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Addis Ababa University

SCHOOL OF GRADUATE STUDIES
DEPARTMENT OF MATHEMATICS

GRADUATE PROJECT REPORT

ON

THE HILLE - YOSIDA THEOREM

SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENT FOR
THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS

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permission

This is to certify that this project is compiled by **Mr. Moges Birhanu Hailesilassie** in the Department of Mathematics Addis Ababa university under my supervision. I here by also confirm that the project can be submitted for evaluation by examiners and eventual defence.

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Abstract

The Hille - Yosida Theorem is one of the most useful tools in the study of existence of a unique solution for evolution problem i.e.,

$$\begin{cases} \dfrac{du}{dt} + Au = 0, \qquad on \quad [0, +\infty), \\ u(0) = u_0, \end{cases}$$

where A is a maximal monotone operator.
The aim of this project is to study the existence and uniqueness of solutions generated by maximal monotone operators in Hilbert spaces by applying Hille-Yosida Theorem for the evolution problem.

Notations

$\|\| A \|\|$	Norm of a bounded linear operator A.
A^*	adjoint operator of A.
$\overline{D(A)}$	Closure of D(A).
$\overline{D(A)}^{\perp}$	Orthogonal complement of a closure D(A).
$C[a,b]$	Space of continuous functions.
$C^1[a,b]$	Space of continuously differentiable functions.
$D(A)$	Domain of an operator A.
$\oplus$	Direct sum.
H	Hilbert spaces.
I	Identity operator.
$L^p[a,b]$	A function space, $1 \leq p < \infty$.
$\ell(H)$	A space of linear operators from H to H.
l^p	A sequence space, $1 \leq p < \infty$.
$R(A)$	Range of an operator A.
sup	Supremum(least upper bound).
(u,v)	Inner product of u and v .

Introduction

When mathematical modelling is used to describe physical, biological or chemical phenomena, one of the most common results is either a differential equation or a system of differential equations, together with appropriate boundary and initial conditions. These differential equations may be ordinary or partial, and finding and interpreting their solutions is at the heart of applied mathematics. In the study of existence of a unique solution for the given evolution problem (or initial value problem), Hille - Yosida Theorem is an essential tool. In this report we have two chapters excluding the preliminary. Of course, in the preliminaries section there are basic definitions and theorems which are used to facilitate the reader in chapter one and two. In chapter one we introduce the basic definitions on monotone operators and maximal monotone operators regarding unbounded linear operators. It is essential to deal with such operators to study the Hille-Yosida Theorem. The second chapter deals with existence of unique solutions to the evolution problems i.e.,

$$\begin{cases} \dfrac{du}{dt} + Au = 0, \qquad \mathit{on} \quad [0, +\infty), \\ u(0) = u_0 \end{cases}$$

In this chapter we will see the Hille-Yosida Theorem which is important to know the well-posedeness of the above evolution problems. This chapter also includes an introduction to semigroups of linear operators, which can be used to solve a large class of problems, commonly known as evolution problems. In addition to this the chapter also includes self-adjoint cases for the given an unbounded linear operator.

Preliminaries

In this section we will review some basic definitions and theorems that are used in this paper. We are not trying to give a complete development, but rather introduce the basic ones only, some theorems are without proof.

Definition 1. *Let X and Y, be normed linear spaces, and suppose that* $A : X \longrightarrow Y$, *we write A(x) or* Ax *to denote the image of an element* $x \in X$.

(a) A *is linear if* $A(\alpha x + \beta y) = \alpha A(x) + \beta A(y)$ *for every* $x, y \in X$ *and* $\alpha, \beta \in \mathbb{C}$

(b) A *is injective if* $A(x) = A(y)$ *implies* $x = y$

(c) *The kernel or nullspace of* A *is* $Ker(A) = \{x \in X : A(x) = 0\}$

(d) *The range of* A *is* $R(A) = \{A(x), x \in X\}$

(e) A *is surjective if range* (A) *or* $R(A) = Y$

(f) A *is bijection if it is both injective and surjective.*

Definition 2. *A linear operator* $A : D(A) \subset H \longrightarrow H$ *is closed if its graph* $G(A) = [u, Au]; u \in D(A) \subset H \times H$ *is closed.*
The closedness of an unbounded linear operator A *can be characterized as follows : If* $u_n \in D(A)$ *such that*

$$u_n \longrightarrow u \text{ and } Au_n \longrightarrow f \text{ in } H \text{ as } n \longrightarrow \infty \text{ then } u \in D(A) \text{ and } Au = f.$$

Definition 3. *Let X be a vector space .*
A mapping $(.,.) : X \times X \longrightarrow \mathbb{K}$ *is said to be an **inner product** on X if and only if for all vectors x,y,z and scalar* α,

$$\begin{aligned} (IP_1) \quad & (x+y,z) = (x,z)+(y,z) \\ (IP_2) \quad & (\alpha x, y) = \alpha(x,y) \\ (IP_3) \quad & (x,y) = \overline{(y,x)} \qquad \textit{(bar denotes complex conjugate)} \\ (IP_4) \quad & (x,x) \geq 0 \quad \textit{and} \quad (x.x) = 0 \Longleftrightarrow x = 0. \end{aligned}$$

The inner product generates the norm

$$|| \; x \; || := \sqrt{(x,x)}.$$

Therefore an inner product space is a normed space.

Theorem 1. ***(Cauchy - Schwartz Inequality)***
Let X be an inner product. Then

$$| \; (x,y) \; | \leq \sqrt{(x,x)}.\sqrt{(y,y)} \quad ,\forall x,y \in X.$$

Proof. For arbitrary scalars λ,

$$0 \leq (x-\lambda y, x-\lambda y) \quad = \quad (x,x) - \overline{\lambda}(x,y) - \lambda\overline{(x,y)} + \lambda\overline{\lambda}(y,y).$$

If $y = 0$, the inequality is obvious. Assume $y \neq$ and let $\lambda = \dfrac{(x,y)}{(y,y)}$.
Then,

$$0 \leq (x,x) - \tfrac{|(x,y)|^2}{(y,y)} - \tfrac{|(x,y)|^2}{(y,y)} + \tfrac{|(x,y)|^2}{(y,y)} = (x,x) - \tfrac{|(x,y)|^2}{(y,y)}$$

and hence

$$| \; (x,y) \; |^2 \leq (x,x).(y,y)$$

□

Definition 4. *A normed linear space X which does have the property that all Cauchy sequences are convergent is said to be **complete**.*

- *A complete normed linear space is called a Banach space.*
- *A complete inner product space is called a Hilbert space.*

Definition 5. *Let E and F be two Banach spaces. An unbounded linear operator from E into F is a linear map* $A : D(A) \subset E \longrightarrow F$ *defined on a linear subspace* $D(A) \subset E$ *with values in F. The set D(A) is called the domain of A.*
A is bounded (or continuous) if $D(A) = E$ *and if there is a constant* $\gamma \geq 0$ *such that*

$$\| Au \| \leq \gamma \| u \| \qquad \forall u \in E.$$

Definition 6. *Let H be a Hilbert space. Let* $A : H \longrightarrow H$ *be a bounded linear map. We define a map* $A^* : H \longrightarrow H$ *by the relation*

$$(Au, v) = (u, A^*v) \text{ for all } u, v \in H.$$

The map A^* *is called the adjoint of A.*

Theorem 2. ***(Weierstrass M -Test)***
Let $u_n(t)$ *be a sequence of functions defined on a set E. Suppose that for all* $n \in \mathbb{N}$ *there exists* $M_n \in \mathbb{R}$ *such that* $| u_n(t) | \leq M_n, \quad \forall t \in E$. *Then if* $\sum M_n$ *converges,* $\sum u_n$ *must converge uniformly.*

Definition 7. *Let E be a Banach space and let* $A : E \longrightarrow E$ *be a function. Then, the function A is said to be* ***Lipschiz function*** *in a domain D i.e., E, if there exists a positive constant L such that*

$$\| Au - Av \| \leq L \| u - v \| \quad , \forall u, v \in E$$

Theorem 3. *(**Cauchy - Peano Theorem**)*
Let $A : E \longrightarrow E$ *be a continuous function in the domain E, and given the initial value problem (IVP)*

$$\begin{cases} \frac{du}{dt}(t) = Au(t) \qquad [0, +\infty), \\ u(0) = u_0 \end{cases} \tag{1}$$

then there exists a solution u in $[0, +\infty; E)$ *which satisfies the given IVP .*

Theorem 4. ***(Picard - Lipschitz Theorem)***
Let $A : E \longrightarrow E$ be a Lipschitz function in the domain E, and given the initial value problem(IVP), then for any $u_0 \in E$ there exists a unique solution $u \in C^1([0, +\infty); E)$ which satisfies the initial value problem (IVP).

Picard Iteration Method

Many problems with a differential equation are often difficult to find whose solutions by standard methods. In such problems, it is sufficient to obtain an approximate solution only.
We shall mention here the Picards iteration method for giving an approximation solution of the initial value problem of the form

$$\frac{du}{dt}(t) = Au(t), \quad u(0) = u_0 \tag{2}$$

By an iteration method we mean a method which consists of a repeated application of exactly the same type of steps where in each step we use the result of preceding step (steps). We now explain the Picard's iteration method. By integrating, we write (2) in the form

$$u(t) = u_0 + \int_0^t A(u(s))ds. \tag{3}$$

In order to obtain a solution u(t) of (3), we proceed stepwise as follows: put $u = u_0 =$ constant, on the right. This gives

$$u_1(t) = u_0 + \textstyle\int_0^t A(u_0(s))ds.$$

We now substitute $u_1(t)$ in the same manner and we get

$$u_2(t) = u_0 + \textstyle\int_0^t A(u_1(s))ds$$

Continuing in this way, at the nth step of iteration process, we shall get

$$u_n(t) = u_0 + \int_0^t A(u_{n-1}(s))ds. \tag{4}$$

Thus we obtain a sequence of approximate solutions

$$u_1(t), u_2(t), u_3(t) \ , ..., \ u_n(t).$$

Chapter 1

Maximal Monotone Operators

In this chapter we will see that the basic definitions of monotone linear operator, maximal monotone operator and its elementary properties. In addition to this we will define the resolvent and Yosida approximation and its properties for the given maximal monotone operator.

1.1 Properties of Maximal Monotone Operators

Definition 8. *Let H be a Hilbert Space. Then, an unbounded linear operator $A : D(A) \subset H \longrightarrow H$ is said to be **monotone** if it satisfies*

$$(Av, v) \geq 0, \, \forall v \in D(A).$$

*It is called **maximal monotone** if, in addition,*

$$R(I + A) = H,$$

i.e., $\forall f \in H, \exists u \in D(A)$ *such that*

$$u + Au = f.$$

Example 1. *Let H be the Hilbert space $L^2(0, 1)$ and let A be the differential operator defined by*

$$D(A) \in C^1(0, 1) \quad \textit{and} \quad Au = u^{'} \qquad \forall u \in C^1(0, 1).$$

and consider the sequence of functions defined by

$$u_n(t) = t^n \qquad n = 1, 2, 3, ...$$

Then A is unbounded maximal monotone linear operator.
Since ,

$$\frac{||Au_n||}{||u_n||} = n(\sqrt{\frac{2n+1}{2n-1}}) \longrightarrow \infty$$

as $n \longrightarrow \infty$, *that is A is unbounded linear operator.*
To show A is monotone : for all $u \in D(A)$ we have

$$(Au, u) = \int_0^1 nt^{n-1}t^n dt = \frac{n}{2n} = \frac{1}{2} > 0 \qquad n \in \mathbb{N}.$$

Hence, A is monotone, and also A is maximal monotone.
Since,
There exists u in $C^1(0, 1)$ such that

$$u + Au = t^n + nt^{n-1}$$
$$\Longrightarrow R(I + A) = L^2(1, 0).$$

So A is maximal monotone.
Remark : $|| u || \leq || f ||$

Proposition 1. *Let A be a maximal monotone operator. Then*

- *(a) $D(A)$ is dense in H,*
- *(b) A is a closed operator,*
- *(c) For every $\lambda > 0$, $(I + \lambda A)$ is bijective from $D(A)$ onto H,*
- *(d) $(I + \lambda A)^{-1}$: $H \longrightarrow H$ is a bounded linear operator and $|| (I + \lambda A)^{-1} ||_{\ell_{(H)}} \leq 1$.*

Proof. (a) Let $f \in \overline{D(A)}^{\perp}$ then $(f, v) = 0, \forall v \in \overline{D(A)}$.
Claim: $\overline{D(A)} = H$, by showing $f = 0$.
Since A is maximal monotone operator, there exists $v_0 \in D(A)$ such that

$$v_0 + Av_0 = f.$$

We have:

$$0 = (f, v_0) = (v_0 + Av_0, v_0) = \| v_0 \|^2 + (Av_0, v_0)$$

since,

$$(Av_0, v_0) \geq 0, \ v_0 = 0 \quad \text{and so} \ \ f = 0 \quad \text{then} \quad \overline{D(A)}^{\perp} = 0.$$

Therefore,

$$H = \overline{D(A)}^{\perp} \oplus \overline{D(A)} = \overline{D(A)} \ .$$

This implies that D(A) is dense in H.

(b) First let us observe that for all f in H, there exists a unique $u \in D(A)$ such that

$$u + Au = f$$

Justification:

since, if v is another solution i.e., if,

$$v \in D(A) \ \text{ and } \ v + Av = f \ \text{ then } \ \ u - v + Au - Av = 0.$$

We need to show that

$$u = v$$

$$0 = (u-v+Au-Av, u-v) = (u-v+A(u-v), u-v) = \| u-v \|^2 + (A(u-v), u-v)$$

Since,

$$(A(u-v), u-v) \geq 0,$$

Thus,

$$\| u - v \|^2 = 0.$$

This implies that

$$u = v$$

Now let us turn to show A is closed. Let (u_n) be a sequence in D(A) such that

$$u_n \longrightarrow u \text{ and } Au_n \longrightarrow f.$$

Claim: $u \in D(A)$ and $Au = f$.
Since,

$$\begin{aligned}
&u_n + Au_n \longrightarrow u + f \Longrightarrow u_n(I + A) \longrightarrow u + f\\
&u_n = (I + A)^{-1}(u_n + Au_n) \longrightarrow (I + A)^{-1}(u + f)\\
&\Longrightarrow u = (I + A)^{-1}(u + f) \Longrightarrow u \in D(A) \quad and \quad (I + A)u = u + f\\
&\Longrightarrow Au = f.
\end{aligned}$$

Therefore, A is closed.
(c) Given, $(I + \lambda A) : D(A) \subset H \longrightarrow H$
We need to show that $(I + \lambda A)$ is bijective i.e.,(injective and surjective)
(i) Injective: Assume $u, v \in D(A)$.
Claim: if $(I + \lambda A)u = (I + \lambda A)v$, then $u = v$.
Assume that,

$$(I + \lambda A)u = (I + \lambda A)v.$$

This implies

$$\begin{gathered}
(I + \lambda A)u - (I + \lambda A)v = 0 \Longrightarrow u - v + \lambda A(u - v) = 0\\
0 = (u - v + \lambda A(u - v), u - v) =|| u - v ||^2 +(\lambda A(u - v), u - v)
\end{gathered}$$

Since λA is a maximal monotone, $\forall \lambda > 0$ and then

$$(\lambda A(u - v), u - v) \geq 0.$$

So we have,

$$|| u - v ||^2 = 0 \Longrightarrow u = v$$

Hence, $(I + \lambda A)$ is injective .
(ii) surjective :
Now, A is maximal monotone operator, so we have

$$(I + A)u = f$$

Similarly for any $\lambda > 0$, λA is a maximal monotone operator. Therefore for all f in H, there exists u in $D(A)$ such that

$$(I + \lambda A)u = f$$

Hence, $I + \lambda A$ is surjective. So from (i) and (ii), $I + \lambda A$ is bijective.
(d) Since, $\| u \| \leq \| f \|$ so we can say that,

$$\begin{aligned} \| u \| \leq \| f \| &\Longrightarrow \| (I + \lambda A)^{-1} f \| \leq \| f \| \\ &\Longrightarrow \| (I + \lambda A)^{-1} f \| \leq \| (I + \lambda A)^{-1} \| \| f \| \leq \| f \| \\ &\Longrightarrow \| (I + \lambda A)^{-1} \| \| f \| \leq \| f \| \end{aligned}$$

Thus,

$$\| (I + \lambda A)^{-1} \| \leq 1.$$

i.e.,

$$(I + \lambda A)^{-1} \qquad \text{is a bounded linear operator.}$$

□

Proposition 2. *If A and B are maximal monotone operators, then $A + B$, defined on $D(A) \cap D(B)$, need not be maximal monotone.*

Proof. By assumption A and B are maximal monotone, so we have,

$$(Au, u) \geq 0 \ , \forall u \in D(A) \text{ and } (Bu, u) \geq 0 \ , \forall u \in D(B).$$

Thus,

$$\begin{aligned} ((A + B)u, u) &= (Au + Bu, u) \\ &= (Au, u) + (Bu, u) \geq 0 \end{aligned}$$

i.e.,

$$((A + B)u, u) \geq 0.$$

Thus, $A + B$ is monotone .
But $A + B$ is not surjective because $D(A) \cap D(B)$ may be zero .
Since for a non zero f in H, $0 = f$, which is wrong.
Therefore A + B is not maximal monotone.

□

Definition 9. *Let A be a maximal monotone operator. For every $\lambda > 0$, set*

$$J_\lambda = (I + \lambda A)^{-1} \quad \textit{and} \quad A_\lambda = \tfrac{1}{\lambda}(I - J_\lambda) \ ;$$

*J_λ is called the **resolvent** of A, and A_λ is the **Yosida approximation**(or regularization) of A.*

Proposition 3. *Let A be a maximal monotone operator . Then:*

$$\begin{aligned}
&(a_1) \quad A_\lambda v = A(J_\lambda v), \quad \forall v \in H \quad \textit{and} \quad \lambda > 0,\\
&(a_2) \quad A_\lambda v = J_\lambda(Av), \quad \forall v \in D(A) \quad \textit{and} \quad \lambda > 0,\\
&(b) \quad \mid A_\lambda v \mid \ \leq \ \mid Av \mid \qquad \forall v \in D(A) \ \textit{and} \quad \lambda > 0,\\
&(c) \quad \lim_{\lambda \longrightarrow 0} J_\lambda v = v, \qquad \forall v \in H,\\
&(d) \quad \lim_{\lambda \longrightarrow 0} A_\lambda v = Av, \quad \forall v \in D(A),\\
&(e) \quad (A_\lambda v, v) \ \geq \ 0, \qquad \forall \quad v \in H \quad \textit{and} \quad \lambda > 0,\\
&(f) \quad \mid A_\lambda v \mid \ \leq \ \frac{1}{\lambda} \mid v \mid, \quad \forall v \in H \quad \textit{and} \quad \lambda > 0.
\end{aligned}$$

Proof. (a_1) To show $A_\lambda v = A(J_\lambda v), \quad \forall \quad v \in H \quad, \lambda > 0,$
we have,

$$J_\lambda v = (I + \lambda A)^{-1} v$$

$$\begin{aligned}
&\Longrightarrow \quad (I + \lambda A) J_\lambda v = v\\
&\Longrightarrow \quad J_\lambda v + \lambda A J_\lambda v = v\\
&\Longrightarrow (I - J_\lambda) v = \lambda A (J_\lambda v)\\
&\Longrightarrow \quad \frac{1}{\lambda}(I - J_\lambda) v = A(J_\lambda v)
\end{aligned}$$

i.e.,

$$A_\lambda v = A(J_\lambda v)$$

$$\begin{aligned}
(a_2) \quad A_\lambda v = \frac{1}{\lambda}(I - J_\lambda) v &= \frac{1}{\lambda}[(I + \lambda A)^{-1}(I + \lambda A) v - (I + \lambda A)^{-1} v]\\
&= \frac{1}{\lambda}(I + \lambda A)^{-1}(v + \lambda A v - v)\\
&= \frac{1}{\lambda} J_\lambda(\lambda A v) = J_\lambda(Av)
\end{aligned}$$

Hence ,

$$A_\lambda v = J_\lambda(Av)$$

(b) To show $\mid A_\lambda v \mid \ \leq \ \mid Av \mid, \quad \forall v \in D(A), \quad \lambda > 0$

$$\begin{aligned} \mid A_\lambda v \mid \ &= \mid J_\lambda(Av) \mid \qquad \text{by} \quad (a_2) \quad \text{above.} \\ &\leq \ \| J_\lambda \| \| Av \mid \ \leq \ \mid Av \mid . \end{aligned}$$

Therefore,

$$\mid A_\lambda v \mid \leq \mid Av \mid$$

(c) Assume first that $v \in D(A)$. Then

$$\begin{aligned} \mid v - J_\lambda v \mid &= \lambda \mid A_\lambda v \mid \\ &\leq \lambda \mid Av \mid \qquad \text{by (b) above .} \end{aligned}$$

This implies that

$$\lim_{\lambda \longrightarrow 0} J_\lambda v = v.$$

Suppose now that v is a general element in H. Given any $\epsilon > 0$, there exists some $v_1 \in D(A)$ such that

$\mid v - v_1 \mid < \epsilon$ (since D(A) is dense in H by proposition 1).

We have,

$$\begin{aligned} \mid J_\lambda v - v \mid &= \mid v - v_1 + v_1 - J_\lambda v_1 + J_\lambda v_1 - J_\lambda v \mid \\ &\leq \mid v - v_1 \mid + \mid v_1 - J_\lambda v_1 \mid + \mid J_\lambda v_1 - J_\lambda v \mid \\ &< \epsilon + \epsilon + \mid v_1 - J_\lambda v_1 \mid \\ &= 2\epsilon + \mid v_1 - J_\lambda v_1 \mid \ . \end{aligned}$$

Thus,

$$\limsup_{\lambda \longrightarrow 0} \mid J_\lambda v - v \mid \leq 2\epsilon \quad , \forall \epsilon > 0,$$

i.e.,

$$\lim_{\lambda \longrightarrow 0} \mid J_\lambda v - v \mid = 0 \implies \lim_{\lambda \longrightarrow 0} J_\lambda v = v, \quad \forall v \in H.$$

(d) To show that,

$$\lim_{\lambda\longrightarrow 0} A_\lambda v = Av, \forall v \in D(A), \lambda > 0.$$

Since,

$$\begin{aligned}\lim_{\lambda\longrightarrow 0} A_\lambda v &= \lim_{\lambda\longrightarrow 0} J_\lambda(Av)\\ &= Av \qquad \text{by} \quad a_2 \text{ and c above.}\end{aligned}$$

(e) To show $(Av, v) \geq 0 \quad \forall v \in H$ and $\forall \lambda > 0$:

$$\begin{aligned}(A_\lambda v, v) &= (A_\lambda v, v - J_\lambda v + J_\lambda v)\\ &= (A_\lambda v, v - J_\lambda v) + (A_\lambda v, J_\lambda v)\\ &= (A_\lambda v, \lambda A_\lambda v) + (A(J_\lambda v), J_\lambda v)\\ &= \lambda(A_\lambda v, A_\lambda v) + (A(J_\lambda v), J_\lambda v)\\ &= \lambda \mid\mid A_\lambda v \mid\mid^2 + (A(J_\lambda v), J_\lambda v)\\ &\geq \lambda \mid\mid A_\lambda v \mid\mid^2 \geq 0\end{aligned}$$

So we have ,

$$(A_\lambda v, v) \geq 0$$

(f) To show $\mid A_\lambda v \mid \leq \frac{1}{\lambda} \mid v \mid, \forall v \in H, \lambda > 0$:
From the proof of (e) we have,

$$(A_\lambda v, v) \geq \lambda \mid A_\lambda v \mid^2 \Longrightarrow \quad \mid A_\lambda v \mid^2 \leq \frac{1}{\lambda}(A_\lambda v, v) \leq \frac{1}{\lambda} \mid A_\lambda v \mid\mid v \mid$$

Hence , $\mid A_\lambda v \mid \leq \frac{1}{\lambda} \mid v \mid$. □

Chapter 2

The Hille - Yosida Theorem

In this chapter we will see the conditions that guarantee the existence of unique solutions for the given evolution problem using the Hille - Yosida theorem. In addition to this we will discuss the concept of C_0 semigroups, symmetric and self-adjoint for the operator A and then we will deduce the existence and uniqueness of solutions.

2.1 Existence and Uniqueness of Solution of Evolution Problem

Theorem 5. ***(Cauchy - Lipschitz - Picard)***
Let E be a Banach space and let $A : E \longrightarrow E$ be a Lipschitz map. Then given any $u_0 \in E$ there exists a unique solution $u \in C^1([0,+\infty);E)$ of the problem,

$$(1) \qquad \begin{cases} \dfrac{du}{dt}(t) = Au(t) \quad on\ [0,+\infty), \\ u(0) = u_0 \end{cases}$$

where u_0 is the initial data .
Given $k > 0$, set

$$X = \{u \in C([0,+\infty);E) : \sup_{t\geq 0} e^{-kt} \mid\mid u(t) \mid\mid < \infty\}.$$

Let us first justify the following claims, before proving the existence and uniqueness of the theorem.

Claim 1 : X is a Banach space with norm

$$\| u \|_X = \sup_{t \geq 0} e^{-kt} \| u(t) \| .$$

Justification:
Let (u_n) be a Cauchy sequence in X and let $\epsilon > 0$.
Then there exists $N > 0$ such that

$$\| u_n - u_m \|_X < \epsilon , \quad \forall n, m \geq N .$$

Thus,

$$\| u_n - u_m \|_X = \sup_{t \geq 0} e^{-kt} \| u_n(t) - u_m(t) \| < \epsilon, \forall n, m \geq N$$

$$\| u_n(t) - u_m(t) \| < \epsilon \tag{2}$$

For every given t, $\{u_n(t)\}$ is Cauchy in E, which converges to u(t) in E. Let,

$$\lim_{m \longrightarrow \infty} u_m(t) =: u$$

We need to show that u_n converges to u.
Choose $\epsilon > 0$. By replacing ϵ with $\frac{\epsilon}{2}$ in eqn.(2) above, we can fined an $N > 0$ such that

$$\| u_n - u_m \| \leq \tfrac{\epsilon}{2} \quad \forall n, m \geq N.$$

Taking the limit as $m \longrightarrow \infty$, we obtain

$$\| u_n - u \| \leq \tfrac{\epsilon}{2}, \quad \forall n \geq N .$$

In other words, we obtain

$$\| u_n - u \| \leq \tfrac{\epsilon}{2} < \epsilon, \quad \forall n > N .$$

This implies that u_n converges to u.

$\therefore$ X is a Banach space.

Claim (2): For every $u \in X$, the function Φu defined by

$$(\Phi u)(t) = u_0 + \int_0^t A(u(s))ds$$

also belongs to X.

Justification:

Since, $\Phi u \in C([0, \infty); E)$

$$\begin{aligned}
\| (\Phi u)(t) \| &= \| u_0 + \int_0^t A(u(s)) \| ds \\
&\leq \| u_0 \| + \int_0^t \| A(u(s)) \| ds \\
&= \| u_0 \| + \int_0^t \| A(u(s)) - A(u_0) + A(u_0) \| ds \\
&\leq \| u_0 \| + \int_0^t \| A(u(s)) - A(u_0) \| ds + \int_0^t \| A(u_0) \| ds \\
&\leq \| u_0 \| + L \int_0^t \| u(s) - u_0 \| ds + \| A(u_0) \| t \\
&\leq \| u_0 \| + L \int_0^t \| u(s) \| ds + L \int_o^t \| u_0 \| ds + \| A(u_0) \| t \\
&= \| u_0 \| + \| u_0 \| Lt + \| A(u_0) \| t + L \int_0^t e^{-ks} \| u(s) \| e^{ks} ds \\
&\leq \| u_0 \| + \| u_0 \| Lt + \| A(u_0) \| t + L \| u \|_X \int_0^t e^{ks} ds \\
&= \| u_0 \| + \| u_0 \| Lt + \| A(u_0) \| t + L \| u \|_X \frac{1}{k}(e^{kt} - 1) \\
&= \| u_0 \| + \| u_0 \| Lt + \| A(u_0) \| t + \frac{L}{k} \| u \|_X (e^{kt} - 1)
\end{aligned}$$

$$\begin{aligned}\Longrightarrow e^{-kt} \parallel (\Phi u)(t) \parallel &\leq [\parallel u_0 \parallel + (\parallel u_0 \parallel L + \parallel A(u_0) \parallel)t]e^{-kt} \\ &+ \frac{L}{k} \parallel u \parallel_X (1-e^{-kt}) \\ &\leq [\parallel u_0 \parallel + (\parallel u_0 \parallel L + \parallel A(u_0) \parallel)t]e^{-kt} + \frac{L}{k} \parallel u \parallel_X\end{aligned}$$

$\Longrightarrow \sup e^{-kt} \parallel (\Phi u)(t) \parallel < \infty.$

Hence, $\Phi u \in X$

Claim(3): $\parallel \Phi u - \Phi v \parallel_X \leq \frac{L}{k} \parallel u - v \parallel_X, \quad \forall u, v \in X.$

Justification:

For every $u, v \in X$, let us define the function Φu and Φv by

$$(\Phi u)(t) = u_0 + \int_0^t A(u(s))ds \;\; and \;\; (\Phi v)(t) = u_0 + \int_0^t A(v(s))ds.$$

Then,

$$\Phi u(t) - \Phi v(t) = \int_0^t (A(u(s)) - A(v(s)))ds$$

$$e^{-kt} \parallel \Phi u(t) - \Phi v(t) \parallel \leq e^{-kt} \int_0^t \parallel A(u(s)) - A(v(s)) \parallel ds$$

$$\leq Le^{-kt} \int_0^t \parallel (u(s) - v(s)) \parallel ds = Le^{-kt} \int_0^t e^{-ks} \parallel (u(s) - v(s)) \parallel e^{ks} ds$$

$$\leq \frac{L}{k} e^{-kt} \parallel u - v \parallel_X (e^{kt} - 1) = \frac{L}{k} \parallel u - v \parallel_X (1 - e^{-kt}) \leq \frac{L}{k} \parallel u - v \parallel_X$$

Hence,

$$\parallel \Phi u - \Phi v \parallel_X \leq \frac{L}{k} \parallel u - v \parallel_X$$

We are now back to prove the theorem.

Proof. (i) Existence, (Using Picard's method:)

Let us consider the iterative sequence

$$u_n(t) = u_0 + \int_0^t A(u_{n-1}(s))ds, \qquad n \in \mathbb{N} \tag{3}$$

Since A is continuous, so we have,

$$\| Au \| \leq c \| u \| = M, \quad \text{for some}\, c > 0. \tag{4}$$

Assume $M = \sup \| Au \|$, and A is satisfies the Lipschitz condition so we have,

$$\| Au - Av \| \leq L \| u - v \|, \forall u, v \in E \tag{5}$$

In order that the initial value problem (IVP) may have a solution, it is necessary that the sequence $u_n(t)$ of functions converges to a limiting function u(t) which is a solution of the IVP or of the equivalent integral equation

$$u(t) = u_0 + \int_0^t A(u(s))ds \tag{6}$$

Now to ensure the existence of the limiting function

$$u(t) = \lim_{n \longrightarrow \infty} u_n(t)$$

we use u_n as a sum of successive differences:

$$u_n = u_0 + \sum_{i=0}^{n-1} (u_{i+1} - u_i)$$

i.e., $u_n(t) = [u_n(t) - u_{n-1}(t)] + [u_{n-1}(t) - u_{n-2}(t)] + ... + [u_1(t) - u_0(t)] + u_0(t)$
using (3) we have

$$u_n(t) = u_0 + \int_0^t A(u_{n-1}(s))ds \quad and \quad u_{n+1}(t) = u_0 + \int_0^t A(u_n(s))ds$$

Therefore,

$$\| u_{n+1}(t) - u_n(t) \| = \| \int_0^t A(u_n(s))ds - \int_0^t A(u_{n-1}(s))ds \| \tag{7}$$

Also from (3)

$$u_1(t) = u_0 + \int_0^t A(u_0)ds$$

$$\begin{aligned}|| u_1(t) - u_0 || &= || \int_0^t A(u_0) ds || \\ &\leq \int_0^t || A(u_0) || \, ds \\ &\leq \int_0^t M ds \\ &= Mt \qquad \text{by (4) above}\end{aligned}$$

Again using Lipschitz condition (5), we get from (6)

$$\begin{aligned}|| u_2(t) - u_1(t) || &\leq \int_0^t || A(u_1(s))ds - A(u_0(s))ds || \\ &\leq \int_0^t L \, || (u_1(s)) - (u_0(s)) || \, ds \\ &\leq \int_0^t LMsds = LM\frac{t^2}{2!}\end{aligned}$$

Similarly

$$|| u_3(t) - u_2(t) || \leq L^2 M \frac{t^3}{3!}.$$

In general we shall have,

$$|| u_n(t) - u_{n-1}(t) || \leq \frac{L^{n-1} M t^n}{n!}$$

Now let us prove the following by induction on n:

$$|| u_{n+1}(t) - u_n(t) || \leq \frac{L^n M t^{n+1}}{(n+1)!}$$

When n is 0,

$$\begin{aligned}|| u_1(t) - u_0(t) || &\leq \int_0^t || A(u_0) || \, ds \\ &\leq \int_0^t M ds = Mt\end{aligned}$$

Assuming the claim is true for $n-1$,

$$\begin{aligned}
\mid\mid u_{n+1}(t) - u_n(t) \mid\mid &\leq \int_0^t \mid\mid A(u_n(s)) - A(u_{n-1}(s)) \mid\mid ds \\
&\leq \int_0^t L \mid\mid u_n(s) - u_{n-1}(s) \mid\mid ds \\
&\leq L \int_0^t \frac{L^{n-1} M s^n}{n!} ds \\
&= \frac{M L^n t^{n+1}}{(n+1)!}.
\end{aligned}$$

Hence, it is valid for all n.

$$\text{Now} \qquad \sum_{n=0}^{\infty} \frac{M L^n t^{n+1}}{(n+1)!} \quad \text{converges.}$$

Therefore by Weierstrass M-Test

$$\sum_{n=0}^{\infty} \mid\mid u_{n+1}(t) - u_n(t) \mid\mid \qquad \text{also converges.}$$

Hence,

$$\lim_{n \longrightarrow \infty} u_n(t) = u_0 + \sum_{n=0}^{\infty} (u_{n+1}(t) - u_n(t)) \qquad \text{converges to a function u(t).}$$

The convergence is uniform in t because,

$$\begin{aligned}
\mid\mid u_{n+1}(t) - u_n(t) \mid\mid &\leq \frac{M L^{n+1} t^{n+1}}{L(n+1)!} \\
\Longrightarrow \quad \mid\mid u(t) - u_N(t) \mid\mid &\leq \sum_{n=0}^{N-1} \mid\mid u_{n+1}(t) - u_n(t) \mid\mid \\
&\leq \frac{M}{L} \sum_{n=N}^{\infty} \frac{L^{n+1} t^{n+1}}{(n+1)!} \\
&\leq \frac{M L^{N+1} t^{N+1}}{L(N+1)!} e^{Lt}
\end{aligned}$$

i.e., it is uniformly convergence.
u(t) is a solution , pick u_n which is close to u.
This is possible since u_n converges uniformly to u.
$\forall\epsilon, \quad \exists N, \quad \forall t \in [0, +\infty), \quad n > N \qquad \Longrightarrow \quad || u_n(t) - u(t) ||< \epsilon.$
Therefore,

$$\begin{aligned} || \int_0^t A(u_n(s))ds - \int_0^t A(u(s))ds || &\leq \int_0^t || A(u_n(s)) - A(u(s)) || \, ds \\ &\leq \int_0^t L \, || u_n(s) - u(s) || \, ds \\ &\leq L\epsilon t \end{aligned}$$

which is as mall as required. So taking the limit for $n \longrightarrow \infty$ for

$$u_n(t) = u_0 + \int_0^t A(u_{n-1}(s))ds$$

gives that u is a solution of the initial value problems. Hence

$$u(t) = u_0 + \int_0^t A(u(s))ds$$

Therefore , the solution exists.
Now it remains to prove uniqueness.
(ii) Uniqueness: Let u and v be solution of the problem. Then we have

$$\begin{aligned} u(t) &= u_0 - \int_0^t A(u(s))ds \\ and \quad v(t) &= u_0 - \int_0^t A(v(s))ds. \end{aligned}$$

Now

$$\begin{aligned} || u(t) - v(t) || &= || \int_0^t (A(u(s)) - A(v(s))) || \, ds \\ &\leq \int_0^t || (A(u(s)) - A(v(s))) || \, ds, \end{aligned}$$

Since A is bounded and satisfies Lipschitz map, so we have

$$\| A(u(t)) \| \leq M \quad \text{and}$$

$$\| A(u(t)) - A(v(t)) \| \leq k \| u - v \| \tag{8}$$

where, $M = \sup \| A(u(t)) \|$.

Now,

$$\begin{aligned} \| u(t) - v(t) \| &\leq \int_0^t \| A(u(s)) - A(v(s)) \| \, ds \\ &\leq \int_0^t (\| A(u(s)) \| + \| A(v(s)) \|) ds \\ &\leq \int_0^t (M + M) ds = 2Mt \end{aligned}$$

This implies that

$$\| u(t) - v(t) \| \leq 2Mt \tag{9}$$

again using the Lipschitz map

$$\| u(t) - v(t) \| \leq \int_0^t k \| u(s) - v(s) \| \, ds \tag{10}$$

Combining (9) and (10) we obtain

$$\| u(t) - v(t) \| \leq \int_0^t 2kms ds = \frac{2mkt^2}{2!} \tag{11}$$

Employing inequality (11) on the right hand side of (10) we have

$$\| u(t) - v(t) \| \leq \int_o^t \frac{2k^2ms^2}{2!} ds = \frac{2mk^2t^3}{3!}$$

Continuing in this way, we shall obtain

$$\| u(t) - v(t) \| \leq \frac{2mk^{n-1}t^n}{n!}, \quad n \in \mathbb{N} \tag{12}$$

Now the right hand side of (12) tends to zero as $n \longrightarrow \infty$.
Thus,

$$\| u(t) - v(t) \| = 0 \quad \text{i.e ., } u(t) = v(t)$$

Hence, the solution is unique. $\square$

Theorem 6. ***(Hille-Yosida)***
Let A be a maximal monotone operator. Then, given any $u_0 \in D(A)$ there exists a unique solution

$$u \in C^1([0, +\infty); H) \cap C([0, +\infty); D(A)),$$

of the problem,

$$\begin{cases} \dfrac{du}{dt} + Au = 0 & \text{on } [0, \infty), \\ u(0) = u_0 \end{cases} \tag{13}$$

Moreover,

$$\| u(t) \| \leq \| u_0 \| \text{ and } \| \frac{du(t)}{dt} \| = \| Au(t) \| \leq \| Au_0 \|, \quad \forall t \geq 0.$$

Proof. First let us prove the next lemma

Lemma 1. *Let $w \in C^1([0, +\infty); H)$ be a function satisfying*

$$\frac{dw}{dt} + A_\lambda w = 0 \text{ ,on } \quad [0, +\infty).$$

Then the function $t \longmapsto \| w(t) \|$ and $t \longmapsto \| \dfrac{dw(t)}{dt} \| = \| A_\lambda w(t) \|$ are nonincreasing on$[0, +\infty)$.

Proof. Since,

$$(\frac{dw}{dt} + A_\lambda w, w) = 0,$$

so we have,

$$(\frac{dw}{dt}, w) + (A_\lambda w, w) = 0$$

By proposition 3 (e) we know that

$$(A_\lambda w, w) \geq 0.$$

This implies that

$$(\frac{dw}{dt}, w) = -(A_\lambda w, w) \leq 0$$

i.e.,
$$(\frac{dw}{dt}, w) \leq 0.$$

But,

$$(\frac{dw}{dt}, w) = \frac{1}{2}\frac{d}{dt} \mid\mid w(t) \mid\mid^2$$
$$\Longrightarrow \frac{1}{2}\frac{d}{dt} \mid\mid w(t) \mid\mid^2 \leq 0$$
$$\Longrightarrow \mid\mid w(t) \mid\mid \leq 0.$$

Hence, $\mid\mid w(t) \mid\mid$ is nonincreasing.

To show $\mid\mid \frac{dw(t)}{dt} \mid\mid$ is nonincreasing on $[0, \infty)$.
Since A_λ is a linear bounded operator, so $w \in C^2([0, +\infty); H)$ and also that

$$\frac{d}{dt}(\frac{dw}{dt}) + A_\lambda(\frac{dw}{dt}) = 0$$

so we have,

$$(\frac{d}{dt}(\frac{dw}{dt}), \frac{dw}{dt}) + (A_\lambda(\frac{dw}{dt}), \frac{dw}{dt}) = 0.$$

Since,

$$(A_\lambda(\tfrac{dw}{dt}), \frac{dw}{dt}) \geq 0.$$

This implies that,

$$(\frac{d}{dt}(\tfrac{dw}{dt}), \frac{dw}{dt}) \leq 0.$$

But,

$$(\frac{d}{dt}(\tfrac{dw}{dt}), \frac{dw}{dt}) = \tfrac{1}{2}\frac{d}{dt} \mid\mid \frac{dw}{dt} \mid\mid^2 \leq 0$$

i.e.,

$$\mid\mid \frac{dw}{dt} \mid\mid \leq 0.$$

Hence, $\frac{dw(t)}{dt}$ is nonincreasing. □

Now let us prove the theorem by breaking it into steps.

Step 1: Uniqueness:

Let u and v be two solution of the problem . Then we have

$$\frac{du}{dt} + Au = 0, \quad u(0) = u_0 \quad \text{and} \quad \frac{dv}{dt} + Av = 0, \quad v(0) = v_0 \quad \text{on} \quad [0, \infty).$$

Then ,

$$\frac{du}{dt} + Au = \frac{dv}{dt} + Av$$

$$\Longrightarrow \frac{du}{dt} - \frac{dv}{dt} + Au - Av = 0 \Longrightarrow \frac{d}{dt}(u - v) + A(u - v) = 0.$$

From this we have,

$$(\frac{d}{dt}(u - v), u - v) + (A(u - v), u - v) = 0$$

$$\Longrightarrow (\frac{d}{dt}(u - v), u - v) = -(A(u - v), u - v) \leq 0.$$

Now let us integrate

$$(\frac{d}{dt}(u - v), u - v) \leq 0,$$

by part both sides to obtain,

$$\int_0^t (\frac{d}{ds}(u - v), u - v)ds = \int_0^t (\frac{d}{ds}(u(s) - v(s))(u(s) - v(s))ds$$

$$= || u(t) - v(t) ||^2 - || u(0) - v(0) ||^2 - \int_0^t (u(s) - v(s))(\frac{d}{ds}(u(s) - v(s)).$$

This implies that

$$|| u(t) - v(t) ||^2 - || u(0) - v(0) ||^2 = 2\int_0^t (u(s) - v(s))\frac{d}{ds}(u(s) - v(s))$$
$$= -2\int_0^t (A(u(s) - v(s)), u(s) - v(s))ds \leq 0.$$

i.e.,

$$|| u(t) - v(t) ||^2 - || u(0) - v(0) ||^2 \leq 0.$$

Since the function $t \longmapsto || u(t) - v(t) ||$ is nonincreasing on $[0, +\infty)$, so we have,

$$|| u(t) - v(t) ||^2 = 0$$

Thus,

$$u(t) = v(t)$$

Hence, the solution is unique.

Step 2:

Claim : $|| \dfrac{du(t)}{dt} || = || Au(t) || \leq || Au_0 ||$, $\quad \forall t \geq 0, \forall \lambda > 0$.

Let u_λ be the solution of the problem

$$(14) \qquad \begin{cases} \dfrac{du_\lambda}{dt} + A_\lambda u_\lambda = 0, \text{on } [0, \infty) \\ u_\lambda(0) = u_0 \in D(A) \end{cases}$$

u_λ is uniquely determined by Cauchy Lipschitz Picard Theorem. By lemma 1 above $t \longmapsto || \dfrac{du_\lambda}{dt} ||$ is decreasing on $[0, \infty)$, and also by proposition 3 (b)

$$|| A_\lambda v || \leq || Av || , \quad \forall v \in D(A) \quad \forall \lambda > 0.$$

So we have,

$$\begin{aligned} || \frac{du_\lambda}{dt}(t) || &= || A_\lambda u_\lambda(t) || \\ &\leq || A_\lambda u_\lambda(0) || = || A_\lambda u_0 || \\ &\leq || Au_0 || . \end{aligned}$$

Step 3:

Claim : $\forall t \geq 0$, $\quad u_\lambda(t)$ converges as $\lambda \longrightarrow 0$, to some limit, denoted by u(t). Moreover, the convergence is uniform on every bounded interval [0,T].

For every $\lambda, \mu \geq 0$ we have

$$\frac{du_\lambda}{dt} - \frac{du_\mu}{dt} + A_\lambda u_\lambda - A_\mu u_\mu = 0$$

$$\Longrightarrow (\frac{du}{dt}(u_\lambda - u_\mu), (u_\lambda - u_\mu)) + (A_\lambda u_\lambda - A_\mu u_\mu, u_\lambda - u_\mu) = 0$$

$$\Longrightarrow (\frac{du}{dt}(u_\lambda - u_\mu), u_\lambda - u_\mu) = -(A_\lambda u_\lambda - A_\mu u_\mu, u_\lambda - u_\mu)$$

Since,

$$(\frac{du}{dt}(u_\lambda - u_\mu), u_\lambda - u_\mu) = \tfrac{1}{2}\frac{d}{dt} \mid\mid u_\lambda(t) - u_\mu(t) \mid\mid^2 .$$

So we have,

$$\begin{aligned}
\tfrac{1}{2}\frac{d}{dt} \mid\mid u_\lambda(t) - u_\mu(t) \mid\mid^2 &= -(A_\lambda u_\lambda(t) - A_\mu u_\mu(t), u_\lambda(t) - u_\mu(t)) \\
&= -(A_\lambda u_\lambda - A_\mu u_\mu, u_\lambda - J_\lambda u_\lambda + J_\lambda u_\lambda - J_\mu u_\mu + J_\mu u_\mu - u_\mu) \\
&= -(A_\lambda u_\lambda - A_\mu u_\mu, \lambda A_\lambda u_\lambda - \mu A_\mu u_\mu) - (A(J_\lambda u_\lambda - J_\mu u_\mu), J_\lambda u_\lambda - J_\mu u_\mu) \\
&\leq -(A_\lambda u_\lambda - A_\mu u_\mu, \lambda A_\lambda u_\lambda - \mu A_\mu u_\mu) \\
&= \lambda \mid\mid A_\lambda u_\lambda \mid\mid^2 + \mu \mid\mid A_\lambda u_\lambda A_\mu u_\mu \mid\mid + \lambda \mid\mid A_\lambda u_\mu A_\mu u_\mu \mid\mid \mu \mid\mid A_\mu u_\mu \mid\mid^2 \\
&\leq \lambda \mid\mid Au_0 \mid\mid^2 + \mu \mid\mid Au_0 Au_0 \mid\mid + \lambda \mid\mid Au_0 Au_0 \mid\mid + \mu \mid\mid Au_0 \mid\mid^2 \\
&= (\lambda + \mu) \mid\mid Au_0 \mid\mid^2 + \mu \mid\mid Au_0 \mid\mid^2 + \lambda \mid\mid Au_0 \mid\mid^2 \\
&= 2(\lambda + \mu) \mid\mid Au_0 \mid\mid^2 .
\end{aligned}$$

Hence,

$$\tfrac{1}{2}\frac{d}{dt} \mid\mid u_\lambda(t) - u_\mu(t) \mid\mid^2 \leq 2(\lambda + \mu) \mid\mid Au_0 \mid\mid^2 .$$

Integrating the inequality, we obtain

$$\mid\mid u_\lambda(t) - u_\mu(t) \mid\mid^2 \leq 4(\lambda + \mu)t \mid\mid Au_0 \mid\mid^2 .$$

i.e.,

$$\mid\mid u_\lambda(t) - u_\mu(t) \mid\mid \leq 2\sqrt{\lambda + \mu t} \mid\mid Au_0 \mid\mid \qquad (15)$$

It follows that for every fixed $t \geq 0$, $u_\lambda(t)$ is a cauchy sequence as $\lambda \longrightarrow 0$ and thus it converges to a limit, denoted by u(t). Passing to the limit in (15) as $\mu \longrightarrow 0$, we have

$$\mid\mid u_\lambda(t) - u_\mu(t) \mid\mid \leq 2\sqrt{\lambda t} \mid\mid Au_0 \mid\mid .$$

Therefore the convergence is uniform in t on every bounded interval [0,T] and so $u \in C([0, +\infty); H)$.

Step 4 :

claim : $\frac{du_\lambda}{dt}(t)$ converges, as $\lambda \longrightarrow 0$, to some limit and that converges is uniform on every bounded interval [0, T]. Assume that $u_0 \in D(A^2)$.

Set

$$v_\lambda = \frac{du_\lambda}{dt} ,$$

so that

$$\frac{dv_\lambda}{dt} + A_\lambda v_\lambda = 0.$$

Now, following the same argument as in step 3, we have :

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} \| v_\lambda - v_\mu \|^2 &\leq (\| A_\lambda v_\lambda(t) \| + \| A_\mu v_\mu(t) \|)(\lambda \| A_\lambda v_\lambda(t) \| + \mu \| A_\mu v_\mu(t) \|) \\
&\leq (\| A_\lambda v_\lambda(0) \| + \| A_\mu v_\mu(0) \|)(\lambda \| A_\lambda v_\lambda(0) \| + \mu \| A_\mu v_\mu(0) \|) \\
&= (\| A_\lambda A_\lambda u_0 \| + \| A_\mu A_\mu u_0 \|)(\lambda \| A_\lambda A_\lambda u_0 \| + \mu \| A_\mu A_\mu u_0 \|) \\
&= (\| J_\lambda^2 A^2 u_0 \| + \| J_\mu^2 A^2 u_0 \|)(\lambda \| J_\lambda^2 A^2 u_0 \| + \mu \| J_\mu^2 A^2 u_0 \|) \\
&\leq (\| A^2 u_0 \| + \| A^2 u_0 \|)(\lambda \| A^2 u_0 \| + \mu \| A^2 u_0 \|) \\
&= \lambda \| A^2 u_0 \|^2 + \mu \| A^2 u_0 \|^2 + \lambda \| A^2 u_0 \|^2 + \mu \| A^2 u_0 \|^2 \\
&= 2\lambda \| A^2 u_0 \|^2 + 2\mu \| A^2 u_0 \|^2 \\
&= 2(\lambda + \mu) \| A^2 u_0 \|^2
\end{aligned}$$

Since,

$$\tfrac{1}{2}\frac{d}{dt} \| v_\lambda - v_\mu \|^2 \quad \leq \quad 2(\lambda + \mu) \| A^2 u_0 \|^2 .$$

Integrating the inequality, we obtain

$$\| v_\lambda - v_\mu \| \quad \leq \quad 2\sqrt{(\lambda + \mu)t} \, \| A^2 u_0 \| \tag{16}$$

It follows that for every fixed $t \geq 0$, $v_\lambda(t)$ is a cauchy sequence as $\lambda \longrightarrow 0$ and thus it converges to a limit denoted by v(t). Passing to the limit in (16) as $\mu \longrightarrow 0$ we have,

$$\| v_\lambda - v_\mu \| \quad \leq \quad 2\sqrt{(\lambda)t} \, \| A^2 u_0 \|.$$

Hence, $v_\lambda(t) = \dfrac{du_\lambda}{dt}(t)$ converges as $\lambda \longrightarrow 0$, to some limit and that the converges is uniform on every bounded interval [0,T].

Step 5 : Assume that $u_0 \in D(A^2)$

Claim : u is a solution of problem (13) (Theorem 6)

Now by step 3 and 4 , we have $u_\lambda(t) \longrightarrow u(t)$ as $\lambda \longrightarrow 0$ uniformly on [0,T], and $\dfrac{du_\lambda}{dt}(t)$ converges as $\lambda \longrightarrow 0$ uniformly on [0,T] for all $T < \infty$.

So it follows that $u \in C^1([0, +\infty); H)$ and that

$$\frac{du_\lambda}{dt}(t) \longrightarrow \frac{du}{dt}(t)$$

as $\lambda \longrightarrow 0$ uniformly on [0,T].

When rewrite (13) as

$$\frac{du_\lambda}{dt}(t) + A(J_\lambda u_\lambda(t)) = 0.$$

Since A has a closed graph, so $u \in D(A), \forall t \geq 0$ and that

$$\frac{du}{dt}(t) + Au(t) = 0.$$

Hence, $u \in C^1([0,+\infty);H)$, the function $t \longmapsto Au(t)$ is continuous from $[0,+\infty)$ onto H and thus $u \in C([0+\infty);D(A))$

Note: $J_\lambda u_\lambda(t) \longrightarrow u(t)$ as $\lambda \longrightarrow 0$.

Since ,

$$\begin{aligned} \| J_\lambda u_\lambda(t) - u(t) \| &\leq \| J_\lambda u_\lambda(t) - J_\lambda u(t) \| + \| J_\lambda u(t) - u(t) \| \\ &= \| u_\lambda(t) - u(t) \| + \| J_\lambda u(t) - u(t) \| \longrightarrow 0 \end{aligned}$$

Step 6: To conclude the proof of the theorem, we shall use the following lemma

Lemma 2. *Let* $u_0 \in D(A)$. *Then,* $\forall \epsilon > 0, \exists \overline{u}_0 \in D(A^2)$ *such that* $\| u_0 - \overline{u}_0 \| < \epsilon$ *and* $\| Au_0 - A\overline{u}_0 \| < \epsilon$.

Let us proof first the lemma :

Proof. Set, $\overline{u}_0$ for some approximate $\lambda > 0$ to be fixed later.

Since A is maximal monotone, so we have $\overline{u}_0 \in D(A)$ and $\overline{u}_0 + \lambda A\overline{u}_0 = u_0$

Thus $A\overline{u}_0 \in D(A)$ *i.e.,* $\overline{u}_0 \in D(A^2)$.

On the other hand by proposition 3, we have

$$\lim_{\lambda \longrightarrow 0} \| J_\lambda u_0 - u_0 \| = 0 \quad , \lim_{\lambda \longrightarrow 0} \| J_\lambda Au_0 - Au_0 \| = 0$$

and $J_\lambda Au_0 = AJ_\lambda u_0$.

Therefore, for every $\epsilon > 0$,

$$\| J_\lambda u_0 - u_0 \| < \epsilon \quad \text{and} \quad \| J_\lambda Au_0 - Au_0 \| < \epsilon \qquad \square$$

Now let us turn to the proof of Theorem 6 .

Given,

$$u_0 \in D(A).$$

We construct (using lemma 2), a sequence u_{0_n} in $D(A^2)$ such that $u_{0_n} \longrightarrow u_0$ and $Au_{0_n} \longrightarrow Au_0$. By step 5, we know that there is a solution u_n of the problem

$$(17) \qquad \begin{cases} \dfrac{du_n}{dt} + Au_n = 0 \quad ,\text{on} \quad [0,+\infty), \\ u_n(0) = u_{0_n}. \end{cases}$$

We have, for all $t \geq 0$,

$$|| u_n(t) - u_m(t) || \leq || u_{0_n} - u_{0_m} || \longrightarrow 0, \quad \text{as} \quad m,n \to \infty$$

$$|| \frac{du_n}{dt}(t) - \frac{du_m}{dt}(t) || \leq || Au_{0_n} - Au_{0_m} || , \quad m,n \to \infty$$

Therefore , $u_{n(t)} \longrightarrow u(t)$ uniformly on, $[0,+\infty)$, $\frac{du_n}{dt}(t) \longrightarrow \frac{du}{dt}$ uniformly on, $[0,+\infty)$, with $u \in C^1([0,+\infty);H)$. Passing to the limit in (17), since A is a closed operator we have, $u(t) \in D(A)$ u satisfies equation (13) (Theorem 6) .

From equation (13) (Theorem 6) we deduce that

$$u \in C([0,+\infty);D(A)).$$

□

Definition 10. *A C_o* ***semigroup(strongly continuous semigroup)*** *is a family $S_A(t)$, $\forall t \in \Re^+$ of bounded linear operators from H to H satisfying:*

$$(i) \quad S_A(t_1+t_2) = S_A(t_1)oS_A(t_2) \qquad \forall t_1,t_2 \geq 0,$$

$$(ii) \quad S_A(0) = I,$$

$$(iii) \quad \lim_{t \longrightarrow 0^+} || S_A(t)u_0 - u_0 || = 0, \quad \forall u_0 \in H.$$

A family of bounded operators $S_A(t),\ 0 \leq t \leq \infty$ *on Hilbert space* H *is called a* ***contraction semigroup*** *if it is a strongly continuous semigroup and ,*

$$\| S_A(t) \| \leq 1 \ , \quad \forall t \in [0, \infty).$$

Semigroups can be use to solve a large class of problems commonly known as an evolution problems. These types of problems appear in many disciplines including, physics , chemistry, biology, engineering, and economics .

Definition 11. *Let* $S_A(t)$ *be a strongly continuous one parameter semigroup. The (infinitesimal) generator of* $S_A(t)$*, denoted by* A *is given by the equation:*

$$Au_0 = \lim_{t \longrightarrow 0^+} \frac{S_A(t)u_0 - u_0}{t}$$

where the limit is evaluated in terms of the norm on H *and* u_0 *is in the domain of* A *if and only if this limit exists.*
Remark:
Given a continuous semigroup of contractions S(t) on H there exists a unique maximal monotone operator A *such that* $S(t) = S_A(t) \quad , \forall t \geq 0.$
Justification*:*
we have,

$$S_A(t) = e^{tA}$$
$$S_A^{'}(t) = Ae^{tA} = AS_A(t), \quad \textit{and} \quad S_A^{'}(0) = A.$$

Hence we can obtain A *by*

$$A = \frac{d}{dt} S_A(t) \mid_{t=0},$$

i.e., (by definition of generator), the generator A *is obtained by differentiating the semigroup* $S_A(t)$*.*
Remark:
Let A *be a maximal monotone operator and let* $\lambda \in \mathbb{R}$*. Then the problem*

$$\begin{cases} \dfrac{du}{dt} + Au + \lambda u = 0 \quad \textit{on} \quad [0, +\infty), \\ u(0) = u_0 \end{cases}$$

reduces to problem (13) i.e., Theorem 6 by setting $v(t) = e^{\lambda t}u(t)$.

Justification:

Let us define $v(t) = e^{\lambda t}u(t)$. Now rearrange v(t) as

$u(t) = \frac{v(t)}{e^{\lambda t}}$

and then differentiating u with respect to t.
We obtain,

$$\frac{du}{dt} = \frac{dv}{dt}e^{-\lambda t} - \lambda v(t)e^{-\lambda t}.$$

From this

$$\frac{du}{dt} + Au + \lambda u = 0$$

becomes,

$$\frac{dv}{dt}e^{-\lambda t} - \lambda v(t)e^{-\lambda t} + Av(t)e^{-\lambda t}\lambda v(t)e^{-\lambda t} = 0$$

$$\implies \quad \frac{dv}{dt} - \lambda v(t) + \lambda v(t) + Av(t) = 0$$

i.e.,

$$\frac{dv}{dt} + Av(t) = 0$$

Hence, v satisfies

$$\begin{cases} \dfrac{dv}{dt} + Av = 0 \quad on \quad [0, +\infty), \\ v(0) = u_0. \end{cases}$$

2.2 The Self - Adjoint

Definition 12. *Let A: D(A)$\subset$ $H \longrightarrow H$ be unbounded linear operator with $\overline{D(A)} = H$ and let $A^* : D(A^*) \subset H \longrightarrow H$ be unbounded linear operator which is called adjoint of A. Then*

(i) A is symmetric if $(Au, v) = (u, Av) \quad \forall u, v \in D(A)$

(ii) A is self-adjoint if $D(A^*) = D(A)$ *and* $A^* = A$

Remark : For bounded operator the notation of symmetric and self-adjoint operators coincide. However if A is unbounded there is a subtle difference between symmetric and self-adjoint.

Example 2. *Let $H = L^2(0,1)$ (with functions taking real values). Let A denote differentiation with domain $D(A) = \{u \in H : u \in C^1[0,1]\}$. such that*

$$Au = u^{'}.$$

Let v be a $C^1[0,1]$ function (so $v \in L^2(0,1)$ with $v(0) = v(1) = 0$;let us use V to denote the subspace of $L^2(0,1)$(it is also dense) Then :

$$(Au, v) = \int_0^1 u^{'}(t)v(t)dt$$

Integrating by parts to find that

$$(Au, v) = u(t)v(t)|_0^1 - \int_0^1 u(t)v^{'}(t)dt = -\int_0^1 u(t)v^{'}(t)dt$$

If we take $A^*u = -u^{'}$, then we have

$(Au, v) = (u, A^*v)$ *,for all* $u \in C^1[0,1]$ *and* $v \in V$.

Hence , $(Au, v) = (u, A^*v)$ *i.e., we have defined an adjoint for A on a dense subspace of H.*

Example 3. *Let $H = L^2(0,1)$ (where the functions may take values in $\mathbb{C}$). Let A be the operator $Au = iu^{'}$ with the domain*

$D(A) = \{u \in H : u \in C^1[0,1], \quad u(0) = u(1)\}$.

The adjoint of A *is defined much as in Example 2 .*
Let v *be a* $C^1[0,1]$ *function (so* $v \in L^2(0,1)$*) with* $v(0) = v(1)$ *(same conditions as u). Then*

$$(Au, v) = \int_0^1 iu^{'}(t)\overline{v(t)}dt$$

Integrate by parts to find that

$$(Au, v) = \int_0^1 iu(t)\overline{v^{'}(t)}dt = (u, Av)$$

In other words, A *is self-adjoint.*

Lemma 3. *An operator* A *is symmetric if and only if* $A \subset A^*$*.*

Proof. The defining relation of A^*(adjoint of A) is

$$(Au, v) = (u, A^* v) \tag{18}$$

for all $u \in D(A)$ and all $v \in D(A^*)$
($\Longleftarrow$) Assume that $A \subset A^*$.
Then,

$$A^* v = Av, \qquad \forall u \in D(A) \quad ,$$

so that equation (18) becomes

$$(Au, v) = (u, Av) \qquad \forall u, v \in D(A).$$

Hence , A is Symmetric.
($\Longrightarrow$) Suppose that A is symmetric i.e.,

$$(Au, v) = (u, Av) \qquad \forall u, v \in D(A).$$

Then comparison with equation (18) i.e.,

$$(Au, v) = (u, A^* v), \quad \forall u \in D(A) \quad \text{and} \quad \forall v \in D(A^*).$$

This shows that

$$D(A) \subset D(A^* \quad \text{and } A = A^* \mid_{D(A)}.$$

By definition of extension this means that A^* is an extension of A i.e., $A \subset A^*$. □

Proposition 4. *Let* A *be a maximal monotone symmetric operator. Then* A *is self-adjoint.*

Proof. Let us first prove that $J_1 = (I + A)^{-1}$ is self- adjoint.
Claim: $(J_1 u, v) = (u, J_1 v) \qquad u, v \in H.$
Set,

$$u_1 = J_1 u \quad \text{and} \quad v_1 = J_1 v$$

so that

$$\begin{aligned} u_1 + Au_1 &= u, \\ v_1 + Av_1 &= v \end{aligned}$$

Since A is symmetric so we have,

$$(u_1, Av_1) = (Au_1, v_1)$$

It follows that

$(u_1, v) = (u, v_1)$ i.e., $(J_1 u, v) = (u, J_1 v)$ and $D(J_1^*) = D(J_1)$ because our operators i.e., $J_1 : H \longrightarrow H$ and $J_1^* : H \longrightarrow H$ have the same domain H. Therefore J_1 is self-adjoint.

Now to show A is self adjoint

(i) Since A is maximal monotone operator so $\forall f \in H, \exists u \in D(A)$ such that

$$u + Au = f$$

and also A is symmetric i.e., $A = A^*$, so we have,

$\forall f \in H, \exists u \in D(A^*)$ such that $u + A^* u = f,$

$\Longrightarrow u + Au = u + A^* u.$

Hence,

$$A = A^*$$

(ii) To show $D(A^*) = D(A)$

Let $u \in D(A)^*$ and set

$$f = u + A^* u.$$

Then we have

$(f, J_1 w) = (J_1 f, w) = (u, w) \qquad \forall w \in H$

$\Longrightarrow u = J_1 f$ and thus $u \in D(A).$

This proves that $D(A^*) = D(A)$, and hence A is self-adjoint. □

Theorem 7. *Let A be a self - adjoint maximal monotone operator. Then for every* $u_0 \in H$ *, there exists a unique solution*

$$u \in C([0,+\infty); H) \cap C^1((0,\infty); H) \cap C((0,+\infty); D(A))$$

such that

$$\begin{cases} \dfrac{du}{dt} + Au = 0, & on(0,+\infty) \\ u(0) = u_0 \end{cases} \tag{19}$$

Moreover, we have

$$\| u(t) \| \leq \| u_0 \| \quad and \quad \| \frac{du}{dt}(t) \| = \| Au(t) \| \leq \tfrac{1}{t} \| u_0 \|, \quad \forall t > 0$$
$$u \in C^k((0,\infty); D(A^l)) \qquad \forall k, l \quad integers.$$

Proof. (i) *Uniqueness*

Let u and v be two solutions of the given problem. Then we have

$$\begin{cases} \dfrac{du}{dt} + Au = 0, & \text{on}(0,+\infty) \\ u(0) = u_0 \end{cases}$$

and

$$\begin{cases} \dfrac{dv}{dt} + Av = 0, & \text{on}(0,+\infty) \\ v(0) = v_0 \end{cases}$$

From this

$$\frac{du}{dt} + Au = \frac{dv}{dt} + Av$$
$$\Longrightarrow \quad \frac{d}{dt}(u-v) + A(u-v) = 0$$
$$\Longrightarrow \quad (\frac{d}{dt}(u-v), (u-v)) + (A(u-v), u-v) = 0$$
$$\Longrightarrow \quad (\frac{d}{dt}(u-v), u-v) = -(A(u-v), u-v) \leq 0$$

i.e.,

$$(\frac{d}{dt}(u-v), u-v) \leq 0 \tag{20}$$

Integrate equation (20) by parts, we obtain

$$\| u(t) - v(t) \|^2 - \| u(0) - v(0) \|^2 = -2 \int_0^t (A(u(s) - v(s), u(s) - v(s)) ds \leq 0.$$

On the other hand $\| u(t)-v(t) \|^2$ is nonincreasing on $[0,\infty)$ and

$$\| u(0)-v(0) \|^2 = 0.$$

This implies that

$$\| u(t)-v(t) \|^2 \leq 0 \ .$$

Therefore,

$$u(t)=v(t)$$

(ii) Existence.The proof is divided is divided into two steps:
Step 1: Assume first that $u_0 \in D(A^2)$ and let u be the solution of (13), given by Theorem 6
Claim : $\| \frac{du}{dt}(t) \| \leq \frac{1}{t} \| u_0 \| \quad \forall t > 0.$
As in the proof of proposition 4 we have,

$$J_\lambda^* = J_\lambda \quad \text{and} \quad A_\lambda^* = A_\lambda, \quad \forall \lambda > 0.$$

Now using approximate problem introduced in the proof of Theorem 6:

$$\frac{du_\lambda}{dt} + A_\lambda u_\lambda = 0 \text{ on } (0,+\infty), \quad u_\lambda(0)=u_0 \tag{21}$$

Taking the scalar product of (21) with u_λ and integrating on [0,T] we obtain

$$\int_0^T (\frac{du_\lambda}{dt}, u_\lambda)dt + \int_0^T (A_\lambda u_\lambda, u_\lambda)dt = 0$$

But,

$$(\frac{du_\lambda}{dt}, u_\lambda) = \tfrac{1}{2}\frac{d}{dt} \| u_\lambda(t) \|^2$$

so

$$\int_0^t \frac{1}{2}\frac{d}{dt} \| u_\lambda(t) \|^2 \, dt + \int_0^T (A_\lambda u_\lambda, u_\lambda)dt = 0$$
$$\Longrightarrow \frac{1}{2}[\| u_\lambda(t) \|^2]_0^T + \int_0^T (A_\lambda u_\lambda, u_\lambda)dt = 0,$$
$$\Longrightarrow \frac{1}{2} \| u_\lambda(t) \|^2 - \frac{1}{2} \| u_\lambda(0) \|^2 + \int_0^T (A_\lambda u_\lambda, u_\lambda)dt = 0$$

i.e.,

$$\frac{1}{2} \mid\mid u_\lambda(t) \mid\mid^2 + \int_0^T (A_\lambda u_\lambda, u_\lambda)dt = \frac{1}{2} \mid\mid u_\lambda(0) \mid\mid^2 \tag{22}$$

Taking the scalar product of (21) with $t\frac{du_\lambda}{dt}$ and integrating over [0,T], we obtain

$$\int_0^T (\frac{du_\lambda}{dt}, \frac{du_\lambda}{dt})tdt + \int_0^T (A_\lambda u_\lambda, \frac{du_\lambda}{dt})tdt = 0$$

This implies that,

$$\int_0^T \mid\mid \frac{du_\lambda}{dt}(t) \mid\mid^2 tdt + \int_0^T (A_\lambda u_\lambda, \frac{du_\lambda}{dt})tdt = 0 \tag{23}$$

But,

$$\frac{d}{dt}(A_\lambda u_\lambda, u_\lambda) = (A_\lambda \frac{d}{dt} u_\lambda, u_\lambda) + (A_\lambda u_\lambda, \frac{d}{dt} u_\lambda) = 2(A_\lambda u_\lambda, \frac{d}{dt} u_\lambda)$$

Hence equation (23) becomes,

$$\int_0^T \mid\mid \frac{du_\lambda}{dt}(t) \mid\mid^2 tdt + \frac{1}{2}\int_0^T \frac{d}{dt}(A_\lambda u_\lambda, u_\lambda)tdt = 0 \tag{24}$$

Now integrating the second integral in equation(24) by parts
i.e.,

$$\begin{aligned}\frac{1}{2}\int_0^T \frac{d}{dt}(A_\lambda u_\lambda, u_\lambda)tdt &= \frac{1}{2}[t(A_\lambda u_\lambda, u_\lambda)]_0^T - \frac{1}{2}\int_0^T (A_\lambda u_\lambda, u_\lambda)dt \\ &= \frac{1}{2}(A_\lambda u_\lambda(T), u_\lambda(T))T - \frac{1}{2}\int_0^T (A_\lambda u_\lambda, u_\lambda)dt\end{aligned}$$

This implies that,

$$\begin{aligned}\int_0^T (A_\lambda u_\lambda, \frac{du_\lambda}{dt})tdt &= \frac{1}{2}\int_0^T \frac{d}{dt}(A_\lambda u_\lambda, u_\lambda)tdt \\ &= \frac{1}{2}(A_\lambda u_\lambda, u_\lambda(T))T - \frac{1}{2}\int_0^T (A_\lambda u_\lambda, u_\lambda)dt\end{aligned}$$

i.e.,

$$\int_0^T (A_\lambda u_\lambda, u_\lambda)dt = (A_\lambda u_\lambda(T), u_\lambda(T))T - 2\int_0^T (A_\lambda u_\lambda, \frac{du_\lambda}{dt})tdt$$

But by equation (23) we have,

$$\int_0^T (A_\lambda u_\lambda, \frac{du_\lambda}{dt})tdt = -\int_0^T \mid\mid \frac{du_\lambda}{dt}(t) \mid\mid^2 tdt$$

Hence,

$$\int_0^T (A_\lambda u_\lambda, u_\lambda)dt = (A_\lambda u_\lambda(T), u_\lambda(T))T + 2\int_0^T \mid\mid \frac{du_\lambda}{dt}(t) \mid\mid^2 tdt$$

The above equation(22) becomes

$$\frac{1}{2} \mid\mid u_\lambda(T) \mid\mid^2 + (A_\lambda u_\lambda(T), u_\lambda(T))T + 2\int_0^T \mid\mid \frac{du_\lambda}{dt}(t) \mid\mid^2 tdt = \frac{1}{2} \mid\mid u_0 \mid\mid^2 \tag{25}$$

Since the function $\mid\mid t \mapsto \frac{du_\lambda}{dt}(t) \mid\mid$ is non increasing by lemma 2, we have

$$\int_0^T \mid\mid \frac{du_\lambda}{dt}(t) \mid\mid^2 tdt \quad \geq \quad \mid\mid \frac{du_\lambda}{dt}(T) \mid\mid^2 \frac{T^2}{2}$$

Hence,

$$\tfrac{1}{2} \mid\mid u_\lambda(T) \mid\mid^2 + T(A_\lambda u_\lambda(T), u_\lambda(T)) + T^2 \mid\mid \frac{du_\lambda}{dt}(T) \mid\mid^2 \quad \leq \quad \tfrac{1}{2} \mid\mid u_0 \mid\mid^2$$

It follows in particular that

$$T^2 \mid\mid \frac{du_\lambda}{dt}(T) \mid\mid^2 \quad \leq \quad \tfrac{1}{2} \mid\mid u_0 \mid\mid^2$$

i.e.,

$$\mid\mid \frac{du_\lambda}{dt}(T) \mid\mid^2 \quad \leq \quad \tfrac{1}{T} \mid\mid u_0 \mid\mid \qquad , \forall T > 0 \tag{26}$$

Finally we pass to the limit in equation(26) as $\lambda \longrightarrow 0$.

Since

$$\frac{du_\lambda}{dt} \longrightarrow \frac{du}{dt}$$

Therefore,

$$|| \frac{du}{dt}(t) || \quad \leq \quad \tfrac{1}{t} \, || u_0 || \qquad \forall t > 0$$

Step 2: Assume now $u_0 \in H$. Let u_{0_n} be a sequence in $D(A^2)$ such that $u_{0_n} \longrightarrow u_0$ recall that $D(A^2)$ is dense in D(A) and that D(A)is dense in H; thus $D(A^2)$ is dense in H). Let u_n be the solution of

$$\begin{cases} \dfrac{du_n}{dt} + Au_n = 0, & [0, +\infty) \\ u_n(0) = u_{0_n}. \end{cases}$$

Using Theorem 5. we have

$|| u_n(t) - u_m(t) || \quad \leq \quad || u_{n_o} - u_{m_0} || \qquad \forall m, n, \quad \forall t \geq 0,$ by step 1 (above).

$$|| \frac{du_n}{dt}(t) - \frac{du_m}{dt}(t) || \quad \leq \quad \tfrac{1}{t} \, || u_{0_n} - u_{0_m} || \qquad \forall m, n, \quad \forall t > 0$$

It follows that u_n converges uniformly on $[0, +\infty)$ to some limit u(t) and that $\dfrac{du_n}{dt}(t)$ converges $\dfrac{du}{dt}(t)$ uniformly on every interval $[\delta, +\infty)$, $\delta > 0$. The limiting function u satisfies

$$u \in C([0, +\infty); H) \cap C^1((0, +\infty); H),$$

$u(t) \in D(A) \quad \forall t > 0$ and $\dfrac{du}{dt}(t) + Au(t) = 0 \qquad \forall > 0,$
(this uses the fact that A is closed)

Now to show $u \in C^k((0, +\infty); D(A^l)) \quad \forall k, l,$ integers. Using induction on $k \geq 2$ that

$$u \in C^{k-j}((0, +\infty); D(A^j)) \qquad \forall j = 0, 1, 2, ..., k. \tag{27}$$

Assume that (27) holds up to order k-1. In particular we have

$$u \in C((0, +\infty); D(A^{k-1})) \tag{28}$$

To prove (27) let us check that

$$u \in C((0, +\infty); D(A^k)) \qquad (29).$$

Let us define the Hilbert space $\tilde{H} = D(A^{k-1})$ and the operator $\tilde{A} : D(\tilde{A}) \subset \tilde{H} \longrightarrow \tilde{H}$ by

$$\begin{cases} D(\tilde{A}) = D(A^k), \\ \tilde{A} = A. \end{cases}$$

Since, $\tilde{A}$ is maximal monotone and symmetric in $\tilde{H}$; thus it is self adjoint . Using the first assertion of Theorem 7 in the space $\tilde{H}$ to the operator $\tilde{A}$, we obtain a unique solution v of the problem

$$\begin{cases} \dfrac{dv}{dt} + Av = 0 \quad on \quad (0, +\infty), \\ v(0) = v_0, \end{cases}$$

given any $v_0 \in \tilde{H}$. That is,

$$v \in C([0, +\infty); \tilde{H}) \cap C^1((0, +\infty); \tilde{H}) \cap C((0, +\infty); D(\tilde{A})).$$

Choose $v_0 = u(\epsilon)$, $\epsilon > 0$ by (28) we have $v_0 \in \tilde{H}$.
Hence,

$$u \in C((\epsilon, +\infty; D(A^k)).$$

Therefore,

$$u \in C((0, +\infty; D(A^k)). \qquad \square$$

Conclusion

Finding a unique solution in differential equation with any given evolution problem is essential. So to determine the existence of unique solution i.e., whether the problem has a unique solution or not, for the evolution problem

$$\begin{cases} \dfrac{du}{dt} + Au = 0, \qquad on \quad [0, +\infty), \\ u(0) = u_0 \end{cases}$$

is guarantee:

i. If A is a Lipschitz map.

ii. if A is maximal monotone linear operator (even if A is unbounded linear operator), there is a unique solution by applying Hille - Yosida Theorem.

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