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Integral Equations and Inequalities with Refinements of Some Classical Results

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This is to certify that the dissertation prepared by **Markos Fisseha Yimer** entitled: "INTEGRAL EQUATIONS AND INEQUALITIES WITH REFINEMENTS OF SOME CLASSICAL RESULTS" submitted in fulfillment of the requirements for the Degree of Doctor of Philosophy in Mathematics complies with the regulations of the University and meets the accepted standards with respect to originality and quality.

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Abstract

In this PhD thesis, we study two closely related mathematical subjects: Integral equations and integral inequalities with refinements. The first part of the thesis deals with various generalizations and refinements of some classical inequalities. We prove and discuss some new Hardy-type inequalities in Banach function space settings. In particular, such a result is proved and applied for a new general Hardy operator, which generalizes the usual Hardy kernel operator. Next, we prove some new refined Hardy-type inequalities again in Banach function space settings. We apply superquadraticity technique to find some refinements of the Jensen, Minkowski and Beckenbach-Dresher inequalities. These results both generalize and unify several results of this type. For the case $0 < p \leq q < \infty$, some new Cochran-Lee inequalities in higher dimensions are proved and good two-sided estimates of the sharp constants are obtained. Using these results a new multidimensional weighted Cochran-Lee inequality with sharp constant is also proved. Further, these results are extended to Pólya-Knopp type inequalities on homogeneous groups using a direct method.

In the second part of the thesis, the Dirichlet and Neumann boundary value problems (BVPs) for the linear second-order scalar elliptic differential equation with variable coefficients in a bounded two-dimensional domain are considered. The right-hand side the PDE belongs to $H^{-1}(\Omega)$ or $\widetilde{H}^{-1}(\Omega)$, when neither classical nor canonical conormal derivatives of solutions are well defined. The two-operator approach and appropriate parametrix (Levi function) are used to reduce each of the problem to two different systems of two-operator boundary-domain integral equations (BDIEs). Although the theory of BDIEs in 3D is well developed, the BDIEs in 2D need a special consideration due to their different equivalence properties. As a result, we need to set conditions on the associated Sobolev spaces or choose appropriate scaling parameter in the parametrix form, to insure the invertibility of the corresponding parametrix-based integral layer potentials and hence the unique solvability of BDIEs. The equivalence of the two-operator BDIE systems to the original problems, BDIE system solvability, solution uniqueness/nonuniqueness and invertibility BDIE system are analyzed in the appropriate Sobolev spaces. It is shown that the BDIE operators for the Neumann BVP are not invertible, and appropriate finite-dimensional perturbations are constructed leading to invertibility of the perturbed operators.

This PhD thesis is written as a monograph. Some of the results, with the candidate as author or coauthor, are already published in international journals (see [15], [19], [94], [132], [133] and the new book [91]). Finally, we give motivation and examples of our future plans to study the application of Hardy type inequalities to solve some interesting problems concerning differential and integral equations.

To Agerie, Nuhamin and Tewodros!

List of Notations and Conventions

Notations

Notation	Meaning
$\lvert\boldsymbol{\alpha}\rvert$	The length, $\alpha_1 + \cdots + \alpha_n$
$\mathbf{x}$	A point in the space $\mathbb{R}^n, (x_1, \ldots, x_n)$
$\mathbf{xt}$	A point in the space $\mathbb{R}^n, (x_1t_1, \ldots, x_nt_n)$
$\mathbf{1/t}$	A point in the space $\mathbb{R}^n, (1/t_1, \ldots, 1/t_n)$
$\mathbf{x}^{\frac{1}{\boldsymbol{\beta}}}$	A point in the space $\mathbb{R}^n, (x_1^{\frac{1}{\beta_1}}, \ldots, x_n^{\frac{1}{\beta_n}})$
$\mathrm{d}\mathbf{x}$	$\mathrm{d}x_1 \cdots \mathrm{d}x_n$
$\mathbb{R}$	The set of real numbers
$\mathbb{Z}$	The set of integers
$\mathbb{N}_0$	The set of nonnegative integers
$\mathbb{N}$	The set of natural numbers
J_n	The set $\{1, \ldots, n\}$, where $n \in \mathbb{N}$
I_n	The set $[0, b_1) \times \cdots \times [0, b_n)$, where $0 < b_i \leq \infty, i \in J_n$
$\mathbb{R}^n_+$	The set $\{(x_1, \ldots, x_n) : x_i \geq 0, i \in J_n\}$
$\mathbf{y} \leq \mathbf{x}$	$y_i \leq x_i$ for all $i \in J_n$
p'	The conjugate parameter of $p \left(\frac{1}{p} + \frac{1}{p'} = 1, p > 1\right)$
χ_D	The characteristic function of the set D
γ^+	The trace operator on $\partial\Omega$ from inside or outside Ω
Ω	Open set in $\mathbb{R}^2$
$\partial\Omega$	The boundary of Ω
$C^\infty(\Omega)$	The set of all infinitely differentiable functions on $\Omega \subseteq \mathbb{R}$
$W^k_p(\Omega)$	The Sobolev spaces order k
$H^s(\Omega)$	Bessel Potential space of order $s \in \mathbb{R}$
$\widetilde{H}^s(\Omega)$	The subspace of $H^s(\mathbb{R}^2), \{g \in H^s(\mathbb{R}^2) : supp(g) \subset \overline{\Omega}\}$
$H^s_{\partial\Omega}$	The subspace of $H^s(\mathbb{R}^2), \{g \in H^s(\mathbb{R}^2) : supp(g) \subset \partial\Omega\}$
$\int_{\mathbb{R}^n_+}$	$\underbrace{\int_0^\infty \cdots \int_0^\infty}_{n \text{ times}}$
$\int_{\mathbf{0}}^{\mathbf{1}}$	$\underbrace{\int_0^1 \cdots \int_0^1}_{n \text{ times}}$
$\int_{\mathbf{0}}^{\mathbf{x}}$	$\int_0^{x_1} \cdots \int_0^{x_n}$
$\int_{\mathbf{y}^{\boldsymbol{\eta}}}^{\mathbf{b}^{\boldsymbol{\beta}}}$	$\int_{y_1^{\eta_1}}^{b_1^{\beta_1}} \cdots \int_{y_n^{\eta_n}}^{b_n^{\beta_n}}$, where $0 \leq y_i < b_i \leq \infty, i \in J_n$.

Conventions

- We consider the measurable spaces (X, μ), (X, λ) and (Y, ν). Moreover, $\mathrm{d}\mu, \mathrm{d}\lambda$ and $\mathrm{d}\nu$ are notations for $\mathrm{d}\mu(x), \mathrm{d}\lambda(x)$ and $\mathrm{d}\nu(y)$, respectively.
- u, v, etc. denote weights, i.e. nonnegative locally integrable functions on $\mathbb{R}^n_+$.
- The functions are assumed to be measurable.

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Chapter 1

Introduction

Inequalities have been very important for the development of different branches of mathematics as functional analysis, interpolation theory, theory of differential and integral equations, probability, harmonic analysis, function theory, fixed point theory, etc. Nowadays the theory of inequalities may be regarded as an independent area of mathematics.

Inequalities involving integrals of a function and its derivatives appear frequently in various branches of mathematics e.g. in the theory and practice of differential equations, in the theory of approximation etc.

Hardy's inequalities are one of the most used integral inequalities in many branches of mathematical analysis and mathematical physics. They have been studied intensively and extensively in the literature due to their important roles since their discovery. The interested reader is referred to the books [65], [71], [73], [113] and the Lecture Notes [103] in P. L. Lions seminar. Other important sources of inspiration have been the PhD thesis [17] and the recent review paper [104].

In this introduction we describe some parts of special interest for this thesis in the more than 100 years of developments of this fascinating area, which today is called Hardy-type inequalities. First we give a short description of the dramatic prehistory. After that we present a fairly new convexity approach to handle Hardy-type inequalities. This convexity approach has partly influenced the research in this thesis. Next, we discuss some results published between 1925-2017, which are of special interest for our new findings. Furthermore, we illustrate the ongoing interest in this area by presenting some of the newest results obtained after 2017 and in this connection especially present the main new contributions in this thesis.

We continue by giving an introduction on integral equation methods to solve boundary value problems (BVPs). The reduction of a BVP given in a domain to integral equations (IEs) on the boundary of the domain is a powerful method that has recently attracted new ideas and developments. Many applications in science and engineering can be modelled by BVPs for Partial Differential Equations (PDEs) with variable coefficients. For instance, a Dirichlet and a Neumann problem on the half line for the time-dependent Schrödinger equation with a space-dependent potential and also Poincaré problem on the quarter plane for a variable coefficient generalization of the Laplace equation can be modelled by variable coefficient BVPs. Reduction of the BVPs with arbitrarily variable coefficients to explicit boundary integral equations is usually not possible, since the fundamental solution needed for such reduction is generally not available in an analytical form (except for some special dependence of the coefficients on coordinates). Nevertheless, for a rather wide class of variable coefficient PDEs it is possible to use instead an explicit parametrix (Levi function) taken as a fundamental solution of corresponding frozen-coefficient PDEs, and reduce BVPs for such PDEs to explicit systems of BDIEs for their further analysis, see e.g. [13, 28, 29] and Chapter 5 of this PhD thesis.

Parts of the mathematical content in this PhD thesis can also be found in the publications [15], [19], [94], [130], [131], [132], and [133] with M. Yimer as the author or coauthor. We also pronounce that this PhD thesis also contains new contributions by M. Yimer which can not be found in these papers (see e.g. Section 4.2)

1.1 Integral Inequalities

1.1.1 The prehistory of the Hardy inequality

The development of the famous Hardy inequality in both discrete and continuous forms during the period 1906 to 1928 has its own history or as we have called it, prehistory. The story started more than 100 years ago namely around 1915 when G. H. Hardy needed an estimate for the arithmetic means of the form

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}a_i\right)^2 \leq C_2\sum_{n=1}^{\infty}a_n^2, \tag{1.1}$$

where $\{a_n\}$ is a sequence of nonnegative real numbers, it was later on proved that the sharp constant in (1.1) is $C_2 = 4$. The first motivation for G. H. Hardy to begin this dramatic history was to find a new and more elementary proof of the Hilbert inequality for double series from 1906:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} \leq \pi\left(\sum_{m=1}^{\infty}a_m^2\right)^{1/2}\left(\sum_{n=1}^{\infty}b_n^2\right)^{1/2}, \tag{1.2}$$

where π is the sharp constant. The sharp constant π was found by I. Schur. In Hilbert's version of (1.2) from early 19^{th} the constant was 2π instead of π.

The inequality (1.1) was then extended to

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}a_i\right)^p \leq \left(\frac{p}{p-1}\right)^p\sum_{n=1}^{\infty}a_n^p, \quad p>1, \tag{1.3}$$

with the continuous (integral) counterpart

$$\int_0^{\infty}\left(\frac{1}{x}\int_0^x f(t)\,\mathrm{d}t\right)^p \mathrm{d}x \leq \left(\frac{p}{p-1}\right)^p\int_0^{\infty}f^p(x)\,\mathrm{d}x, \quad p>1, \tag{1.4}$$

where f is a nonnegative p-integrable function on $(0, \infty)$. This result was stated and proved by G. H. Hardy in his 1925 paper [49]. The inequality (1.4) is usually called the classical Hardy inequality in the literature and it has been extensively studied and used as a model for the investigation of more general integral inequalities. Later in 1928, G. H. Hardy himself proved the following weighted version of (1.4):

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,\mathrm{d}t\right)^p x^\alpha\,\mathrm{d}x \leq \left(\frac{p}{p-1-\alpha}\right)^p \int_0^\infty f^p(x)x^\alpha\,\mathrm{d}x, \tag{1.5}$$

with $p \geq 1$ and $\alpha < p-1$ for all measurable nonnegative functions f on $(0, \infty)$, where the constant $\left(\frac{p}{p-1-\alpha}\right)^p$ is sharp, see [48]. The constants in (1.3)-(1.5) are all sharp and the discrete inequalities follows from the continuous ones. After this, a lot of generalizations and complementary results have been published. These are called today Hardy-type inequalities. See e.g. the monographs [50], [70], [71], [73], [102], [113] and the references therein.

During the subsequent decades, approximately until the 1980-ies, inequality (1.5) was extended to the *general Hardy inequality* of the form:

$$\left(\int_a^b \left(\int_a^x f(t)\,\mathrm{d}t\right)^q u(x)\,\mathrm{d}x\right)^{1/q} \leq C_{p,q}\left(\int_a^b f^p(x)v(x)\,\mathrm{d}x\right)^{1/p}$$

with parameters a, b, p, q such that $-\infty \leq a < b \leq \infty$, $0 < q \leq \infty$, $1 \leq p \leq \infty$, and with $u(x), v(x)$ given weight functions.

The main research question in connection with the Hardy inequality is to determine conditions on the parameters p, q and on the weights u and v under which the inequality holds for some classes of functions and good estimates of the sharp constant exists.

In this PhD thesis we derive and discuss various generalizations and refinements of the Hardy and related Pólya-Knopp inequalities. Moreover, we also prove some new generalizations also of some other classical inequalities namely those by Jensen, Minkowski and Beckenbach-Dresher.

Remark 1.1. Concerning the dramatic prehistory before G. H. Hardy discovered his inequalities (1.3)-(1.5) we refer to [70] and [71]. Some contributions of other mathematicians than G. H. Hardy, such as E. Landau, G. Pólya, E. Schur and M. Riesz are also important here.

1.1.2 Pólya-Knopp's inequality

The well known Pólya-Knopp's inequality

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x f(t)\,\mathrm{d}t\right)\mathrm{d}x \leq e\int_0^\infty f(x)\,\mathrm{d}x. \tag{1.6}$$

where e is the sharp constant can be considered as a limiting case of the classical Hardy inequality (1.4). Indeed, by replacing $f(x)$ with $(f(x))^{1/p}$ in (1.4), we obtain

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f^{\frac{1}{p}}(t)\,\mathrm{d}t\right)^p \mathrm{d}x \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)\,\mathrm{d}x. \tag{1.7}$$

Now, if we assume that f is positive a.e., then

$$\lim_{p\to\infty}\left(\frac{1}{x}\int_0^x f^{\frac{1}{p}}(t)\,\mathrm{d}t\right)^p = \exp\left(\frac{1}{x}\int_0^x \log f(t)\,\mathrm{d}t\right),$$

(cf. [121], page 344; 5c and also the power mean argument presented below) and

$$\lim_{p\to\infty}\left(\frac{p}{p-1}\right)^p = e.$$

Thus, it follows from Fatou's lemma and the inequality (1.7) that the inequality (1.6) holds. Moreover, by making a similar limiting procedure in (1.5) we obtain the following weighted version of (1.6):

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x f(t)\,\mathrm{d}t\right)x^\alpha\,\mathrm{d}x \le e^{(1+\alpha)}\int_0^\infty f(x)x^\alpha\,\mathrm{d}x, \tag{1.8}$$

for $\alpha > -1$. Moreover, the constant $e^{(1+\alpha)}$ is sharp.

Remark 1.2. Sometimes the inequality (1.6) is referred to as the Knopp inequality with reference to the paper [63] from 1928. But, it is clear that it was known before, e.g. in his 1925 paper [49] G. H. Hardy informed that G. Pólya had pointed out this inequality to him, via the limit argument above. This is the main reason why the inequality has got the name Pólya-Knopp inequality.

Remark 1.3. The Hardy operator

$$H : (Hf)(x) = \frac{1}{x}\int_0^x f(y)\,\mathrm{d}y$$

is an arithmetic mean operator while

$$G : (Gf)(x) = \exp\left(\frac{1}{x}\int_0^x \ln f(y)\,\mathrm{d}y\right)$$

is the corresponding geometric mean operator.

In this PhD thesis we have proved some new mapping properties (Hardy-type inequalities) of multidimensional geometric mean operators even in a much more general setting than discussed so far. It is also worth to mention that the operators H and G are special cases of a more general scale of Hardy-type averaging operators called power means $P_\alpha(f,\mu)$, $-\infty < \alpha < \infty$, of a function f on a finite measure space (Ω,μ). They are defined as follows (see e.g. [103]):

$$P_\alpha(f,\mu) := \begin{cases} \left(\frac{1}{\mu(\Omega)}\int_\Omega |f|^\alpha\,\mathrm{d}\mu\right)^{1/\alpha}, \alpha \neq 0 \\ \exp\left(\frac{1}{\mu(\Omega)}\int_\Omega \log|f|\,\mathrm{d}\mu\right), \alpha = 0 \end{cases}.$$

Note that $H = P_1$ and $G = P_0$, when $\Omega = [0,x]$ and $\mathrm{d}\mu = \mathrm{d}x$. The operator H_{-1} is the harmonic mean operator.

1.1.3 A newer convexity proof and its consequences

Today there exist more than 20 different proofs of Hardy's original inequality (in addition to Hardy's original proof where he essentially used partial integration). Here we will only present a fairly new convexity proof, which already has got some surprising consequences and even could have changed the further history if G. H. Hardy had discovered it himself. See [106] and the references therein. But first, we present the following remark, which can be proved by elementary calculations.

Remark 1.4. The "fundamental" Hardy inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x g(t)\,\mathrm{d}t\right)^p \frac{\mathrm{d}x}{x} \leq 1\cdot \int_0^\infty g^p(t)\,\frac{\mathrm{d}t}{t}, \quad p>1 \tag{1.9}$$

is equivalent to both (1.4) and (1.5), see e.g. the new book [73] and the references therein. This equivalence follows by just doing the substitutions $f(x) = g\left(x^{\frac{p-1}{p}}\right)x^{-\frac{1}{p}}$ and $f(x) = g\left(x^{\frac{p-\alpha-1}{p}}\right)x^{-\frac{\alpha+1}{p}}$ respectively. These facts imply especially the following:

(a) Hardy's inequalities (1.4) and (1.5) hold also for $p<0$ (because the function $\varphi(u) = u^p$ is convex also for $p<0$) and hold in the reverse direction for $0<p<1$ with sharp constants $\left(\frac{p}{1-p}\right)^p$ and $\left(\frac{p}{\alpha+1-p}\right)^p$, $\alpha > p-1$, respectively.

(b) The inequalities (1.4) and (1.5) are equivalent for $p>1$, since both are equivalent to the inequality (1.9).

Remark 1.5. Note that the proof of (1.9) consists of only an application of Jensen's inequality and Fubini's theorem. The power weighted inequalities are more or less equivalent with the basic inequality. See Theorem 1.1 (see also [106] and [73, Theorem 7.10]).

Hardy-Knopp's inequality

The simple proof of (1.9) which we pointed out in Remark 1.5 can be repeated for any convex function Φ. Precisely, if Φ is positive and convex function on the range of f, then

$$\int_0^\infty \Phi\left(\frac{1}{x}\int_0^x f(t)\,\mathrm{d}t\right)\frac{\mathrm{d}x}{x} \leq 1\cdot\int_0^\infty \Phi\left(f(t)\right)\frac{\mathrm{d}t}{t}. \tag{1.10}$$

Sometimes, this inequality is called the Hardy-Knopp type inequality since it directly implies

(a) the inequality (1.9), and, thus, of (1.4) and (1.5), see Remark 1.4, when applied to $\Phi(u) = u^p$, $p\geq 1$.

(b) the inequality (1.6) when applied to $\Phi(u) = \exp(u)$ and replacing $f(x)$ by $\ln xf(x)$.

(c) the inequality (1.8) when applied to $\Phi(u) = \exp(u)$ and replacing $f(x)$ by $\ln\left(x^{\alpha+1}f(x)\right)$.

Remark 1.6. The above results show that the inequality (1.10) implies all of the inequalities (1.4) – (1.6) and (1.8) – (1.9).

Some consequences of this convexity approach

It is known that the Hardy inequalities (1.4) and (1.9) still hold with the same sharp constants if the interval $(0,\infty)$ is replaced by a finite interval $(0,\ell)$, $0<\ell<\infty$. But here the inequality can be essentially improved by considering another function space on the right hand side with strictly smaller norm. More generally, the following remarkable equivalence theorem was recently proved in [106] (see also [73, Theorem 7.10]) using essentially this convexity idea.

One crucial point in the proof of Theorem 1.1 is the following Lemma of independent interest. It is sometimes called "fundamental form of classical Hardy inequalities".

Lemma 1.1. Let g be a nonnegative and measurable function on $(0,\ell), 0<\ell\leq\infty$.

a) If $p<0$ or $p\geq 1$, then

$$\int_0^\ell \left(\frac{1}{x}\int_0^x g(t)\,\mathrm{d}t\right)^p \frac{\mathrm{d}x}{x} \leq 1\cdot\int_0^\ell g^p(x)\left(1-\frac{x}{\ell}\right)\frac{\mathrm{d}x}{x}. \tag{1.11}$$

(In the case $p<0$ we assume that $g(x)>0$).

b) If $0<p\leq 1$, then (1.11) holds in the reversed direction.

c) The constant $C=1$ is sharp in both a) and b).

By using this lemma and straightforward calculations we can prove the following theorem:

Theorem 1.1. Let $0<\ell\leq\infty$, let $p\in\mathbb{R}\setminus\{0\}$ and let f be a nonnegative and measurable function. Then

a) the inequality

$$\begin{aligned}&\int_0^\ell \left(\frac{1}{x}\int_0^x f(t)\,\mathrm{d}t\right)^p x^\alpha\,\mathrm{d}x\\ \leq&\left(\frac{p}{p-1-\alpha}\right)^p\int_0^\ell f^p(x)x^\alpha\left(1-\left(\frac{x}{\ell}\right)^{\frac{p-\alpha-1}{p}}\right)\mathrm{d}x,\end{aligned} \tag{1.12}$$

holds for all f, $0<\ell\leq\infty$ and α in the following cases:

$$\begin{aligned}&(a_1)\quad p\geq 1,\alpha<p-1,\\ &(a_2)\quad p<0,\alpha>p-1.\end{aligned}$$

b) For the case $0<p<1, \alpha<p-1$, the inequality (1.12) holds in the reversed direction.

c) The inequality

$$\begin{aligned}&\int_\ell^\infty \left(\frac{1}{x}\int_x^\infty f(t)\,\mathrm{d}t\right)^p x^{\alpha_0}\,\mathrm{d}x\\ \leq&\left(\frac{p}{\alpha_0+1-p}\right)^p\int_\ell^\infty f^p(x)x^{\alpha_0}\left(1-\left(\frac{\ell}{x}\right)^{\frac{\alpha_0+1-p}{p}}\right)\mathrm{d}x,\end{aligned} \tag{1.13}$$

holds for all f, $0 \le \ell < \infty$ and α_0 in the following cases:

$$\begin{aligned}&(c_1)\quad p \ge 1, \alpha_0 > p-1,\\&(c_2)\quad p < 0, \alpha_0 < p-1.\end{aligned}$$

d) For the case $0 < p \le 1$, inequality (1.13) holds in the reversed direction.

e) All inequalities above are sharp.

f) Let $p \ge 1$ or $p < 0$. Then, the statements in *a)* and *c)* are equivalent for all permitted α and α_0 because they are in all cases equivalent to (1.11) via substitutions.

g) Let $0 < p < 1$. Then, the statements in *b)* and *d)* are equivalent for all permitted α and α_0.

Remark 1.7. Note that in the theory of (weighted) Hardy-type inequalities we usually have good estimates of the sharp constant (the operator norm). However, in some cases we can even find the sharp constant and this is especially interesting and maybe regarded as an art of its own. For example the constants in all inequalities in Theorem 1.1 are sharp. Here, we also refer to the new paper [107], where even some of the inequalities in Theorem 1.1 are proved in the reversed direction on the cone of monotone functions and also here with both sharp constant and optimal"target function" e.g. $\left(1-\left(\frac{x}{e}\right)^{\frac{p-\alpha-1}{p}}\right)$.

Remark 1.8. Note that inequality (1.5) (or (1.12)) has no meaning when $\alpha = p-1$. However, by restricting to finite intervals and involving some suitable logarithms, C. Bennett first prove such a result when he developed well-known theory for real interpolation between the (fairly close) spaces L and $L\log^+ L$ (see [21]).

The next result is for the exceptional case $\alpha = p-1$, where the authors succeeded to (see [20]), by using another convexity argument, prove the following essential improvement of the mentioned Bennett's Hardy-type inequality in [21]

Theorem 1.2. Let $\alpha, p > 0$ and f be a non-negative and measurable function on $[0,1]$.

(a) If $p > 1$, then

$$\begin{aligned}\alpha^{p-1}\left(\int_0^1 f(x)\,\mathrm{d}x\right)^p + \alpha^p\int_0^1 [\log(e/x)]^{\alpha p-1}\left(\int_0^x f(y)\,\mathrm{d}y\right)^p\frac{\mathrm{d}x}{x}&\\ \le \int_0^1 x^p\,[\log(e/x)]^{(1+\alpha)p-1} f^p(x)\frac{\mathrm{d}x}{x}.&\end{aligned} \tag{1.14}$$

Both constants α^{p-1} and α^p in (1.14) are sharp. Equality is never attained unless f is identically zero.

(b) If $0 < p < 1$, then (1.14) holds in the reverse direction and the constant is sharp. Equality is never attained unless f is identically zero.

(c) If $p = 1$, then we have equality in (1.14) for any measurable function f and any $\alpha > 0$.

Remark 1.9. In Bennett's original paper [21] only the case (a) was considered and the first term on the left hand side did not appear. Moreover, the sharpness was even not discussed and the proof was not connected to convexity at all. Hence, also this result is remarkable from many points of view and especially because it is one of the few inequalities involving two constants and both of them are sharp.

1.2 Integral Equations

The reduction of boundary value problems (BVPs) given in a domain to integral equations (IEs) on the boundary of the domain is a powerful method that has recently attracted new ideas and developments.

1.2.1 Boundary Integral Equations

The boundary integral equation (BIE) method has been intensively developed in the recent decades both concerning theory and engineering applications. Its popularity is due to the possibility of reducing a boundary value problem for a partial differential equation in a domain to an integral equation on the boundary of the domain. This approach reduces the problem dimensionality by one, which is very important for the construction of various numerical algorithms using small computer resources. The main ingredient necessary for efficient reduction of a BVP to a BIE is a fundamental solution to the original partial differential equation, available in analytical form and/or cheaply calculated.

1.2.2 Boundary-Domain Integral Equations

Many applications in science and engineering can be modelled by BVPs for PDEs with variable coefficients. Reduction of the BVPs with arbitrarily variable coefficients to explicit BIEs is usually not possible, since the fundamental solution needed for such reduction is generally not available in an analytical form (except for some special dependence of the coefficients on coordinates). Nevertheless, for a rather wide class of variable coefficient PDEs it is possible to use instead an explicit parametrix (Levi function) taken as a fundamental solution of the corresponding frozen-coefficient PDEs, and reduce BVPs for such PDEs to explicit systems of boundary-domain Integral equations (BDIEs) for their further analysis.

As in the Láme system of anisotropic elasticity, the one-operator approach used in [27–29, 83, 85, 86] does not work when the fundamental solution of the frozen-coefficient PDE is not known explicitly. To overcome this difficulty, one can apply the so-called two-operator approach, formulated in [84] for a certain non-linear problem, that employs a parametrix of another (second) PDE, not related with the PDE in question, for reducing the BVP to a BDIE system and further analysis of the latter. Since the second PDE is rather arbitrary, one can always choose it in such a way, that its parametrix is known explicitly. The simplest choice for the second PDE is the one with an explicit fundamental solution. The BDIE analysis is useful for discretization and numerical solution of the BDIEs and thus of the associated BVPs. For a more complete description of these fundamental ideas and fact we refer to Chapter 5.

1.3 The main objectives of this PhD thesis

The main objectives are to generalize and derive refinements of some classical inequalities and to investigate and complement the theory of some important integral equations. More exactly, the following main objectives are addressed in this thesis:

(1) To prove and discuss some new Hardy-type inequalities in a Banach function space setting.

(2) To derive and prove some new refinements of some classical inequalities such as Jensen's, Minkowski's, Beckenbach-Dresher's type and Hardy's.

(3) To derive and investigate the necessary and sufficient condition for multidimensional Cochran-Lee type inequalities with general weight functions with parameters $0 < p \leq q < \infty$.

(4) To state and prove a sharp multidimensional Cochran-Lee type inequalities with particular power weight functions. Even the case with homogeneous groups involving sharp constants shall be considered.

(5) To derive and investigate two-operator boundary domain integral equations (BDIEs) for variable-coefficient boundary value problems (BVPs) with general right-hand side on a two dimensional bounded domain.

1.4 Methodology Used

To achieve the desired objectives we use several classical new methodologies:

(1) We use the concept of superquadratic function rather than convex function to prove the refinement of some classical inequalities (see e.g. Section 3.1).

(2) We apply a direct method rather than limiting procedure to prove the higher dimensional Cochran-Lee inequalities (see Chapter 4). We also develop the theory of Hardy-type inequalities on groups so we can prove our results in this more general context.

(3) The two-operator approach and appropriate parametrix (Levi function) are used to reduce each of the BVPs to two different systems of two-operator boundary-domain integral equations (BDIEs). We set conditions on the associated Sobolev spaces or choose appropriate scaling parameter in the parametrix form, to insure the invertibility of the corresponding parametrix-based integral layer potentials and hence the unique solvability of BDIEs. The equivalence of the two-operator BDIE systems to the original problems, BDIE system solvability, solution uniqueness/nonuniqueness and invertibility BDIE system are analyzed in the appropriate Sobolev spaces.

1.5 Organization of the thesis

The remaining part of this PhD thesis is organized as follows:

Chapter 2 contains *preliminaries and a literature review*, here we give some basic concepts, definitions and theorems that play important roles in the discussions of the main results.

In Chapter 3 we to address objectives (1) and (2). Some new Hardy-type inequalities in Banach function space involving the classical Hardy operator are stated and proved in Section 3.1 (see Theorems 3.1, 3.2 and 3.3). In particular, such a result is proved and applied for a new general Hardy operator which generalizes the usual Hardy kernel operator. These results generalize and unify several classical Hardy-type inequalities. In Section 3.2, to address objective (2), we state, prove and apply some new refinements of the Minkowski inequality and new Beckenbach-Dresher type inequalities (see Theorems 3.10 and 3.12). Moreover, we derive some corresponding refinements of Hardy's inequality even in a Banach function space setting (see Theorem 3.13). These results both generalize and unify several results of this type.

In Chapter 4 is devoted address objectives (3) and (4). Here we state and prove a multidimensional Cochran-Lee type inequality in more general case with good two-sided estimates of the sharp constants (see Theorems 4.2 and 4.4). Moreover, we state and prove Pólya-Knopp type inequalities on homogeneous groups $\mathbb{G}$ (see Theorems 4.7 and 4.8).

In Chapter 5 we address objective (5). Using the two-operator approach and an appropriate parametrix (Levi function) we reduce each of the BVPs into two different systems of two-operator BDIEs. We also set conditions on the associated Sobolev spaces or choose appropriate scaling parameter in the parametrix form to insure the invertibility of corresponding parametrix-based integral layer potentials and hence the unique solvability of BDIEs. The equivalence of the two-operator BDIE systems to the original problems, BDIE system solvability, solution uniqueness/nonuniqueness and invertibility BDIE system are analyzed in the appropriate Sobolev spaces (see, Theorems 5.8, 5.9, 5.11, 5.16 and 5.17).

Finally, in Chapter 6 we conclude the thesis with a summary of the obtained results and future plan. We also include a number of open questions related to this PhD thesis, which are of interest for a broad audience. These questions also give some hints for directions of continued collaboration between the research group in Ethiopia, Sweden and Belgium, which in one or other way have been involved in the research presented in this PhD thesis.

Chapter 2

Preliminaries and Literature Review

2.1 Preliminaries

2.1.1 Convex and concave functions

Convexity is one of the natural and fundamental concepts which plays the most important role in many areas of mathematics. Convex functions were introduced by Jensen in 1905 and have received a remarkable attention in the literature due to their applications in different scientific fields such as Mathematical analysis and Mathematical physics. As our main reference we mention the very cited recent book [90] and the references therein. We begin this section by giving the definition of convex function.

Definition 2.1. A function $\varphi : X \to \mathbb{R}$ is said to be convex if

$$\varphi\big(\lambda x + (1-\lambda)y\big) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y) \tag{2.1}$$

for all $x, y \in X$ and $\lambda \in (0,1)$. Moreover, a function φ is concave if the inequality (2.1) holds in the reversed direction.

Example 2.1. The following are examples of convex (concave) function:

(a) $\varphi(x) = x^p$ on $[0,\infty)$ for all $p \geq 1$ or $p < 0$ (for all $0 < p \leq 1$)

(b) $\varphi(x) = \exp(x)$ on $[0,\infty)$ ($\varphi(x) = \ln x$ on $(0,\infty)$)

Sometimes it is not so easy to check the convexity of a function, but we do have some useful conditions to check the convexity of a function. We present here one of the simplest condition:

Theorem 2.1 (J.L.W.V. Jensen, [58])**.** Let $\varphi : I \to \mathbb{R}$ be a continuous function. Then φ is convex if and only if φ is midpoint convex, that is,

$$\varphi\left(\frac{x+y}{2}\right) \leq \frac{\varphi(x)+\varphi(y)}{2}, \text{ for all } x, y \in I.$$

2.1.2 Jensen's inequality

Jensen's inequality involving convex function is the most significant classical inequalities in mathematical analysis. In fact, Jensen's inequality is more or less equivalent to the concept of convexity (c.f. Corollary 2.1 below) and implies most of the other classical inequalities (e.g. those by Hölder, Minkowski, Beckenbach-Drecher, etc.) Here we give the following version:

Theorem 2.2 (Jensen's inequality)**.** Let (Ω, μ) be a measure space, where $\mu(\Omega) = 1$ (i.e. μ is a probability measure). If $\varphi : [0, \infty) \to \mathbb{R}$ is a convex function, then

$$\varphi\left(\int_{\Omega} f \, \mathrm{d}\mu\right) \leq \int_{\Omega} \varphi \circ f \, \mathrm{d}\mu \tag{2.2}$$

for all nonnegative, μ-integrable functions f. Moreover, if φ is concave function, then the inequality (2.2) holds in the reversed direction. In particular, the inequality (2.2) reduces to equality when $\varphi(x) = x$.

Corollary 2.1 (The discrete case of Jensen's inequality)**.** A real valued function φ defined on an interval I is convex if and only if for all $x_1, ..., x_n$ in I and all scalars $\lambda_1, ..., \lambda_n$ in $[0, 1]$ with $\sum_{i=1}^{n} \lambda_i = 1$ we have

$$\varphi\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i \varphi(x_i). \tag{2.3}$$

Remark 2.1. If (2.2) holds for all probability measures (but φ not necessarily convex), then obviously (2.3) holds. Indeed, just apply (2.2) with the measure $\mathrm{d}\mu = \sum_{i=1}^{n} \lambda_i \delta_i$, where δ_i are the Dirac functions at $i = 1, 2, ..., n$ so (2.3) holds with $f(i) = x_i$. This means that φ is convex. On the other hand we know by Theorem 2.2 that if φ is convex then (2.2) holds. We conclude that Jensen's inequality correctly formulated is more or less equivalent.

2.1.3 Minkowski's integral inequalities

Minkowski's inequality is the other important inequality in the study of mathematical analysis. For instance, Minkowski's inequality plays a basic role in the proof of L^p spaces are normed vector spaces.

Theorem 2.3 (Minkowski's integral inequality)**.** Let f be a nonnegative measurable function on $X \times Y$ with respect to the measure $\mu \times \nu$, and let $p \geq 1$. Then

$$\left(\int_X \left(\int_Y f \mathrm{d}\nu\right)^p \mathrm{d}\mu\right)^{\frac{1}{p}} \leq \int_Y \left(\int_X f^p \mathrm{d}\mu\right)^{\frac{1}{p}} \mathrm{d}\nu. \tag{2.4}$$

For the case $0 < p < 1$, (2.4) holds in the reversed direction.

Remark 2.2. Theorem 2.3 is maybe the first example of so called continuous form of classical inequalities. A theory of this fascinating area is presented in the new book [91]. When restricting to a special case we obtain the following standard form of Minkowski's inequality.

Example 2.2. Let $Y = [0, 2], \mathrm{d}\nu = \mathrm{d}x, f = f_1$ on $[0, 1)$ and $f = f_2$ on $[1, 2]$. Then (2.4) reads:

$$\left(\int_X (f_1 + f_2)^p \mathrm{d}\mu\right)^{1/p} \leq \left(\int_X f_1^p \mathrm{d}\mu\right)^{1/p} + \left(\int_X f_2^p \mathrm{d}\mu\right)^{1/p},$$

i.e. the standard form of the Minkowski inequality.

Remark 2.3. It is known that (2.4) follows very easily from Jensen's inequality (2.2). See e.g. the Lecture Notes [103].

The following special case of (2.4) is very useful for several applications.

Corollary 2.2 (Minkowski integral inequality of Fubini type)**.** Let the positive kernel $K(x, y)$ be measurable. If $p \geq 1$, then

$$\begin{aligned} &\left(\int_a^b \left(\int_a^x K(x, y)\Psi(y)\mathrm{d}y\right)^p \Phi(x)\mathrm{d}x\right)^{\frac{1}{p}} \\ &\leq \int_a^b \left(\int_y^b \Phi(x)K^p(x, y)\mathrm{d}x\right)^{\frac{1}{p}} \Psi(y)\mathrm{d}y. \end{aligned} \tag{2.5}$$

Remark 2.4. If $K(x, y) \equiv 1$ in Corollary 2.2, then the inequality (2.5) becomes

$$\left(\int_a^b \left(\int_a^x \Psi(y)\mathrm{d}y\right)^p \Phi(x)\mathrm{d}x\right)^{\frac{1}{p}} \leq \int_a^b \left(\int_y^b \Phi(x)\mathrm{d}x\right)^{\frac{1}{p}} \Psi(y)\mathrm{d}y.$$

In order to prove some of the main results in this PhD thesis, we also need the following forms of Minkowski's integral inequality in n-dimension:

Proposition 2.1. Let $n \in \mathbb{Z}_+$, let $r > 1, -\infty \leq a_i < b_i \leq \infty$ for all $i = 1, ..., n$. If Φ and Ψ are positive measurable functions on $[a_1, b_1] \times \cdots \times [a_n, b_n]$, then

$$\left(\int_{\mathbf{a}}^{\mathbf{b}} \Phi(\mathbf{x}) \left(\int_{\mathbf{a}}^{\mathbf{x}} \Psi(\mathbf{y})\, \mathrm{d}\mathbf{y}\right)^r \mathrm{d}\mathbf{x}\right)^{\frac{1}{r}} \leq \int_{\mathbf{a}}^{\mathbf{b}} \Psi(\mathbf{y}) \left(\int_{\mathbf{y}}^{\mathbf{b}} \Phi(\mathbf{x})\, \mathrm{d}\mathbf{x}\right)^{\frac{1}{r}} \mathrm{d}\mathbf{y}. \tag{2.6}$$

For a proof see e.g. [127, Remark 5.2].

Remark 2.5. For $r = 1$ the inequality in (2.6) is reduced to equality according to the Fubini theorem.

2.1.4 Superquadratic and subquadratic functions

The concept of superquadratic function was formally introduced in 2004 by S. Abramovich, G. Jameson and G. Sinnamon, but the idea seems to be known before in 2001. (See [2] and [117]). The idea of a superquadratic function is a modification of the idea of a convex function.

First, we present the definition of superquadratic function and then conditions which are useful to determine the superquadraticity of a function.

Definition 2.2. (See [2, Definition 2.1].) A function $\varphi : [0,\infty) \to \mathbf{R}$ is superquadratic provided that for all $x \geq 0$ there exists a constant $C_x \in \mathbf{R}$ such that

$$\varphi(y) - \varphi(x) - \varphi(|y-x|) \geq C_x(y-x) \tag{2.7}$$

for all $y \geq 0$. We say that f is subquadratic if $-f$ is superquadratic.

Remark 2.6. Inequality (2.7) holds for all $\varphi(x) = x^p, x \geq 0, p \geq 2$ and it holds in reverse direction for all $\varphi(x) = x^p, x \geq 0, 1 < p \leq 2$. Moreover, the inequality (2.7) reduces to equality when $\varphi(x) = x^2$ with $C_x = 2x$.

Remark 2.7. If $\varphi(x)$ is a superquadratic function, then $\varphi(0) \leq 0$ and if $\varphi(0) = \varphi'(0) = 0$, then $C_x = \varphi'(x)$ whenever φ is differentiable at $x > 0$.

Likewise convexity, sometimes it is not so easy to check the superquadraticity of a function, but we do have some useful criteria to check with. We present here some of the conditions which help us to check the superquadraticity of a function.

Definition 2.3. A function $f : [0,\infty) \to \mathbb{R}$ is superadditive provided $f(x+y) \geq f(x) + f(y)$ for all $x, y \geq 0$. If the reverse inequality holds, then f is said to be subadditive.

Lemma 2.1. Suppose $\varphi : [0,\infty) \to \mathbb{R}$ is continuously differentiable and $\varphi(0) \leq 0$. If φ' is superadditive or $\dfrac{\varphi'(x)}{x}$ is nondecreasing, then φ is superquadratic.

A proof can be found in [2, Lemma 3.1].

Lemma 2.2. Suppose φ is differentiable and $\varphi(0) = \varphi'(0) = 0$. If φ is superquadratic, then $\dfrac{\varphi(x)}{x^2}$ is nondecreasing on $(0,\infty)$.

For a proof see e.g. [2, Lemma 3.2].

Lemma 2.3. A non-positive, non-increasing, superadditive function is a superquadratic function.

A proof can be found in [2, Lemma 4.1].

Example 2.3. (See [2, Example 4.2]) Let

$$\varphi_p(x) = -\left(1 + x^{1/p}\right)^p.$$

Then φ_p is superquadratic for $p > 0$ and $1 + \varphi_p$ is superquadratic for $p \geq 1/2$.

The next result was stated and proved in [2, Theorem 4.3].

Theorem 2.4. Suppose that (Ω, ν) is a measure space, and that f and g are non-negative functions such that f^p and g^q are ν-integrable. Set

$$h = \left| g^p - f^p \frac{\int_\Omega g^p \, d\nu}{\int_\Omega f^p \, d\nu} \right|^{1/p}.$$

If $0 < p \leq 1$, then

$$\left(\int_\Omega f^p \,\mathrm{d}\nu\right)^{1/p} + \left(\int_\Omega g^p \,\mathrm{d}\nu\right)^{1/p} \leq \left(\int_\Omega (f+g)^p \,\mathrm{d}\nu\right)^{1/p}$$

and

$$\left(\int_\Omega (f+g)^p \,\mathrm{d}\nu - \int_\Omega (f+h)^p \mathrm{d}\nu\right)^{1/p} \leq \left(\int_\Omega f^p \,\mathrm{d}\nu\right)^{1/p} + \left(\int_\Omega g^p \,\mathrm{d}\nu\right)^{1/p}.$$

If $p \geq 1/2$, then

$$\left(\int_\Omega (f+g)^p \,\mathrm{d}\nu - \int_\Omega (f+h)^p \mathrm{d}\nu + \int_\Omega f^p \mathrm{d}\nu\right)^{1/p} \leq \left(\int_\Omega f^p \,\mathrm{d}\nu\right)^{1/p} + \left(\int_\Omega g^p \,\mathrm{d}\nu\right)^{1/p}.$$

2.1.5 Basics on homogeneous Lie groups

In this subsection, we recall the basics of homogeneous groups, which is a necessary preparation for our Section 4.2. For more details on homogeneous groups as well as several inequalities on homogeneous groups, we refer to the monographs [39], [40], [113] and the references therein.

Definition 2.4. A Lie group $\mathbb{G}$ (identified with $(\mathbb{R}^N, \circ)$) is called a homogeneous group if it is equipped with a dilation mapping

$$D_\lambda : \mathbb{R}^N \to \mathbb{R}^N, \ \lambda > 0,$$

defined as

$$D_\lambda(x) = (\lambda^{v_1} x_1, \lambda^{v_2} x_2, ..., \lambda^{v_N} x_N), v_1, v_2, ..., v_n > 0$$

which is an automorphism of the group $\mathbb{G}$ for each $\lambda > 0$.

Here and in the sequel, we will denote the image of $x \in \mathbb{G}$ under D_λ by $\lambda(x)$ or, simply λx. The homogeneous dimension Q of a homogeneous Lie group $\mathbb{G}$ is defined by

$$Q = v_1 + v_2 + \cdots + v_N.$$

Remark 2.8. It is well known that a homogeneous group is necessarily nilpotent and unimodular.

The Haar measure $\mathrm{d}x$ on $\mathbb{G}$ is nothing but the Lebesgue measure on $\mathbb{R}^N$.

Theorem 2.5. Let ω be a measurable subset of $\mathbb{G}$. Then, for $\lambda > 0$

$$|D_\lambda(\omega)| = \lambda^Q |\omega| \text{ and } \int_\mathbb{G} f(\lambda x)\mathrm{d}x = \lambda^{-Q} \int_\mathbb{G} f(x)\mathrm{d}x,$$

where $|\omega|$ is the volume of ω.

Proof. Let $\omega \subset \mathbb{G}$. Then

$$|D_\lambda(\omega)| = \int_{\mathbb{G}} \chi_{D_\lambda(\omega)}(y)\mathrm{d}y = \lambda^{v_1+v_2+\cdots+v_N} \int_{\mathbb{G}} \chi_\omega(x)\mathrm{d}x = \lambda^Q|\omega|.$$

On the other hand

$$\int_{\mathbb{G}} f(\lambda x)\mathrm{d}x = \int_{\mathbb{G}} f(y)\mathrm{d}(\lambda^{-1}y) = \lambda^{-Q}\int_{\mathbb{G}} f(x)\mathrm{d}x.$$

This complete the proof. □

Definition 2.5. A quasi-norm on $\mathbb{G}$ is any continuous function $|\cdot| : \mathbb{G} \to [0,\infty)$ satisfying the following conditions:

(i) $|x| = |x^{-1}|$ for all $x \in \mathbb{G}$

(ii) $|\lambda x| = \lambda|x|$ for all $x \in \mathbb{G}$ and $\lambda > 0$

(iii) $|x| = 0 \Longleftrightarrow x = 0$.

Proposition 2.2 (Polar decomposition)**.** Let

$$\mathfrak{S} = \{x \in \mathbb{G} : |x| = 1\}$$

be the unit sphere with respect to the quasi-norm $|\cdot|$. Then there is a unique Radon measure σ on $\mathfrak{S}$ such that for all $f \in L^1(\mathbb{G})$,

$$\int_{\mathbb{G}} f(x)\mathrm{d}x = \int_0^\infty \int_{\mathfrak{S}} f(ry)r^{Q-1}\mathrm{d}\sigma(y)\mathrm{d}r. \tag{2.8}$$

For the proof we refer to [113, Proposition 1.2.10].

Here and in the sequel, we use the following notations. The letters u and v are weights on the homogeneous group $\mathbb{G}$. A quasi-ball in the homogeneous group $\mathbb{G}$ with radius $|x|$, $x \in \mathbb{G}$, and centred at the origin will be denoted by $B(0,|x|)$. We denote the surface measure of the unit sphere $\mathfrak{S}$ in $\mathbb{G}$ by $|\mathfrak{S}|$. The Haar measure of the unit quasi-ball $B(0,|x|)$, denoted by $|B(0,|x|)|$, can be calculated by using (2.8) as

$$|B(0,|x|)| = \int_{B(0,|x|)} \mathrm{d}y = \int_{\mathfrak{S}} \left(\int_0^{|x|} r^{Q-1}\mathrm{d}r\right)\mathrm{d}\sigma(t) = \frac{|\mathfrak{S}|}{Q}|x|^Q,$$

where $\mathfrak{S} = \{x \in \mathbb{G} : |x| = 1\}$ is the unit sphere with respect to the quasi-norm $|\cdot|$.

Theorem 2.6 (Trace theorem)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. The trace operator,

$$\gamma^+ : H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega)$$

is continuous for $s > \frac{1}{2}$.

This useful theorem is proved in [78].

Theorem 2.7. ([78, Theorem 3.29]) If Ω is a C^0 domain, then

(i) $\mathcal{D}(\overline{\Omega})$ is dense in $W^s(\Omega)$ for $s \geq 0$,

(ii) $\mathcal{D}(\overline{\Omega})$ is dense in $H^s(\Omega)$ for $s \in \mathbb{R}$.

Definition 2.6. The subspace $H^{1,0}(\Omega; A)$ of $H^1(\Omega)$ is defined as

$$H^{1,0}(\Omega; A) := \{g \in H^1(\Omega) : Ag \in L^2(\Omega)\},$$

with the norm $\|g\|^2_{H^{1,0}(\Omega;A)} := \|g\|^2_{H^1(\Omega)} + \|Ag\|^2_{L^2(\Omega)}$ (see [33] and [82]).

2.2 Literature Review

2.2.1 A superquadraticity technique to prove refined versions of inequalities

In the recent book [90] a number of variations of convexity are discussed and applied for proving classical inequalities in continuous forms and their refinements. For this PhD thesis especially the concept of superquadraticity is of interest.

In particular, the function $f(u) = u^p$ is superquadratic if $p \geq 2$ but subquadratic for $1 < p < 2$. By using, essentially the convexity (concavity) replaced by superquadraticity (subquadraticity) the authors of [96] proved the following example of a refined classical inequality, namely the Hardy inequality:

Theorem 2.8. Let $p > 1, k > 1, 0 < b \leq \infty$ and let the function f be a positive locally integrable on $(0, b)$ such that $\int_0^b x^{p-k} f^p(x)\,\mathrm{d}x < \infty$.

(i) If $p \geq 2$, then

$$\int_0^b x^{-k}\left(\int_0^x f(t)\mathrm{d}t\right)^p \mathrm{d}x + \frac{k-1}{p}\times$$
$$\int_0^b\int_t^b \left|\frac{p}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{p}} f(t) - \frac{1}{x}\int_0^x f(s)\mathrm{d}s\right|^p x^{p-k-\frac{k-1}{p}}\mathrm{d}x\, t^{\frac{k-1}{p}-1}\mathrm{d}t \tag{2.9}$$
$$\leq \left(\frac{p}{k-1}\right)^p \int_0^b \left[1 - \left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right] x^{p-k} f^p(x)\,\mathrm{d}x.$$

(ii) If $1 < p \leq 2$, then the inequality (2.9) holds in the reversed direction.

In Chapter 3.2 of this thesis we use this technique to prove a related result, now with both a general superquadratic function involved (and not only a power function) and in a much more general Banach function space setting. For example, the following was proved:

Theorem 2.9. Let $0 < b \leq \infty, a < c$, let φ be a positive and superquadratic function on (a, c) and E be a Banach function space on $[0, b)$. If E has the Fatou property and $a < f(x) < c$, then

$$\left\|\varphi\left(\frac{1}{x}\int_0^x f(t)\,\mathrm{d}t\right)\right\|_E$$
$$\leq \int_0^b \varphi\left(f(t)\right)\left\|\left(1 - \frac{\varphi\left(\left|f(t) - \frac{1}{x}\int_0^x f(s)\mathrm{d}s\right|\right)}{\varphi\left(f(t)\right)}\right)\frac{1}{x}\chi_{[t,b)}(x)\right\|_E \mathrm{d}t,$$

provided that both sides have sense.

Remark 2.9. To show the power of this technique also new refinements of other classical inequalities (e.g. those by Jensen and Minkowski) are proved and applied in Chapter 3.2. A description of these results is given in Section 3.2.2 – 3.2.5.

2.2.2 Examples of the developments during 1925 – 2017

One important early question was the following: For which weights u and v does it hold that

$$\left(\int_0^b \left(\int_0^x f(t)\,\mathrm{d}t\right)^q u(x)\,\mathrm{d}x\right)^{1/q} \leq C\left(\int_0^b f^p(x)v(x)\,\mathrm{d}x\right)^{1/p},$$

where $0 < b \leq \infty$, for some finite constant C?

During the last 80 years it has been a lot of activities to answer this and more general questions concerning Hardy type inequalities and a lot of interesting results have been proved.

Just as one example we mention the following well known result:

Theorem 2.10. Let $1 < p \leq q < \infty$ and u and v be weight functions on $\mathbb{R}_+$. Then each of the following conditions are necessary and sufficient for the inequality

$$\left(\int_0^b \left(\int_0^x f(t)\,\mathrm{d}t\right)^q u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \leq C\left(\int_0^b f^p(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p}} \tag{2.10}$$

to hold for all possible and measurable functions on $\mathbb{R}_+$:

a) The Muckenhoupt-Bradley-type condition,

$$A_{MB} := \sup_{x>0}\left(\int_x^b u(t)\,\mathrm{d}t\right)^{\frac{1}{q}}\left(\int_0^x v^{1-p'}(t)\,\mathrm{d}t\right)^{\frac{1}{p'}} < \infty \tag{2.11}$$

with $C \in [A_{MB}, \lambda A_{MB}]$, for the best constant C in (2.10), where

$$\lambda = \min\{p^{1/q}(p')^{1/p'}, q^{1/q}(q')^{1/p'}\};$$

b) The Persson-Stepanov condition

$$A_{PS} := \sup_{x>0} V^{-\frac{1}{p}}(x)\left(\int_0^x u(t)V^q(t)\,\mathrm{d}t\right)^{\frac{1}{q}} < \infty, V(x) := \int_0^x v^{1-p'}(t)\,\mathrm{d}t, \tag{2.12}$$

with $C \in [A_{PS}, p'A_{PS}]$, for the best constant in (2.10).

Remark 2.10. A simple proof of the condition (2.11) was given by B. Muckenhoupt in 1972 for $p = q$ (see [89]) and by J. S. Bradley in 1978 for $p \leq q$ (see [25]). In 2002, L. E. Persson and V. D. Stepanov presented an elementary proof of the alternative condition (2.12) (see [108]). In this connection it should be mentioned even the earlier papers by G. Talenti (see [122]) and G. Tomaselli (see [123]) (the case p=q), respectively G. Sinnamon and V. D. Stepanov (see [118]). Just note that the newer characterization (2.12) indeed gives the sharp constant $(p/(p-1))^p$ in the original form of Hardy's inequality while the estimates in the Muckenhoupt-Bradley description can never give that.

Remark 2.11. The motivation for the authors of [108] to derive the alternative PS condition was to find a characterization of the limit Pólya-Knopp inequality, see more details in Section 2.2.5.

Remark 2.12. It has recently been discovered that also these two conditions to characterize (2.10) are not unique and can even be replaced by infinite many equivalent conditions, in fact even by so called scales of conditions (for details see [73, Section 7.3.3] and our Subsection 2.2.9).

2.2.3 On the sharp constant for the power weighted case when $1 < p < q < \infty$

By applying the general results (see Theorem 2.10 and the corresponding dual result) for the power weighted case we get the following:

Example 2.4. The inequality

$$\left(\int_0^\infty \left(\int_0^x f(t)\,\mathrm{d}t\right)^q x^\alpha\,\mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\int_0^\infty f^p(x)x^\beta\,\mathrm{d}x\right)^{\frac{1}{p}} \tag{2.13}$$

holds for some constant $C > 0$, for all positive and measurable functions $f(t)$ on $(0,\infty)$ and $1 < p \le q < \infty$ if and only if

$$\beta < p-1 \text{ and } \frac{\alpha+1}{q} = \frac{\beta+1}{p} - 1. \tag{2.14}$$

For the case $p = q > 1$ the sharp constant C in (2.13) was pointed out in Theorem 1.1 $a), e)$ (the case $\ell = \infty$). However, for the case $1 < p < q < \infty$ this sharpness question is much more delicate.

The next result was proved in 2015 by L. E. Persson and S. Samko, see [105]. Indeed, this result gave a final answer to an old open question, where G. A. Bliss in 1930 found the best constant C for the case $\beta = 0$ and $\alpha = -\left(\frac{q}{p'}+1\right)$, $1 < p < q < \infty$, in (2.13), see [24].

Theorem 2.11. Let $1 < p < q < \infty$ and the parameters α and β satisfy (2.14). Then the sharp constant in (2.13) is $C = C^*_{pq}$, where

$$C^*_{pq} = \left(\frac{p-1}{p-1-\beta}\right)^{\frac{1}{p'}+\frac{1}{q}} \left(\frac{p'}{q}\right)^{\frac{1}{p}} \left(\frac{\frac{q-p}{p}\Gamma\left(\frac{pq}{q-p}\right)}{\Gamma\left(\frac{p}{q-p}\right)\Gamma\left(\frac{p(q-1)}{q-p}\right)}\right)^{\frac{1}{p}-\frac{1}{q}}.$$

The relation (2.13) holds with the equality when

$$f(x) = \frac{cx^{-\frac{\beta}{p-1}}}{\left(dx^{\frac{p-1-\beta}{p-1}\cdot\left(\frac{q}{p}-1\right)}+1\right)^{\frac{q}{q-p}}} a.e.,$$

where c and d are positive constants. Moreover,

$$C^*_{pq} \to \frac{p}{p-1-\beta} \text{ as } q \to p. \tag{2.15}$$

Note that (2.15) shows that we have the expected continuity in the sharpness results in Theorem 1.1 $a), e)$ (the case $\ell = \infty$) and Theorem 2.11.

2.2.4 On the multidimensional case

Next we want to pronounce that there is a much less developed theory for the multidimensional case than in the one-dimensional case. We also remark that many such results up to 2017 can be found in the book [73, Chapter 7.7]. Here we will only mention a possibility with relation to this thesis namely the case of "rectangular" Hardy operators e.g.

$$H_2 : (H_2 f)(x, y) = \int_0^x \int_0^y f(s, t)\, \mathrm{d}s\mathrm{d}t.$$

The basic (at that time surprising and first not so well understood) result by E. Sawyer from 1985 reads (see [116]):

Theorem 2.12. Let $1 < p \leq q < \infty$ and u and v be weights on $\mathbb{R}^2_+$. Then, the inequality

$$\begin{aligned} &\left(\int_0^\infty \int_0^\infty \left(\int_0^{x_1} \int_0^{x_2} f(t_1, t_2)\, \mathrm{d}t_1 \mathrm{d}t_2 \right)^q u(x_1, x_2)\, \mathrm{d}x_1 \mathrm{d}x_2 \right)^{\frac{1}{q}} \\ &\qquad \leq C \left(\int_0^\infty \int_0^\infty f^p(x_1, x_2) v(x_1, x_2)\, \mathrm{d}x_1 \mathrm{d}x_2 \right)^{\frac{1}{p}} \end{aligned} \tag{2.16}$$

holds for all nonnegative and measurable functions on $\mathbb{R}^2_+$, if and only if the following three conditions are satisfied:

$$\sup_{(y_1,y_2)\in\mathbb{R}^2_+} \left(\int_{y_1}^\infty \int_{y_2}^\infty u(x_1, x_2)\, \mathrm{d}x_1 \mathrm{d}x_2 \right)^{\frac{1}{q}} \left(\int_0^{y_1} \int_0^{y_2} v(x_1, x_2)^{1-p'}\, \mathrm{d}x_1 \mathrm{d}x_2 \right)^{\frac{1}{p'}} < \infty, \tag{2.17}$$

$$\sup_{(y_1,y_2)\in\mathbb{R}^2_+} \frac{\left(\int_0^{y_1} \int_0^{y_2} \left(\int_0^{x_1} \int_0^{x_2} v(t_1, t_2)^{1-p'} \mathrm{d}t_1 \mathrm{d}t_2 \right)^q u(x_1, x_2)\, \mathrm{d}x_1 \mathrm{d}x_2 \right)^{\frac{1}{q}}}{\left(\int_0^{y_1} \int_0^{y_2} v(x_1, x_2)^{1-p'} \mathrm{d}x_1 \mathrm{d}x_2 \right)^{\frac{1}{p}}} < \infty, \tag{2.18}$$

$$\sup_{(y_1,y_2)\in\mathbb{R}^2_+} \frac{\left(\int_{y_1}^\infty \int_{y_2}^\infty \left(\int_{x_1}^\infty \int_{x_2}^\infty u(t_1, t_2)\, \mathrm{d}t_1 \mathrm{d}t_2 \right)^{p'} v(x_1, x_2)^{1-p'}\, \mathrm{d}x_1 \mathrm{d}x_2 \right)^{\frac{1}{p'}}}{\left(\int_{y_1}^\infty \int_{y_2}^\infty u(x_1, x_2)\, \mathrm{d}x_1 \mathrm{d}x_2 \right)^{\frac{1}{q'}}} < \infty. \tag{2.19}$$

Remark 2.13. Note that (2.17) corresponds to the Muckenhoupt-Bradley condition (2.11), (2.18) corresponds to the Persson-Stepanov condition (2.12) and (2.19) corresponds to the dual condition of (2.12).

Remark 2.14. In the PhD theses by A. Wedestig (see [128]) and E. Ushakova (see [124]) (and in related papers) it was proved that if the weight to the left or to the right in (2.16) is of product type, then the inequality (2.16) can be characterized by using only one condition. Moreover, in this case the results could be extended to the general n-dimensional setting.

Remark 2.15. E. Sawyer proved also that none of the conditions (2.17), (2.18) or (2.19) could be removed in general.

Concerning Sawyer's Theorem 2.12 we want to pronounce some recent remarkable complements. In fact, in the paper [120] it was proved that for the case

$1 < p < q < \infty$ the inequality (2.16) can be characterized by using only one condition, namely the first Muckenhoupt-Bradley type condition. And in fact, they even proved this result in a general n-dimensional setting, see also the newer complementary paper [120]. Moreover, Hardy-type inequalities have also been developed in the time-scale setting even for multidimensional cases, see [38] and the references therein. The results in paper [132] can be regarded as limit cases of a scale of Hardy-type inequalities (even if they are not proved so) and therefore it can be an interesting research question to investigate if also these inequalities can be proved in the time-scale setting.

2.2.5 On a limit case involving the geometric mean operator

We remind some facts we already indicated in Section 1.1.2. Indeed, by replacing $f(x)$ with $(f(x))^{1/p}$ and letting $p \to \infty$ in (1.5), we obtain the following weighted Pólya-Knopp's inequality

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \log f(t)\,\mathrm{d}t\right)x^\alpha\,\mathrm{d}x \le e^{(1+\alpha)}\int_0^\infty f(x)x^\alpha\,\mathrm{d}x,$$

for $\alpha > -1$ and f is a positive and measurable function on $(0,\infty)$. Moreover, the constant $e^{(1+\alpha)}$ is sharp. Of course, it is tempting to use this simple idea to obtain the limit inequality also in the general weighted case as described in our Theorem 2.10. But it is maybe surprising that this is not possible by using the standard Muckenhoupt-Bradley condition (2.11). This was the reason why the authors of [108] derived the alternative characterization (2.12), see [108, Theorem 1]. Also the paper [110] was important in this investigation (the authors of [108] even called it the Pick and Opic scheme). Indeed by using this new characterization it was proved that we obtain the following limit (Pólya-Knopp type) inequality (see [108, Theorem 2]):

Theorem 2.13. Let $0 < p \le q < \infty$ and let $v(x)$ and $u(x)$ be weight functions. Then, the inequality

$$\left(\int_0^\infty \left[\exp\left(\frac{1}{x}\int_0^x \log f(t)\,\mathrm{d}t\right)\right]^q u(x)\,\mathrm{d}x\right)^{1/q} \le C\left(\int_0^\infty f^p(x)v(x)\,\mathrm{d}x\right)^{1/p} \tag{2.20}$$

holds for all positive and measurable functions f on $(0,\infty)$ if and only if

$$D := \sup_{t>0} t^{-1/p}\left(\int_0^t w(x)\,\mathrm{d}x\right)^{1/q} < \infty,$$

where

$$w(x) := \left[\exp\left(\frac{1}{x}\int_0^x \log v^{-1}(t)\,\mathrm{d}t\right)\right]^{\frac{q}{p}} u(x).$$

Moreover, the sharp constant C in (2.20) satisfies

$$D \le C \le e^{1/p}D.$$

Open Question 2.1. Is it possible to do a similar limit procedure to derive the characterizations of limit (Pólya-Knopp type) inequalities in weighted multidimensional cases by using known characterizations of Hardy-type inequalities for such cases (cf. Remark 2.14, [124], [128] and later results in [120])? If so we get an alternative to prove some results obtained later on in this PhD thesis (see Theorem 4.2).

However, multidimensional Pólya-Knopp type inequalities are proved by using direct methods. For example the following result was proved in [127, Theorem 4.1] (see also [128]):

Theorem 2.14. Let $0 < p \leq q < \infty$, and let u, v and f be positive and measurable functions on $\mathbb{R}^2_+$. If $0 < b_1, b_2 \leq \infty$, then the inequality

$$\left(\int_0^{b_1}\int_0^{b_2}\left[\exp\left(\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2}\ln f(y_1,y_2)\,\mathrm{d}y_1\mathrm{d}y_2\right)\right]^q u(x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2\right)^{\frac{1}{q}} \leq C\left(\int_0^{b_1}\int_0^{b_2} f^p(x_1,x_2)v(x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2\right)^{\frac{1}{p}} \tag{2.21}$$

holds if and only if

$$D_W(s_1,s_2,p,q) := \sup_{\substack{y_1\in(0,b_1)\\ y_2\in(0,b_2)}} y_1^{\frac{s_1-1}{p}} y_2^{\frac{s_2-1}{p}} \left(\int_{y_1}^{b_1}\int_{y_2}^{b_2} x_1^{-\frac{s_1q}{p}} x_2^{-\frac{s_2q}{p}} w(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2\right)^{\frac{1}{q}} < \infty,$$

where $s_1, s_2 > 1$ and

$$w(x_1,x_2) = \left[\exp\left(\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2}\ln v^{-1}(t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2\right)\right]^{\frac{q}{p}} u(x_1,x_2),$$

and the best possible constant C in (2.21) can be estimated in the following way:

$$\sup_{s_1,s_2>1}\left(\frac{e^{s_1}(s_1-1)}{e^{s_1}(s_1-1)+1}\right)^{\frac{1}{p}}\left(\frac{e^{s_2}(s_2-1)}{e^{s_2}(s_2-1)+1}\right)^{\frac{1}{p}} D_W(s_1,s_2,p,q) \leq C \leq \inf_{s_1,s_2>1} e^{\frac{s_1+s_2-2}{p}} D_W(s_1,s_2,p,q). \tag{2.22}$$

Remark 2.16. Another characterization of (2.21) for the case $b_1 = b_2 = \infty$ and $p = q = 1$, but without explicit estimates of the best constant like in (2.22), was earlier proved in [52]. In [132, Theorem 3], it was proved that Theorem 2.14 indeed holds in a general n-dimensional setting ($n = 2, 3, ...$).

Remark 2.17. The first paper where the sharp constant in a multidimensional Pólya-Knopp inequality was discussed seems to be [54]. In particular, in [54, Theorem 2.2], the authors stated that the inequality

$$\int_0^\infty\int_0^\infty \exp\left(\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2}\ln f(y_1,y_2)\,\mathrm{d}y_1y_2\right)x_1^a x_2^a\,\mathrm{d}x_1\mathrm{d}x_2 \leq e^{2(1+a)}\int_0^\infty\int_0^\infty f(x_1,x_2)x_1^a x_2^a\,\mathrm{d}x_1\mathrm{d}x_2$$

holds and that the constant $e^{2(1+a)}$ is sharp. Here $a \in (0, \infty)$ and f is a positive and measurable function. This result is correct but the proof in [54] contains a gap, a gap which is corrected in [132, Theorem 2] (see also the proof of the more general case in [132, Theorem 4]).

Remark 2.18. In paper [132] both the general descriptions and the question concerning sharp constants are essentially improved and unified for multidimensional Pólya-Knopp type inequalities.

2.2.6 Cochran-Lee type inequalities

H. P. Heinig stated and proved in his 1975 paper [51] the following Cochran-Lee type inequalities:

$$\int_0^\infty x^\lambda \exp\left[px^{-p}\int_0^x t^{p-1}\log|x^{-s}f(t)|\mathrm{d}t\right]\mathrm{d}x \le e^{1/p}A\int_0^\infty t^{\lambda-s}|f(t)|\mathrm{d}t, \tag{2.23}$$

where $A = p/(p+s-\lambda)$ with p, s and λ real numbers satisfying $p+s > \lambda$, $p > 0$;

$$\int_0^\infty x^\lambda \exp\left[p^2x^{-p}\int_0^x t^{p-1}\log|x^{-s}f(t)|\mathrm{d}t\right]\mathrm{d}x \le eB\int_0^\infty t^{\lambda-sp}|f(t)|^p\mathrm{d}t, \tag{2.24}$$

where $B = p/(2p+sp-\lambda-1)$ with p, s and λ real numbers satisfying $2p+sp > \lambda+1$, $p > 0$. In both inequalities, the right hand sides are assumed to be finite. He also showed that these results have application in the estimates of the Laplace transform. But, no one of these inequalities are sharp.

In [31], by taking an appropriate limit in a classical result of Hardy, J. A. Cochran and C.-S. Lee proved that for $p > 0$ and γ real the inequality

$$\int_0^\infty x^\gamma \exp\left[px^{-p}\int_0^x t^{p-1}\log|f(t)|\mathrm{d}t\right]\mathrm{d}x \le \exp\left(\frac{\gamma+1}{p}\right)\int_0^\infty t^\gamma|f(t)|\mathrm{d}t. \tag{2.25}$$

holds for any measurable function f. Moreover, the constant $\exp\left(\frac{\gamma+1}{p}\right)$ is sharp. This result improve the Heinig inequalities (2.23) and (2.24), since both of the inequalities (2.23) and (2.24) are variants of the same inequality.

In Chapter 4 of this PhD thesis we use the direct method to prove the higher dimensional version of Cochran-Lee inequality (2.25). Moreover, we derive also some new Pólya-Knopp type inequalities on homogeneous groups in Section 4.2.

2.2.7 On the case with more general Hardy operators

Hardy type inequalities with kernel operators

In [73, Chapter 2], in particular, characterizations of the following more general Hardy-type inequality

$$\|Tf\|_{q,u} \le C\|f\|_{p,v}, \tag{2.26}$$

or, more precisely,

$$\left(\int_0^b |(Tf)(x)|^q u(x)\,\mathrm{d}x\right)^{1/q} \leq C\left(\int_0^b |f(x)|^p v(x)\,\mathrm{d}x\right)^{1/p}$$

are considered, where T is an operator of the form

$$Tf(x) := \int_0^x k(x,y)f(y)\,\mathrm{d}x,$$

with $k(x,y)$ a given positive kernel, u,v weight functions was discussed. If $k(x,y) \equiv \frac{1}{x}$, then T coincides with the usual Hardy operator H defined by

$$Hf(x) := \frac{1}{x}\int_0^x f(y)\,\mathrm{d}y.$$

Some facts:

(i) Without restrictions on the kernel $k(x,y)$ the problem is open, in general.

(ii) The solution of this problem is known for a number of special cases and parameters.

The following result was recently proved for the general kernel operator case (see [74] and Theorem 7.44 in [73]):

Theorem 2.15. Let $1 < p \leq q < \infty, 0 < b \leq \infty$, u and v are weights. Let $k(x,y)$ be a nonnegative kernel.

(a) Then (2.26) holds if

$$A_s := \sup_{0<y<b}\left(\int_y^b k^q(x,y)u(x)V^{\frac{q(p-s-1)}{p}}(x)\,\mathrm{d}x\right)^{\frac{1}{q}} V^{s/p}(y) < \infty \tag{2.27}$$

for any $s < p-1$.

(b) The condition (2.27) can not be improved in general for $s > 0$ because for product kernels of type $k(x,y) \equiv A(x)B(y)$, it is even necessary and sufficient for (2.26) to hold.

(c) For the best constant C in (2.26) we have the following estimate

$$C \leq \inf_{s<p-1}\left(\frac{p}{p-s-1}\right)^{1/p'} A_s.$$

Here we use the following notations

$$U(x) := \int_x^b u(y)\,\mathrm{d}y, \quad V(x) := \int_0^x v^{1-p'}(y)\,\mathrm{d}y. \tag{2.28}$$

Remark 2.19. This result opens a possibility that the condition (2.27) can be a candidate to solve the open question we have pointed out in (i) above.

We consider the following general Hardy-type kernel operator A_k defined by

$$A_k f(x) := \frac{1}{K(x)} \int_0^x k(x,y) f(y)\,\mathrm{d}y, \tag{2.29}$$

where $k(x,y)$ is nonnegative measurable function and

$$K(x) := \int_0^x k(x,y)\,\mathrm{d}y. \tag{2.30}$$

With some restrictions on the kernels (e.g. so called Oinarov kernels, homogeneous kernels, product kernels) it is fairly much knowledge about Hardy-type inequalities even in this case, see [73, Chapter 2 and Section 7.5] and the references therein. However, without such restrictions there remain many open questions, see [73, Section 7.5]. But also in this general case there exist some results e.g. the following (cf. [59, Theorem 4.4]):

Theorem 2.16. Let $1 < p \leq q < \infty$, $0 < b \leq \infty$, $s \in (1,p)$, let Φ be a positive and convex function on (a,c), and let A_k be the operator defined by (2.29). Then, the inequality

$$\left(\int_0^b \Phi^q \left(A_k f(x) \right) u(x) \frac{\mathrm{d}x}{x} \right)^{1/q} \leq C \left(\int_0^b \Phi^p \left(f(x) \right) v(x) \frac{\mathrm{d}x}{x} \right)^{1/p} \tag{2.31}$$

holds for all functions $f(x)$, $a < f(x) < c$, $x \in [0,b]$, and some constant $C > 0$ if

$$A(s) := \sup_{0<t<b} \left(\int_t^b \left(\frac{k(x,t)}{K(x)} \right)^q u(x) V(x)^{\frac{q(p-s)}{p}} \frac{\mathrm{d}x}{x} \right)^{\frac{1}{q}} V(t)^{\frac{s-1}{p}} < \infty,$$

where $u(x)$ and $v(x)$ are general weight functions and $V(x) = \int_0^x \frac{v^{1-p'}(t)}{t^{1-p'}} \mathrm{d}t$. Moreover, if C is the sharp constant in (2.31), then

$$C \leq \inf_{1<s<p} \left(\frac{p-1}{p-s} \right)^{\frac{1}{p'}} A(s).$$

Remark 2.20. In Chapter 3 of this thesis we consider mapping properties of a more general Hardy type operator T defined as follows:

$$Tf(x) = \frac{1}{\sigma_x(S)} \int_S f(t) \mathrm{d}\sigma_x(t),$$

where f is defined on S with values in $(a,b), -\infty \leq a < b \leq \infty$, and $\sigma_x(S) = \int_S \mathrm{d}\sigma_x(t)$. For detail description of the obtained results see Section 3.1.3.

2.2.8 On the case with more general function spaces

Concerning generalization of Hardy type inequalities in more general function spaces up to 2017, see the book [73, Chapter 7.6]. Such results are known for the following cases: Orlicz, Lorentz, rearrangement invariant, Morrey-type, Hölder-type and variable $L^{p}(\cdot)$ spaces. However, very little is known for general function spaces, metric spaces or more general cases.

Remark 2.21. In Chapter 3 of this PhD thesis we investigate the mapping properties in function spaces not considered before e.g. Banach function spaces and even with more general Hardy type operators of type (2.29) involved.

2.2.9 On some scales of conditions to characterize the modern form of Hardy's inequality

The following new result was already mentioned for the general kernel operator case. The first results of this type were discovered in connection to the work in the PhD thesis [128]. After that also this theory was developed very much, see [73, Section 7.3]. As an example connected to Theorem 2.10 we mention the following result:

Theorem 2.17. Let $1 < p \leq q < \infty, 0 < s < \infty$, and define, for the weight functions u, v, the functions U and V by (2.28). Then (2.10) can be characterized by any of the conditions $A_i(s) < \infty, i = 1, 2, 3, 4$, where

$$A_1(s) := \sup_{0<x<b} \left(\int_x^b u(t) V^{q\left(\frac{1}{p'}-s\right)}(t)\, \mathrm{d}t \right)^{1/q} V^s(x);$$

$$A_2(s) := \sup_{0<x<b} \left(\int_0^x v^{1-p'}(t) U^{p'\left(\frac{1}{q}-s\right)}(t)\, \mathrm{d}t \right)^{1/p'} U^s(x);$$

$$A_3(s) := \sup_{0<x<b} \left(\int_0^x u(t) V^{q\left(\frac{1}{p'}+s\right)}(t)\, \mathrm{d}t \right)^{1/q} V^{-s}(x);$$

$$A_4(s) := \sup_{0<x<b} \left(\int_x^b v^{1-p'}(t) U^{p'\left(\frac{1}{q}+s\right)}(t)\, \mathrm{d}t \right)^{1/p'} U^{-s}(x).$$

Remark 2.22. Note that A_{MB} and A_{PS} conditions are just two different points on these scales, namely

$$A_{MB} = A_1\left(\frac{1}{p'}\right) \text{ and } A_{PS} = A_3\left(\frac{1}{p}\right).$$

Also all other known alternative previous conditions can be expressed in terms of these fundamental scales of conditions.

Remark 2.23. The scales on conditions in Theorem 2.17 can even (equivalently) be complemented with 10 more scales of condition. Very surprising. Also all other known alternative conditions are just points on these 14 scales of conditions.

In view of Subsections 2.2.4 – 2.2.5 above and the ideas in this thesis, it is natural to ask the following:

Open Question 2.2. Is it possible to develop a similar theory concerning scales of conditions for the following cases:

a) multidimensional Hardy type inequalities,

b) limit Pólya-Knopp type inequalities both in the one-dimensional and multidimensional cases?

Concerning question *b)* we even have a first result in this PhD thesis see Theorem 4.5.

We aim to return to this question in a forthcoming paper.

2.3 Examples of the latest development 2018 and later

As mentioned before some of the most important results up to 2017 have been presented in some books, see e.g. [65], [71], [73], [113] and the references given there. But the interest in this area has continued to be great and even increasing also after 2017 and several interesting results have been obtained. In this section we give a brief descriptions of some of these newest developments and applications after 2017, which are or can be of interest in relation to the results obtained in this PhD thesis (see e.g. the raised open questions). In Chapter 3 – 5, we describe in detail the new results obtained in this PhD thesis.

Hardy-type inequalities in more general function spaces and more general Hardy operators

Remark 2.24. In Chapter 3 both some new Hardy-type inequalities in Banach function spaces are proved and with a general Hardy-type operator involved. See also [19].

Remark 2.25. Inspired by the information in Remark 2.24 it is of interest to investigate if also other parts of the theory of Hardy-type inequalities can be generalized in some or both of these directions.

Refinements of Hardy-type inequalities

For results up to 2017 see our Section 2.2.1. Our new main results in this PhD thesis are presented in detail in Chapter 3.2. Other recent refinements are proved and discussed in [95]. Also in the limit case described in Theorem 1.2 further refinements can be done, see [97] (cf. also [99]). We remark that in these papers and this thesis we have mainly used the following variants of concept of convexity: superquadraticity and strong convexity. However, there are also many other possibilities to derive refinements of inequalities e.g. to use γ-convexity (see e.g. [4]) and other direct methods (see e.g. [92]). It is an interesting research question to try to further generalize and complement some of the results in this PhD thesis by using also these and other related methods.

Further developments of the multidimensional case

For the results up to 2017 we refer to our Subsection 2.2.4. Concerning the simple idea to use polar coordinates we just mention the new results in [55] (cf. also [43] and [61]), where this simple idea is developed also for bilinear and iterated Hardy-type inequalities and even with the new idea of “scales of conditions” involved.

Remark 2.26. An important development of polar coordinate idea to be able to even handle some metric spaces has recently been presented and applied by M. Ruzhansky and collaborators. The basic idea is as follows:

Consider a metric spaces $\mathbb{X}$ with a Borel measure $\mathrm{d}x$, allowing for the following polar decomposition at $a \in \mathbb{X}$: we assume that there is a locally integrable function

$\lambda \in L^1_{loc}$ such that for all $f \in L^1(\mathbb{X})$ we have

$$\int_{\mathbb{X}} f(x)\,\mathrm{d}x = \int_0^\infty \int_{\Sigma_r} f(r,\omega)\lambda(r,\omega)\,\mathrm{d}\omega_r \mathrm{d}r,$$

for the $\Sigma_r = \{x \in \mathbb{X} : d(x,a) = r\} \subset \mathbb{X}$ with a measure on it denoted by $\mathrm{d}\omega = \mathrm{d}\omega_r$, and $\omega \to a$ as $r \to 0$. See for example the recent paper [115], and especially the new impressive book [113] on Hardy inequalities on homogeneous groups.

Remark 2.27. Another important development of polar coordinate idea on homogeneous group $\mathbb{G}$ has been applied on some Hardy type inequalities by M. Ruzhansky and others. The basic idea is as follows:

Let $\mathbb{G}$ be a homogeneous group. If $\mathfrak{S} = \{x \in \mathbb{G} : |x| = 1\} \subset \mathbb{G}$ is the unit sphere with respect to the quasi-norm, then there is a unique Radon measure σ on $\mathfrak{S}$ such that for all $f \in L^1(\mathbb{G})$, we have the following polar decomposition

$$\int_{\mathbb{G}} f(x)\mathrm{d}x = \int_0^\infty \int_{\mathfrak{S}} r^{Q-1} f(ry)\mathrm{d}\sigma(y)\mathrm{d}r,$$

where Q is the homogeneous dimension of a homogeneous group $\mathbb{G}$. (See [113]).

New applications

One reason why Hardy-type inequalities have survived as an important area of research in more than 100 years is heavily depending on its importance for applications. First, we mention the fact that the development concerning idea of "the scales of conditions" was recently applied and developed in Fourier analysis to prove a result where some Fourier inequalities could be characterized by using any of infinitely many different (but equivalent) conditions, even scales of conditions, where only one of such condition was known before. See [72].

The second application we want to mention is also related to Fourier analysis. Indeed, Hardy-type inequalities are applied, and partly complemented, in several situations in the new book [109] on the most modern form of Fourier analysis.

The third application to the theory of differential equations can be found in the papers [60] and [100], see also our Section 6.2.

An interesting research question can be to investigate the possibility to find some more general applications in all three cases above by applying and modifying the results in this PhD thesis.

New proofs

As mentioned before to find new methods of proofs can be even more important for the development than to obtained new results. One important such example we want to mention is the new proof presented in [44]. This proof is very surprising due to the following reasons:

(a) Only Fubini's theorem, Hölder's inequality, Minkowski's integral inequality and Hardy's lemma are used.

(b) It unifies the proofs of all cases including the convex case $1 < p \leq q$ and the non-convex case $p > q$.

Here we also want to mention some new proof techniques (usually called discretization/antidiscretization) developed and applied by A Gogatishvili and collaborators. For a very new paper using this technique we mention [45].

It seems of interest to investigate if this new "universal" proof and discretization/antidiscretization techniques can be generalized to some of the more general cases studied in this thesis.

Sharp constants

In the classical theory of Hardy-type inequalities we have good estimates for the sharp constants. This fact is very important for applications since the sharp constant is equal to the operator norm of the corresponding mapping. And therefore, especially, to derive the sharp constant is of special interest. In the paper [107] most of the known results for the one-dimensional case can be found and also new results are derived especially for cones of monotone functions where even two-sided estimates are obtained and both involved constants are sharp. In particular, these results imply some two-sided sharp inequalities for different (quasi)norms in Lorentz spaces.

Inspired by these results and some results in [132] it is of interest to describe and further develop the corresponding situation for the multidimensional case.

Discrete Hardy-type inequalities

As motivated in Section 1.1.1, in the simplest classical cases the continuous Hardy inequality (1.4) implies the corresponding discrete inequality (1.3). However, this is not as obvious in the general weighted Hardy-type inequalities. But there are many such results in the literature where discrete Hardy-type inequalities are proved by using the corresponding continuous result. As a very recent result of this type we refer to [45]. As just another such recent example of new result for the discrete case we mention the paper [101]. Since many results in this PhD thesis are formulated and proved for a general measure they can also be applied to obtain some new discrete Hardy-type inequalities.

2.4 Review of BDIEs for variable-coefficient BVPs

Partial Differential Equations (PDEs) with variable coefficients often arise in mathematical modelling of inhomogeneous media (e.g. functionally graded materials or materials with damage induced inhomogeneity) in solid mechanics, electromagnetics, thermo-conductivity, fluid flows through porous media, and other areas of physics and engineering.

Generally, explicit fundamental solutions are not available if the PDE coefficients are not constant, preventing reduction of boundary value problems (BVPs) for such PDEs to explicit boundary integral equations (BIEs), which could be effectively solved numerically. Nevertheless, for a rather wide class of variable-coefficient PDEs it is possible to use instead an explicit parametrix (Levi function) associated with the fundamental solution of the corresponding frozen-coefficient PDEs, and reduce BVPs for such PDEs to systems of boundary-domain integral equations (BDIEs) for further numerical solution of the latter, see e.g. [27, 29, 83, 85, 86] and references therein. However this (one-operator) approach does not work when the fundamental solution of the frozen-coefficient PDE is not known explicitly (as e.g. in the Lamé system of anisotropic elasticity). To overcome this difficulty, one can apply the so-called two-operator approach, formulated in [84] for a certain non-linear problem, that employs a parametrix of another (second) PDE, not related with the PDE in question, for reducing the BVP to a BDIE system. Since the second PDE is rather arbitrary, one can always chose it in such a way, that its parametrix is known explicitly. The simplest choice for the second PDE is the one with an explicit fundamental solution.

In [13, 14], one of the linear versions of the two-operator approach is applied to the mixed (Dirichlet-Neumann) BVP for a linear second-order scalar elliptic variable-coefficient PDE with *square integrable right-hand side*. Using appropriate parametrix the problem is reduced to four different two-operator BDIE systems. The BDIE systems are nonstandard systems of equations containing integral operators defined on the domain under consideration and potential type and pseudo-differential operators defined on open sub-manifolds of the boundary. Using results in [29], a rigorous analysis of the two-operator BDIEs is given in appropriate Sobolev spaces.

As described in [79, 80] for a function from the Sobolev space $H^1(\Omega)$, a classical conormal derivative in the sense of traces may not exist. However, when this function satisfies a second order PDE with a right-hand side from $H^{-1}(\Omega)$, the generalized conormal derivative can be defined in the weak sense, associated with the first Green identity and an extension of the PDE right-hand side to $\widetilde{H}^{-1}(\Omega)$ (see, e.g., [78, Lemma 4.3], [82, Definition 3.1]). Since the extension is non-unique, the conormal derivative appears to be a non-unique operator, which is also non-linear in u unless a linear relation between u and the PDE right-hand side extension is enforced. This creates some difficulties in formulating the BDIEs. These difficulties are addressed in [79, 80] presenting formulation and analysis of direct segregated BDIE systems equivalent to the Dirichlet and Neumann problems for the divergent-type PDE with a variable scalar coefficient and a general right-hand side from the space $H^{-1}(\Omega)$ extended when necessary to the space $\widetilde{H}^{-1}(\Omega)$. In particular, this required to derive a non-trivial generalization of the third Green identity and its conormal derivative for such functions, which extends the approach implemented in [27–29, 81, 83] for the PDE right-hand from $L_2(\Omega)$.

In [7], using the two-operator approach in settings different from the one in [13, 14], a generalization of the two-operator third Green identity and its conormal derivative is derived and the two-operator BDIE systems for variable-coefficient mixed BVPs are investigated.

Nowadays, the theory of BDIEs in 3D is well developed, cf. [27–29, 84, 86], the BDIEs in 2D need a special consideration due to their different equivalence properties. As a result, we need to set conditions on the associated Sobolev spaces or choose appropriate scaling parameter in parametrix form to insure the invertibility of the corresponding parametrix-based integral layer potentials and hence the unique solvability of BDIEs.

In [6], extending the results of [8], the mixed BVP for the linear second-order scalar elliptic differential equation with variable coefficients in a bounded two-dimensional domain with general data is considered and a condition is set only on the domain to ensure the invertibility of the layer potentials. The PDE right-hand side belongs to $H^{-1}(\Omega)$ or $\widetilde{H}^{-1}(\Omega)$ when neither classical nor canonical conormal derivatives of the solutions are well defined. The two-operator approach and appropriate parametrix (Levi function) are used to reduce the problem to four different systems of BDIEs. Using similar approach as in [7, 8, 11] rigorous analysis of the newly obtained BDIEs is given.

We devote the second part of this PhD thesis to the analysis of two-operator BDIEs for variable-coefficient Dirichlet and Neumann problems in two-dimensional bounded domain with general right-hand side and extends the results in [6]. Finally, in Chapter 6 we shorty describe the close relations between these two parts.

Chapter 3

Some Hardy-type inequalities in Banach function spaces and Refinements of some classical inequalities

3.1 Hardy-type inequalities

3.1.1 Introduction

In this Chapter, we prove some new Hardy-type inequalities in a Banach function space setting, which, in particular, generalize and unify several classical Hardy-type inequalities.

For the presentations of these results we need some notations and definitions.

Let (Ω, Σ, μ) be a complete $\sigma-$finite measure space and $L^0(\mu) = L^0(\Omega, \Sigma, \mu)$ denote the space of (equivalence classes) of $\mu-$measurable real-valued functions endowed with the topology of convergence in measure relative to each set of finite measure.

Definition 3.1. A Banach space $E \subset L^0(\mu)$ is called a Banach function space on (Ω, Σ, μ) if there exists a $u \in E$ such that $u > 0$ a.e. and E satisfies the following ideal property:

$$x \in L^0(\mu), y \in E, |x| \leq |y| \, \mu - a.e. \Rightarrow x \in E \text{ and } \|x\|_E \leq \|y\|_E.$$

Remark 3.1. A Banach function space E is said to have the *Fatou property* if whenever (f_n) is a norm-bounded sequence in E such that

$$0 \leq f_n \uparrow f \in L^0(\mu), \text{ then } f \in E \text{ and } \|f_n\| \to \|f\|.$$

The non-increasing rearrangement $f^*(t), 0 < t < \infty$, of a $\mu-$measurable function f on Ω is defined by

$$f^*(t) := \inf\{\lambda > 0 : \mu_f(\lambda) \leq t\},$$

where $\mu_f(\lambda)$ is the distribution function defined by

$$\mu_f(\lambda) := \mu\left(\{t \in \Omega : |f(t)| > \lambda\}\right).$$

Definition 3.2. The Lorentz spaces $L^{p,q}, 0 < p < \infty$ and $0 < q \leq \infty$ are defined by the quasi-norm $\|f\|_{L^{p,q}}$ defined by

$$\|f\|_{L^{p,q}} = \left(\int_0^\infty \left(t^{\frac{1}{p}} f^*(t) \right)^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}},$$

with the usual modification when $q = \infty$.

If we replace $t^{\frac{1}{p}}$ by a more general weight function $w(t)$ we arrive at the more general weighted Lorentz spaces $\Lambda^q(w)$. For the case $p = q$, the Lorentz space $L^{p,q}$ coincides with the usual Lebesgue space L^p with the norm

$$\|f\|_{L^p} = \left(\int_\Omega |f|^p \mathrm{d}\mu \right)^{\frac{1}{p}}.$$

For the proofs of some main results we need the following Lemma (see [93]):

Lemma 3.1. Assume that the Banach function space E has the Fatou property. Let $f(x,t) \geq 0$ on $\Omega \times T$ and let for almost every $t \in T$, $f(x,t) \in E$. If the function $\|f^r(x,t)\|_E^{\frac{1}{r}}$ is integrable on T, then, for $r \geq 1$,

$$\left\| \left(\int_T f(x,t)\mathrm{d}t \right)^r \right\|_E^{\frac{1}{r}} \leq \int_T \|f^r(x,t)\|_E^{\frac{1}{r}} \mathrm{d}t.$$

A proof can also be found in [66, Chapter 2].

3.1.2 The classical Hardy operator

In this section we state and prove the main results in Banach function setting involving the classical Hardy operator (see Theorems 3.1, 3.2 and 3.3). In particular, we then cover, unify and generalize all results mentioned above in Sections 1.1.1 – 1.1.3.

Our first main result reads:

Theorem 3.1. Let $0 < b \leq \infty, -\infty \leq a < c \leq \infty$, let Φ be a positive and convex function on (a,c) and E be a Banach function space on $[0,b)$. If E has the Fatou property and $a < f(x) < c$, then

$$\left\| \Phi\left(\frac{1}{x} \int_0^x f(t)\mathrm{d}t \right) \right\|_E \leq \int_0^b \Phi\left(f(t)\right) \left\| \frac{1}{x} \chi_{[t,b]}(x) \right\|_E \mathrm{d}t, \tag{3.1}$$

provided both sides have sense.

Proof. Let $D = \{(x,t) : 0 \leq x \leq b, 0 \leq t \leq x\}$. Then

$$\chi_D(x,t) = \chi_{[0,x]}(t) = \chi_{[t,b]}(x). \tag{3.2}$$

By using Jensen's inequality, the lattice property of E, Lemma 3.1 with $r = 1$ and (3.2) we find that

$$\begin{aligned}
\left\| \Phi \left[\frac{1}{x} \int_0^x f(t)\mathrm{d}t \right] \right\|_E &\leq \left\| \int_0^x \frac{\Phi\left(f(t)\right)}{x} \mathrm{d}t \right\|_E = \left\| \int_0^b \frac{\Phi\left(f(t)\right)}{x} \chi_{[0,x]}(t)\mathrm{d}t \right\|_E \\
&= \left\| \int_0^b \frac{\Phi\left(f(t)\right)}{x} \chi_D(x,t)\mathrm{d}t \right\|_E \leq \int_0^b \left\| \frac{\Phi\left(f(t)\right)}{x} \chi_D(x,t) \right\|_E \mathrm{d}t \\
&= \int_0^b \left\| \frac{\Phi\left(f(t)\right)}{x} \chi_{[t,b]}(x) \right\|_E \mathrm{d}t = \int_0^b \Phi\left(f(t)\right) \left\| \frac{1}{x} \chi_{[t,b]}(x) \right\|_E \mathrm{d}t.
\end{aligned}$$

The proof is complete. □

Next we state the following Pólya-Knopp type inequality for Lorentz spaces:

Corollary 3.1. Let $0 < b \leq \infty, -\infty \leq a < c \leq \infty, 1 \leq p < \infty$ and $0 < q < \infty$. Moreover, let Φ be a positive and convex function on (a,c) and $a < f(x) < c$. Then

$$\begin{aligned}
&\left[\int_0^b \left[\left(\Phi \left(\frac{1}{x} \int_0^x f(t)\mathrm{d}t \right) \right)^* \right]^q x^{\frac{q}{p}-1} \mathrm{d}x \right]^{\frac{1}{q}} \\
&\leq \int_0^b \Phi(f(t)) \left[\int_0^{b-t} \left(\frac{x^{\frac{1}{p}}}{x+t} \right)^q \frac{\mathrm{d}x}{x} \right]^{\frac{1}{q}} \mathrm{d}t.
\end{aligned} \tag{3.3}$$

Proof. It is well-known that the Banach function spaces $E = L^{p,q}$ satisfy the Fatou property (see e.g. [22]). Denote $k(x) = \frac{1}{x}\chi_{[t,b]}(x)$. Then $k^*(x) = \frac{1}{x+t}$ for $x \in [0, b-t)$ and $k^*(x) = 0$ for $b - t \leq x \leq b$. Therefore

$$\left\| \frac{1}{x} \chi_{[t,b]}(x) \right\|_E = \left(\int_0^b \left(x^{\frac{1}{p}} k^*(x) \right)^q \frac{\mathrm{d}x}{x} \right)^{\frac{1}{q}} = \left(\int_0^{b-t} \left(\frac{x^{\frac{1}{p}}}{x+t} \right)^q \frac{\mathrm{d}x}{x} \right)^{\frac{1}{q}},$$

so (3.3) coincides with (3.1). The proof is complete. □

Remark 3.2. In some cases the last integral on the right hand side of (3.3) can be calculated exactly so (3.3) can be written in a more explicit form. For example, if $p = q \geq 1$ i.e. $E = L^p\left((0,b), \frac{\mathrm{d}x}{x}\right)$, then we have

$$\left\| \frac{1}{x} \chi_{[t,b]}(x) \right\|_E = \left(\int_t^b \frac{1}{x^{p+1}} \mathrm{d}x \right)^{\frac{1}{p}} = \left(\frac{1}{p} \right)^{\frac{1}{p}} \left(t^{-p} - b^{-p} \right)^{\frac{1}{p}},$$

so that

$$\left[\int_0^b \Phi^p \left(\frac{1}{x} \int_0^x f(t)\mathrm{d}t \right) \frac{\mathrm{d}x}{x} \right]^{\frac{1}{p}} \leq \left(\frac{1}{p} \right)^{\frac{1}{p}} \int_0^b \Phi\left(f(x)\right) \left[1 - \left(\frac{x}{b} \right)^p \right]^{\frac{1}{p}} \frac{\mathrm{d}x}{x}, p \geq 1. \tag{3.4}$$

For some more such examples see our Remark 3.12.

Remark 3.3. For $p = 1$, (3.4) was proved in [30] (see also [59]). Moreover, by modifying the proof we see that (3.4) holds in the reversed direction if Φ instead is a positive and concave function.

Example 3.1. By using the fact that $\Phi(u) = u^p$ is convex for the cases $p < 0$ and $p \geq 1$ and concave for $0 < p < 1$ and making the same substitutions as in Remark 1.4 we obtain the following fairly recent sharp generalization of (1.4) and (1.5) yielding also for finite intervals (see [73, Theorem 7.10] and the references therein):

$$\int_0^b \left(\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right)^p x^\alpha \mathrm{d}x \leq \left(\frac{p}{p-1-\alpha}\right)^p \int_0^b f^p(x)x^\alpha \left[1-\left(\frac{x}{b}\right)^{\frac{p-\alpha-1}{p}}\right]\mathrm{d}x, \quad (3.5)$$

for $0 < b \leq \infty, p \geq 1, \alpha < p-1$ or $p < 0, \alpha > p-1$. Moreover, (3.5) holds in the reversed direction if $0 < p < 1$, $\alpha < p-1$. In all cases the inequalities are sharp. Hence, we have a new proof of the inequalities in Theorem 1.1 *a*) and *b*).

Remark 3.4. By replacing $f(x)$ by $(f(x))^{\frac{1}{p}}$ in (3.5) and performing the standard limiting procedure as $p \to \infty$ we obtain the following weighted sharp version of (1.8):

$$\int_0^b \exp\left(\frac{1}{x}\int_0^x \log f(t)\mathrm{d}t\right) x^\alpha \mathrm{d}x \leq e^{1+\alpha}\int_0^b f(x)x^\alpha \left(1-\frac{x}{b}\right)\mathrm{d}x, \alpha > -1,$$

yielding also for finite intervals.

Next we state the following complement of Theorem 3.1 for $b = \infty$:

Theorem 3.2. Let $-\infty \leq a < c \leq \infty$, let Φ be a positive and convex function on (a, c) and $E = E[0, \infty)$ be a Banach function space with Fatou property. Then, for $p > 1$,

$$\left\|\Phi^p\left(\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right)\right\|_E \leq \left(\frac{p}{p-1}\right)^{p-1}\int_0^\infty t^{1-\frac{1}{p}}\Phi^p(f(t))\left\|\frac{1}{x^{2-\frac{1}{p}}}\chi_{[t,\infty)}(x)\right\|_E \mathrm{d}t. \quad (3.6)$$

Proof. For the proof we use the Jensen inequality, Hölder's inequality and Lemma 3.1 for the function $g(x, t) = \chi_D(x, t)$ and we apply (3.2) for $b = \infty$. Indeed, we have that

$$\begin{aligned}
\left\|\Phi^p\left(\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right)\right\|_E &\leq \left\|\left(\frac{1}{x}\int_0^x \Phi\left(f(t)\right)\mathrm{d}t\right)^p\right\|_E \\
&\leq \left\|x^{-p}\int_0^x t^{\frac{p-1}{p}}\Phi^p\left(f(t)\right)\mathrm{d}t\cdot\left(\int_0^x t^{\frac{(1-p)p'}{p^2}}\mathrm{d}t\right)^{\frac{p}{p'}}\right\|_E \\
&= \left(\frac{p}{p-1}\right)^{p-1}\left\|\int_0^\infty x^{-2+\frac{1}{p}}\chi_{[0,x]}(t)t^{1-\frac{1}{p}}\Phi^p\left(f(t)\right)\mathrm{d}t\right\|_E \\
&= \left(\frac{p}{p-1}\right)^{p-1}\left\|\int_0^\infty x^{-2+\frac{1}{p}}\chi_D(x,t)t^{1-\frac{1}{p}}\Phi^p\left(f(t)\right)\mathrm{d}t\right\|_E \\
&\leq \left(\frac{p}{p-1}\right)^{p-1}\int_0^\infty t^{1-\frac{1}{p}}\Phi^p\left(f(t)\right)\left\|\frac{1}{x^{2-\frac{1}{p}}}\chi_D(x,t)\right\|_E \mathrm{d}t \\
&= \left(\frac{p}{p-1}\right)^{p-1}\int_0^\infty t^{1-\frac{1}{p}}\Phi^p\left(f(t)\right)\left\|\frac{1}{x^{2-\frac{1}{p}}}\chi_{[t,\infty)}(x)\right\|_E \mathrm{d}t.
\end{aligned}$$

The proof is complete. □

Remark 3.5. It is a continuity between Theorems 3.2 and 3.1. Indeed, since $\left(\frac{p}{p-1}\right)^{p-1} \to 1$ when $p \to 1^+$ we have that (3.6) tends to (3.1) with $b = \infty$.

Remark 3.6. By applying Theorem 3.2 for the case $E = L^{1,q}$ we can state another variant of Hardy-type inequality than that in Corollary 3.1.

Remark 3.7. It is obvious that all results in this section can be obtained in the "dual" situation when $\int_0^x f(t)\mathrm{d}t$ is replaced by $\int_x^\infty f(t)\mathrm{d}t$, $0 \leq b \leq x < \infty$. It turns out that it is convenient to just consider the Hardy type operator $H^* : H^*f(x) = x\int_x^\infty \frac{f(t)}{t^2}\mathrm{d}t$. As an example we formulate the following "dual" version of Theorem 3.1:

Theorem 3.3. Let $-\infty \leq a < c \leq \infty$, let Φ be a positive and convex function on (a, c) and E be a Banach function space on $[b, \infty)$, $b \geq 0$, with the Fatou property. Then, whenever $a < f(x) < c$,

$$\left\|\Phi\left(x\int_x^\infty \frac{f(t)}{t^2}\,\mathrm{d}t\right)\right\|_E \leq \int_b^\infty \frac{\Phi\left(f(t)\right)}{t^2}\left\|x\chi_{[b,t)}(x)\right\|_E \mathrm{d}t.$$

Proof. Let $D = \{(x, t) : x \geq b, x \leq t < \infty\}$. Then

$$\chi_D(x,t) = \chi_{[x,\infty)}(t) = \chi_{[b,t)}(x). \tag{3.7}$$

By using (3.7) and the same arguments as in the proof of Theorem 3.1 we obtain that

$$\begin{aligned}
\left\|\Phi\left(x\int_x^\infty \frac{f(t)}{t^2}\,\mathrm{d}t\right)\right\|_E &\leq \left\|\int_x^\infty \frac{x}{t^2}\Phi\left(f(t)\right)\,\mathrm{d}t\right\|_E = \left\|\int_b^\infty \frac{x}{t^2}\Phi\left(f(t)\right)\chi_{[x,\infty)}(t)\,\mathrm{d}t\right\|_E \\
&= \left\|\int_b^\infty \frac{x}{t^2}\Phi\left(f(t)\right)\chi_D(x,t)\,\mathrm{d}t\right\|_E \\
&\leq \int_b^\infty \left\|\frac{x}{t^2}\Phi\left(f(t)\right)\chi_D(x,t)\right\|_E \mathrm{d}t \\
&= \int_b^\infty \left\|\frac{x}{t^2}\Phi\left(f(t)\right)\chi_{[b,t)}(x)\right\|_E \mathrm{d}t \\
&= \int_b^\infty \frac{\Phi\left(f(t)\right)}{t^2}\left\|x\chi_{[b,t)}(x)\right\|_E \mathrm{d}t.
\end{aligned}$$

The proof is complete. □

Remark 3.8. It is possible to write a similar Corollary for Lorentz spaces as that in Corollary 3.1. In particular, by applying Theorem 3.3 with $E = L_1\left((b,\infty), \frac{\mathrm{d}x}{x}\right), b \geq 0$, and the function $\Phi(u) = u^p, p < 0$ or $p \geq 1$ we obtain the following "dual" version of the well-known sharp inequality (see [73])

$$\int_0^b \left(\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right)^p \frac{\mathrm{d}x}{x} \leq 1\int_0^b f^p(x)\left(1 - \frac{x}{b}\right)\frac{\mathrm{d}x}{x} \tag{3.8}$$

when $0 < b \leq \infty, p < 0$ or $p \geq 1$ (For the case $b = \infty$ this is just (1.9) and for $0 < p < 1$ (3.8) holds in the reversed direction).

Corollary 3.2. Let $0 \leq b < \infty$. Then

$$\int_b^\infty \left(x \int_x^\infty \frac{f(t)}{t^2} \mathrm{d}t \right)^p \frac{\mathrm{d}x}{x} \leq 1 \int_b^\infty f^p(x) \left(1 - \frac{b}{x}\right) \frac{\mathrm{d}x}{x} \tag{3.9}$$

whenever $p < 0$ or $p \geq 1$. Moreover, (3.9) holds in the reversed direction when $0 < p < 1$ (for the case $p < 0$ we as usual assume that $f(x) > 0$). The inequality (3.9) is sharp.

Proof. Let $p < 0$ or $p \geq 1$. Apply Theorem 3.3 with $E = L_1\left((b, \infty), \frac{\mathrm{d}x}{x}\right)$ and $\Phi(u) = u^p$. Note that

$$\|x\chi_{[b,t)}(x)\|_E = \int_b^t \mathrm{d}x = t - b.$$

Hence,

$$\int_b^\infty \frac{f^p(t)}{t^2} \|x\chi_{[b,t)}(x)\|_E \, \mathrm{d}t = \int_b^\infty f^p(t) \left(1 - \frac{b}{t}\right) \frac{\mathrm{d}t}{t}.$$

Moreover, in this case

$$\left\| \Phi\left(x \int_x^\infty \frac{f^p(t)}{t^2} \mathrm{d}t \right) \right\|_E = \int_b^\infty \left(x \int_x^\infty \frac{f(t)}{t^2} \mathrm{d}t \right)^p \frac{\mathrm{d}x}{x}$$

and (3.9) follows from Theorem 3.3. The case $0 < p < 1$ can be proved by using reversed Jensen inequality for this special case in the proof of Theorem 3.3. The sharpness of the inequality (3.9) is more or less obvious but can be done in detail by repeating the arguments of the proof of the sharpness of (3.8) (see [73, proof of Lemma 7.8]). □

As an application of Corollary 3.2, by making substitutions similar to those in Remark 1.4 we obtain the following (see [73, Theorem 7.10]).

Example 3.2. Let f be a positive function on $[b, \infty)$, $b \geq 0$. Then the sharp inequality

$$\int_b^\infty \left(\frac{1}{x} \int_x^\infty f(t)\mathrm{d}t \right)^p x^\alpha \mathrm{d}x \leq \left(\frac{p}{\alpha + 1 - p} \right)^p \times \int_b^\infty f^p(x) x^\alpha \left[1 - \left(\frac{b}{x}\right)^{\frac{\alpha+1-p}{p}} \right] \mathrm{d}x \tag{3.10}$$

holds whenever $p \geq 1, \alpha > p - 1$ or $p < 0, \alpha < p - 1$. Moreover, (3.10) holds in the reversed direction if $0 < p < 1$, $\alpha > p - 1$. Thus, in particular, we have presented a new proof of the inequalities in Theorem 1.1 *c*) and *d*).

3.1.3 The generalized Hardy operators

In this section we introduce a new generalized Hardy operator covering the case with kernel operators described above. We state and prove some new Hardy-type inequalities involving these operators (see Theorems 3.4 and 3.5). In particular, Theorem 2.16 appears as a special case. In order to be able to cover also situations involving more general Hardy-type operators like that in Theorem 2.16, we consider the following more general situation: Let σ be a positive measure on the measure space S and let σ_x denote a $\sigma-$finite positive measure on S such that $\sigma_x(S) < \infty$. Moreover, we suppose that σ_x is absolutely continuous with respect to σ. We define the general Hardy type operator T as follows:

$$Tf(x) = \frac{1}{\sigma_x(S)} \int_S f(t) \mathrm{d}\sigma_x(t),$$

where f is defined on S with values in $(a, b), -\infty \leq a < b \leq \infty$, and $\sigma_x(S) = \int_S \mathrm{d}\sigma_x(t)$.

Our first main result in this section reads:

Theorem 3.4. Let $-\infty \leq a < b \leq \infty$ and let Φ be a positive and convex function on (a, b), where $\Phi(f)$ is measurable on S. Moreover, let E be a Banach function space on S with Fatou property. Then

$$\|\Phi(Tf(\cdot))\|_E \leq \int_S \Phi\left(f(y)\right) \left\| \frac{1}{\sigma_x(S)} \frac{\mathrm{d}\sigma_x(y)}{\mathrm{d}\sigma(y)} \right\|_E \mathrm{d}\sigma(y) \tag{3.11}$$

for any f defined on S with values in (a, b) such that the right hand side is finite.

Proof. First we use Jensen's inequality and the lattice property of the norm and get that

$$\begin{aligned} \|\Phi(Tf(\cdot))\|_E &= \left\| \Phi\left(\frac{1}{\sigma_x(S)} \int_S f(y) \mathrm{d}\sigma_x(y) \right) \right\|_E \\ &\leq \left\| \int_S \frac{1}{\sigma_x(S)} \Phi\left(f(y)\right) \mathrm{d}\sigma_x(y) \right\|_E . \end{aligned} \tag{3.12}$$

Since σ_x is absolutely continuous with respect to σ, for any x, we have, by the Radon-Nikodym theorem, that

$$\int_S \frac{1}{\sigma_x(S)} \Phi\left(f\right) \mathrm{d}\sigma_x = \int_S \frac{1}{\sigma_x(S)} \Phi\left(f\right) \frac{\mathrm{d}\sigma_x}{\mathrm{d}\sigma} \mathrm{d}\sigma.$$

Hence, by using this equality and Lemma 3.1 with $r = 1$, we find that

$$\left\| \int_S \frac{1}{\sigma_x(S)} \Phi\left(f\right) \mathrm{d}\sigma_x \right\|_E \leq \int_S \Phi\left(f\right) \left\| \frac{1}{\sigma_x(S)} \frac{\mathrm{d}\sigma_x}{\mathrm{d}\sigma} \right\|_E \mathrm{d}\sigma.$$

The proof of (3.11) follows by just combining the last inequality with (3.12). □

Next, we point out the following illustrative application (cf. Theorem 4.1 in [59]):

Corollary 3.3. Let u be a weight function on $(0, b), 0 < b \leq \infty$ and let $k(x, y) \geq 0$ be a measurable function on $(0, b) \times (0, b)$. Assume that $k(x, y)$ is locally integrable on $(0, b)$ for every fixed $y \in (0, b)$ and define v by

$$v(y) = y \int_y^b \frac{k(x, y)}{K(x)} u(x) \frac{dx}{x} < \infty, y \in (0, b).$$

If Φ is a positive and convex function on $(a,c), -\infty < a < c < \infty$, then

$$\int_0^b \Phi\left(A_k f\left(x\right)\right) u(x) \frac{\mathrm{d}x}{x} \leq \int_0^b \Phi\left(f\left(y\right)\right) v(y) \frac{\mathrm{d}y}{y},$$

for all f with $a < f(x) < c, 0 \leq x \leq b$. ($A_k(\cdot)$ and $K(x)$ are defined by (2.29) and (2.30), respectively).

Proof. Just apply Theorem 3.4 with $E = L^1\left((0,b); u(x)\frac{\mathrm{d}x}{x}\right), S = (0,b), \mathrm{d}\sigma = \mathrm{d}y$ and $\mathrm{d}\sigma_x(y) = \chi_{[0,x]}(y)k(x,y)\mathrm{d}y$ and make some standard calculations. □

Example 3.3. By applying Corollary 3.3 with $\Phi(u) = e^u$ and replacing $f(x)$ by $\log f^p(x)$, $p > 0$, we obtain the following kernel Pólya-Knopp inequality

$$\int_0^b \left[\exp\left(\frac{1}{K(x)}\int_0^x k(x,y)\log f(y)\mathrm{d}y\right)\right]^p u(x)\frac{\mathrm{d}x}{x} \leq \int_0^b f^p(x)v(x)\frac{\mathrm{d}x}{x}, \tag{3.13}$$

for $p > 0$, where $k(x,y), K(x), u(x)$ and $v(x)$ are defined as in Corollary 3.3.

Remark 3.9. In particular, by applying (3.13) with $p = 1, u(x) = 1, k(x,y) = 1$, so that $K(x) = x$ and making some obvious calculations we rediscover (1.6).

Our next aim is to derive a generalization of Theorem 2.16 to a partly Banach function setting (we keep the same notations $k(x,y), K(x)$ and A_k as in Theorem 2.16).

Theorem 3.5. Let $1 < p \leq q < \infty, 0 < b \leq \infty$ and $s \in (1,p)$. Let E be a Banach function space on $[0,b)$, which has Fatou property and let Φ be a positive and convex function on $(a,c), -\infty \leq a < c \leq \infty$. Then

$$\left\|\Phi^q\left(A_k f\left(x\right)\right) u(x)\right\|_E^{\frac{1}{q}} \leq C\left(\int_0^b \Phi^p\left(f\left(x\right)\right) v(x)\frac{\mathrm{d}x}{x}\right)^{\frac{1}{p}} \tag{3.14}$$

holds for all functions $f(x), a < f(x) < c$, and some constant $C > 0$ if

$$\overline{A}(s) := \sup_{0<t<b} \left\| \left(\frac{k(x,t)}{K(x)}\right)^q u(x)V(x)^{\frac{q(p-s)}{p}}\chi_{[t,b]}(x)\right\|_E^{\frac{1}{q}} V(t)^{\frac{s-1}{p}} < \infty, \tag{3.15}$$

where $u(x)$ and $v(x)$ are weight functions and $V(t) = \int_0^t \frac{v^{1-p'}(x)}{x^{1-p'}}\mathrm{d}x$. Moreover, if C is the best possible constant in the above inequality, then

$$C \leq \inf_{1<s<p}\left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}}\overline{A}(s). \tag{3.16}$$

Proof. For simplicity we introduce the notation $\mathrm{d}\sigma_x = \mathrm{d}\sigma_x(t) = \chi_{[0,x)}(t)k(x,t)\mathrm{d}t$. First we use Jensen's inequality and the lattice property of the norm and get that

$$\begin{aligned}\left\|\Phi^q\left(A_k f\left(\cdot\right)\right) u\right\|_E^{\frac{1}{q}} &= \left\|\left(\Phi\left(\frac{1}{K(x)}\int_S f(t)\mathrm{d}\sigma_x(t)\right)\right)^q u(x)\right\|_E^{\frac{1}{q}} \\ &\leq \left\|\left(\frac{1}{K(x)}\int_S \Phi(f(t))\mathrm{d}\sigma_x(t)\right)^q u(x)\right\|_E^{\frac{1}{q}} := B. \end{aligned} \tag{3.17}$$

Let $\Phi^p(f(t))\frac{v(t)}{t} = \Phi(g(t))$. Then B can be written as

$$B = \left\| \left(\int_S \Phi^{\frac{1}{p}}(g(t)) V(t)^{\frac{s-1}{p}} V(t)^{-\frac{s-1}{p}} v(t)^{-\frac{1}{p}} t^{\frac{1}{p}} \frac{\mathrm{d}\sigma_x(t)}{\mathrm{d}\sigma(t)} \mathrm{d}\sigma(t) \right)^q \frac{u(x)}{K(x)^q} \right\|_E^{\frac{1}{q}}. \tag{3.18}$$

Next we apply the Hölder inequality with power $p > 1$ and obtain that

$$\begin{aligned}
B &\leq \left\| \left(\int_0^x k^p(x,t)\Phi(g(t))V(t)^{s-1}\mathrm{d}t \right)^{\frac{q}{p}} \times \right. \\
&\quad \left. \left(\int_0^x V(t)^{-\frac{p'(s-1)}{p}} v(t)^{1-p'} t^{p'-1} \mathrm{d}t \right)^{\frac{q}{p'}} \frac{u(x)}{K(x)^q} \right\|_E^{\frac{1}{q}} \\
&= \left(\frac{p-1}{p-s} \right)^{\frac{1}{p'}} \left\| \left(\int_0^x k^p(x,t)\Phi(g(t))V(t)^{s-1}\mathrm{d}t \right)^{\frac{q}{p}} V(x)^{\frac{q(p-s)}{p}} \frac{u(x)}{K(x)^q} \right\|_E^{\frac{1}{q}} \\
&= \left(\frac{p-1}{p-s} \right)^{\frac{1}{p'}} \left\| \left(\int_0^b k^p(x,t)\Phi(g(t))V(t)^{s-1}V(x)^{(p-s)} \frac{u(x)^{\frac{p}{q}}}{K(x)^p} \chi_{(0,x)}(t)\mathrm{d}t \right)^{\frac{q}{p}} \right\|_E^{\frac{1}{q}}.
\end{aligned}$$

Since $\frac{q}{p} > 1$, also the space $E^{\frac{q}{p}}$ has the Fatou property and we can apply Lemma 3.1 with $r = \frac{q}{p}$ and use (3.2) to find that

$$\begin{aligned}
B &\leq \left[\frac{p-1}{p-s} \right]^{\frac{1}{p'}} \left(\int_0^b \left\| \left[\frac{k(x,t)}{K(x)} \right]^q \Phi^{\frac{q}{p}}(g(t)) V(t)^{\frac{(s-1)q}{p}} V(x)^{\frac{q(p-s)}{p}} \times \right. \right. \\
&\quad \left. \left. u(x)\chi_{(t,b)}(x) \right\|_E^{\frac{p}{q}} \mathrm{d}t \right)^{\frac{1}{p}} \\
&\leq \left(\frac{p-1}{p-s} \right)^{\frac{1}{p'}} \overline{A}(s) \left[\int_0^b \Phi(g(t))\mathrm{d}t \right]^{\frac{1}{p}} \\
&= \left(\frac{p-1}{p-s} \right)^{\frac{1}{p'}} \overline{A}(s) \left[\int_0^b \Phi^p(f(t)) \frac{v(t)}{t} \mathrm{d}t \right]^{\frac{1}{p}}.
\end{aligned} \tag{3.19}$$

By just combining (3.15) with (3.17)-(3.19) we conclude that (3.14) and also the estimate (3.16) hold. The proof is complete. □

Remark 3.10. Theorem 2.16 is just the special case of Theorem 3.5 when $E = L^1\left((0,b); \frac{\mathrm{d}x}{x}\right)$.

Remark 3.11. Since the results in this section can be applied for kernel operators the results in this section may be seen as complements and further generalizations of some results in [59], [68](cf. also [67]) and [98]). We just give one such example of application of Theorem 3.4 (see [59, Theorem 2.1] and cf. also [98, Proposition 2.1]):

Example 3.4. Let $0 < b_i \leq \infty, i = 1, 2, ..., n \, (n \in \mathbb{N}), -\infty \leq a < c \leq \infty$ and if Φ is a positive and convex function on (a,c), then

$$\begin{aligned}
&\int_0^{b_1} \cdots \int_0^{b_n} \Phi\left(\frac{1}{x_1 \cdots x_n} \int_0^{x_1} \cdots \int_0^{x_n} f(t_1, \ldots, t_n)\, \mathrm{d}t_1 \cdots \mathrm{d}t_n \right) \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_n}{x_1 \cdots x_n} \\
&\qquad \leq \int_0^{b_1} \cdots \int_0^{b_n} \Phi\big(f(t_1, \ldots, t_n)\big) \left(1 - \frac{t_1}{b_1}\right) \cdots \left(1 - \frac{t_n}{b_n}\right) \frac{\mathrm{d}t_1 \cdots \mathrm{d}t_n}{t_1 \cdots t_n}.
\end{aligned}$$

3.1.4 Concluding examples and remarks

Our first concluding remark reads:

Remark 3.12. (cf. Remark 3.2). Some more cases when the last integral on the right hand side of (3.3) can be calculated exactly:

(a) $p = q = 2$. Then

$$\left[\int_0^{b-t} \frac{1}{(x+t)^2}\,\mathrm{d}x\right]^{\frac{1}{2}} = \left(\frac{b-t}{tb}\right)^{\frac{1}{2}} \to \frac{1}{\sqrt{t}} \text{ as } b \to \infty.$$

(b) $p = 2, q = 1$. Then

$$\int_0^{b-t} \frac{1}{(x+t)\sqrt{x}}\,\mathrm{d}x = \frac{2}{\sqrt{t}} \arctan \sqrt{\frac{b-t}{t}} \to \frac{\pi}{\sqrt{t}} \text{ as } b \to \infty.$$

(c) $p = \frac{17}{5}, q = \frac{17}{6}$. Then

$$\left(\int_0^{b-t} \frac{1}{x^{\frac{1}{6}}(x+t)^{\frac{17}{6}}}\,\mathrm{d}x\right)^{\frac{6}{17}} = \left(\frac{6(b-t)^{\frac{5}{6}}(6b+5t)}{55t^2 b^{\frac{11}{6}}}\right)^{\frac{6}{17}}$$
$$\to \left(\frac{36}{55t^2}\right)^{\frac{6}{17}} \text{ as } b \to \infty.$$

In each of these cases the inequality (3.3) can be stated more explicitly like

Example 3.5. Let Φ be a positive and convex function. Consider the following integrals

$$A = \int_0^\infty \Phi(f(x))\,\frac{\mathrm{d}x}{\sqrt{x}}, \quad B = \int_0^\infty \frac{1}{x}\left(\int_0^x \Phi(f(t))\,\mathrm{d}t\right)\frac{\mathrm{d}x}{\sqrt{x}}$$
$$C = \int_0^\infty \frac{1}{\sqrt{x}}\Phi\left(\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right)\mathrm{d}x \text{ and } D = \int_0^\infty \frac{1}{\sqrt{x}}\left(\Phi\left(\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right)\right)^*\mathrm{d}x.$$

By using (b) of Remark 3.12 we get that

$$D \leq \pi A.$$

By using the Hardy-Littlewood inequality and Jensen's inequality, we find that

$$C \leq D \text{ and } C \leq B,$$

respectively. Moreover, by using (1.5) with $p = 1, \alpha = -\frac{1}{2}$ and f replaced by $\Phi(f)$ we obtain the sharp inequality

$$B \leq 2A.$$

Hence, we have the following inequalities

$$C \leq D \leq \pi A \text{ and } C \leq B \leq 2A. \tag{3.20}$$

Remark 3.13. By assuming that $f(x)$ is non-increasing and Φ is non-decreasing (or $f(x)$ is non-decreasing and Φ is non-increasing) we have $A \leq C$ and then from (3.20) we get the following chain of inequalities:

$$A \leq C \leq B \leq 2A \leq 2C \leq 2D \leq 2\pi A,$$

so, indeed, the integrals A, B, C and D are equivalent in this case.

Example 3.6. Another example of a Banach function space with Fatou property is the space $E = L_1(0,\infty) + L_\infty(0,\infty)$, which is an important space in particular in real interpolation theory (see e.g. [22] or [66]), with the usual norm

$$\|g\|_E = \int_0^1 g^*(t)\,\mathrm{d}t.$$

Then

$$\left\| \frac{1}{x}\chi_{[t,\infty)}(x) \right\|_E = \int_0^1 \frac{1}{(x+t)}\mathrm{d}x = \log\frac{t+1}{t}.$$

Therefore, from (3.1), we have the following new inequality in this condition:

$$\int_0^1 \left(\Phi\left(\frac{1}{x}\int_0^x f(t)\mathrm{d}t \right) \right)^* \mathrm{d}x \leq \int_0^\infty \Phi\left(f(t)\right) \log\left(\frac{t+1}{t} \right) \mathrm{d}t,$$

yielding for any positive and convex function Φ and where $f(x)$ is in the definition set of Φ.

Remark 3.14. All main results in this paper have been developed for Banach function spaces over a set with general measure $\mathrm{d}\mu$. However, all applications have been given in the continuous case with Lebesgue measure. But similar applications can be given in the discrete case with counting measure $\mathrm{d}\delta = \sum\limits_{n=0}^{\infty} \delta_n$ implying the corresponding discrete inequalities.

In particular, by applying Theorem 3.1 with $E = L^p(\mathrm{d}\delta) = l^p$ we get the following discrete inequality:

Corollary 3.4. Let $1 \leq k \leq \infty$ and for $p > 1$. Then

$$\left(\sum_{n=1}^{k} \Phi^p \left(\frac{1}{n} \sum_{i=1}^{n} a_i \right) \right)^{\frac{1}{p}} \leq \sum_{i=1}^{k} \Phi(a_i) \left(\sum_{n=i}^{k} \frac{1}{n^p} \right)^{\frac{1}{p}},$$

whenever $\{a_i\}_1^k$ is a non-negative sequence and Φ is a positive and convex function on this sequence.

Remark 3.15. (Concerning sharpness) As seen all our results in this section gives the sharp constants in the special cases we have pointed out. Hence, we can claim that our main results in this section are in this sense sharp. Also Theorem 3.4 is sharp in the same sense (see Example 3.3 and Remark 3.9). However, Theorem 3.5 depends on a not necessarily sharp constant C.

3.2 Refinements of some classical inequalities via superquadraticity

3.2.1 Introduction

Classical inequalities are of great importance for the development of several areas both within the mathematical sciences and beyond. Hence, it is not surprising that the area "Inequalities" has been developed to an independent area of increasing interest. Several wonderful generalizations, sharpening and applications have been presented. In particular, fairly lately even refinements of these inequalities are derived. See e.g. [1], [16], [62], [69], [91], [102] and the references given there.

In this section we derive new such refinements of some classical inequalities (e.g. the Jensen, Minkowski, Beckenbach-Dresher and Hardy inequalities) using the concept of superquadratic and subquadratic functions introduced by Abramovich, Jameson and Sinnamon in [2] (see also [3]). We cite the following result, which is very useful in the proofs of the main results in this Chapter (see [2], [3] and [117] for further details).

Theorem 3.6. (See [2, Theorem 2.3].) Let (Ω, μ) be a probability measure space. The inequality

$$\varphi\left(\int_\Omega f(s)\mathrm{d}\mu(s)\right) \leq \int_\Omega \varphi(f(s))\mathrm{d}\mu(s) - \int_\Omega \varphi\left(\left|f(s) - \int_\Omega f(s)\mathrm{d}\mu(s)\right|\right)\mathrm{d}\mu(s) \tag{3.21}$$

holds for all probability measures μ and all nonnegative $\mu-$integrable functions f if and only if φ is superquadratic. Moreover, (3.21) holds in the reversed direction if and only if φ is subquadratic.

The proof is available in [2, Theorem 2.3].

Remark 3.16. If φ is a nonnegative superquadratic function, then φ is convex (see [2, Lemma 2.2]) and inequality (3.21) is a refinement of the Jensen inequality for a convex function which states

$$\varphi\left(\int_\Omega f(s)\,\mathrm{d}\mu(s)\right) \leq \int_\Omega \varphi\left(f(s)\right)\,\mathrm{d}\mu(s).$$

3.2.2 Further refinements of Jensen's inequality

In this section we derive some new refinements of Jensen's inequality (see Theorems 3.7 and 3.8). We need the following useful special case of the refinement presented in Theorem 3.6.

Lemma 3.2. Let φ be a superquadratic function and let $t(s)$ be a nonnegative measurable function such that $T = \int_\Omega t(s)\mathrm{d}s$. The inequality

$$\varphi\left(\overline{f}\right) \leq \frac{1}{T}\int_\Omega t(s)\varphi\left(f(s)\right)\mathrm{d}s - \frac{1}{T}\int_\Omega t(s)\varphi\left(\left|f(s) - \overline{f}\right|\right)\mathrm{d}s, \tag{3.22}$$

holds for all nonnegative functions f, where $\overline{f} = \dfrac{1}{T}\int_\Omega t(s)f(s)\mathrm{d}s$. Moreover, (3.22) holds in the reversed direction if φ is subquadratic.

Proof. Set $\mathrm{d}\mu(s) = \dfrac{t(s)}{T}\mathrm{d}s$. Then (3.22) follows from (3.21). The proof is complete. □

Example 3.7. Let φ be a superquadratic function, let x_1, x_2 be two nonnegative real numbers and $\lambda \in [0,1]$. Then

$$\begin{aligned} &\varphi(\lambda x_1 + (1-\lambda)x_2) \\ &\le \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2) - \lambda\varphi\big((1-\lambda)|x_1 - x_2|\big) - (1-\lambda)\varphi\big(\lambda|x_1 - x_2|\big). \end{aligned} \tag{3.23}$$

Moreover, (3.23) holds in the reversed direction if φ is subquadratic.

In fact, by taking $\Omega = [0,1]$, $t(s) = 1$ and

$$f(s) = \begin{cases} x_1 : s \in [0,\lambda] \\ x_2 : s \in (\lambda, 1] \end{cases},$$

we see that (3.23) follows from (3.22).

Consider the nonnegative measurable functions $\alpha(s)$ and $\beta(s)$ satisfying

$$\alpha(s) + \beta(s) = 1, \text{ for all } s \in \Omega.$$

Denote

$$T = \int_\Omega t(s)\mathrm{d}s, \quad Q = \int_\Omega \alpha(s)t(s)\mathrm{d}s, \quad R = \int_\Omega \beta(s)t(s)\mathrm{d}s.$$

Our first main result in this section reads as follows:

Theorem 3.7. Let $\varphi : [0,\infty) \to \mathbf{R}$ be a superquadratic function and let f be a nonnegative and measurable function. Then the following refined variant of Jensen type inequality

$$\begin{aligned} \varphi\left(\overline{f}\right) &\le \frac{Q}{T}\varphi\left(\overline{f}_Q\right) + \frac{R}{T}\varphi\left(\overline{f}_R\right) - \frac{Q}{T}\varphi\left(\frac{R}{T}|\overline{f}_Q - \overline{f}_R|\right) - \frac{R}{T}\varphi\left(\frac{Q}{T}|\overline{f}_Q - \overline{f}_R|\right) \\ &= \frac{Q}{T}\varphi\left(\overline{f}_Q\right) + \frac{R}{T}\varphi(\overline{f}_R) - \frac{Q}{T}\varphi\left(|\overline{f}_Q - \overline{f}|\right) - \frac{R}{T}\varphi\left(|\overline{f}_R - \overline{f}|\right), \end{aligned} \tag{3.24}$$

holds, where

$$\overline{f} = \frac{1}{T}\int_\Omega t(s)f(s)\mathrm{d}s, \overline{f}_Q = \frac{1}{Q}\int_\Omega \alpha(s)t(s)f(s)\mathrm{d}s \text{ and } \overline{f}_R = \frac{1}{R}\int_\Omega \beta(s)t(s)f(s)\mathrm{d}s.$$

Moreover, (3.24) holds in the reversed direction if φ is subquadratic.

Proof. Set $x_1 = \overline{f}_Q, x_2 = \overline{f}_R$ and $\lambda = \dfrac{Q}{T}$. It is clear that

$$1 - \lambda = \frac{R}{T} \text{ and } \lambda x_1 + (1-\lambda)x_2 = \overline{f}.$$

Then, from Example 3.7 it follows that

$$\varphi\left(\overline{f}\right) \le \frac{Q}{T}\varphi\left(\overline{f}_Q\right) + \frac{R}{T}\varphi\left(\overline{f}_R\right) - \frac{Q}{T}\varphi\left(\frac{R}{T}|\overline{f}_Q - \overline{f}_R|\right) - \frac{R}{T}\varphi\left(\frac{Q}{T}|\overline{f}_Q - \overline{f}_R|\right).$$

Moreover, using the fact that

$$\frac{R}{T}|\overline{f}_Q - \overline{f}_R| = |\overline{f}_Q - \overline{f}| \text{ and } \frac{Q}{T}|\overline{f}_Q - \overline{f}_R| = |\overline{f}_R - \overline{f}|,$$

we obtain inequality (3.24). The proof is complete, since the proof of the reversed inequality is similar to the proof above and we can omit the details. □

By making a further restriction of φ we can also state the following version of Theorem 3.7:

Theorem 3.8. Let $\varphi : [0,\infty) \to \mathbf{R}$ be a nondecreasing and superquadratic function such that

$$\varphi(a+b) \leq c\big(\varphi(a) + \varphi(b)\big), \text{ for some } c > 0. \tag{3.25}$$

Then the following refined variant of Jensen type inequality

$$\varphi\left(\overline{f}\right) \leq I \leq \frac{1}{T}\int_\Omega t(s)\varphi\big(f(s)\big)\mathrm{d}s - \frac{1}{cT}\int_\Omega t(s)\varphi\big(|f(s) - \overline{f}|\big)\mathrm{d}s,$$

holds for all measurable functions f, where

$$I = \frac{Q}{T}\varphi\left(\overline{f}_Q\right) + \frac{R}{T}\varphi\big(\overline{f}_R\big) - \frac{Q}{T}\varphi\left(|\overline{f}_Q - \overline{f}|\right) - \frac{R}{T}\varphi\big(|\overline{f}_R - \overline{f}|\big).$$

Proof. We proved the first inequality $\varphi\left(\overline{f}\right) \leq I$ in Theorem 3.7 so we only need to prove the second inequality.

By applying Lemma 3.2 in the first two terms of I, we get that

$$I \leq \frac{1}{T}\int_\Omega t(s)\varphi\big(f(s)\big)\mathrm{d}s - A_1 - B_1,$$

where

$$A_1 = \frac{Q}{T}\varphi\left(|\overline{f}_Q - \overline{f}|\right) + \frac{R}{T}\varphi\big(|\overline{f}_R - \overline{f}|\big)$$

and

$$B_1 = \frac{1}{T}\int_\Omega \alpha(s)t(s)\varphi\big(|f(s) - \overline{f}_Q|\big)\mathrm{d}s + \frac{1}{T}\int_\Omega \beta(s)t(s)\varphi\big(|f(s) - \overline{f}_R|\big)\mathrm{d}s.$$

To finish the proof, it is enough to prove that

$$\frac{1}{cT}\int_\Omega t(s)\varphi\big(|f(s) - \overline{f}|\big)\mathrm{d}s \leq A_1 + B_1. \tag{3.26}$$

Now, by using the triangle inequality, nondecreasing property of φ and (3.25), we obtain that

$$\begin{aligned} &\frac{1}{T}\int_\Omega \alpha(s)t(s)\varphi\big(|f(s) - \overline{f}|\big)\mathrm{d}s \\ &\leq c\left(\frac{1}{T}\int_\Omega \alpha(s)t(s)\varphi\big(|f(s) - \overline{f}_Q|\big)\mathrm{d}s + \frac{Q}{T}\varphi\big(|\overline{f}_Q - \overline{f}|\big)\right), \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{T}\int_\Omega \beta(s)t(s)\varphi\big(|f(s) - \overline{f}|\big)\mathrm{d}s \\ &\leq c\left(\frac{1}{T}\int_\Omega \beta(s)t(s)\varphi\big(|f(s) - \overline{f}_R|\big)\mathrm{d}s + \frac{R}{T}\varphi\big(|\overline{f}_R - \overline{f}|\big)\right). \end{aligned}$$

Hence (3.26) follows as a sum of the above two inequalities. The proof is complete. □

3.2.3 Refinements of Minkowski's inequality

In this section we state, prove and apply some new refinements of the Minkowski's inequality (see Theorem 3.10). First, we remind about the following interesting refinement of the Hölder inequality by G. Sinnamon [117, Theorem 1.1].

Lemma 3.3. Let $p \geq 2$ and $\dfrac{1}{p}+\dfrac{1}{q}=1$. Then

$$\int_\Omega fg\mathrm{d}\nu \leq \left(\int_\Omega (f^p - h^p)\,\mathrm{d}\nu\right)^{\frac{1}{p}} \left(\int_\Omega g^q\,\mathrm{d}\nu\right)^{\frac{1}{q}} \tag{3.27}$$

holds for any two nonnegative $\nu-$measurable functions f and g, where

$$h = \left|f - \frac{g^{q-1}\int_\Omega fg\mathrm{d}\nu}{\int_\Omega g^q \mathrm{d}\nu}\right|.$$

Moreover, (3.27) holds in the reversed direction if $1 < p \leq 2$.

The continuous Minkowski inequality reads as follows (see [76, p. 41]).

Theorem 3.9. Let f be a nonnegative measurable function on $X \times Y$ with respect to the measure $\mu \times \nu$, and let $p \geq 1$. Then

$$\left(\int_X \left(\int_Y f\,\mathrm{d}\nu\right)^p \mathrm{d}\mu\right)^{\frac{1}{p}} \leq \int_Y \left(\int_X f^p\,\mathrm{d}\mu\right)^{\frac{1}{p}} \mathrm{d}\nu.$$

Our first main result in this section is the following refinement of the continuous Minkowski inequality:

Theorem 3.10. Let f be a nonnegative measurable function on $X\times Y$ with respect to the measure $\mu \times \nu$ and let $p \geq 2$. Then

$$\left(\int_X \left(\int_Y f\mathrm{d}\nu\right)^p \mathrm{d}\mu\right)^{\frac{1}{p}} \leq \int_Y \left(\int_X (f^p - h^p)\,\mathrm{d}\mu\right)^{\frac{1}{p}} \mathrm{d}\nu, \tag{3.28}$$

where

$$h = \left|f - \frac{H\int_X fH^{p-1}\mathrm{d}\mu}{\int_X H^p \mathrm{d}\mu}\right|, \quad H(x) = \int_Y f(x,y)\mathrm{d}\nu.$$

If $1 < p \leq 2$, then (3.28) holds in the reversed direction.

Proof. Let $H(x) = \int_Y f(x,y)\mathrm{d}\nu$. Let $p \geq 2$ and $\dfrac{1}{p}+\dfrac{1}{q}=1$. By applying Lemma 3.3, by replacing $f(x)$ and $g(x)$ with $f(x,y)$ and $H^{p-1}(x)$, respectively, we get that

$$\int_X fH^{p-1}\mathrm{d}\mu \leq \left(\int_X (f^p - h^p)\mathrm{d}\mu\right)^{\frac{1}{p}} \left(\int_X H^p\mathrm{d}\mu\right)^{\frac{1}{q}}, \tag{3.29}$$

where

$$h(x,y) = \left|f - \frac{H\int_X fH^{p-1}\mathrm{d}\mu}{\int_X H^p\mathrm{d}\mu}\right|.$$

We integrate inequality (3.29) over Y, apply Fubini's theorem on the left side of the inequality to find that

$$\left(\int_X H^p \mathrm{d}\mu\right) \leq \left(\int_X H^p \mathrm{d}\mu\right)^{\frac{1}{q}} \int_Y \left(\int_X (f^p - h^p)\mathrm{d}\mu\right)^{\frac{1}{p}} \mathrm{d}\nu.$$

Since $H(x) = \int_Y f(x,y)\mathrm{d}\nu$ and $1 - \dfrac{1}{q} = \dfrac{1}{p}$, we deduce that

$$\left(\int_X \left(\int_Y f \mathrm{d}\nu\right)^p \mathrm{d}\mu\right)^{\frac{1}{p}} \leq \int_Y \left(\int_X (f^p - h^p)\,\mathrm{d}\mu\right)^{\frac{1}{p}} \mathrm{d}\nu.$$

The proof of the case $1 < p \leq 2$ is similar so we omit the details and the proof is complete. □

Next, we point out the following useful special case of Theorem 3.10:

Corollary 3.5. Let $p \geq 2$ and let $f_1, f_2, ..., f_n$ be nonnegative $\mu-$measurable functions. Then

$$\left(\int_X \left(\sum_{i=1}^n f_i\right)^p \mathrm{d}\mu\right)^{\frac{1}{p}} \leq \sum_{i=1}^n \left(\int_X (f_i^p - h_i^p)\,\mathrm{d}\mu\right)^{\frac{1}{p}}, \tag{3.30}$$

where

$$h_i = \left| f_i - \frac{H \int_X f_i H^{p-1}\mathrm{d}\mu}{\int_X H^p \mathrm{d}\mu} \right|, \forall i = 1, ..., n, \text{ and } H(x) = \sum_{i=1}^n f_i(x).$$

If $1 < p \leq 2$, then (3.30) holds in the reversed direction.

Proof. Let $Y = \bigcup\limits_{i=1}^n Y_i$, where $Y_i = [i-1, i)$, for all $i \in J_n$ and let $\mathrm{d}\nu = \mathrm{d}y$ be the Lebesgue measure.

Define $f(x,y) = \sum\limits_{i=1}^n f_i(x)\chi_{Y_i}(y)$. Then

$$H(x) = \int_Y f(x,y)\mathrm{d}\nu = \sum_{i=1}^n f_i(x)$$

and

$$\begin{aligned} h(x,y) &= \left| f - \frac{H \int_X f H^{p-1}\mathrm{d}\mu}{\int_X H^p \mathrm{d}\mu} \right| = \sum_{i=1}^n \chi_{Y_i}(y) \left| f_i - \frac{H \int_X f_i H^{p-1}\mathrm{d}\mu}{\int_X H^p \mathrm{d}\mu} \right| \\ &= \sum_{i=1}^n \chi_{Y_i}(y) h_i(x), \end{aligned}$$

where

$$h_i = \left| f_i - \frac{H \int_X f_i H^{p-1}\mathrm{d}\mu}{\int_X H^p \mathrm{d}\mu} \right|.$$

Therefore, by applying Theorem 3.10, one can complete the proof. □

3.2.4 Beckenbach-Dresher's inequality

In this section we state and prove a new Beckenbach-Dresher type inequality (see Theorem 3.12). The continuous form of the Beckenbach-Dresher inequality was first derived in [47, Theorem 3.1] (see also [125]). It has the following form.

Theorem 3.11. Let f and u be nonnegative measurable functions on $X \times Y$ with respect to the measures $\mu \times \nu$ and $\lambda \times \nu$, respectively, and let

(i) $s \geq 1, q \leq 1 \leq p, (q \neq 0)$ or

(ii) $s \leq 0, p \leq 1 \leq q, (p \neq 0)$.

Then

$$\frac{\left(\int_X \left(\int_Y f \, \mathrm{d}\nu\right)^p \mathrm{d}\mu\right)^{\frac{s}{p}}}{\left(\int_X \left(\int_Y u \, \mathrm{d}\nu\right)^p \mathrm{d}\lambda\right)^{\frac{s-1}{q}}} \leq \int_Y \frac{\left(\int_X f^p \, \mathrm{d}\mu\right)^{\frac{s}{p}}}{\left(\int_X u^q \, \mathrm{d}\lambda\right)^{\frac{s-1}{q}}} \, \mathrm{d}\nu, \tag{3.31}$$

provided all occurring integrals exist.

If $0 < s \leq 1, p \leq 1$, and $q \leq 1 (p, q \neq 0)$, then inequality (3.31) is reversed.

Our new result to the continuous Beckenbach-Dresher type inequality reads as follows.

Theorem 3.12. Let f and u be nonnegative measurable functions on $X \times Y$ with respect to the measures $\mu \times \nu$ and $\lambda \times \nu$, respectively, and let $1 < q \leq 2 \leq p$, $s \geq 1$. Then

$$\frac{\left(\int_X \left(\int_Y f \mathrm{d}\nu\right)^p \mathrm{d}\mu\right)^{\frac{s}{p}}}{\left(\int_X \left(\int_Y u \mathrm{d}\nu\right)^q \mathrm{d}\lambda\right)^{\frac{s-1}{q}}} \leq \int_Y \frac{\left(\int_X \left(f^p - h^p\right) \mathrm{d}\mu\right)^{\frac{s}{p}}}{\left(\int_X \left(u^q - r^q\right) \mathrm{d}\lambda\right)^{\frac{s-1}{q}}} \mathrm{d}\nu,$$

where

$$h = \left| f - \frac{H \int_X f H^{p-1} \mathrm{d}\mu}{\int_X H^p \mathrm{d}\mu} \right|, \quad H(x) = \int_Y f(x,y) \mathrm{d}\nu$$

$$r = \left| u - \frac{\hat{H} \int_X u \hat{H}^{q-1} \mathrm{d}\mu}{\int_X \hat{H}^q \mathrm{d}\mu} \right|, \quad \hat{H}(x) = \int_Y u(x,y) \mathrm{d}\nu.$$

Proof. Let $1 < q \leq 2 \leq p$. Then, in view of Theorem 3.10 for $p \geq 2$ and $1 < q \leq 2$ we have that

$$\begin{aligned} \frac{\left(\int_X \left(\int_Y f \mathrm{d}\nu\right)^p \mathrm{d}\mu\right)^{\frac{s}{p}}}{\left(\int_X \left(\int_Y u \mathrm{d}\nu\right)^q \mathrm{d}\lambda\right)^{\frac{s-1}{q}}} &\leq \frac{\left(\int_Y \left(\int_X \left(f^p - h^p\right) \mathrm{d}\mu\right)^{\frac{1}{p}} \mathrm{d}\nu\right)^s}{\left(\int_Y \left(\int_X \left(u^q - r^q\right) \mathrm{d}\lambda\right)^{\frac{1}{q}} \mathrm{d}\nu\right)^{s-1}} \\ &= \left(\int_Y a^{\frac{1}{s}} \mathrm{d}\nu\right)^s \left(\int_Y b^{\frac{1}{1-s}} \mathrm{d}\nu\right)^{1-s} \\ &\leq \int_Y ab \, \mathrm{d}\nu, \end{aligned}$$

where $a^{\frac{1}{s}} = \left(\int_X \left(f^p - h^p\right) \mathrm{d}\mu\right)^{\frac{1}{p}}$ and $b^{\frac{1}{1-s}} = \left(\int_X \left(u^q - r^q\right) \mathrm{d}\lambda\right)^{\frac{1}{q}}$. In the last inequality we used the reverse Hölder inequality for two functions a and b when one exponent $(1-s)$ is negative and the other exponent s is positive. The proof is complete. $\square$

By using Theorem 3.12 and similar arguments as those in the proof of Corollary 3.5 we can also derive the following version:

Corollary 3.6. Let $1 < q \leq 2 \leq p, s \geq 1$, $f_i, u_i : X \to [0, \infty)$, $f_i^p, u_i^q \in L^1$, for all $i = 1, ..., n$. Then

$$\frac{\left(\int_X \left(\sum_{i=1}^n f_i\right)^p \mathrm{d}\mu\right)^{\frac{s}{p}}}{\left(\int_X \left(\sum_{i=1}^n u_i\right)^q \mathrm{d}\lambda\right)^{\frac{s-1}{q}}} \leq \sum_{i=1}^n \frac{\left(\int_X (f_i^p - h_i^p)\, \mathrm{d}\mu\right)^{\frac{s}{p}}}{\left(\int_X (u_i^q - r_i^q)\, \mathrm{d}\lambda\right)^{\frac{s-1}{q}}},$$

where

$$h_i = \left| f_i - \frac{H \int_X f_i H^{p-1} \mathrm{d}\mu}{\int_X H^p \mathrm{d}\mu} \right|, \quad H(x) = \sum_{i=1}^n f_i(x)$$

$$r_i = \left| u_i - \frac{\hat{H} \int_X u_i \hat{H}^{q-1} \mathrm{d}\mu}{\int_X \hat{H}^q \mathrm{d}\mu} \right|, \quad \hat{H}(x) = \sum_{i=1}^n u_i(x).$$

Proof. The proof is similar to that of Corollary 3.5 so we omit the details. □

As an application of Corollary 3.6, by making the substitution $s = \dfrac{p}{p-q}, p \neq q$, we obtain the following Beckenbach-Dresher type inequality.

Example 3.8. Let $1 < q \leq 2 \leq p, q \neq p, f_i, u_i : X \to [0, \infty)$, $f_i^p, u_i^q \in L^1$, for all $i = 1, ..., n$. Then

$$\left(\frac{\int_X \left(\sum_{i=1}^n f_i\right)^p \mathrm{d}\mu}{\int_X \left(\sum_{i=1}^n u_i\right)^q \mathrm{d}\lambda}\right)^{\frac{1}{p-q}} \leq \sum_{i=1}^n \left(\frac{\int_X (f_i^p - h_i^p)\, \mathrm{d}\mu}{\int_X (u_i^q - r_i^q)\, \mathrm{d}\lambda}\right)^{\frac{1}{p-q}},$$

where h_i and r_i are as in Corollary 3.6.

3.2.5 Refinements of Hardy's inequality

In this section we derive some corresponding refinements of Hardy's inequality even in a Banach function space setting. The results maybe seen as complements and further generalizations of some results in [69] and [96]. [In Section 3.1.2, the following Hardy-type inequality was given (see Theorem 3.1)]

To prove our main results in this section we need to apply Lemma 3.1. Our first main result in this section reads as follows:

Theorem 3.13. Let $0 < b \leq \infty, -\infty \leq a < c \leq \infty$, let φ be a positive and superquadratic function on (a, c) and E be a Banach function space on $[0, b)$. If E has the Fatou property and $a < f(x) < c$, then

$$\left\| \varphi\left(\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right) \right\|_E$$

$$\leq \int_0^b \varphi\left(f(t)\right) \left\| \left(1 - \frac{\varphi\left(\left|f(t) - \frac{1}{x}\int_0^x f(s)\mathrm{d}s\right|\right)}{\varphi\left(f(t)\right)}\right) \frac{1}{x}\chi_{[t,b)}(x) \right\|_E \mathrm{d}t,$$

provided that both sides have sense.

Proof. Let $D = \{(x,t) : 0 \le x \le b, 0 \le t \le x\}$. Then

$$\chi_D(x,t) = \chi_{[0,x]}(t) = \chi_{[t,b]}(x). \tag{3.32}$$

By using Theorem 3.6, the lattice property of E, Lemma 3.1 with $r = 1$ and (3.32) we find that

$$\begin{aligned}
\left\|\varphi\left(\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right)\right\|_E &\le \left\|\int_0^x \frac{\varphi\left(f(t)\right) - \varphi\left(\left|f(t) - \frac{1}{x}\int_0^x f(s)\mathrm{d}s\right|\right)}{x}\mathrm{d}t\right\|_E \\
&= \left\|\int_0^b \frac{\varphi\left(f(t)\right) - \varphi\left(\left|f(t) - \frac{1}{x}\int_0^x f(s)\mathrm{d}s\right|\right)}{x}\chi_{[0,x]}(t)\mathrm{d}t\right\|_E \\
&= \left\|\int_0^b \frac{\varphi\left(f(t)\right) - \varphi\left(\left|f(t) - \frac{1}{x}\int_0^x f(s)\mathrm{d}s\right|\right)}{x}\chi_D(x,t)\mathrm{d}t\right\|_E \\
&\le \int_0^b \left\|\frac{\varphi\left(f(t)\right) - \varphi\left(\left|f(t) - \frac{1}{x}\int_0^x f(s)\mathrm{d}s\right|\right)}{x}\chi_D(x,t)\right\|_E \mathrm{d}t \\
&= \int_0^b \left\|\frac{\varphi\left(f(t)\right) - \varphi\left(\left|f(t) - \frac{1}{x}\int_0^x f(s)\mathrm{d}s\right|\right)}{x}\chi_{[t,b]}(x)\right\|_E \mathrm{d}t \\
&= \int_0^b \varphi\left(f(t)\right)\left\|\left(1 - \frac{\varphi\left(\left|f(t) - \frac{1}{x}\int_0^x f(s)\mathrm{d}s\right|\right)}{\varphi\left(f(t)\right)}\right)\frac{1}{x}\chi_{[t,b]}(x)\right\|_E \mathrm{d}t.
\end{aligned}$$

The proof is complete. □

Here we just give one example of application of Theorem 3.13 (cf. [96, Proposition 2.1] and [69, Theorem 2.3]):

Corollary 3.7. Let $0 < b \le \infty, u : (0,b) \to \mathbf{R}$ be a nonnegative weight function such that the function $x \mapsto \frac{u(x)}{x^2}$ is locally integrable on $(0,b)$, and define the weight function v by

$$v(t) = t\int_t^b \frac{u(x)}{x^2}\mathrm{d}x, t \in (0,b).$$

If the real-valued function φ is positive and superquadratic on $(a,c), 0 \le a < c \le \infty$, then the inequality

$$\begin{aligned}
&\int_0^b u(x)\varphi\left(\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right)\frac{\mathrm{d}x}{x} \le \\
&\int_0^b v(t)\varphi\left(f(t)\right)\frac{\mathrm{d}t}{t} - \int_0^b\int_t^b \varphi\left(\left|f(t) - \frac{1}{x}\int_0^x f(s)\mathrm{d}s\right|\right)\frac{u(x)}{x^2}\mathrm{d}x\mathrm{d}t
\end{aligned} \tag{3.33}$$

holds for all f with $a < f(x) < c, 0 < x \le b$.

Proof. It is known that $E = L^1\left((0,b), \frac{u(x)}{x}\mathrm{d}x\right)$ satisfy the Fatou property (see e.g. [22]). Moreover,

$$\begin{aligned}
&\left\|\left(1 - \frac{\varphi\left(\left|f(t) - \frac{1}{x}\int_0^x f(s)\mathrm{d}s\right|\right)}{\varphi\left(f(t)\right)}\right)\frac{1}{x}\chi_{[t,b]}(x)\right\|_E \\
&= \int_t^b \frac{u(x)}{x^2}\mathrm{d}x - \frac{1}{\varphi\left(f(t)\right)}\int_t^b \varphi\left(\left|f(t) - \frac{1}{x}\int_0^x f(s)\mathrm{d}s\right|\right)\frac{u(x)}{x^2}\mathrm{d}x \\
&= \frac{v(t)}{t} - \frac{1}{\varphi\left(f(t)\right)}\int_t^b \varphi\left(\left|f(t) - \frac{1}{x}\int_0^x f(s)\mathrm{d}s\right|\right)\frac{u(x)}{x^2}\mathrm{d}x.
\end{aligned} \tag{3.34}$$

Therefore, (3.33) follows from (3.34) and Theorem 3.13. The proof is complete. □

Next we state a "dual" version of Theorem 3.13. Note that the natural dual operator of the Hardy operator $H : H(f)(x) = \frac{1}{x}\int_0^x f(t)\mathrm{d}t$ is $\hat{H} : \hat{H}(f)(x) = \int_x^\infty \frac{f(t)}{t}\mathrm{d}t$, but here we use its alternative $H^* : H^*(f)(x) = x\int_x^\infty \frac{f(t)}{t^2}\mathrm{d}t$.

Theorem 3.14. Let $-\infty \leq a < c \leq \infty$, let φ be a positive and superquadratic function on (a, c) and E be a Banach function space on $[b, \infty), b \geq 0$, with the Fatou property. Then, whenever $a < fx) < c$,

$$\left\|\varphi\left(x\int_x^\infty \frac{f(t)}{t^2}\mathrm{d}t\right)\right\|_E \leq$$
$$\int_b^\infty \varphi\left(f(t)\right)\left\|\left(1 - \frac{\varphi\left(\left|f(t) - x\int_x^\infty f(s)\frac{\mathrm{d}s}{s^2}\right|\right)}{\varphi\left(f(t)\right)}\right)x\chi_{[b,t]}(x)\right\|_E \frac{\mathrm{d}t}{t^2}.$$

Proof. Let $D = \{(x, t) : b \leq x,\ x \leq t < \infty\}$. Then

$$\chi_D(x, t) = \chi_{[x,\infty)}(t) = \chi_{[b,t]}(x). \tag{3.35}$$

By using (3.35) and the same arguments as in the proof of Theorem 3.13 we obtain that

$$\begin{aligned}
\left\|\varphi\left(x\int_x^\infty \frac{f(t)}{t^2}\mathrm{d}t\right)\right\|_E &\leq \left\|\int_x^\infty \left(\varphi\left(f(t)\right) - \varphi\left(\left|f(t) - x\int_x^\infty f(s)\frac{\mathrm{d}s}{s^2}\right|\right)\right)x\frac{\mathrm{d}t}{t^2}\right\|_E \\
&= \left\|\int_b^\infty \left(\varphi\left(f(t)\right) - \varphi\left(\left|f(t) - x\int_x^\infty f(s)\frac{\mathrm{d}s}{s^2}\right|\right)\right)x\chi_{[x,\infty)}(t)\frac{\mathrm{d}t}{t^2}\right\|_E \\
&= \left\|\int_b^\infty \left(\varphi\left(f(t)\right) - \varphi\left(\left|f(t) - x\int_x^\infty f(s)\frac{\mathrm{d}s}{s^2}\right|\right)\right)x\chi_D(x,t)\frac{\mathrm{d}t}{t^2}\right\|_E \\
&\leq \int_b^\infty \left\|\left(\varphi\left(f(t)\right) - \varphi\left(\left|f(t) - x\int_x^\infty f(s)\frac{\mathrm{d}s}{s^2}\right|\right)\right)x\chi_D(x,t)\right\|_E \frac{\mathrm{d}t}{t^2} \\
&= \int_b^\infty \left\|\left(\varphi\left(f(t)\right) - \varphi\left(\left|f(t) - x\int_x^\infty f(s)\frac{\mathrm{d}s}{s^2}\right|\right)\right)x\chi_{[b,t]}(x)\right\|_E \frac{\mathrm{d}t}{t^2} \\
&= \int_b^\infty \varphi\left(f(t)\right)\left\|\left(1 - \frac{\varphi\left(\left|f(t) - x\int_x^\infty f(s)\frac{\mathrm{d}s}{s^2}\right|\right)}{\varphi\left(f(t)\right)}\right)x\chi_{[b,t]}(x)\right\|_E \frac{\mathrm{d}t}{t^2}.
\end{aligned}$$

The proof is complete. □

We give the following example of application of Theorem 3.14 (cf. [96, Proposition 2.2]).

Corollary 3.8. Let $0 \leq b < \infty, u : (b, \infty) \to \mathbf{R}$ be a nonnegative locally integrable function on (b, ∞), and define the function v by

$$v(t) = \frac{1}{t}\int_b^t u(x)\,\mathrm{d}x, t \in (b, \infty).$$

If the real-valued function φ is positive and superquadratic on $(a, c), 0 \leq a < c \leq \infty$, then the inequality

$$\int_b^\infty u(x)\varphi\left(x\int_x^\infty f(t)\frac{\mathrm{d}t}{t^2}\right)\frac{\mathrm{d}x}{x} \leq$$
$$\int_b^\infty v(t)\varphi\left(f(t)\right)\frac{\mathrm{d}t}{t} - \int_b^\infty\int_b^t \varphi\left(\left|f(t) - x\int_x^\infty f(s)\frac{\mathrm{d}s}{s^2}\right|\right)u(x)\mathrm{d}x\frac{\mathrm{d}t}{t^2} \tag{3.36}$$

holds for all f with $a < f(x) < c, x \geq b$.

Proof. It is known that $E = L^1\left([b,\infty), \frac{u}{x}\mathrm{d}x\right)$ satisfy the Fatou property (see e.g. [22]). Moreover,

$$\begin{aligned}
&\left\| \left(1 - \frac{\varphi\left(|f(t) - x\int_x^\infty f(s)\frac{\mathrm{d}s}{s^2}|\right)}{\varphi\left(f(t)\right)}\right) x\chi_{[b,t]}(x) \right\|_E \\
&= \int_b^t u(x)\mathrm{d}x - \frac{1}{\varphi\left(f(t)\right)}\int_b^t \varphi\left(\left|f(t) - x\int_x^\infty f(s)\frac{\mathrm{d}s}{s^2}\right|\right) u(x)\mathrm{d}x \\
&= tv(t) - \frac{1}{\varphi\left(f(t)\right)}\int_t^b \varphi\left(\left|f(t) - x\int_x^\infty f(s)\frac{\mathrm{d}s}{s^2}\right|\right) u(x)\mathrm{d}x. \qquad (3.37)
\end{aligned}$$

Therefore, (3.36) follows from (3.37) and Theorem 3.14, so the proof is complete. □

3.2.6 Concluding remarks and results

In this section we give some concluding remarks and results, which in particular put our results to a more general context.

Remark 3.17. The natural "turning point" in Minkowski and Beckenbach-Dresher type inequalities is 1 but in our refined versions of these inequalities we have proved the first inequalities of this type with turning point 2 (see Theorems 3.10 and 3.12).

Our first new result of this type in this section is the following improved version of the inequality in [134, Theorem 1.2].

Proposition 3.1. Let p, s and t be different real numbers such that $s \geq 2, t \geq 2$ and $(s-t)/(p-t) > 1$. Then, for any positive $\mu-$measurable functions $f_1, ..., f_n$,

$$\left(\int_X \left(\sum_{i=1}^n f_i\right)^p \mathrm{d}\mu\right)^{s-t} \leq \frac{\left(\sum\limits_{i=1}^n \left(\int_X (f_i^s - h_i^s)\mathrm{d}\mu\right)^{\frac{1}{s}}\right)^{s(p-t)}}{\left(\sum\limits_{i=1}^n \left(\int_X (f_i^t - r_i^t)\mathrm{d}\mu\right)^{\frac{1}{t}}\right)^{t(p-s)}} \qquad (3.38)$$

where

$$h_i = \left|f_i - \frac{H\int_X f_i H^{s-1}\mathrm{d}\mu}{\int_X H^s \mathrm{d}\nu}\right|, H = \sum_{i=1}^n f_i$$

and

$$r_i = \left|f_i - \frac{H\int_X f_i H^{t-1}\mathrm{d}\mu}{\int_X H^t \mathrm{d}\mu}\right|.$$

Moreover, if $p \neq 0, 1 < t < 2, 1 < s < 2$ and $(s-t)/(p-t) < 1$, then (3.38) holds in the reversed direction.

Proof. Let $s \geq 2, t \geq 2$ such that $\dfrac{s-t}{p-t} > 1$. Then, by Hölder's inequality,

$$\begin{aligned}
\int_X \left(\sum_{i=1}^n f_i\right)^p \mathrm{d}\mu &= \int_X \left[(f_1 + \cdots + f_n)^s\right]^{\frac{p-t}{s-t}} \left[(f_1 + \cdots + f_n)^t\right]^{\frac{s-p}{s-t}} \mathrm{d}\mu \\
&\leq \left(\int_X (f_1 + \cdots + f_n)^s \mathrm{d}\mu\right)^{\frac{p-t}{s-t}} \left(\int_X (f_1 + \cdots + f_n)^t \mathrm{d}\mu\right)^{\frac{s-p}{s-t}}.
\end{aligned}$$

In view of Corollary 3.5, the above inequality becomes

$$\int_X \left(\sum_{i=1}^n f_i\right)^p \mathrm{d}\mu \le \left(\sum_{i=1}^n \left(\int_X (f_i^s - h_i^s)\mathrm{d}\mu\right)^{\frac{1}{s}}\right)^{s\frac{p-t}{s-t}} \left(\sum_{i=1}^n \left(\int_X (f_i^t - r_i^t)\mathrm{d}\mu\right)^{\frac{1}{t}}\right)^{t\frac{s-p}{s-t}},$$

where

$$h_i = \left| f_i - \frac{H\int_X f_i H^{s-1}\mathrm{d}\mu}{\int_X H^s \mathrm{d}\mu}\right|, H = \sum_{i=1}^n f_i$$

and

$$r_i = \left| f_i - \frac{H\int_X f_i H^{t-1}\mathrm{d}\mu}{\int_X H^t \mathrm{d}\mu}\right|.$$

The proof of the other case is similar so we omit the details and the proof is complete. □

Remark 3.18. In [134, Theorem 1.2] only the case $n = 2$ was considered so Proposition 3.1 is both a generalization and refinement of this result.

Proposition 3.2. Suppose that ν is a measure, and f_1 and f_2 are nonnegative measurable functions such that f_i^p are $\nu-$integrable, for $i = 1, 2$.

(a) If $p > 0$, then

$$\left\|\sum_{i=1}^2 f_i\right\|_{L^p(\nu)}^p \le \left(\sum_{i=1}^2 \|f_i\|_{L^p(\nu)}\right)^p + \frac{1}{2}\sum_{i=1}^2 \|f_i + h_i\|_{L^p(\nu)}^p$$

(b) If $p \ge \dfrac{1}{2}$, then

$$\left\|\sum_{i=1}^2 f_i\right\|_{L^p(\nu)}^p \le \left(\sum_{i=1}^2 \|f_i\|_{L^p(\nu)}\right)^p + \frac{1}{2}\sum_{i=1}^2 \left(\|f_i + h_i\|_{L^p(\nu)}^p - \|f_i\|_{L^p(\nu)}^p\right),$$

where

$$h_i = \left| g_i^p - \frac{f_i^p \|g_i\|_{L^p(\nu)}^p}{\|f_i\|_{L^p(\nu)}^p}\right|^{\frac{1}{p}}, \ g_i = (f_1 + f_2) - f_i,$$

for $i = 1, 2$.

Proof. In view of [2, Theorem 4.3], we have

$$\begin{aligned} -\left(1 + \left(\int F\mathrm{d}\mu\right)^{\frac{1}{p}}\right)^p \le &-\int \left(1 + F^{\frac{1}{p}}\right)^p \mathrm{d}\mu \\ &+ \int \left(1 + \left|F - \int F\mathrm{d}\mu\right|^{\frac{1}{p}}\right)^p \mathrm{d}\mu \end{aligned} \tag{3.39}$$

for $p > 0$ and

$$-\left(1+\left(\int F\mathrm{d}\mu\right)^{\frac{1}{p}}\right)^{p}+1 \le -\int\left(1+F^{\frac{1}{p}}\right)^{p}\mathrm{d}\mu + \int\left(1+\left|F-\int F\mathrm{d}\mu\right|^{\frac{1}{p}}\right)^{p}\mathrm{d}\mu \tag{3.40}$$

for $p \ge \frac{1}{2}$.

By substituting $F = \frac{f_2^p}{f_1^p}$ and $\mathrm{d}\mu = \frac{f_1^p}{\int f_1^p \mathrm{d}\nu}\mathrm{d}\nu$ in both of the inequalities (3.39) and (3.40), we obtain that

$$\|f_1+f_2\|_{L^p(\nu)}^p \le \left(\|f_1\|_{L^p(\nu)}+\|f_2\|_{L^p(\nu)}\right)^p + \|f_1+h_1\|_{L^p(\nu)}^p, \tag{3.41}$$

for $p > 0$ and

$$\|f_1+f_2\|_{L^p(\nu)}^p \le \left(\sum_{i=1}^{2}\|f_i\|_{L^p(\nu)}\right)^p + \|f_1+h_1\|_{L^p(\nu)}^p - \|f_1\|_{L^p(\nu)}^p, \tag{3.42}$$

for $p \ge \frac{1}{2}$, where

$$h_1 = \left|f_2^p - \frac{f_1^p\|f_2\|_{L^p(\nu)}^p}{\|f_1\|_{L^p(\nu)}^p}\right|^{\frac{1}{p}}.$$

By interchanging the role of f_1 and f_2 in the above discussion, we have that

$$\|f_1+f_2\|_{L^p(\nu)}^p \le \left(\|f_1\|_{L^p(\nu)}+\|f_2\|_{L^p(\nu)}\right)^p + \|f_2+h_2\|_{L^p(\nu)}^p, \tag{3.43}$$

for $p > 0$ and

$$\|f_1+f_2\|_{L^p(\nu)}^p \le \left(\sum_{i=1}^{2}\|f_i\|_{L^p(\nu)}\right)^p + \|f_2+h_2\|_{L^p(\nu)}^p - \|f_2\|_{L^p(\nu)}^p, \tag{3.44}$$

for $p \ge \frac{1}{2}$, where

$$h_2 = \left|f_1^p - \frac{f_2^p\|f_1\|_{L^p(\nu)}^p}{\|f_2\|_{L^p(\nu)}^p}\right|^{\frac{1}{p}}.$$

Consequently, by taking the sum of inequalities (3.41) and (3.43), we get the result in (a). The inequality in (b) follows similarly from (3.42) and (3.44) so the proof is complete. □

Remark 3.19. The concept of superquadratic function was formally introduced in [2, 3] but this idea seems to be known even before (see e.g. [117] and the references therein)

Remark 3.20. The first important book in the area of inequalities is (the bible) [50] but after that more than 30 books or monographs in this area have been published. The really last one concerning continuous inequalities is [91] where even some results in this PhD thesis are mentioned.

Remark 3.21. Concerning Hardy type inequalities in Section 3.2.5 the first result was proved in 1925 (see [49]). The dramatic history and prehistory up to 2007 is described in the book [65] (see also [102]). The corresponding history up to 2017 is given in detail in the book [73]. Our results in Section 3.2.5 is just one example of the fact that the development of the theory of this fascinating inequality still continues. For the development of classical inequalities in continuous and/or Banach function space setting we refer to [93] and the references given there, see also [18], [19], [23] and the book [91].

Chapter 4

Cochran-Lee and Hardy type inequalities

4.1 Multidimensional Cochran-Lee and Hardy type inequalities

4.1.1 Introduction

For the purpose of this section we just mention the following generalization of (1.8) by J. A. Cochran and C. S. Lee (see [31]):

$$\int_0^\infty x^a \exp\left[\beta x^{-\beta} \int_0^x t^{\beta-1} \log f(t)\,\mathrm{d}t\right] \mathrm{d}x \le e^{(a+1)/\beta} \int_0^\infty x^a f(x)\,\mathrm{d}x,$$

where $\beta > 0, a \in \mathbb{R}$ and the constant $e^{(a+1)/\beta}$ is sharp. This means that the geometric mean operator G, defined by

$$(Gf)(x) := \exp\left(\frac{1}{x}\int_0^x \log f(t)\,\mathrm{d}t\right)$$

is replaced by the more general weighted geometric mean operator G_β, defined by

$$(G_\beta f)(x) := \exp\left(\beta x^{-\beta}\int_0^x t^{\beta-1}\log f(t)\,\mathrm{d}t\right)$$

for any $\beta > 0$. Later on a number of results are proved concerning more general weighted versions of (1.8):

$$\left(\int_0^\infty \left[\exp\left(\frac{1}{x}\int_0^x \log f(t)\,\mathrm{d}t\right)\right]^q u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\int_0^\infty f^p(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p}}$$

for various parameters p and q, weights $u(x), v(x)$ and some constant $C > 0$. See e.g. [30], [56], [57], [59], [75] and the references therein.

Moreover, some results for the corresponding two-dimensional cases are also known, see e.g. [52], [54], [127], [128] and the references therein. In particular, the result in [127] (see also [128]) is of special interest for this chapter since also good estimates are given of the sharp constant C in the inequality.

First we state the following result (see [127, Theorem 4.1] and also [128]):

Theorem 4.1. Let $0 < p \leq q < \infty$, and let u, v and f be positive and measurable functions on $\mathbb{R}_+^2$. If $0 < b_1, b_2 \leq \infty$, then

$$\left(\int_0^{b_1}\int_0^{b_2}\left[\exp\left(\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2}\ln f(y_1,y_2)\,\mathrm{d}y_1\mathrm{d}y_2\right)\right]^q u(x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2\right)^{\frac{1}{q}} \leq C\left(\int_0^{b_1}\int_0^{b_2} f^p(x_1,x_2)v(x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2\right)^{\frac{1}{p}} \tag{4.1}$$

if and only if

$$D_W(s_1,s_2,p,q) := \sup_{\substack{y_1\in(0,b_1)\\ y_2\in(0,b_2)}} y_1^{\frac{s_1-1}{p}} y_2^{\frac{s_2-1}{p}}\left(\int_{y_1}^{b_1}\int_{y_2}^{b_2} x_1^{-\frac{s_1q}{p}} x_2^{-\frac{s_2q}{p}} w(x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2\right)^{\frac{1}{q}} < \infty,$$

where $s_1, s_2 > 1$ and

$$w(x_1,x_2) = \left[\exp\left(\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2}\ln v^{-1}(t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2\right)\right]^{\frac{q}{p}} u(x_1,x_2),$$

and the best possible constant C in (4.1) can be estimated in the following way:

$$\sup_{s_1,s_2>1}\left(\frac{e^{s_1}(s_1-1)}{e^{s_1}(s_1-1)+1}\right)^{\frac{1}{p}}\left(\frac{e^{s_2}(s_2-1)}{e^{s_2}(s_2-1)+1}\right)^{\frac{1}{p}} D_W(s_1,s_2,p,q) \leq C \leq \inf_{s_1,s_2>1} e^{\frac{s_1+s_2-2}{p}} D_W(s_1,s_2,p,q).$$

Concerning the general n-dimensional case ($n \in \mathbb{Z}_+$) the only known results so far seems to be those in the recent paper by M. F. Yimer (see [132]), but here we state and prove an even more general result (see Theorem 4.2, Remark 4.1 and c.f. also [133]).

By using this result for the power weighted case (and $p = q$) we prove in Section 4.1.3 a general new n-dimensional Cochran-Lee inequality with sharp constant (see Theorem 4.4). This result generalizes several results in the literature including one in [132].

Finally, in Section 4.1.4 we give some concluding remarks and applications, including both well-known and new Hardy-type inequalities. In particular, we point out that Theorem 4.2 and its proof can be used to formulate the first example where a multidimensional Hardy-type inequality can be characterized not only by one condition but by infinite many (equivalent) conditions, even by a scale of condition (see Theorem 4.5). For the one-dimensional case this fairly new idea in the theory of Hardy-type inequalities is described and applied in [73, Section 7.3].

4.1.2 New multidimensional Cochran-Lee type inequality

In this section we prove a generalization of Theorem 4.1 (see Theorem 4.2) general n-dimensional case ($n \in \mathbb{Z}_+$) but we do it in a more general frame where the standard n-dimensional geometric operator is replaced by a more general geometric mean operator G_β (see (4.2)) so we can cover also the Cochran-Lee situation.

First we introduce the following multidimensional geometric mean operator G_β:

$$(G_{\boldsymbol{\beta}} f)(\mathbf{x}) := \exp\left(\prod_{i=1}^n \beta_i x_i^{-\beta_i} \int_0^{x_1} \cdots \int_0^{x_n} \prod_{i=1}^n t_i^{\beta_i-1} \ln f(\mathbf{t})\,\mathrm{d}\mathbf{t}\right), \tag{4.2}$$

for any nonnegative and measurable function $f(t)$ on $\mathbb{R}_+^n := (0,\infty)^n$, where $\beta_i > 0$, $i \in J_n$, $n \in \mathbb{Z}_+$.

We investigate the multidimensional weighted generalized geometric mean inequality

$$\left(\int_0^{b_1} \cdots \int_0^{b_n} \left[(G_{\boldsymbol{\beta}} f)(\mathbf{x})\right]^q u(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}} \le C \left(\int_0^{b_1} \cdots \int_0^{b_n} f^p(\mathbf{x}) v(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{p}}, \tag{4.3}$$

where u and v are weight functions, $0 < p \le q < \infty$, $0 < b_i \le \infty$, $\beta_i > 0$ ($i \in J_n$), and C is a positive constant independent of f.

Our main theorem in this section reads as follows:

Theorem 4.2. Let $n \in \mathbb{Z}_+$, $0 < p \le q < \infty$, $\beta_i > 0$, $i \in J_n$, and let u, v and f be positive and measurable functions on $\mathbb{R}_+^n$. Then, the inequality (4.3) holds for some finite constant C if and only if for any $\alpha_i > 0$, $i = 1, \ldots, n$,

$$A_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) := \sup_{\substack{t_i \in (0,b_i) \\ i \in J_n}} \prod_{i=1}^n t_i^{\frac{\alpha_i+\beta_i-1}{p}} \left(\int_{\mathbf{t}}^{\mathbf{b}} \prod_{i=1}^n x_i^{-(\alpha_i+\beta_i)\frac{q}{p}} w(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}} < \infty, \tag{4.4}$$

where

$$w(\mathbf{x}) = u(\mathbf{x}) \left[G_{\boldsymbol{\beta}} v^{-1}(\mathbf{x})\right]^{\frac{q}{p}}. \tag{4.5}$$

Moreover, if C is the best possible constant in (4.3), then

$$\begin{aligned} \sup_{\substack{\alpha_i > 0 \\ i \in J_n}} \prod_{i=1}^n \left(\frac{(\beta_i+\alpha_i-1)\exp\left(1+\frac{\alpha_i}{\beta_i}\right)}{1+(\beta_i+\alpha_i-1)\exp\left(1+\frac{\alpha_i}{\beta_i}\right)}\right)^{\frac{1}{p}} A_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) \\ \le C \le \inf_{\substack{\alpha_i > 0 \\ i \in J_n}} \prod_{i=1}^n \left(\beta_i \exp\frac{\alpha_i}{\beta_i}\right)^{\frac{1}{p}} A_{\boldsymbol{\beta}}(\boldsymbol{\alpha}). \end{aligned} \tag{4.6}$$

Proof. Sufficiency. Let $g(\mathbf{x}) = f^p(\mathbf{x}) v(\mathbf{x})$. Then the inequality (4.3) is equivalent to the inequality

$$\left(\int_{\mathbf{0}}^{\mathbf{b}} \left[(G_{\boldsymbol{\beta}} g)(\mathbf{x})\right]^{\frac{q}{p}} w(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}} \le C \left(\int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{p}}, \tag{4.7}$$

where $w(x)$ is defined by (4.5).

Let $t_i = x_i y_i$, $i \in J_n$. Then the inequality (4.7) becomes

$$\begin{aligned} \left(\int_{\mathbf{0}}^{\mathbf{b}} \left[\exp\left(\prod_{i=1}^n \beta_i \int_{\mathbf{0}}^{\mathbf{1}} \prod_{i=1}^n y_i^{\beta_i-1} \ln g(\mathbf{x}\mathbf{y})\,\mathrm{d}\mathbf{y}\right)\right]^{\frac{q}{p}} w(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}} \\ \le C \left(\int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{p}}. \end{aligned} \tag{4.8}$$

For $\alpha_i > 0$, $i = 1, \ldots, n$, we trivially have that

$$\exp\left(\sum_{i=1}^{n} \frac{\alpha_i}{\beta_i}\right) \exp\left(\prod_{i=1}^{n} \beta_i \int_{\mathbf{0}}^{\mathbf{1}} \prod_{i=1}^{n} y_i^{\beta_i - 1} \log \prod_{i=1}^{n} y_i^{\alpha_i} \, \mathrm{d}\mathbf{y}\right) = 1. \tag{4.9}$$

By applying the identity (4.9) and then using Jensen's inequality, we find that the left hand side of (4.8) can be written and estimated as follows:

$$\begin{aligned} &\exp\left(\sum_{i=1}^{n} \frac{\alpha_i}{p\beta_i}\right) \left(\int_{\mathbf{0}}^{\mathbf{b}} \left[\exp\left(\prod_{i=1}^{n} \beta_i \int_{\mathbf{0}}^{\mathbf{1}} \prod_{i=1}^{n} y_i^{\beta_i - 1} \log g(\mathbf{x}\mathbf{y}) \prod_{i=1}^{n} y_i^{\alpha_i} \, \mathrm{d}\mathbf{y}\right)\right]^{\frac{q}{p}} w(\mathbf{x}) \, \mathrm{d}\mathbf{x}\right)^{\frac{1}{q}} \\ &\quad \leq \left(\prod_{i=1}^{n} \beta_i \exp \frac{\alpha_i}{\beta_i}\right)^{\frac{1}{p}} \left(\int_{\mathbf{0}}^{\mathbf{b}} \left(\int_{\mathbf{0}}^{\mathbf{1}} \prod_{i=1}^{n} y_i^{\beta_i + \alpha_i - 1} g(\mathbf{x}\mathbf{y}) \, \mathrm{d}\mathbf{y}\right)^{\frac{q}{p}} w(\mathbf{x}) \, \mathrm{d}\mathbf{x}\right)^{\frac{1}{q}} \end{aligned} \tag{4.10}$$

$$= \left(\prod_{i=1}^{n} \beta_i \exp \frac{\alpha_i}{\beta_i}\right)^{\frac{1}{p}} \left(\int_{\mathbf{0}}^{\mathbf{b}} \left(\int_{\mathbf{0}}^{\mathbf{x}} g(\mathbf{t}) \prod_{i=1}^{n} t_i^{\beta_i + \alpha_i - 1} \, \mathrm{d}\mathbf{t}\right)^{\frac{q}{p}} \prod_{i=1}^{n} x_i^{-(\beta_i + \alpha_i)\frac{q}{p}} w(\mathbf{x}) \, \mathrm{d}\mathbf{x}\right)^{\frac{1}{q}} := I.$$

Therefore, by using Minkowski's integral inequality when $p < q$ or Fubini's theorem when $p = q$ (cf. [127, Remark 5.2]), we have that

$$I \leq \left(\prod_{i=1}^{n} \beta_i \exp \frac{\alpha_i}{\beta_i}\right)^{\frac{1}{p}} A_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) \left(\int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{t}) \, \mathrm{d}\mathbf{t}\right)^{\frac{1}{p}}, \tag{4.11}$$

where $A_\beta(\alpha)$ is defined by (4.4)-(4.5).

By combining (4.10)-(4.11) we find that (4.8) and, thus, (4.7) holds. Moreover, since (4.3) is equivalent to (4.7), we conclude that (4.3) holds and that the best constant C in (4.3) satisfies

$$C \leq \inf_{\substack{\alpha_i > 0 \\ i \in J_n}} \prod_{i=1}^{n} \left(\beta_i \exp \frac{\alpha_i}{\beta_i}\right)^{\frac{1}{p}} A_{\boldsymbol{\beta}}(\boldsymbol{\alpha}).$$

Necessity. Assume that (4.3), or equivalently (4.8), holds. In order to prove that (4.8) implies (4.4)-(4.5), we define the test function g by

$$g(\mathbf{x}) := \prod_{i=1}^{n} \left(t_i^{-1} \chi_{[0,t_i]}(x_i) + \frac{t_i^{\beta_i + \alpha_i - 1}}{x_i^{(\beta_i + \alpha_i)}} \exp\left(-\frac{\beta_i + \alpha_i}{\beta_i}\right) \chi_{(t_i, b_i)}(x_i)\right),$$

for fixed $t_i, 0 < t_i < b_i$ $(i = 1, \ldots, n)$. Then

$$\begin{aligned}
\int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{x})\,\mathrm{d}\mathbf{x} &= \int_{\mathbf{0}}^{\mathbf{b}} \prod_{i=1}^{n} \left(t_i^{-1}\chi_{[0,t_i]}(x_i) + \frac{t_i^{\beta_i+\alpha_i-1}}{x_i^{(\beta_i+\alpha_i)}} \exp\left(-\frac{\beta_i+\alpha_i}{\beta_i}\right)\chi_{(t_i,b_i)}(x_i)\right)\,\mathrm{d}\mathbf{x} \\
&= \prod_{i=1}^{n} \left(\int_0^{t_i} t_i^{-1}\,\mathrm{d}x_i + \exp\left(-\frac{\beta_i+\alpha_i}{\beta_i}\right) t_i^{\beta_i+\alpha_i-1} \int_{t_i}^{b_i} x_i^{-(\beta_i+\alpha_i)}\,\mathrm{d}x_i \right) \\
&= \prod_{i=1}^{n} \left(1 + \frac{1}{\beta_i+\alpha_i-1} \exp\left(-\frac{\beta_i+\alpha_i}{\beta_i}\right)\left(1 - \left(\frac{t_i}{b_i}\right)^{\beta_i+\alpha_i-1}\right)\right) \\
&\leq \prod_{i=1}^{n} \left(\frac{1 + (\beta_i+\alpha_i-1)\exp\left(\frac{\beta_i+\alpha_i}{\beta_i}\right)}{(\beta_i+\alpha_i-1)\exp\left(\frac{\beta_i+\alpha_i}{\beta_i}\right)} \right).
\end{aligned}$$

This implies that

$$\left(\int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{p}} \leq \prod_{i=1}^{n} \left(\frac{1 + (\beta_i+\alpha_i-1)\exp\left(\frac{\beta_i+\alpha_i}{\beta_i}\right)}{(\beta_i+\alpha_i-1)\exp\left(\frac{\beta_i+\alpha_i}{\beta_i}\right)} \right)^{\frac{1}{p}} < \infty. \tag{4.12}$$

Trivially, for $\mathbf{0} \leq \mathbf{t} < \mathbf{b}$, we have that

$$\left(\int_{\mathbf{t}}^{\mathbf{b}} \left[(G_{\boldsymbol{\beta}} g)(\mathbf{x})\right]^{\frac{q}{p}} w(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}} \leq \left(\int_{\mathbf{0}}^{\mathbf{b}} \left[(G_{\boldsymbol{\beta}} g)(\mathbf{x})\right]^{\frac{q}{p}} w(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}}. \tag{4.13}$$

Moreover, for $\mathbf{t} \leq \mathbf{x} < \mathbf{b}$, we find that

$$\begin{aligned}
&\int_{\mathbf{0}}^{\mathbf{x}} \prod_{j=1}^{n} y_j^{\beta_j-1} \ln g(\mathbf{y})\,\mathrm{d}\mathbf{y} \\
&= \sum_{i=1}^{n} \int_{\mathbf{0}}^{\mathbf{x}} \prod_{j=1}^{n} y_j^{\beta_j-1} \ln\left(t_i^{-1}\chi_{[0,t_i]}(y_i) + \frac{t_i^{\beta_i+\alpha_i-1}}{y_i^{(\beta_i+\alpha_i)}} \exp\left(-\frac{\beta_i+\alpha_i}{\beta_i}\right)\chi_{(t_i,b_i)}(y_i)\right)\,\mathrm{d}\mathbf{y} \\
&= \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{x_j^{\beta_j}}{\beta_j} \int_0^{x_i} y_i^{\beta_i-1} \ln\left(t_i^{-1}\chi_{[0,t_i]}(y_i) + \frac{t_i^{\beta_i+\alpha_i-1}}{y_i^{(\beta_i+\alpha_i)}} \exp\left(-\frac{\beta_i+\alpha_i}{\beta_i}\right)\chi_{(t_i,b_i)}(y_i)\right)\,\mathrm{d}y_i \\
&= \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{x_j^{\beta_j}}{\beta_j} \left(\frac{x_i^{\beta_i}}{\beta_i} \ln\left(t_i^{\beta_i+\alpha_i-1} x_i^{-(\beta_i+\alpha_i)}\right)\right) = \left(\prod_{j=1}^{n} \frac{x_j^{\beta_j}}{\beta_j}\right) \sum_{i=1}^{n} \ln\left(t_i^{\beta_i+\alpha_i-1} x_i^{-(\beta_i+\alpha_i)}\right),
\end{aligned}$$

and, hence,

$$\left(\int_{\mathbf{t}}^{\mathbf{b}} \left[(G_{\boldsymbol{\beta}} g)(\mathbf{x})\right]^{\frac{q}{p}} w(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}} = \prod_{i=1}^{n} t_i^{\frac{\beta_i+\alpha_i-1}{p}} \left(\int_{\mathbf{t}}^{\mathbf{b}} \prod_{i=1}^{n} x_i^{-(\beta_i+\alpha_i)\frac{q}{p}} w(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}}. \tag{4.14}$$

It follows from (4.8) and (4.12)-(4.14) that

$$A_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) = \sup_{\substack{t_i\in(0,b_i)\\ i\in J_n}} \prod_{i=1}^{n} t_i^{\frac{\beta_i+\alpha_i-1}{p}} \left(\int_{\mathbf{t}}^{\mathbf{b}} \prod_{i=1}^{n} x_i^{-(\beta_i+\alpha_i)\frac{q}{p}} w(\mathbf{x})\,\mathrm{d}\mathbf{x} \right)^{\frac{1}{q}}$$
$$\leq C \prod_{i=1}^{n} \left(\frac{1+(\beta_i+\alpha_i-1)\exp\left(\frac{\beta_i+\alpha_i}{\beta_i}\right)}{(\beta_i+\alpha_i-1)\exp\left(\frac{\beta_i+\alpha_i}{\beta_i}\right)} \right)^{\frac{1}{p}}.$$

Thus, since $C<\infty$, we conclude that indeed (4.4)-(4.5) holds and that the left hand side inequality of (4.6) holds. Thus, also the necessity part is proved. The proof is complete including the fact that (4.6) holds. The proof is complete. □

Remark 4.1. Note that for the case $n=2, \beta_i=1, \alpha_i=s_i-1$, $(i=1,2)$, we obtain Theorem 4.1 so in particular, we can conclude that Theorem 4.1 holds also in a general n-dimensional setting ($n\in\mathbb{Z}_+$).

As in the classical situation, by doing suitable substitutions, we can also derive a dual version of Theorem 4.2 (where integrals $\int_0^t$ are replaced by $\int_t^\infty$).

Theorem 4.3. Let $0<p\leq q<\infty$, $\beta_i>0$ ($i\in J_n$), and let u, v and f be positive and measurable functions on $\mathbb{R}_+^n$. Then, for $n=2,3,\ldots$,

$$\left(\int_{\mathbb{R}_+^n} \left[\exp\left(\prod_{i=1}^{n} \beta_i x_i^{\beta_i} \int_{\mathbf{x}}^{\infty} \prod_{i=1}^{n} t_i^{-(\beta_i+1)} \ln f(\mathbf{t})\,\mathrm{d}\mathbf{t} \right) \right]^q u(\mathbf{x})\,\mathrm{d}\mathbf{x} \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}_+^n} f^p(\mathbf{x}) v(\mathbf{x})\,\mathrm{d}\mathbf{x} \right)^{\frac{1}{p}} \tag{4.15}$$

holds for some finite C if and only if for any $\alpha_i>0$ ($i=1,\ldots,n$),

$$B_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) := \sup_{\substack{t_i>0\\ i\in J_n}} \prod_{i=1}^{n} t_i^{\frac{\beta_i+\alpha_i-1}{p}} \left(\int_{\mathbf{t}}^{\infty} \prod_{i=1}^{n} x_i^{-(\beta_i+\alpha_i)\frac{q}{p}} W(\mathbf{x})\,\mathrm{d}\mathbf{x} \right)^{\frac{1}{q}} < \infty, \tag{4.16}$$

where

$$W(\mathbf{x}) = U(\mathbf{x}) \left[G_{\boldsymbol{\beta}} V^{-1}(\mathbf{x}) \right]^{\frac{q}{p}}, \tag{4.17}$$

and

$$U(\mathbf{x}) = u(\mathbf{1}/\mathbf{x}) \prod_{i=1}^{n} x_i^{-2} \text{ and } V(\mathbf{x}) = v(\mathbf{1}/\mathbf{x}) \prod_{i=1}^{n} x_i^{-2}. \tag{4.18}$$

Moreover, if C is the best possible constant in (4.15), then

$$\sup_{\substack{\alpha_i>0\\ i\in J_n}} \prod_{i=1}^{n} \left(\frac{(\beta_i+\alpha_i-1)\exp\left(1+\frac{\alpha_i}{\beta_i}\right)}{1+(\beta_i+\alpha_i-1)\exp\left(1+\frac{\alpha_i}{\beta_i}\right)} \right)^{\frac{1}{p}} B_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) \leq C \leq \inf_{\substack{\alpha_i>0\\ i\in J_n}} \prod_{i=1}^{n} \left(\beta_i \exp\frac{\alpha_i}{\beta_i} \right)^{\frac{1}{p}} B_{\boldsymbol{\beta}}(\boldsymbol{\alpha}). \tag{4.19}$$

Proof. First we note that by using the substitutions first $y_i = 1/t_i$, and then $z_i = 1/x_i$ ($i \in J_n$) and $g(\mathbf{t}) = f(\mathbf{1}/\mathbf{t})$, elementary calculations show that (4.15) is equivalent to the inequality

$$\left(\int_{\mathbb{R}^n_+}\left[\exp\left(\prod_{i=1}^n \beta_i z_i^{-\beta_i}\int_{\mathbf{0}}^{\mathbf{z}}\prod_{i=1}^n t_i^{\beta_i-1}\ln g(\mathbf{t})\,\mathrm{d}\mathbf{t}\right)\right]^q U(\mathbf{z})\,\mathrm{d}\mathbf{z}\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbb{R}^n_+} g^p(\mathbf{z})V(\mathbf{z})\,\mathrm{d}\mathbf{z}\right)^{\frac{1}{p}}, \tag{4.20}$$

where $U(x)$ and $V(x)$ are defined by (4.18).

In view of Theorem 4.2, the inequality (4.20) holds for some finite C if and only if for any $\alpha_i > 0$ ($i = 1, \dots, n$),

$$B_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) = \sup_{\substack{t_i>0\\ i\in J_n}}\prod_{i=1}^n t_i^{\frac{\beta_i+\alpha_i-1}{p}}\left(\int_{\mathbf{t}}^{\infty}\prod_{i=1}^n x_i^{-(\beta_i+\alpha_i)\frac{q}{p}}W(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}} < \infty,$$

where

$$W(\mathbf{x}) = U(\mathbf{x})\left[\left(G_{\boldsymbol{\beta}}V^{-1}\right)(\mathbf{x})\right]^{\frac{q}{p}},$$

with $U(x)$ and $V(x)$ are defined by (4.18).

Since the inequality (4.3) is equivalent to (4.15) with $U(x)$ and $V(x)$ are defined by (4.18), we can, by Theorem 4.2, conclude that (4.15) holds if and only if (4.16)-(4.18) holds. Moreover, for the sharp constant in (4.15) we have the estimates (4.19). The proof is complete. □

4.1.3 Multidimensional Cochran-Lee inequalities with sharp constants

First we note that Theorem 4.2 implies the following inequality for power weights:

Proposition 4.1. Let $0 < p \leq q < \infty$, and let $\beta_i > 0, \eta_i, \gamma_i > -1$ ($i \in J_n$). Then, the inequality

$$\left(\int_{\mathbf{0}}^{\mathbf{b}}\left[\left(G_{\boldsymbol{\beta}}f\right)(\mathbf{x})\right]^q\prod_{i=1}^n x_i^{\eta_i}\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbf{0}}^{\mathbf{b}} f^p(\mathbf{x})\prod_{i=1}^n x_i^{\gamma_i}\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{p}} \tag{4.21}$$

holds if and only if

$$\frac{1+\eta_i}{q} = \frac{1+\gamma_i}{p}, \tag{4.22}$$

for all $i \in J_n$.

Proof. By applying Theorem 4.2 with these power weights we easily obtain the proof since in this case $A_\beta(\alpha)$ (see (4.4)) is of the form

$$A_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) = \exp\left(\sum_{i=1}^{n} \frac{\gamma_i}{p\beta_i}\right) \sup_{\substack{t_i\in(0,b_i)\\ i\in J_n}} \prod_{i=1}^{n} t_i^{\frac{1+\eta_i}{q}-\frac{1+\gamma_i}{p}} \left[\frac{1-\left(\frac{t_i}{b_i}\right)^{(\alpha_i+\gamma_i+\beta_i)\frac{q}{p}-(1+\eta_i)}}{(\alpha_i+\gamma_i+\beta_i)\frac{q}{p}-(1+\eta_i)}\right]^{\frac{1}{q}}$$

$$\leq \exp\left(\sum_{i=1}^{n} \frac{\gamma_i}{p\beta_i}\right) \sup_{\substack{t_i\in(0,b_i)\\ i\in J_n}} \prod_{i=1}^{n} \frac{t_i^{\frac{1+\eta_i}{q}-\frac{1+\gamma_i}{p}}}{\left[(\alpha_i+\gamma_i+\beta_i)\frac{q}{p}-(1+\eta_i)\right]^{\frac{1}{q}}},$$

provided that $\beta_i > \frac{p}{q}(1+\eta_i) - (\alpha_i+\gamma_i)(\ i \in J_n)$. We omit the details. $\square$

Remark 4.2. For the case $p < q$ we judge that it is a difficult and open question to find the sharp constant C in (4.21). See Remark 4.5. However, in our next main theorem we will derive the sharp constant for the case $p = q$ and thus obtain a genuine generalization of the Cochran-Lee inequality to a multidimensional setting. Moreover, since $(G_\beta f^p)(x) = [(G_\beta f)(x)]^p$ it is sufficient to prove this fact for the case $p = q = 1$. See also Remark 4.4.

Theorem 4.4. Let $\beta_i > 0$, $\eta_i > -1$ $(i \in J_n)$, and let f be a positive and measurable function defined on $\mathbb{R}^n_+$, $n \in \mathbb{Z}_+$. Then the inequality

$$\int_{\mathbb{R}^n_+} (G_{\boldsymbol{\beta}} f)(\mathbf{x}) \prod_{i=1}^{n} x_i^{\eta_i}\, d\mathbf{x} \leq \exp\left(\sum_{i=1}^{n} \frac{1+\eta_i}{\beta_i}\right) \int_{\mathbb{R}^n_+} f(\mathbf{x}) \prod_{i=1}^{n} x_i^{\eta_i}\, d\mathbf{x} \tag{4.23}$$

holds and the constant $\exp\left(\sum_{i=1}^{n} \frac{1+\eta_i}{\beta_i}\right)$ is sharp.

Proof. In view of Proposition 4.1, the inequality (4.23) holds with $\exp\left(\sum_{i=1}^{n} \frac{1+\eta_i}{\beta_i}\right)$ replaced by some finite $C > 0$. Now, we prove that the best constant $C = \exp\left(\sum_{i=1}^{n} \frac{1+\eta_i}{\beta_i}\right)$. From (4.6) in Theorem 4.2, it follows that the best constant C satisfies

$$C \leq \exp\left(\sum_{i=1}^{n} \frac{\eta_i}{\beta_i}\right) \inf_{\substack{\alpha_i>0\\ i\in J_n}} \left(\prod_{i=1}^{n} \frac{\exp\left(\frac{\alpha_i}{\beta_i}\right)}{\left(\frac{\beta_i+\alpha_i-1}{\beta_i}\right)}\right).$$

The infimum in the above inequality is attained at $\alpha_i = 1$ $(i = 1, ..., n)$. Hence, we find that

$$C \leq \exp\left(\sum_{i=1}^{n} \frac{1+\eta_i}{\beta_i}\right). \tag{4.24}$$

It only remains to prove that the inequality (4.24) holds also in the reversed direction. Consider the test function

$$f(\mathbf{x}) = \prod_{i=1}^{n} \left(\chi_{[0,e^{\frac{1}{\beta_i}}]}(x_i) + x_i^{-\gamma_i} \chi_{(e^{\frac{1}{\beta_i}},\infty)}(x_i)\right),$$

where $\gamma_i > 1 + \eta_i$ $(i \in J_n)$. Next we note that then the integral part on the right hand side of (4.23) becomes

$$\int_{\mathbb{R}_+^n} f(\mathbf{x}) \prod_{i=1}^n x_i^{\eta_i} \, \mathrm{d}\mathbf{x} = \prod_{i=1}^n \int_{\mathbb{R}_+} x_i^{\eta_i} \left(\chi_{[0, e^{\frac{1}{\beta_i}}]}(x_i) + x_i^{-\gamma_i} \chi_{(e^{\frac{1}{\beta_i}}, \infty)}(x_i) \right) \mathrm{d}x_i$$

$$= \prod_{i=1}^n \exp\left(\frac{1+\eta_i}{\beta_i}\right) \left(\frac{1}{1+\eta_i} + \frac{\exp\left(-\frac{\gamma_i}{\beta_i}\right)}{\gamma_i - (1+\eta_i)} \right). \tag{4.25}$$

Moreover, the left hand side of (4.23) is equal to

$$\prod_{i=1}^n \int_{\mathbb{R}_+} x_i^{\eta_i} \exp\left(\beta_i x_i^{-\beta_i} \int_0^{x_i} t_i^{\beta_i - 1} \ln \left(\chi_{[0, e^{\frac{1}{\beta_i}}]}(t_i) + t_i^{-\gamma_i} \chi_{(e^{\frac{1}{\beta_i}}, \infty)}(t_i) \right) \mathrm{d}t_i \right) \mathrm{d}x_i$$

$$= \prod_{i=1}^n \exp\left(\frac{1+\eta_i}{\beta_i}\right) \left(\frac{\gamma_i}{(1+\eta_i)(\gamma_i - (1+\eta_i))} \right) \tag{4.26}$$

It follows from (4.23) with best constant C, (4.25) and (4.26) that

$$\prod_{i=1}^n \frac{\exp\left(\frac{\gamma_i}{\beta_i}\right)}{\left(\frac{1+\eta_i}{\gamma_i} + \left(1 - \frac{1+\eta_i}{\gamma_i}\right) \exp\left(\frac{\gamma_i}{\beta_i}\right) \right)} \leq C.$$

By letting $\gamma_i \to (1+\eta_i)^+$ $(i \in J_n)$, we find that

$$\exp\left(\sum_{i=1}^n \frac{1+\eta_i}{\beta_i} \right) = \prod_{i=1}^n \exp \frac{1+\eta_i}{\beta_i} \leq C. \tag{4.27}$$

Therefore, the sharpness of the constant in (4.23) is proved by just combining (4.24) and (4.27). The proof is complete. $\square$

Next we consider the special case $\beta_i = a$ and $\gamma_i = c$ $(i \in J_n)$. First we point out the following immediate consequence of Theorem 4.4:

Corollary 4.1. Let $n \in \mathbb{Z}_+$, let $a > 0$, $c > -1$, and let f be positive and measurable function defined on $\mathbb{R}_+^n$. Then, the inequality

$$\int_{\mathbb{R}_+^n} \exp\left(a^n \prod_{i=1}^n x_i^{-a} \int_{\mathbf{0}}^{\mathbf{x}} \prod_{i=1}^n t_i^{a-1} \ln f(\mathbf{t}) \mathrm{d}\mathbf{t} \right) \prod_{i=1}^n x_i^c \, \mathrm{d}\mathbf{x}$$

$$\leq \exp\left(n \frac{1+c}{a} \right) \int_{\mathbb{R}_+^n} f(\mathbf{x}) \prod_{i=1}^n x_i^c \, \mathrm{d}\mathbf{x}$$

holds and the constant $\exp\left(n \frac{1+c}{a} \right)$ is sharp.

But Theorem 4.4 also implies the following less obvious weighted multidimensional generalization of the Cochran-Lee inequality:

Corollary 4.2. Let $n \in \mathbb{Z}_+$, let $a > 0, c \in \mathbb{R}$ and $k \in \mathbb{N}$ such that $ck > -1$, and let f be a positive and measurable function defined on $\mathbb{R}^n_+$. Then, the inequality

$$\begin{aligned}\int_{\mathbb{R}^n_+} \exp\left(a^n \prod_{i=1}^n x_i^{-a} \int_{\mathbf{0}}^{\mathbf{x}} \prod_{i=1}^n t_i^{a-1} \ln f(\mathbf{t})\, \mathrm{d}\mathbf{t}\right) \left(\sum_{i=1}^n x_i^c\right)^k \mathrm{d}\mathbf{x} \\ \leq \exp\left(\frac{n+ck}{a}\right) \int_{\mathbb{R}^n_+} f(\mathbf{x}) \left(\sum_{i=1}^n x_i^c\right)^k \mathrm{d}\mathbf{x}\end{aligned} \tag{4.28}$$

holds and the constant $\exp\left(\frac{n+ck}{a}\right)$ is sharp.

Proof. In view of Theorem 4.4 and by applying the multinomial theorem twice, we have that

$$\begin{aligned}&\int_{\mathbb{R}^n_+} \exp\left(a^n \prod_{i=1}^n x_i^{-a} \int_{\mathbf{0}}^{\mathbf{x}} \prod_{i=1}^n t_i^{a-1} \ln f(\mathbf{t})\, \mathrm{d}\mathbf{t}\right) \left(\sum_{i=1}^n x_i^c\right)^k \mathrm{d}\mathbf{x} \\ &= \sum_{\substack{m_1+\cdots+m_n=k \\ m_i \in \mathbb{N}_0}} \binom{k}{m_1, \ldots, m_n} \int_{\mathbb{R}^n_+} \exp\left(a^n \prod_{i=1}^n x_i^{-a} \int_{\mathbf{0}}^{\mathbf{x}} \prod_{i=1}^n t_i^{a-1} \ln f(\mathbf{t})\, \mathrm{d}\mathbf{t}\right) \prod_{i=1}^n x_i^{cm_i}\, \mathrm{d}\mathbf{x} \\ &\leq \sum_{\substack{m_1+\cdots+m_n=k \\ m_i \in \mathbb{N}_0}} \binom{k}{m_1, \ldots, m_n} \exp\left(\sum_{i=1}^n \frac{1+cm_i}{a}\right) \int_{\mathbb{R}^n_+} f(\mathbf{x}) \prod_{i=1}^n x_i^{cm_i}\, \mathrm{d}\mathbf{x} \\ &= \exp\left(\frac{n+ck}{a}\right) \int_{\mathbb{R}^n_+} f(\mathbf{x}) \sum_{\substack{m_1+\cdots+m_n=k \\ m_i \in \mathbb{N}_0}} \binom{k}{m_1, \ldots, m_n} \prod_{i=1}^n x_i^{cm_i}\, \mathrm{d}\mathbf{x} \\ &= \exp\left(\frac{n+ck}{a}\right) \int_{\mathbb{R}^n_+} f(\mathbf{x}) \left(\sum_{i=1}^n x_i^c\right)^k \mathrm{d}\mathbf{x},\end{aligned}$$

where

$$\binom{k}{m_1, \ldots, m_n} = \frac{k!}{m_1! \cdots m_n!}$$

is a multinomial coefficient. Since the constant in Theorem 4.4 is sharp, then the sharpness of the constant $\exp\left(\frac{n+ck}{a}\right)$ in (4.28) is guaranteed so the proof is complete. □

We conclude this section by pointing out that the fact that Theorem 4.4 also implies the following multidimensional Cochran-Lee type inequality, which, in particular, generalizes a result in [59] (see Example 4.1 and Remark 4.7):

Corollary 4.3. Let $n \in \mathbb{Z}_+$, let $\beta_i > 0$ $(i \in J_n)$, let f be positive and measurable functions on $\mathbb{R}^n_+$ and let the weights $u(x)$ and $v(x)$ be related by $u(x) = (G_\beta v)(x)$. Then, the inequality

$$\int_{\mathbb{R}^n_+} [(G_{\boldsymbol{\beta}} f)(\mathbf{x})]\, u(\mathbf{x})\, \mathrm{d}\mathbf{x} \leq \exp\left(\sum_{i=1}^n \frac{1}{\beta_i}\right) \int_{\mathbb{R}^n_+} f(\mathbf{x}) v(\mathbf{x})\, \mathrm{d}\mathbf{x} \tag{4.29}$$

holds and the constant $\exp\left(\sum_{i=1}^n \frac{1}{\beta_i}\right)$ is sharp.

Proof. Let $g(\mathbf{x}) = f(\mathbf{x})v(\mathbf{x})$. Then, the inequality (4.29) is equivalent to

$$\int_{\mathbb{R}_+^n} \left[\left(G_{\boldsymbol{\beta}} g \right) (\mathbf{x}) \right] \mathrm{d}\mathbf{x} \leq \exp \left(\sum_{i=1}^{n} \frac{1}{\beta_i} \right) \int_{\mathbb{R}_+^n} g(\mathbf{x}) \, \mathrm{d}\mathbf{x}. \tag{4.30}$$

In view of Theorem 4.4 with $\eta_i = 0$ $(i = 1, ..., n)$, we have that indeed the inequality (4.30) holds and the constant $\exp\left(\sum\limits_{i=1}^{n} \frac{1}{\beta_i}\right)$ is sharp. Therefore, from the equivalence of (4.29) and (4.30), we can conclude that (4.29) holds and the constant $\exp\left(\sum\limits_{i=1}^{n} \frac{1}{\beta_i}\right)$ is sharp. The proof is complete. □

4.1.4 Concluding Remarks and Results

Remark 4.3. Some authors referred early to the inequality (1.8) with $\alpha = 0$ as Knopp's inequality with reference to [63] but it was later on discovered that G. H. Hardy in his famous paper 1925 paper [49] mentioned that his friend G. Pólya had pointed out to him that this inequality is just a limit case of his original inequality. By applying our results in Sections 4.1.2 and 4.1.3 with $\beta_i = 1, i = 1, ..., n$, we obtain as special cases most of us known multidimensional Pólya-Knopp's inequalities, and especially all concerning sharp constants, see especially the recent paper [132], and the references therein.

Remark 4.4. Let $0 < p \leq q < \infty$. If the condition (4.22) in Proposition 4.1 holds, then the best possible constant C in (4.21) satisfies

$$C \leq \exp \left(\frac{n}{q} - \frac{n}{p} \right) \prod_{i=1}^{n} \beta_i^{\frac{1}{p} - \frac{1}{q}} \exp \left(\frac{1 + \gamma_i}{p \beta_i} \right). \tag{4.31}$$

In particular, if $p = q$, $0 < p < \infty$, and $b_i = \infty$ $(i = 1, \dots, n)$, then by replacing $f(x)$ by $f^p(x)$ in Theorem 4.4, we obtain that the inequality

$$\left(\int_{\mathbb{R}_+^n} \left[\left(G_{\boldsymbol{\beta}} f \right) (\mathbf{x}) \right]^p \prod_{i=1}^{n} x_i^{\eta_i} \, \mathrm{d}\mathbf{x} \right)^{\frac{1}{p}} \leq C \left(\int_{\mathbb{R}_+^n} f^p(\mathbf{x}) \prod_{i=1}^{n} x_i^{\eta_i} \, \mathrm{d}\mathbf{x} \right)^{\frac{1}{p}}, \tag{4.32}$$

holds with the sharp constant $C = \prod\limits_{i=1}^{n} \exp\left(\frac{1+\gamma_i}{p\beta_i}\right)$. Hence, (4.32) is a formal generalization of (4.23) which is the case $p = 1$.

Open Question 4.1. Find the sharp constant in the inequality (4.21) for the case $0 < p < q < \infty$.

Remark 4.5. We believe that this open question is not so easy to solve. Our motivation for that is that the corresponding question in the theory of Hardy-type inequalities (with our geometric mean operator G_β replaced by the corresponding Hardy arithmetic mean operator H) was an especially long lasting question even in the one dimensional case. It was finally solved only in 2015 in the paper [105] by L. E. Persson and S. Samko.

Remark 4.6. For $\beta = 1$ an one dimensional analogue of the estimate (4.21) in Proposition 4.1 was stated in [108, Example on page 744]. However, the inequality (4.31) gives a better estimate of the sharp constant than that in [106].

The next result follows from Theorem 4.4 and, in particular, it generalizes the result in [75, Theorem C]:

Corollary 4.4. Let $\beta_i > 0$, $\eta_i < -1 (i = 1 \in J_n)$, and let f be a positive measurable function defined on $\mathbb{R}^n_+$. Then, the inequality

$$\int_{\mathbb{R}^n_+} \exp\left(\prod_{i=1}^n \beta_i x_i^{\beta_i} \int_{\mathbf{x}}^{\infty} \prod_{i=1}^n t_i^{-(\beta_i+1)} \ln f(\mathbf{t})\,\mathrm{d}\mathbf{t}\right) \prod_{i=1}^n x_i^{\eta_i}\,\mathrm{d}\mathbf{x} \le C \int_{\mathbb{R}^n_+} f(\mathbf{x}) \prod_{i=1}^n x_i^{\eta_i}\,\mathrm{d}\mathbf{x} \tag{4.33}$$

holds for some finite $C > 0$ and the constant $C = \exp\left(\sum_{i=1}^n \frac{-(1+\eta_i)}{\beta_i}\right)$ is sharp.

Proof. By using the substitutions first $y_i = 1/t_i$, and then $z_i = 1/x_i$ $(i \in J_n)$ and $g(\mathbf{t}) = f(\mathbf{1}/\mathbf{t})$, elementary calculations show that (4.33) is equivalent to the inequality

$$\int_{\mathbb{R}^n_+} \exp\left(\prod_{i=1}^n \beta_i z_i^{-\beta_i} \int_{\mathbf{0}}^{\mathbf{z}} \prod_{i=1}^n y_i^{(\beta_i-1)} \ln g(\mathbf{y})\,\mathrm{d}\mathbf{y}\right) \prod_{i=1}^n z_i^{-\eta_i-2}\,\mathrm{d}\mathbf{z} \le C \int_{\mathbb{R}^n_+} g(\mathbf{z}) \prod_{i=1}^n z_i^{-\eta_i-2}\,\mathrm{d}\mathbf{z}. \tag{4.34}$$

In view of Theorem 4.4, the inequality 4.34 holds with sharp constant

$$C = \exp\left(\sum_{i=1}^n \frac{-(1+\eta_i)}{\beta_i}\right).$$

Therefore, from the equivalence of (4.33) and (4.34), we can conclude that (4.33) holds and the constant $C = \exp\left(\sum_{i=1}^n \frac{-(1+\eta_i)}{\beta_i}\right)$ is sharp. The proof is complete. □

In order to relate our results to another result in the literature (see Remark 4.7) we present the following consequence of Corollary 4.3:

Example 4.1. Let $n \in \mathbb{Z}_+$, let $\beta_i > 0$, and let η_i and γ_i be real numbers such that $\beta_i + \gamma_i > 0$ $(i \in J_n)$. Then, the inequality

$$\int_{\mathbb{R}^n_+} \left[\left(G_{\boldsymbol{\beta}} f\right)(\mathbf{x})\right] \exp\left(\sum_{i=1}^n \frac{\beta_i \eta_i}{\beta_i + \gamma_i} x_i^{\gamma_i}\right) \mathrm{d}\mathbf{x} \le \exp\left(\sum_{i=1}^n \frac{1}{\beta_i}\right) \int_{\mathbb{R}^n_+} f(\mathbf{x}) \exp\left(\sum_{i=1}^n \eta_i x_i^{\gamma_i}\right) \mathrm{d}\mathbf{x}$$

holds and the constant $\exp\left(\sum_{i=1}^n \frac{1}{\beta_i}\right)$ is sharp.

Remark 4.7. The one-dimensional analogue of Example 4.1 was discussed in [59, Corollary 1.6] for the case $\beta_i = \gamma_i = 1$ $(i = 1, \ldots, n)$ but without the estimate of the sharp constant.

Next we note that it is possible to derive also reversed Cochran-Lee type inequalities on the cone of non-increasing functions. This fact follows from the following elementary fact:

If a function f, defined on $\mathbb{R}^n_+$, is nonnegative and nonincreasing in all the variables, then

$$\prod_{i=1}^{n} \beta_i x_i^{-\beta_i} \int_{\mathbf{0}}^{\mathbf{x}} \prod_{i=1}^{n} t_i^{\beta_i - 1} f(t)\,\mathrm{d}t \geq f(\mathbf{x}), \tag{4.35}$$

where $\beta_i > 0$ $(i = 1, ..., n)$.

Remark 4.8. Several of the results in this paper can be given also in the reversed direction on the cone of nondecreasing functions but in this case it is not always clear that the constant 1 is sharp. Next, we present such a reversed Cochran-Lee inequality where indeed the constant is sharp.

Proposition 4.2. Let $n \in \mathbb{Z}_+$, let $\beta_i > 0$ and $\eta_i > -1$ $(i = 1, \ldots, n)$. Then, the inequality

$$\begin{aligned} \int_{\mathbb{R}^n_+} \exp\left(\prod_{i=1}^{n} \beta_i x_i^{-\beta_i} \int_{\mathbf{0}}^{\mathbf{x}} \prod_{i=1}^{n} t_i^{\beta_i - 1} \ln f(t)\mathrm{d}t\right) \prod_{i=1}^{n} x_i^{\eta_i}\,\mathrm{d}\mathbf{x} \\ \geq 1 \cdot \int_{\mathbb{R}^n_+} f(\mathbf{x}) \prod_{i=1}^{n} x_i^{\eta_i}\,\mathrm{d}\mathbf{x} \end{aligned} \tag{4.36}$$

holds for all nonnegative and non-increasing functions f defined on $\mathbb{R}^n_+$ and the constant 1 is sharp.

Proof. Clearly, (4.35) implies (4.36). The sharpness follows by considering the test function $f_{\boldsymbol{\delta}}$, defined as

$$f_{\boldsymbol{\delta}}(\mathbf{x}) = \prod_{i=1}^{n} \left(\chi_{[0,1]}(x_i) + e^{-\frac{\delta_i}{\beta_i}} x_i^{-\delta_i} \chi_{(1,\infty)}(x_i)\right),$$

where $\delta_i > 1$ and letting $\delta_i \to \infty$ $(i = 1, \ldots, n)$. We omit the details. □

Remark 4.9. The one and two-dimensional analogues of inequality (4.36) with $\beta_i = 1$ $(i = 1, \ldots, n)$ are given in [56, Example 5.1] and [54], respectively.

Next, we prove that inequalities (4.3) and inequality (10) in [132] are equivalent. This equivalence in one and two dimensions were proved in [56] and [54], respectively.

Proposition 4.3. Let $0 < p \leq q < \infty$, $0 < b_i \leq \infty, \beta_i > 0$ $(i = 1, ..., n)$, and let u, v and f be positive functions on $\mathbb{R}^n_+$. Then, the inequality

$$\begin{aligned} \left(\int_{\mathbf{0}}^{\mathbf{b}} \left[\exp\left(\prod_{i=1}^{n} \beta_i x_i^{-\beta_i} \int_{\mathbf{0}}^{\mathbf{x}} \prod_{i=1}^{n} t_i^{\beta_i - 1} \ln f(\mathbf{t})\,\mathrm{d}\mathbf{t}\right)\right]^q u(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}} \\ \leq C \left(\int_{\mathbf{0}}^{\mathbf{b}} f^p(\mathbf{x}) v(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{p}} \end{aligned} \tag{4.37}$$

is equivalent to the inequality

$$\left(\int_{\mathbf{0}}^{\mathbf{b}^{\boldsymbol{\beta}}}\left[\exp\left(\prod_{i=1}^{n}x_i^{-1}\int_{\mathbf{0}}^{\mathbf{x}}\ln g(\mathbf{t})\,\mathrm{d}\mathbf{t}\right)\right]^q u_{\boldsymbol{\beta}}(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}} \le C\left(\int_{\mathbf{0}}^{\mathbf{b}^{\boldsymbol{\beta}}} g^p(\mathbf{x})v_{\boldsymbol{\beta}}(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{p}}, \tag{4.38}$$

where C is a finite constant,

$$u_{\boldsymbol{\beta}}(\mathbf{x}) = u(\mathbf{x}^{\frac{1}{\boldsymbol{\beta}}})\prod_{i=1}^{n}\frac{x_i^{\frac{1-\beta_i}{\beta_i}}}{\beta_i},\;\; v_{\boldsymbol{\beta}}(\mathbf{x}) = v(\mathbf{x}^{\frac{1}{\boldsymbol{\beta}}})\prod_{i=1}^{n}\frac{x_i^{\frac{1-\beta_i}{\beta_i}}}{\beta_i},$$

and

$$g(\mathbf{x}) = f(\mathbf{x}^{\frac{1}{\boldsymbol{\beta}}}).$$

Proof. By making the substitution $z_i = x_i^{\beta_i}$ $(i = 1, \ldots, n)$, we find that the inequality (4.37) is equivalent to

$$\left(\int_{\mathbf{0}}^{\mathbf{b}^{\boldsymbol{\beta}}}\left[\exp\left(\prod_{i=1}^{n}\beta_i z_i^{-1}\int_{\mathbf{0}}^{\mathbf{z}^{\frac{1}{\boldsymbol{\beta}}}}\prod_{i=1}^{n}t_i^{\beta_i-1}\ln f(\mathbf{t})\,\mathrm{d}\mathbf{t}\right)\right]^q u(\mathbf{z}^{\frac{1}{\boldsymbol{\beta}}})\prod_{i=1}^{n}\frac{z_i^{\frac{1-\beta_i}{\beta_i}}}{\beta_i}\,\mathrm{d}\mathbf{z}\right)^{\frac{1}{q}} \le C\left(\int_{\mathbf{0}}^{\mathbf{b}^{\boldsymbol{\beta}}} f^p(\mathbf{z}^{\frac{1}{\boldsymbol{\beta}}})v(\mathbf{z}^{\frac{1}{\boldsymbol{\beta}}})\prod_{i=1}^{n}\frac{z_i^{\frac{1-\beta_i}{\beta_i}}}{\beta_i}\,\mathrm{d}\mathbf{z}\right)^{\frac{1}{p}}. \tag{4.39}$$

Moreover, by making the variable transformation $t_i = y_i^{1/\beta_i}$, $(i = 1, \ldots, n)$ in (4.39) we can conclude that (4.37) is equivalent to (4.38). The proof is complete. □

Our final remark is related to a fairly new development in the theory of Hardy-type inequalities, namely the following: In the classical situation a Hardy-type inequality was usually characterized by using one condition (e.g. the famous Muckenhoupt condition for the case $1 < p \le q < \infty$). However, it was discovered that this condition is not unique and can be replaced by infinite many equivalent conditions, even by scales of conditions. See [73, Section 7.3]. However, for multidimensional Hardy-type inequalities no such scales of characterizing conditions are known. But correctly interpreted our proof of Theorem 4.2 shows that we indeed have such scale of conditions to characterize the multidimensional limit Hardy-type inequality (4.3). Indeed, Theorem 4.2 can be reformulated as follows:

Theorem 4.5. Let $n \in \mathbb{Z}_+$, let $0 < p \le q < \infty, \beta_i > 0$ $(i = 1, ..., n)$ and let u, v and f be positive and measurable functions. Then the inequality (4.3) holds if and only if any condition on the scale of conditions ($0 < \alpha_i < \infty$) $(i = 1, ..., n)$

$$A_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) := \sup_{\substack{t_i\in(0,b_i)\\ i\in J_n}}\prod_{i=1}^{n}t_i^{\frac{\alpha_i+\beta_i-1}{p}}\left(\int_{\mathbf{t}}^{\mathbf{b}}\prod_{i=1}^{n}x_i^{-(\alpha_i+\beta_i)\frac{q}{p}}w(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}} < \infty,$$

holds. Moreover, the sharp constant in (4.3) can be estimated as follows:

$$\sup_{\substack{\alpha_i>0\\ i\in J_n}} \prod_{i=1}^{n} \left(\frac{(\beta_i+\alpha_i-1)\exp\left(1+\frac{\alpha_i}{\beta_i}\right)}{1+(\beta_i+\alpha_i-1)\exp\left(1+\frac{\alpha_i}{\beta_i}\right)} \right)^{\frac{1}{p}} A_{\boldsymbol{\beta}}(\boldsymbol{\alpha})$$
$$\leq C \leq \inf_{\substack{\alpha_i>0\\ i\in J_n}} \prod_{i=1}^{n} \left(\beta_i \exp\frac{\alpha_i}{\beta_i} \right)^{\frac{1}{p}} A_{\boldsymbol{\beta}}(\boldsymbol{\alpha}).$$

4.2 Pólya-Knopp type inequalities on homogeneous groups

4.2.1 Introduction

For the purpose of this section we need to mention the following characterization of the general weight functions u and v on a homogeneous group for a maximal integral weighted Hardy inequality to hold (see [113, Theorem 5.4.1] and [114, Theorem 5.1]):

Theorem 4.6 (Maximal integral weighted Hardy inequality)**.** Let $\mathbb{G}$ be a homogeneous group with the homogeneous dimension Q equipped with a homogeneous quasi-norm $|\cdot|$. Let u and v be positive functions defined on $\mathbb{G}$. Then there exists a constant $C>0$ such that

$$\int_{\mathbb{G}} \exp\left(\frac{1}{|B(0,|x|)|}\int_{B(0,|x|)} \ln f(y)\,\mathrm{d}y\right) u(x)\,\mathrm{d}x \leq C\int_{\mathbb{G}} f(x)v(x)\,\mathrm{d}x,$$

holds for all $f\geq 0$ if and only if

$$A := \sup_{R>0} R^Q \int_{|x|\geq R} \frac{u(x)}{|x|^{2Q}} \exp\left(\frac{1}{|B(0,|x|)|}\int_{B(0,|x|)} \ln\left(\frac{1}{v(y)}\right)\mathrm{d}y\right)\mathrm{d}x < \infty.$$

Remark 4.10. Inequalities of the type as those in Theorem 4.6 when $\mathbb{G}$ is replaced by

$$I_n = [0,b_1)\times\cdots[0,b_n)\subseteq \mathbb{R}^n_+,$$

with $0<b_i\leq\infty$, $i=1,...,n$ and the means are considered over a hyperrectangle, were studied in [132] for the multidimensional case with $0<p\leq q<\infty$. Moreover, estimates of the sharp constant were also discussed (see also [54], [128] and the references therein).

The main purpose of this section is to study the generalized Pólya-Knopp type inequality (or Maximal weighted integral Hardy inequality) on homogeneous group $\mathbb{G}$ for the case $0<p\leq q<\infty$.

4.2.2 General Pólya-Knopp type inequalities on homogeneous groups

Our main theorem in this section reads as follows.

Theorem 4.7. Let $\mathbb{G}$ be a homogeneous group with the homogeneous dimension Q equipped with a quasi-norm $|\cdot|$ and let $0 < p \leq q < \infty$. Suppose that u and v are positive functions on $\mathbb{G}$. Then

$$\begin{aligned}\left(\int_{\mathbb{G}}\left[\exp\left(\frac{1}{|B(0,|x|)|}\int_{B(0,|x|)}\ln f(y)\,\mathrm{d}y\right)\right]^{q}u(x)\,\mathrm{d}x\right)^{\frac{1}{q}}&\\ \leq C\left(\int_{\mathbb{G}}f^{p}(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p}}&\end{aligned}\tag{4.40}$$

holds for some finite C and for all positive functions f if and only if for some $\alpha > 0$

$$A(\alpha) := \sup_{|\xi|>0}|\xi|^{\frac{\alpha}{p}}\left(\int_{\mathbb{G}\setminus B(0,|\xi|)}|x|^{-(Q+\alpha)\frac{q}{p}}w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} < \infty,\tag{4.41}$$

where

$$w(x) = u(x)\left[\exp\left(\frac{1}{|B(0,|x|)|}\int_{B(0,|x|)}\ln v^{-1}(y)\,\mathrm{d}y\right)\right]^{\frac{q}{p}}.\tag{4.42}$$

Moreover, the best constant C in (4.40) satisfies

$$\begin{aligned}&\left(\frac{1}{|B(0,1)|}\right)^{1/p}\sup_{\alpha>0}A(\alpha)\left(1+\frac{e^{-(1+\frac{\alpha}{Q})}}{\frac{\alpha}{Q}}\right)^{-1/p}\\ &\leq C \leq \left(\frac{1}{|B(0,1)|}\right)^{1/p}\inf_{\alpha>0}A(\alpha)\exp\left(\frac{\alpha}{pQ}\right).\end{aligned}\tag{4.43}$$

Furthermore, if $A(\alpha) < \infty$ for some $\alpha > 0$, then $A(\alpha) < \infty$ for all $\alpha > 0$.

Proof. Sufficiency. Let $g(x) = f^p(x)v(x)$. Then the inequality (4.40) is equivalent to

$$\begin{aligned}\left(\int_{\mathbb{G}}\left[\exp\left(\frac{1}{|B(0,|x|)|}\int_{B(0,|x|)}\ln g(y)\,\mathrm{d}y\right)\right]^{\frac{q}{p}}w(x)\,\mathrm{d}x\right)^{\frac{1}{q}}&\\ \leq C\left(\int_{\mathbb{G}}g(x)\,\mathrm{d}x\right)^{\frac{1}{p}},&\end{aligned}\tag{4.44}$$

where $w(x)$ is defined by (4.42). Clearly,

$$\exp\left(\frac{1}{|B(0,|x|)|}\int_{B(0,|x|)}\ln g(y)\,\mathrm{d}y\right) = \exp\left(\frac{1}{|B(0,1)|}\int_{B(0,1)}\ln g(|x|\xi)\,\mathrm{d}\xi\right),\tag{4.45}$$

and for any $\alpha > 0$, we trivially have that

$$\exp\left(\frac{\alpha}{Q}\right)\exp\left(\frac{1}{|B(0,1)|}\int_{B(0,1)}\ln(|\xi|^{\alpha})\,\mathrm{d}\xi\right) = 1.\tag{4.46}$$

We apply (4.45) and the identity (4.46) in the left hand side of (4.44), and then, by Jensen's inequality, the left hand side of (4.44) becomes

$$\exp\left(\frac{\alpha}{pQ}\right)\left(\int_{\mathbb{G}}\left[\exp\left(\frac{1}{|B(0,1)|}\int_{B(0,1)}\ln\left(|\xi|^{\alpha}g(|x|\xi)\right)\,\mathrm{d}\xi\right)\right]^{\frac{q}{p}}w(x)\,\mathrm{d}x\right)^{\frac{1}{q}}$$

$$\leq\left(\frac{\exp\left(\frac{\alpha}{Q}\right)}{|B(0,1)|}\right)^{1/p}\left(\int_{\mathbb{G}}w(x)\left(\int_{B(0,1)}|\xi|^{\alpha}g(|x|\xi)\,\mathrm{d}\xi\right)^{\frac{q}{p}}\mathrm{d}x\right)^{\frac{1}{q}}$$

$$=\left(\frac{\exp\left(\frac{\alpha}{Q}\right)}{|B(0,1)|}\right)^{1/p}\left(\int_{\mathbb{G}}|x|^{-(Q+\alpha)\frac{q}{p}}w(x)\left(\int_{B(0,|x|)}|\xi|^{\alpha}g(\xi)\,\mathrm{d}\xi\right)^{\frac{q}{p}}\mathrm{d}x\right)^{\frac{1}{q}}.$$

Therefore, by using Minkowski's integral inequality when $p<q$ and Fubini's theorem when $p=q$, the later expression is less than or equal to

$$\left(\frac{\exp\left(\frac{\alpha}{Q}\right)}{|B(0,1)|}\right)^{1/p}\left(\int_{\mathbb{G}}g(\xi)\cdot|\xi|^{\alpha}\left(\int_{\mathbb{G}\backslash B(0,|\xi|)}|x|^{-(Q+\alpha)\frac{q}{p}}w(x)\,\mathrm{d}x\right)^{\frac{p}{q}}\mathrm{d}\xi\right)^{\frac{1}{p}}$$
$$\leq\left(\frac{1}{|B(0,1)|}\right)^{1/p}\exp\left(\frac{\alpha}{pQ}\right)A(\alpha)\left(\int_{\mathbb{G}}g(x)\,\mathrm{d}x\right)^{\frac{1}{p}},\tag{4.47}$$

so that (4.44) follows from (4.41) and (4.47). Since (4.40) is equivalent to (4.44), we conclude that (4.40) holds and the best constant C satisfies

$$C\leq\left(\frac{1}{|B(0,1)|}\right)^{1/p}\inf_{\alpha>0}A(\alpha)\exp\left(\frac{\alpha}{pQ}\right).$$

Necessity. To prove that (4.40), or equivalently (4.44), implies (4.41), we define the test function g on $\mathbb{G}$ by

$$g(x)=|\xi|^{-Q}\chi_{[0,|\xi|]}(|x|)+e^{-\left(\frac{Q+\alpha}{Q}\right)}|\xi|^{\alpha}|x|^{-(Q+\alpha)}\chi_{(|\xi|,\infty)}(|x|),$$

for fixed $|\xi|>0$. Then, the right hand side of (4.44) becomes

$$\left(\int_{\mathbb{G}}g(x)\,\mathrm{d}x\right)^{1/p}=|B(0,1)|^{1/p}\left(1+\frac{e^{-(1+\frac{\alpha}{Q})}}{\frac{\alpha}{Q}}\right)^{1/p}.\tag{4.48}$$

On the other hand, for $|\xi|>0$, we have that

$$\left(\int_{\mathbb{G}\backslash B(0,|\xi|)}\left[\exp\left(\frac{1}{|B(0,|x|)|}\int_{B(0,|x|)}\ln g(y)\mathrm{d}y\right)\right]^{\frac{q}{p}}w(x)\,\mathrm{d}x\right)^{\frac{1}{q}}$$
$$\leq\left(\int_{\mathbb{G}}\left[\exp\left(\frac{1}{|B(0,|x|)|}\int_{B(0,|x|)}\ln g(y)\,\mathrm{d}y\right)\right]^{\frac{q}{p}}w(x)\,\mathrm{d}x\right)^{\frac{1}{q}}.\tag{4.49}$$

Moreover, for $0 < |\xi| < |x|$,

$$\int_{B(0,|x|)} \ln g(y)\mathrm{d}y = |\mathfrak{S}| \int_0^{|x|} r^{Q-1} \ln \left(|\xi|^{-Q} \chi_{[0,|\xi|]}(r) + e^{-(\frac{Q+\alpha}{Q})} \frac{|\xi|^\alpha}{r^{(Q+\alpha)}} \chi_{(|\xi|,\infty)}(r) \right) \mathrm{d}r$$
$$= |B(0,|x|)| \ln \left(|\xi|^\alpha |x|^{-(Q+\alpha)} \right). \tag{4.50}$$

It follows from (4.44) and (4.48) – (4.50) that

$$|\xi|^{\frac{\alpha}{p}} \left(\int_{\mathbb{G}\setminus B(0,|\xi|)} |x|^{-(Q+\alpha)\frac{q}{p}} w(x)\mathrm{d}x \right)^{\frac{1}{q}}$$
$$= \left(\int_{\mathbb{G}\setminus B(0,|\xi|)} \left[\exp\left(\frac{1}{|B(0,|x|)|} \int_{B(0,|x|)} \ln g(y)\mathrm{d}y \right) \right]^{\frac{q}{p}} w(x)\mathrm{d}x \right)^{\frac{1}{q}}$$
$$\leq C|B(0,1)|^{1/p} \left(1 + \frac{e^{-(1+\frac{\alpha}{Q})}}{\frac{\alpha}{Q}} \right)^{1/p}.$$

Therefore,

$$A(\alpha) = \sup_{|\xi|>0} |\xi|^{\frac{\alpha}{p}} \left(\int_{\mathbb{G}\setminus B(0,|\xi|)} |x|^{-(Q+\alpha)\frac{q}{p}} w(x)\mathrm{d}x \right)^{\frac{1}{q}}$$
$$\leq C|B(0,1)|^{1/p} \left(1 + \frac{e^{-(1+\frac{\alpha}{Q})}}{\frac{\alpha}{Q}} \right)^{1/p} < \infty.$$

We conclude that (4.41) holds and that the sharp constant C satisfies (4.43). The proof is complete. ☐

Theorem 4.7 implies the following inequality for power weights:

Example 4.2. Let $\mathbb{G}$ be a homogeneous group with the homogeneous dimension Q equipped with a quasi-norm $|\cdot|$. Let $0 < p \leq q < \infty$ and $a, b \in \mathbb{R}$. Then the inequality

$$\left(\int_{\mathbb{G}} \left[\exp\left(\frac{1}{|B(0,|x|)|} \int_{B(0,|x|)} \ln f(y)\, \mathrm{d}y \right) \right]^q |B(0,|x|)|^a \, \mathrm{d}x \right)^{\frac{1}{q}}$$
$$\leq C \left(\int_{\mathbb{G}} f^p(x) |B(0,|x|)|^b \, \mathrm{d}x \right)^{\frac{1}{p}} \tag{4.51}$$

holds for all positive functions f if and only if

$$\frac{1+a}{q} = \frac{1+b}{p}. \tag{4.52}$$

Moreover, the best constant C in (4.51) satisfies

$$\left(\frac{p}{q}\right)^{1/q} \exp\left(\frac{b}{p}\right) \sup_{\alpha>0} \frac{\left(1 + \frac{e^{-(1+\frac{\alpha}{Q})}}{\frac{\alpha}{Q}} \right)^{-1/p}}{\left(\frac{\alpha}{Q}\right)^{1/q}} \leq C \leq \exp\left(\frac{1}{q} + \frac{b}{p} \right). \tag{4.53}$$

This fact is obvious since in this case (4.41) is of the form

$$A(\alpha) = \left(\frac{p}{q}\right)^{1/q} |B(0,1)|^{\frac{1+a}{q}-\frac{b}{p}} \frac{\exp\left(\frac{b}{p}\right) \sup\limits_{|\xi|>0} |\xi|^{Q\left(\frac{1+a}{q}-\frac{1+b}{p}\right)}}{[(b+1)+\alpha/Q-(1+a)\frac{p}{q}]^{1/q}},$$

provided that $\alpha > Qp\left(\frac{1+a}{q} - \frac{1+b}{p}\right)$. Furthermore, if condition (4.52) is satisfied, then we have

$$A(\alpha) = \left(\frac{p}{q}\right)^{1/q} |B(0,1)|^{1/p} \frac{\exp\left(\frac{b}{p}\right)}{\left(\frac{\alpha}{Q}\right)^{1/q}}. \tag{4.54}$$

Consequently, from (4.43) and (4.54), we have

$$\begin{aligned} &\left(\frac{p}{q}\right)^{1/q} \exp\left(\frac{b}{p}\right) \sup_{\alpha>0} \frac{\left(1+\frac{e^{-(1+\frac{\alpha}{Q})}}{\frac{\alpha}{Q}}\right)^{-1/p}}{\left(\frac{\alpha}{Q}\right)^{1/q}} \\ &\leq C \leq \left(\frac{p}{q}\right)^{1/q} \exp\left(\frac{b}{p}\right) \left[\inf_{\alpha>0} \frac{\exp\left(\frac{q}{p}\frac{\alpha}{Q}\right)}{\left(\frac{\alpha}{Q}\right)}\right]^{1/q}. \end{aligned} \tag{4.55}$$

Moreover,

$$\inf_{\alpha>0} \frac{\exp\left(\frac{q}{p}\frac{\alpha}{Q}\right)}{\left(\frac{\alpha}{Q}\right)} = \frac{q}{p}e. \tag{4.56}$$

Therefore, (4.53) follows from (4.55) and (4.56).

4.2.3 Pólya-Knopp type inequalities on homogeneous groups with sharp constant

We state and prove the following particular result of Theorem (4.7):

Theorem 4.8. Let $\mathbb{G}$ be a homogeneous group with the homogeneous dimension Q equipped with a quasi-norm $|\cdot|$, and let $a \in \mathbb{R}$. Then the inequality

$$\begin{aligned} \int_{\mathbb{G}} \exp\left(\frac{1}{|B(0,|x|)|}\int_{B(0,|x|)} \ln f(y)\,\mathrm{d}y\right)|B(0,|x|)|^a\,\mathrm{d}x \\ \leq e^{(a+1)} \int_{\mathbb{G}} f(x)|B(0,|x|)|^a\,\mathrm{d}x \end{aligned} \tag{4.57}$$

holds for all positive functions f on $\mathbb{G}$, and the constant $e^{(a+1)}$ in (4.57) is sharp.

Proof. In view of Example 4.2, the inequality (4.57) holds. Now, we prove that the sharp constant $C = e^{(a+1)}$. From Theorem 4.7, in particular from inequality (4.43), it follows that the sharp constant C satisfies

$$C \leq e^a \inf_{\alpha>0} \frac{\exp(\alpha/Q)}{(\alpha/Q)}.$$

The infimum in the above inequality is attained at $\alpha = Q$. Hence,

$$C \leq e^{(1+a)}. \tag{4.58}$$

It only remains to prove that the inequality (4.58) also holds in the reverse direction. Consider the test function

$$f(x) = \chi_{[0,e^{1/Q}]}(|x|) + |x|^{-\beta}\chi_{(e^{1/Q},\infty)}(|x|),$$

where $\beta > Q(a+1)$. Then, the integral part of the right hand side of (4.57) becomes

$$\begin{aligned}\int_{\mathbb{G}} f(x)|B(0,|x|)|^a \, \mathrm{d}x &= \int_0^\infty \int_{\mathfrak{S}} r^{Q-1} f(ry)|B_r|^a \, \mathrm{d}\sigma(y)\mathrm{d}r \\ &= \frac{|\mathfrak{S}|^{a+1}}{Q^a} \int_0^\infty r^{Q(a+1)-1}\big(\chi_{[0,e^{1/Q}]}(r) + r^{-\beta}\chi_{(e^{1/Q},\infty)}(r)\big)\,\mathrm{d}r \\ &= |B(0,1)|^{a+1}e^{(a+1)}\left(\frac{1}{a+1} + \frac{e^{-\beta/Q}}{\beta/Q-(a+1)}\right),\end{aligned} \tag{4.59}$$

and the left hand side of (4.57) becomes

$$\begin{aligned}&\int_{\mathbb{G}} \exp\left(\frac{1}{|B(0,|x|)|}\int_{B(0,|x|)} \ln f(y)\,\mathrm{d}y\right)|B(0,|x|)|^a \, \mathrm{d}x \\ &= \int_0^\infty \int_{\mathfrak{S}} r^{Q-1} \exp\left(\frac{1}{|B(0,r)|}\int_{B(0,r)} \ln f(y)\,\mathrm{d}y\right)|B(0,r)|^a \mathrm{d}\sigma(y)\mathrm{d}r \\ &= \frac{|\mathfrak{S}|^{a+1}}{Q^a}\int_0^\infty r^{Q(a+1)-1}\exp\left(\frac{Q}{r^Q}\int_0^r s^{Q-1}\ln\left(\chi_{[0,e^{\frac{1}{Q}}]}(s) + \frac{1}{s^\beta}\chi_{(e^{\frac{1}{Q}},\infty)}(s)\right)\mathrm{d}s\right)\mathrm{d}r \\ &= |B(0,1)|^{a+1}e^{(a+1)}\left(\frac{1}{a+1} + \frac{1}{\beta/Q-(a+1)}\right).\end{aligned} \tag{4.60}$$

It follows from (4.57), (4.59) and (4.60) that

$$\frac{e^{\beta/Q}}{\frac{Q}{\beta}(a+1) + \left(1-\frac{Q}{\beta}(a+1)\right)e^{\beta/Q}} \leq C.$$

By letting $\frac{\beta}{Q} \to (a+1)^+$, we find that

$$e^{(a+1)} \leq C. \tag{4.61}$$

Therefore, the sharpness of the constant in (4.57) follows by combining (4.58) and (4.61). The proof is complete. ∎

4.2.4 Dual version

Theorem 4.9. Let $\mathbb{G}$ be a homogeneous group with the homogeneous dimension Q equipped with a quasi-norm $|\cdot|$, and let $0<p\leq q<\infty$. Suppose that u and v are positive functions on $\mathbb{G}$. Then

$$\left(\int_{\mathbb{G}}\left[\exp\left(|B(0,|x|)|\int_{\mathbb{G}\backslash B(0,|x|)}|B(0,|y|)|^{-2}\ln f(y)\mathrm{d}y\right)\right]^{q}u(x)\mathrm{d}x\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbb{G}}f^{p}(x)v(x)\mathrm{d}x\right)^{\frac{1}{p}} \tag{4.62}$$

holds for some finite C and for all positive functions f if and only if for some $\alpha>0$

$$\tilde{A}(\alpha):=\sup_{|\xi|>0}|\xi|^{\frac{\alpha}{p}}\left(\int_{\mathbb{G}\backslash B(0,|\xi|)}|x|^{-(Q+\alpha)\frac{q}{p}}\tilde{w}(x)\mathrm{d}x\right)^{\frac{1}{q}}<\infty, \tag{4.63}$$

where

$$\tilde{w}(x)=\tilde{u}(x)\left[\exp\left(\frac{1}{|B(0,|x|)|}\int_{B(0,|x|)}\ln\frac{1}{\tilde{v}(y)}\mathrm{d}y\right)\right]^{\frac{q}{p}}, \tag{4.64}$$

and

$$\tilde{u}(ry)=|ry|^{-2Q}u\left(\frac{1}{r}y\right) \text{ and } \tilde{v}(ry)=|ry|^{-2Q}v\left(\frac{1}{r}y\right) \tag{4.65}$$

for $r>0$. Moreover, the best constant C satisfies

$$\left(\frac{1}{|B(0,1)|}\right)^{1/p}\sup_{\alpha>0}\left(1+\frac{e^{-(\frac{Q+\alpha}{Q})}}{\frac{\alpha}{Q}}\right)^{-1/p}\tilde{A}(\alpha) \leq C\leq\left(\frac{1}{|B(0,1)|}\right)^{1/p}\inf_{\alpha>0}\exp\left(\frac{\alpha}{pQ}\right)\tilde{A}(\alpha). \tag{4.66}$$

Proof. By making variable transformations we have that

$$\int_{\mathbb{G}}\left[\exp\left(|B(0,|x|)|\int_{\mathbb{G}\backslash B(0,|x|)}|B(0,|y|)|^{-2}\ln f(y)\mathrm{d}y\right)\right]^{q}u(x)\mathrm{d}x = \int_{\mathbb{G}}\left[\exp\left(\frac{1}{|B(0,|x|)|}\int_{B(0,|x|)}\ln g(y)\mathrm{d}y\right)\right]^{q}\tilde{u}(x)\mathrm{d}x \tag{4.67}$$

and

$$\int_{\mathbb{G}}f^{p}(x)v(x)\mathrm{d}x=\int_{\mathbb{G}}g^{p}(x)\tilde{v}(x)\mathrm{d}x, \tag{4.68}$$

where $g(ry)=f\left(\frac{1}{r}y\right)$ for $r>0$ with $\tilde{u}(x)$ and $\tilde{v}(x)$ are defined by (4.65). It follows from (4.67) and (4.68) that, the inequality (4.62) is equivalent to the inequality

$$\left(\int_{\mathbb{G}}\left[\exp\left(\frac{1}{|B(0,|x|)|}\int_{B(0,|x|)}\ln g(y)\mathrm{d}y\right)\right]^{q}\tilde{u}(x)\mathrm{d}x\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbb{G}}g^{p}(x)\tilde{v}(x)\mathrm{d}x\right)^{\frac{1}{p}}, \tag{4.69}$$

where $g(ry) = f(\frac{1}{r}y)$ with $\tilde{u}(x)$ and $\tilde{v}(x)$ are defined by (4.65).

In view of Theorem 4.7, the inequality (4.69) holds for some finite C if and only if for any $\alpha > 0$,

$$\tilde{A}(\alpha) := \sup_{|\xi|>0} |\xi|^{\frac{\alpha}{p}} \left(\int_{\mathbb{G}\setminus B(0,|\xi|)} |x|^{-(Q+\alpha)\frac{q}{p}} \tilde{w}(x) \mathrm{d}x \right)^{\frac{1}{q}} < \infty,$$

where

$$\tilde{w}(x) = \tilde{u}(x) \left[\exp\left(\frac{1}{|B(0,|x|)|} \int_{B(0,|x|)} \ln \frac{1}{\tilde{v}(y)} \mathrm{d}y \right) \right]^{\frac{q}{p}}$$

with $\tilde{u}(x)$ and $\tilde{v}(x)$ are defined by (4.65). Since the inequality (4.62) is equivalent to (4.69) with $\tilde{u}(x)$ and $\tilde{v}(x)$ are defined by (4.65), we can, by Theorem 4.7, conclude that (4.62) holds if and only if (4.63)–(4.65) holds. Moreover, the sharp constant C in (4.62) satisfies the estimates (4.66). The proof is complete. $\square$

Theorem 4.10. Let $\mathbb{G}$ be a homogeneous group with the homogeneous dimension Q equipped with a quasi-norm $|\cdot|$, and let $a \in \mathbb{R}$. Suppose that u and v are positive functions on $\mathbb{G}$. Then

$$\begin{aligned} \int_{\mathbb{G}} \exp\left(|B(0,|x|)| \int_{\mathbb{G}\setminus B(0,|x|)} |B(0,|y|)|^{-2} \ln f(y) \mathrm{d}y \right) |B(0,|x|)|^{a} \mathrm{d}x \\ \leq e^{-(a+1)} \int_{\mathbb{G}} f(x) |B(0,|x|)|^{a} \mathrm{d}x \end{aligned} \tag{4.70}$$

holds for all positive functions f, and the constant $e^{-(a+1)}$ in (4.70) is sharp.

Proof. Using similar approach as we did in Theorem 4.9, we can show that (4.70) is equivalent to

$$\begin{aligned} \int_{\mathbb{G}} \exp\left(\frac{1}{|B(0,|x|)|} \int_{B(0,|x|)} \ln g(y) \mathrm{d}y \right) |B(0,|x|)|^{-(a+2)} \mathrm{d}x \\ \leq e^{-(a+1)} \int_{\mathbb{G}} g(y) |B(0,|x|)|^{-(a+2)} \mathrm{d}x. \end{aligned} \tag{4.71}$$

In view of Theorem 4.8, the inequality (4.71) holds with sharp constant $C = e^{-(a+1)}$. Therefore, from the equivalence of (4.70) and (4.71), we can conclude that (4.70) holds for some finite $C > 0$ and the constant $C = e^{-(a+1)}$ is sharp. $\square$

Chapter 5

Analysis of two-operator boundary-domain integral equations for variable-coefficient Dirichlet and Neumann problems in 2D with general right-hand side

5.1 Introduction

Partial differential equations (PDEs) with variable coefficients often arise in mathematical modelling of inhomogeneous media (e.g. functionally graded materials or materials with damage induced inhomogeneity) in solid mechanics, electromagnetics, thermal conductivity, fluid flows through porous media, and other areas of physics and engineering. Generally, explicit fundamental solutions are not available if the PDE coefficients are not constant, preventing reduction of boundary value problems (BVPs) for such PDEs to explicit boundary integral equations (BIEs), which could be effectively solved numerically. Nevertheless, for a rather wide class of variable-coefficient PDEs it is possible to use instead an explicit parametrix (Levi function) associated with the fundamental solution of the corresponding frozen-coefficient PDEs, and reduce BVPs for such PDEs to systems of boundary-domain integral equations (BDIEs) for further numerical solution of the latter, see for example [27], [29], [83], [85], [86]. Still this (one-operator) approach does not work when the fundamental solution of the frozen-coefficient PDE is not known explicitly (as e.g. in the Lamé system of anisotropic elasticity). To overcome this difficulty, one can apply the so-called two-operator approach, formulated in [84] for a certain nonlinear problem, that employs a parametrix of another (second) PDE, not related with the PDE in question, for reducing the BVP to a BDIE system. Since the second PDE is rather arbitrary, one can always choose it in such a way, that its parametrix is known explicitly. The simplest choice for the second PDE is the one with an explicit fundamental solution.

For a function from the Sobolev space $H^1(\Omega)$, a classical conormal derivative in the sense of traces may not exist [80, Appendix A]. However, when this function satisfies a second order PDE with a right-hand side from $H^{-1}(\Omega)$, the generalized conormal derivative can be defined in the weak sense, associated with the first Green identity and an extension of the PDE right-hand side to $\widetilde{H}^{-1}(\Omega)$, see [78, Lemma 4.3] and [82, Definition 3.1]. Since the extension is not unique, the conormal derivative appears to be an operator that is not unique, which is also nonlinear in u unless a linear relation between u and the PDE right-hand side extension is enforced. This creates some difficulties in formulating the BDIEs. These difficulties are addressed in [79], [80] presenting formulation and analysis of direct segregated BDIE systems equivalent to the Dirichlet and Neumann problems for the divergent-type PDE with a variable scalar coefficient and a general right-hand side. This needed a non-trivial generalization of the third Green identity and its conormal derivative for such functions, which extends the approach implemented in [27], [28], [29], [81], [83] for the PDE right-hand from $L_2(\Omega)$. In [7], using the two-operator approach in settings different from those in [13], [14], a generalization of the two-operator third Green identity and its conormal derivative is derived and the two-operator BDIEs for variable-coefficient Dirichlet, Neumann and mixed BVPs are analyzed in 3D.

Nowadays, the theory of BDIEs in 3D is well developed [27], [28], [29], [84], [86], but the BDIEs in 2D need a special consideration due to their different equivalence properties. As a result, we need to set conditions on the associated Sobolev spaces or choose appropriate scaling parameter in the parametrix form, to insure the invertibility of the corresponding parametrix-based integral layer potentials and hence the unique solvability of BDIEs [6], [8], [10], [11], [12], [37].

In this chapter, we extend the results in [6], [15], and consider the Dirichlet and Neumann BVPs for the linear second-order scalar elliptic differential equation with variable coefficient in a two-dimensional bounded domain. The PDE right-hand side belongs to $H^{-1}(\Omega)$ or $\widetilde{H}^{-1}(\Omega)$ when neither classical nor canonical conormal derivatives of solutions are well defined. The two-operator approach and appropriate parametrix (Levi function) are used to reduce each problem into two different systems of BDIEs. The properties of the corresponding potential operators are investigated. The equivalence of the two-operator BDIE systems to the original problems, BDIE system solvability, solution uniqueness/nonuniqeuness and invertibility BDIE system are analyzed in the appropriate Sobolev spaces. It is shown that the BDIE operators for the Neumann BVP are not invertible, and appropriate finite-dimensional perturbations are constructed leading to invertibility of the perturbed operators.

5.2 Preliminaries

5.2.1 Conormal derivatives

Let Ω be a domain in $\mathbb{R}^2$ bounded by a smooth curve $\partial\Omega$. Consider the scalar elliptic differential equation, which for sufficiently smooth function u and $x \in \Omega$ has the following strong form,

$$Au(x) := A(x, \partial_x)u(x) = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(a(x)\frac{\partial u(x)}{\partial x_i}\right) = \tilde{f}(x), \tag{5.1}$$

where u is unknown function and $\tilde{f}$ is a given function in Ω. We assume that $a \in C^\infty(\mathbb{R}^2)$ and

$$0 < a_{\min} \le a(x) \le a_{\max} < \infty, \quad \forall\, x \in \mathbb{R}^2.$$

In what follows $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$, $H^s(\Omega) = H_2^s(\Omega)$, $H^s(\partial\Omega) = H_2^s(\partial\Omega)$ are the Bessel potential spaces, where $s \in \mathbb{R}$ is an arbitrary real number [77], [78]. We recall that H^s coincides with the Sobolev-Slobodetski spaces W_2^s for any nonnegative s. We denote by $\widetilde{H}^s(\Omega)$ the subspace of $H^s(\mathbb{R}^2)$,

$$\widetilde{H}^s(\Omega) := \{g : g \in H^s(\mathbb{R}^2),\ \mathrm{supp}(g) \subset \overline{\Omega}\}$$

while $H^s(\Omega)$ denotes the space of restriction on Ω of distributions from $H^s(\mathbb{R}^2)$,

$$H^s(\Omega) = \{r_\Omega g : g \in H^s(\mathbb{R}^2)\}$$

where r_Ω denotes the restriction operator on Ω. We will also use the notation $g|_\Omega := r_\Omega g$. We denote by $H^s_{\partial\Omega}$ the following subspace of $H^s(\mathbb{R}^2)$ (and $\widetilde{H}^s(\Omega)$),

$$H^s_{\partial\Omega} := \{g : g \in H^s(\mathbb{R}^2),\ \mathrm{supp}(g) \subset \partial\Omega\}. \tag{5.2}$$

From the trace theorem [36], [77], [78] for $u \in H^1(\Omega)$, it follows that $\gamma^+ u \in H^{\frac{1}{2}}(\partial\Omega)$, where $\gamma^+ = \gamma^+_{\partial\Omega}$ is the trace operator on $\partial\Omega$ from Ω. Let also $\gamma^{-1} : H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$ denote a (non-unique) continuous right inverse to the trace operator γ^+, i.e., $\gamma^+_{\partial\Omega}\gamma^{-1}_{\partial\Omega} w = \gamma^+\gamma^{-1} w = w$ for any $w \in H^{\frac{1}{2}}(\partial\Omega)$, and $(\gamma^{-1})^* : \widetilde{H}^{-1}(\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is continuous operator dual to γ^{-1}, i. e., $\langle(\gamma^{-1})^*\tilde{f}, w\rangle_{\partial\Omega} := \langle\tilde{f}, \gamma^{-1}w\rangle_\Omega$ for any $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ and $w \in H^{\frac{1}{2}}(\partial\Omega)$.

For $u \in H^2(\Omega)$, we denote by T_a^+ the corresponding canonical (strong) conormal derivative operator on $\partial\Omega$ in the sense of traces,

$$T_a^+ u := \sum_{i=1}^{2} a(x) n_i(x) \gamma^+ \frac{\partial u(x)}{\partial x_i} = a(x)\gamma^+ \frac{\partial u(x)}{\partial n(x)},$$

where $n(x)$ is the outward to Ω unit normal vector at the point $x \in \partial\Omega$. However, the classical conormal derivative operator is generally, not well defined if $u \in H^1(\Omega)$, see [80, Appendix A]. For $u \in H^1(\Omega)$, the PDE Au in (5.1) is understood in the sense of distributions,

$$\langle Au, v\rangle_\Omega := -\mathcal{E}_a(u, v), \qquad \forall v \in \mathcal{D}(\Omega), \tag{5.3}$$

where

$$\mathcal{E}_a(u, v) := \int_\Omega a(x)\nabla u(x) \cdot \nabla v(x) dx$$

is a symmetric bilinear form and the duality brackets $\langle g, \cdot\rangle_\Omega$ denote the value of a linear functional (distribution) g, extending the usual L_2 inner product. Since the set $\mathcal{D}(\Omega)$ is dense in $\widetilde{H}^1(\Omega)$, the above formula defines a continuous operator $A : H^1(\Omega) \to H^{-1}(\Omega) = [\widetilde{H}^1(\Omega)]^*$,

$$\langle Au, v\rangle_\Omega := -\mathcal{E}_a(u, v), \forall u \in H^1(\Omega),\ \forall v \in \widetilde{H}^1(\Omega).$$

Let us consider also the operator

$$\check{A} : H^1(\Omega) \to \widetilde{H}^{-1}(\Omega) = [H^1(\Omega)]^*$$

such that

$$\langle \check{A}u, v\rangle_\Omega := -\mathcal{E}_a(u,v) = -\int_\Omega a(x)\nabla u(x)\cdot\nabla v(x)dx = -\int_{\mathbb{R}^2} \mathring{E}[a\nabla u](x)\cdot\nabla V(x)dx$$
$$= \langle \nabla\cdot\mathring{E}[a\nabla u], V\rangle_{\mathbb{R}^2}\langle \nabla\cdot\mathring{E}[a\nabla u], v\rangle_\Omega, \forall u\in H^1(\Omega), \quad \forall v\in H^1(\Omega),$$

which is evidently continuous and can be written as:

$$\check{A}u = \nabla\cdot\mathring{E}[a\nabla u]. \tag{5.4}$$

Here $V \in H^1(\mathbb{R}^2)$ is such that $r_\Omega V = v$ and $\mathring{E}$ denotes the operator of extension of the functions, defined in Ω, by zero outside Ω in $\mathbb{R}^2$. For any $u \in H^1(\Omega)$, the functional $\check{A}u$ belongs to $\widetilde{H}^{-1}(\Omega)$ and is the extension of the functional $Au \in H^{-1}(\Omega)$, which domain is thus extended from $\widetilde{H}^1(\Omega)$ to the domain $H^1(\Omega)$ for $\check{A}u$.

Inspired by the first Green identity for smooth functions, we can define *the generalized conormal derivative* as in [64, Lemma 2.2], [78, Lemma 4.3] and [82, Definition 3.1].

Definition 5.1. Let $u \in H^1(\Omega)$ and $Au = r_\Omega\tilde{f}$ in Ω for some $\tilde{f}\in\widetilde{H}^{-1}(\Omega)$. Then the generalized conormal derivative $T_a^+(\tilde{f},u)\in H^{-\frac{1}{2}}(\partial\Omega)$ is defined as

$$\langle T_a^+(\tilde{f},u), w\rangle_{\partial\Omega} := \langle \tilde{f}, \gamma^{-1}w\rangle_\Omega + \mathcal{E}_a(u,\gamma^{-1}w) = \langle \tilde{f}-\check{A}u, \gamma^{-1}w\rangle_\Omega, \quad \forall w\in H^{\frac{1}{2}}(\partial\Omega),$$

that is

$$T^+(\tilde{f},u) := (\gamma^{-1})^*(\tilde{f}-\check{A}u). \tag{5.5}$$

Due to [78, Lemma 4.3] and [82, Theorem 3.2], we have the estimate

$$\|T_a^+(\tilde{f},u)\|_{H^{-\frac{1}{2}}(\partial\Omega)} \le C_1\|u\|_{H^1(\Omega)} + C_2\|\tilde{f}\|_{\widetilde{H}^{-1}(\Omega)}, \tag{5.6}$$

and for $u\in H^1(\Omega)$ such that $Au = r_\Omega\tilde{f}$ in Ω for some $\tilde{f}\in\widetilde{H}^{-1}(\Omega)$ the first Green identity holds in the following form:

$$\langle T_a^+(\tilde{f},u), \gamma^+v\rangle_{\partial\Omega} := \langle \tilde{f}, v\rangle_\Omega + \mathcal{E}_a(u,v) = \langle \tilde{f}-\check{A}u, v\rangle_\Omega, \quad \forall v\in H^1(\Omega). \tag{5.7}$$

As follows from Definition 5.1, the generalised conormal derivative is nonlinear with respect to u for a fixed $\tilde{f}$, but linear with respect to the couple $(\tilde{f},u)$, i.e.,

$$\alpha_1 T_a^+(\tilde{f}_1,u_1) + \alpha_2 T_a^+(\tilde{f}_2,u_2) = T_a^+(\alpha_1\tilde{f}_1,\alpha_1 u_1) + T_a^+(\alpha_2\tilde{f}_2,\alpha_2 u_2)$$
$$= T_a^+(\alpha_1\tilde{f}_1+\alpha_2\tilde{f}_2, \alpha_1u_1+\alpha_2u_2) \tag{5.8}$$

for any real numbers α_1,α_2.

Let us also define some subspaces of $H^s(\Omega)$, cf. [33], [46], [81], [82].

Definition 5.2. Let $s\in\mathbb{R}$ and $A_*: H^s(\Omega)\to\mathcal{D}^*(\Omega)$ be a linear operator. For $t\ge -\frac{1}{2}$ we introduce the space

$$H^{s,t}(\Omega;A_*) := \{g\in H^s(\Omega) : \text{there exists } \tilde{f}_g\in\widetilde{H}^t(\Omega) \text{ such that } A_*g|_\Omega = \tilde{f}_g|_\Omega\}$$

endowed with the norm

$$\|g\|_{H^{s,t}(\Omega;A_*)} := \left(\|g\|^2_{H^s(\Omega)} + \|\tilde{f}_g\|^2_{\widetilde{H}^t(\Omega)}\right)^{\frac{1}{2}}$$

and the inner product

$$(g,h)_{H^{s,t}(\Omega;A_*)} = (g,h)_{H^s(\Omega)} + (\tilde{f}_g,\tilde{f}_h)_{\widetilde{H}^t(\Omega)}.$$

The distribution $\tilde{f}_g \in \widetilde{H}^t(\Omega),\ t \geq -\frac{1}{2}$, in the above definition is an extension of the distribution $A_* g|_\Omega \in H^t(\Omega)$, and the extension is unique (if it does exist) since any distribution from the space $H^t(\mathbb{R}^2)$ with support in $\partial\Omega$ is identically zero if $t \geq -\frac{1}{2}$, see [78, Lemma 3.39] and [82, Theorem 2.10]. We denote this extension as an operator $\tilde{A}_*$, i.e., $\tilde{A}_* g = \tilde{f}_g$. The uniqueness implies that the norm $\|g\|_{H^{s,t}(\Omega;A_*)}$ is well defined.

We will mostly use the operators A, B or Δ as A_* in the above definition. Note that since $Au - a\Delta u = \nabla a \cdot \nabla u \in L_2(\Omega)$, for $u \in H^1(\Omega)$, we have $H^{1,0}(\Omega; A) = H^{1,0}(\Omega; \Delta)$.

Definition 5.3. For $u \in H^{1,-\frac{1}{2}}(\Omega; A)$, we define the canonical conormal derivative $T_a^+ u \in H^{-\frac{1}{2}}(\partial\Omega)$ as

$$\langle T_a^+ u, w\rangle_{\partial\Omega} := \langle \tilde{A}u, \gamma^{-1}w\rangle_\Omega + \mathcal{E}_a(u, \gamma^{-1}w) = \langle \tilde{A}u - \check{A}u, \gamma^{-1}w\rangle_\Omega, \quad \forall w \in H^{\frac{1}{2}}(\partial\Omega),$$
$$i.\ e., \qquad T_a^+ u := (\gamma^{-1})^*(\tilde{A}u - \check{A}u).$$

The canonical conormal derivative $T_a^+ u$ is independent of (non-unique) choice of the operator γ^{-1}, the operator $T_a^+ : H^{1,-\frac{1}{2}}(\Omega; A) \to H^{-\frac{1}{2}}(\partial\Omega)$ is continuous, and the first Green identity holds in the following form,

$$\langle T_a^+ u, \gamma^+ v\rangle_{\partial\Omega} := \langle \tilde{A}u, v\rangle_\Omega + \mathcal{E}_a(u, v)\ , \forall v \in H^1(\Omega).$$

The operator $T_a^+ : H^{1,\ t}(\Omega; A) \to H^{-\frac{1}{2}}(\partial\Omega)$ in Definition 5.3 is continuous for $t \geq -\frac{1}{2}$. The canonical conormal derivative is defined by the function u and the operator A and does not depend separately on the right-hand side $\tilde{f}$ (i.e. its behavior on the boundary), unlike the generalised conormal derivative defined in (5.5), and the operator T_a^+ is linear. Note that the canonical conormal derivative coincides with classical conormal derivative $T_a^+ u = a\frac{\partial u}{\partial n}$ if the latter does exist in the trace sense, see [82, Corollary 3.14 and Theorem 3.16].

Let $u \in H^{1,-\frac{1}{2}}(\Omega; A)$. Then Definitions 5.1 and 5.3 imply that the generalised conormal derivative for arbitrary extension $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ of the distribution Au can be expressed as

$$\langle T_a^+(\tilde{f}, u), w\rangle_{\partial\Omega} \quad := \quad \langle T_a^+ u, w\rangle_{\partial\Omega} \ + \ \langle \tilde{f} \ - \ \check{A}u, \gamma^{-1}w\rangle_\Omega, \quad \forall w \quad \in \quad H^{\frac{1}{2}}(\partial\Omega).$$

Let us consider the auxiliary linear elliptic partial differential operator B defined by

$$Bu(x) := B(x, \partial_x)u(x) = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\Big(b(x)\frac{\partial u(x)}{\partial x_i}\Big), \tag{5.9}$$

where $b \in C^\infty(\overline{\Omega})$ and $b(x) > 0$ for $x \in \overline{\Omega}$. Since for $u \in H^{1,0}(\Omega, \Delta),\ Au - Bu = (a-b)\Delta u + \nabla(a-b)\cdot\nabla u \in L_2(\Omega)$, we have, $H^{1,0}(\Omega; A) = H^{1,0}(\Omega; B)$. Let $u \in H^1(\Omega)$ and $v \in H^{1,0}(\Omega; B)$. Then we write the first Green identity for operator B in the form

$$\mathcal{E}_b(u, v) + \int_\Omega u(x)Bv(x)dx = \langle T_b^+ v, \gamma^+ u\rangle_{\partial\Omega}, \tag{5.10}$$

where

$$\mathcal{E}_b(u, v) = \int_\Omega b(x)\nabla u(x)\cdot\nabla v(x)dx.$$

If, in addition, $Au = \tilde{f}$ in Ω, where $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$, then according to the definition of $T_a^+(\tilde{f}, u)$, in (5.5), the *two-operator second Green identity* can be written as

$$\langle \tilde{f}, v\rangle_\Omega - \int_\Omega u(x)Bv(x)dx + \int_\Omega [a(x) - b(x)]\nabla u(x) \cdot \nabla v(x)dx$$
$$= \langle T_a^+(\tilde{f}, u), \gamma^+ v\rangle_{\partial\Omega} - \langle T_b^+ v, \gamma^+ u\rangle_{\partial\Omega}. \quad (5.11)$$

Moreover, for $u, v \in H^{1,0}(\Omega; A) = H^{1,0}(\Omega; B)$ (5.11) becomes

$$\int_\Omega [v(x)Au(x) - u(x)Bv(x)]dx + \int_\Omega [a(x) - b(x)]\nabla u(x) \cdot \nabla v(x)dx$$
$$= \langle T_a^+ u, \gamma^+ v\rangle_{\partial\Omega} - \langle T_b^+ v, \gamma^+ u\rangle_{\partial\Omega}.$$

5.2.2 Parametrix, remainder and potential type operators

Definition 5.4. We will say, a function $P_b(x, y)$ of two variables $x, y \in \Omega$ is a parametrix (Levi function) for the operator $B(x, \partial_x)$ in $\mathbb{R}^2$ [53], [86], [88], [111], [112] if

$$B(x, \partial_x)P_b(x, y) = \delta(x - y) + R_b(x, y),$$

where δ is the Dirac-delta distribution, while $R_b(x, y)$ is a remainder possessing at most a weak singularity at $x = y$.

For some positive constant r_0 and $x, y \in \mathbb{R}^2$, the parametrix and hence the corresponding remainder in 2D can be chosen as in [86],

$$P_b(x, y) = \frac{1}{2\pi b(y)} \ln\left(\frac{|x - y|}{r_0}\right), \quad (5.12)$$

$$R_b(x, y) = \sum_{i=1}^{2} \frac{x_i - y_i}{2\pi b(y)|x - y|^2} \frac{\partial b(x)}{\partial x_i}. \quad (5.13)$$

The parametrix $P_b(x, y)$ in (5.12) is the fundamental solution to the operator $B(y, \partial_x) := b(y)\Delta_x$ with "frozen" coefficient $b(x) = b(y)$, and

$$B(y, \partial_x)P_b(x, y) = \delta(x - y).$$

Let $b \in C^\infty(\mathbb{R}^2)$ and $b(x) > 0$ a.e. in $\mathbb{R}^2$. For some scalar function g the parametrix-based Newtonian and the remainder volume potential operators, corresponding to the parametrix (5.12) and the remainder (5.13) are given by

$$\mathbf{P}_b g(y) := \int_{\mathbb{R}^2} P_b(x, y)g(x)dx, y \in \mathbb{R}^2, \quad (5.14)$$

$$\mathcal{P}_b g(y) := \int_\Omega P_b(x, y)g(x)dx, \quad y \in \Omega, \quad (5.15)$$

$$\mathcal{R}_b g(y) := \int_\Omega R_b(x, y)g(x)dx, \quad y \in \Omega. \quad (5.16)$$

For $g \in H^s(\mathbb{R}^2)$, $s \in \mathbb{R}$, (5.14) is understood as $\mathbf{P}_b g = \frac{1}{b}\mathbf{P}_\Delta g$, where the Newtonian potential operator $\mathbf{P}_\Delta$ for Laplacian Δ is well defined in terms of the Fourier transform (i.e., as pseudodifferential operator), on any space $H^s(\mathbb{R}^2)$. For $g \in \widetilde{H}^s(\Omega)$, and any $s \in \mathbb{R}$, definitions in (5.15) and (5.16) can be understood as

$$\begin{aligned} \mathcal{P}_b g &= \frac{1}{b} r_\Omega \mathbf{P}_\Delta g, \;\; \mathcal{P}_b g = \frac{a}{b} r_\Omega \mathbf{P}_a g, \\ \mathcal{R}_b g &= -\frac{1}{b} r_\Omega \nabla \cdot \mathbf{P}_\Delta (g \nabla b), \end{aligned} \tag{5.17}$$

while for $g \in H^s(\Omega), -\frac{1}{2} < s < \frac{1}{2}$, as (5.17) with g replaced by $\widetilde{E}g$, where $\widetilde{E} : H^s(\Omega) \to \widetilde{H}^s(\Omega), -\frac{1}{2} < s < \frac{1}{2}$, is the unique extension operator related with the operator $\mathring{E}$ of extension by zero, cf. [82, Theorem 16].

For $y \notin \partial\Omega$, the single layer and the double layer surface potential operators, corresponding to the parametrix (5.12) are defined as

$$V_b g(y) := -\int_{\partial\Omega} P_b(x,y) g(x) dS_x, \tag{5.18}$$

$$W_b g(y) := -\int_{\partial\Omega} [T_b(x, n(x), \partial_x) P_b(x,y)] g(x) dS_x, \tag{5.19}$$

where g is some scalar density function. The integrals are understood in the distributional sense if g is not integrable, while V_Δ and W_Δ are the single layer and double layer potentials corresponding to the Laplacian Δ. The corresponding boundary integral (pseudodifferential) operators of direct surface values to the single and the double layer potentials,$\mathcal{V}_b$ and $\mathcal{W}_b$ when $y \in \partial\Omega$, are

$$\mathcal{V}_b g(y) := -\int_{\partial\Omega} P_b(x,y) g(x) dS_x, \tag{5.20}$$

$$\mathcal{W}_b g(y) := -\int_{\partial\Omega} T_b(x, n(x), \partial_x) P_b(x,y) g(x) dS_x, \tag{5.21}$$

where $\mathcal{V}_\Delta$ and $\mathcal{W}_\Delta$ are respectively the direct values of the single and double layer potentials corresponding to the Laplacian Δ.

We can also calculate at $y \in \partial\Omega$ the conormal derivatives, associated with the operator A, of the single layer potential and of the double layer potential:

$$T_a^{\pm} V_b g(y) = \frac{a(y)}{b(y)} T_b^{\pm} V_b g(y), \tag{5.22}$$

$$\mathcal{L}_{ab}^{\pm} g(y) := T_a^{\pm} W_b g(y) = \frac{a(y)}{b(y)} T_b^{\pm} W_b g(y). \tag{5.23}$$

The direct value operators associated with (5.22) are

$$\mathcal{W}'_{ab} g(y) := -\int_{\partial\Omega} [T_a(y, n(y), \partial_y) P_b(x,y)] g(x) dS_x, \tag{5.24}$$

$$\mathcal{W}'_b g(y) := -\int_{\partial\Omega} [T_b(y, n(y), \partial_y) P_b(x,y)] g(x) dS_x. \tag{5.25}$$

From equations (5.14)-(5.25) we deduce representations of the parametrix-based surface potential boundary operators in terms of their counterparts for $b = 1$, that is, associated with the fundamental solution $P_\Delta = \frac{1}{2\pi} \ln\left(\frac{|x-y|}{r_0}\right)$ of the Laplace operator Δ.

$$\mathbf{P}_a g = \frac{1}{a} \mathbf{P}_\Delta g, \quad \mathbf{P}_b g = \frac{1}{b} \mathbf{P}_\Delta g, \mathcal{P}_a g = \frac{1}{a} \mathcal{P}_\Delta g, \quad \mathcal{P}_b g = \frac{1}{b} \mathcal{P}_\Delta g. \tag{5.26}$$

$$\frac{a}{b}V_a g = V_b g \; = \; \frac{1}{b}V_\Delta g, \frac{a}{b}W_a\Big(\frac{bg}{a}\Big) \; = \; W_b g \; = \; \frac{1}{b}W_\Delta\big(bg\big). \tag{5.27}$$

$$\frac{a}{b}\mathcal{V}_a g = \mathcal{V}_b g \; = \; \frac{1}{b}\mathcal{V}_\Delta g, \frac{a}{b}\mathcal{W}_a\Big(\frac{bg}{a}\Big) \; = \; \mathcal{W}_b g \; = \; \frac{1}{b}\mathcal{W}_\Delta\big(bg\big). \tag{5.28}$$

$$\mathcal{W'}_{ab} g = \frac{a}{b}\mathcal{W'}_b g = \; \frac{a}{b}\Big\{\mathcal{W'}_\Delta g + \Big[b\frac{\partial}{\partial n}\Big(\frac{1}{b}\Big)\Big]\mathcal{V}_\Delta g\Big\}, \tag{5.29}$$

$$\mathcal{L}^{\pm}_{ab} g = \frac{a}{b}\mathcal{L}^{\pm}_{b} g = \; \frac{a}{b}\Big\{\mathcal{L}_\Delta(bg) + \Big[b\frac{\partial}{\partial n}\Big(\frac{1}{b}\Big)\Big]\gamma^{\pm}W_\Delta(bg)\Big\}, \tag{5.30}$$

$$\hat{\mathcal{L}}_b g := T^{+}_{\Delta}W_\Delta(bg) = T^{-}_{\Delta}W_\Delta(bg) = \hat{\mathcal{L}}_\Delta(bg) \quad \text{on} \quad \partial\Omega. \tag{5.31}$$

It is taken into account that b and its derivatives are continuous in $\mathbb{R}^2$ and

$$\mathcal{L}_\Delta(bg) := \mathcal{L}^{+}_{\Delta}(bg) = \mathcal{L}^{-}_{\Delta}(bg)$$

by the Liapunov-Tauber theorem. Hence,

$$\Delta(bV_b g) = 0, \;\; \Delta(bW_b g) = 0 \quad \text{in} \quad \Omega, \; \forall g \in H^s(\partial\Omega), \quad \forall s \in \mathbb{R}, \tag{5.32}$$

$$\Delta(b\mathcal{P}_b g) = g \quad \text{in } \Omega, \forall g \in \widetilde{H}^s(\Omega), \quad \forall s \in \mathbb{R}. \tag{5.33}$$

The mapping properties of the operators (5.14)-(5.25) follow from relations (5.26)-(5.31) and are described in detail in [13, Appendix A]. Particularly, we have the following jump relations:

Theorem 5.1. For $g_1 \in H^{-\frac{1}{2}}(\partial\Omega)$, and $g_2 \in H^{\frac{1}{2}}(\partial\Omega)$. Then there hold the following relations on $\partial\Omega$,

$$\gamma^{\pm}V_b g_1 = \mathcal{V}_b g_1, \tag{5.34}$$

$$\gamma^{\pm}W_b g_2 = \mp\frac{1}{2}g_2 + \mathcal{W}_b g_2, \tag{5.35}$$

$$T^{\pm}_{a}V_b g_1 = \pm\frac{1}{2}\frac{a}{b}g_1 + \mathcal{W'}_{ab} g_1. \tag{5.36}$$

5.3 The two-operator third Green identity and integral relations

Applying some limiting procedures [53, Section 3.8], [88], and we obtain the parametrix based third Green identities.

Theorem 5.2. (i) If $u \in H^1(\Omega)$, then the following third Green identity holds,

$$u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b\gamma^{+}u = \mathcal{P}_b\check{A}u \quad \text{in} \quad \Omega, \tag{5.37}$$

where the operator $\check{A}$ is defined in (5.4), and for $u \in C^1(\overline{\Omega})$,

$$\begin{aligned}\mathcal{P}_b\check{A}u(y) &:= \langle\check{A}u, P_b(.,y)\rangle_\Omega = -\mathcal{E}_a(u, P_b(.,y)) \\ &= -\int_\Omega a(x)\nabla u(x)\cdot\nabla_x P_b(x,y)dx\end{aligned} \tag{5.38}$$

and

$$\mathcal{Z}_b u = -\int_\Omega [a(x)-b(x)]\nabla_x P_b(x,y)\cdot\nabla u(x)dx$$
$$= \frac{1}{b(y)}\sum_{j=1}^{2}\partial_j\mathcal{P}_\Delta\left[(a-b)\partial_j u\right] \quad \text{in} \quad \Omega. \quad (5.39)$$

(ii) If $Au = r_\Omega \tilde{f}$ in Ω, where $\tilde{f}\in\widetilde{H}^{-1}(\Omega)$, then recalling the definition of $T_a^+(\tilde{f},u)$, in (5.5), we arrive at the generalised two-operator third Green identity in the following form,

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b T_a^+(\tilde{f},u) + W_b\gamma^+ u = \mathcal{P}_b\tilde{f} \quad \text{in} \quad \Omega, \qquad (5.40)$$

where it was taken into account that

$$\langle T_a^+(\tilde{f},u), P_b(x,y)\rangle_{\partial\Omega} \quad = \quad -V_b T_a^+(\tilde{f},u) \text{ and } \langle\tilde{f}, P_b(x,y)\rangle_\Omega \quad = \quad \mathcal{P}_b\tilde{f}.$$

Proof. (i) Let first $u\in D(\overline{\Omega})$. Let $y\in\Omega$, $B_\epsilon(y)\subset\Omega$ be a ball centred at y with sufficiently small radius ϵ, and $\Omega_\epsilon := \Omega\setminus\overline{B_\epsilon(y)}$. For the fixed y, evidently, $P_b(.,y)\in D(\overline{\Omega_\epsilon})\subset H^{1,0}(A;\Omega_\epsilon)$ and has the coinciding classical and cannonical conormal derivatives on $\partial\Omega_\epsilon$. Then from (5.12) and the first Green identity (5.10) employed for Ω_ϵ with $v = P_b(.,y)$ we obtain

$$-\int_{\partial B_\epsilon(y)} T_x^+ P_b(x,y)\gamma^+u(x)ds_x - \int_{\partial\Omega} T_x P_b(x,y)\gamma^+u(x)ds_x + \int_{\Omega_\epsilon} u(x)R_b(x,y)dx$$
$$= -\int_{\Omega_\epsilon} b(x)\nabla u(x)\cdot\nabla_x P_b(x,y)dx,$$

which we rewrite as

$$-\int_{\partial B_\epsilon(y)} T_x^+ P_b(x,y)\gamma^+u(x)ds_x - \int_{\partial\Omega} T_x P_b(x,y)\gamma^+u(x)ds_x$$
$$-\int_{\Omega_\epsilon}[a(x)-b(x)]\,\nabla u(x)\cdot\nabla_x P_b(x,y)dx + \int_{\Omega_\epsilon} u(x)R_b(x,y)dx$$
$$= -\int_{\Omega_\epsilon} a(x)\nabla u(x)\cdot\nabla_x P_b(x,y)dx. \quad (5.41)$$

Taking the limit as $\epsilon\to 0$, (5.41) reduces to the third Green identity (5.37)–(5.38) for any $u\in\mathcal{D}(\overline{\Omega})$. Taking into account the density of $\mathcal{D}(\overline{\Omega})$ in $H^1(\Omega)$, and the mapping properties of the integral potentials, see [13, Appendix A], we obtain that (5.37)–(5.38) hold true also for any $u\in H^1(\Omega)$.
(ii) Let $\{\tilde{f}_k\}\in\mathcal{D}(\Omega)$ be a sequence of converging to $\tilde{f}$ in $\widetilde{H}^{-1}(\Omega)$ as $k\to\infty$. Then, according to [80, Theorem B.1] there exists a sequence $\{u_k\}\in\mathcal{D}(\overline{\Omega})$ converging to u in $H^1(\Omega)$ such that $Au_k = r_\Omega\tilde{f}_k$ and $T_a^+(u_k) = T_a^+(\tilde{f}_k,u_k)$ converge to $T_a^+(\tilde{f},u)$ in

$H^{-\frac{1}{2}}(\partial\Omega)$. For such u_k by (5.38) and (5.5), we have

$$\mathcal{P}_b\check{A}u_k(y) = \frac{1}{b(y)}\nabla_y\cdot\int_\Omega a(x)P_\Delta(x,y)\nabla u_k(x)dx = -\frac{1}{b(y)}\lim_{\epsilon\to 0}\int_{\Omega_\epsilon} a(x)\nabla u_k(x)P_\Delta(x,y)dx$$

$$= -\lim_{\epsilon\to 0}\mathcal{E}_{\Omega_\epsilon}(u_k, P_b(.,y))$$

$$= -\lim_{\epsilon\to 0}\int_{\Omega_\epsilon}\tilde{f}_k P_b(x,y)dx + \lim_{\epsilon\to 0}\int_{\partial B_\epsilon(y)} P_b(x,y)T_a^+u_k(x)dS(x)$$

$$+\lim_{\epsilon\to 0}\int_{\partial\Omega} P_b(x,y)T_a^+u_k(x)dS(x) = \mathcal{P}_b\tilde{f}_k + V_bT_a^+u_k(y). \quad (5.42)$$

Taking the limits as $k\to\infty$ in (5.42), we obtain $\mathcal{P}_b\check{A}u(y) = \mathcal{P}_b\tilde{f} + V_bT_a^+(\tilde{f},u)$, and substitution to (5.37) gives (5.40). The proof is complete. □

Below we shall state and prove Corollary 3 from [8] for completeness.

Corollary 5.1. Using the Gauss divergence theorem, we can rewrite $\mathcal{Z}_bu(y)$ in the form that does not involve derivatives of u,

$$\mathcal{Z}_bu(y) := \left[\frac{a(y)}{b(y)} - 1\right]u(y) + \widehat{\mathcal{Z}}_bu(y), \quad (5.43)$$

$$\widehat{\mathcal{Z}}_bu(y) := \frac{a(y)}{b(y)}W_a\gamma^+u(y) - W_b\gamma^+u(y) + \frac{a(y)}{b(y)}\mathcal{R}_au(y) - \mathcal{R}_bu(y), \quad (5.44)$$

which allows to call $\mathcal{Z}_b$ integral operator in spite of its integro-differential representation (5.39).

Proof. As in the proof of Theorem 5.2 item (i), let first $u\in D(\overline{\Omega})$. Let $y\in\Omega$, $B_\epsilon(y)\subset\Omega$ be a ball centred at y with sufficiently small radius ϵ, and $\Omega_\epsilon := \Omega\setminus\overline{B_\epsilon}(y)$. For the fixed y, evidently, $P_b(.,y)\in D(\overline{\Omega_\epsilon})\subset H^{1,0}(A;\Omega_\epsilon)$ and has the coinciding classical and canonical conormal derivatives on $\partial\Omega_\epsilon = \partial\Omega\cup\partial B_\epsilon(y)$. Further, let us denote

$$\mathcal{Z}_b^\epsilon u(y) = -\int_{\Omega_\epsilon}[a(x)-b(x)]\nabla_xP_b(x,y)\cdot\nabla u(x)\mathrm{d}x,$$

which can be rewritten as

$$\mathcal{Z}_b^\epsilon u(y) = \int_{\Omega_\epsilon}\big[\nabla\big(a(x)-b(x)\big)\cdot\nabla_xP_b(x,y)\big]u(x)dx$$
$$-\int_{\Omega_\varepsilon}\nabla\big[\big(a(x)-b(x)\big)u(x)\big]\cdot\nabla_xP_b(x,y)dx.$$

Observe that

$$I_1(y,\epsilon) = \int_{\Omega_\epsilon}\big[\nabla\big(a(x)-b(x)\big)\cdot\nabla_xP_b(x,y)\big]u(x)\mathrm{d}x = \int_{\Omega_\varepsilon}\big[\nabla a(x)\cdot\nabla_xP_b(x,y)\big]u(x)\mathrm{d}x$$
$$-\int_{\Omega_\epsilon}\big[\nabla b(x)\cdot\nabla_xP_b(x,y)\big]u(x)\mathrm{d}x = \frac{a(y)}{b(y)}\int_{\Omega_\epsilon}\big[\nabla a(x)\cdot\nabla_xP_a(x,y)\big]u(x)\mathrm{d}x$$
$$-\int_{\Omega_\varepsilon}\big[\nabla b(x)\cdot\nabla_xP_b(x,y)\big]u(x)\mathrm{d}x$$

and

$$I_2(y,\epsilon) = -\int_{\Omega_\epsilon} \nabla\big[\big(a(x){-}b(x)\big)u(x)\big]{\cdot}{\cdot}\nabla_x P_b(x,y)dx = \int_{\Omega_\varepsilon} [a(x){-}b(x)]u(x)\Delta_x P_b(x,y)dx$$

$$-\int_{\partial\Omega_\epsilon} [a(x){-}b(x)]\gamma^+u(x)\nabla_x P_b(x,y){\cdot}n(x)dS_x = -\frac{a(y)}{b(y)}\int_{\partial\Omega} a(x)\nabla_x P_a(x,y){\cdot}n(x)\gamma^+u(x)dS_x$$

$$+\int_{\partial\Omega} b(x)\nabla_x P_b(x,y)\cdot n(x)\gamma^+u(x)dS_x - \frac{a(y)}{b(y)}\int_{\partial B(y,\epsilon)} a(x)\nabla_x P_a(x,y)\cdot n(x)u(x)dS_x$$

$$+\int_{\partial B_\epsilon(y)} b(x)\nabla_x P_b(x,y)\cdot n(x)u(x)dS_x = \frac{a(y)}{b(y)}W_a\gamma^+u(y) - W_b\gamma^+u(y)$$

$$-\frac{1}{b(y)}\int_{\partial B_\epsilon(y)} a(x)\nabla_x P_\Delta(x,y)\cdot n(x)\gamma^+u(x)dS_x$$

$$+\frac{1}{b(y)}\int_{\partial B_\epsilon(y)} b(x)\nabla_x P_\Delta(x,y)\cdot n(x)\gamma^+u(x)dS_x.$$

Taking the limit as $\epsilon \to 0$ we obtain

$$\begin{aligned}\mathcal{Z}_b u(y) &= \lim_{\epsilon\to 0}\mathcal{Z}_b^\epsilon u(y) = \lim_{\epsilon\to 0}[I_1(y,\epsilon)+I_2(y,\epsilon)] \\ &= \frac{a(y)}{b(y)}\mathcal{R}_a u(y) - \mathcal{R}_b u(y) + \frac{a(y)}{b(y)}W_a\gamma^+u(y) - W_b\gamma^+u(y) + \Big[\frac{a(y)}{b(y)} - 1\Big]u(y)\end{aligned}$$

which is as in (5.43) and (5.44). The proof is complete. □

Note that the operator $\mathcal{Z}_b$ does not vanish unless operators A and B are equal. For some functions $\tilde{f}$, Ψ, Φ let us consider a more general "indirect" integral relation, associated with (5.40).

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b\Psi + W_b\Phi = \mathcal{P}_b\tilde{f} \quad \text{in} \quad \Omega. \tag{5.45}$$

Lemma 5.1. Let $u \in H^1(\Omega), \Psi \in H^{-\frac{1}{2}}(\partial\Omega), \Phi \in H^{\frac{1}{2}}(\partial\Omega)$ and $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ satisfy (5.45). Then

$$Au = r_\Omega\tilde{f} \quad \text{in} \quad \Omega, \tag{5.46}$$

$$r_\Omega V_b(\Psi - T_a^+(\tilde{f},u)) - r_\Omega W_b(\Phi - \gamma^+u) = 0 \quad \text{in} \quad \Omega, \tag{5.47}$$

$$\gamma^+u + \gamma^+\mathcal{Z}_b u + \gamma^+\mathcal{R}_b u - \mathcal{V}_b\Psi - \frac{1}{2}\Phi + \mathcal{W}_b\Phi = \gamma^+\mathcal{P}_b\tilde{f} \quad \text{on} \quad \partial\Omega, \tag{5.48}$$

$$\begin{aligned}T_a^+(\tilde{f},u) + T_a^+\mathcal{Z}_b u + T_a^+\mathcal{R}_b u - \frac{a}{2b}\Psi - \mathcal{W}'_{ab}\Psi + \mathcal{L}_{ab}^+\Phi \\ = T_a^+(\tilde{f} + \mathring{E}\mathcal{R}_*^b\tilde{f}, \mathcal{P}_b\tilde{f}) \quad \text{on} \quad \partial\Omega,\end{aligned} \tag{5.49}$$

where

$$\mathcal{R}_*^b\tilde{f}(y) := -\sum_{j=1}^{2}\partial_j[(\partial_j b)\mathcal{P}_b\tilde{f}]. \tag{5.50}$$

Proof. Subtracting (5.45) from identity (5.37), we obtain

$$V_b\Psi(y) - W_b(\Phi - \gamma^+ u)(y) = \mathcal{P}_b[\check{A}u(y) - \tilde{f}](y), \quad y \in \Omega. \tag{5.51}$$

Multiplying equality (5.51) by $b(y)$, applying the Laplace operator Δ and taking into account equations (5.32) and (5.33), we get $r_\Omega \tilde{f} = r_\Omega(\check{A}u) = Au$ in Ω. This means that $\tilde{f}$ is an extension of the distribution $Au \in H^{-1}(\Omega)$ to $\widetilde{H}^{-1}(\Omega)$, and u satisfies (5.46). Then (5.5) implies that

$$\begin{aligned}\mathcal{P}_b[\check{A}u - \tilde{f}](y) = \langle \check{A}u - \tilde{f}, P_b(.,y)\rangle_\Omega = -\langle T_a^+(\tilde{f},u), P_b(.y)\rangle_{\partial\Omega} \\ = V_b T_a^+(\tilde{f},u), \quad y \in \Omega. \quad (5.52)\end{aligned}$$

Substituting (5.52) into (5.51) leads to (5.47). Equation (5.48) follows from (5.45) and jump relations in (5.34) and (5.35). To prove (5.49), let us first remark that for $u \in H^1(\Omega)$, we have $H^1(\Omega; A) = H^1(\Omega; \Delta) = H^1(\Omega; B)$ and

$$B\mathcal{P}_b\tilde{f} = \tilde{f} + \mathcal{R}_*^b\tilde{f} \ \text{ in } \Omega, \tag{5.53}$$

due to (5.46), which implies that $B(\mathcal{P}_b\tilde{f} - u) = \mathcal{R}_*^b\tilde{f}$ in Ω, with $\mathcal{R}_*^b\tilde{f}$ given by (5.50), and thus $\mathcal{R}_*^b\tilde{f} \in L_2(\Omega)$. Then $B(\mathcal{P}_b\tilde{f} - u)$ can be canonically extended (by zero) to

$$\widetilde{B}(\mathcal{P}_b\tilde{f} - u) = \mathring{E}\mathcal{R}_*^b\tilde{f} \in \widetilde{H}^0(\Omega) \subset \widetilde{H}^{-1}(\Omega).$$

Thus there exists a canonical conormal derivative $T_b^+(\mathcal{P}_b\tilde{f} - u)$ written as (see, e.g., [79, Eq. (4.23)], [80, Eq. (4.14)].)

$$T_b^+(\mathcal{P}_b\tilde{f} - u) = T_b^+(\tilde{f} + \mathring{E}\mathcal{R}_*^b\tilde{f}, \mathcal{P}_b\tilde{f}) - T_b^+(\tilde{f}, u), \tag{5.54}$$

and hence

$$\begin{aligned}T_a^+\big(\mathcal{P}_b\tilde{f} - u\big) = \frac{a}{b}T_b^+\big(\mathcal{P}_b\tilde{f} - u\big) = \frac{a}{b}\big[T_b^+(\tilde{f} + \mathring{E}\mathcal{R}_*^b\tilde{f}, \mathcal{P}_b\tilde{f}) - T_b^+(\tilde{f}, u)\big] \\ = T_a^+\big(\tilde{f} + \mathring{E}\mathcal{R}_*^b\tilde{f}, \mathcal{P}_b\tilde{f}\big) - T_a^+(\tilde{f}, u).\end{aligned}$$

From (5.45) it follows that

$$\mathcal{P}_b\tilde{f} - u = \mathcal{Z}_b u + \mathcal{R}_b u - V_b\Psi + W_b\Phi \quad \text{in} \quad \Omega.$$

Substituting this on the left-hand side of (5.54) and taking into account (5.30) and the jump relation (5.36), we arrive at (5.49). The proof is complete. □

Remark 5.1. If $\tilde{f} \in \widetilde{H}^{-\frac{1}{2}}(\Omega) \subset \widetilde{H}^{-1}(\Omega)$, then $\tilde{f} + \mathring{E}\mathcal{R}_*^b\tilde{f} \in \widetilde{H}^{-\frac{1}{2}}(\Omega)$ as well, which implies that

$$\tilde{f} + \mathring{E}\mathcal{R}_*^b\tilde{f} = \tilde{A}\mathcal{P}_b\tilde{f}$$

and

$$T_a^+(\tilde{f} + \mathring{E}\mathcal{R}_*^b\tilde{f}, \mathcal{P}_b\tilde{f}) = T_a^+(\tilde{B}\mathcal{P}_b\tilde{f}, \mathcal{P}_b\tilde{f}) = T_a^+\mathcal{P}_b\tilde{f}. \tag{5.55}$$

Furthermore, if the hypotheses of Lemma 5.1 are satisfied, then (5.46) implies $u \in H^{1,-\frac{1}{2}}(\Omega; A)$ and $T_a^+(\tilde{f}, u) = T_a^+(\tilde{A}u, u) = T_a^+u$. Henceforth (5.49), takes the familiar form, cf. [13, equation (3.23)],

$$T_a^+u + T_a^+\mathcal{Z}_b u + T_a^+\mathcal{R}_b u - \frac{a}{2b}\Psi - \mathcal{W}'_{ab}\Psi + \mathcal{L}_{ab}^+\Phi = T_a^+\mathcal{P}_b\tilde{f} \quad \text{on} \quad \partial\Omega.$$

Remark 5.2. Let $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ and a sequence $\{\phi_i\} \in \widetilde{H}^{-1}(\Omega)$ converge to $\tilde{f}$ in $\widetilde{H}^{-1}(\Omega)$. By the continuity of operators (see [80, C.1 and C.2]) estimate (5.6) and relation (5.55) for ϕ_i, we obtain that

$$T_a^+(\tilde{f}+\mathring{E}\mathcal{R}_*^b\tilde{f}, \mathcal{P}_b\tilde{f}) = \lim_{i\to\infty} T_a^+(\phi_i+\mathring{E}\mathcal{R}_*^b\phi_i, \mathcal{P}_b\phi_i) = \lim_{i\to\infty} T_a^+\mathcal{P}_b\phi_i, \quad \text{in} \quad H^{-\frac{1}{2}}(\partial\Omega),$$

see [80, Theorem B.1].

Lemma 5.1 and the third Green identity (5.40) imply the following assertion.

Corollary 5.2. If $u \in H^1(\Omega)$ and $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ are such that $Au = r_\Omega\tilde{f}$ in Ω, then

$$\frac{1}{2}\gamma^+u + \gamma^+\mathcal{Z}_bu + \gamma^+\mathcal{R}_bu - \mathcal{V}_bT_a^+(\tilde{f},u) + \mathcal{W}_b\gamma^+u = \gamma^+\mathcal{P}_b\tilde{f}, \text{ on } \partial\Omega, \tag{5.56}$$

$$\left(1-\frac{a}{2b}\right)T_a^+(\tilde{f},u) + T_a^+\mathcal{Z}_bu + T_a^+\mathcal{R}_bu - \mathcal{W}'_{ab}T_a^+(\tilde{f},u) + \mathcal{L}_{ab}^+\gamma^+u$$
$$= T_a^+(\tilde{f}+\mathring{E}\mathcal{R}_*^b\tilde{f}, \mathcal{P}_b\tilde{f}) \quad \text{on} \quad \partial\Omega. \tag{5.57}$$

Note that if $\mathcal{P}_b$ is not only the parametrix but also the fundamental solution of the operator B, then the remainder operator $\mathcal{R}_b$ vanishes in (5.40) and (5.56)-(5.57) (and everywhere in this Chapter), while the operator $\mathcal{Z}_b$ stays unless $A = B$. The following statement is proved in [80, Lemma 4.6]:

Theorem 5.3. Let $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$. A function $u \in H^1(\Omega)$ is a solution of PDE $Au = r_\Omega\tilde{f}$ in Ω if and only if it is a solution of (5.40).

Proof. If $u \in H^1(\Omega)$ solves the PDE $Au = r_\Omega\tilde{f}$ in Ω, then it satisfies (5.40). On the other hand, if u solves (5.40), then using Lemma 5.1 for $\Psi = T_a^+(\tilde{f},u)$, $\Phi = \gamma^+u$ completes the proof. □

5.4 Invertibility of single layer potential operator

The boundary integral operator, $\mathcal{V}_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is a Fredholm operator of index zero, see, e.g.,[78, Theorem 7.6]. Thus the first relation in (5.28) leads to the same result for the single layer potential $\mathcal{V}_b$. For the case of 3D, [13, Lemma 3.2(i)] asserts that for $\Psi^* \in H^{-\frac{1}{2}}(\partial\Omega)$, if $V_b\Psi^* = 0$ in Ω, then $\Psi^* = 0$ in Ω. This fact implies the invertibility of single layer potential operator V_b mapping from $H^{-\frac{1}{2}}(\partial\Omega)$ to $H^{\frac{1}{2}}(\partial\Omega)$. But this is not the case for 2D. It is well-known, see e.g. [32, Remark 1.42(ii)] and [119, Theorem 6.22] that for some 2D domains the kernel of the operator $\mathcal{V}_\Delta$ is nontrivial, thus due to the first relation in (5.28), the kernel of operator $\mathcal{V}_b$ is nontrivial as well for the same domains. To ensure the invertibility of the single layer potential operator in 2D, for $s \in \mathbb{R}$, let us define the subspace of $H^s(\partial\Omega)$, cf. e.g., [119, p. 147],

$$H^s_{**}(\partial\Omega) := \{g \in H^s(\partial\Omega) : \langle g, 1\rangle_{\partial\Omega} = 0\}.$$

The following result is proved in [37, Theorem 4], see also [9, Theorem 1].

Theorem 5.4. If $\psi \in H^{-\frac{1}{2}}_{**}(\partial\Omega)$ satisfies $\mathcal{V}_b\psi = 0$ on $\partial\Omega$, then $\psi = 0$.

Following [78, Theorem 8.15], there exists a unique real-valued distribution $\psi_{eq} \in H^{-\frac{1}{2}}(\partial\Omega)$ called equilibrium density for $\partial\Omega$ such that $\mathcal{V}_\Delta \psi_{eq}$ is constant on $\partial\Omega$, and $(1, \psi_{eq})_{\partial\Omega} = 1$. For $n = 2$ the constant $\mathcal{V}_\Delta \psi_{eq}$ is not always positive and one introduces the *logarithmic capacity*, $\mathrm{Cap}_{\partial\Omega}$ using the relation

$$\mathcal{V}_\Delta \psi_{eq} = \frac{1}{2\pi} \ln \left(\frac{r_0}{\mathrm{Cap}_{\partial\Omega}} \right),$$

for some positive constant r_0 as in equation (5.12). In particular $\mathcal{V}_\Delta \psi_{eq} = 0$ if and only if $r_0 = \mathrm{Cap}_{\partial\Omega}$.

The following statement is proved in [78, Theorem 8.16].

Theorem 5.5. Let r_0 be some positive constant.

(i) The operator $\mathcal{V}_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$, is $H^{-\frac{1}{2}}(\partial\Omega)$- elliptic, i.e., $\langle \mathcal{V}_\Delta \psi, \psi \rangle_{\partial\Omega} \geq c\|\psi\|_{H^{-\frac{1}{2}}(\partial\Omega)}$ for all $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$, if and only if $r_0 > \mathrm{Cap}_{\partial\Omega}$.

(ii) The operator $\mathcal{V}_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$, has a bounded inverse if and only if $r_0 \neq \mathrm{Cap}_{\partial\Omega}$.

The following theorem ensures the invertibility of the single layer potential operator $\mathcal{V}_b$ in 2D.

Theorem 5.6. Let $\Omega \subset \mathbb{R}^2$ with $r_0 > \mathrm{diam}(\Omega)$. Then the single layer potential $\mathcal{V}_b : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is invertible.

Proof. Since $\mathrm{Cap}_{\partial\Omega} \leq \mathrm{diam}(\Omega)$, see, [129, p.553, properties 1 and 3], then $r_0 > \mathrm{diam}(\Omega)$ implies $r_0 > \mathrm{Cap}_{\partial\Omega}$. For the case $a = b$ the assertion is proved in [37, Theorem 5]. Due to the first relation in (5.28) and Theorem 5.5(ii) follows the invertibility of the single layer potential operator $\mathcal{V}_b$ for the case $a \neq b$ as well, see also [9, Theorem 2]. □

As in [8], we shall restrict ourselves to Theorem 5.6 while discussing about the invertibility of single layer potential V_b in 2D. Because, choosing an appropriate parameter r_0, one can get the zero kernel for $\mathcal{V}$ not only on the subspace $H^{-1/2}_{**}(\partial\Omega)$ but also on the entire space $H^{-1/2}(\partial\Omega)$. The proof to the following result is due to [9, Lemma 1] and [7, Lemma 2].

Lemma 5.2. (i) Let $r_0 > \mathrm{diam}(\Omega)$. If $\Psi^* \in H^{-\frac{1}{2}}(\partial\Omega)$ and $r_\Omega V_b \Psi^* = 0$ in Ω, then $\Psi^* = 0$.

(ii) If $\Phi^* \in H^{\frac{1}{2}}(\partial\Omega)$ and $r_\Omega W_b \Phi^* = 0$ in Ω, then $\Phi^* = 0$.

5.5 Two-operator BDIE systems for Dirichlet problem

Let Ω be a domain in $\mathbb{R}^2$ bounded by a smooth curve $\partial\Omega$. We shall derive and investigate the two-operator BDIE systems for the following Dirichlet problem: *for $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $f \in H^{-1}(\Omega)$, find a function $u \in H^1(\Omega)$ satisfying*

$$Au = f \quad \text{in} \quad \Omega, \tag{5.58}$$

$$\gamma^+ u = \varphi_0 \quad \text{on} \quad \partial\Omega. \tag{5.59}$$

Here equation (5.58) is understood in the distributional sense (5.3) and the Dirichlet boundary condition (5.59) is understood in the trace sense. The following assertion is well-known and can be proved e.g. using variational settings and the Lax-Milgram lemma:

Theorem 5.7. The Dirichlet problem (5.58)-(5.59) is uniquely solvable in $H^1(\Omega)$. The solution is $u = (\mathcal{A}^D)^{-1}(f, \varphi_0)^\top$, where the inverse operator, $(\mathcal{A}^D)^{-1} : H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$, to the left-hand side operator, $\mathcal{A}^D : H^1(\Omega) \to H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, of the Dirichlet problem (5.58)-(5.59), is continuous.

5.5.1 BDIE system formulation to the Dirichlet problem

Following similar procedure as in [80], let us reduce the Dirichlet problem (5.58)-(5.59) with $f \in H^{-1}(\Omega)$, for $u \in H^1(\Omega)$, to two different systems of *segregated two-operator* BDIEs.

Let $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ be an extension of $f \in H^{-1}(\Omega)$ (i.e., $f = r_\Omega \tilde{f}$), which always exists, see, [80, Lemma 2.15 and Theorem 2.16]. We represent in (5.40), (5.56) and (5.57) the generalized conormal derivative and the trace of the function u as

$$T_a^+(\tilde{f}, u) = \psi, \quad \gamma^+ u = \varphi_0$$

respectively, and will regard the new unknown function $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$ as formally segregated of u. Thus we will look for the couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$.

BDIE system (D1). To reduce BVP (5.58)-(5.59) to one of BDIE systems we will use equation (5.40) in Ω and equation (5.56) on $\partial\Omega$. Then we arrive at the system of BDIEs (D1),

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b \psi = \mathcal{F}_1^{D1} \quad \text{in} \quad \Omega, \tag{5.60}$$

$$\gamma^+ \mathcal{Z}_b u + \gamma^+ \mathcal{R}_b u - \mathcal{V}_b \psi = \mathcal{F}_2^{D1} \quad \text{on} \quad \partial\Omega, \tag{5.61}$$

where $F_0 := \mathcal{P}_b \tilde{f} - W_b \varphi_0$ and

$$\mathcal{F}^{D1} := \begin{bmatrix} \mathcal{F}_1^{D1} \\ \mathcal{F}_2^{D1} \end{bmatrix} = \begin{bmatrix} F_0 \\ \gamma^+ F_0 - \varphi_0 \end{bmatrix}. \tag{5.62}$$

For $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$, we have the inclusions $\mathcal{F}_1^{D1} = F_0 \in H^1(\Omega)$ if $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ and due to the mapping properties of operators involved in (5.62), we have the inclusion $\mathcal{F}^{D1} \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.

Remark 5.3. Let $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ and $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$. Then $\mathcal{F}^{D1} = 0$ if and only if $(\tilde{f}, \varphi_0) = 0$.

Proof. The later equality implies the former. Conversely, let $\mathcal{F}^{D1} = 0$, that is, $F_0 = \mathcal{P}_b \tilde{f} - W_b \varphi_0 = 0$ in Ω and $\gamma^+ F_0 - \varphi_0 = 0$ on $\partial\Omega$. Then $\varphi_0 = 0$ on $\partial\Omega$ and $\mathcal{P}_b \tilde{f} = 0$ in Ω. Multiplying the later by b, we get $\mathcal{P}_\Delta \tilde{f} = 0$ in Ω and applying Laplace operator gives $\tilde{f} = 0$ in $\mathbb{R}^2$. □

BDIE system (D2). To obtain a BDIE system of *the second kind*, we will use equation (5.40) in Ω and equation (5.57) on $\partial\Omega$. Then we arrive at the system of BDIEs (D2),

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b\psi = \mathcal{P}_b\tilde{f} - W_b\varphi_0 \quad \text{in} \quad \Omega, \tag{5.63}$$

$$\left(1 - \frac{a}{2b}\right)\psi + T_a^+\mathcal{Z}_b u + T_a^+\mathcal{R}_b u - \mathcal{W}'_{ab}\psi = T_a^+(\tilde{f} + \mathring{E}\mathcal{R}_*^b\tilde{f}, \mathcal{P}_b\tilde{f}) - \mathcal{L}_{ab}^+\varphi_0 \quad \text{on} \quad \partial\Omega, \tag{5.64}$$

where

$$\mathcal{F}^{D2} := \begin{bmatrix} \mathcal{F}_1^{D2} \\ \mathcal{F}_2^{D2} \end{bmatrix} = \begin{bmatrix} \mathcal{P}_b\tilde{f} - W_b\varphi_0 \\ T_a^+(\tilde{f} + \mathring{E}\mathcal{R}_*^b\tilde{f}, \mathcal{P}_b\tilde{f}) - \mathcal{L}_{ab}^+\varphi_0 \end{bmatrix}. \tag{5.65}$$

Due to the mapping properties of operators involved in (5.65), we have the inclusion $\mathcal{F}^{D2} \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$. In similar way as in Remark 5.3, we can prove the following remark.

Remark 5.4. Let $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ and $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$. Then $\mathcal{F}^{D2} = 0$ if and only if $(\tilde{f}, \varphi_0) = 0$.

Proof. The later equality implies the former. Conversely, let $\mathcal{F}^{D2} = 0$, that is, $F_0 = \mathcal{P}_b\tilde{f} - W_b\varphi_0 = 0$ in Ω and $T_a^+(\tilde{f} + \mathring{E}\mathcal{R}_*^b\tilde{f}, \mathcal{P}_b\tilde{f}) - \mathcal{L}_{ab}^+\varphi_0$ on $\partial\Omega$. Multiplying the first relation by b, we get $\mathcal{P}_\Delta\tilde{f} - W_\Delta(b\varphi_0) = 0$ in Ω. Taking into account that $bW_b(\varphi_0) = W_\Delta(b\varphi_0)$ is harmonic and applying Laplace operator gives $\tilde{f} = 0$ in $\mathbb{R}^2$, and hence $W_b\varphi_0 = 0$ in Ω. Then by Lemma 5.2(ii), $\varphi_0 = 0$ on $\partial\Omega$. □

5.5.2 BDIE systems equivalence to the Dirichlet problem

Theorem 5.8. Let $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega), f \in H^{-1}(\Omega)$ and $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ is such that $r_\Omega\tilde{f} = f$. Then

(i) If $u \in H^1(\Omega)$ solves the BVP (5.58)-(5.59), then the couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\Omega)$, where

$$\psi = T_a^+(\tilde{f}, u), \qquad \text{on } \partial\Omega, \tag{5.66}$$

solves the BDIE systems (D1) and (D2).

(ii) If a couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves BDIE system (D1) and $r_0 >$ diam(Ω), then this solution is unique and solves BDIEs (D2), while u solves the Dirichlet problem (5.58)-(5.59), and ψ satisfies (5.66).

(iii) If a couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves BDIE system (D2), then this solution is unique and solves BDIEs (D1), while u solves the Dirichlet problem (5.58)-(5.59), and ψ satisfies (5.66).

Proof. (i) Let $u \in H^1(\Omega)$ be a solution to BVP (5.58)–(5.59). Due to Theorem 5.7 it is unique. Setting ψ by (5.66) evidently implies, $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$. From Theorem 5.3 and relations (5.56)–(5.57) follows that the couple (u, ψ) satisfies the BDIE systems (D1) and (D2), with the right-hand sides (5.62) and (5.65) respectively, which completes the proof of item (i).

Let now the couple $(u,\psi) \in H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)$ solve BDIE system (D1) or (D2). Due to Theorem 5.3, the hypothesis of Lemma 5.1 are satisfied implying that u solves PDE (5.58) in Ω, while relations in (5.46) and (5.47) also hold.

(ii) Let the couple $(u,\psi) \in H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)$ solve BDIE system (D1). Taking trace of (5.60) on $\partial\Omega$ and subtracting (5.61) from it we obtain

$$\gamma^+ u = \varphi_0 \qquad \text{on } \partial\Omega, \tag{5.67}$$

that is, u satisfies the Dirichlet condition (5.59). (5.60) and Lemma 5.1 with $\Psi = \psi$, $\Phi = \varphi_0$ imply that $V_b\Psi^* - W_b\Phi^* = 0$, in Ω, where $\Psi^* = \psi - T_a^+(\tilde{f},u)$ and $\Phi^* = \varphi_0 - \gamma^+ u$. Due to (5.67), $\Phi^* = 0$. Then Lemma 5.2(i) implies $\Psi^* = 0$, which proves condition (5.66). Thus u obtained from the solution of BDIE system (D1) solves the Dirichlet problem and hence, by item (i) of the theorem, (u,ψ) solve also BDIE system (D2).

(iii) Let now the couple $(u,\psi) \in H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)$ solve BDIE system (D2). Taking generalized conormal derivative of (5.63) and subtracting (5.64) from it, we get $\psi = T_a^+(\tilde{f},u)$ on $\partial\Omega$. Then substituting this in (5.47) gives $W_b(\varphi_0 - \gamma^+ u) = 0$ in Ω and Lemma 5.2(ii) then implies $\varphi_0 = \gamma^+ u$ on $\partial\Omega$. Due to (5.62), the BDIE system (5.60)-(5.61) with zero right-hand side can be considered as obtained for $\tilde{f} = 0$, $\varphi_0 = 0$, where $\tilde{f} \in \widetilde{H}(\Omega)$ is an extension of $f \in H^{-1}(\Omega)$, that is, $f = r_\Omega \tilde{f}$, implying that its solution is given by a solution of the homogeneous problem (5.58)-(5.59), which is zero by Theorem 5.7. This implies uniqueness of the solution of the inhomogeneous BDIE system (D1). Similar arguments work for the BDIE system (D2). The proof is complete. □

5.5.3 BDIE system operators invertibility for the Dirichlet problem

The BDIE systems (D1) and (D2) can be written as

$$\mathfrak{D}^1 \mathcal{U}^D = \mathcal{F}^{D1} \quad \text{and} \quad \mathfrak{D}^2 \mathcal{U}^D = \mathcal{F}^{D2},$$

respectively. Here $\mathcal{U}^D := (u,\psi)^\top \in H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)$,

$$\mathfrak{D}^1 \; := \; \begin{bmatrix} I + \mathcal{Z}_b + \mathcal{R}_b & -V_b \\ \gamma^+\mathcal{Z}_b + \gamma^+\mathcal{R}_b & -\mathcal{V}_b \end{bmatrix},$$

$$\mathfrak{D}^2 := \begin{bmatrix} I + \mathcal{Z}_b + \mathcal{R}_b & -V_b \\ T_a^+\mathcal{Z}_b + T_a^+\mathcal{R}_b & \left(1 - \frac{a}{2b}\right) I - \mathcal{W}'_{ab} \end{bmatrix},$$

while $\mathcal{F}^{D1}$ and $\mathcal{F}^{D2}$ are given by (5.62) and (5.65) respectively. Due to the mapping properties of the operators involved in the definitions of the operators $\mathfrak{D}^1$ and $\mathfrak{D}^2$ as well as the right-hand sides $\mathcal{F}^{D1}$ and $\mathcal{F}^{D2}$ (see, e.g., [13, Appendix A], we have $\mathcal{F}^{D1} \in H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$, $\mathcal{F}^{D2} \in H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)$, while the operators

$$\mathfrak{D}^1 : H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega) \tag{5.68}$$

$$\mathfrak{D}^2 : H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega) \tag{5.69}$$

are continuous. Due to Theorem 5.8(ii)-(iii), operators (5.68) and (5.69) are injective.

Lemma 5.3. For any couple $(\mathcal{F}_1, \mathcal{F}_2) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$, there exists a unique couple $(\tilde{f}_{**}, \Phi_*) \in \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ such that

$$\mathcal{F}_1 = \mathcal{P}_b \tilde{f}_{**} - W_b \Phi_* \tag{5.70}$$

$$\mathcal{F}_2 = T_a^+(\tilde{f}_{**} + \mathring{E}\mathcal{R}_*^b \tilde{f}_{**}, \mathcal{P}_b \tilde{f}_{**}) - \mathcal{L}_{ab}^+ \Phi_* \tag{5.71}$$

Moreover, $(\tilde{f}_{**}, \Phi_*) = \mathcal{C}_{**}(\mathcal{F}_1, \mathcal{F}_2)$ with $\mathcal{C}_{**} : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ a linear continuous operator given by

$$\tilde{f}_{**} = \check{\Delta}(b\mathcal{F}_1) + \gamma^*(\mathcal{F}_2 + (\gamma^+\mathcal{F}_1)\partial_n b) \tag{5.72}$$

$$\Phi_* = \frac{1}{b}\Big(-\frac{1}{2}I + \mathcal{W}_\Delta\Big)^{-1} \gamma^+ \Big\{ - b\mathcal{F}_1 + \mathcal{P}_\Delta \Big[\check{\Delta}(b\mathcal{F}_1) + \gamma^*\Big(\frac{b}{a}\mathcal{F}_2 + (\gamma^+\mathcal{F}_1)\partial_n b\Big)\Big]\Big\} \tag{5.73}$$

where $\check{\Delta}(b\mathcal{F}_1) = \nabla \cdot \check{E}\nabla(b\mathcal{F}_1)$.

Proof. Let us first assume that there exist $(\tilde{f}_{**}, \Phi_*) \in \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ satisfying equations (5.70)-(5.71) and find their expression in terms of $\mathcal{F}_1$ and $\mathcal{F}_2$. Let us rewrite (5.70) as

$$\mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**} = -W_b \Phi_* \quad \text{in } \Omega. \tag{5.74}$$

Multiplying (5.74) by b and applying Laplacian to it, we obtain,

$$\Delta(b\mathcal{F}_1 - \mathcal{P}_\Delta \tilde{f}_{**}) = \Delta(b\mathcal{F}_1) - \tilde{f}_{**} = -\Delta(W_\Delta(b\Phi_*)) = 0 \quad \text{in } \Omega, \tag{5.75}$$

which means that

$$\Delta(b\mathcal{F}_1) = r_\Omega \tilde{f}_{**} \quad \text{in } \Omega, \tag{5.76}$$

and $b\mathcal{F}_1 - \mathcal{P}_\Delta \tilde{f}_{**} \in H^{1,0}(\Omega, \Delta)$ and hence $\mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**} \in H^{1,0}(\Omega, B) = H^{1,0}(\Omega, A)$. The latter imply that the canonical conormal derivatives $T_b^+(\mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**})$ and $T_a^+(\mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**})$ are well defined and can be also written in terms of their generalized conormal derivatives

$$\begin{aligned}\frac{b}{a} T_a^+(\mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**}) &= T_b^+(\mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**}) = T_b^+(\tilde{B}(\mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**}), \mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**}) \\ &= T_b^+(\mathring{E}\nabla \cdot (b\nabla(\mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**})), \mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**}) = T_b^+(\mathring{E}\Delta(b\mathcal{F}_1 - \mathcal{P}_\Delta \tilde{f}_{**}) \\ &- \mathring{E}\nabla \cdot ((\mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**})\nabla b), \mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**}) = T_b^+(-\mathring{E}\nabla \cdot (\mathcal{F}_1 \nabla b) - \mathring{E}\mathcal{R}_*^b \tilde{f}_{**}, \mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**})\end{aligned}$$

and therefore,

$$T_a^+(\mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**}) = \;= T_a^+(-\mathring{E}\nabla \cdot (\mathcal{F}_1 \nabla b) - \mathring{E}\mathcal{R}_*^b \tilde{f}_{**},\ \mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**}) \tag{5.77}$$

where (5.53) and (5.76) were taken into account. Applying the conormal derivative operator T_a^+ to both sides of equation (5.74), substituting their (5.77), taking into account (5.8), we obtain,

$$T_a^+(\tilde{f}_{**} - \mathring{E}\nabla \cdot (\mathcal{F}_1 \nabla b),\ \mathcal{F}_1) - T_a^+(\tilde{f}_{**} + \mathring{E}\mathcal{R}_*^b \tilde{f}_{**}, \mathcal{P}_b \tilde{f}_{**}) = -\mathcal{L}_{ab}^+ \Phi_*, \ \text{on } \partial\Omega. \tag{5.78}$$

Subtracting (5.78) from (5.71), we get,

$$\mathcal{F}_2 = T_a^+\big(\tilde f_{**} - \mathring{E}\nabla\cdot(\mathcal{F}_1\nabla b), \mathcal{F}_1\big) \qquad \text{on} \quad \partial\Omega. \tag{5.79}$$

Due to (5.76), we can represent

$$\tilde f_{**} = \check{\Delta}(b\mathcal{F}_1) + \tilde f_{1*} = \nabla\cdot\mathring{E}\nabla(b\mathcal{F}_1) - \gamma^*\Psi_{**} \tag{5.80}$$

where $\tilde f_{1*} \in H^{-1}_{\partial\Omega}$ is defined by (5.2) and hence, due to e.g. [82, Theorem 2.10] can be in turn represented as $\tilde f_{1*} = -\gamma^*\Psi_{**}$, with some $\Psi_{**} \in H^{-\frac{1}{2}}(\partial\Omega)$. Then (5.76) is satisfied and

$$\begin{aligned}
&\frac{b}{a}T_a^+\big(\tilde f_{**} - \mathring{E}\Delta\cdot(\mathcal{F}_1\nabla b), \mathcal{F}_1\big) = T_b^+\big(\tilde f_{**} - \mathring{E}\Delta\cdot(\mathcal{F}_1\nabla b), \mathcal{F}_1\big)\\
&= (\gamma^{-1})^*[\tilde f_{**} - \mathring{E}\nabla\cdot(\mathcal{F}_1\nabla b) - \check{B}\mathcal{F}_1] = (\gamma^{-1})^*[\tilde f_{**} - \mathring{E}\nabla\cdot(\mathcal{F}_1\nabla b) - \nabla\cdot\mathring{E}(b\nabla\mathcal{F}_1)]\\
&\qquad = (\gamma^{-1})^*[\nabla\cdot\mathring{E}\nabla(b\mathcal{F}_1) - \nabla\cdot\mathring{E}(b\nabla\mathcal{F}_1) - \gamma^*\Psi_{**} - \mathring{E}\nabla\cdot(\mathcal{F}_1\nabla b)]\\
&\qquad = (\gamma^{-1})^*[\nabla\cdot\mathring{E}(\mathcal{F}_1\nabla b) - \gamma^*\Psi_{**} - \mathring{E}\nabla\cdot(\mathcal{F}_1\nabla b)] = -\Psi_{**} - (\gamma^+\mathcal{F}_1)\partial_n b
\end{aligned}$$

for which

$$T_a^+(\tilde f_{**} - \mathring{E}\Delta\cdot(\mathcal{F}_1\nabla b), \mathcal{F}_1) = \frac{a}{b}\left[-\Psi_{**} - (\gamma^+\mathcal{F}_1)\partial_n b\right] \tag{5.81}$$

because

$$\begin{aligned}
&\langle(\gamma^{-1})^*[\nabla\cdot\mathring{E}(\mathcal{F}_1\nabla b) - \gamma^*\Psi_{**} - \mathring{E}\nabla\cdot(\mathcal{F}_1\nabla b)], w\rangle_{\partial\Omega}\\
&\qquad = \langle[\nabla\cdot\mathring{E}(\mathcal{F}_1\nabla b) - \gamma^*\Psi_{**} - \mathring{E}\nabla\cdot(\mathcal{F}_1\nabla b)], \gamma^{-1}w\rangle_\Omega\\
&\qquad = \langle[\nabla\cdot\mathring{E}(\mathcal{F}_1\nabla b), \gamma^{-1}w\rangle_{\mathbb{R}^2} - \gamma^*\Psi_{**} - \langle\mathring{E}\nabla\cdot(\mathcal{F}_1\nabla b)], \gamma^{-1}w\rangle_\Omega\\
&= -\langle[\mathring{E}(\mathcal{F}_1\nabla b), \nabla(\gamma^{-1}w)\rangle_{\mathbb{R}^2} - \gamma^*\Psi_{**} + \langle(\mathcal{F}_1\nabla b), \nabla(\gamma^{-1}w)\rangle_\Omega - -\langle n\cdot\gamma^+(\mathcal{F}_1\nabla b), \gamma^+\gamma^- w\rangle_\Omega\\
&\qquad = -\langle(\gamma^+(\mathcal{F}_1)\nabla b), w\rangle_{\partial\Omega} - \Psi_{**}.
\end{aligned}$$

Hence (5.79) reduces to

$$\Psi_{**} = -\frac{b}{a}\mathcal{F}_2 - (\gamma^+\mathcal{F}_1)\partial_n b = -T_b^+\mathcal{F}_1 - (\gamma^+\mathcal{F}_1)\partial_n b, \tag{5.82}$$

and (5.80) to (5.72).

Now (5.74) can be written in the form

$$W_\Delta(b\Phi_*) = \mathcal{F}_\Delta \quad \text{in} \ \ \Omega, \tag{5.83}$$

where

$$\mathcal{F}_\Delta := -b\mathcal{F}_1 + \mathcal{P}_\Delta\tilde f_{**} = -b\mathcal{F}_1 + \mathcal{P}_\Delta\Big[\check{\Delta}(b\mathcal{F}_1) + \gamma^*\Big(\frac{b}{a}\mathcal{F}_2 + (\gamma^+\mathcal{F}_1)\partial_n b\Big)\Big] \tag{5.84}$$

is harmonic function in Ω due to (5.75). The trace of equation (5.84) gives

$$-\frac{1}{2}b\Phi_* + \mathcal{W}_\Delta(b\Phi_*) = \gamma^+\mathcal{F}_\Delta \quad \text{on} \ \ \partial\Omega. \tag{5.85}$$

It is well known that the operator $\left[-\frac{1}{2}I + \mathcal{W}_\Delta\right]$ is an isomorphism, see [119, Lemmas 6.10 and 6.11], this implies that

$$\begin{aligned}\Phi_* &= \frac{1}{b}\Big(-\frac{1}{2}I + \mathcal{W}_\Delta\Big)^{-1}\gamma^+\mathcal{F}_\Delta \\ &= \frac{1}{b}\Big(-\frac{1}{2}I + \mathcal{W}_\Delta\Big)^{-1}\gamma^+\Big\{-b\mathcal{F}_1 + \mathcal{P}_\Delta\Big[\check{\Delta}(b\mathcal{F}_1) + \gamma^*\Big(\frac{b}{a}\mathcal{F}_2 + (\gamma^+\mathcal{F}_1)\partial_n b\Big)\Big]\Big\},\end{aligned}$$

which is equation (5.73). Relations (5.72), (5.73) can be written as $(\tilde{f}_{**}, \Phi_*) = \mathcal{C}_{**}(\mathcal{F}_1, \mathcal{F}_2)$, where $\mathcal{C}_{**} : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ is a linear continuous operator, as required. We still have to check that the functions $\tilde{f}_{**}$ and Φ_*, given by (5.72) and (5.73), satisfy equations (5.70) and (5.71). Indeed, Φ_* given by (5.73) satisfies equation (5.85) and thus $\gamma^+ W_\Delta(a\Phi_*) = \gamma^+\mathcal{F}_\Delta$. Since both $W_\Delta(a\Phi_*)$ and $\mathcal{F}_\Delta$ are harmonic functions, this implies (5.83)-(5.84) and by (5.72) also (5.70). Finally, (5.72) implies by (5.81) that (5.79) is satisfied, and adding (5.78) to it leads to (5.71). Let us prove that the operator $\mathcal{C}_{**}$ is unique. Indeed, let a couple $(\tilde{f}_{**}, \Phi_*) \in \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ be a solution of linear system (5.70)-(5.71) with $\mathcal{F}_1 = 0$ and $\mathcal{F}_2 = 0$. Then (5.76) implies that $r_\Omega \tilde{f}_{**} = 0$ in Ω, that is $\tilde{f}_{**} \in H^{-1}_{\partial\Omega} \subset \widetilde{H}^{-1}(\Omega)$. Hence (5.79) reduces to

$$0 = T_a^+(\tilde{f}_{**}, 0) \qquad \text{on } \partial\Omega.$$

By the first Green identity (5.7), this gives,

$$0 = \langle T_a^+(\tilde{f}_{**}, 0), \gamma^+ v\rangle_{\partial\Omega} = \langle \tilde{f}_{**}, v\rangle_\Omega, \forall v \in H^1(\Omega), \tag{5.86}$$

which implies that $\tilde{f}_{**} = 0$ in $\mathbb{R}^2$. Finally, (5.73) gives $\Phi_* = 0$. Hence any solution of non-homogeneous linear system (5.70) – (5.71) has only one solution, which implies that the uniqueness of the operator $\mathcal{C}_{**}$. The proof is complete. □

The following assertion is [85, Lemma 19] generalized to a wider space in 2D.

Lemma 5.4. For any couple $(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, there exists a unique couple $(\tilde{f}_{**}, \Phi_*) \in \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ such that

$$\tilde{\mathcal{F}}_1 = \mathcal{P}_b\tilde{f}_{**} - W_b\Phi_* \tag{5.87}$$

$$\tilde{\mathcal{F}}_2 = \gamma^+(\mathcal{P}_b\tilde{f}_{**} - W_b\Phi_*) \tag{5.88}$$

Moreover, $(\tilde{f}_{**}, \Phi_*) = \tilde{\mathcal{C}}_{**}(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)$ with $\tilde{\mathcal{C}}_{**} : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to \widetilde{H}^{-1}(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ a linear continuous operator is given by

$$\tilde{f}_{**} = \check{\Delta}(b\tilde{\mathcal{F}}_1) + \gamma^*(T_b^+\tilde{\mathcal{F}}_1 + \tilde{\mathcal{F}}_2)\partial_n b) \tag{5.89}$$

$$\Phi_* = \frac{1}{b}\big(-\frac{1}{2}I + \mathcal{W}_\Delta\big)^{-1}\big(-b\tilde{\mathcal{F}}_2 + \gamma^+\mathcal{P}_\Delta[\check{\Delta}(b\tilde{\mathcal{F}}_1) + \gamma^*(T_b^+\tilde{\mathcal{F}}_1 + \tilde{\mathcal{F}}_2)\partial_n b)]\big) \tag{5.90}$$

where $\check{\Delta}(b\tilde{\mathcal{F}}_1) = \nabla \cdot \check{E}\nabla(b\tilde{\mathcal{F}}_1)$.

Proof. Let us first assume that there exist $(\tilde{f}_{**}, \Phi_*) \in \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ satisfying equations (5.87)-(5.88) and find their expression in terms of $\tilde{\mathcal{F}}_1$ and $\tilde{\mathcal{F}}_2$. Let us re write (5.87) as

$$\tilde{\mathcal{F}}_1 - \mathcal{P}_b \tilde{f}_{**} = -W_b \Phi_* \quad \text{in } \Omega. \tag{5.91}$$

Multiplying (5.91) by b and applying Laplacian to it, we obtain,

$$\Delta(b\tilde{\mathcal{F}}_1 - \mathcal{P}_\Delta \tilde{f}_{**}) = \Delta(b\tilde{\mathcal{F}}_1) - \tilde{f}_{**} = -\Delta(W_\Delta(b\Phi_*)) = 0 \quad \text{in } \Omega, \tag{5.92}$$

which means that

$$\Delta(b\tilde{\mathcal{F}}_1) = r_\Omega \tilde{f}_{**} \quad \text{in } \Omega, \tag{5.93}$$

and $b\tilde{\mathcal{F}}_1 - \mathcal{P}_\Delta \tilde{f}_{**} \in H^{1,0}(\Omega, \Delta)$, while $\tilde{\mathcal{F}}_1 - \mathcal{P}_b \tilde{f}_{**} \in H^{1,0}(\Omega, B) = H^{1,0}(\Omega, A)$. The latter imply that the canonical conormal derivatives $T_b^+(\tilde{\mathcal{F}}_1 - \mathcal{P}_b \tilde{f}_{**})$ and $T_a^+(\tilde{\mathcal{F}}_1 - \mathcal{P}_b \tilde{f}_{**})$ are well defined and $T_a^+(\tilde{\mathcal{F}}_1 - \mathcal{P}_b \tilde{f}_{**}) = \frac{b}{a} T_b^+(\tilde{\mathcal{F}}_1 - \mathcal{P}_b \tilde{f}_{**})$.

Due to (5.93) and using $\tilde{f}_{1*} = -\gamma^* \Psi_{**}$ with some $\Psi_{**} \in H^{-\frac{1}{2}}(\partial\Omega)$ as in (5.82), we can represent

$$\tilde{f}_{**} = \check{\Delta}(b\tilde{\mathcal{F}}_1) + \tilde{f}_{1*} = \nabla \cdot \mathring{E} \nabla(b\tilde{\mathcal{F}}_1) - \gamma^* \Psi_{**} \tag{5.94}$$

where $\tilde{f}_{1*} \in H^{-1}_{\partial\Omega}$. Then (5.93) is satisfied. Replacing $\mathcal{F}_2$ by $T_a^+(\tilde{\mathcal{F}}_1, u)$ in Lemma 5.3, relation (5.82) yields,

$$\Psi_{**} = -\frac{b}{a} T_a^+ \tilde{\mathcal{F}}_1 - (\gamma^+ \tilde{\mathcal{F}}_1)\partial_n b = -T_b^+ \tilde{\mathcal{F}}_1 - \tilde{\mathcal{F}}_2 \partial_n b$$

and (5.94) reduces to (5.89). Now (5.91) can be written in the form

$$W_\Delta(b\Phi_*) = \mathcal{Q}_\Delta \quad \text{in } \Omega, \tag{5.95}$$

where

$$\mathcal{Q}_\Delta := -b\tilde{\mathcal{F}}_1 + \mathcal{P}_\Delta \tilde{f}_{**} = -b\tilde{\mathcal{F}}_1 + \mathcal{P}_\Delta[\check{\Delta}(b\tilde{\mathcal{F}}_1) + \gamma^*(T_b^+ \tilde{\mathcal{F}}_1 + (\gamma^+ \tilde{\mathcal{F}}_1)\partial_n b)] \tag{5.96}$$

is harmonic function in Ω due to (5.92). The trace of equation (5.96) gives

$$-\frac{1}{2} b\Phi_* + \mathcal{W}_\Delta(b\Phi_*) = \gamma^+ \mathcal{Q}_\Delta \quad \text{on } \partial\Omega. \tag{5.97}$$

By similar argument as in Lemma 5.3, the operator $-\frac{1}{2}I + \mathcal{W}_\Delta : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is an isomorphism this implies

$$\begin{aligned} \Phi_* &= \frac{1}{b}\Big(-\frac{1}{2}I + \mathcal{W}_\Delta\Big)^{-1} \gamma^+ \mathcal{Q}_\Delta \\ &= \frac{1}{b}\Big(-\frac{1}{2}I + \mathcal{W}_\Delta\Big)^{-1} \gamma^+ \{-b\tilde{\mathcal{F}}_1 + \mathcal{P}_\Delta[\check{\Delta}(b\tilde{\mathcal{F}}_1) + \gamma^*(T_b^+ \tilde{\mathcal{F}}_1 + (\gamma^+ \tilde{\mathcal{F}}_1)\partial_n b)]\} \end{aligned}$$

which is equation (5.90). Relations (5.89), (5.90) can be written as $(\tilde{f}_{**}, \Phi_*) = \tilde{\mathcal{C}}_{**}(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)$, where $\tilde{\mathcal{C}}_{**} : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ is a linear continuous operator, as required. We still have to check that the functions $\tilde{f}_{**}$ and Φ_*, given by (5.89) and (5.90), satisfy equations (5.87) and (5.88). Indeed, Φ_* given by

(5.90) satisfies equation (5.97) and thus $\gamma^+ W_\Delta(a\Phi_*) = \gamma^+ \mathcal{Q}_\Delta$. Since both $W_\Delta(a\Phi_*)$ and $\mathcal{Q}_\Delta$ are harmonic functions, this implies (5.95)-(5.96) and by (5.89) also (5.87) while (5.88) follows from Eqs.(5.89) and (5.95). Let us prove that the operator $\tilde{\mathcal{C}}_{**}$ is unique. Indeed, let a couple $(\tilde{f}_{**}, \Phi_*) \in \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ be a solution of linear system (5.87)-(5.88) with $\tilde{\mathcal{F}}_1 = 0$ and $\tilde{\mathcal{F}}_2 = 0$. Then (5.93) implies that $r_\Omega \tilde{f}_{**} = 0$ in Ω, that is $\tilde{f}_{**} \in H^{-1}_{\partial\Omega} \subset \widetilde{H}^{-1}(\Omega)$. Hence (5.79) reduces to

$$0 = T_a^+(\tilde{f}_{**}, 0) \qquad \text{on } \partial\Omega.$$

By the first Green identity (5.7), this gives relation (5.86), which implies that $\tilde{f}_{**} = 0$ in $\mathbb{R}^2$. Finally, (5.90) gives $\Phi_* = 0$. Hence any solution of nonhomogeneous linear system (5.87) − (5.88) has only one solution, which implies the uniqueness of the operator $\tilde{\mathcal{C}}_{**}$. The proof is complete. □

Theorem 5.9. Let $r_0 > \mathrm{diam}(\Omega)$. The operators (5.68) and (5.69) are continuous and continuously invertible.

Proof. The continuity of operators (5.68) and (5.69) is proved above. To prove the invertibility of operator (5.68), let us consider the BDIE system (D1) with arbitrary right-hand side

$$\mathcal{F}_*^{D1} = (\mathcal{F}_{*1}^{D1}, \mathcal{F}_{*2}^{D1})^T \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega).$$

Take $\tilde{\mathcal{F}}_1 = \mathcal{F}_{*1}^{D1}$ and $\Phi_* = \gamma^+\mathcal{F}_{*1}^{D1} - \mathcal{F}_{*2}^{D1}$ in Lemma 5.4, to obtain the representation of $\mathcal{F}_*^{D1}$ as:

$$\mathcal{F}_{*1}^{D1} = \tilde{\mathcal{F}}_1 \qquad \mathcal{F}_{*2}^{D1} = \gamma^+\tilde{\mathcal{F}}_1 - \Phi_*$$

where the couple

$$(\tilde{f}_*, \Phi_*) = \tilde{\mathcal{C}}_{**}(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2) \in \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{5.98}$$

is unique and the operator

$$\tilde{\mathcal{C}}_{**} : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{5.99}$$

is linear and continuous. If $r_0 > \mathrm{diam}(\Omega)$, then taking into account [80, Remark 5.3] and applying Theorem 5.7 with $f = r_\Omega\tilde{f} = r_\Omega\tilde{f}_*$, $\Phi_* = \varphi_0$, we obtain that BDIE system (D1) is uniquely solvable and its solution is: $\mathcal{U}_1 = (\mathcal{A}^D)^{-1}(r_\Omega\tilde{f}, \varphi_0)^\top$, $\mathcal{U}_2 = \gamma^+\mathcal{U}_1 - \varphi_0$, where the inverse operator, $(\mathcal{A}^D)^{-1} : H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$, to the left-hand side operator, $\mathcal{A}^D : H^1(\Omega) \to H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, of the Dirichlet problem (5.58)–(5.59), is continuous. Representation (5.98) and continuity of the operator (5.99) imply invertibility of (5.68). To prove the invertibility of operator (5.69), let us consider the BDIE system (D2) with arbitrary right-hand side

$$\mathcal{F}_*^{D2} = (\mathcal{F}_{*1}^{D2}, \mathcal{F}_{*2}^{D2})^\top \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega).$$

Take $\mathcal{F}_1 = \mathcal{F}_{*1}^{D2}$ and $\mathcal{F}_2 = T_a^+(\mathcal{F}_1, u) = \mathcal{F}_{*2}^{D2}$ in Lemma 5.3 to represent $\mathcal{F}_*^{D2}$ as

$$\mathcal{F}_{*1}^{D2} = \mathcal{F}_1 \qquad \mathcal{F}_{*2}^{D2} = T_a^+(\mathcal{F}_1, u) = \mathcal{F}_2$$

and the couple

$$(\tilde{f}_{**}, \Phi_*) = \tilde{\mathcal{C}}_{**}(\mathcal{F}_1, \mathcal{F}_2) \in \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

is unique and the operator

$$\tilde{\mathcal{C}}_{**} : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{5.100}$$

is linear and continuous. Taking into account [80, Remark 5.3] and applying Theorem 5.7 with $\tilde{f} = \tilde{f}_{**}$, $\Phi_* = \varphi_0$, we obtain that BDIE system (D2) is uniquely solvable and its solution is: $\mathcal{U}_1 = (\mathcal{A}^D)^{-1}(r_\Omega \tilde{f}, \varphi_0)^\top$, $\mathcal{U}_2 = T_a^+(r_\Omega \tilde{f}, \mathcal{U}_1)$, where the inverse operator, $(\mathcal{A}^D)^{-1} : H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$, to the left-hand side operator, $\mathcal{A}^D : H^1(\Omega) \to H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, of the Dirichlet problem (5.58)–(5.59), is continuous. Representation (5.98) and continuity of the operator (5.100) imply invertibility of (5.69). The proof is complete. □

Let us finish this section by giving the reason why we need to study the two-operator approach. One-operator approach does *not* work when the fundamental solution of the frozen-coefficient PDE is *not* known explicitly. To overcome this difficultly, one can apply the so-called two-operator approach that employs a parametrix of another (second) PDE not related with the given PDE. Since the second PDE in (5.9) is rather arbitrary, one can always choose it in such a way that its parametrix is known explicitly. The simplest choice for the second PDE is the one with an explicit fundamental solution. We consider the following particular example:

Example 5.1. Let $B = \Delta$ in (5.9). Then **BDIE system (D1)** is written as:

$$\begin{aligned} u + \mathcal{Z}_\Delta u - V_\Delta \psi &= \mathcal{F}^{D1}_{\Delta 1} \quad \text{in} \quad \Omega, \\ \gamma^+ \mathcal{Z}_\Delta u - \mathcal{V}_\Delta \psi &= \mathcal{F}^{D1}_{\Delta 2} \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where

$$\mathcal{Z}_\Delta u := (a-1)\, u + W_\Delta \left((a-1)\gamma^+ u\right) + a\mathcal{R}_a u,$$

and

$$\mathcal{F}^{D1}_\Delta := \begin{bmatrix} \mathcal{F}^{D1}_{\Delta 1} \\ \mathcal{F}^{D1}_{\Delta 2} \end{bmatrix} = \begin{bmatrix} F_{\Delta 0} \\ \gamma^+ F_{\Delta 0} - \varphi_0 \end{bmatrix}, \quad F_{\Delta 0} := \mathcal{P}_\Delta \tilde{f} - W_\Delta \varphi_0.$$

Similarly, **BDIE system (D2)** is written as:

$$\begin{aligned} u + \mathcal{Z}_\Delta u - V_\Delta \psi &= \mathcal{F}^{D2}_{\Delta 1} \quad \text{in} \quad \Omega, \\ \left(1 - \frac{a}{2}\right)\psi + T_a^+ \mathcal{Z}_\Delta u - a\mathcal{W}'_\Delta \psi &= \mathcal{F}^{D2}_{\Delta 2} \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where

$$\mathcal{F}^{D2}_\Delta := \begin{bmatrix} \mathcal{F}^{D2}_{\Delta 1} \\ \mathcal{F}^{D2}_{\Delta 2} \end{bmatrix} = \begin{bmatrix} \mathcal{P}_\Delta \tilde{f} - W_\Delta \varphi_0 \\ T_a^+(\tilde{f}, \mathcal{P}_\Delta \tilde{f}) - a\mathcal{L}_\Delta \varphi_0 \end{bmatrix}.$$

5.6 Two-operator BDIE systems for Neumann problem

Let Ω be a domain in $\mathbb{R}^2$ bounded by a smooth curve $\partial\Omega$. We shall derive and investigate the two-operator BDIE systems for the following Neumann problem: *for* $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ *and* $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$, *find a function* $u \in H^1(\Omega)$ *satisfying*

$$Au = r_\Omega \tilde{f} \qquad \text{in} \quad \Omega, \tag{5.101}$$

$$T_a^+(\tilde{f}, u) = \psi_0 \qquad \text{on} \quad \partial\Omega. \tag{5.102}$$

Here equation (5.101) is understood in the distributional sense (5.3) and the Neumann boundary condition (5.102) in the weak sense (5.7). The following assertion is well-known and can be proved e.g. using variational settings and the Lax-Milgram lemma.

Theorem 5.10. (i) The homogeneous Neumann problem (5.101)-(5.102) admits only linearly independent solution $u^0 = 1$ in $H^1(\Omega)$.

(ii) The nonhomogeneous Neumann problem (5.101)-(5.102) is solvable if and only if the following solvability condition is satisfied.

$$\langle \tilde{f}, u^0\rangle_\Omega - \langle \psi_0, \gamma^+ u^0\rangle_{\partial\Omega} = 0 \tag{5.103}$$

5.6.1 BDIE system formulation for the Neumann problem

We explore different possibilities of reducing the Neumann problem (5.101)–(5.102) with $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$, for $u \in H^1(\Omega)$, to two different *segregated* boundary-domain integral equations (BDIE) systems. Corresponding formulations for the Neumann problem for $u \in H^{1,0}(\Omega, \Delta)$ with $f \in L_2(\Omega)$ in 2D were introduced and analysed in [12]. Let us represent in (5.40), (5.56) and (5.57) the generalised conormal derivative and the trace of the function u as

$$T_a^+(\tilde{f}, u) = \psi_0, \quad \gamma^+ u = \varphi,$$

and will regard the new unknown function $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$ as formally segregated of u. Thus we will look for the couple $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.

BDIE system (N1). To reduce BVP (5.101)-(5.102) to a BDIE system in this section we will use equation (5.40) in Ω and equation (5.57) on $\partial\Omega$. Then we arrive at the following system, (N1), of two boundary-domain integral equations for the couple of unknowns,(u, φ),

$$\begin{aligned} u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b\varphi &= \mathcal{F}_1^{N1}, \quad \text{in} \quad \Omega, \\ T_a^+\mathcal{Z}_b u + T_a^+\mathcal{R}_b u + \mathcal{L}_{ab}^+\varphi &= \mathcal{F}_2^{N1}, \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{5.104}$$

where

$$\mathcal{F}^{N1} := \begin{bmatrix} \mathcal{F}_1^{N1} \\ \mathcal{F}_2^{N1} \end{bmatrix} = \begin{bmatrix} \mathcal{P}_b\tilde{f} + V_b\psi_0 \\ T_a^+(\tilde{f} + \mathring{E}\mathcal{R}_*^b\tilde{f}, \mathcal{P}_b\tilde{f}) - \psi_0 + \frac{a}{2b}\psi_0 + \mathcal{W}'_{ab}\psi_0 \end{bmatrix} \tag{5.105}$$

Due to the mapping properties of operators involved in (5.105) we have $\mathcal{F}^{N1} \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ and $\mathcal{F}_0^N := \mathcal{P}_b\tilde{f} + V_b\psi_0 \in H^1(\Omega)$.

Remark 5.5. Let $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$, $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ and $r_0 > \text{diam}(\Omega)$. Then $\mathcal{F}^{N1} = 0$ if and only if $(\tilde{f}, \psi_0) = 0$.

Proof. The later equality implies the former. Conversely, let $\mathcal{F}^{N1} = 0$, that is, $\mathcal{P}_b\tilde{f} + V_b\psi_0 = 0$ in Ω and $T_a^+(\tilde{f} + \mathring{E}\mathcal{R}_*^b\tilde{f}, \mathcal{P}_b\tilde{f}) - \psi_0 + \frac{a}{2b}\psi_0 + \mathcal{W}'_{ab}\psi_0 = 0$ on $\partial\Omega$. Multiplying the first relation by b gives $\mathcal{P}_\Delta\tilde{f} + V_\Delta\psi_0 = 0$ in Ω. Further, taking into account that $bV_b\psi_0 = V_\Delta\psi_0$ is harmonic and applying Laplace operator we get $\tilde{f} = 0$ in $\mathbb{R}^2$ and hence $V_b\psi_0 = 0$ in Ω. Then due to Lemma 5.2(i), we get $\psi_0 = 0$ on $\partial\Omega$. □

BDIE system (N2). To obtain a segregated BDIE system of *the second kind*, we will use equation (5.40) in Ω and equation (5.56) on $\partial\Omega$. Then we arrive at the following system, (D2), of boundary-domain integral equation systems,

$$u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b \varphi = \mathcal{P}_b \tilde{f} + V_b \psi_0, \quad \text{in} \quad \Omega, \tag{5.106}$$

$$\frac{1}{2}\varphi + \gamma^+ \mathcal{Z}_b u + \gamma^+ \mathcal{R}_b u + \mathcal{W}_b \varphi = \gamma^+ \mathcal{P}_b \tilde{f} + \mathcal{V}_b \psi_0, \quad \text{on} \quad \partial\Omega, \tag{5.107}$$

where

$$\mathcal{F}^{N2} := \begin{bmatrix} \mathcal{F}_1^{N2} \\ \mathcal{F}_2^{N2} \end{bmatrix} = \begin{bmatrix} \mathcal{P}_b \tilde{f} + V_b \psi_0 \\ \gamma^+ \mathcal{P}_b \tilde{f} + \mathcal{V}_b \psi_0 \end{bmatrix}. \tag{5.108}$$

Due to the mapping properties of operators involved in (5.108), we have the inclusion $\mathcal{F}_1^{N2} = \mathcal{P}_b \tilde{f} + V_b \psi_0 \in H^1(\Omega)$ and $\mathcal{F}^{N2} \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.

Remark 5.6. Let $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$, $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ and $r_0 > \operatorname{diam}(\Omega)$. Then $\mathcal{F}^{N2} = 0$ if and only if $(\tilde{f}, \psi_0) = 0$.

Proof. The later equality implies the former. Conversely, let $\mathcal{F}^{N2} = 0$, that is, $\mathcal{P}_b \tilde{f} + V_b \psi_0 = 0$ in Ω and $\gamma^+ \mathcal{P}_b \tilde{f} + \mathcal{V}_b \psi_0 = 0$ on $\partial\Omega$. Multiplying the first relation by b gives $\mathcal{P}_\Delta \tilde{f} + V_\Delta \psi_0 = 0$ in Ω. Further, taking into account that $bV_b\psi_0 = V_\Delta \psi_0$ is harmonic and applying Laplace operator we get $\tilde{f} = 0$ in $\mathbb{R}^2$ and hence $V_b \psi_0 = 0$ in Ω. Then due to Lemma 5.2(i), we get $\psi_0 = 0$ on $\partial\Omega$. □

5.6.2 BDIE systems equivalence to the Neumann problem

Theorem 5.11. Let $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ and $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$.

(i) If a function $u \in H^1(\Omega)$ solves the BVP (5.101)-(5.102), then the couple (u, φ), where

$$\varphi = \gamma^+ u \tag{5.109}$$

solves the BDIE systems (N1) and (N2).

(ii) If a couple $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solves the BDIE system (N1), then the u solves BDIE system (N2) and u solves the Neumann problem (5.101)-(5.102) and φ satisfies (5.109).

(iii) If a couple $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solves the BDIE system (N2) and $r_0 > \operatorname{diam}(\Omega)$, then the u solves BDIE system (N1) and Neumann problem (5.101)-(5.102) and φ satisfies (5.109).

(iv) The homogeneous BDIE systems (N1) and (N2) have unique linearly independent solution spanned by $\mathcal{U}_0 = (u^0, \varphi^0)^\top = (1, 1)^\top$ in $H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$. Condition (5.103) is necessary and sufficient for solvability of the nonhomogeneous BDIE systems (N1) and, if $r_0 > \operatorname{diam}(\Omega)$, also of the system (N2), in $H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.

Proof. (i) Let $u \in H^1(\Omega)$ be a solution to the Neumann BVP (5.101)–(5.102). It immediately follows from Theorem 5.40 and relations (5.56)–(5.57) that the couple (u, φ) with $\varphi = \gamma^+ u$ satisfies the BDIE systems (N1) and (N2), which proves item (i).

(ii) Let now a couple $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solve BDIE system (N1) or (N2). Due to the first equations in the BDIE systems, the hypotheses of Lemma 5.1 are satisfied implying that u is a solution of equation (5.101) in Ω, and equations (5.46)-(5.49) hold for $\Psi = \psi_0$ and $\Phi = \varphi$.

If a couple $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solve the system (N1) then subtracting (5.49) from (5.104) gives $T_a^+(\tilde{f}, u) = \psi_0$ on $\partial\Omega$. Thus Neumann (5.102) is satisfied. Further, from (5.47) we derive $W_b(\gamma^+ u - \varphi) = 0$ in Ω, where $\gamma^+ u = \varphi$ on $\partial\Omega$ by Lemma 5.2 completing the proof of item (ii).

(iii) Let now couple $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solve BDIE system (N2). Further, taking the trace of (5.106) on $\partial\Omega$ and comparing the results with (5.107), we easily derive that $\gamma^+ u = \varphi$ on $\partial\Omega$. Lemma 5.1 for equation (5.106) implies that u is a solution of equation (5.101), while equations (5.46)-(5.49) hold for $\Psi = \psi_0$ and $\Phi = \varphi$. Further, from (5.47) we derive

$$V_b(\psi_0 - T_a(\tilde{f}, u)) = 0 \quad \text{in} \quad \Omega,$$

whence $T_a(\tilde{f}, u) = \psi_0$ on $\partial\Omega$ due to Lemma 5.2 (i) and u solves Neumann problem (5.101)-(5.102) which completes the proof of item (iii).

(iv) Theorem 5.10 along with items (i) and (ii) imply the claims of item (iii) for BDIE system (N2) and (N1). The proof is complete. □

5.6.3 Properties of BDIE system operators for the Neumann problem

BDIE systems (N1) and (N2) can be written respectively, as

$$\mathfrak{R}^1 \mathcal{U}^N = \mathcal{F}^{N1}, \qquad \mathfrak{R}^2 \mathcal{U}^N = \mathcal{F}^{N2}, \tag{5.110}$$

where $\mathcal{U}^N = (u, \varphi)^T \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega_D)$, while $\mathcal{F}^{N1}$ and $\mathcal{F}^{N2}$ are given by Eqs. (5.105) and (5.108) respectively. Due to the mapping properties of potentials in (5.105) and (5.108), $\mathcal{F}^{N1} \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ and $\mathcal{F}^{N2} \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.

$$\mathfrak{R}^1 := \begin{bmatrix} I + \mathcal{Z}_b + \mathcal{R}_b & W_b \\ T_a^+ \mathcal{Z}_b + T_a^+ \mathcal{R}_b & \mathcal{L}_{ab}^+ \end{bmatrix},$$
$$\mathfrak{R}^2 := \begin{bmatrix} I + \mathcal{Z}_b + \mathcal{R}_b & W_b \\ \gamma^+ \mathcal{Z}_b + \gamma^+ \mathcal{R}_b & \frac{1}{2} I + \mathcal{W}_b \end{bmatrix}.$$

Due to the mapping properties of potentials in (5.105) and (5.108), the right hand sides of BDIE systems (N1)and (N2) are such that $\mathcal{F}^{N1} \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ and $\mathcal{F}^{N2} \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.

Theorem 5.12. The operators

$$\mathfrak{R}^1 : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega), \tag{5.111}$$
$$\mathfrak{R}^2 : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega), \tag{5.112}$$

are continuous. They have one-dimensional null spaces, $\ker \mathfrak{R}^1 = \ker \mathfrak{R}^2$, in $H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, spanned over the element $(u^0, \varphi^0) = (1, 1)$.

Proof. The mapping properties of the potentials imply continuity of the operators (5.111) and (5.112). The claims that $\ker \mathfrak{R}^1$ and $\ker \mathfrak{R}^2$ are one-dimensional and the couple $(u^0, \varphi^0) = (1, 1)$ belong to $\ker \mathfrak{R}^1 = \ker \mathfrak{R}^2$ directly follows from Theorem 5.11(iii). □

To describe in more details the range of operators (5.111) and (5.112), i.e., to give more information about the co-kernels of these operators, we will need several auxiliary assertions. First of all, let us remark that for any $v \in H^{s-\frac{3}{2}}(\partial\Omega)$, $s < \frac{3}{2}$, the single layer potential can be defined as follows:

$$V_b v(y) := -\langle \gamma P_b(.,y), v\rangle_{\partial\Omega} = -\langle P_b(.,y), \gamma^* v\rangle_{\mathbb{R}^3} = -\mathbf{P}_b \gamma^* v(y), \quad y \in \mathbb{R}^2 \setminus \partial\Omega. \tag{5.113}$$

where $\gamma^* : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-2}_{\partial\Omega}$, $s < \frac{3}{2}$, is the operator adjoined to the trace operator $\gamma : H^{2-s}(\mathbb{R}^3) \to H^{\frac{3}{2}-s}(\partial\Omega)$, and the space $H^s_{\partial\Omega}$ is defined by (5.2).

Lemma 5.5. Let $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$, $s > \frac{1}{2}$ and $r_0 > \operatorname{diam}(\Omega)$. If

$$r_\Omega \mathbf{P}_b \tilde{f} = 0 \quad \text{in } \Omega, \tag{5.114}$$

then $\tilde{f} = 0$ in $\mathbb{R}^2$.

Proof. Multiplying (5.114) by b, taking into account the first relation in (5.26) and applying the Laplace operator, we obtain $r_\Omega \tilde{f} = 0$, which means that $\tilde{f} \in H^{s-2}_{\partial\Omega}$. If $s \geq \frac{3}{2}$, then $\tilde{f} = 0$ by [82, Theorem 2.10]. If $\frac{1}{2} < s < \frac{3}{2}$, then by the same theorem there exists $v \in H^{s-\frac{3}{2}}(\partial\Omega)$ such that $\tilde{f} = \gamma^* v$. This gives $\mathbf{P}_b \tilde{f} = \mathbf{P}_b \gamma^* v = -V_b v$ in $\mathbb{R}^2$. Then (5.114) reduces to $V_b v = 0$ in Ω, which by Lemma 5.2(i) (for $s = 1$, which can be generalized to $\frac{1}{2} < s < \frac{3}{2}$) implies $v = 0$ on $\partial\Omega$ and thus $\tilde{f} = 0$ in $\mathbb{R}^2$. □

Theorem 5.13. Let $\frac{1}{2} < s < \frac{3}{2}$ and $r_0 > \operatorname{diam}(\Omega)$. The operator

$$\mathbf{P}_b : \widetilde{H}^{s-2}(\Omega) \to H^s(\Omega) \tag{5.115}$$

and its inverse

$$(\mathbf{P}_b)^{-1} : H^s(\Omega) \to \widetilde{H}^{s-2}(\Omega)$$

are continuous and

$$(\mathbf{P_b})^{-1} g = \Big[\Delta \mathring{E}(I - r_\Omega V_\Delta \mathcal{V}_\Delta^{-1} \gamma^+) - \gamma^* \mathcal{V}_\Delta^{-1} \gamma^+\Big](bg) \qquad \text{in} \quad \mathbb{R}^2, \quad \forall g \in H^s(\Omega). \tag{5.116}$$

Proof. The continuity of equation (5.115) follows from [29, Theorem 3.8]. By Lemma 5.5 operator (5.115) is injective. Let us prove its surjectivity. To this end, for arbitrary $g \in H^s(\Omega)$ let us consider the following equation with respect to $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$,

$$\mathbf{P}_\Delta \tilde{f} = g \quad \text{in} \quad \Omega. \tag{5.117}$$

Let $g_1 \in H^s(\Omega)$ be the (unique) solution of the following Dirichlet problem:

$$\Delta g_1 = 0 \text{ in } \Omega, \quad \gamma^+ g_1 = \gamma^+ g,$$

which by [9, Theorem 2] the single layer potential $\mathcal{V}_\Delta^{-1}$ exists and due to [33] or [82, Lemma 2.6] can be particularly presented as $g_1 = V_\Delta \mathcal{V}_\Delta^{-1} \gamma^+ g$. Let $g_0 := g - g_1$. Then

$g_0 \in H^s(\Omega)$ and $\gamma^+ g_0 = 0$ and thus g_0 can be uniquely extended to $\mathring{E} g_0 \in \widetilde{H}^s(\Omega)$, where $\mathring{E}$ is the operator of extension by zero outside Ω . Thus, by (5.113), the equation (5.117) takes the form

$$r_\Omega \mathbf{P}_\Delta[\tilde{f} + \gamma^* \mathcal{V}_\Delta^{-1} \gamma^+ g] = g_0 \quad \text{in} \quad \Omega. \tag{5.118}$$

Any solution $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$ of the corresponding equation on $\mathbb{R}^2$

$$\mathbf{P}_\Delta[\tilde{f} + \gamma^* \mathcal{V}_\Delta^{-1} \gamma^+ g] = \mathring{E} g_0 \quad \text{in } \mathbb{R}^2, \tag{5.119}$$

solves (5.118). If $\tilde{f}$ solves (5.119), then acting with the Laplace operator on (5.119) we obtain

$$\begin{aligned} \tilde{f} = \tilde{Q} g &:= \Delta \mathring{E} g_0 - \gamma^* \mathcal{V}_\Delta^{-1} \gamma^+ g \\ &= \Delta \mathring{E}(g - r_\Omega V_\Delta \mathcal{V}_\Delta^{-1} \gamma^+ g) - \gamma^* \mathcal{V}_\Delta^{-1} \gamma^+ g \quad \text{in} \quad \mathbb{R}^2. \end{aligned} \tag{5.120}$$

On the other hand, substituting $\tilde{f}$ given by (5.120) to (5.119) and taking into account that $\mathbf{P}_\Delta \Delta \tilde{h} = \tilde{h}$ for any $\tilde{h} \in \widetilde{H}^s(\Omega)$, $s \in \mathbb{R}$, we obtain that $\tilde{Q} g$ is indeed a solution of equation (5.119) and thus (5.118). By Lemma 5.5 the solution of (5.119) is unique, which means that the operator $\tilde{Q}$ is inverse to the operator (5.115), i.e., $\tilde{Q} = (r_\Omega \mathbf{P}_b)^{-1}$. Since Δ is a continuous operator from $\widetilde{H}^s(\Omega)$ to $\widetilde{H}^{s-2}(\Omega)$, equation (5.72) implies that the operator $(r_\Omega \mathbf{P})_b^{-1} = \tilde{Q} : H^s(\Omega) \to \widetilde{H}^{s-2}(\Omega)$ is continuous. The relations $\mathbf{P}_b = \frac{1}{b} \mathbf{P}_\Delta$ and $b(x) > c > 0$ then imply the invertibility of the operator (5.115) and anstatz (5.116). The proof is complete. □

Theorem 5.14. The co-kernel of the operator (5.111) is spanned over the functional

$$g^{*1} := ((\gamma^+)^* \partial_n b, 1)^\top \tag{5.121}$$

in $\widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, that is, $g^{*1}(\mathcal{F}_1, \mathcal{F}_2) = \langle (\gamma^+ \mathcal{F}_1) \partial_n b + \mathcal{F}_2, \gamma^+ u^0 \rangle_{\partial\Omega}$, where $u^0 = 1$.

Proof. The proof follows from the proof of [80, Theorem 6.7] and Lemma 5.3. Indeed, let us consider the first equation in (5.110), i.e. the equation $\mathfrak{R}^1 \mathcal{U} = (\mathcal{F}_1, \mathcal{F}_2)^\top$, representing the BDIE system (N1)

$$u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b \varphi = \mathcal{F}_1 \quad \text{in} \quad \Omega, \tag{5.122}$$

$$T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u + \mathcal{L}_{ab}^+ \varphi = \mathcal{F}_2 \quad \text{on } \partial\Omega, \tag{5.123}$$

with arbitrary right hand side $(\mathcal{F}_1, \mathcal{F}_2)^\top \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$, for $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$. By Lemma 5.3 the right-hand side of the system has the form (5.70)-(5.71), that is, system (5.122)-(5.123) reduces to

$$u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b(\varphi + \Phi_*) = \mathcal{P}_b \tilde{f}_{**} \quad \text{in} \quad \Omega, \tag{5.124}$$

$$T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u + \mathcal{L}_{ab}^+(\varphi + \Phi_*) = T_a^+(\tilde{f}_{**} + \mathring{E} \mathcal{R}_*^b \tilde{f}_{**}, \mathcal{P}_b \tilde{f}_{**}) \quad \text{on} \quad \partial\Omega, \tag{5.125}$$

where the couple $(\tilde{f}_{**}, \Phi_*) \in \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ is given by (5.70)-(5.71). Up to the notations (5.124)-(5.125) is the same as in (5.105) with $\psi_0 = 0$. Then, Theorems

5.11(iii) and 5.13 imply that the BDIE system (5.124)-(5.125) and hence (5.122)-(5.123) is solvable if and only if

$$\begin{aligned}\langle \tilde{f}_{**}, u^0\rangle_\Omega &= \langle(\check{\Delta} b\mathcal{F}_1) + \gamma^*(\mathcal{F}_2 + (\gamma^+\mathcal{F}_1)\partial_n b), u^0\rangle_\Omega \\ &= \langle(\nabla\cdot\check{E}\nabla(b\mathcal{F}_1) + \gamma^*(\mathcal{F}_2(\gamma^+\mathcal{F}_1)\partial_n b), u^0\rangle_{\mathbb{R}^2} \\ &= \langle(\nabla\cdot\check{E}\nabla(b\mathcal{F}_1, \nabla u^0\rangle_{\mathbb{R}^2} + \langle(\mathcal{F}_2 + (\gamma^+\mathcal{F}_1)\partial_n b), \gamma^+u^0\rangle_{\partial\Omega} \\ &= \langle(\mathcal{F}_2 + (\gamma^+\mathcal{F}_1)\partial_n b), \gamma^+u^0\rangle_{\partial\Omega} = 0,\end{aligned}$$

where we took into account that $\nabla u^0 = 0$ in $\mathbb{R}^2$. Thus the functional g^{*1} defined by (5.121) generates the necessary and sufficient solvability condition for the first equation in (5.110). Hence g^{*1} is basis of the co-kernel of $\mathfrak{R}^1$. The proof is complete. □

Theorem 5.15. Let $r_0 > \operatorname{diam}(\Omega)$. Then the co-kernel of operator (5.112) is spanned over

$$g^{*2} := \begin{pmatrix} -b\gamma^{+*}(\frac{1}{2} + \mathcal{W}'_\Delta)\mathcal{V}_\Delta^{-1}\gamma^+u^0 \\ -b(\frac{1}{2} - \mathcal{W}'_\Delta)\mathcal{V}_\Delta^{-1}\gamma^+u^0 \end{pmatrix} \tag{5.126}$$

in $\widetilde{H}^{-1}(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$, that is,

$$g^{*2}(\mathcal{F}_1, \mathcal{F}_2) = \Big\langle -b\gamma^{+*}\Big(\frac{1}{2}+\mathcal{W}'_\Delta\Big)\mathcal{V}_\Delta^{-1}\gamma^+u^0, \mathcal{F}_1\Big\rangle_\Omega + \Big\langle -b\Big(\frac{1}{2}-\mathcal{W}'_\Delta\Big)\mathcal{V}_\Delta^{-1}\gamma^+u^0, \mathcal{F}_2\Big\rangle_{\partial\Omega},$$

where $u^0 = 1$.

Proof. The proof follows from the proof of [80, Theorem 6.8], [9, Theorem 2] and Lemma 5.3. Indeed, let us consider the first equation in (5.110), i.e. the equation $\mathfrak{R}^1\mathcal{U} = (\mathcal{F}_1, \mathcal{F}_2)^\top$, representing the BDIE system (N1)

$$u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b\varphi = \mathcal{F}_1 \quad \text{in} \quad \Omega, \tag{5.127}$$

$$\frac{1}{2}\varphi + \gamma^+\mathcal{Z}_b u + \gamma^+\mathcal{R}_b u + \mathcal{W}_b\varphi = \mathcal{F}_2 \quad \text{on} \quad \partial\Omega, \tag{5.128}$$

with arbitrary $(\mathcal{F}_1, \mathcal{F}_2)^T \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$, for $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.

Introducing the new variable , $\varphi' = \varphi - (\mathcal{F}_2 - \gamma^+\mathcal{F}_1)$, BDIE system (5.127)-(5.128) takes the form

$$u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b\varphi = \mathcal{F}'_1 \quad \text{in} \quad \Omega, \tag{5.129}$$

$$\frac{1}{2}\varphi' + \gamma^+\mathcal{Z}_b u + \gamma^+\mathcal{R}_b u + \mathcal{W}_b\varphi' = \mathcal{F}'_2 \quad \text{on} \quad \partial\Omega, \tag{5.130}$$

where

$$\mathcal{F}'_1 = \mathcal{F}_1 - W_b(\mathcal{F}_2 - \gamma^+\mathcal{F}_1) \in H^1(\Omega).$$

Let us recall that $\mathcal{P}_b = r_\Omega\mathbf{P}_b : \widetilde{H}^{s-2}(\Omega) \to H^s(\Omega)$ and then by Theorem 5.13, the operator $\mathcal{P}_b^{-1} = (\mathbf{P}_b)^{-1} : H^s(\Omega) \to \widetilde{H}^{s-2}(\Omega)$ is continuous for $\frac{1}{2} < s < \frac{3}{2}$, while $\mathcal{V}_\Delta^{-1}$ exists [9, Theorem 2]. Hence we always represent $\mathcal{F}_1 = \mathcal{P}_b\tilde{f}_*$, with

$$\tilde{f}_* = [\Delta\mathring{E}(I - r_\Omega V_\Delta\mathcal{V}_\Delta^{-1}\gamma^+) - \gamma^*\mathcal{V}_\Delta^{-1}\gamma^+](b\mathcal{F}'_1) \in \widetilde{H}^{-1}(\Omega).$$

For $\mathcal{F}_1' = \mathcal{P}_b\tilde{f}_*$, the right hand side of BDIE system (5.129)-(5.130) is the same as in (5.108) with $f = \tilde{f}_*$ and $\psi_0 = 0$. Then Theorems 5.11(iii) implies that the BDIE system (5.129)-(5.130) and hence (5.127)-(5.128) is solvable if and only if

$$\langle \tilde{f}_*, u^0\rangle_\Omega = \langle [\Delta\mathring{E}(I - r_\Omega V_\Delta \mathcal{V}_\Delta^{-1}\gamma^+) - \gamma^{+*}\mathcal{V}_\Delta^{-1}\gamma^+](b\mathcal{F}_1'), u^0\rangle_{\mathbb{R}^2}$$
$$= \langle \mathring{E}(I - r_\Omega V_\Delta \mathcal{V}_\Delta^{-1}\gamma^+)(b\mathcal{F}_1'), \Delta u^0\rangle_{\mathbb{R}^2} - \langle(\gamma^{+*}\mathcal{V}_\Delta^{-1}\gamma^+)(b\mathcal{F}_1'), u^0\rangle_{\mathbb{R}^2} - \langle \gamma^+(b\mathcal{F}_1'), \mathcal{V}_\Delta^{-1}\gamma^+u^0\rangle_{\partial\Omega}$$
$$= -\langle \frac{1}{2}[\gamma^+(b\mathcal{F}_1) + (b\mathcal{F}_2)] - \mathcal{W}_\Delta[b(\mathcal{F}_2 - \gamma^+\mathcal{F}_1)]\mathcal{V}_\Delta^{-1}\gamma^+u^0\rangle_{\partial\Omega}$$
$$= \langle -b\gamma^{+*}\Big(\frac{1}{2} + \mathcal{W}_\Delta'\Big)\mathcal{V}_\Delta^{-1}\gamma^+u^0, \mathcal{F}_1\rangle_\Omega + \langle -b\Big(\frac{1}{2} + \mathcal{W}_\Delta'\Big)\mathcal{V}_\Delta^{-1}\gamma^+u^0, \mathcal{F}_2\rangle_{\partial\Omega} = 0.$$

Thus the functional g^{*2} defined by (5.126) generates the necessary and sufficient solvability condition of the equation $\mathfrak{R}^2\mathcal{U} = (\mathcal{F}_1, \mathcal{F}_2)^\top$. Hence g^{*2} is basis of the cokernel of $\mathfrak{R}^2$. The proof is complete. □

5.6.4 Perturbed segregated BDIE systems for Neumann problem

Theorem 5.11 implies, that even when the solvability condition (5.103) is satisfied, the solutions of both BDIE systems, (N1) and (N2), are not unique. By Theorem 5.12, in turn, the BDIE left hand side operators, $\mathfrak{R}^1$ and $\mathfrak{R}^2$, have non-zero kernels and thus are not invertible. To find a solution (u, φ) from uniquely solvable BDIE system with continuously invertible left hand side operators, let us consider, following [87], some BDIE systems obtained form (N1) and (N2) by finite-dimensional operator perturbations (cf. [7] for the three-dimensional case). Below we use the notations $\mathcal{U} = (u, \varphi)^\top$ and $|\partial\Omega| := \int\limits_{\partial\Omega} dS$.

Perturbation of BDIE system (N1)

Let us introduce the perturbed counterparts of the BDIE system (N1),

$$\hat{\mathfrak{R}}^1\mathcal{U}^N = \mathcal{F}^{N1}, \tag{5.131}$$

where

$$\hat{\mathfrak{R}}^1 := \hat{\mathfrak{R}}^1 + \mathring{\mathfrak{R}}^1 \text{ and } \mathring{\mathfrak{R}}^1\mathcal{U}^N(y) := g^0(\mathcal{U}^N)\mathcal{G}^1(y) = \frac{1}{|\partial\Omega|}\int_{|\partial\Omega|}\varphi(x)dS \begin{pmatrix} b^{-1}(y) \\ 0 \end{pmatrix},$$

that is,

$$g^0(\mathcal{U}^N) := \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\varphi(x)dS, \quad \mathcal{G}^1(y) := \begin{pmatrix} b^{-1}(y) \\ 0 \end{pmatrix}.$$

For the functional g^{*1} given by (5.121) in Theorem 5.14, $g^{*1}(\mathcal{G}^1) = |\partial\Omega|$, while $g^0(\mathcal{U}^0) = 1$. Hence [80, Theorem D.1 in Appendix] and [12] imply the following assertion.

Theorem 5.16. Let $r_0 > \text{diam}(\Omega)$, then

(i) The operator $\hat{\mathfrak{R}}^1 : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ is continuous and continuously invertable.

(ii) If condition $g^{*1}(\mathcal{F}^{N1}) = 0$ or condition (5.103) for $\mathcal{F}^{N1}$ in form (5.111) is satisfied, then the unique solution of perturbed BDIDE system (5.131) gives a solution of original BDIE system (N1) such that

$$g^0(\mathcal{U}^N) = \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\gamma^+ u dS = \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\varphi dS = 0.$$

Perturbation of BDIE system (N2)

Let us introduce the perturbed counterparts of the BDIE system (N2)

$$\hat{\mathfrak{R}}^2\mathcal{U}^N = \mathcal{F}^{N2}, \tag{5.132}$$

where

$$\hat{\mathfrak{R}}^2 := \mathfrak{R}^2 + \mathring{\mathfrak{R}}^2 \quad \text{and} \quad \mathring{\mathfrak{R}}^2\mathcal{U}^N(y) := g^0(\mathcal{U}^N)\mathcal{G}^2(y) = \frac{1}{|\partial\Omega|}\int_{|\partial\Omega|}\varphi(x)ds\begin{pmatrix} b^{-1}(y) \\ \gamma^+ b^{-1}(y)\end{pmatrix},$$

that is,

$$g^0(\mathcal{U}^N) := \frac{1}{|\partial\Omega|}\int_{|\partial\Omega|}\varphi(x)ds, \mathcal{G}^2(y) := \begin{pmatrix} (b^{-1}u^0)(y) \\ \gamma^+(b^{-1}u^0)(y)\end{pmatrix}.$$

For the functional g^{*2} given by (5.126) in Theorem 5.15, since the operator $\mathcal{V}_\Delta^{-1}$: $H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is positive definite (with additional condition $r_0 > \operatorname{diam}(\Omega)$) and $u^0(x) = 1$, there exists a positive constant C such that

$$\begin{aligned} g^{*2}(\mathcal{G}^2) &= \langle -b\gamma^{+*}(\frac{1}{2}+\mathcal{W}'_\Delta)\mathcal{V}_\Delta^{-1}\gamma^+u^0, b^{-1}u^0\rangle_\Omega + \langle -b(\frac{1}{2}-\mathcal{W}'_\Delta)\mathcal{V}_\Delta^{-1}\gamma^+u^0, \gamma^+\big(b^{-1}u^0\big)\rangle_{\partial\Omega} \\ &= -\langle(\frac{1}{2}+\mathcal{W}'_\Delta)\mathcal{V}_\Delta^{-1}\gamma^+u^0 + +(\frac{1}{2}-\mathcal{W}')\mathcal{V}_\Delta^{-1}\gamma^+u^0, \gamma^+u^0\rangle_{\partial\Omega} \\ &= -\langle\mathcal{V}_\Delta^{-1}\gamma^+u^0, \gamma^+u^0\rangle_{\partial\Omega} \le -C\|\gamma^+u^0\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \le -C\|\gamma^+u^0\|^2_{H^{\frac{1}{2}}(\partial\Omega)} = -C|\partial\Omega|^2 < 0. \end{aligned} \tag{5.133}$$

Due to (5.133) and $g^0(\mathcal{U}^0) = 1$, [12, Theorem 7] and [80, Theorem D.1] imply the following assertion.

Theorem 5.17. Let $r_0 > \operatorname{diam}(\Omega)$, then

(i) The operator $\hat{\mathfrak{R}}^2 : H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$ is continuous and continuously invertible.

(ii) If condition $g^{*2}(\mathcal{F}^2) = 0$ or condition (5.103) for $\mathcal{F}^{N2}$ in form (5.112) is satisfied, then the unique solution of perturbed BDIDE system (5.132) gives a solution of original BDIE system (N2) such that

$$g^0(\mathcal{U}^N) = \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\gamma^+ u dS = \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\varphi dS = 0.$$

Chapter 6

Conclusion and future plans

6.1 Conclusion

In this PhD thesis, we study two closely related mathematical subjects: Integral equations and integral inequalities with refiements. The first part of the thesis deals with integral inequalities with refinements. First, we consider some of the most important inequalities: the Hardy, Pólya-Knopp, Cochran-Lee, Jensen, Minkowski and Beckenbach-Dresher inequalities. Then various generalizations and refinements of these classical inequalities in different function spaces are obtained with the main focus on the Hardy inequality and its limiting case Pólya-Knopp inequality, and even more general Cochran-Lee inequality. In Banach space settings, some new Hardy-type inequalities are proved and applied in the case when the classical Hardy kernel operator is generalized to a new general Hardy operator. The superquadraticity technique is also used to obtain some new generalized and refined forms of the Jensen, Minkowski and Beckenbach-Dresher inequalities. For the case $0 < p \leq q < \infty$, some new Cochran-Lee inequalities in higher dimensions are proved and good two-sided estimates of the sharp constants are obtained. Using these results a new multidimensional weighted Cochran-Lee inequality with sharp constant is also proved. Further, these results are extended to Pólya-Knopp type inequalities on homogeneous groups using a direct method.

In the second part of the thesis, the Dirichlet and Neumann BVPs for the linear second-order scalar elliptic PDE with variable coefficients in a bounded two-dimensional domain are considered. The right-hand side of the PDE belongs to $H^{-1}(\Omega)$ or $\widetilde{H}^{-1}(\Omega)$, when neither classical nor canonical conormal derivatives of solutions are well defined. The two-operator approach and appropriate parametrix (Levi function) are used to reduce each of the problems to two different systems of two-operator BDIEs. Although the theory of BDIEs in 3D is well developed, the BDIEs in 2D need a special consideration due to their different equivalence properties. As a consequence of this fact, we need to set conditions on the associated Sobolev spaces or choose appropriate scaling parameter in the parametrix form, to insure the invertibility of corresponding parametrix-based integral layer potentials and, hence, the unique solvability of BDIEs. The equivalence of the two-operator BDIE systems to the original problems, BDIE system solvability, solution uniqueness/nonuniqueness and invertibility BDIE system are analyzed in the appropriate Sobolev spaces. It is shown that the BDIE operators for the Neumann BVP are not invertible, and appropriate finite-dimensional perturbations are constructed leading to invertibility of the perturbed operators.

Finally, we give examples on the rôle of the Hardy inequality in the study of BVPs for linear and nonlinear PDEs and set direction for future activities (see also our Section 6.3).

6.2 Examples and motivation

Let us begin by recalling the various Hardy inequalities and their rôles in the study of BVPs for linear and nonlinear PDEs (see [34]). Let Ω be a bounded domain in $\mathbb{R}^n$ containing the origin and where $n \geq 3$. Then Hardy's inequality is given by (see [102]):

$$\int_\Omega |\nabla u|^2 \mathrm{d}x \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} \mathrm{d}x, \tag{6.1}$$

for all $u \in H_0^1(\Omega)$ (and ∇u is the gradient of u). Moreover the constant $\left(\frac{n-2}{2}\right)^2$ is optimal and not attained. An analogous result asserts that for any bounded convex domain $\Omega \subset \mathbb{R}^n$ with smooth boundary and $d(x) : dist(x, \partial\Omega)$ (the Euclidean distance from x to $\partial\Omega$), there holds (see [26]):

$$\int_\Omega |\nabla u|^2 \mathrm{d}x \geq \frac{1}{4} \int_\Omega \frac{u^2}{d^2} \mathrm{d}x, \tag{6.2}$$

for all $u \in H_0^1(\Omega)$. Moreover the constant $\frac{1}{4}$ is optimal and not attained. We will refer to this inequality as Hardy's boundary inequality. Note that in the simplest case when $\Omega = (0, \infty)$ and $u(x) = \int_0^x f(t)\,\mathrm{d}t$, then (6.2) coincides with the standard Hardy inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,\mathrm{d}t\right)^2 \mathrm{d}x \leq 4 \int_0^\infty f^2(t)\,\mathrm{d}t,$$

with the sharp constant 4.

In the last few years improved versions of the above inequalities have been obtained, in the sense that nonnegative terms are added to the right hand sides of the inequalities (see [26] and the references therein). One common type of improvement for the above Hardy inequalities are the so called potentials; we call $0 \leq V(x)$, defined in Ω, a potential for (6.1) provided

$$\int_\Omega |\nabla u|^2 \mathrm{d}x - \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} \mathrm{d}x \geq \int_\Omega V(x) u^2 \mathrm{d}x, \qquad u \in H_0^1(\Omega). \tag{6.3}$$

Most of the results in this direction are explicit examples of potentials V, where, in the best results, V is an infinite series involving complicated inductively defined functions. Very recently Ghoussoub and Moradifam [42] gave the following necessary and sufficient conditions for a radial function $V(x) = v(|x|)$ to be a potential in the case of Hardy's inequality (6.1) on a radial domain Ω:

V is a potential if and only if there exists a positive function $y(r)$ which solves $y'' + \frac{y'}{r} + vy = 0$ in $\left(0, \sup\limits_{x\in\Omega} |x|\right)$.

In another direction people have considered Hardy inequalities for operators more general Laplacian. One case of this is the results obtained by Adimurthi and A. Sekar [5]:

Suppose that Ω is a smooth domain in $\mathbb{R}^n$ which contains the origin, $A(x) = ((a^{i,j}(x)))$ denotes a symmetric, uniformly positive definite matrix with suitably smooth coefficients and for $\xi \in \mathbb{R}^n$ we define $|\xi|_A^2 := |\xi|_{A(x)}^2 := A(x)\xi \cdot \xi$. Now suppose that E is a solution of $\mathcal{L}_{A,p}(E) := -div\left(|\nabla E|_A^{p-2} A\nabla E\right) = \delta_0$ in Ω with $E = 0$ on $\partial\Omega$ and where δ_0 is the Dirac mass at 0. Then, for all $u \in W_0^{1,p}(\Omega)$, we have that

$$\int_\Omega |\nabla u|_A^p \mathrm{d}x - \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|\nabla E|_A^p}{E^p}|u|^p \mathrm{d}x \geq 0. \tag{6.4}$$

Example 6.1. (Fundamental solutions and Hardy's inequality) Let Δ as usual denote the Laplace operator.

1. **Hardy's inequality in 2D:** Now suppose Ω is a domain in $\mathbb{R}^2$ which contains the origin. Put $E(x) := \log\left(R^{-1}|x|\right)$ where $R := \sup_\Omega |x|$. Then $\Delta E = c\delta_0$ where $c > 0$ and δ_0 is the Dirac mass at 0. Putting E into (6.4) with $p = 2$ we find that

$$\int_\Omega |\nabla u|^2 \mathrm{d}x - \frac{1}{4}\int_\Omega \frac{|\nabla E|^2}{E^2} u^2 \mathrm{d}x \geq 0, \quad u \in H_0^1(\Omega) \tag{6.5}$$

so that

$$\int_\Omega |\nabla u|^2 \mathrm{d}x \geq \frac{1}{4}\int_\Omega \frac{u^2}{|x|^2 \log^2\left(R^{-1}|x|\right)} \mathrm{d}x, \quad u \in C_c^\infty(\Omega).$$

2. **Hardy's inequality in 3D and higher dimensions:** Let Ω denote a domain in $\mathbb{R}^n (n \geq 3)$ which contains the origin and set $E(x) := -|x|^{2-n}$. Then $\Delta E = c\delta_0$ where $c > 0$. Also $\frac{|\nabla E|^2}{4E^2} = \left(\frac{n-2}{2}\right)^2 \frac{1}{|x|^2}$ and putting E into (6.5) gives the Hardy's inequality (6.1) i.e. that

$$\int_\Omega |\nabla u|^2 \mathrm{d}x \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} \mathrm{d}x.$$

(See [34]).

Example 6.2. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain where $n \geq 3$ and $0 \in \Omega$. Let us consider the heat equation with zero Dirichlet boundary condition (cf. [41]):

$$\begin{cases} u_t - \Delta u = \lambda \dfrac{u}{|x|^2} & \text{in } \Omega \times (0,T) \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) \geq 0, & \text{in } \Omega \end{cases} . \tag{6.6}$$

In [126], the authors study the well-posedness and describe the asymptotic behavior of solutions of the heat equation with inverse-square potentials for the Cauchy-Dirichlet problem (6.6) in a bounded domain and also for the Cauchy problem in $\mathbb{R}^n$.

In the case of bounded domain the authors used an improved form of the so-called Hardy-Poincare inequality and proved exponential stabilization towards a solution in separated variables. In $\mathbb{R}^n$ the authors first establish a new weighted version of the Hardy-Poincaré inequality, and then show the stabilization towards a radially symmetric solution in self-similar variables with a polynomial decay.

The result is as follows: Problem (6.6) has a global solution if $\lambda \leq \lambda_* = \left(\frac{n-2}{2}\right)^2$ and no solution, even locally in time, if $\lambda > \lambda_*$. The solution when $\lambda > \lambda_*$ tends to infinity for all $(x,t) \in \Omega \times (0,\infty)$ (so-called complete instantaneous blow up). Moreover, the authors pointed that there is more discussion if the sign restriction on u_0 or u is dropped. While the standard variational analysis applies to the case $\lambda < \lambda_*$, so that for every $u_0 \in L^2(\Omega)$ there exists a solution $u \in C\left([0,\infty); L^2(\Omega)\right) \cap L^2\left(0,\infty; H_0^1(\Omega)\right)$, which is global in time, which is not true for the limit case $\lambda = \lambda_*$.

Example 6.3. In [35], the author proved some Hardy-type inequalities related to quasilinear second-order degenerate elliptic differential operators

$$L_p u := -\nabla_L^* \left(|\nabla_L u|^{p-2} \nabla_L u\right).$$

If ϕ is a positive weight such that $-L_p\phi \geq 0$, then the Hardy-type inequality

$$c \int_\Omega \frac{|u|^p}{\phi^p} |\nabla_L \phi|^p \mathrm{d}\xi \leq \int_\Omega |\nabla_L u|^p \mathrm{d}\xi \qquad \left(u \in C_0^1(\Omega)\right)$$

holds.

Example 6.4. In the recent paper [100] the authors proved a new Hardy-type inequality in order to be able to exactly describe the oscillatory and spectral properties of a class of fourth-order differential operator.

Finally, we pronounce that in the very well-cited book [73] several motivations for the importance of Hardy-type inequalities are given. The authors even initiate the book with the following example:

Example 6.5. The non-linear ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(v(x)\left|\frac{\mathrm{d}}{\mathrm{d}x}\right|^{p-2}\frac{\mathrm{d}y}{\mathrm{d}x}\right) + \lambda u(x)|y(x)|^{q-2}y(x) = 0 \text{ on } (a,b) \tag{6.7}$$

together with the homogeneous boundary conditions

$$y(a) := \lim_{x\to a^+} y(x) = 0, y(b) := \lim_{x\to b^-} y(x) = 0. \tag{6.8}$$

Here, $-\infty \leq a < b \leq \infty, p > 1, q > 1, u = u(x), v = v(x)$ are weight functions, i.e., functions which are measurable and positive a.e. in (a,b).

A function $y = y(x)$ is a weak solution of (6.7)-(6.8), if the identity

$$\int_a^b |y'(x)|^{p-2} y'(x) z'(x) v(x) \mathrm{d}x = \lambda \int_a^b |y'(x)|^{q-2} y(x) z(x) u(x) \mathrm{d}x \tag{6.9}$$

holds for every function $z = z(x) \in C_0^\infty(a,b)$. [Notice that (6.9) can be obtained, multiplying (6.7) by $z(x)$ and integrating by parts.]

Putting in (6.9) $z = y$, we obtain that

$$\int_a^b |y'(x)|^p v(x)\mathrm{d}x = \lambda \int_a^b |y(x)|^q u(x)\mathrm{d}x. \tag{6.10}$$

If we introduce the weighted Lebesgue space $L^r(w) = L^r(w; a, b)$ with $r > 1$ and the weight $w = w(x)$ as

$$L^r(w) := \{y = y(x), x \in (a,b), \|y\|_{r,w} := \left(\int_a^b |y(x)|^r w(x)\mathrm{d}x\right)^{1/r} < \infty\},$$

then we can rewrite (6.10) as

$$\|y'\|_{p,v}^p = \lambda \|y\|_{q,u}^q. \tag{6.11}$$

Suppose that we have an inequality of the form

$$\left(\int_a^b |f(x)|^q u(x)\mathrm{d}x\right)^{1/q} \leq C_{p,q} \left(|f'(x)|^p v(x)\mathrm{d}x\right)^{1/p} \tag{6.12}$$

or shortly

$$\|f\|_{q,u} \leq C_{p,q} \|f'\|_{p,v}, \tag{6.13}$$

which should hold for all functions f on (a, b) such that $\|f\|_{p,v} < \infty$ and satisfying some additional conditions (like, e.g., $f(a) = 0$ or $f(b) = 0$).

Inequality (6.12) is called the Hardy inequality or Hardy's inequality in differential form (since the function f is estimated by its derivative f').

Comparing (6.11)-(6.13) with $f = y$ we obtain after normalization (i.e., taking $\|y'\|_{p,v} = 1$) that

$$\lambda \geq \frac{1}{C_{p,q}^q}.$$

Consequently, the Hardy inequality (6.12) [more precisely, the (optimal) constant $C_{p,q}$ in this inequality] provides us with an estimate from below of the possible eigenvalues of the problem (6.7)-(6.8). Moreover, the Hardy inequality can give more information about the spectrum of differential operators like that in (6.7), and also about more-dimensional operators like the weighted $p-$Laplacean

$$div(v(x)|\nabla y(x)|^{p-2}\nabla y(x)), x \in \Omega \subset \mathbb{R}^n.$$

This is just one of the classical motivations on why to investigate the Hardy inequality.

6.3 Future plans and open questions

The research in this PhD thesis has implied some new possibilities and open questions in these interesting areas, we mention the following:

1. A new characterization of vis-a-vis the standard one by Muckenhoupt-Bradley of the Hardy inequality for $1 < p \leq q < \infty$ was derived in [108, Theorem 2] (see also Theorem 2.13). By using the standard limit procedure the authors derived a characterization of the limit Pólya-Knopp inequality. In this connection, we pose the following open problem: Is it possible to do a similar limit procedure to derive the characterizations of the limit (Pólya-Knopp type) inequality (4.3) in weighted multidimensional cases by using the known characterizations of Hardy-type inequalities? See Open Question 2.1.

2. In view of Subsections 2.2.4 – 2.2.5, in this PhD thesis, it is natural to ask the following: Is it possible to develop a similar theory concerning scales of conditions for the following cases?

   (a) multidimensional Hardy type inequalities,

   (b) limit Pólya-Knopp type inequalities both in the one-dimensional and multidimensional cases? See Open Question 2.2.

3. In Theorem 4.4, we obtained the sharp constant C of the inequality (4.21) for the case $0 < p = q < \infty$. Further, even if this is a much more challenging question, we hope to be able to derive the sharp constant C of the multidimensional Cochran-Lee type inequality (4.21) with general or particular weight functions for the case $0 < p < q < \infty$. Also the question of estimating the sharp constants C of the multidimensional Cochran-Lee type inequality (4.21) in the cases when $p = 1$ and $p = \infty$ are of great interest. See Open Question 4.1.

4. In Section 6.2, some of the known results concerning the important role of the Hardy inequality when characterising the behaviour of solutions to BVPs are given. We plan to identify further such applications of Hardy-type inequalities (including the newly obtained ones) when investigating BDIEs for variable-coefficient BVPs and their rôle in BDIE-based numerical analysis.

5. This successful PhD education has been done in collaboration between research groups in Ethiopia, Sweden, Belgium. The positive experience of this collaboration implies that we plan to further develop it with more researches and PhD students involved.

List of publications

1. S. Barza, L. Nikolova, L.-E. Persson and M. Yimer, *Some Hardy-type inequalities in Banach function spaces*, Math. Inequal. Appl. **24**(4) (2021), 1001 – 1016.
2. L. Nikolova, L. E. Persson, S. Varošanec and M. F. Yimer, *Refinements of some classical inequalities via superquadraticity*, J. Inequal. Appl. (2022), Paper No. 86, 15 pp.
3. M. F. Yimer, L. E. Persson and T. G. Ayele, *Some new multidimensional Cochran-Lee and Hardy type inequalities*, Math. Inequal. Appl. **26** (4) (2023), 887-903.
4. M. F. Yimer and T. G. Ayele, *Analysis of two-operator boundary domain integral equations for variable coefficient Dirchlet and Neumann problems in 2D with general right-hand side*, to appear in Ethiop. J. Sci., 46(2): 109–130, 2023.

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