation, we obtained the explicit Scheme (2.4) and the stability analysis of this scheme showed that $\gamma_x + \gamma_y \leq \frac{1}{2}$ the scheme is stable and converge to the numerical solution of the initial boundary value problem. This means it is stable if

$$\Delta t \leq \frac{[O(x^2)][O(y^2)]}{2\nu[O(x^2) + O(y^2)]}$$

on the other hand, if $\gamma_x + \gamma_y > \frac{1}{2}$ or

$$\Delta t > \frac{[O(x^2)][O(y^2)]}{2\nu[O(x^2) + O(y^2)]}$$

the scheme is unstable.

Bibliography

[1] Randalls J.LeVeque,Finite Difference Method for Ordinary and Partial Differential Equations,Philadelphia,Washington,2007.

[2] J.W Thomas.,Numerical Partial Equations: Finite Difference Methods, Springer-Verlag, New York,1995.

[3] K.W.Marton and David Mayers,Numerical Solution of Partial Differential Equations, Cambridge University Press, New York, second edition, 2005

[4] Mark H.Holmes, Intrduction to Numerical Methods in Differential Equation, Rensselar Polytechnic Institute, Springer science+Bussiness, New York, 2007

[5] Manolis Georgoulis, Computational Method for Partial Differential Equation, Department of Mathematics University of Leicester, January 2009