**ADDIS ABABA UNIVERSITY**

**ADDIS ABABA INSTITUTE OF TECHNOLOGY**

**SCHOOL OF CIVIL AND ENVIRONMENTAL ENGINEERING**


# Comparison of the Approximate and Exact Second Order Elastic Analysis Methods in Relation with Eurocode 8's interstorey drift sensitivity Coefficient (θ) limit

**A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Science in Structural Engineering**


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**Addis Ababa, Ethiopia**


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**ABSTRACT**

Stability of structures is a major issue that must be checked at the first stage of the design. It is the ability of a structure to keep its equilibrium position after being disturbed by external actions. One of the parameters used to check the stability of structures in code of practices is a second order effect. The Eurocode families, Eurocode-2 and Eurocode-8 set the criteria for neglecting second order effect by considering different parameters into account. In the EC8 approach to the second order effect, inerstory drift sensitivity coefficient θ is the most important quantity governing the analysis. The code allows to neglect the second order effect if θ is less than 0.1 and to use an amplification factor (1/1- θ) if θ does not exceed 0.2. It restricts the value of θ not to exceed 0.3.

The second order effect is the phenomenon of additional force and moments produced by the vertical load acting on the laterally deformed structure. Second order elastic analysis can be evaluated in one of the two methods, **approximate** and **"exact"** second order elastic analysis method. The accuracy limit of approximate second order elastic analysis method is illustrated by comparing the results of the two methods in relation to the limit of interstory drift sensitivity coefficient θ.

The procedure is done in two cases using simple elastic portal frames. In case I axial load is applied only on the weak column of the frame, and in case II the load is applied on both strong and weak columns. In both cases five portal frames with a different relative stiffness value of columns are used to study the effect of the relative stiffness of columns on the bracing action of strong column. These frames clearly show cases where the validity of the basic assumption behind the approximate methods becomes uncertain.

The results of the investigation reveal that, the approximate method gives relatively good results up to the value of θ=0.3 in the case of supported sway column. After this point, the P-δ effect, which is neglected by the approximate method, becomes very high and the supported sway column deflects in an elastic failure mode which is very different from the basic assumption of approximate method. As a result, the approximate method becomes less accurate for θ values greater than 0.3 compared to the "exact" one.

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**My Families and My Advisor Dr.-Ing. Adil Zekaria**

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# CHAPTER 1 INTRODUCTION

## 1.1 Background

In the design process of any structure, a check on the stability is the major step which must be ensured by the reserve of stiffness against the expected actions on the structure. In general terms, structural stability can be defined as*; **the power to recover equilibrium.*** A structure is said to be stable at an equilibrium position if it returns to its original position after being disturbed by an external action. On the other hand, instability is loss of equilibrium of the deformed structure.

One of the parameters used to check the stability of structures is a second order effect. Most compression members are subjected to simultaneous flexure, caused by transverse loads or by end moments owing to continuity in addition to the axial load. When an element is subjected to an axial load combined with a moment, it will deflect. This deflection will increase the moment at any section in the element by an amount equal to the axial force multiplied by the deflection at that point. This extra moment will cause the resistance of the element to be reduced below that calculated ignoring the deflections. This phenomenon is called second order effect or P-delta effect.

P-delta effect is sub divided further as, p-Δ and p-δ effect. P-Δ effect is caused by the lateral deflections or sidesway of the beam-column joints from their original undeflected position; whereas, p-δ effect is caused by the deflection of the axis of the individual bent column away from the chord joining the ends of the columns.

Failure of structures due to structural instability is a total story phenomenon. A single individual member cannot fail by instability failure without all the members in the same story also fail by instability. To insure local and global stability of structure, both p-δ and p-Δ effect must be checked in the design of structures.

Code of practices aim to eliminate failure of structures due to structural instability by reducing the second order effect, thereby allowing more simplified method if the second order effect is below some specified limits.

From the Eurocode families, Eurocode -2 (EN 1992-1-1) and Euro code-8 (EN 1998-1:2004) states criteria for neglecting second order effect by considering different parameters into account.

Eurocode -2 section 5.8.2 (6) states the general principle that second order effects may be ignored if they are less than 10% of the corresponding first- order effects. In section 5.8.3 (1) a simplified criteria for an isolated members is given so that a second order effect can be ignored when the slenderness of the member is below certain value, $\lambda_{\text{lim}}$. For the global effect in a building, section 5.8.3.3(1) of the code gives an alternative method to ignore second order effect.

Eurocode -8 in its turn in section 4.4.2.2 (2), (3), (4) state that, Second-order effects may be neglected, if the following conditions is fulfilled in all stories.

$$\theta = \frac{P_{tot}.d_r}{V_{tot}.h} \leq 0.1 \quad \text{.............................................} \quad (1.1)$$

Where:

$\theta$ is the inerstorey drift sensitivity coefficient;

$P_{tot}$ is the total gravity load at and above the storey considered in the seismic design situation.

$d_r$ is the design interstorey drift, evaluated as the difference of the average lateral displacement d, at the top and bottom of the storey under consideration and calculated in accordance with section 4.3.4 of the code;

$V_{tot}$ is the total seismic storey shear; and

h is the interstory height

If the value of $\theta$ does not exceed 0.2 at any storey, the code allows second order effects to be taken into account approximately without a second order analysis, by multiplying all first order action effects due to the horizontal component of the seismic action by 1/(1-θ). The code restricts the value of θ not to exceed 0.3.

Different structural analysis method uses different approach to consider second order effect caused by material and/or geometric non-linearity. Among them, the second order elastic analysis includes geometric non-linearity only and assumes elastic response of structures. And it is further divided as approximate and "exact" second order elastic analysis method.

This study is aimed to investigate the accuracy limit of approximate second order elastic analysis method by comparing it with the “exact” one. Through the comparison of the two methods the accuracy limit of the approximate method is presented in relation to the limit of interstorey drift sensitivity coefficient θ set by Eurocode-8.

## 1.2 Objective

From the various second order analysis methods, second order elastic analysis is the most widely used one. Consideration of the frame geometric effect (p-Δ effect) and/or member geometric effect (p-δ effect) is used to subdivide the method as approximate and “exact”.

The main objective of this study is to show the accuracy limit of approximate second order analysis method compared to the “exact” one and to see the relation of this accuracy limit to the limit of interstorey drift sensitivity coefficient θ set by Eurocode-8.

## 1.3 Scope of the research

- In this study, second order elastic analysis method which considers geometric nonlinearity only is used.
- Elastic response of the structure is assumed to avoid the greater complexity of an analysis that includes material nonlinearity.
- In comparison of “Exact” and approximate second order analysis method, the main difference in the two methods comes from the basic assumption involved in the derivation of the approximate method. In this study only simple two dimensional portal frame models which show clearly the concept of the basic assumption is used.

# CHAPTER 2 LITERATURE REVIEW

## 2.1 Stability of Structures

Failures of many engineering structures fall into one of two simple categories: material failure and structural instability (stability failure). Material failure can usually be adequately predicted by analyzing the structure on the basis of equilibrium conditions or equations of motion that are written for the initial, undeformed configuration of the structure. By contrast, the prediction of failures due to structural instability requires equations of equilibrium or motion to be formulated on the basis of the deformed configuration of the structure. Since the deformed configuration is not known in advance but depends on the deflections to be solved, the problem is in principle nonlinear, although frequently it can be linearized in order to facilitate analysis.

Failure of structures which are caused by material failure is governed by the value of the material strength or yield limit, which is independent of structural geometry and size. While, Failures of structures due to structural instability are independent of the material strength or yield limit; it depends on structural geometry and size, especially slenderness, which are primarily governed by the stiffness of the material.[5]

Structural instability or stability failure can be subdivided in to two types of failures that may be encountered in a frame analysis. One is known as "bifurcation of equilibrium," or buckling, and occurs when the applied axial load reaches the critical buckling load. The other type is referred to as "instability through disturbance of equilibrium" and occurs because equilibrium between external and internal forces cannot be achieved due to such things as imperfections and reduction in stiffness. This type of instability occurs for an axial load smaller than the bifurcation load of the member. [4]

To illustrate the concept of stability, consider an ideal column without geometrical or material imperfections. Furthermore, assume that there are no lateral loads, and that the column remains elastic regardless of the force magnitude. If the axial force is slowly increased, the column will undergo axial deformation, and no lateral displacements will occur. However, when the applied forces reach a certain magnitude called the critical load ($P_{cr}$), significant lateral displacements may be observed as shown in fig.2.1.

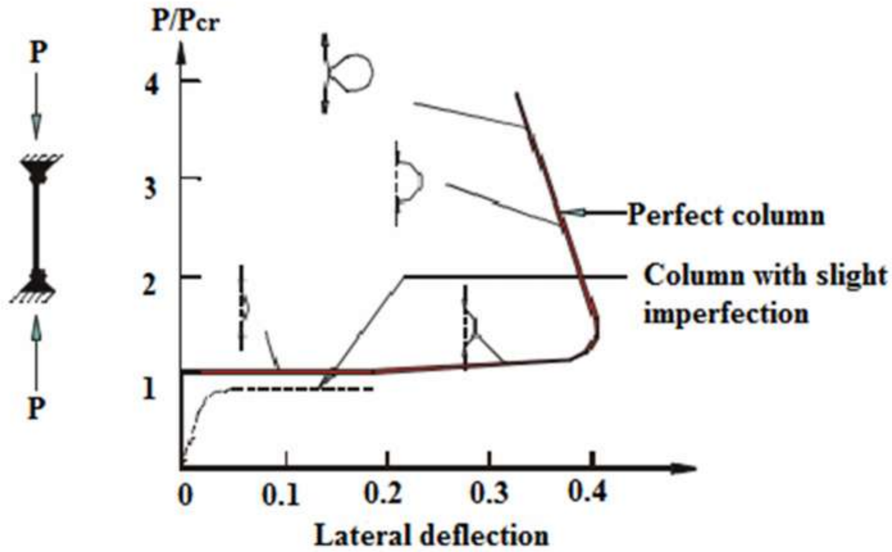



**Figure 2.1: Load – Deflection behavior for an ideal elastic column**

It is important to notice that when the magnitude of axial force exceeds *Pcr*, there are two possible paths of equilibrium: one along the original path, with no lateral displacements, and one with lateral displacements. However, equilibrium along the original path is not stable, and any slight disturbance can cause a change in the equilibrium position and significant lateral displacements. The force *Pcr* is called the bifurcation load or first critical load of the system and the type of stability is referred as "bifurcation of equilibrium". For this ideal column reaching the bifurcation point does not imply failure simply because it was assumed that it will remain elastic regardless of the deflection magnitude.

However, in a real column, such large deformations can cause yielding, stiffness reduction, and failure. In a structural system, buckling of critical members and the corresponding large lateral displacements can cause a major redistribution of forces and overall collapse of the system. It is important to note that the bifurcation point exists only for perfectly symmetric members (ideal embers) under pure axial forces.

If the same ideal column is simultaneously subjected to lateral loads, or if asymmetry of material or geometric imperfections are present, as they are in any real system, lateral displacements would be observed from very early stages of loading.

When a frame under constant gravity load is subjected to slowly increasing lateral loads, the lateral displacement of the system slowly increases, until it reaches a stage that in order to maintain static equilibrium, a reduction in the gravity or lateral loads is necessary. This corresponds to the region with negative slope on the force-displacement diagram (Fig.2.2). If the loads are not reduced, the system will fail by "instability through disturbance of equilibrium".

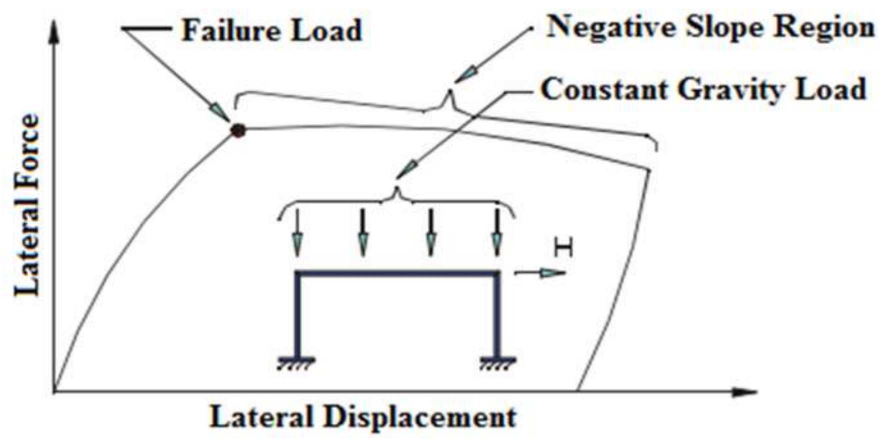



**Figure 2.2: Load – Deflection relationship for a frame subjected to combined gravity and lateral loads**

The simplest way to minimize lateral stability problems is to limit the expected lateral displacement or drift of the structure. It is believed that, in most cases, observance of proper drift limitations will provide the necessary safeguard against the overall lateral stability failure of the structure. [11]

### 2.1.1 Factors Affecting Lateral Stability

In general, the magnitude of the gravity loads and factors that increase lateral displacement, affects lateral stability of the structure. Chief among these factors are rotation at the base of the structure, any significant rotation at any level above the base (as that caused by formation of plastic hinges in the columns or walls), and significant asymmetry or torsion in the structure. [11]

## 2.2 Sway - non-sway /braced - unbraced structures

In the analysis of structures, there are two modes of deflection which are normally possible: sidesway of the whole structure, and deflection of a single column without sidesway. These two possibilities are shown schematically in Fig. 2.3.

The two modes give very different forms of bending moment diagram in the columns. In the sidesway mode it is important to note that all the columns within the storey are subjected to the same deflection.

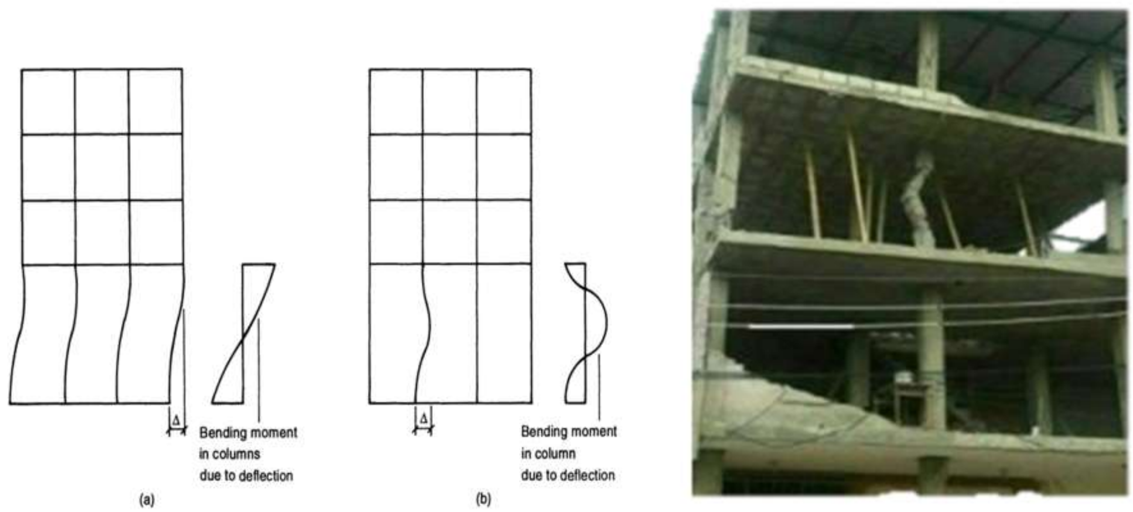



**Figure 2.3: Modes of deflection of columns in a structure; (a) sidesway of whole structure; (b) deflection of a single column**

In order to establish the deflection modes which should be considered, Eurocode 2 (EN 1992-1-1) uses the following classification system. Structures are classified as:

• Sway or non-sway and

• Braced or unbraced.

A sway structure is one where sidesway (global second-order effects) is likely to be significant as in Fig. 2.3 a, while in non-sway structures; sidesway is unlikely to be significant. 'Significance' is defined in Eurocode context as a situation where the lateral displacement of the ends of the columns increase the critical bending moments by more than 10% above the moment calculated ignoring the displacements.

A braced structure is one which contains bracing elements. These are vertical elements (usually walls) which are so stiff relative to the other vertical elements that they may be assumed to attract all horizontal forces. Unbraced structure is the vice versa. [6]

However, the terms sway – non-sway has been omitted in the final draft of Eurocode 2, after many comments for or against. The words in themselves are misleading, since all structures are more or less “sway”. These terms are now replaced by unbraced – braced in the code. No simple limits are given in EN 1992-1-1 for sway frames, but it seems reasonable to assume that any structure that is classifiable as a sway frame will need to be designed for the effects of deformations. [10]

Having classified the structure, the following can be said;

- Non-sway structures:

Only the deformation of individual columns of the type illustrated in Fig. 2.3b need be considered, by definition, sidesway will be insignificant.

- Sway structures:

– Braced structures. Braced structures may normally be assumed to be non-sway unless the bracing elements are relatively flexible. If they are, then the bracing elements or structure should be analysed for the effects of sidesway but the braced elements within the structure may be assumed to deform, as illustrated in Fig. 2.3b, and sidesway may be ignored.

– Unbraced structures. These structures should first be designed to take account of sidesway of the whole structure as illustrated in Fig. 2.3a, and then each column in turn should be analyzed for the possibility of its deformation, as illustrated in Fig. 2.3b. [6]

### 2.2.1 Sway Moments Ms vs. Non-sway Moments Mns

Two different types of moments occur in frames:

**i)** Nonsway moments $\mathrm{M_{ns}}$, are the factored end moments on a column due to loads that cause no appreciable sidesway, as computed by a first-order elastic frame analysis. These moments result from gravity loads.

Nonsway moments $\mathrm{M_{ns}}$, are magnified when the column deflects by an amount δ relative to its original straight axis as shown in fig 2.3b, using $(1/1 - P/P_{cr})$ as stated in Eurocode-2 section 5.8.7.3, equation 5.30.

**ii)** Sway moments $M_s$ , are the factored end moments on a column due to loads which cause appreciable sidesway, calculated by a first-order elastic frame analysis. These moments result from either lateral loads or large unsymmetrical gravity loads, or gravity loads on highly unsymmetrical frames.

Sway moments, Ms, are magnified by the P-Δ moments resulting from the sway deflections Δ at joints in the frame, as shown in fig 2.3a, using (1/1-θ) as stated in Eurocode-8 section 4.4.2.2 (3). [12]

## 2.3 Behavior of Elastic Sway Frames

Figure 2.4 (a) shows a column, which can be any column in a frame, subjected to internal forces acting at the ends. The joints of the column are deflected laterally from their original undeflected location by an amount equal to Δ and the axis of this bent column is deflected from the line joining the ends of the column by an amount equal to δ. With the application of the axial load, these deflections cause secondary effects in the column, which is termed as second order effects or p-delta effects.

The straight line joining the two ends of the column will form an angle equal to $\Delta/L$ with respect to the vertical (fig 2.4 (c)). The axial load P may be replaced with its horizontal and inclined components (fig 2.4 (b)). The first of these is equal to $P\Delta/L$ the second one acts parallel to the line joining the ends of the column and, assuming small deformations, is equal to P . Consequently, the total shear acting at the ends of the column is the sum of the original end-shear acting at the end of the column V resisting the external lateral loads and the P-Δ shear $(P\Delta/L)$ resulting from the moments induced by P acting through the deflection Δ.

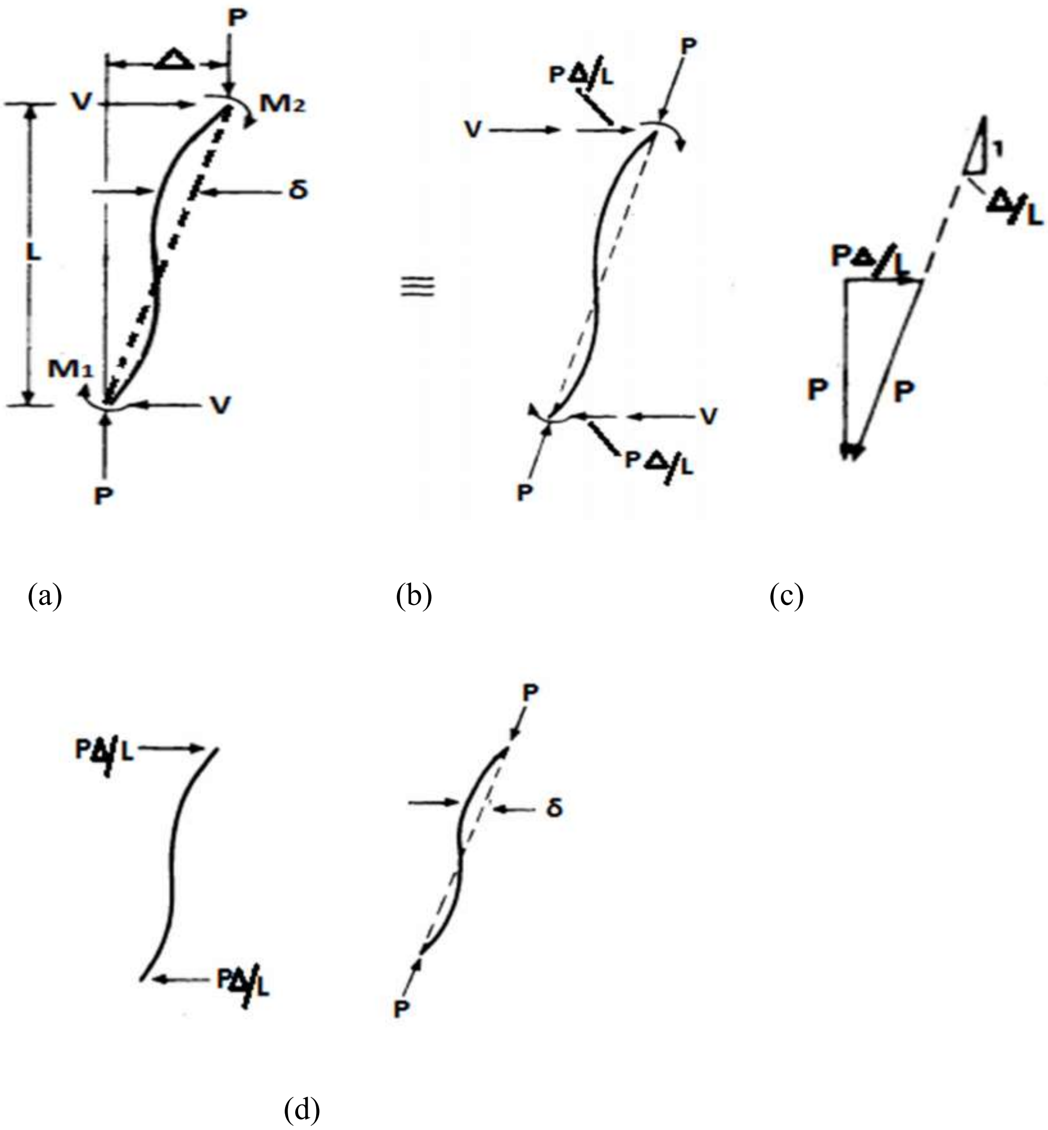



**Figure 2.4: Geometric effects due to axial loads**

Accordingly, the second order effects or geometric effects due to the axial load can be decomposed into two types shown in fig 2.4(d). First are effects due to the P-Δ shear, which are termed as the **P-Δ effects**. The P-Δ shear produce an overturning moment in the direction of lateral displacement, and therefore it tends to increase the lateral displacement. The second type of geometric effects occurs due to secondary moments produced by P times the displacements from the chord line δ; which is termed as **P-δ effect** and causes a redistribution of the total shears (lateral load shear plus P-Δ shears) and column end-moments according to the changing stiffness of individual columns. [1]

### 2.3.1 Interactions between Elements of sway Frame Systems

Sidesway mode is a total story phenomenon; a single column cannot fail by sidesway without all the columns in the same story also fail in a sway mode. So that it is important to know the horizontal and vertical interaction between frame elements due to p-delta effects.

#### 2.3.1.1 Horizontal interaction

To explain the horizontal interaction due to P-delta effects see fig 2.5 below. The columns in the frame are loaded with axial load such that the buckling load of the two exterior columns is not reached while the interior columns reach their independent buckling load.

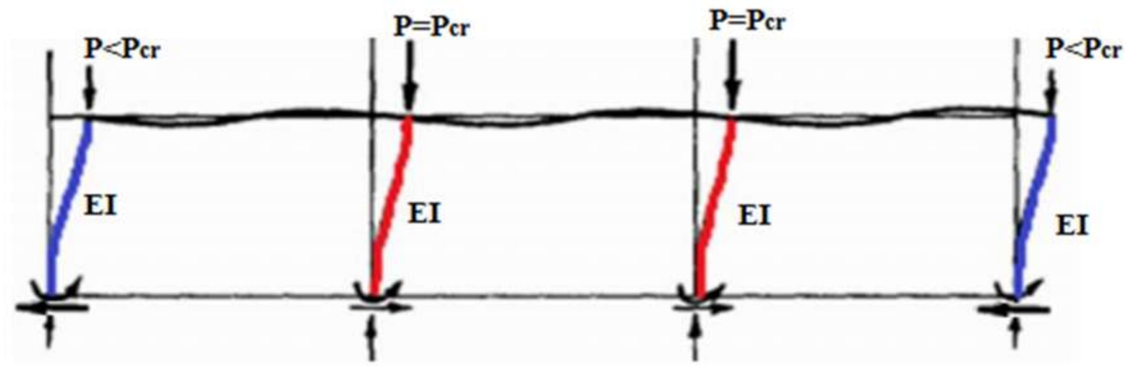



**Figure 2.5: Horizontal interaction of frame element**

When the system undergoes a lateral deflection it will not fail because the interior columns reach their independent buckling load. Rather shear resistance will be developed in the exterior columns which counteract the sidesway tendency. In this frame the exterior columns “brace” the interior columns. Only when the gravity loading is increased enough to offset the stabilizing effect of the lightly loaded exterior columns will sideways buckling occur (the frame fails in a sway mode).[3]

From this, columns in a sway frame can be categorized as: **“supporting sway columns”** and **“supported sway columns”**. Supporting sway columns are those columns which contribute to resisting lateral loads and to bracing other columns, while supported sway columns are columns which need lateral support from the frame in order not to fail sidesways. [1]

A quantitative description of frame sidesway buckling is presented in fig 2.6. The column sizes were chosen so that both columns buckle at the individual loads shown and in each case the effective length is 2. The frame will not fail by sidesway until a total base moment of PΔ is produced. For the given situation, the total load on the frame is 600 and the PΔ moments total 600Δ.

Suppose that the column on the right supports a load of 300, a reduction of 200 from its individual unbraced strength. The loading configuration is shown in fig2.6 (b).This column will not fail by sidesway buckling until a moment of 500Δ is reached at the base, so the column has the capability to sustain an additional moment of 200Δ from another source. In other words, the right column has a reserve of strength which can be utilized to provide a bracing force to prevent sidesway buckling of the left-hand column. Since the column on the left is now braced by the right column, it can support an additional load of 200 (total load=300). This is 200 greater than the individual unbraced column capacity. Note that the total frame load at buckling is still 600, the same as in fig 2.6 (a): however, the load distribution on the individual column is different.

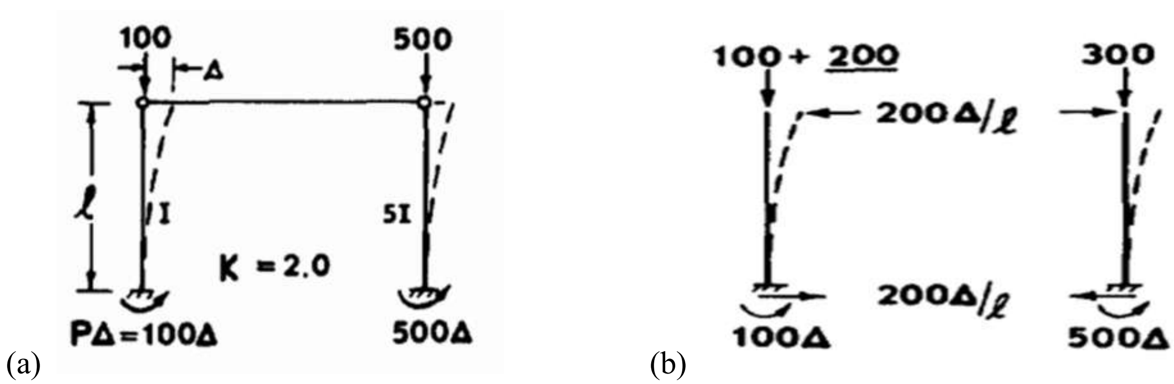



**Figure 2.6: Quantitative description of frame instability**

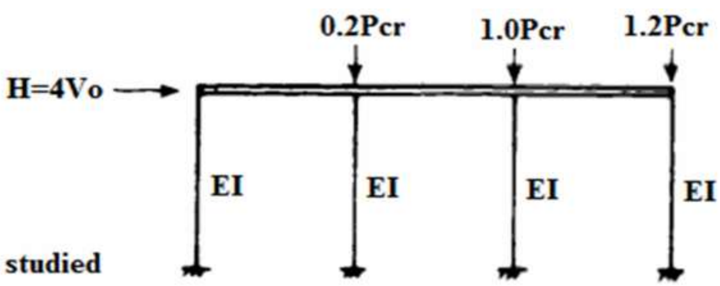
To summarize, the shear capacity of the right column can be replaced (approximately) by an equivalent axial load carried by the weak column.

In general, the total gravity loads which produce sidesway buckling can be distributed among the columns in a story in any manner. Sidesway buckling will not occur until the total frame load on a story reaches the sum of the potential individual column loads for the unbraced frame. [3]

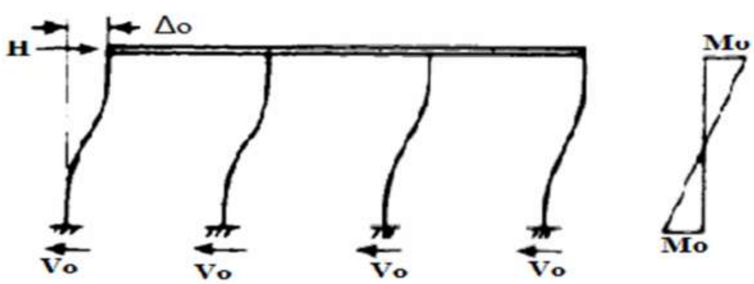
A more detailed quantitative example of frame interaction is given below which shows the mechanical behavior of geometric nonlinearity using a simple single-story frame subjected to lateral and vertical loads (Fig. 2.7(a)) to illustrate P-Δ and P-δ effect discussed in section 2.3.

It is assumed that the axial load in a column is equal to the vertical load above it, the beam is rigid, and the stiffness parameter EI is constant for all the columns. The first order lateral stiffness is therefore equal for all the columns.

In the absence of vertical loads, i.e., first-order effects only, a lateral load $H = 4V_o$ produces a shear of $V_o$ in each of the columns, and the frame undergoes a lateral deflection $\Delta_o$ as shown in fig 2.7(b). The end- moment $M_o$ is equal in all the columns.




(a)Frame studied




(b) First-order effects

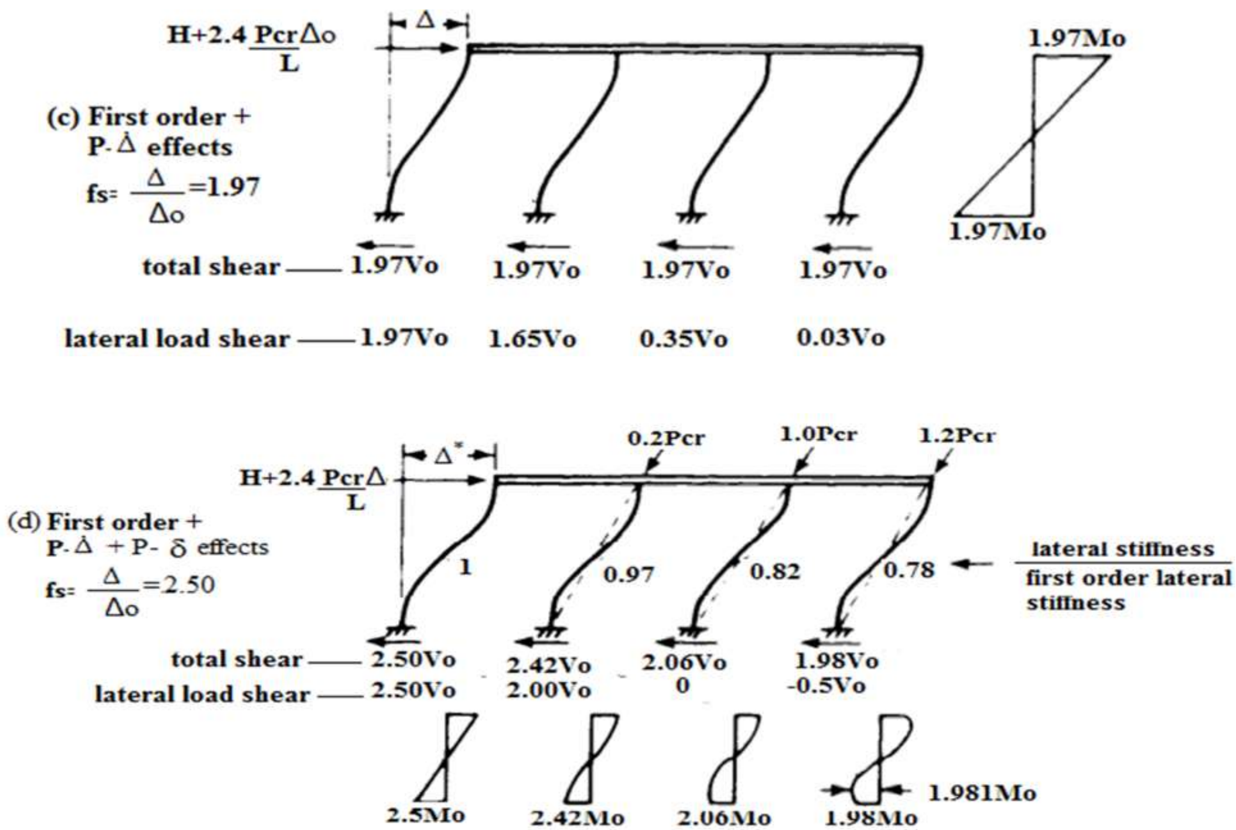



**Figure 2.7: single story frame subjected to vertical and lateral load**

To study the P-Δ effects alone, the P-δ effects are first neglected. The vertical loads are replaced by a horizontal force equal to the P-Δ shears from all the columns as shown in fig. 2.7 (c). With this additional force, the sway of the frame is increased to $\Delta = f_s\Delta_o$. And the shear resisting the total horizontal force in each column becomes $f_sV_o$, and the end-moment in each column is also equal to $f_sM_o$.

The shear resisting the lateral load H in each column can be obtained by subtracting the P-Δ shear for that particular column from the total shear $f_sV_o$ . As the P-Δ shear is different for each column, the lateral load shears have been redistributed compared to the first order shears. Since the total shear $f_sV_o$ in each column is the same, the capacity of a column to resist the lateral load becomes diminished with a higher axial load.

When both P-Δ and P-δ effects are incorporated in the analysis, i.e. an "exact" analysis, the sway of the frame is increased further to $\Delta^* = f_s\Delta_o$because the lateral stiffnesses of those columns subjected to axial loads are reduced to the values shown in fig.2.7 (d). The stiffness reduction increases with higher axial loads. Since the total horizontal forces are resisted by the columns in proportion to their relative lateral stiffness, the total shear in each column is different. The shear in the column without any axial load is equal to $f_s V_o$, whereas the total shear in each of the axially loaded columns is less than $f_s V_o$. The weakest column (i.e. the most highly loaded column) resists the least amount of shear.

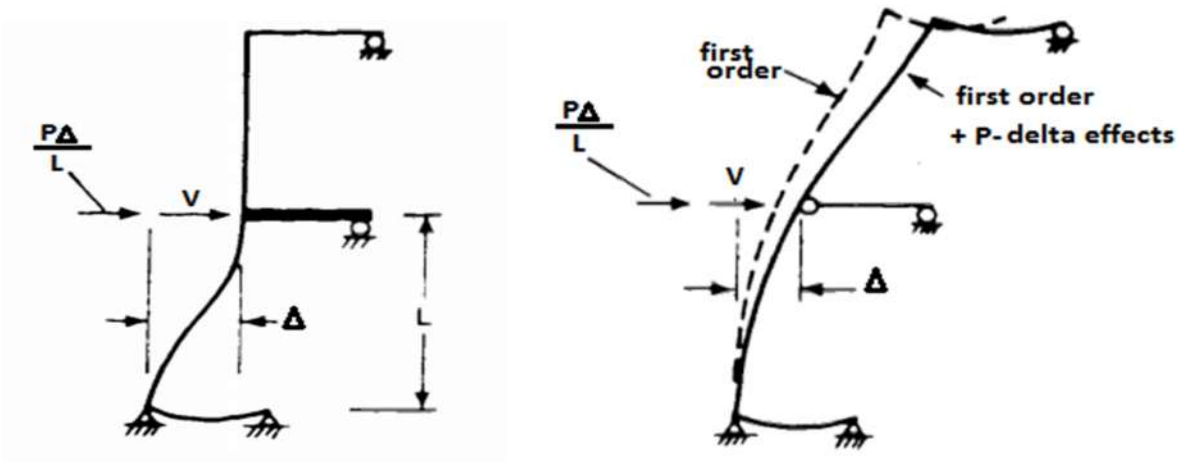
The moment diagrams for the axially loaded columns are non-linear due to the P-δ effects, compared to the linear moment distribution which results if only the P-Δ term is considered.

The lateral load shear in each column is also presented in fig.2.7 (d). As stated earlier, the reduced stiffness of the axially loaded columns reduces their ability to resist the lateral loads. Consequently, more lateral load shear is added to the stronger columns. In fact, the columns with vertical loads equal to $1.0\,P_{cr}$ and $1.2\,P_{cr}$ would have failed, had they been free to sway independently of the other columns. However, elastic failure of these columns is prevented since all columns must undergo an equal lateral displacement Δ. In this process the lateral load shear in the weaker columns will be redistributed to the stronger columns (fig.2.7 (c) and (d)).

For the column with a vertical load of $1.0P_{cr}$, the column does not offer any resistance to the lateral load, because $P_{cr}$ is equal to the free- to- sway critical load of the column. For the column with a vertical load greater than $P_{cr}$, the lateral load shear has reversed direction, indicating that a negative shear is required to brace it from falling laterally; in other words it becomes **supported** sway column which need lateral support from **supporting** (strong) columns. [1]

### 2.3.1.2 Vertical interaction

The vertical interaction due to the P-delta effects is explained with the aid of fig 2.8. A two story frame with a completely rigid beam connecting the two stories is shown in fig 2.8(a). Only the bottom story is loaded. It can be seen that the P-delta effects are localized to the bottom story and do not affect the upper story. In fig 2.8(b) another extreme case is represented by a completely flexible beam (pin-ended) connecting the two stories. Again, only the bottom story is loaded, but in this case the p-delta effects obviously increase the deformation in the upper story. In fact, the p-delta effects in the bottom story are also affected by the lateral stiffness of the upper story. If the top beam were made stiffer, for example, the deformation in the bottom story will also be reduced. These effects suggest that the stories tend to assist each other to resist the geometric effects.



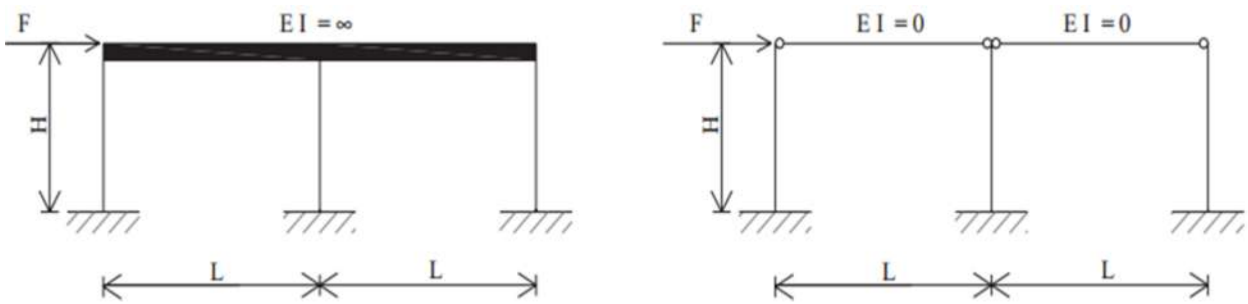

(a) Rigid connecting beam (b)flexible connecting beam

**Figure 2.8: Vertical interaction due to p-delta effects**

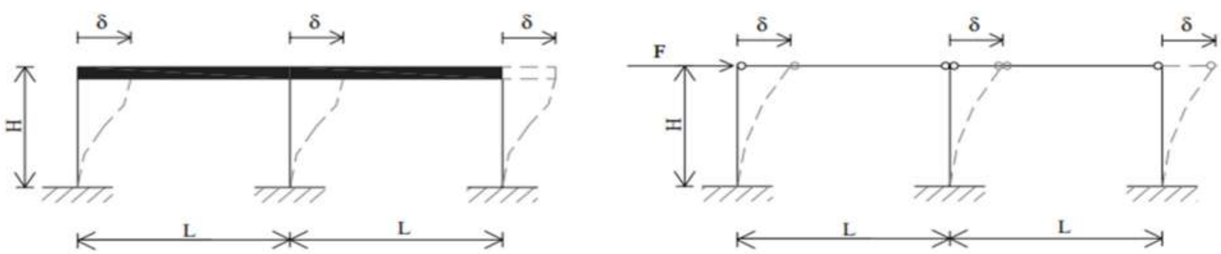
In general, when the beams that connect the stories are stiff the geometric behavior in one story is independent of the behavior in other stories. However, when the beams are flexible, the storys tend to interact each other. [1]

### 2.3.2 Failure mechanism of frames

The failure mechanism of frames as shown in fig. 2.9 is governed by the relative stiffness of beams to columns. In addition to the importance of absolute stiffness, the relative stiffness of members within a structural system is of significance especially in seismic assessment, because it influences the distribution of actions and deformations. For example, beams with very low flexural stiffness, e.g., flat beams (fig. 2.9), do not restrain the rotation of the columns connected to them. On the other hand, deep beams provide effective restrain for columns in framed structures.




a) Frame layout




b) Frame deformation

**Figure 2.9: Effects of relative stiffness of beams and columns on the distribution of actions and deformations in single - story frames**

In general using the relative stiffness of beams to columns, we can classify a frame system or failure mechanism of frames as; "frames with strong column-weak beam" or "beam-sway mechanism" and "frames with weak column-strong beam" or " column-sway mechanism fig 2.10.

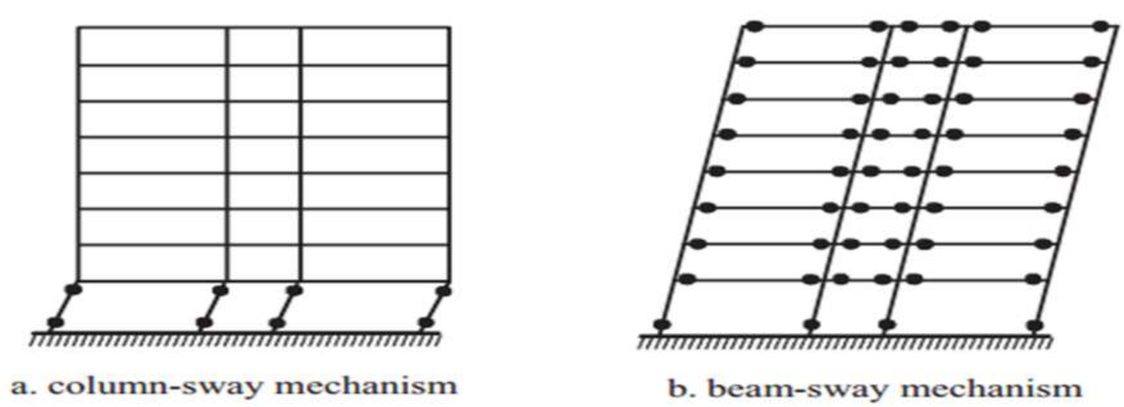



**Figure 2.10: Failure mechanisms of frame system**

To dissipate a large amount of energy, favorable failure modes are global mechanisms with plastic hinges in beams rather than in columns – referred to as 'beam - sway' mechanism. [15]

## 2.4 Stiffness of members

If deformations under the action of lateral force are to be reliably quantified and subsequently controlled, designers must make a realistic estimate of the relevant property - stiffness.

Stiffness is the ability of a component or an assembly of components to resist deformations when subjected to actions. This quantity relates loads or forces to the ensuing structural deformations. Familiar relationships are readily established from first principles of structural mechanics using geometric properties of members, $I$, and the modulus of elasticity for the material, E.

Under the action of lateral loads it is important that the distribution of member forces be based on realistic stiffness values, applying at close to member yield forces, as this will ensure that the hierarchy of formation of member yield conforms to assumed distributions, and that member ductilities are reasonably uniformly distributed through the frame.

At any section, moment of inertia (second moment of area), $I$ will be influenced by the magnitude and sign of moment, and the amount of flexural reinforcement, as well as by the section geometry and the axial load. It is also known that flexural cracking varies along the element length, therefore, $I$, also varies along element length. Due to these factors the stiffness will vary along the length and material non-linearity will be introduced in the system. [13]

In order to avoid the grater complexity of an analysis that include material non-linearity, several studies and code practices propose a simplified consideration of material non-linearity through the reduction of initial stiffness of the sections (effective stiffness). [20]

Eurocode 8 (Pr EN 1998-1: 2003(E)) section 4.3.1 (7) says that "unless a more accurate analysis of the cracked element is performed the elastic flexural and shear stiffness properties of concrete and masonry elements may be taken to be equal to one half of the corresponding stiffness of the uncracked elements. $(EI)_{eff} = 0.5(EI)$ gross.

However, in principle, the reduction of stiffness depends on many factors as mentioned above, such as: the type of constructive element, the percentage of reinforcement and the level of normal stress caused by the influence of gravity loads. For this reason in Eurocode 8, rather than giving one value of stiffness reduction for all structural elements, a more detailed guideline should be given for different type of structural elements. [21]

When an estimate of the effective stiffness on the low side is used in the analysis, it leads to an increased second-order effects. And lateral drifts and P-delta effects computed on the basis of overly high stiffness values may be seriously underestimated. [7]

## 2.5 Methods of structural analysis

The two major difficulties in the analysis and design of structures especially in slender reinforced concrete frames are result from:

i/ the “ material non-linearity”, caused by the inelastic properties of materials and,

ii/ the “geometric non-linearity”, caused by the effect of displacement on the equilibrium of individual members and of the whole structures.

Based on the non-linearities included, structural analysis can be grouped in to three groups.

A structural analysis which includes both material and geometric non-linearities is referred to as **Second-order Inelastic Analysis.**

A structural analysis which includes the geometric non-linearity only and assumes elastic response of the structure is referred to as **Second-order Elastic Analysis**.

The second-order **elastic** analysis is derived by modifying the second-order inelastic analysis and the first-order elastic analysis. The inelastic analysis is modified by using effective EI, which accounts the inelastic property of material approximately, and the first-order elastic analysis is modified to account for the geometric-non linearity.

The simplest type of structural analysis which neglects both non-linearities is referred as **First-order Elastic Analysis.** [1]

The analysis methods mentioned above are discussed below in detail starting from the simplest to the more complex one.

**First-order Elastic Analysis**

The first and most common approach to structural analysis is the first-order elastic analysis, which is also called simply elastic analysis. In this case, any inelastic behavior of the material is ignored and deformations are assumed to be small so that action effects are calculated without consideration of the effect of structural deformations, but including geometric imperfections.

**Second-order Elastic Analysis.**

When the equations of equilibrium are written with reference to the deformed configuration of the structure and the deflections corresponding to a given set of loads are determined, the resulting analysis is a second-order elastic analysis. In this analysis method, material nonlinearity is considered approximately by taking reduced stiffness, Effective stiffness. This is the analysis generally referred to as a P-delta analysis. Two components of second-order effects should be included in this analysis.

When the sidesway effects, lateral deflections of the beam–column joints from their original undeflected locations, are included, it is said that **the *P*-Δ effects or *frame effects*** are included.

When the influence of member curvature, deflections of the axis of the bent column away from the chord joining the ends of the column, is included, it is said that the ***P*-δ effects or *member effects*** are included. [16], [12]

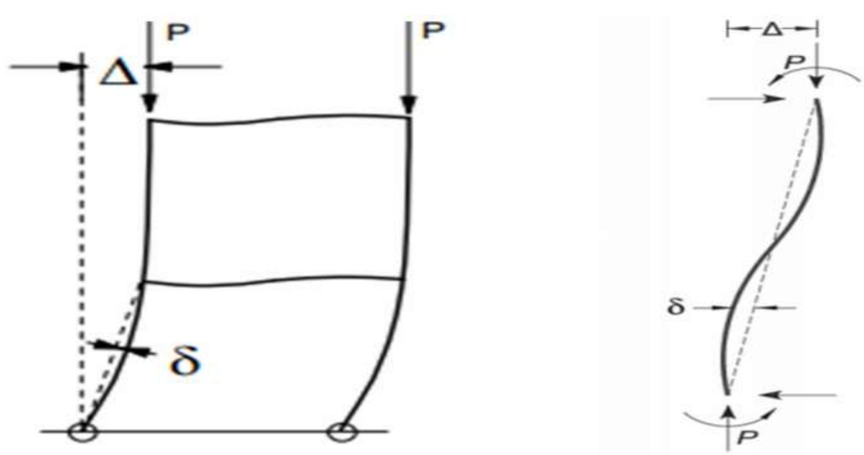



**Figure 2.11: The P-Δ and P – δ effect**

Based on which p - delta effect (P-Δ or/and P – δ effect) is considered, second - order elastic analysis can be generally subdivide in to two methods, **approximate** and **"exact" second order elastic analysis** method. These methods are discussed in depth in section 2.6.

**Second-order Inelastic Analysis**

This analytical approach combines the same principles of second-order analysis discussed previously with the plastic hinge analysis. This category of analysis is more complex than any of the other methods of analysis discussed thus far. It does, however, yield a more complete and accurate picture of the behavior of the structure, depending on the completeness of the model that is used. This type of analysis is often referred to as "advanced analysis."

The behavior of the analysis methods discussed above are shown with the aid of load displacement curve in Figure 2.12 below.

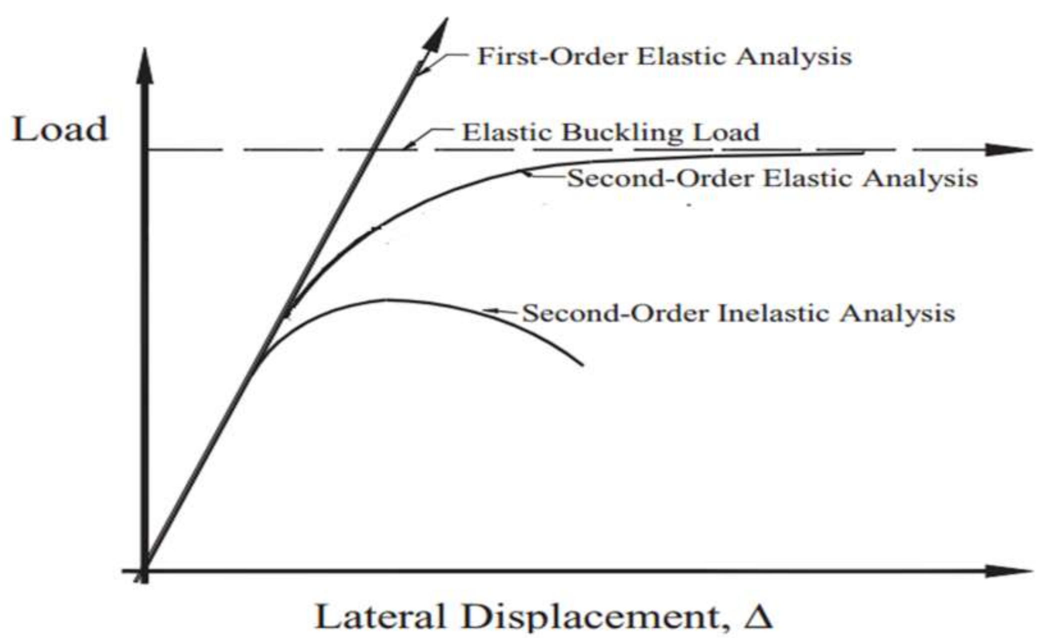



**Figure 2.12: Load - displacement history [16]**

In summary, it can be seen that as more realistic and hence more complex behavior is taken into account in the analysis, the predicted critical load level is reduced or the calculated lateral displacement for a given load is increased. Thus, designers need to be aware of the assumptions utilized in any analytical approach that they employ. This is particularly important when using commercially available software. [16]

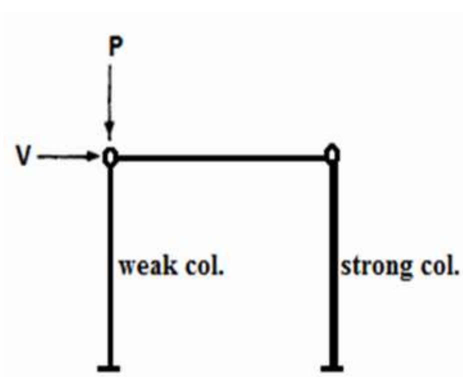
## 2.6 Methods of Second-Order Elastic Analysis

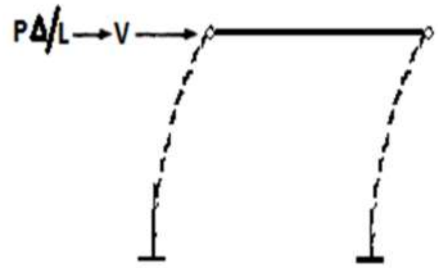
The approach taken for design of structures must be consistent with the approach chosen for analysis. Modern building codes attempt to predict stability failures by means of simple approximate methods, due to the complexity in carrying out an exact/rigorous/ second-order analysis. [4]

In the approximate second-order elastic analysis methods, only p-Δ effect is considered and p-δ effect is neglected. In “exact” second-order elastic analysis method however, both p-Δ and p-δ effects are considered.

### 2.6.1 Approximate Second-Order Elastic Analysis

The basic assumption behind all approximate methods is that, “***The deformed shape of the frame subjected to lateral loads and vertical loads can be represented by the deformed shape due to lateral loads plus sway forces”.***




(a) frame subjected to vertical and lateral load (b) deflected shape due to V and a P-Δ shear

However, the validity of the basic assumption is uncertain in the case of supported sway column (the weak column in this case). As the applied axial load on the weak column is increasing higher, the weak column reaches a deflected shape very different from the basic assumption; a shape in the approach of elastic failure mode (fig. 2.13 (c). [1]

In this work the concept of ‘elastic failure mode of a supported sway column’ and ‘deflection shape of a frame in the approach of basic assumption’ is mentioned and used in the analysis by directly adopting from the work of Lai,Shu-Ming Albert(1982)[1]. His paper is trusted in this way because it is still the most referred document in the topic of the limit of θ even in the last version of ACI (ACI 318-11/318R-11).

From the various types of the approximate methods, only Iterative and Direct p-Δ analysis methods are discussed here. As mentioned above in both type of this methods, only p-Δ effect will be considered, while p-δ effect is neglected.

#### 2.6.1.1 Iterative p-Δ analysis

Based on the basic assumption for all approximate methods, the idea behind this procedure is that; the moments produced by the total vertical load p, acting through a lateral deflection Δ at a certain level in a frame may be replaced by equivalent lateral shears applied at floor levels.

The iterative P-Δ analysis method is based on the simple idea of correcting first order displacements by adding the P-Δ shears to the applied story shears.

When a frame is displaced sideways under the action of lateral and vertical loads, as shown in Figs. 2.14 and 2.15, the column end moments must equilibrate the lateral loads and a moment equal to ($\sum P\Delta$) ; that is,

$$\sum\left(M_{top} + M_{btm}\right) = V l_c + \sum P\Delta \qquad (2.1)$$

Where, Δ is the lateral deflection of the top of the story relative to the bottom of the story (story drift).

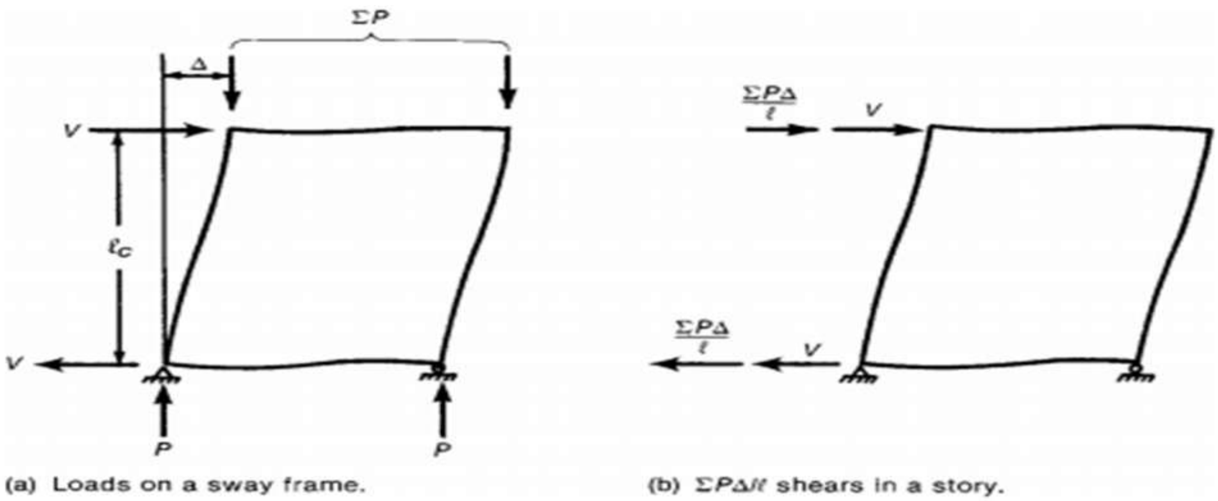
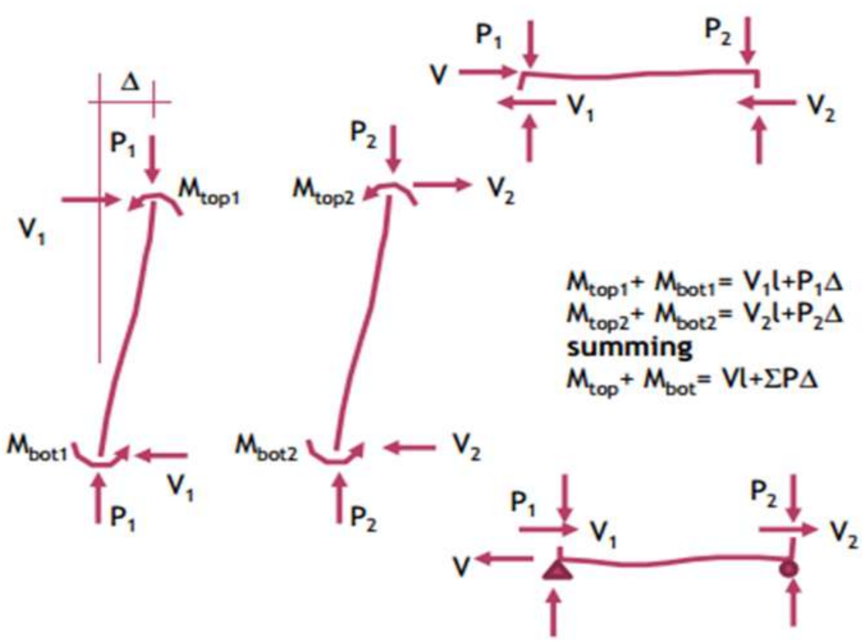



**Figure 2.14: Iterative p-delta analysis**

Using internal force diagram, the steps in this method are discussed below.

i. Initial first order analysis is made with external horizontal loading ,V ;
ii. The resulting horizontal deflection, Δ are then used in conjunction with the gravity loading to compute at each floor level an equivalent increment of horizontal load (shear force) $\sum P\Delta/l$ ;



iii. This increment is added to the initial horizontal load, $V + \sum P\Delta/l$, and the analysis is repeated.

iv. The resulting increased deflection are then used in conjunction with the gravity loads to compute another set of equivalent horizontal loading;

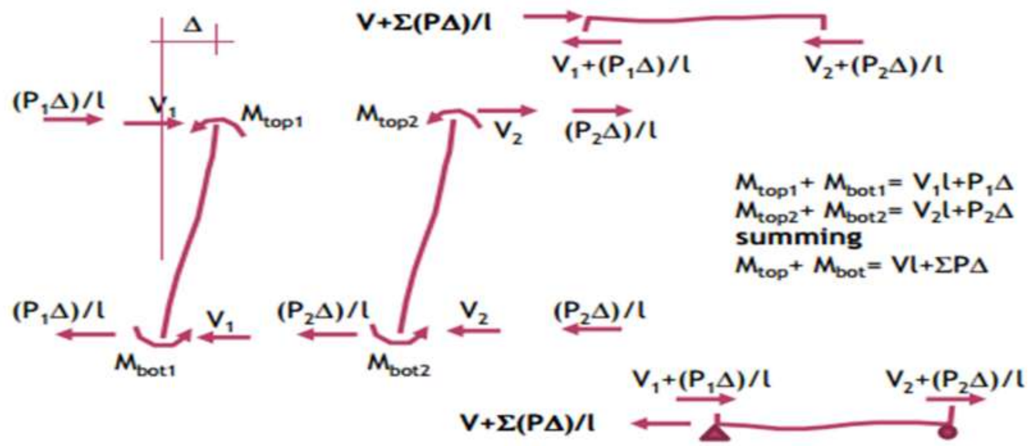



**Figure 2.15: Iterative p-delta analysis**

v. The new increment is again added to the initial horizontal load for reanalysis.

vi. The iterations are continued until increases in the deflection become negligible.

Fig.2.16 shows the story shears in two different stories. The algebraic sum of the story shears from the columns above and below a given floor gives rise to a sway force acting on that floor. At the $j^{th}$ floor, the sway force is:

$$\text{Sway force}_j = (\sum P_i \Delta_i/l_i) - (\sum P_j \Delta_j/l_j) \ldots\ldots\ldots\ldots\ldots \text{(2.2)}$$

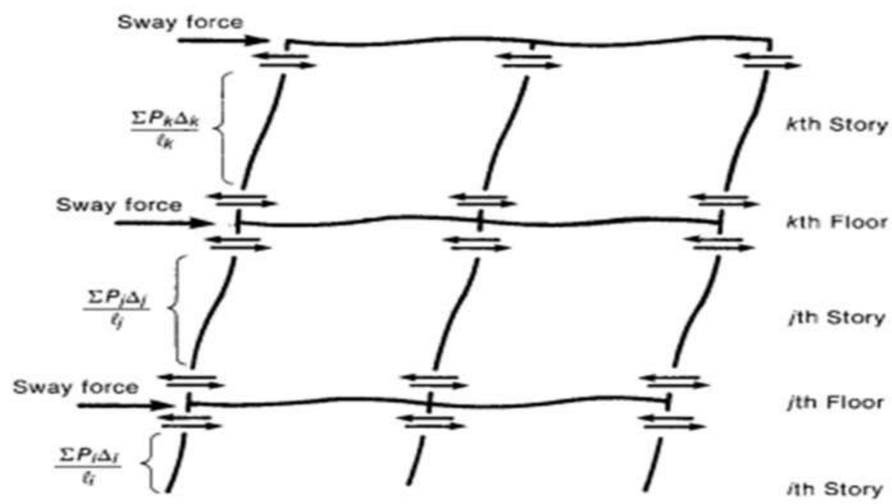



**Figure 2.16: Iterative p-delta analysis**

The sway forces are added to the applied lateral loads at each floor level, and the structure is reanalyzed, giving new lateral deflections and larger column moments. This process is continued until convergence is obtained (until the increased lateral deflection become negligible). [19][12]

S.E Hage (1974) [4] in his study show that the rate of convergence of the iteration process could be used as an indication as to the possibility of a stability failure occurring.

### 2.6.1.2 Direct p-Δ Analysis

The above iterative analysis may be expressed in a more convenient form. Let $v_0$ and $\Delta_0$ represent the applied lateral load and the corresponding first order deflection, respectively. Also, let the appropriate axial load be $p$, the deflection caused by a unit lateral load be $k_s$ and let $v_i$ (i=2, 3…∞) be the sum of the applied and additional lateral loads in the i$^{th}$ cycle.

Then the iteration process may be expressed in the following manner;

1$^{st}$ iteration: $\Delta_0 = k_s v_0$

2$^{nd}$ iteration: $\Delta_1 = k_s v_1 = k_s v_o\,(1 + \frac{p}{l} k_s)$

3$^{rd}$ iteration: $\Delta_2 = k_s v_2 = k_s v_o\,(1 + \frac{p}{l} k_s + \left(\frac{p}{l}\right)^2 k_s{}^2)$

and the general term for the i^th iteration is;

$$\Delta_i = k_s v_o \left[1 + \frac{p}{l} k_s + \left(\frac{p}{l}\right)^2 k_s{}^2 + \cdots + \left(\frac{p}{l}\right)^{i-2} k_s{}^{i-2} + \left(\frac{p}{l}\right)^{i-1} k_s{}^{i-1}\right]$$

This geometric series will converge if $\frac{p}{l} k_s < 1.0$ and in that case the sum of the infinite series is;

$$\frac{1}{1 - \frac{p}{l} k_s} = \frac{1}{1 - \frac{p\Delta_o}{v_o l}}$$

And since $k_s v_0 = \Delta_0$, the process converges to the final deflection including p-Δ effect i.e. second–order deflection, $\Delta_{final}$;

$$\Delta_{final} = \frac{\Delta_o}{1 - \frac{p\Delta_o}{v_o l}} \qquad (2.3)$$

This equation shows that the second order deflections may be computed directly from the result of a first-order analysis, so that the method is called ***direct p-Δ analysis.***

The clear advantage of using equation 2.3 rather than the actual iteration process is that only two first order analyses are required to obtain the second-order moments and forces in the elastic structure. [4]

From equation 2.3 it can be seen that, the first order deflection, $\Delta_o$ is magnified by the term $\frac{1}{1 - \frac{p\Delta_o}{v_o l}}$ in order to gate the final deflection including p-Δ effect.

Late the term, $\frac{1}{1 - \frac{p\Delta_o}{v_o l}}$ in equation 2.3 as $f_s$, and rewrite $f_s$ gives;

$$f_s = \frac{\Delta_{final}}{\Delta_o} \qquad (2.4)$$

$f_s$ , is referred as the **deflection magnifier.**

Eurocode 8 Section 4.4.2.2, (2) defines the stability index (interstorey drift sensitivity coefficient) for a story as

$$\theta = \frac{p_{tot} d_r}{v_{tot} h}$$

Substituting this into equation 2.3 gives;

$$\Delta_{final} = \frac{\Delta_o}{1-\theta} \rightarrow \frac{\Delta_{final}}{\Delta_o} = \frac{1}{1-\theta}$$

and the deflection magnifier, $f_s = \frac{1}{1-\theta}$ ……………………(2.5)

The mathematical expression for the deflection magnifier $f_s$ in equation 2.5 agree with Eurocode-8 expression for the case when θ is below 0.2*; "If 0,1 < θ ≤ 0,2, the second-order effects may approximately be taken into account by multiplying all first order action effects due to horizontal component of the seismic action by a factor equal to 1/(1 – θ).*

In section 4.1.2 of this study, a graph of deflection magnifier, $f_s = \Delta_{final}/\Delta_o$ versus the normalized applied force $P/P_{cr}$ is used to show the limit of the accuracy of approximate method relative to "Exact method".

## 2.6.2 "Exact" Second-order Elastic Analysis

This elastic second order analysis method is termed as **"exact"** and the values obtained from it are referred as **"exact values**" , for simplicity and since it include geometric nonlinearity (both p-Δ and p-δ) by means of the geometric stiffness matrix, and the material non-linearity is taken approximately by means of using effective stiffness.

In this method the elastic frame analysis is performed by means of the finite element method. A frame may be visualized as an assemblage of elements interconnected at their ends which are referred to as nodal points or nodes. If the force-displacement relations for each element are known, the equilibrium configuration of the complete structure can be expressed in terms of the nodal displacements.

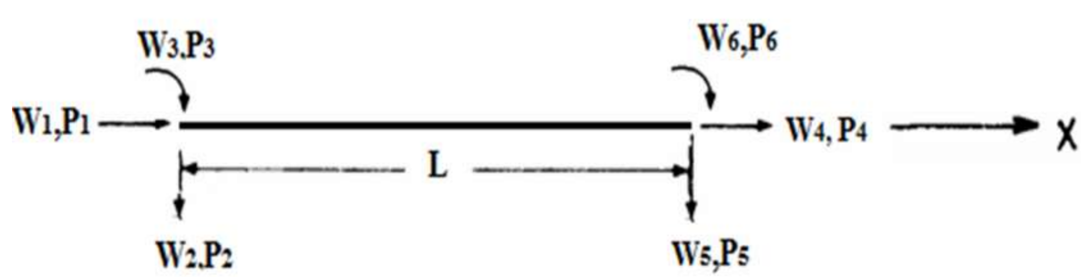



**Figure 2.17: Frame element**

The force-displacement relationship for the element shown in Fig. 2.17 can be written as;

$$[K]\{W\} = \{P\},$$

Where, {W} is the displacement vector of the element and {P} the corresponding force vector.

$$\{W\} = \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \\ W_5 \\ W_6 \end{Bmatrix} \qquad \{P\} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix}$$

Applying standard finite element techniques, the stiffness matrix [K] can be written as

$$[K] = [K_o] + [K_g]$$

Where $[K_o]$ is the first order stiffness matrix, (element stiffness matrix)

$[K_g]$ is the non-linear geometrical stiffness matrix,

$[K_o]$ and $[K_g]$ are given in Fig 2.18

$$Ko = \frac{EI}{L^3}\begin{bmatrix} \frac{AL^2}{I} & 0 & 0 & \frac{-AL^2}{I} & 0 & 0 \\ 0 & 12 & 6L & 0 & -12 & 6L \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\ \frac{-AL^2}{I} & 0 & 0 & \frac{AL^2}{I} & 0 & 0 \\ 0 & -12 & -6L & 0 & 12 & -6L \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix} \qquad Kg = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-6P}{5L} & \frac{-P}{10} & 0 & \frac{6P}{5L} & \frac{-P}{10} \\ 0 & \frac{-P}{10} & \frac{-2LP}{15} & 0 & \frac{P}{10} & \frac{LP}{30} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6P}{5L} & \frac{P}{10} & 0 & \frac{-6P}{5L} & \frac{P}{10} \\ 0 & \frac{-P}{10} & \frac{LP}{30} & 0 & \frac{P}{10} & \frac{-2LP}{15} \end{bmatrix}$$

**Figure 2.18: Element stiffness matrix $K_o$ and geometrical stiffness matrix $K_g$**

The additional stiffness matrix $[K_g]$ is dependent on the axial forces P in the elements. The axil forces depend in turn of the external loads, so that a non-linear relation between displacements and loads results. Due to the fact that the geometrical stiffness matrix $[K_g]$ depends on the axial forces in the elements, an iterative procedure results. [2]

It can be seen that the final stiffness matrix $[K]$ is obtained by reducing the non-linear geometric stiffness matrix $[K_g]$ from the first order stiffness matrix $[K_o]$, to include the geometric non linearity induced by the axial force, this results in a reduced load carrying capacity of the element. The derivation of the geometric stiffness matrix is not discussed in this work since it is not the subject matter of this study. However, it is shown in Appendix A, which is given by J.S Przemieniecki (1968) [12] for reference purpose.

This method is adopted directly in this work because it was used by Lai Shu Ming Albert (1982) in his PhD. work by referring the work of Aas-jakobsen (1973) who tested the accuracy of the approach.

# CHAPTER 3 METHOD OF MODELING AND PROCEDURES

## 3.1 General

In this study the accuracy limit of approximate second order elastic analysis method is illustrated by comparing its results with the results of "exact" method. It is further illustrated using the relation between the P-delta effect and the lateral load resisting capacity of the columns in a simple portal frame.

The limit of interstorey drift sensitivity coefficient $\theta$ set by Eurocode-8 is also shown in relation to the accuracy limit of approximate second order elastic analysis method.

## 3.2 Description of the Study

Approximate elastic second order analysis method is primarily based on the **basic assumption** that; " *the deformed shape of the frame subjected to lateral loads and vertical loads can be represented by the deformed shape due to lateral loads plus sway forces"*. The validity of this assumption becomes uncertain in the case of supported sway columns.

This problem is examined with the aid of 5 simple elastic portal frames having a different relative stiffness value of columns (EI $v_s$2EI, 4EI $v_s$ 8EI, EI $v_s$ 5EI, EI $v_s$8EI, EI $v_s$10EI) to see the effect of relative stiffness of columns on their interaction.

Simple elastic portal frame is selected because it clearly shows cases where the validity of the basic assumption behind the approximate methods becomes uncertain.

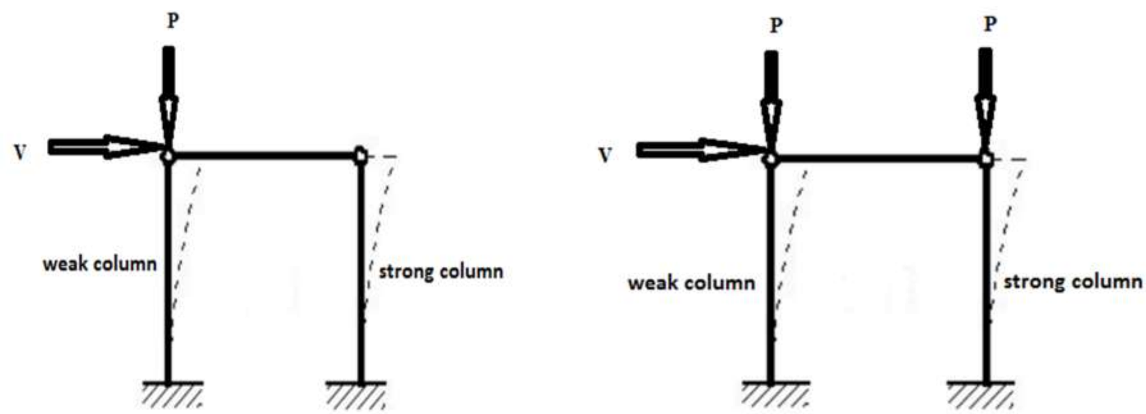
## 3.3 Description of the structure

The selected frames consist of a much stiffer (strong) column and a weak column to examine the problem using supporting and supported sway columns.

The beam is connected to the columns by hinge connection so that the columns can be treated as an individual cantilever column fixed at the base with effective length factor 2. This type of joint connection is preferred in order to show the potential bracing capacity of columns in a story with no restrain action of the beam on the rotation of the columns. This type of connection is the most critical for P-Δ effects because of largest flexibility of the buildings under seismic actions.

It is safe to treat separately each column to which beams are rigidly attached. However, Joseph A.Yura (1971) [2] has stated in his work that, in some instances this usual approach may be unduly conservative.

In analyzing the frame, the axial load p is applied to the weak column only in the first case, and on both the weak and strong column in the second case (fig. 3.1 (a) and (b). In both cases, a horizontal lateral load V is applied first to get a first order deflection.




(a) Case I

(b) Case II

**Figure 3.1: frame loading in case I (a) and case II (b)**

## 3.4 Exact method procedures

1. Solve the first order displacement $\Delta_o$ of the frame due to the horizontal lateral load V;
2. Solve the element stiffness matrix for all members according to their member type;
3. Assemble the element stiffness matrix of the members to get structure stiffness matrix $[K_o]$ using the usual FEM procedures;
4. Solve the geometric stiffness matrix only for the weak column (in the first case) and for both columns (in the second case) where axial force p is applied;
5. Assemble the geometric stiffness matrix of the columns to get the structure geometric stiffness matrix $[K_g]$;
6. Assemble the structure stiffness matrix $[K_o]$ and geometric stiffness matrix $[K_g]$ to get the complete structure stiffness matrix [K], $[K] = [K_o] + [K_g]$

7. Using the load displacement relation [K]{W} = {P}, solve for the displacement vector {w} which include the geometric effect due to the axial force p applied on the deformed frame, to get the final deflection $\Delta_{final}$ ;
8. Plot the load-deformation graph of the frame with $P/P_{cr}$ values in the X-axis and $\Delta_{final}/\Delta_o$ in the Y-axis.

## 3.5 Approximate Method Procedures

To use the approximate method for the analysis of frame structures, the frame must be replaced by an equivalent cantilever column. This is because that the method is derived and valid only for a one-story cantilever column. An extension of its application even from a single-story one-bay frame (frame with two columns), the index θ begins to exhibit divergence.

For the frame under consideration, the two columns can be added directly. Since the beam-column joint is a hinge connection, we can ignore the restraining action of the beam and the columns can be treated as individual cantilever columns.

1. Convert the frame into an equivalent cantilever column;
2. Calculate the first order deflection $\Delta_o$ for the equivalent cantilever column due to the horizontal lateral load V applied at the top of the column;
3. Using direct P-Δ analysis method (eq. 2.1) $\Delta_{final}= \frac{\Delta_o}{1-\frac{p\Delta_o}{v_o l}}$ calculate the final deflection due to the axial load p applied to the deformed column;
4. Plot the load-deformation graph of the frame with $P/P_{cr}$ values in the X-axis and $\Delta_{final}/\Delta_o$ in the Y-axis.

# CHAPTER 4 RESULTS & DISCUSSION

## 4.1 Case I: Load Applied on the Weak Column Only

### 4.1.1 Approximate method vs. "exact" method

The results of the "exact" method are compared to that of approximate method for 5 portal frames with different column relative stiffness, using the graph of $P/P_{cr}$ $v_s$ $\Delta_{final}/\Delta_o$ and percentage difference tables. Only the graph of EI $v_s$ 2EI and EI $v_s$ 10EI is shown below since the graph of all the 5 frames is similar.

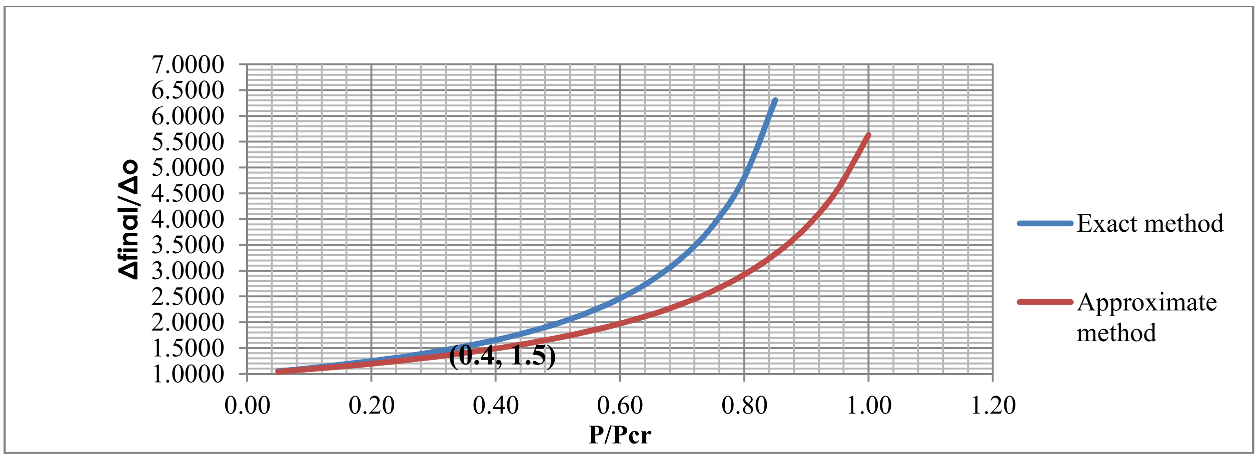



**Figure 4.1: Exact and approximate method for frame with relative stiffness EI vs 2EI**

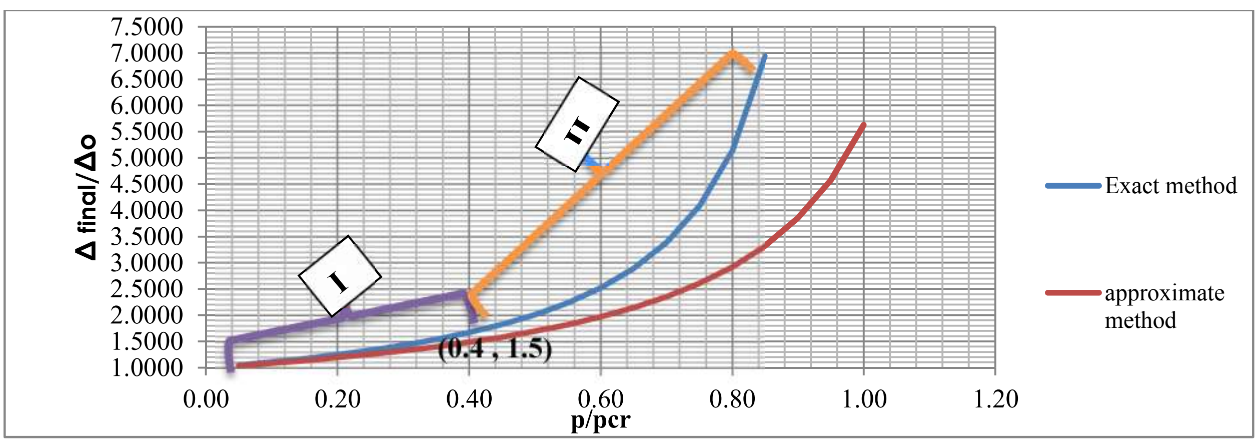



**Figure 4.2: Exact and approximate method for frame with relative stiffness EI $v_s$ 10EI**

It can be seen from the graph that, initially (in region I) the two curves agree reasonably well, gives almost equal result. This means that P-Δ effect is the more governing one than P-δ effect; so that the results of approximate method is very close to that of "exact" one. For $f_s$ >1.5, (in region II) P-δ effect become more significant and the two curves diverges each other because the approximate method doesn't include P-δ effect.

The point where the difference between the two curves becomes distinguishable is when $f_s$ is equal to1.5 (θ=0.3). Therefore, for θ<0.3, the approximate method gives a good result compared to the "exact" one.

To avoid uncertainty in reading the graph, the two methods are compared each other using

% difference table. Interstorey drift sensitivity coefficient θ is calculated for each level of loading.

| TABLE 4.1 : EI Vs 2EI, Case I | | Comparison of Exact & Approximate method and location of θ | | | |
|---|---|---|---|---|---|
| EXACT M | | APPROXIMATE M | | % Diffrence | θ |
| $(P/P_{cr})$ frame | $\Delta_{final}/\Delta_0$ | $(P/P_{cr})$ frame | $\Delta_{final}/\Delta_0$ | | |
| 0.05 | 1.052 | 0.05 | 1.043 | 1% | 0.04 |
| 0.10 | 1.110 | 0.10 | 1.090 | 2% | 0.08 |
| 0.15 | 1.174 | 0.15 | 1.141 | 3% | 0.12 |
| 0.20 | 1.247 | 0.20 | 1.197 | 4% | 0.16 |
| 0.25 | 1.329 | 0.25 | 1.259 | 5% | 0.21 |
| 0.30 | 1.422 | 0.30 | 1.328 | 7% | 0.25 |
| 0.35 | 1.530 | 0.35 | 1.404 | 8% | 0.29 |
| 0.40 | 1.655 | 0.40 | 1.490 | 10% | 0.33 |
| 0.45 | 1.803 | 0.45 | 1.588 | 12% | 0.37 |
| 0.50 | 1.980 | 0.50 | 1.698 | 14% | 0.41 |
| 0.55 | 2.195 | 0.55 | 1.826 | 17% | 0.45 |
| 0.60 | 2.463 | 0.60 | 1.974 | 20% | 0.49 |
| 0.65 | 2.804 | 0.65 | 2.149 | 23% | 0.53 |
| 0.70 | 3.256 | 0.70 | 2.357 | 28% | 0.58 |
| 0.75 | 3.882 | 0.75 | 2.610 | 33% | 0.62 |
| 0.80 | 4.805 | 0.80 | 2.924 | 39% | 0.66 |
| 0.85 | 6.305 | 0.85 | 3.323 | 47% | 0.70 |
| 0.90 | 9.165 | 0.90 | 3.849 | 58% | 0.74 |
| 0.95 | 16.775 | 0.95 | 4.573 | 73% | 0.78 |
| 1.00 | 98.816 | 1.00 | 5.633 | 94% | 0.82 |

| TABLE 4.2 : 4EI Vs 8EI, Case I | | Comparison of Exact & Approximate method and location of θ | | | |
|---|---|---|---|---|---|
| EXACT M | | APPROXIMATE M | | % Diffrence | θ |
| $(P/P_{cr})$ frame | $\Delta_{final}/\Delta_0$ | $(P/P_{cr})$ frame | $\Delta_{final}/\Delta_0$ | | |
| 0.05 | 1.053 | 0.05 | 1.043 | 1% | 0.04 |
| 0.10 | 1.111 | 0.10 | 1.090 | 2% | 0.08 |
| 0.15 | 1.176 | 0.15 | 1.141 | 3% | 0.12 |
| 0.20 | 1.250 | 0.20 | 1.197 | 4% | 0.16 |
| 0.25 | 1.333 | 0.25 | 1.259 | 6% | 0.21 |
| 0.30 | 1.428 | 0.30 | 1.328 | 7% | 0.25 |
| 0.35 | 1.537 | 0.35 | 1.404 | 9% | 0.29 |
| 0.40 | 1.665 | 0.40 | 1.490 | 10% | 0.33 |
| 0.45 | 1.816 | 0.45 | 1.588 | 13% | 0.37 |
| 0.50 | 1.997 | 0.50 | 1.698 | 15% | 0.41 |
| 0.55 | 2.218 | 0.55 | 1.826 | 18% | 0.45 |
| 0.60 | 2.495 | 0.60 | 1.974 | 21% | 0.49 |
| 0.65 | 2.850 | 0.65 | 2.149 | 25% | 0.53 |
| 0.70 | 3.322 | 0.70 | 2.357 | 29% | 0.58 |
| 0.75 | 3.983 | 0.75 | 2.610 | 34% | 0.62 |
| 0.80 | 4.972 | 0.80 | 2.924 | 41% | 0.66 |
| 0.85 | 6.614 | 0.85 | 3.323 | 50% | 0.70 |
| 0.90 | 9.875 | 0.90 | 3.849 | 61% | 0.74 |
| 0.95 | 19.478 | 0.95 | 4.573 | 77% | 0.78 |
| 1.00 | 708.636 | 1.00 | 5.633 | 99% | 0.82 |

| TABLE 4.3 : EI Vs 5EI, Case I | | Comparison of Exact & Approximate method and location of θ | | | |
|---|---|---|---|---|---|
| EXACT M | | APPROXIMATE M | | % Diffrence | θ |
| $(P/P_{cr})$ frame | $\Delta_{final}/\Delta_o$ | $(P/P_{cr})$ frame | $\Delta_{final}/\Delta_o$ | | |
| 0.05 | 1.052 | 0.05 | 1.043 | 1% | 0.04 |
| 0.10 | 1.111 | 0.10 | 1.090 | 2% | 0.08 |
| 0.15 | 1.176 | 0.15 | 1.141 | 3% | 0.12 |
| 0.20 | 1.249 | 0.20 | 1.197 | 4% | 0.16 |
| 0.25 | 1.332 | 0.25 | 1.259 | 5% | 0.21 |
| 0.30 | 1.426 | 0.30 | 1.328 | 7% | 0.25 |
| 0.35 | 1.535 | 0.35 | 1.404 | 9% | 0.29 |
| 0.40 | 1.662 | 0.40 | 1.490 | 10% | 0.33 |
| 0.45 | 1.812 | 0.45 | 1.588 | 12% | 0.37 |
| 0.50 | 1.992 | 0.50 | 1.698 | 15% | 0.41 |
| 0.55 | 2.212 | 0.55 | 1.826 | 17% | 0.45 |
| 0.60 | 2.485 | 0.60 | 1.974 | 21% | 0.49 |
| 0.65 | 2.837 | 0.65 | 2.149 | 24% | 0.53 |
| 0.70 | 3.303 | 0.70 | 2.357 | 29% | 0.58 |
| 0.75 | 3.954 | 0.75 | 2.610 | 34% | 0.62 |
| 0.80 | 4.923 | 0.80 | 2.924 | 41% | 0.66 |
| 0.85 | 6.522 | 0.85 | 3.323 | 49% | 0.70 |
| 0.90 | 9.659 | 0.90 | 3.849 | 60% | 0.74 |
| 0.95 | 18.614 | 0.95 | 4.573 | 75% | 0.78 |
| 1.00 | 255.228 | 1.00 | 5.633 | 98% | 0.82 |

| TABLE 4.4: EI Vs 8EI, Case I | | Comparison of Exact & Approximate method and location of θ | | | |
|---|---|---|---|---|---|
| EXACT M | | APPROXIMATE M | | % Diffrence | θ |
| $(P/P_{cr})$ frame | $\Delta_{final}/\Delta_o$ | $(P/P_{cr})$ frame | $\Delta_{final}/\Delta_o$ | | |
| 0.05 | 1.053 | 0.05 | 1.0429 | 0.9% | 0.04 |
| 0.10 | 1.111 | 0.10 | 1.0896 | 2.0% | 0.08 |
| 0.15 | 1.177 | 0.15 | 1.1407 | 3.1% | 0.12 |
| 0.20 | 1.251 | 0.20 | 1.1969 | 4.3% | 0.16 |
| 0.25 | 1.334 | 0.25 | 1.2588 | 5.7% | 0.21 |
| 0.30 | 1.430 | 0.30 | 1.3276 | 7.2% | 0.25 |
| 0.35 | 1.541 | 0.35 | 1.4042 | 8.8% | 0.29 |
| 0.40 | 1.669 | 0.40 | 1.4903 | 10.7% | 0.33 |
| 0.45 | 1.822 | 0.45 | 1.5876 | 12.9% | 0.37 |
| 0.50 | 2.005 | 0.50 | 1.6985 | 15.3% | 0.41 |
| 0.55 | 2.229 | 0.55 | 1.8260 | 18.1% | 0.45 |
| 0.60 | 2.510 | 0.60 | 1.9743 | 21.3% | 0.49 |
| 0.65 | 2.871 | 0.65 | 2.1487 | 25.1% | 0.53 |
| 0.70 | 3.353 | 0.70 | 2.3570 | 29.7% | 0.58 |
| 0.75 | 4.031 | 0.75 | 2.6099 | 35.2% | 0.62 |
| 0.80 | 5.051 | 0.80 | 2.9238 | 42.1% | 0.66 |
| 0.85 | 6.763 | 0.85 | 3.3233 | 50.9% | 0.70 |
| 0.90 | 10.233 | 0.90 | 3.8494 | 62.4% | 0.74 |
| 0.95 | 21.008 | 0.95 | 4.5734 | 78.2% | 0.78 |
| 1.00 | -395.880 | 1.00 | 5.6328 | 101.4% | 0.82 |

| TABLE 4.5: EI Vs 10EI, Case I | | Comparison of Exact & Approximate method and location of θ | | | |
|---|---|---|---|---|---|
| EXACT M | | APPROXIMATE M | | % Diffrence | θ |
| $(P/P_{cr})$ frame | $\Delta_{final}/\Delta_o$ | $(P/P_{cr})$ frame | $\Delta_{final}/\Delta_o$ | | |
| 0.05 | 1.053 | 0.05 | 1.043 | 0.93% | 0.04 |
| 0.10 | 1.112 | 0.10 | 1.090 | 1.98% | 0.08 |
| 0.15 | 1.178 | 0.15 | 1.1410 | 3.14% | 0.12 |
| 0.20 | 1.252 | 0.20 | 1.1970 | 4.39% | 0.16 |
| 0.25 | 1.336 | 0.25 | 1.2590 | 5.76% | 0.21 |
| 0.30 | 1.433 | 0.30 | 1.3280 | 7.33% | 0.25 |
| 0.35 | 1.544 | 0.35 | 1.4040 | 9.07% | 0.29 |
| 0.40 | 1.674 | 0.40 | 1.4903 | 10.97% | 0.33 |
| 0.45 | 1.828 | 0.45 | 1.588 | 13.13% | 0.37 |
| 0.50 | 2.014 | 0.50 | 1.698 | 15.67% | 0.41 |
| 0.55 | 2.241 | 0.55 | 1.826 | 18.52% | 0.45 |
| 0.60 | 2.526 | 0.60 | 1.974 | 21.84% | 0.49 |
| 0.65 | 2.894 | 0.65 | 2.149 | 25.75% | 0.53 |
| 0.70 | 3.388 | 0.70 | 2.357 | 30.43% | 0.58 |
| 0.75 | 4.084 | 0.75 | 2.610 | 36.09% | 0.62 |
| 0.80 | 5.141 | 0.80 | 2.924 | 43.13% | 0.66 |
| 0.85 | 6.936 | 0.85 | 3.323 | 52.09% | 0.70 |
| 0.90 | 10.658 | 0.90 | 3.849 | 63.88% | 0.74 |
| 0.95 | 22.997 | 0.95 | 4.573 | 80.11% | 0.78 |
| 1.00 | -145.799 | 1.00 | 5.633 | 103.86% | 0.82 |

In the above tables (Table 4.1 to 4.5), the results of the two methods are compared to each other. When the error in approximate method is about 5% than the exact one, θ becomes 0.2. When the error reaches 10%, the difference between the two methods becomes distinguishable, and θ reaches 0.3. At this point the weak column (supported sway column) reaches a deflected shape very different from the basic assumption of the approximate method due to the larger member deflection (δ) of the weak column than the deflection of the whole frame (Δ). Above this point the approximate method becomes less accurate compared to the "exact" one.

The above scenario is illustrated more using the effect of geometric nonlinearity caused by the P-delta effects on the lateral load capacity of the frame in the next section.

### 4.1.2 P-Delta effects on the lateral load resisting capacity of columns in a frame

Table 4.6 to 4.10 shows the effect of the axial load p applied to the deformed frame on the lateral force resisting capacity of the column; and location of θ=0.3 using the procedures explained in section 2.3.1.1, fig.2.7.

The first order deflection $\Delta_o$ is calculated by applying a lateral load on the frame in the absence of vertical load, shown in the first row of each table. With the application of the axial load P on the deformed frame, the final deflection $\Delta_f$ is calculated using "exact" second-order analysis method which accounts the geometric nonlinearities caused by both P-Δ and P-δ effect.

| TABLE 4.6 EI Vs 2 EI , Case I | comparision of total shear and lateral force shear resisted by the columns | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $(P/P_{cr})$ of weak c | P | Δ | $f_s = \Delta_f/\Delta_0$ | Total shear force on the frame $V_{tot} = V_0 * f_s$ | Total shear force resisted by weak col. $V_1$ | Total shear force resisted by strong col $V_2 = 2V_1$ | Lateral force shear(weak c) $H_1 = V_1 - P\Delta/l$ | Lateral force shear(Strong c) $H_2 = V_2 - 0$ | θ |
| - | - | 0.9926 | | 9105.55 | 3035.183 | 6070.367 | 3035.183 | 6070.367 | - |
| 0.10 | 754.476 | 1.026 | 1.034 | 9416.241 | 3138.747 | 6277.494 | 2880.596 | 6277.494 | 0.03 |
| 0.20 | 1508.953 | 1.063 | 1.071 | 9748.891 | 3249.630 | 6499.261 | 2715.088 | 6499.261 | 0.05 |
| 0.30 | 2263.429 | 1.102 | 1.110 | 10105.906 | 3368.635 | 6737.271 | 2537.458 | 6737.271 | 0.08 |
| 0.40 | 3017.906 | 1.144 | 1.152 | 10490.063 | 3496.688 | 6993.375 | 2346.323 | 6993.375 | 0.11 |
| 0.50 | 3772.382 | 1.189 | 1.198 | 10904.581 | 3634.860 | 7269.721 | 2140.084 | 7269.721 | 0.14 |
| 0.60 | 4526.859 | 1.238 | 1.247 | 11353.204 | 3784.401 | 7568.803 | 1916.874 | 7568.803 | 0.16 |
| 0.70 | 5281.335 | 1.291 | 1.300 | 11840.327 | 3946.776 | 7893.551 | 1674.511 | 7893.551 | 0.19 |
| 0.80 | 6035.811 | 1.349 | 1.359 | 12371.124 | 4123.708 | 8247.416 | 1410.417 | 8247.416 | 0.22 |
| 0.90 | 6790.288 | 1.412 | 1.422 | 12951.745 | 4317.248 | 8634.497 | 1121.533 | 8634.497 | 0.25 |
| 1.00 | 7544.764 | 1.481 | 1.492 | 13589.552 | 4529.851 | 9059.701 | 804.198 | 9059.701 | 0.27 |
| 1.10 | 8299.241 | 1.558 | 1.570 | 14293.428 | 4764.476 | 9528.952 | 453.989 | 9528.952 | 0.30 |
| 1.20 | 9053.717 | 1.643 | 1.655 | 15074.206 | 5024.735 | 10049.470 | 65.520 | 10049.470 | 0.33 |
| 1.30 | 9808.194 | 1.738 | 1.751 | 15945.207 | 5315.069 | 10630.138 | -367.841 | 10630.138 | 0.36 |
| 1.40 | 10562.670 | 1.845 | 1.859 | 16923.041 | 5641.014 | 11282.027 | -854.353 | 11282.027 | 0.38 |
| 1.50 | 11317.146 | 1.965 | 1.980 | 18028.639 | 6009.546 | 12019.093 | -1404.434 | 12019.093 | 0.41 |
| 1.60 | 12071.623 | 2.103 | 2.118 | 19288.791 | 6429.597 | 12859.194 | -2031.414 | 12859.194 | 0.44 |
| 1.70 | 12826.099 | 2.261 | 2.278 | 20738.347 | 6912.782 | 13825.565 | -2752.628 | 13825.565 | 0.47 |
| 1.80 | 13580.576 | 2.444 | 2.463 | 22423.471 | 7474.490 | 14948.981 | -3591.049 | 14948.981 | 0.49 |
| 1.90 | 14335.052 | 2.661 | 2.680 | 24406.676 | 8135.559 | 16271.117 | -4577.776 | 16271.117 | 0.52 |
| 2.00 | 15089.529 | 2.919 | 2.940 | 26774.712 | 8924.904 | 17849.808 | -5755.975 | 17849.808 | 0.55 |

TABLE 4.7

4EI Vs 8 EI, Case I

**comparision of total shear and lateral force shear resisted by the columns**

| $(P/P_{cr})$ of weak c | P | Δ | $f_s = \Delta_f/\Delta_0$ | Total shear force on the frame $V_{tot} = V_0 * f_s$ | Total shear force resisted by weak col. $V_1$ | Total shear force resisted by strong c $V_2 = 2V_1$ | Lateral force shear(weak c) $H_1 = V_1 - P\Delta/l$ | Lateral force shear(Strong c) $H_2 = V_2 - 0$ | θ |
|---|---|---|---|---|---|---|---|---|---|
| - | - | 0.9999 | | 36693.333 | 12231.111 | 24462.222 | 12231.111 | 24462.222 | - |
| 0.15 | 4526.859 | 1.053 | 1.053 | 38625.555 | 12875.185 | 25750.370 | 11286.932 | 25750.370 | 0.04 |
| 0.30 | 9053.717 | 1.111 | 1.111 | 40768.054 | 13589.351 | 27178.703 | 10236.649 | 27178.703 | 0.08 |
| 0.45 | 13580.576 | 1.176 | 1.176 | 43162.196 | 14387.399 | 28774.797 | 9063.009 | 28774.797 | 0.12 |
| 0.60 | 18107.434 | 1.250 | 1.250 | 45855.077 | 15285.026 | 30570.052 | 7742.922 | 30570.052 | 0.16 |
| 0.75 | 22634.293 | 1.333 | 1.333 | 48906.330 | 16302.110 | 32604.220 | 6247.155 | 32604.220 | 0.21 |
| 0.90 | 27161.151 | 1.428 | 1.428 | 52392.604 | 17464.201 | 34928.403 | 4538.137 | 34928.403 | 0.25 |
| 1.05 | 31688.010 | 1.537 | 1.537 | 56414.060 | 18804.687 | 37609.373 | 2566.764 | 37609.373 | 0.29 |
| 1.20 | 36214.868 | 1.665 | 1.665 | 61104.187 | 20368.062 | 40736.124 | 267.601 | 40736.124 | 0.33 |
| 1.35 | 40741.727 | 1.816 | 1.816 | 66644.876 | 22214.959 | 44429.917 | -2448.521 | 44429.917 | 0.37 |
| 1.50 | 45268.586 | 1.997 | 1.997 | 73290.582 | 24430.194 | 48860.388 | -5706.336 | 48860.388 | 0.41 |
| 1.65 | 49795.444 | 2.218 | 2.219 | 81408.488 | 27136.163 | 54272.325 | -9685.844 | 54272.325 | 0.45 |
| 1.80 | 54322.303 | 2.495 | 2.495 | 91548.725 | 30516.242 | 61032.483 | -14656.726 | 61032.483 | 0.49 |
| 1.95 | 58849.161 | 2.850 | 2.850 | 104574.516 | 34858.172 | 69716.344 | -21042.148 | 69716.344 | 0.53 |
| 2.10 | 63376.020 | 3.322 | 3.323 | 121921.894 | 40640.631 | 81281.263 | -29546.067 | 81281.263 | 0.58 |
| 2.25 | 67902.878 | 3.983 | 3.984 | 146169.204 | 48723.068 | 97446.136 | -41432.430 | 97446.136 | 0.62 |
| 2.40 | 72429.737 | 4.972 | 4.972 | 182455.134 | 60818.378 | 121636.756 | -59220.284 | 121636.756 | 0.66 |
| 2.55 | 76956.595 | 6.614 | 6.614 | 242705.873 | 80901.958 | 161803.915 | -88756.019 | 161803.915 | 0.70 |
| 2.70 | 81483.454 | 9.875 | 9.876 | 362367.867 | 120789.289 | 241578.578 | -147415.952 | 241578.578 | 0.74 |
| 2.85 | 86010.312 | 19.478 | 19.480 | 714775.595 | 238258.532 | 476517.063 | -320171.020 | 476517.063 | 0.78 |
| 3.00 | 90537.171 | 708.636 | 708.706 | 26004802.894 | 8668267.631 | 17336535.262 | -12717686.844 | 17336535.262 | 0.82 |

| TABLE 4.8<br>EI Vs 5 EI, Case I | comparision of total shear and lateral force shear resisted by the columns | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $(P/P_{cr})$ of weak c | P | $\Delta$ | $f_s = \Delta_f/\Delta_0$ | Total shear force on the frame $V_{tot} = V_0 * f_s$ | Total shear force resisted by weak col $V_1$ | Total shear force resisted by strong c $V_2 = 5V_1$ | Lateral force shear(weak c) $H_1 = V_1 - P\Delta/l$ | Lateral force shear(Strong c) $H_2 = V_2 - 0$ | $\theta$ |
| - | - | 1.00 | | 18346.669 | 3057.778 | 15288.891 | 3057.778 | 15288.891 | |
| 0.15 | 1131.715 | 1.026 | 1.026 | 18815.206 | 3135.868 | 15679.339 | 2748.996 | 15679.339 | 0.02 |
| 0.30 | 2263.429 | 1.052 | 1.052 | 19308.302 | 3218.050 | 16090.252 | 2424.028 | 16090.252 | 0.04 |
| 0.45 | 3395.144 | 1.081 | 1.081 | 19827.939 | 3304.656 | 16523.282 | 2081.570 | 16523.282 | 0.06 |
| 0.60 | 4526.859 | 1.111 | 1.111 | 20376.317 | 3396.053 | 16980.264 | 1720.168 | 16980.264 | 0.08 |
| 0.75 | 5658.573 | 1.142 | 1.142 | 20955.892 | 3492.649 | 17463.243 | 1338.208 | 17463.243 | 0.10 |
| 0.90 | 6790.288 | 1.176 | 1.176 | 21569.403 | 3594.900 | 17974.502 | 933.882 | 17974.502 | 0.12 |
| 1.05 | 7922.002 | 1.211 | 1.211 | 22219.921 | 3703.320 | 18516.600 | 505.169 | 18516.600 | 0.14 |
| 1.20 | 9053.717 | 1.249 | 1.249 | 22910.895 | 3818.482 | 19092.412 | 49.792 | 19092.412 | 0.16 |
| 1.35 | 10185.432 | 1.289 | 1.289 | 23646.224 | 3941.037 | 19705.186 | -434.816 | 19705.186 | 0.19 |
| 1.50 | 11317.146 | 1.332 | 1.332 | 24430.320 | 4071.720 | 20358.600 | -951.563 | 20358.600 | 0.21 |
| 1.65 | 12448.861 | 1.377 | 1.377 | 25268.198 | 4211.366 | 21056.831 | -1503.754 | 21056.831 | 0.23 |
| 1.80 | 13580.576 | 1.426 | 1.426 | 26165.590 | 4360.932 | 21804.658 | -2095.168 | 21804.658 | 0.25 |
| 1.95 | 14712.290 | 1.479 | 1.479 | 27129.073 | 4521.512 | 22607.561 | -2730.137 | 22607.561 | 0.27 |
| 2.10 | 15844.005 | 1.535 | 1.535 | 28166.221 | 4694.370 | 23471.851 | -3413.655 | 23471.851 | 0.29 |
| 2.25 | 16975.720 | 1.596 | 1.596 | 29285.825 | 4880.971 | 24404.854 | -4151.513 | 24404.854 | 0.31 |
| 2.40 | 18107.434 | 1.662 | 1.662 | 30498.121 | 5083.020 | 25415.101 | -4950.458 | 25415.101 | 0.33 |
| 2.55 | 19239.149 | 1.734 | 1.734 | 31815.116 | 5302.519 | 26512.597 | -5818.405 | 26512.597 | 0.35 |
| 2.70 | 20370.863 | 1.812 | 1.812 | 33250.987 | 5541.831 | 27709.156 | -6764.696 | 27709.156 | 0.37 |
| 2.85 | 21502.578 | 1.898 | 1.898 | 34822.592 | 5803.765 | 29018.827 | -7800.439 | 29018.827 | 0.39 |
| 3.00 | 22634.293 | 1.992 | 1.992 | 36550.131 | 6091.689 | 30458.443 | -8938.949 | 30458.443 | 0.41 |

TABLE 4.9 comparision of total shear and lateral force shear resisted by the columns

EI Vs 8 EI, Case I

| $(P/P_{cr})$ of weak c | P | Δ | $f_s = \Delta_f/\Delta_0$ | Total shear force on the frame $V_{tot} = V_0 * f_s$ | Total shear force esisted by weak col $V_1$ | Total shear force resisted by strong c $V_2 = 8V_1$ | Lateral force shear(weak c) $H_1 = V_1 - P\Delta/l$ | Lateral force shear(Strong c) $H_2 = V_2 - 0$ | θ |
|---|---|---|---|---|---|---|---|---|---|
| - | - | 0.9972 | | 27442.9462 | 3049.216 | 24393.730 | 3049.216 | 24393.730 | |
| 0.23 | 1697.572 | 1.023 | 1.026 | 28148.435 | 3127.604 | 25020.831 | 2548.825 | 25020.831 | 0.02 |
| 0.45 | 3395.144 | 1.050 | 1.053 | 28891.153 | 3210.128 | 25681.025 | 2022.027 | 25681.025 | 0.04 |
| 0.68 | 5092.716 | 1.078 | 1.081 | 29674.128 | 3297.125 | 26377.002 | 1466.676 | 26377.002 | 0.06 |
| 0.90 | 6790.288 | 1.108 | 1.111 | 30500.724 | 3388.969 | 27111.755 | 880.386 | 27111.755 | 0.08 |
| 1.13 | 8487.860 | 1.140 | 1.143 | 31374.690 | 3486.077 | 27888.614 | 260.496 | 27888.614 | 0.10 |
| 1.35 | 10185.432 | 1.174 | 1.177 | 32300.218 | 3588.913 | 28711.305 | -395.966 | 28711.305 | 0.12 |
| 1.58 | 11883.004 | 1.209 | 1.213 | 33282.011 | 3698.001 | 29584.010 | -1092.336 | 29584.010 | 0.14 |
| 1.80 | 13580.576 | 1.247 | 1.251 | 34325.360 | 3813.929 | 30511.431 | -1832.366 | 30511.431 | 0.16 |
| 2.03 | 15278.148 | 1.288 | 1.291 | 35436.241 | 3937.360 | 31498.881 | -2620.296 | 31498.881 | 0.19 |
| 2.25 | 16975.720 | 1.331 | 1.334 | 36621.431 | 4069.048 | 32552.383 | -3460.931 | 32552.383 | 0.21 |
| 2.48 | 18673.292 | 1.377 | 1.381 | 37888.645 | 4209.849 | 33678.795 | -4359.744 | 33678.795 | 0.23 |
| 2.70 | 20370.863 | 1.426 | 1.430 | 39246.696 | 4360.744 | 34885.952 | -5322.989 | 34885.952 | 0.25 |
| 2.93 | 22068.435 | 1.479 | 1.483 | 40705.724 | 4522.858 | 36182.866 | -6357.854 | 36182.866 | 0.27 |
| 3.15 | 23766.007 | 1.536 | 1.541 | 42277.422 | 4697.491 | 37579.930 | -7472.633 | 37579.930 | 0.29 |
| 3.38 | 25463.579 | 1.598 | 1.602 | 43975.365 | 4886.152 | 39089.213 | -8676.955 | 39089.213 | 0.31 |
| 3.60 | 27161.151 | 1.665 | 1.669 | 45815.399 | 5090.600 | 40724.799 | -9982.061 | 40724.799 | 0.33 |
| 3.83 | 28858.723 | 1.738 | 1.742 | 47816.139 | 5312.904 | 42503.234 | -11401.153 | 42503.234 | 0.35 |
| 4.05 | 30556.295 | 1.817 | 1.822 | 49999.603 | 5555.511 | 44444.092 | -12949.848 | 44444.092 | 0.37 |
| 4.28 | 32253.867 | 1.904 | 1.909 | 52392.019 | 5821.335 | 46570.683 | -14646.749 | 46570.683 | 0.39 |
| 4.50 | 33951.439 | 1.999 | 2.005 | 55024.888 | 6113.876 | 48911.011 | -16514.200 | 48911.011 | 0.41 |

| TABLE 4.10<br>EI Vs 10 EI, Case I | comparision of total shear and lateral force shear resisted by the columns | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $(P/P_{cr})$ of weak c | P | Δ | $f_s = \Delta_f/\Delta_o$ | Total shear force on the frame $V_{tot} = V_o * f_s$ | Total shear force resisted by weak col. $V_1$ | Total shear force resisted by strong col $V_2 = 10V_1$ | Lateral force shear(weak c) $H_1 = V_1 - P\Delta/l$ | Lateral force shear(Strong c) $H_2 = V_2 - 0$ | θ |
| - | - | 1.00 | | 33635.558 | 3057.778 | 30577.780 | 3057.778 | 30577.780 | |
| 0.14 | 1037.405 | 1.013 | 1.013 | 34064.284 | 3096.753 | 30967.530 | 2746.544 | 30967.530 | 0.01 |
| 0.28 | 2074.810 | 1.026 | 1.026 | 34504.075 | 3136.734 | 31367.341 | 2427.273 | 31367.341 | 0.02 |
| 0.41 | 3112.215 | 1.039 | 1.039 | 34955.374 | 3177.761 | 31777.612 | 2099.650 | 31777.612 | 0.03 |
| 0.55 | 4149.620 | 1.053 | 1.053 | 35418.636 | 3219.876 | 32198.760 | 1763.343 | 32198.760 | 0.04 |
| 0.69 | 5187.025 | 1.067 | 1.067 | 35894.340 | 3263.122 | 32631.219 | 1418.003 | 32631.219 | 0.05 |
| 0.83 | 6224.431 | 1.082 | 1.082 | 36382.998 | 3307.545 | 33075.453 | 1063.259 | 33075.453 | 0.06 |
| 0.96 | 7261.836 | 1.097 | 1.097 | 36885.143 | 3353.195 | 33531.948 | 698.724 | 33531.948 | 0.07 |
| 1.10 | 8299.241 | 1.112 | 1.112 | 37401.345 | 3400.122 | 34001.222 | 323.985 | 34001.222 | 0.08 |
| 1.24 | 9336.646 | 1.128 | 1.128 | 37932.198 | 3448.382 | 34483.816 | -61.391 | 34483.816 | 0.09 |
| 1.38 | 10374.051 | 1.144 | 1.144 | 38478.338 | 3498.031 | 34980.308 | -457.865 | 34980.308 | 0.10 |
| 1.51 | 11411.456 | 1.161 | 1.161 | 39040.436 | 3549.131 | 35491.305 | -865.921 | 35491.305 | 0.11 |
| 1.65 | 12448.861 | 1.178 | 1.178 | 39619.199 | 3601.745 | 36017.454 | -1286.077 | 36017.454 | 0.12 |
| 1.79 | 13486.266 | 1.196 | 1.196 | 40215.379 | 3655.944 | 36559.436 | -1718.877 | 36559.436 | 0.13 |
| 1.93 | 14523.671 | 1.214 | 1.214 | 40829.777 | 3711.798 | 37117.979 | -2164.902 | 37117.979 | 0.14 |
| 2.06 | 15561.076 | 1.233 | 1.233 | 41463.239 | 3769.385 | 37693.853 | -2624.767 | 37693.853 | 0.15 |
| 2.20 | 16598.481 | 1.252 | 1.252 | 42116.666 | 3828.788 | 38287.879 | -3099.126 | 38287.879 | 0.16 |
| 2.34 | 17635.886 | 1.272 | 1.272 | 42791.016 | 3890.092 | 38900.923 | -3588.675 | 38900.923 | 0.17 |
| 2.48 | 18673.292 | 1.293 | 1.293 | 43487.315 | 3953.392 | 39533.923 | -4094.156 | 39533.923 | 0.19 |
| 2.61 | 19710.697 | 1.314 | 1.314 | 44206.649 | 4018.786 | 40187.863 | -4616.360 | 40187.863 | 0.20 |
| 2.75 | 20748.102 | 1.336 | 1.336 | 44950.180 | 4086.380 | 40863.800 | -5156.130 | 40863.800 | 0.21 |
| 2.89 | 21785.507 | 1.359 | 1.359 | 45719.149 | 4156.286 | 41562.863 | -5714.367 | 41562.863 | 0.22 |
| 3.03 | 22822.912 | 1.383 | 1.383 | 46514.886 | 4228.626 | 42286.260 | -6292.037 | 42286.260 | 0.23 |
| 3.16 | 23860.317 | 1.407 | 1.407 | 47338.816 | 4303.529 | 43035.287 | -6890.172 | 43035.287 | 0.24 |
| 3.30 | 24897.722 | 1.433 | 1.433 | 48192.459 | 4381.133 | 43811.326 | -7509.879 | 43811.326 | 0.25 |
| 3.44 | 25935.127 | 1.459 | 1.459 | 49077.454 | 4461.587 | 44615.868 | -8152.346 | 44615.868 | 0.26 |
| 3.58 | 26972.532 | 1.486 | 1.486 | 49995.560 | 4545.051 | 45450.509 | -8818.851 | 45450.509 | 0.27 |
| 3.71 | 28009.937 | 1.515 | 1.515 | 50948.674 | 4631.698 | 46316.977 | -9510.768 | 46316.977 | 0.28 |
| 3.85 | 29047.342 | 1.544 | 1.544 | 51938.831 | 4721.712 | 47217.119 | -10229.579 | 47217.119 | 0.29 |
| 3.99 | 30084.747 | 1.575 | 1.575 | 52968.241 | 4815.295 | 48152.946 | -10976.883 | 48152.946 | 0.30 |
| 4.13 | 31122.153 | 1.607 | 1.607 | 54039.278 | 4912.662 | 49126.616 | -11754.409 | 49126.616 | 0.31 |
| 4.26 | 32159.558 | 1.640 | 1.640 | 55154.525 | 5014.048 | 50140.477 | -12564.028 | 50140.477 | 0.32 |
| 4.40 | 33196.963 | 1.674 | 1.674 | 56316.772 | 5119.707 | 51197.065 | -13407.767 | 51197.065 | 0.33 |

As a horizontal force is applied to the frame, it is resisted by each column in proportion to their lateral stiffness and the frame undergoes a lateral deflection of $\Delta_o$.

With continuous application of axial load on the displaced frame, the stiffness of the weak column is reduced due to the geometric non-linearity caused by the axial load. As a result, the member deflection (δ) of the weak column increases faster than the deflection of the whole frame (Δ).

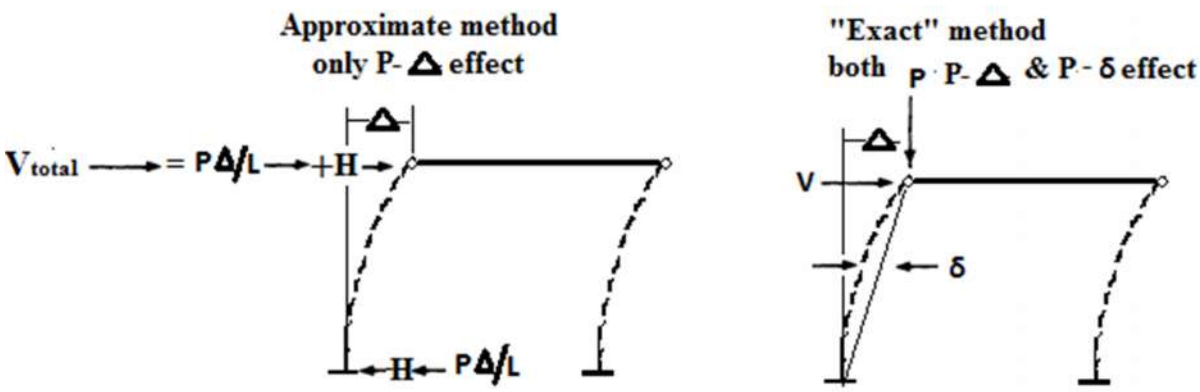



**Figure 4.3: Approximate and "Exact" method**

The total shear on the frame is the sum of the external lateral load shear (H) and the P-delta shear (PΔ/l) due to the axial load on the deformed frame. As the load increases the P-delta shear on the weak column become greater than the lateral load resisting capacity of the column. Consequently, the shear resisting the lateral force H becomes negative, indicating that the column need lateral support from the frame in order not to fall sidesway as shown in fig.4.4 (a).

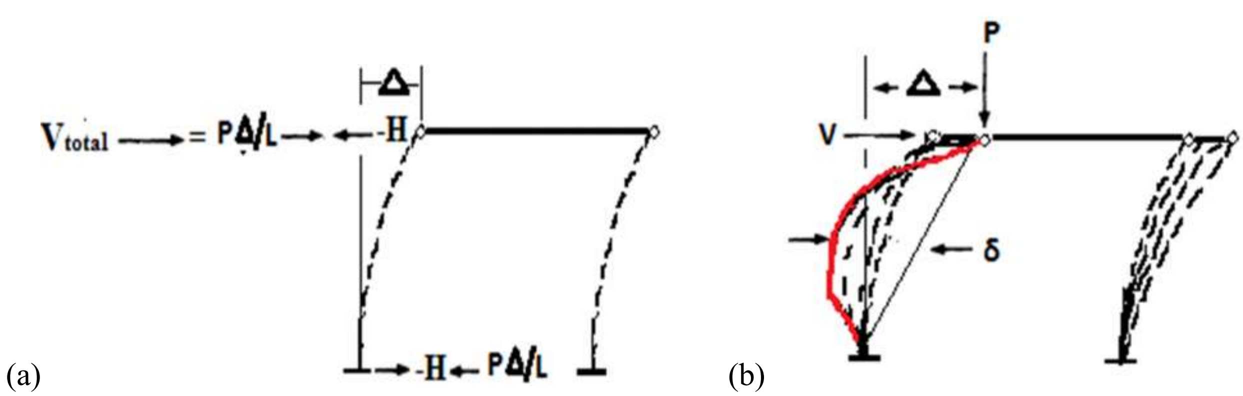



**Figure 4.4: (a) supported sway column (b) deflection shape in the approach of elastic failure mode**

The required negative lateral load shear must be provided by the strong column to maintain the equilibrium of the weak column as well as the whole frame. At this time the weak column is supported sway column, which is braced by the supporting sway column, the strong column (Fig 4.4 (a)).

As the application of the axial load increases, the deflected shape of the weak column approaches the shape of the elastic failure mode (buckling shape) of a column, which is different from the deflection shape of the basic assumption of approximate method (Fig 4.4 (b)). This is due to the high member deflection of the weak column than the total frame deflection as stated above. This will make the approximate method (which considers only P-Δ effect) less accurate compared to the "exact" one (which consider both P-Δ and P-δ). And as the shape of the weak column approaches the shape of elastic failure mode (buckling shape) of a column, the deflection calculated using approximate method is 10% different from the deflection calculated using the exact method and θ reaches 0.3 as shown in tables 4.1-4.5.

The bracing action of the strong column is increased as its stiffness is increased from 2EI to 10EI as it can be observed from tables 4.6-4.10.

For frames with strong column stiffness > 5EI, the strong column brace sufficiently so that it decreases both the deflection of the frame and the member deflection of the weak column. As a result, the weak column can carry axial load far beyond its critical load after it becomes supported sway column and before its deflection approach elastic failure mode or before θ reaches 0.3.

For frames with relative stiffness of EI $v_s$ 2EI and 4EI $v_s$ 8EI however, the deflected shape of the weak column approach elastic failure mode shape just as it becomes supported sway column and θ reaches 0.3 before this happens.

From the above discussion it can be seen that the point where the weak column becomes supported sway column is not necessarily the same as the point where its deflection shape reaches a shape different from the basic assumption of the approximate method; rather it is based on / affected by the stiffness of the strong column relative to the weak one.

## 4.2 Case II: Load Applied Both on the Weak and Strong Column

In case II, the axial load P is applied to both strong and weak columns in proportion to their stiffness. The point where θ=0.3 is illustrated using table 4.11 to 4.15 which shows the P-delta effect on the lateral load resisting capacity of the column. The tables are developed by the same procedures as in case I.

TABLE 4.11 comparision of total shear and lateral force shear resisted by the columns
EI Vs 2 EI , case II

| $(P/P_{cr})$ of the frame | $P_1$ | $\Delta$ | $P_2 = 2P_1$ | $f_s = \Delta_f/\Delta_0$ | Total shear force on the frame $V_{tot} = V_0 * f_s$ | Total shear force resisted by weak col. $V_1$ | Total shear force resisted by strong col. $V_2 = 2V_1$ | Lateral force shear res. by weak c $H_1 = V_1 - P_1\Delta/l$ | Lateral force shear res. by strong c $H_2 = V_2 - P_2\Delta/l$ | $\theta$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | - | 0.9926 | - | | 9105.55 | 3035.183 | 6070.367 | 3035.183 | 6070.367 | - |
| 0.04 | 301.791 | 1.033 | 603.581 | 1.041 | 9479.789 | 3159.930 | 6319.859 | 3055.972 | 6111.944 | 0.03 |
| 0.10 | 754.476 | 1.101 | 1508.953 | 1.110 | 10102.632 | 3367.544 | 6735.088 | 3090.575 | 6513.513 | 0.08 |
| 0.16 | 1207.162 | 1.179 | 2414.325 | 1.188 | 10813.074 | 3604.358 | 7208.716 | 3130.044 | 6971.559 | 0.13 |
| 0.22 | 1659.848 | 1.268 | 3319.696 | 1.277 | 11630.995 | 3876.998 | 7753.997 | 3175.484 | 7498.900 | 0.18 |
| 0.28 | 2112.534 | 1.372 | 4225.068 | 1.382 | 12582.779 | 4194.260 | 8388.519 | 3228.360 | 8112.548 | 0.23 |
| 0.34 | 2565.220 | 1.494 | 5130.440 | 1.505 | 13704.219 | 4568.073 | 9136.146 | 3290.663 | 8835.579 | 0.28 |
| 0.40 | 3017.906 | 1.640 | 6035.811 | 1.652 | 15045.113 | 5015.038 | 10030.076 | 3365.156 | 9700.099 | 0.33 |
| 0.46 | 3470.592 | 1.818 | 6941.183 | 1.832 | 16676.869 | 5558.956 | 11117.912 | 3455.809 | 10752.148 | 0.38 |
| 0.52 | 3923.277 | 2.039 | 7846.555 | 2.054 | 18705.635 | 6235.212 | 12470.423 | 3568.518 | 12060.163 | 0.43 |
| 0.58 | 4375.963 | 2.322 | 8751.927 | 2.339 | 21296.370 | 7098.790 | 14197.580 | 3712.448 | 13730.499 | 0.48 |
| 0.64 | 4828.649 | 2.695 | 9657.298 | 2.715 | 24720.113 | 8240.038 | 16480.076 | 3902.656 | 15937.903 | 0.53 |
| 0.70 | 5281.335 | 3.211 | 10562.670 | 3.235 | 29455.583 | 9818.528 | 19637.056 | 4165.737 | 18991.022 | 0.58 |
| 0.76 | 5734.021 | 3.972 | 11468.042 | 4.001 | 36435.244 | 12145.081 | 24290.163 | 4553.495 | 23491.049 | 0.63 |
| 0.82 | 6186.707 | 5.205 | 12373.413 | 5.244 | 47749.827 | 15916.609 | 31833.218 | 5182.082 | 30785.947 | 0.67 |
| 0.88 | 6639.393 | 7.550 | 13278.785 | 7.606 | 69256.797 | 23085.599 | 46171.198 | 6376.912 | 44652.227 | 0.72 |
| 0.94 | 7092.078 | 13.737 | 14184.157 | 13.839 | 126015.245 | 42005.082 | 84010.163 | 9530.154 | 81246.340 | 0.77 |
| 1.00 | 7544.764 | 76.121 | 15089.529 | 76.688 | 698285.389 | 232761.796 | 465523.593 | 41322.892 | 450208.480 | 0.82 |
| 1.06 | 7997.450 | -21.495 | 15994.900 | -21.655 | -197184.634 | -65728.211 | -131456.423 | -8425.367 | -127131.680 | 0.87 |
| 1.12 | 8450.136 | -9.418 | 16900.272 | -9.488 | -86394.152 | -28798.051 | -57596.102 | -2270.350 | -55701.266 | 0.92 |
| 1.18 | 8902.822 | -6.030 | 17805.644 | -6.075 | -55314.857 | -18438.286 | -36876.571 | -543.725 | -35663.381 | 0.97 |

**TABLE 4.12** **comparision of total shear and lateral force shear resisted by the columns**

**4EI Vs 8EI, Case II**

| $(P/P_{cr})$ of the frame | $P_1$ | $\Delta$ | $P_2 = 2P_1$ | $f_s = \Delta_f/\Delta_o$ | Total shear force on the frame $V_{tot} = V_o * f_s$ | Total shear force resisted by weak col. $V_1$ | Total shear force resisted by strong col. $V_2 = 2V_1$ | Lateral force shear res. by weak c $H_1 = V_1 - P_1\Delta/l$ | Lateral force shear res. by strong c $H_2 = V_2 - P_2\Delta/l$ | $\theta$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | - | 1.000 | - | | 36693.333 | 12231.111 | 24462.222 | 12231.111 | 24462.222 | - |
| 0.03 | 754.476 | 1.025 | 1508.953 | 1.025 | 37621.608 | 12540.536 | 25081.072 | 12282.682 | 24565.363 | 0.02 |
| 0.10 | 3017.906 | 1.110 | 6035.811 | 1.110 | 40711.385 | 13570.462 | 27140.923 | 12454.336 | 26582.860 | 0.08 |
| 0.18 | 5281.335 | 1.209 | 10562.670 | 1.209 | 44354.086 | 14784.695 | 29569.391 | 12656.709 | 28961.395 | 0.14 |
| 0.25 | 7544.764 | 1.328 | 15089.529 | 1.328 | 48712.715 | 16237.572 | 32475.143 | 12898.855 | 31807.400 | 0.21 |
| 0.33 | 9808.194 | 1.472 | 19616.387 | 1.472 | 54021.330 | 18007.110 | 36014.220 | 13193.777 | 35273.707 | 0.27 |
| 0.40 | 12071.623 | 1.652 | 24143.246 | 1.652 | 60628.508 | 20209.503 | 40419.005 | 13560.843 | 39587.923 | 0.33 |
| 0.48 | 14335.052 | 1.883 | 28670.104 | 1.883 | 69077.115 | 23025.705 | 46051.410 | 14030.210 | 45104.516 | 0.39 |
| 0.55 | 16598.481 | 2.187 | 33196.963 | 2.187 | 80261.595 | 26753.865 | 53507.730 | 14651.570 | 52407.521 | 0.45 |
| 0.63 | 18861.911 | 2.610 | 37723.821 | 2.610 | 95767.610 | 31922.537 | 63845.074 | 15513.015 | 62532.312 | 0.51 |
| 0.70 | 21125.340 | 3.235 | 42250.680 | 3.235 | 118699.601 | 39566.534 | 79133.067 | 16787.015 | 77505.959 | 0.58 |
| 0.78 | 23388.769 | 4.253 | 46777.538 | 4.253 | 156071.676 | 52023.892 | 104047.784 | 18863.241 | 101908.387 | 0.64 |
| 0.85 | 25652.198 | 6.208 | 51304.397 | 6.208 | 227790.586 | 75930.195 | 151860.390 | 22847.625 | 148737.886 | 0.70 |
| 0.93 | 27915.628 | 11.486 | 55831.255 | 11.486 | 421464.063 | 140488.021 | 280976.042 | 33607.262 | 275198.703 | 0.76 |
| 1.00 | 30179.057 | 76.690 | 60358.114 | 76.690 | 2814006.240 | 938002.080 | 1876004.160 | 166526.285 | 1837430.371 | 0.82 |
| 1.08 | 32442.486 | -16.398 | 64884.973 | -16.398 | -601702.705 | -200567.568 | -401135.137 | -23235.338 | -392887.126 | 0.88 |
| 1.15 | 34705.916 | -7.407 | 69411.831 | -7.407 | -271793.410 | -90597.803 | -181195.607 | -4907.040 | -177469.922 | 0.95 |
| 1.23 | 36969.345 | -4.784 | 73938.690 | -4.784 | -175543.881 | -58514.627 | -117029.254 | 440.154 | -114622.936 | 1.01 |
| 1.30 | 39232.774 | -3.533 | 78465.548 | -3.533 | -129636.166 | -43212.055 | -86424.111 | 2990.583 | -84647.086 | 1.07 |
| 1.38 | 41496.203 | -2.801 | 82992.407 | -2.801 | -102762.119 | -34254.040 | -68508.079 | 4483.586 | -67099.439 | 1.13 |
| 1.45 | 43759.633 | -2.320 | 87519.265 | -2.320 | -85117.048 | -28372.349 | -56744.698 | 5463.868 | -55577.932 | 1.19 |

TABLE 4.13
EI Vs 5EI, Case II

## comparision of total shear and lateral force shear resisted by the columns

| $(P/P_{cr})$ of the frame | $P_1$ | $\Delta$ | $P_2 = 5P_1$ | $f_s = \Delta_f/\Delta_o$ | Total shear force on the frame $V_{tot} = V_o * f_s$ | Total shear force resisted by weak col. $V_1$ | Total shear force resisted by strong col. $V_2 = 5V_1$ | Lateral force shear res. by weak c $H_1 = V_1 - P_1\Delta/l$ | Lateral force shear res. by strong c $H_2 = V_2 - P_2\Delta/l$ | $\theta$ |
|---|---|---|---|---|---|---|---|---|---|---|
| - | - | 1.000 | - | | 18346.669 | 3057.778 | 15288.891 | 3057.778 | 15288.891 | - |
| 0.04 | 301.791 | 1.041 | 1508.953 | 1.041 | 19100.735 | 3183.456 | 15917.280 | 3078.724 | 15393.622 | 0.03 |
| 0.10 | 754.476 | 1.110 | 3772.382 | 1.110 | 20355.695 | 3392.616 | 16963.079 | 3113.584 | 16405.017 | 0.08 |
| 0.16 | 1207.162 | 1.188 | 6035.811 | 1.188 | 21787.159 | 3631.193 | 18155.966 | 3153.347 | 17558.659 | 0.13 |
| 0.22 | 1659.848 | 1.277 | 8299.241 | 1.277 | 23435.178 | 3905.863 | 19529.315 | 3199.126 | 18886.827 | 0.18 |
| 0.28 | 2112.534 | 1.382 | 10562.670 | 1.382 | 25352.921 | 4225.487 | 21127.434 | 3252.396 | 20432.369 | 0.23 |
| 0.34 | 2565.220 | 1.505 | 12826.099 | 1.505 | 27612.500 | 4602.083 | 23010.417 | 3315.162 | 22253.404 | 0.28 |
| 0.40 | 3017.906 | 1.652 | 15089.529 | 1.652 | 30314.254 | 5052.376 | 25261.879 | 3390.211 | 24430.796 | 0.33 |
| 0.46 | 3470.592 | 1.832 | 17352.958 | 1.832 | 33602.066 | 5600.344 | 28001.721 | 3481.539 | 27080.502 | 0.38 |
| 0.52 | 3923.277 | 2.054 | 19616.387 | 2.054 | 37689.810 | 6281.635 | 31408.175 | 3595.088 | 30374.887 | 0.43 |
| 0.58 | 4375.963 | 2.339 | 21879.816 | 2.339 | 42909.861 | 7151.643 | 35758.217 | 3740.089 | 34581.819 | 0.48 |
| 0.64 | 4828.649 | 2.715 | 24143.246 | 2.715 | 49808.330 | 8301.388 | 41506.941 | 3931.713 | 40141.418 | 0.53 |
| 0.70 | 5281.335 | 3.235 | 26406.675 | 3.235 | 59349.792 | 9891.632 | 49458.160 | 4196.754 | 47831.052 | 0.58 |
| 0.76 | 5734.021 | 4.001 | 28670.104 | 4.001 | 73413.057 | 12235.509 | 61177.547 | 4587.400 | 59164.887 | 0.63 |
| 0.82 | 6186.707 | 5.244 | 30933.533 | 5.244 | 96210.730 | 16035.122 | 80175.609 | 5220.669 | 77537.937 | 0.67 |
| 0.88 | 6639.393 | 7.606 | 33196.963 | 7.606 | 139545.001 | 23257.500 | 116287.501 | 6424.398 | 112461.796 | 0.72 |
| 0.94 | 7092.078 | 13.839 | 35460.392 | 13.839 | 253907.441 | 42317.907 | 211589.534 | 9601.133 | 204628.519 | 0.77 |
| 1.00 | 7544.764 | 76.689 | 37723.821 | 76.689 | 1406991.754 | 234498.626 | 1172493.128 | 41631.261 | 1133919.655 | 0.82 |
| 1.06 | 7997.450 | -21.655 | 39987.251 | -21.655 | -397303.942 | -66217.324 | -331086.618 | -8488.071 | -320194.307 | 0.87 |
| 1.12 | 8450.136 | -9.488 | 42250.680 | -9.488 | -174074.441 | -29012.407 | -145062.034 | -2287.252 | -140289.685 | 0.92 |
| 1.18 | 8902.822 | -6.075 | 44514.109 | -6.075 | -111453.203 | -18575.534 | -92877.670 | -547.774 | -89822.117 | 0.97 |

**TABLE 4.14**
**EI Vs 8EI, Case II**

**comparision of total shear and lateral force shear resisted by the columns**

| $(P/P_{cr})$ of the frame | $P_1$ | $\Delta$ | $P_2 = 8P_1$ | $f_s = \Delta_f/\Delta_0$ | Total shear force on the frame $V_{tot} = V_0 * f_s$ | Total shear force resisted by weak col. $V_1$ | Total shear force resisted by strong col. $V_2 = 8V_1$ | Lateral force shear res. by weak c $H_1 = V_1 - P_1\Delta/l$ | Lateral force shear res. by strong c $H_2 = V_2 - P_2\Delta/l$ | $\theta$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | - | 0.9972 | | | 27442.9462 | 3049.216 | 24393.730 | 3049.216 | 24393.730 | - |
| 0.12 | 893.459 | 1.129 | 7147.671 | 1.132 | 31074.882 | 3452.765 | 27622.117 | 3116.474 | 24931.795 | 0.10 |
| 0.17 | 1317.852 | 1.205 | 10542.815 | 1.208 | 33159.409 | 3684.379 | 29475.031 | 3155.077 | 26604.239 | 0.14 |
| 0.23 | 1742.245 | 1.292 | 13937.959 | 1.295 | 35543.712 | 3949.301 | 31594.411 | 3199.231 | 28517.197 | 0.19 |
| 0.29 | 2166.638 | 1.392 | 17333.103 | 1.396 | 38297.460 | 4255.273 | 34042.186 | 3250.226 | 30726.566 | 0.24 |
| 0.34 | 2591.031 | 1.508 | 20728.247 | 1.513 | 41513.733 | 4612.637 | 36901.096 | 3309.786 | 33307.025 | 0.28 |
| 0.40 | 3015.424 | 1.647 | 24123.391 | 1.651 | 45319.752 | 5035.528 | 40284.224 | 3380.268 | 36360.646 | 0.33 |
| 0.46 | 3439.817 | 1.813 | 27518.535 | 1.818 | 49894.091 | 5543.788 | 44350.304 | 3464.978 | 40030.699 | 0.37 |
| 0.51 | 3864.210 | 2.017 | 30913.679 | 2.022 | 55495.515 | 6166.168 | 49329.347 | 3568.708 | 44524.797 | 0.42 |
| 0.57 | 4288.603 | 2.272 | 34308.823 | 2.278 | 62513.694 | 6945.966 | 55567.728 | 3698.675 | 50155.576 | 0.47 |
| 0.62 | 4712.996 | 2.600 | 37703.967 | 2.608 | 71563.955 | 7951.551 | 63612.404 | 3866.272 | 57416.721 | 0.51 |
| 0.68 | 5137.389 | 3.041 | 41099.111 | 3.049 | 83678.260 | 9297.584 | 74380.675 | 4090.611 | 67136.191 | 0.56 |
| 0.74 | 5561.782 | 3.660 | 44494.254 | 3.671 | 100729.733 | 11192.193 | 89537.541 | 4406.379 | 80816.816 | 0.61 |
| 0.79 | 5986.175 | 4.597 | 47889.398 | 4.610 | 126508.988 | 14056.554 | 112452.434 | 4883.773 | 101499.858 | 0.65 |
| 0.85 | 6410.568 | 6.178 | 51284.542 | 6.195 | 170021.854 | 18891.317 | 151130.537 | 5689.566 | 136410.815 | 0.70 |
| 0.91 | 6834.961 | 9.417 | 54679.686 | 9.444 | 259160.129 | 28795.570 | 230364.559 | 7340.275 | 207927.649 | 0.75 |
| 0.96 | 7259.354 | 19.795 | 58074.830 | 19.851 | 544770.030 | 60530.003 | 484240.026 | 12629.349 | 437076.305 | 0.79 |
| 1.02 | 7683.747 | -193.961 | 61469.974 | -194.506 | -5337804.396 | -593089.377 | -4744715.019 | -96307.232 | -4282592.093 | 0.84 |
| 1.07 | 8108.140 | -16.440 | 64865.118 | -16.486 | -452422.864 | -50269.207 | -402153.657 | -5837.188 | -362984.935 | 0.88 |
| 1.13 | 8532.533 | -8.584 | 68260.262 | -8.608 | -236222.251 | -26246.917 | -209975.335 | -1833.473 | -189524.282 | 0.93 |
| 1.19 | 8956.926 | -5.808 | 71655.406 | -5.824 | -159839.393 | -17759.933 | -142079.460 | -418.975 | -128241.290 | 0.98 |

**TABLE 4.15** **comparision of total shear and lateral force shear resisted by the columns**

**EI Vs 10EI, Case II**

| $(P/P_{cr})$ of the frame | $P_1$ | $\Delta$ | $P_2 = 10P_1$ | $f_s = \Delta_f/\Delta_o$ | Total shear force on the frame $V_{tot} = V_o * f_s$ | Total shear force resisted by weak col. $V_1$ | Total shear force resisted by strong col. $V_2 = 10V_1$ | Lateral force shear res. by weak c $H_1 = V_1 - P_1\Delta/l$ | Lateral force shear res. by strong c $H_2 = V_2 - P_2\Delta/l$ | $\theta$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | - | 1.000 | | | 33635.558 | 3057.778 | 30577.780 | 3057.778 | 30577.780 | - |
| 0.070 | 528.614 | 1.074 | 5286.141 | 1.074 | 36134.246 | 3284.931 | 32849.315 | 3095.637 | 30956.370 | 0.06 |
| 0.125 | 943.576 | 1.141 | 9435.761 | 1.141 | 38371.915 | 3488.356 | 34883.560 | 3129.541 | 32873.392 | 0.10 |
| 0.180 | 1358.538 | 1.216 | 13585.381 | 1.216 | 40905.023 | 3718.638 | 37186.384 | 3167.921 | 35043.516 | 0.15 |
| 0.235 | 1773.500 | 1.302 | 17735.002 | 1.302 | 43796.218 | 3981.474 | 39814.743 | 3211.727 | 37520.416 | 0.19 |
| 0.290 | 2188.462 | 1.401 | 21884.622 | 1.401 | 47127.197 | 4284.291 | 42842.907 | 3262.197 | 40374.081 | 0.24 |
| 0.345 | 2603.424 | 1.516 | 26034.242 | 1.516 | 51006.571 | 4636.961 | 46369.610 | 3320.975 | 43697.558 | 0.28 |
| 0.400 | 3018.386 | 1.652 | 30183.863 | 1.652 | 55581.909 | 5052.901 | 50529.008 | 3390.298 | 47617.270 | 0.33 |
| 0.455 | 3433.348 | 1.815 | 34333.483 | 1.815 | 61058.955 | 5550.814 | 55508.141 | 3473.284 | 52309.480 | 0.37 |
| 0.510 | 3848.310 | 2.014 | 38483.103 | 2.014 | 67733.417 | 6157.583 | 61575.833 | 3574.412 | 58027.521 | 0.42 |
| 0.565 | 4263.272 | 2.261 | 42632.724 | 2.261 | 76046.152 | 6913.287 | 69132.866 | 3700.363 | 65149.079 | 0.46 |
| 0.620 | 4678.234 | 2.577 | 46782.344 | 2.577 | 86684.736 | 7880.431 | 78804.305 | 3861.554 | 74263.201 | 0.51 |
| 0.675 | 5093.196 | 2.996 | 50931.964 | 2.996 | 100784.059 | 9162.187 | 91621.872 | 4075.180 | 86342.155 | 0.56 |
| 0.730 | 5508.158 | 3.578 | 55081.585 | 3.578 | 120360.805 | 10941.891 | 109418.913 | 4371.797 | 103113.640 | 0.60 |
| 0.785 | 5923.120 | 4.441 | 59231.205 | 4.441 | 149376.347 | 13579.668 | 135796.679 | 4811.426 | 127971.386 | 0.65 |
| 0.840 | 6338.083 | 5.852 | 63380.825 | 5.852 | 196825.286 | 17893.208 | 178932.078 | 5530.350 | 168621.103 | 0.69 |
| 0.895 | 6753.045 | 8.576 | 67530.446 | 8.576 | 288450.799 | 26222.800 | 262227.999 | 6918.615 | 247117.091 | 0.74 |
| 0.950 | 7168.007 | 16.045 | 71680.066 | 16.045 | 539681.889 | 49061.990 | 490619.899 | 10725.147 | 462347.891 | 0.78 |
| 1.005 | 7582.969 | 124.348 | 75829.686 | 124.348 | 4182504.612 | 380227.692 | 3802276.920 | 65919.430 | 3583170.429 | 0.83 |
| 1.060 | 7997.931 | -21.626 | 79979.307 | -21.626 | -727399.113 | -66127.192 | -661271.921 | -8473.050 | -623166.077 | 0.87 |
| 1.115 | 8412.893 | -9.948 | 84128.927 | -9.948 | -334603.336 | -30418.485 | -304184.851 | -2521.599 | -286656.176 | 0.92 |

In this case since both columns carry axial load in proportion to their stiffness, the increase in the deflection of the frame is produced by the axial load in both columns. The two columns are supporting sway columns (supports the load on theme independently); i.e., their total strength is required to support their own gravity loads, leaving no reserve which might be counted upon to provide a bracing force for each other. This will lead to a rapid increase in frame deflection (Δ) than the individual member deflection (δ) (Figure 4.5 (b) & (c)).

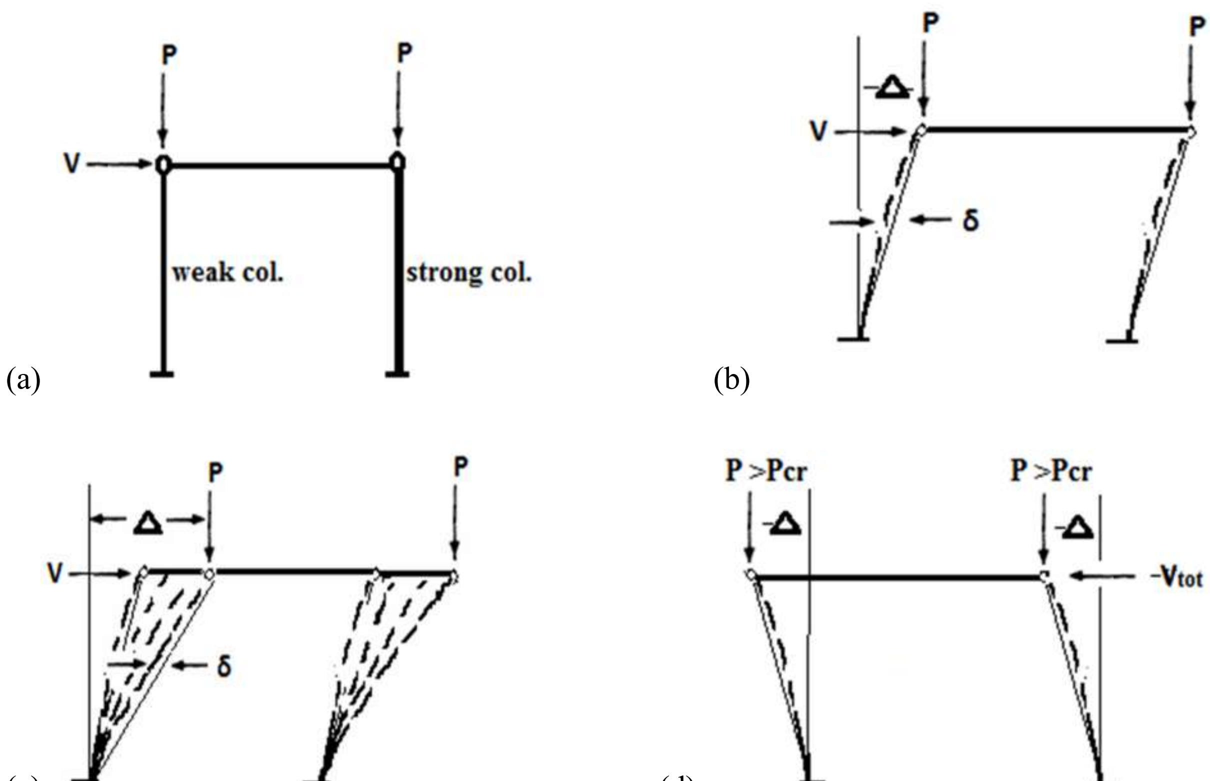



**Figure 4.5: Mechanics of frame deflection in case II**

This small member deflection than frame deflection make approximate method in good agreement with the exact one in the case of a frame where most columns are supporting sway columns up to θ greater than 0.3 as shown in tables 4.11- 4.15.

As the total axial load on the two columns exceeds the critical load of the frame, the whole frame deflects in a reversed direction (Fig. 4.5 (d)).

# CHAPTER 5 CONCLUSION & RECOMMENDATION

## 5.1 Conclusion

In this research approximate and "exact" second order elastic analysis methods are discussed in detail. The accuracy of the approximate method in relation to the "exact" one is shown by using the graph of $P/P_{cr}$ $v_s$ $\Delta_{final}/\Delta_o$. To avoid uncertainty in reading the graph, the results of the two methods are compared with each other using percentage difference table.

The analysis is done using simple elastic portal frames with different stiffness value of columns in two cases. In case I, continuously increasing axial force P is applied only on the weak column and in case II both the weak and strong column carries axial force. In both cases a horizontal lateral load V is applied first to get a first order deflection.

The relation between the limit of θ and the accuracy limit of approximate method is further illustrated using the effect of geometric nonlinearity or P-delta effects on the lateral load resisting capacity of the columns of a frame.

Based on the analysis result, evaluation and discussion made in the previous chapter, the following conclusions were made.

1. In case I, when the values of θ is less than 0.3, the P-Δ effect is more governing one than P-δ effect; so that the results of approximate method is very close to that of "exact" one. When θ is greater than 0.3, the high member deflection of the weak column (P-δ effect) than the total frame deflection (P-Δ effect) makes the approximate method less accurate compared to the "exact" one. This is due to the reason that approximate method considers only P-Δ effect while "exact" method considers both P-Δ and P-δ effect.
2. The "**basic assumption"** of the approximate methods is considered valid in the case of supported sway columns when θ is less than 0.3.
3. In case II, the small member deflection of the columns than the whole frame deflection makes approximate method in good agreement with the "exact" one up to θ greater than 0.3.
4. In the case of frames where most columns are supporting sway columns, approximate method gives results very close to the "exact" one up to θ greater than 0.3.

## 5.2 Recommendation

This thesis focuses on elastic second order analysis methods in which only the geometric non-linearity is considered. The study does not consider the effect of the material non-linearity.

Further study is recommended to investigate the effects of both material and geometric nonlinearities. Inelastic second order analysis methods are used to consider both types of nonlinearity and to study the performance of the structure in the inelastic range. Such level of study would be helpful to get an insight for the upper limit of the interstory drift sensitivity coefficient.

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# APPENDICES

## Appendix A: Derivation of Geometric Stiffness Matrix

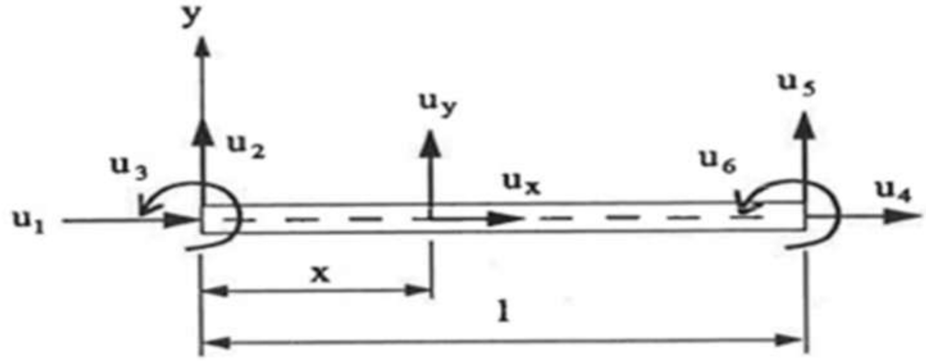



**Figure A.1: Beam element with six d.o.f.**

The displacement functions for this element are given by the following:

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1-\xi & 6\left(\xi-\xi^2\right)\eta & \left(-1+4\xi-3\xi^2\right)l\eta & \xi & 6\left(-\xi+\xi^2\right)\eta & \left(2\xi-3\xi^2\right)l\eta \\ 0 & 1-3\xi^2+2\xi^3 & \left(\xi-2\xi^2+\xi^3\right)l & 0 & 3\xi^2-2\xi^3 & \left(-\xi^2+\xi^3\right)l \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}$$

(A.1)

Where $u_{1........}u_6$ are the end displacements as shown in Figure A.1. the non-dimensional parameters are given by:

$$\xi = \frac{x}{l} \qquad \eta = \frac{y}{l} \tag{A.2}$$

The normal strain $\varepsilon_{xx}$ at a distance x given by:

$$\varepsilon_{xx} = \frac{\partial u_0}{\partial x} - \frac{\partial^2 u_y}{\partial x^2} y + \frac{1}{2}\left(\frac{\partial u_y}{\partial x}\right)^2 \tag{A.3}$$

Where y is measured from the neutral axis of the beam and $u_o$ denotes the x displacement at y=0. In the calculation of the strain energy $u_i$ , shearing strains are neglectedand only the normal strains are considered.

$$u_i = \frac{E}{2}\int_v {\varepsilon_{xx}}^2\, dv$$

$$= \frac{E}{2}\int_v \left[\frac{\partial u_o}{\partial x} - \frac{\partial^2 u_y}{\partial x^2} y + \frac{1}{2}\left(\frac{\partial u_y}{\partial x}\right)^2\right]^2 dv$$

$$= \frac{E}{2}\int_{x=0}^{l}\int_A \left(\frac{\partial u_o}{\partial x}\right)^2 + \left(\frac{\partial^2 u_y}{\partial x^2}\right)^2 y^2 + \frac{1}{4}\left(\frac{\partial u_y}{\partial x}\right)^4 - 2\frac{\partial u_o}{\partial x}\frac{\partial^2 u_y}{\partial x^2} y - \frac{\partial^2 u_y}{\partial x^2}\left(\frac{\partial u_y}{\partial x}\right)^2 y + \frac{\partial u_o}{\partial x}\left(\frac{\partial u_y}{\partial x}\right)^2 dx\, dA$$

The higher-order term $\frac{1}{4}\left(\frac{\partial u_y}{\partial x}\right)^4$ can be neglected in this expression. Integrating over the cross-sectional area A and nothing that since y is measured from the neutral axis, all integrals of the form ∫ydA must vanish, hence

$$u_{i=} \frac{EA}{2}\int_0^1 \left(\frac{\partial u_o}{\partial x}\right)^2 dx + \frac{EI}{2}\int_0^1 \left(\frac{\partial^2 u_y}{\partial x^2}\right) dx + \frac{EA}{2}\int_0^1 \frac{\partial u_o}{\partial x}\left(\frac{\partial u_y}{\partial x}\right)^2 dx \quad \text{(A.4)}$$

Where I is the moment of inertia of the cross section. The first two integrals in Equation A.4 are contributions from the linear strain energy and the third integral is the contribution from the non-linear strain component. The following expressions are obtained from Equation A.1:

$$\frac{\partial u_o}{\partial x} = \frac{1}{l}(-u_1 + u_4) \quad \text{(A.5)}$$

$$\frac{\partial u_y}{\partial x} = \frac{1}{l}[6(-\xi + \xi^2)u_2 + (1 - 4\xi + 3\xi^2)lu_3 + 6(\xi - \xi^2)u_5 + (-2\xi + 3\xi^2)lu_6] \quad \text{(A.6)}$$

$$\frac{\partial^2 u_y}{\partial x^2} = \frac{1}{l^2}[6(-1 + 2\xi)u_2 + 2(-2 + 3\xi)lu_3 + 6(1 - 2\xi)u_5 + 2(-1 + 3\xi)lu_6] \quad \text{(A.7)}$$

Substituting Equations A.5,A.6 and A.7 into A.4 gives the following:

$$u_{i=} \frac{EA}{2l}({u_1}^2 - 2u_1 u_4 + {u_4}^2) + \frac{2EI}{l^3}(3{u_2}^2 + l^2 {u_3}^2 + 3{u_5}^2 + l^2{u_6}^2 + 3lu_2u_3 - 6u_2u_5 + 3lu_2u_6 - 3lu_3u_5 + l^2u_3u_6 - 3lu_5u_6) + \frac{EA}{l^2}(u_4 - u_1)(\frac{3}{5}{u_2}^2 + \frac{1}{15}l^2{u_3}^2 + \frac{3}{5}{u_5}^2 + \frac{1}{15}l^2{u_6}^2 + \frac{1}{10}lu_2u_3 - \frac{6}{5}u_2u_5 + \frac{1}{10}lu_2u_6 - \frac{1}{10}lu_2u_6 - \frac{1}{10}lu_3u_5 - \frac{1}{30}l^2u_3u_6 - \frac{1}{10}lu_5u_6)$$

(A.8)

Equation A.9 is obtained by nothing the elemental axial force is constant.

$$F = \frac{EA}{l}(u_4 - u_1) = constant \quad \text{(A.9)}$$

By applying Castigliano's theorem (part I) to the strain energy expression of Equation A.8, the following element force-displacement equation is obtained.

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} \frac{AL^2}{I} & 0 & 0 & \frac{-AL^2}{I} & 0 & 0 \\ 0 & 12 & 6L & 0 & -12 & 6L \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\ \frac{-AL^2}{I} & 0 & 0 & \frac{AL^2}{I} & 0 & 0 \\ 0 & -12 & -6L & 0 & 12 & -6L \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6F}{5L} & \frac{F}{10} & 0 & \frac{-6F}{5L} & \frac{F}{10} \\ 0 & \frac{F}{10} & \frac{2LF}{15} & 0 & \frac{-F}{10} & \frac{-LF}{30} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-6F}{5L} & \frac{-F}{10} & 0 & \frac{6F}{5L} & \frac{-F}{10} \\ 0 & \frac{F}{10} & \frac{-LF}{30} & 0 & \frac{-F}{10} & \frac{2LF}{15} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{bmatrix} \quad \text{(A.10)}$$

This last equation can be written as:

$$S = (K_e + K_g)u \quad \text{(A.11)}$$

Where $K_e$ is the first order linear elastic matrix, and $K_g$ is the geometric siffness matrix. The force and displacement vectorsare given by S and U respectively.

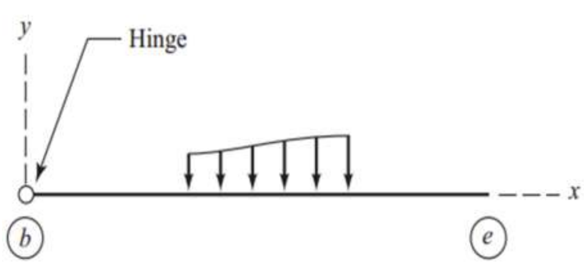
## Appendix B: Stiffness Matrix

Element stiffness matrix of frames connected by a rigid connection is given by;

$$K = \frac{EI}{L^3}\begin{bmatrix} \frac{AL^2}{I} & 0 & 0 & \frac{-A\,L^2}{I} & 0 & 0 \\ 0 & 12 & 6L & 0 & -12 & 6L \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\ \frac{-A\,L^2}{I} & 0 & 0 & \frac{AL^2}{I} & 0 & 0 \\ 0 & -12 & -6L & 0 & 12 & -6L \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix}$$

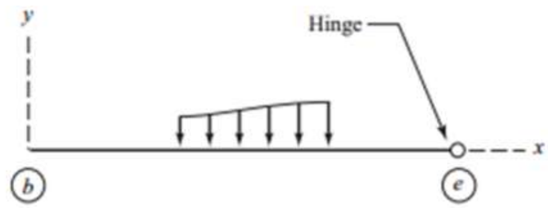
For Frames connected by hinge connection, the element stiffness matrix is modified as shown below.

**1/ Members with a Hinge at the Beginning (Member type one, MT=1)**



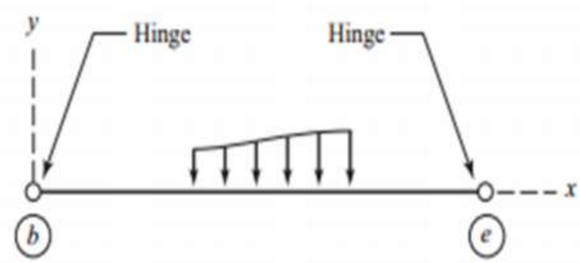

$$\mathbf{k} = \frac{EI}{L^3}\begin{bmatrix} \frac{AL^2}{I} & 0 & 0 & -\frac{AL^2}{I} & 0 & 0 \\ 0 & 3 & 0 & 0 & -3 & 3L \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{AL^2}{I} & 0 & 0 & \frac{AL^2}{I} & 0 & 0 \\ 0 & -3 & 0 & 0 & 3 & -3L \\ 0 & 3L & 0 & 0 & -3L & 3L^2 \end{bmatrix}$$

**2/ Members with a Hinge at the End (Member type two, MT = 2)**




$$\mathbf{k} = \frac{EI}{L^3}\begin{bmatrix} \frac{AL^2}{I} & 0 & 0 & -\frac{AL^2}{I} & 0 & 0 \\ 0 & 3 & 3L & 0 & -3 & 0 \\ 0 & 3L & 3L^2 & 0 & -3L & 0 \\ -\frac{AL^2}{I} & 0 & 0 & \frac{AL^2}{I} & 0 & 0 \\ 0 & -3 & -3L & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**3/ Members with Hinges at Both Ends (Member type three, MT = 3)**




$$\mathbf{k} = \frac{EA}{L}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$