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Addis Ababa University
School of Graduate Studies
Department of Mathematics

Project Report

On

Integral Representation of Harmonic functions

On a Disc and Upper Half Plane

By

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A.A.U

Declaration

I declare that this project has been composed by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship or any other similar title to me.

Permission

This is to certify that this project is compiled by Biniyam Shimelis in the Department of mathematics, Addis Ababa University, under my supervision. I hereby also confirm that the project can be submitted for evaluation by examiners and eventual defense.

Seid Mohammed (PhD) -----

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ABSTRACT

The most important in the theory of harmonic functions is that of finding a harmonic function with given boundary values; it is known as the Dirichlet problem. The Dirichlet problem consists in determining all regions G such that for any continuous function $f: \partial G \rightarrow \mathbb{R}$ there is a continuous function $u: \bar{G} \rightarrow \mathbb{R}$ such that $u(z) = f(z)$ for z in ∂G and u is harmonic on G . To study the Dirichlet problem we are concerned with two main questions. Does a solution exist, and if so, is it uniquely determined by the given boundary values? To solve the boundary value problem the major tool is to develop the Poisson integral formula which is integral representation of harmonic functions. We are, in fact, able to show the Poisson integral of harmonic functions on a disk and upper half plane. The theory of harmonic function on the upper half plane H develop by transforming the theory of harmonic function of a unit disc on to upper half plane H by conformal mapping.

The purpose of the project is to study the integral representation of harmonic functions in a disc and upper half plane and then compiled as reading material. It means that for a harmonic function u on a disc $D := \{z \in \mathbb{C}: |z - a| < r\}$ has a Poisson integral formula $Pu(a + re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} p_r(\theta - \varphi) u(a + Re^{i\varphi}) d\varphi$ and for the upper half plane H , using Möbius transformation which maps D to upper half plane H . Then we have to develop the integral representation of harmonic functions on the upper half plane H . For a harmonic function u on the upper half plane $H := \{z \in \mathbb{C}: \text{Im}z > 0\}$ has a Poisson integral formula and denoted by $P\bar{u}(w) = \int_{-\infty}^{\infty} \bar{u}(t) p_y(x - t) dt$.

Key words: Harmonic function, Poisson integral, Dirichlet problem

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INTRODUCTION

This project is devoted to the study of integral representation of harmonic functions on a disc and upper half plane. Harmonic functions are closely connected to analytic functions since the real and imaginary parts of analytic function are harmonic functions. The study of harmonic functions is important in physics and engineering, and there are many results in the theory of harmonic functions that are not connected directly with complex analysis. However, in this project we consider that part of the theory of integral representation of harmonic functions that grows out of the Cauchy theory and Möbius transformation. One of the most important aspects of harmonic functions is that they arise as functions that solve a boundary value problem for holomorphic functions (it is known as the Dirichlet problem). An example is the problem of finding a function continuous in a closed disc D (upper half plane H) that assumes certain known values on the boundary of the disc $D(H)$ and is harmonic in the interior of the disc $D(H)$. An important tool to solve the Dirichlet problem is the Poisson formula of the harmonic functions. This paper is organized in to three chapters.

In the first chapter we define harmonic functions, some properties of harmonic functions such that mean value property(MVP) and the maximum principle, the role of conformal mapping, convolution and the Laplacian of a function. In the second we obtain integral representations for harmonic functions on a disk and upper half plane using the Cauchy integral formula and Möbius transformation and also define the properties of Poisson kernels on a disk and upper half plane. And lastly we have to show the application of integral representation of harmonic functions on a disc and upper half plane.

Chapter One

1. PRELIMINARIES

In this chapter we consider basic properties of complex function theory with some topological concept. Several results and techniques of this chapter frequently used in the next chapter.

1.1 DEFINITIONS AND NOTATIONS

Let \mathbb{C} denote the set of complex numbers and \mathbb{R} denote the set of real numbers. The symbol ‘:=’ denotes ‘equals by definition’; it is used to stress that an equation is defining something and also as convenient short hand. We denote the end of a proof by customary, ■. Then \mathbb{C}^n denote the Eculidean space of n dimension for $n > 1$. However in this project we use for $n = 2$, there are many interesting concept on \mathbb{C}^n for $n > 1$. Throughout this project the letter G always will denote plane open set. In this work we consider \mathbb{C}^2 , let $z \in \mathbb{C}$ we identify a point (x, y) with $z = x + iy$. Now consider the following notations:

If a is a point in space \mathbb{C}^n and $r > 0$, then:

The set

- i) $D(a, r) = \{z \in \mathbb{C}^n : |z - a| < r\}$ is called an open ball (open disc if $n = 2$) center at a and radius r .
- ii) $D[a, r] = \{z \in \mathbb{C}^n : |z - a| \leq r\}$ is called a closed ball (closed disc if $n = 2$) center at a and radius r .
- iii) $S = \{z \in \mathbb{C}^n : |z - a| = r\}$ is called a sphere (circle if $n = 2$) center at a and radius r .
- iv) The open unit ball in \mathbb{C}^n is the set $D(a, 1) = \{z \in \mathbb{C}^n : |z - a| < 1\}$
- v) The closed unit ball is given by $D[a, 1] = \{z \in \mathbb{C}^n : |z - a| \leq 1\}$
- vi) The upper half plane, denoted by H , consists of those complex numbers with positive imaginary; that is $H = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$

Definition 1.1: A real valued function u defined on an open subset G of \mathbb{C} will be called harmonic in G if it has first and second order partial derivatives continuous on G and satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ in } G.$$

Example: Let: $\mathbb{C} \rightarrow \mathbb{R}$ by $u(x + iy) = e^x \cos y$

Solution:

Clearly u is two times continuously differentiable and

$$\frac{\partial u}{\partial x} = e^x \cos y \text{ and } \frac{\partial u}{\partial y} = -e^x \sin y$$

$$\frac{\partial^2 u}{\partial x^2} = e^x \cos y \text{ and } \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \cos y - e^x \cos y = 0$$

So, it satisfies the Laplace equation and u is harmonic.

Note that: - A function is harmonic at a point if it is harmonic in a neighborhood of that point

- It is clear that $u_1(z)$ and $u_2(z)$ are harmonic functions so are $u_1 + u_2$ and ku , where k is arbitrary real numbers.
- The product, however, of two functions which are harmonic need not be harmonic.

Example: Let $u_1(z) = x$ and $u_2(z) = e^x \cos y$ where $z = x + iy$

Solution:

Since u_1 and u_2 are harmonic but $u_1(z)u_2(z) = xe^x \cos y$ is not satisfy Laplace equation

Let $V(z) = u_1(z)u_2(z) = xe^x \cos y$

$$\frac{\partial V}{\partial x} = e^x \cos y + x e^x \cos y \text{ and } \frac{\partial V}{\partial y} = -x e^x \sin y$$

$$\frac{\partial^2 V}{\partial x^2} = e^x \cos y + e^x \cos y + x e^x \cos y = 2e^x \cos y + x e^x \cos y \text{ and } \frac{\partial^2 V}{\partial y^2} = -x e^x \cos y$$

Then

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 2e^x \cos y + x e^x \cos y - x e^x \cos y = 2e^x \cos y \neq 0$$

$\therefore u_1(z)u_2(z)$ is not harmonic.

The differentiability of $F(z) = u(z) + iv(z)$, where $z = x + iy$ with respect to z implies the differentiability with respect to x and y separately. In particular $u_x(z)$ and $u_y(z)$ exists and continuous.

Definition 1.2: suppose f is a complex function defined on G , if $z_0 \in G$ and if $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists for $z \in G$, we denote this limit by $f'(z_0)$. If $f'(z_0)$ exists for every $z_0 \in G$, then we say that f is holomorphic (analytic) in G . The class of all holomorphic functions in G will be denoted by $H(G)$.

If $f(z) \in H(G)$, then for every $z_0 \in G$, $f'(z_0)$ exists. This implies $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exist in any direction of $z \rightarrow z_0$. In particular consider, $z \rightarrow z_0$ horizontally and vertically

Then we obtain

$$f'(z_0) = \left[\frac{\partial f}{\partial x} \right]_{z=z_0} = \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \right]_{z=z_0} \dots \dots \dots (1)$$

where $f(z) = u(x, y) + iv(x, y)$ and

$$f'(z_0) = \frac{1}{i} \left[\frac{\partial f}{\partial y} \right]_{z=z_0} = \frac{1}{i} \left[\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right]_{z=z_0} \dots \dots \dots (2)$$

Now from equation (1) and (2) we have that

$$\left[\frac{\partial f}{\partial x} \right]_{z=z_0} = \frac{1}{i} \left[\frac{\partial f}{\partial y} \right]_{z=z_0}$$

In terms of real and imaginary parts of $f(z)$, the equation becomes

$$u_x + iv_x = -i(u_y + iv_y).$$

Then we get

$$u_x(x_0, y_0) = v_y(x_0, y_0) \text{ and } u_y(x_0, y_0) = -v_x(x_0, y_0) \dots \dots \dots (3)$$

Equation (3) is known as the **Cauchy Riemann equations**.

The real and imaginary parts of analytic function are not independent of each other. Since they satisfy the Cauchy Riemann equations states above.

Now differentiating the left side of equation (3) with respect to x and the right side with respect to y and adding them we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

From this we conclude that the real (imaginary) part of analytic functions are harmonic.

Definition 1.3: Let u be harmonic in D where $u: D \rightarrow R$ then $v: D \rightarrow R$ is a harmonic conjugate of u in D if

- v is harmonic and
- $u + iv$ is differentiable in D in the complex sense.

Note: - There is an intimate connection between harmonic functions and analytic functions as shown by the following theorem.

Theorem 1.1: A necessary and sufficient condition for a function

$f(z) = u(x, y) + iv(x, y)$ to be analytic on a domain G is that, it's real part $u(x, y)$ and imaginary part $v(x, y)$ be conjugate harmonic functions on G .

Proof:

To prove the necessity, Suppose $f(z)$ is analytic on G

This implies f is differentiable at every point of G . Also the functions, $u(x, y)$ and $v(x, y)$ are differentiable and satisfy the Cauchy Riemann equations

That is,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \dots \dots \dots (4) \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \dots \dots \dots (5)$$

Differentiating (4) with respect to x and (5) with respect to y , we have

$$\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial y} \right) \Leftrightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \dots \dots \dots (6)$$

$$\text{and } \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial y} \left(-\frac{\partial v}{\partial x} \right) \Leftrightarrow \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \dots \dots \dots (7)$$

Since a differentiable function is continuous and for continuous functions

$$\frac{\partial}{\partial x \partial y} = \frac{\partial}{\partial y \partial x} \text{ then from (6) and (7) we have}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} + \left(-\frac{\partial^2 v}{\partial y \partial x} \right) = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0$$

Hence, u is harmonic

Similarly, differentiating (4) with respect to y and (5) with respect to x we have

$$\frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial y} \left(\frac{\partial v}{\partial y} \right) \Leftrightarrow \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \dots \dots \dots (8)$$

$$\text{and } \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial x} \left(-\frac{\partial v}{\partial x} \right) \Leftrightarrow \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2} \dots \dots \dots (9)$$

Since a differentiable function is continuous and for continuous functions

$$\frac{\partial}{\partial x \partial y} = \frac{\partial}{\partial y \partial x} \text{ then from (8) and (9) we have}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x} + \left(\frac{-\partial^2 v}{\partial x \partial y} \right) = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0$$

Hence, v is harmonic

Conversely, suppose that u and v are harmonic and satisfy Cauchy Riemann equations

Claim $f'(z_0)$ exists

For this, u and v are harmonic implies that their partial derivatives through second order continuous (particularly their first order partial derivatives continuous)

From the derivatives of functions of two variables

$$\Delta u = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) = \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

Where $\partial u / \partial x$ and $\partial u / \partial y$ are the values of the partial derivatives at the point (x_0, y_0) and ε_1 and ε_2 approach zero as both Δx and Δy approach zero.

That is,

$$\varepsilon_1 \xrightarrow{\Delta x \rightarrow 0} 0 \text{ and } \varepsilon_2 \xrightarrow{\Delta y \rightarrow 0} 0$$

Similarly,

$$\Delta v = v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0) = \frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y + \varepsilon_3 \Delta x + \varepsilon_4 \Delta y$$

Where $\partial v / \partial x$ and $\partial v / \partial y$ are the values of the partial derivatives at the point (x_0, y_0) and ε_3 and ε_4 approach zero as both Δx and Δy approach zero.

That is,

$$\varepsilon_3 \xrightarrow{\Delta x \rightarrow 0} 0 \text{ and } \varepsilon_4 \xrightarrow{\Delta y \rightarrow 0} 0$$

Then

$$\Delta f = f(z_0 + \Delta z) - f(z_0) = \Delta u + i\Delta v$$

$$\Delta f = \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y + i \left(\frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y + \varepsilon_3 \Delta x + \varepsilon_4 \Delta y \right) \dots \dots \dots (10)$$

From Cauchy Riemann equations, replace $\partial v/\partial y$ by $\partial u/\partial x$ and $\partial u/\partial y$ by $-\partial v/\partial x$ in equation (10)

$$\begin{aligned} \Delta f &= \frac{\partial u}{\partial x} \Delta x - \frac{\partial v}{\partial x} \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y + i \left(\frac{\partial v}{\partial x} \Delta x + \frac{\partial u}{\partial x} \Delta y + \varepsilon_3 \Delta x + \varepsilon_4 \Delta y \right) \\ &= \frac{\partial u}{\partial x} (\Delta x + i\Delta y) + \frac{\partial v}{\partial x} (\Delta x + i\Delta y) + \delta_1 \Delta x + \delta_2 \Delta y \end{aligned}$$

where $\delta_1 \rightarrow 0, \delta_2 \rightarrow 0$ as $\varepsilon_1 + \varepsilon_2 \rightarrow 0$ and $\varepsilon_3 + \varepsilon_4 \rightarrow 0$

Then

$$\frac{\Delta f}{\Delta z} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} + \delta_1 \frac{\Delta x}{\Delta z} + \delta_2 \frac{\Delta y}{\Delta z}$$

Since $|\Delta x| \leq |\Delta z|$ and $|\Delta y| \leq |\Delta z|$

then $\left| \frac{\Delta x}{\Delta z} \right| \leq 1$ and $\left| \frac{\Delta y}{\Delta z} \right| \leq 1$ which is bounded

So that

$$\delta_1 \frac{\Delta x}{\Delta z} \xrightarrow{\Delta z \rightarrow 0} 0 \text{ and } \delta_2 \frac{\Delta y}{\Delta z} \xrightarrow{\Delta z \rightarrow 0} 0 \text{ since } \delta_1 \rightarrow 0 \text{ and } \delta_2 \rightarrow 0$$

Then,

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{\Delta f}{\Delta z} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

This shows that $f'(z_0)$ exists

As z_0 is an arbitrary point, f' is analytic. ■.

Theorem 1.2: Let G be either the whole plane \mathbb{C} or some open disk. If $u: G \rightarrow \mathbb{R}$ is a harmonic function, then u has a harmonic conjugate.

Proof:

Let $G = B(0, R), 0 < R < \infty$ and $u: G \rightarrow \mathbb{R}$ be harmonic function

Claim to find a harmonic function v such that u and v satisfy the Cauchy Riemann equation

So define

$$v(x, y) = \int_0^y u_x(x, t) dt + \varphi(x)$$

And determine φ so that $v_x = -u_y$

$$\begin{aligned} v_x(x, y) &= \int_0^y u_{xx}(x, t) dt + \varphi'(x) \\ &= \int_0^y -u_{yy}(x, t) dt + \varphi'(x) \quad (\text{from } u_{xx} + u_{yy} = 0) \\ &= -u_y(x, y) + u_y(x, 0) + \varphi'(x) \end{aligned}$$

This shows that $\varphi'(x) = -u_y(x, 0)$

Then $\varphi(x) = -\int_0^x u_y(s, 0) ds$

Therefore $v(x, y) = \int_0^y u_x(x, t) dt - \int_0^x u_y(s, 0) ds$ ■.

Example: Let $u(x, y) = e^x \cos y$ then find the harmonic conjugate of u and the analytic function f .

Solution:

Let v be the harmonic conjugate of u . Then, their first order partial derivative satisfies the Cauchy Riemann equations.

That is,

$$u_x = v_y \dots \dots (i) \text{ and } u_y = -v_x \dots \dots (ii)$$

From (i) $u_x = e^x \cos y = v_y$

$$\Rightarrow v_y = e^x \cos y \text{ then } v = \int e^x \cos y dy$$

$$v = e^x \sin y + \varphi(x)$$

Where φ is the constant of the integral with respect to x

From (ii) $u_y = -v_x$

$$-e^x \sin y = -(e^x \sin y + \varphi'(x))$$

$$-e^x \sin y = -e^x \sin y - \varphi'(x)$$

$$\Rightarrow \varphi'(x) = 0 \text{ then } \varphi(x) = c \text{ where } c \text{ is a constant}$$

$$\therefore v(x, y) = e^x \sin y + c \text{ and } f(z) = e^x \cos y + ie^x \sin y$$

Proposition 1.3: Two harmonic conjugates of a harmonic function differ by a constant.

Proof:

Let u be a harmonic function and v_1 and v_2 are harmonic conjugates of u . Then

$u + iv_1$ and $u + iv_2$ are analytic function by theorem 1.1 where, u , v_1 and v_2 are real.

But the sum and difference of analytic functions are analytic

This implies,

$(u + iv_1) - (u + iv_2) = (u - u) + i(v_1 - v_2) = 0 + i(v_1 - v_2) = i(v_1 - v_2)$ This is analytic with range along the imaginary axis.

Then, the Cauchy Riemann equation satisfied

That is,

$$(v_1 - v_2)_x = 0 \Rightarrow v_1 - v_2 = \varphi(x) \text{ (}\varphi \text{ is constant with respect to } y \text{)}$$

$$(v_1 - v_2)_y = \varphi'(y) \Rightarrow 0 = \varphi'(y)$$

$$\varphi(y) = c \Rightarrow v_1 - v_2 = c$$

$$\therefore v_1 = v_2 + c \quad \blacksquare$$

Proposition 1.4: Let $u(x, y)$ be harmonic on D which is an interior of a circle for a and b are constants. Let the translation $x' = x + a, y' = y + b$ map D in to D' and let $u(x, y)$ transformed in to $u(x', y')$. Then, $u(x', y')$ is harmonic on D' (Invariant under translations)

Proof:

Claim To show $u_{x'x'} + u_{y'y'} = 0$

For this

$$\begin{aligned} \frac{\partial u}{\partial x'} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial x'} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial x'} \quad (\text{chain rule}) \\ &= \frac{\partial u}{\partial x} \quad \text{since} \left(\frac{\partial x}{\partial x'} = 1, \frac{\partial y}{\partial x'} = 0 \right) \end{aligned}$$

This implies

$$u_{x'} = u_x \text{ and } u_{x'x'} = u_{xx}$$

Similarly,

$$\frac{\partial u}{\partial y'} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial y'} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial y'} = \frac{\partial u}{\partial y} \quad \text{since} \left(\frac{\partial x}{\partial y'} = 0, \frac{\partial y}{\partial y'} = 1 \right)$$

That is,

$$u_{y'} = u_y \text{ and } u_{y'y'} = u_{yy}$$

Since $u(x, y)$ is harmonic, we have

$$u_{xx} + u_{yy} = 0 \Leftrightarrow u_{x'x'} + u_{y'y'} = 0$$

Hence $u(x', y')$ is harmonic ■

Theorem 1.5: Let f and g have continuous first partial derivatives on an open unit disc D with center $z = a + ib$. Let $\partial f(z)/\partial y = \partial g(z)/\partial x$ for all $z \in D$. Then there is a function h which has continuous second order partial derivatives on D such that $\partial h(z)/\partial y = g(z)$ and $\partial h(z)/\partial x = f(z)$ for all $z \in D$.

Proof:

For each $z = x + iy$ on D

$$\text{Set } h(x, y) = \int_a^x f(t, b)dt + \int_b^y g(x, s)ds$$

By fundamental theorem of calculus we have that

$$\frac{\partial h(z)}{\partial y} = g(x, y) \text{ and } f(x, b) = \frac{\partial}{\partial x} \int_a^x f(t, b)dt \dots \dots \dots (1)$$

More over since g has continuous first partial derivatives on an open unit disc D , differentiation under integral sign is possible and hence

$$\frac{\partial}{\partial x} \int_b^y g(x, s)ds = \int_b^y \frac{\partial}{\partial x} g(x, s)ds = \int_b^y \frac{\partial}{\partial y} f(x, s)ds = f(x, y) - f(x, b) \dots \dots \dots (2)$$

Now,

$$\frac{\partial h(x, y)}{\partial x} = \frac{\partial}{\partial x} \int_a^x f(t, b)dt + \frac{\partial}{\partial x} \int_b^y g(x, s)ds$$

From equation (1) and (2) we have that

$$\frac{\partial h(x, y)}{\partial x} = f(x, b) + f(x, y) - f(x, b) = f(x, y)$$

Since

$$\frac{\partial h(x, y)}{\partial y} = g(z) \text{ and } \frac{\partial h(x, y)}{\partial x} = f(z) \forall z \in D$$

h has continuous second order partial derivatives on D and h is the required function. ■.

Definition 1.4: A simply connected domain D is a domain having the following property. If γ is any simple closed contour lying in D , then the domain interior to γ lies wholly in D .

Note that if f is analytic in a simply connected domain D and γ is any closed contour in D , then $\int_{\gamma} f(z)dz = 0$ (Cauchy theorem). Given f analytic inside and on the simple closed contour, we know from Cauchy theorem that $\int_{\gamma} f(z)dz = 0$. However, if we consider the integral $\int_{\gamma} \frac{f(z)dz}{z-z_0}$, where z_0 is a point in the interior of γ , then there is no reason to expect that this integral is zero, because the integrand has singularity inside the contour γ .

Theorem 1.6 (Cauchy's integral formula): Let γ be a simply closed positively oriented contour. If f is analytic in a simply connected domain D containing γ and z_0 is any point inside γ , then

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)dz}{z-z_0}$$

1.2. PROPERTIES OF HARMONIC FUNCTIONS

1.2.1. MEAN VALUE PROPERTY (MVP)

Mean value property is one of the properties of harmonic function which is analogous to the Cauchy integral formula. It is used to find the value of a harmonic function at the center of the disc from its value at the boundary as well as on the surface of the disc.

Definition 1.5: Let G be an open subset of \mathbb{C} . A continuous function $u: G \rightarrow \mathbb{R}$ has the mean value property (MVP) if whenever a closed ball with center a radius r , $B(a, r) \subset G$

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) d\theta$$

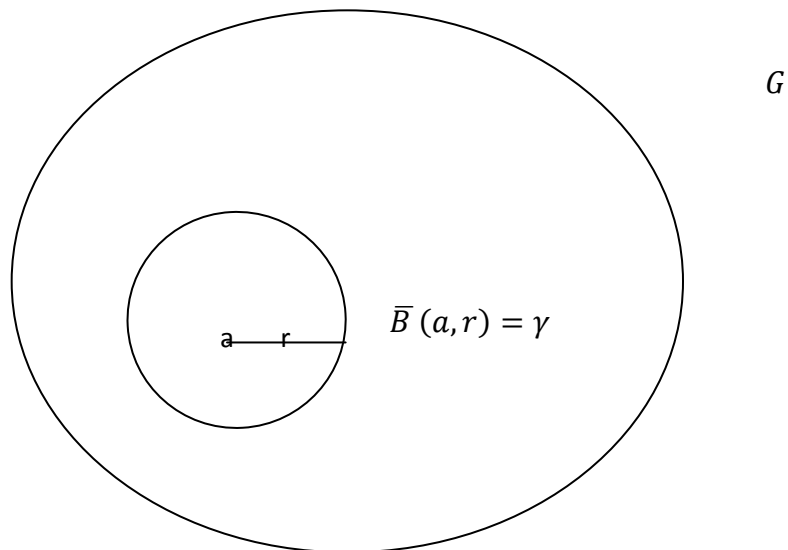
Theorem 1.7 (Mean value theorem): Let $u: G \rightarrow \mathbb{R}$ be a harmonic function and let $\bar{B}(a, r)$ be a closed disc contained in G . If γ is the circle $|z - a| = r$ then

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) d\theta$$

Proof:

We apply a Cauchy integral formula (if $f(z)$ is analytic with in or on a simple closed curve γ and a is any interior point of γ , then $f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)dz}{z-a}$)

Let D be a disc such that $\bar{B}(a, r) \subset D \subset G$ and let f be an analytic function on D such that $f = u + iv$ (u and v are real)



Then, from Cauchy integral formula

$$f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)dz}{z - a}$$

Where $c: |z - a| = r \Rightarrow z = a + re^{i\theta}$ and $dz = ire^{i\theta} d\theta$

$$f(a) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a + re^{i\theta})}{a + re^{i\theta} - a} ire^{i\theta} d\theta$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a + re^{i\theta})}{re^{i\theta}} ire^{i\theta} d\theta$$

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) d\theta$$

$$u(a) + iv(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) + iv(a + re^{i\theta}) d\theta \text{ where } f(a) = u(a) + iv(a)$$

$$u(a) + iv(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) d\theta + \frac{i}{2\pi} \int_0^{2\pi} v(a + re^{i\theta}) d\theta$$

Now comparing the real and imaginary parts of both sides, we get

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) d\theta \quad \blacksquare.$$

1.2.2. THE MAXIMUM PRINCIPLE FOR HARMONIC FUNCTION

Definition 1.6: A real valued function $f(z)$ defined in a region G of open complex plane is said to satisfy the maximum condition in G if it is either reduces to a constant in G or else does not attain its “upper bound” at any point of G .

Theorem 1.8: A function $u(z)$ harmonic in G satisfies maximum principle in G .

Proof:

Let M be the maximum of u in \bar{G} , the maximum is at some point of \bar{G} since \bar{G} is compact and u is continuous in G

Let E_1 and E_2 be the sets of points defined by

$$E_1 = \{z \in G | u(z) < M\} \text{ and } E_2 = \{z \in G | u(z) = M\}$$

Clearly $G = E_1 \cup E_2$ and $\emptyset = E_1 \cap E_2$

Also since u is continuous, E_1 is open. Now let's show that E_2 is open.

Let $a \in E_2$ and suppose that $D(a, r) \subseteq G$. Then $u(a) = M$

Then we need to show that $D(a, r) \subseteq E_2$ for some $r > 0$

Now from the Mean value theorem

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) d\theta$$

Then

$$0 = \frac{1}{2\pi} \int_0^{2\pi} \{u(a) - u(a + re^{i\theta})\} d\theta \text{ for } 0 < r < R \dots \dots \dots (1)$$

Since the integrand is non-negative and continuous, the integral equation (1) can be zero only if the integrand is zero.

Thus

$$u(a) = u(a + re^{i\theta}) = M \text{ for } 0 < r < \rho \text{ and } 0 \leq \theta \leq 2\pi$$

Since $u(a + re^{i\theta}) = M$ for $z \in D(a; r)$. That is, $u(z) = M, E_2$ is open

Now since G is a region (connected) and both E_1 and E_2 are open, either E_1 or E_2 empty and the other coincides with all of G . That is, either $u(z) < M \forall z \in G$ or $u(z) = M$ for all $z \in G$ and this proves the theorem. ■.

Corollary 1.9: Let $G \subseteq \mathbb{C}$ be a bounded domain and let $u: \bar{G} \rightarrow \mathbb{R}$ is harmonic function on G . Then $\max_{\bar{G}} u = \max_{\partial G} u$

Proof:

Since u is harmonic on \bar{G} , a compact set, there is a point $z \in \bar{G}$ at which $u(z)$ has maximum point. If $z \in G$, then by the above theorem (the maximum principle), u is constant and hence $\max_{\bar{G}} u = \max_{\partial G} u$. If $z \in \partial G$, then also $\max_{\bar{G}} u = \max_{\partial G} u$ ■.

Corollary 1.10: Let $G \subseteq \mathbb{C}$ be bounded domain. Let $u_1, u_2: \bar{G} \rightarrow \mathbb{R}$ be harmonic functions and $u_1 = u_2$ on ∂G , then $u_1 \equiv u_2$ on G .

Proof:

Let $V = u_1 - u_2$ on G . Since u_1 and u_2 are harmonic functions on G and $V = 0$ on the boundary of G (∂G), then by above corollary $\max_{\bar{G}} V = \max_{\partial G} V = 0$ and since the value of V on ∂G is identically zero, by maximum principle $V = 0$ on G then $u_1 \equiv u_2$ on G . ■.

1.3. THE ROLE OF CONFORMAL MAPPING

The main object of study in this topic are analytic functions $f: U \rightarrow V$, with U and V is open in \mathbb{C} , that are one to one and onto. Such analytic function is called a conformal mapping. A conformal map $f: U \rightarrow V$ from one open set to another can be used to transfer analytic function on U to V and vice versa: that is, $h: v \rightarrow \mathbb{C}$ is analytic if and only if $h \circ f$ is analytic on U ; and $g: U \rightarrow \mathbb{C}$ is analytic if and only if $g \circ f^{-1}$ is analytic on V . Thus, if there is a conformal mapping from U to V , then U and V are essentially indistinguishable from the view point of complex function theory. On a practical level, one can often study analytic function on a rather complicated region open set by first mapping that open set to some simpler open set, then transforming the analytic function as indicated. The philosophy behind the conformal map under lies the use of (invertible) conformal maps in problem involving harmonic functions. Suppose that we have a region G and a conformal map g of G on to a simpler region, say $D(0,1)$. Significantly, g and g^{-1} goes beyond pure geometry. Because the maps are conformal, it turns out that they transfer harmonicity backwards and forwards too. Hence a boundary value problem for G is converted to an equivalent boundary value problem for $D(0,1)$.

Definition 1.7: A bijective analytic function $f: U \rightarrow V$ is called a conformal map. Given such a mapping f , we say that U and V are conformally equivalent.

Remark: If $f: U \rightarrow V$ is analytic and injective, then $f'(z) \neq 0$ for all $z \in U$. In particular the inverse of f defined on its range is analytic, and thus the inverse of a conformal map is also analytic.

A remarkable fact, which at first seems surprising, is that the unbounded upper half plane H is conformally equivalent to the unit disc D . Indeed, let $F(z) = (i - z)/(i + z)$ and $G(z) = i(1 - w)/(1 + w)$ then the map $F: H \rightarrow D$ is a conformal map with inverse $G: D \rightarrow H$. An interesting aspect about these functions is their behavior on the boundary of our open set. Observe that F is analytic on \mathbb{C} except at $z = -i$, and in particular it is continuous everywhere on the boundary of H , namely in the real line. If we take $z = x$ real, then the distance from x to i is the same as the distance from x to $-i$. Therefore $|F(x)| = \left| \frac{i-x}{i+x} \right| = 1$. Thus F maps \mathbb{R} on to the boundary of the unit circle D .

We get more information if we write $F(x) = \frac{i-x}{i+x} = \frac{1-x^2}{1+x^2} + i \frac{2x}{1+x^2}$ and parametrizing the real line by $x = \tan t$ with $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Since $\sin 2t = \frac{2 \tan t}{1 + \tan^2 t}$ and $\cos 2t = \frac{1 - \tan^2 t}{1 + \tan^2 t}$, we have

$$\begin{aligned} F(x) &= \frac{1-x^2}{1+x^2} + i \frac{2x}{1+x^2} = \frac{1-\tan^2 t}{1+\tan^2 t} + i \frac{2 \tan t}{1+\tan^2 t} \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ &= \cos 2t + i \sin 2t = e^{i2t} \end{aligned}$$

Hence the image of the real line is the arc containing of the circle omitting the point -1 . Moreover as x travels from $-\infty$ to ∞ , $F(x)$ travels along that arc of the circle starting at -1 and first going through the part of the circle that lies on the lower half plane. The point -1 on the circle corresponds to the point of infinity of the upper half plane.

Definition 1.8: A Möbius (Moebius) transformation (same times known as a fractional linear transformation or bilinear transformation) is any function of the form $w = f(z) = \frac{az+b}{cz+d}$ with the restriction that $ad \neq bc$ (so that w is not a constant function)

Notice that since $f'(z) = \frac{ad-bc}{(cz+d)^2}$ does not vanish, the Möbius transformation $f(z)$ is conformal at every point except at its pole $z = \frac{-d}{c}$.

Analytic mapping of one disc to another:

Here we consider the special map of unit disc $D(0,1)$ in to itself. The following lemma supplies us the important class of examples.

Lemma 1.11: For $a \in \mathbb{C}, |a| < 1$ we define $\phi_a(z) = \frac{z-a}{1-\bar{a}z}$. Then each ϕ_a is a conformal self map of the unit disc.

Proof:

Since ϕ_a is fractional linear transformation, ϕ_a is conformal mapping. ϕ_a is one to one continuous map and its inverse is ϕ_{-a} . That is $\phi_{-a} = \frac{z+a}{1+\bar{a}z}$. Also ϕ_a is analytic $\mathbb{C} \setminus \left\{ \frac{1}{\bar{a}} \right\}$. Thus ϕ_a is analytic on $\bar{D}(0,1)$.

Now to see ϕ_a map $|z| = 1$ to itself, we compute, for $|z| = 1$,

$$\left| \frac{z-a}{1-\bar{a}z} \right| = \left| \frac{z-a}{\bar{z}(1-\bar{a}z)} \right| = \left| \frac{z-a}{\bar{z}-\bar{a}} \right| = 1$$

Thus by maximum principle ϕ_a maps, $D(0,1)$ in to $D(0,1)$

Since $\phi_a^{-1} = \phi_{-a}$, $\phi_{-a}(\phi_a(z)) = z$ and $|a| < 1$ iff $|-a| < 1$, it follows that ϕ_a maps D on to D and map ∂D on to ∂D ■.

In general for $a \in \mathbb{C}, |a| < 1$. Then the analytic function $\phi_a(z) = \frac{z-a}{1-\bar{a}z}$ has the following properties

- i) ϕ_a is analytic and invertible on a neighborhood of $\bar{D}(0,1)$
- ii) $\phi_a: D(0,1) \rightarrow D(0,1)$ is one to one and onto
- iii) $\phi_a^{-1} = \phi_{-a}$
- iv) $\phi_a(a) = 0$

Theorem 1.12: A harmonic function is transformed in to a harmonic function under a transformation $w = f(z)$ which is analytic. That is,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = |f'(z)|^2 \left(\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2} \right), \text{ where } w = f(z) \text{ is analytic and } f'(z) \neq 0.$$

Proof:

Let $\phi(x, y)$ be harmonic function in G and suppose that G is mapped in to G' of the $w -$ plane. By the conformal mapping $w = f(z)$ so that $x = x(u, v), y = y(u, v)$.

Then the function $\phi(x, y)$ transformed in to a function $\phi[x(u, v), y(u, v)]$ by the conformal map. By differentiation we have

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= \frac{\partial \phi}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial y} \\ \frac{\partial^2 \phi}{\partial x^2} &= \frac{\partial \phi}{\partial u} \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial u} \right) + \frac{\partial \phi}{\partial v} \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial v} \right) \\ &= \frac{\partial \phi}{\partial u} \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \left[\frac{\partial}{\partial u} \left(\frac{\partial \phi}{\partial u} \right) \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial \phi}{\partial u} \right) \frac{\partial v}{\partial x} \right] + \frac{\partial \phi}{\partial v} \frac{\partial^2 v}{\partial x^2} \\ &\quad + \frac{\partial v}{\partial x} \left[\frac{\partial}{\partial u} \left(\frac{\partial \phi}{\partial v} \right) \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial \phi}{\partial v} \right) \frac{\partial v}{\partial x} \right] \\ &= \frac{\partial \phi}{\partial u} \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \left[\frac{\partial^2 \phi}{\partial u^2} \frac{\partial u}{\partial x} + \frac{\partial^2 \phi}{\partial v \partial u} \frac{\partial v}{\partial x} \right] + \frac{\partial \phi}{\partial v} \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} \left[\frac{\partial^2 \phi}{\partial u \partial v} \frac{\partial u}{\partial x} + \frac{\partial^2 \phi}{\partial v^2} \frac{\partial v}{\partial x} \right] \dots \dots \dots (1) \end{aligned}$$

Similarly

$$\begin{aligned} \frac{\partial^2 \phi}{\partial y^2} &= \frac{\partial \phi}{\partial u} \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} \left[\frac{\partial^2 \phi}{\partial u^2} \frac{\partial u}{\partial y} + \frac{\partial^2 \phi}{\partial v \partial u} \frac{\partial v}{\partial y} \right] + \frac{\partial \phi}{\partial v} \frac{\partial^2 v}{\partial y^2} \\ &+ \frac{\partial v}{\partial y} \left[\frac{\partial^2 \phi}{\partial u \partial v} \frac{\partial u}{\partial y} + \frac{\partial^2 \phi}{\partial v^2} \frac{\partial v}{\partial y} \right] \dots \dots \dots (2) \end{aligned}$$

Adding equation (1) and (2)

$$\begin{aligned} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} &= \frac{\partial \phi}{\partial u} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \frac{\partial \phi}{\partial v} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \frac{\partial^2 \phi}{\partial u^2} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] \\ &+ 2 \frac{\partial^2 \phi}{\partial u \partial v} \left[\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right] + \frac{\partial^2 \phi}{\partial v^2} \left[\left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] \dots \dots \dots (3) \end{aligned}$$

Since u and v are harmonic,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ and } \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Also by Cauchy Riemann equation,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Then

$$\begin{aligned} \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 &= \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 = \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 = \left| \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial y} \right|^2 \\ &= |f'(z)|^2 \text{ and } \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} = 0 \end{aligned}$$

Hence equation (3) becomes

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = |f'(z)|^2 \left(\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2} \right)$$

From this harmonic function $\phi(x, y)$ remains harmonic under this map,

Since

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \text{ and } f'(z) \neq 0 \text{ then } \frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2} = 0 \quad \blacksquare.$$

Proposition 1.13: (Conformal invariance) Let $f: D \rightarrow R$ be an analytic function and u is harmonic on $f(D)$, then $u \circ f$ is harmonic on D .

Proof:

Let u is locally the real part of analytic function g . Since the composition of analytic function is analytic, $g \circ f$ is analytic. Since $u \circ f$ is the real part of analytic function $g \circ f$, $u \circ f$ is harmonic on D . \blacksquare .

1.4. CONVOLUTION

Definition of the Laplace transforms

Definition 1.9: Let $f(t)$ be a function on $[0, \infty)$. The Laplace transform of f is the function F defined by the integral

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt \dots \dots \dots (1)$$

The domain of $F(s)$ is all the values of s for which the integral in (1) exists. The Laplace transform of f is denoted by both F and $\mathcal{L}\{f\}$.

Example: Determine the Laplace transform of the constant function $f(t) = 1, t \geq 0$.

Solution:

Using the definition of the transform, we compute

$$\begin{aligned} F(s) &= \int_0^{\infty} e^{-st} f(t) dt = \int_0^{\infty} e^{-st} * 1 dt = \int_0^{\infty} e^{-st} dt \\ &= \lim_{N \rightarrow \infty} \int_0^N e^{-st} dt = \lim_{N \rightarrow \infty} \left[\frac{-e^{-st}}{s} \right]_{t=0}^{t=N} \end{aligned}$$

$$F(s) = \lim_{N \rightarrow \infty} \left[\frac{1}{s} - \frac{e^{-sN}}{s} \right]$$

Since $e^{-sN} \rightarrow 0$ when $s > 0$ is fixed and $N \rightarrow \infty$, we get

$$F(s) = \frac{1}{s} \text{ for } s > 0$$

When $s < 0$ the integral $\int_0^{\infty} e^{-st} dt$ diverges

Hence $F(s) = \frac{1}{s}$, with the domain of $F(s)$ being $s > 0$.

Definition 1.10: A function $f(t)$ on $[a, b]$ is said to have a jump discontinuity at $t_0 \in (a, b)$ if $f(t)$ is discontinuous at t_0 , but the one-sided limits $\lim_{t \rightarrow t_0^-} f(t)$ and $\lim_{t \rightarrow t_0^+} f(t)$ exists as finite numbers.

If the discontinuity occurs at an end point, $t_0 = a$ (or b), a jump discontinuity occurs if the one-sided limit of $f(t)$ as $t \rightarrow a^+$ ($t \rightarrow b^-$) exists as a finite number.

Definition 1.11: A function $f(t)$ is said to be piecewise continuous on a finite interval $[a, b]$ if $f(t)$ is continuous at every point in $[a, b]$, except possibly for a finite number of points at which $f(t)$ has a jump discontinuity.

A function $f(t)$ is said to be piecewise continuous on $[0, \infty)$ if $f(t)$ is piecewise continuous on $[a, N]$ for all $N > 0$.

Definition 1.12: A function $f(t)$ is said to be of exponential order α if there exists positive constant T and M such that $|f(t)| \leq Me^{\alpha t}, \forall t \geq T \dots \dots (1)$

Example: $f(t) = e^{5t} \sin 2t$ is of exponential order $\alpha = 5$ since $|e^{5t} \sin 2t| \leq e^{5t}$ and hence it holds with $M = 1$ and T any positive constant.

Theorem 1.14: (Conditions for existence of the transform) If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α . Then $\mathcal{L}\{f\}(s)$ exists for $s > \alpha$.

Proof:

We need to show that the integral $\int_0^{\infty} e^{-st} f(t) dt$ converges for $s > \alpha$. we begin by breaking up this integral in to two separate integrals.

$$\int_0^T e^{-st} f(t) dt + \int_T^{\infty} e^{-st} f(t) dt \dots \dots \dots (2)$$

Where T is chosen so that inequality (1) holds. The first integral in (2) exists because $f(t)$ and hence $e^{-st} f(t)$ are piecewise continuous on the interval $[0, T]$ for any fixed s . To see that the second integral in (2) converges, we use the comparison test for improper integrals.

Since $f(t)$ is of exponential order α , we have for $t \geq T, |f(t)| \leq M e^{\alpha t}$, and hence

$$|e^{-st} f(t)| = e^{-st} |f(t)| \leq M e^{-(s-\alpha)t}, \text{ for all } t \geq T. \text{ Now for } s > \alpha$$

$$\int_T^{\infty} M e^{-(s-\alpha)t} dt = M \int_T^{\infty} e^{-(s-\alpha)t} dt = \frac{M e^{-(s-\alpha)T}}{s - \alpha} < \infty$$

$$\text{since } |e^{-st} f(t) dt| \leq M e^{-(s-\alpha)t} \text{ for } t \geq T \text{ and}$$

The improper integral of the larger function converges for $s > \alpha$, then, by the comparison test, the integral $\int_T^{\infty} e^{-st} f(t) dt$ converges for $s > \alpha$.

Finally, because the two integrals in (2) exist, the Laplace transform $\mathcal{L}\{f\}(s)$ exists for $s > \alpha$. ■.

Theorem 1.15: (Translation) If the Laplace transforms $\mathcal{L}\{f\}(s) = F(s)$ exists for $s > \alpha$. then $\mathcal{L}\{e^{at} f(t)\}(s) = F(s - a)$ for $s > \alpha + a$.

Proof:

We simply compute

$$\mathcal{L}\{e^{at} f(t)\}(s) = \int_0^{\infty} e^{-st} e^{at} f(t) dt = \int_0^{\infty} e^{-(s-a)t} f(t) dt = F(s - a) \quad \blacksquare.$$

Definition 1.13: The unit step function $u(t)$ is defined by

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & 0 < t \end{cases}$$

By shifting the argument of $u(t)$, the jump can be moved to a different location. That is,

$$u(t - a) = \begin{cases} 0, & t - a < 0 \\ 1, & 0 < t - a \end{cases} = \begin{cases} 0, & t < a \\ 1, & a < t \end{cases}$$

has its jump at $t = a$.

Theorem 1.16: (Translation) Let $\mathcal{L}\{f\}(s) = F(s)$ exists for $s > \alpha \geq 0$. If a is a positive constant, then

$$\mathcal{L}\{f(t - a)u(t - a)\}(s) = e^{-as}F(s)$$

Proof:

By the definition of the Laplace transform, we have

$$\mathcal{L}\{f(t - a)u(t - a)\}(s) = \int_0^{\infty} e^{-st} f(t - a) u(t - a) dt = \int_a^{\infty} e^{-st} f(t - a) dt \dots (1)$$

Where, in the last equation, we used the fact that $u(t - a)$ is zero for $t < a$ and equals 1 for $a < t$.

Now let $v = t - a$. Then we have $dv = dt$ and equation (1) becomes

$$\mathcal{L}\{f(t - a)u(t - a)\}(s) = \int_0^{\infty} e^{-as} e^{-sv} f(v) dv = e^{-as} \int_0^{\infty} e^{-sv} f(v) dv = e^{-as} F(s) \blacksquare.$$

Definition 1.14: (Convolution) Let $f(t)$ and $g(t)$ be piecewise continuous on $[0, \infty)$.

The convolution of $f(t)$ and $g(t)$, denoted $f * g$, is defined by

$$(f * g)(t) = \int_0^t f(t - v)g(v)dv$$

Example: Find the convolution of t and t^2

Solution:

Let $f(t) = t$ and $g(t) = t^2$

$$(f * g)(t) = \int_0^t f(t-v)g(v)dv$$

$$t * t^2 = \int_0^t (t-v)v^2 dv = \int_0^t tv^2 - v^3 dv = \left(\frac{tv^3}{3} - \frac{v^4}{4}\right)_0^t = \frac{t^4}{3} - \frac{t^4}{4} = \frac{t^4}{12}$$

Note that: convolution is certainly different from ordinary multiplication. For example, $1 * 1 = t \neq 1$ and in general $1 * f \neq f$. However, convolution does satisfy some of the same properties as multiplication.

Theorem 1.17: (Properties of convolution) Let $f(t), g(t)$ and $h(t)$ be piecewise continuous on $[0, \infty)$. Then,

- i) $f * g = g * f$
- ii) $f * (g + h) = (f * g) + (f * h)$
- iii) $(f * g) * h = f * (g * h)$

Proof:

- i) To prove equation i) we begin with the definition

$$(f * g)(t) = \int_0^t f(t-v)g(v)dv$$

Now using the change of variables $w = t - v$, we have

$$(f * g)(t) = \int_t^0 f(w)g(t-w)(-dw) = \int_0^t g(t-w)f(w)dw = (g * f)(t)$$

- ii) Now from the definition we know that

$$\begin{aligned} (f * (g + h))(t) &= \int_0^t f(t-v)((g + h)v)dv \\ &= \int_0^t f(t-v)(g(v) + h(v))dv \end{aligned}$$

$$\begin{aligned}
 (f * (g + h))(t) &= \int_0^t f(t - v)g(v) + f(t - v)h(v)dv \\
 &= \int_0^t f(t - v)g(v)dv + \int_0^t f(t - v)h(v)dv \\
 &= (f * g)(t) + (f * h)(t)
 \end{aligned}$$

iii) Simply follows the same as i) and ii) ■.

Theorem 1.18: (Convolution theorem) Let $f(t)$ and $g(t)$ be piecewise continuous on $[0, \infty)$ and of exponential order α and set

$$\mathcal{L}\{f\}(s) = F(s) \text{ and } \mathcal{L}\{g\}(s) = G(s). \text{ Then } \mathcal{L}\{f * g\}(s) = F(s)G(s) \dots \dots \dots (1)$$

Proof:

Starting with the left-hand side of (1), we use the definition of convolution to write for $s > \alpha$

$$\mathcal{L}\{f * g\}(s) = \int_0^\infty e^{-st} \left[\int_0^t f(t - v)g(v)dv \right] dt$$

To simplify the evaluation of this iterated integral, we introduce the unit step function $u(t - v)$ and write

$$\mathcal{L}\{f * g\}(s) = \int_0^\infty e^{-st} \left[\int_0^\infty u(t - v)f(t - v)g(v)dv \right] dt$$

Where we have used the fact that $u(t - v) = 0$ if $v > t$, reversing the order of integration gives

$$\mathcal{L}\{f * g\}(s) = \int_0^\infty g(v) \left[\int_0^\infty e^{-st} u(t - v)f(t - v)dt \right] dv \dots \dots \dots (2)$$

Now recall from the translation property that the integral in brackets in equation (2) equals $e^{-sv}F(s)$

Hence $\mathcal{L}\{f * g\}(s) = \int_0^\infty g(v)e^{-sv}F(s)dv = F(s) \int_0^\infty e^{-sv}g(v)dv = F(s)G(s)$ ■.

Chapter Two

2. INTEGRAL REPRESENTATION OF HARMONIC FUNCTIONS

2.1 THE POISSON'S INTEGRAL ON A DISC

In this section we introduce the Poisson kernel function and we develop the Poisson integral formula which is integral representation of harmonic function on the unit disc. Then we extend the result for arbitrary disc. The Poisson integral formula will show, among other things, that every real harmonic functions is locally the real part of analytic function and it will yield information about the boundary behavior of certain classes of analytic functions in an open disc.

Definition 2.1: Poisson kernel is the function defined by

$$p_r(\theta) = p(r, \theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} \quad \text{where } 0 \leq r < 1, \theta \text{ is real}$$

The mean value property for harmonic function shows that the value of the function at the center of the circle is the average of the values on the boundary of the circle. That is,

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) d\theta.$$

This formula says that under the assumption that $u(z)$ is harmonic in D and continuous on \bar{D} , the value of the function at the center of the circle $D(0,1)$ is the average of its value on the circle. However the Poisson integral formula gives the average value for any arbitrary points inside the circle.

Here now we use Möbius transformation;

Suppose a is an arbitrary point in D ; from section 1.2.3, we have

$$\phi_a(z) = w := Tz = \frac{z-a}{1-\bar{a}z} \quad \text{where } T \text{ is Möbius transformation}$$

Then Möbius transformation T maps the unit disc conform ally on to itself and the boundary on to boundary, mapping point ' a ' to the origin (center). Therefore

$(u \circ T^{-1})(w) = u(T^{-1}(w)) = u(z)$ is harmonic on $D(0,1)$ and continuous on \bar{D} . So we apply the Mean value property. Then we obtain

$$(u \circ T^{-1})(0) = u(a) = \frac{1}{2\pi} \int_0^{2\pi} (u \circ T^{-1})(e^{i\varphi}) d\varphi = \frac{1}{2\pi} \int_0^{2\pi} u(T^{-1}e^{i\varphi}) d\varphi \dots \dots \dots (1)$$

Since T is bijection of the boundary circle to it's self (and $u(T^{-1}e^{i\varphi})$ is 2π - periodic with respect to φ), we may change the variable of integration by setting

$$e^{i\theta} = T^{-1}e^{i\varphi}. \text{ That is, } e^{i\varphi} = Te^{i\theta} = \frac{e^{i\theta} - a}{1 - \bar{a}e^{i\theta}} \dots \dots \dots (2)$$

Differentiating both sides of equation (2)

$$\frac{d}{d\theta} \left(e^{i\varphi} = \frac{e^{i\theta} - a}{1 - \bar{a}e^{i\theta}} \right)$$

$$ie^{i\varphi} \frac{d\varphi}{d\theta} = \frac{ie^{i\theta}(1 - \bar{a}e^{i\theta}) + i\bar{a}e^{i\theta}(e^{i\theta} - a)}{(1 - \bar{a}e^{i\theta})^2}$$

$$e^{i\varphi} \frac{d\varphi}{d\theta} = \frac{(1 - a\bar{a})e^{i\theta}}{(1 - \bar{a}e^{i\theta})^2}$$

$$\frac{e^{i\theta} - a}{1 - \bar{a}e^{i\theta}} d\varphi = \frac{(1 - a\bar{a})e^{i\theta}}{(1 - \bar{a}e^{i\theta})^2} d\theta$$

$$d\varphi = \frac{(1 - a\bar{a})e^{i\theta}}{(1 - \bar{a}e^{i\theta})(e^{i\theta} - a)} d\theta = \frac{1 - |a|^2}{(1 - \bar{a}e^{i\theta})(1 - ae^{-i\theta})} d\theta = \frac{1 - |a|^2}{|1 - ae^{-i\theta}|^2} d\theta$$

Now taking $a = re^{it}$ for $0 \leq r < 1$, then

$$d\varphi = \frac{1 - r^2}{|1 - re^{-i(\theta-t)}|^2} d\theta = \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - t)} d\theta \dots \dots \dots (3)$$

Hence from equation (1), (2) and (3), we get that

$$\begin{aligned} u(a) &= \frac{1}{2\pi} \int_0^{2\pi} u(T^{-1}e^{i\varphi}) \frac{1-r^2}{1+r^2-2rcos(\theta-t)} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) \frac{1-r^2}{1+r^2-2rcos(\theta-t)} d\theta \dots \dots \dots (4) \end{aligned}$$

If we make change of variable $\theta \rightarrow \theta - t$ and use the 2π -periodicity of $p_r(\theta)$, we obtain an alternative form of equation (4),

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i(\theta-t)}) \frac{1-r^2}{1+r^2-2rcos(\theta)} d\theta$$

Since

$$p_r(\theta) = \frac{1-r^2}{1+r^2-2rcos(\theta)} \text{ is a Poisson kernel, we have}$$

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i(\theta-t)}) p_r(\theta) d\theta \dots \dots \dots (5)$$

The integral given in equation (5) is called Poisson integral formula and denoted by Pu .

Lemma 2.1: If $0 \leq r < 1$ and $\theta \in R$ then

$$\begin{aligned} p_r(\theta) &= \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} = Re \left(\frac{1+re^{i\theta}}{1-re^{i\theta}} \right) = \frac{1-r^2}{1-2rcos(\theta)+r^2} \\ &= \frac{1-r^2}{(1-rcos(\theta))^2+r^2sin^2\theta} \end{aligned}$$

Proof:

We use the fact that $z + \bar{z} = 2Re(z)$ and the formula for the sum of a geometric series, then we rationalize the denominator and finally complete the square in the denominator.

$$p_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} = 1 + \sum_{n=1}^{\infty} r^n (e^{in\theta} + e^{-in\theta})$$

$$\begin{aligned}
 p_r(\theta) &= 1 + 2\operatorname{Re}\left(\sum_{n=1}^{\infty} r^n e^{in\theta}\right) \\
 &= 1 + 2\operatorname{Re}\left(\frac{re^{i\theta}}{1 - re^{i\theta}}\right) \text{ since it is a gometric series} \\
 &= \operatorname{Re}\left(1 + \frac{2re^{i\theta}}{1 - re^{i\theta}}\right) = \operatorname{Re}\left(\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right) \\
 &= \operatorname{Re}\left(\frac{1 + re^{i\theta}}{1 - re^{i\theta}} * \frac{1 - re^{-i\theta}}{1 - re^{-i\theta}}\right) \\
 &= \operatorname{Re}\left(\frac{1 + re^{i\theta} - re^{-i\theta} - r^2}{1 - re^{i\theta} - re^{-i\theta} + r^2}\right) \\
 &= \operatorname{Re}\left(\frac{1 + 2irsin(\theta) - r^2}{1 - 2rcos(\theta) + r^2}\right) \\
 &= \frac{1 - r^2}{1 - 2rcos(\theta) + r^2} = \frac{1 - r^2}{(1 - rcos(\theta))^2 + r^2sin^2\theta} \text{ (for unit disc) } \blacksquare.
 \end{aligned}$$

Definition 2.2:- Let u be a continuous function on the circle $|z - z_0| = R$ then for any $0 \leq r < R$,

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} p_r(\theta - \varphi) u(Re^{i\varphi}) d\varphi \text{ is defined in } D(0, R)$$

is called Poisson integral of u .

Now we extend the result for arbitrary disc.

Theorem 2.2: Let $f(z)$ is analytic in $D(a; R)$ and $f(z)$ is continuous on $\bar{D}(a; R)$, then the Poisson integral formula is given by,

$$Pf(a + re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} f(a + Re^{i\varphi}) \frac{R^2 - r^2}{R^2 + r^2 - 2Rrcos(\theta - \varphi)} d\varphi \dots \dots \dots (1)$$

Proof:

First we consider the center at the origin, and then we translate the center to arbitrary point a . Now equation (1) for center at the origin becomes,

$$Pf(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\varphi}) \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\theta - \varphi)} d\varphi$$

Let w be any point on the domain $\bar{D}(0; R)$ and let $a = re^{i\theta}, 0 \leq r < R$ be the interior point of $D(0; R)$. Then from Cauchy integral formula we have that;

$$f(a) = \frac{1}{2\pi i} \int_{|w|=R} \frac{f(w)}{w - a} dw \dots \dots \dots (2)$$

Let a^* be the a point outside of the domain $D(0; R)$ which is the inverse of a with respect to the circle, then the inverse of a is the point lying outside of the circle on the same ray as a and satisfying the condition $|a||a^*| = R^2, a^* = \frac{R^2}{\bar{a}}$

Applying Cauchy integral formula for a^* , we get that

$$0 = \frac{1}{2\pi i} \int_{|w|=R} \frac{f(w)}{w - a^*} dw = \frac{1}{2\pi i} \int_{|w|=R} \frac{f(w)}{w - \frac{R^2}{\bar{a}}} dw \dots \dots \dots (3)$$

Subtracting equation (2) from (3), we have that

$$\begin{aligned} f(a) &= \frac{1}{2\pi i} \int_{|w|=R} \left(\frac{f(w)}{w - a} - \frac{f(w)}{w - \frac{R^2}{\bar{a}}} \right) dw \\ f(a) &= \frac{1}{2\pi i} \int_{|w|=R} \left(\frac{1}{w - a} - \frac{1}{w - \frac{R^2}{\bar{a}}} \right) f(w) dw \\ &= \frac{1}{2\pi i} \int_{|w|=R} \left(\frac{1}{w - a} - \frac{\bar{a}}{w\bar{a} - R^2} \right) f(w) dw \\ &= \frac{1}{2\pi i} \int_{|w|=R} \frac{w\bar{a} - R^2 - w\bar{a} + a\bar{a}}{(w - a)(w\bar{a} - R^2)} f(w) dw \end{aligned}$$

since $a = re^{i\theta}$, $0 \leq r < R$, $\bar{a}a = r^2$ and $w = Re^{i\varphi}$, $dw = iRe^{i\varphi} d\varphi$

$$\begin{aligned}
 f(a) &= \frac{1}{2\pi i} \int_0^{2\pi} f(Re^{i\varphi}) iRe^{i\varphi} \frac{r^2 - R^2}{(Re^{i\varphi} - re^{i\theta})(Rre^{-i(\theta-\varphi)} - R^2)} d\varphi \\
 &= \frac{1}{2\pi i} \int_0^{2\pi} f(Re^{i\varphi}) iRe^{i\varphi} \frac{R^2 - r^2}{(Re^{i\varphi} - re^{i\theta})(R^2 - Rre^{-i(\theta-\varphi)})} d\varphi \\
 &= \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\varphi}) Re^{i\varphi} \frac{R^2 - r^2}{Re^{i\varphi}(R - re^{i(\theta-\varphi)})(R - re^{-i(\theta-\varphi)})} d\varphi \\
 &= \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\varphi}) \frac{R^2 - r^2}{(R - re^{i(\theta-\varphi)})(R - re^{-i(\theta-\varphi)})} d\varphi \\
 &= \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\varphi}) \frac{R^2 - r^2}{|R - re^{i(\theta-\varphi)}|^2} d\varphi \\
 &= \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\varphi}) \frac{R^2 - r^2}{R^2 + r^2 - 2Rrcos(\theta - \varphi)} d\varphi \\
 \therefore Pf(re^{i\theta}) &= \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\varphi}) \frac{R^2 - r^2}{R^2 + r^2 - 2Rrcos(\theta - \varphi)} d\varphi \dots \dots \dots (4)
 \end{aligned}$$

Now for arbitrary disc centered at a , equation (4) becomes,

$$Pf(a + re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} f(a + Re^{i\varphi}) \frac{R^2 - r^2}{R^2 + r^2 - 2Rrcos(\theta - \varphi)} d\varphi \dots \dots \dots (5)$$

If we take the real part of equation (5), we have that,

$$Pu(a + re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} u(a + Re^{i\varphi}) \frac{R^2 - r^2}{R^2 + r^2 - 2Rrcos(\theta - \varphi)} d\varphi$$

Note that

$$\frac{R^2 - r^2}{R^2 + r^2 - 2Rrcos(\theta - \varphi)}$$

is a Poisson kernel for arbitrary disc. ■.

Proposition 2.3: The Poisson kernel satisfies the following properties

- i) $\frac{1}{2\pi} \int_{-\pi}^{\pi} p_r(\theta) d\theta = 1$
- ii) For each fixed $\rho < 1$, the series $p_r(\theta) = \sum_{k=-\infty}^{\infty} r^{|k|} e^{ik\theta}$ where $0 \leq r < 1$, θ is real converges uniformly for $r < \rho$ and $-\pi \leq \theta \leq \pi$
- iii) $p_r(\theta) > 0, -\pi \leq \theta \leq \pi$
- iv) $p_r(\theta) = p_r(-\theta), -\pi \leq \theta \leq \pi$ and $p_r(\theta)$ is periodic in θ with 2π
- v) $p_r(\theta)$ is increasing on $-\pi \leq \theta \leq 0$ and decreasing on $0 \leq \theta \leq \pi$
- vi) For each $\delta > 0, \lim_{r \rightarrow 1} p_r(\theta) = 0$ uniformly in θ for $\delta \leq \theta \leq \pi$ and $\max_{\delta \leq |\theta| \leq \pi} \{p_r(\theta)\} \rightarrow 0$ as $r \rightarrow 1$

Proof:

- i) For a fixed value of $r, 0 \leq r < 1$, the series $\sum_{k=-\infty}^{\infty} r^{|k|} e^{ik\theta}$ converges uniformly in θ by ii) so,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} p_r(\theta) d\theta = \sum_{k=-\infty}^{\infty} r^{|k|} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} d\theta = r^0 \frac{1}{2\pi} \int_0^{2\pi} d\theta = \frac{1}{2\pi} * 2\pi = 1$$

- ii) $p_r(\theta)$ is given by $\sum_{k=-\infty}^{\infty} r^{|k|} e^{ik\theta}$.

Since $|r^{|k|} e^{ik\theta}| = r^{|k|}$ and $r < \rho, r^{|k|} \leq \rho^{|k|}$ and $\sum_{k=-\infty}^{\infty} \rho^{|k|}$ is uniformly convergent, by Weierstrass M Test $p_r(\theta) = \sum_{k=-\infty}^{\infty} r^{|k|} e^{ik\theta}$ is uniformly convergent

- iii) We have that

$$p_r(\theta) = \frac{1 - r^2}{1 - 2rcos(\theta) + r^2}$$

since $1 + r^2 - 2rcos(\theta) \geq 1 + r^2 - 2r = (1 - r)^2 > 0$ as $0 \leq r < 1$,

$$\frac{1 - r^2}{1 - 2rcos(\theta) + r^2} > 0$$

$$\therefore p_r(\theta) > 0$$

- iv) $p_r(\theta)$ is given by

$$p_r(\theta) = \frac{1 - r^2}{1 - 2rcos(\theta) + r^2}$$

Since $\cos\theta$ is even,

$$p_r(-\theta) = \frac{1-r^2}{1-2r\cos(-\theta)+r^2} = \frac{1-r^2}{1-2r\cos(\theta)+r^2} = p_r(\theta)$$

And since $\cos\theta$ is periodic in θ with 2π , $p_r(\theta)$ is periodic in θ with 2π .

v) For $\theta_1, \theta_2 \in [-\pi, 0]$ such that $\theta_1 < \theta_2$;

we have $\cos(\theta_1) < \cos(\theta_2)$ on $[-\pi, 0]$ and hence

$$1+r^2-2r\cos(\theta_1) \geq 1+r^2-2r\cos(\theta_2)$$

from this we get that $p_r(\theta_1) \leq p_r(\theta_2)$

Hence $p_r(\theta)$ is increasing on $-\pi \leq \theta \leq 0$

The other case is similar to this.

vi) From v) we have that $p_r(\theta)$ is decreasing on $[0, \pi]$. So for $0 < \delta \leq \theta \leq \pi$ we have $p_r(\theta) \leq p_r(\delta)$

$$0 < \lim_{r \rightarrow 1} p_r(\theta) \leq \lim_{r \rightarrow 1} p_r(\delta) = \lim_{r \rightarrow 1} \left(\frac{1-r^2}{1-2r\cos(\delta)+r^2} \right) = 0$$

$\therefore \lim_{r \rightarrow 1} p_r(\theta) = 0$ Uniformly in θ for $\delta \leq \theta \leq \pi$

From this we get directly $\max_{\delta \leq |\theta| \leq \pi} \{p_r(\theta)\} \rightarrow 0$ as $r \rightarrow 1$ ■.

Note that the Poisson integral also yields information about sequence of harmonic functions

Theorem 2.4: Let $\{u_n\}$ be a sequence of harmonic functions in a region G . If $u_n \rightarrow u$ uniformly on a compact subset of G , then u is harmonic in G .

Proof:

Assume that $C(z, R) \subseteq G$ and replace u by u_n in the equation of

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + Re^{i\theta}) d\theta \quad (\text{Mean value property})$$

$$u_n(z) = \frac{1}{2\pi} \int_0^{2\pi} u_n(z + Re^{i\theta}) d\theta$$

Since $\{u_n\}$ converges to u on every compact subset of G , taking the limit as $n \rightarrow \infty$ in $D(a, R)$ we have

$$\begin{aligned}
 u(z) &= \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_0^{2\pi} u_n(z + re^{i\theta}) d\theta \\
 u(z) &= \frac{1}{2\pi} \int_0^{2\pi} \lim_{n \rightarrow \infty} u_n(z + re^{i\theta}) d\theta \\
 &= \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta
 \end{aligned}$$

Thus u has the Mean value property. Since u is continuous at every point, u is harmonic by the converse of Mean value property. ■.

2.2 THE POISSON INTEGRAL FOR THE UPPER HALF-PLANE

In this section we study harmonic functions defined on the upper half plane.

$H = \{z \in \mathbb{C}: \text{Im}(z) > 0\}$. The theory of harmonic function on the upper half plane H develops by transforming the theory of harmonic function of a unit disc on to upper half-plane H by conformal mapping.

Method 1:

We seek a function Poisson kernel on upper half-plane H analogous to the Poisson kernel for the disc. Thus, for each fixed $t \in R$, we would like Poisson kernel to be a positive harmonic function on H having the appropriate approximation identity properties.

To develop the Poisson kernel for upper half- plane H , we use Möbries transformation which maps D to upper half-plane H . Now consider the linear fractional transformation $z = \frac{w-i}{w+i}$ carrying the upper half-plane $\text{Im}(w) > 0$ on to the disc $|z| < 1$ and the circle $|z| = 1$ is mapped to the real axis $\text{Im}(w) = 0$. Under this transformation, for each real θ , then there is a unique t (areal number or ∞) such that

$$e^{i\theta} = \frac{t - i}{t + i} \dots \dots \dots (1)$$

If θ is restricted to $(0, 2\pi]$, then θ is uniquely determined by w . To “lift” the Poisson integral formula from the disc to the half-plane we simply substitute $\frac{t-i}{t+i}$ for $e^{i\theta}$ and $\frac{w-i}{w+i}$ for z in the Poisson kernel of the disc.

If $z = re^{i\varphi}$ ($r < 1$) and $w = x + iy$, then we have

$$\begin{aligned} p_r(\varphi - \theta) &= \operatorname{Re} \left[\frac{e^{i\theta} + z}{e^{i\theta} - z} \right] = \operatorname{Re} \left[\frac{\frac{t-i}{t+i} + \frac{w-i}{w+i}}{\frac{t-i}{t+i} - \frac{w-i}{w+i}} \right] \\ &= \operatorname{Re} \left[\frac{(t-i)(w+i) + (t+i)(w-i)}{(t-i)(w+i) - (t+i)(w-i)} \right] \\ p_r(\varphi - \theta) &= \operatorname{Re} \left[\frac{2(wt + 1)}{2i(t-w)} \right] \end{aligned}$$

Substitute $w = x + iy$, we have that

$$p_r(\varphi - \theta) = \frac{(t^2 + 1)y}{(x-t)^2 + y^2} \dots \dots \dots (2)$$

Thinking θ as a function of t and differentiating both side of equation (1)

$$\begin{aligned} \frac{d}{dt} \left[e^{i\theta} = \frac{t-i}{t+i} \right] \\ ie^{i\theta} \frac{d\theta}{dt} = \frac{2i}{(t+i)^2} \\ i \left(\frac{t-i}{t+i} \right) \frac{d\theta}{dt} = \frac{2i}{(t+i)^2} \\ \frac{d\theta}{dt} = \frac{2}{(t+i)(t-i)} = \frac{2}{t^2 + 1} \dots \dots \dots (3) \end{aligned}$$

Now from equation (2) and (3) we have that

$$\frac{1}{2\pi} p_r(\varphi - \theta) \frac{d\theta}{dt} = \frac{1}{2\pi} \left[\frac{(t^2 + 1)y}{(x - t)^2 + y^2} \right] \left(\frac{2}{t^2 + 1} \right) = \frac{1}{\pi} \frac{y}{(x - t)^2 + y^2} \dots \dots \dots (4)$$

The right side of this equation is the Poisson kernel for the upper half-plane. The notation we use for the Poisson kernel for upper half-plane is $p_z(t)$ or $p_y(x - t)$.

Remark:

- i) For $z = x + iy \in H$ and $t \in R$, $p_z(t) = \frac{1}{\pi} \frac{y}{|z-t|^2}$
- ii) $p_z(t) = \frac{1}{\pi} \text{Im} \left(\frac{1}{t-z} \right)$
- iii) Note that the Poisson integral formula for D over to H , we see that

$$Pu(z) = \int p_z(t) u(t) dt = \int p_y(x - t) u(t) dt$$

Whenever the function $Pu(z)$ is continuous on $\bar{H} \cup \{\infty\}$ and harmonic in H . We use these to write the Poisson integral of a piecewise continuous function u on the unit circle as an integral over the real line. The function \tilde{u} defined on $\text{Im}(w) = 0$ by

$$\tilde{u}(t) = u \left(\frac{t - i}{t + i} \right)$$

is piecewise continuous on $\text{Im}(w) = 0$ and for $w = x + iy$ and $z = \frac{t-i}{t+i}$, $|z| < 1$, from equation (3) and (4) we have that,

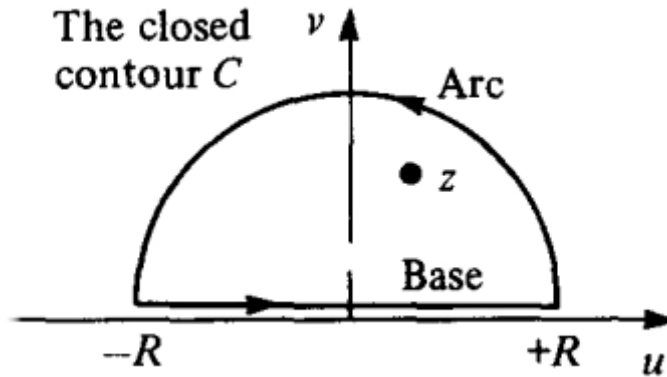
$$\begin{aligned} Pu(z) &= \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) p_r(\varphi - \theta) d\theta \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} u \left(\frac{t - i}{t + i} \right) \left[\frac{(t^2 + 1)y}{(x - t)^2 + y^2} \right] \left(\frac{2}{t^2 + 1} \right) dt \\ P\tilde{u}(w) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \tilde{u}(t) \frac{y}{(x - t)^2 + y^2} dt = \int_{-\infty}^{\infty} \tilde{u}(t) p_y(x - t) dt \\ &= (p_y * u)(t) \text{ from the definition of convolution} \end{aligned}$$

Where we have let the final integral $P\tilde{u}(w)$ is the Poisson integral for upper half-plane and denoted by $P\tilde{u}$. ■.

Method 2:

Now we use the Cauchy integral formula to get the Poisson integral formula for the upper half-plane. Suppose a function $f(w)$ is holomorphic and bounded in modulus in the upper half plane.

let $f(w) = \phi(u, v) + i\psi(u, v)$ is analytic for $v \geq 0$



Now we know that from Cauchy integral formula

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw \dots \dots \dots (1)$$

And we know that \bar{z} is not in the semicircle then

$$0 = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - \bar{z}} dw \dots \dots \dots (2)$$

Now subtracting (2) from (1)

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw - \frac{1}{2\pi i} \int_C \frac{f(w)}{w - \bar{z}} dw \\ &= \frac{1}{2\pi i} \int_C \left(\frac{f(w)}{w - z} - \frac{f(w)}{w - \bar{z}} \right) dw \\ f(z) &= \frac{1}{2\pi i} \int_C f(w) \left(\frac{1}{w - z} - \frac{1}{w - \bar{z}} \right) dw \end{aligned}$$

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_C f(w) \left(\frac{(w - \bar{z}) - (w - z)}{(w - z)(w - \bar{z})} \right) dw \\ &= \frac{1}{2\pi i} \int_C f(w) \left(\frac{z - \bar{z}}{(w - z)(w - \bar{z})} \right) dw \end{aligned}$$

Now let $w = u + iv$ and $z = x + iy$ then $z - \bar{z} = (x + iy) - (x - iy) = 2iy$

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_C f(w) \left(\frac{2iy}{(w - z)(w - \bar{z})} \right) dw \\ &= \frac{y}{\pi} \int_C \left(\frac{f(w)}{(w - z)(w - \bar{z})} \right) dw \\ &= \frac{y}{\pi} \int_{-R}^R \frac{f(w)}{(w - z)(w - \bar{z})} dw + \frac{y}{\pi} \int_{\gamma} \frac{f(w)}{(w - z)(w - \bar{z})} dw \end{aligned}$$

Now we have to show that

$$\frac{y}{\pi} \int_{\gamma} \frac{f(w)}{(w - z)(w - \bar{z})} dw = 0$$

Since f is analytic and bounded $|f(z)| \leq M$, i. e, M is an upper bound of $|f(z)|$ on C .

$$\begin{aligned} \text{Now } \left| \int_{\gamma} \frac{yf(w)}{(w - z)(w - \bar{z})} dw \right| &\leq \int_{\gamma} \left| \frac{yf(w)}{(w - z)(w - \bar{z})} \right| dw \\ &\leq \int_{\gamma} \frac{|y||f(w)|}{|w - z||w - \bar{z}|} dw \\ &\leq \int_{\gamma} \frac{|y|M}{|w - z||w - \bar{z}|} dw \end{aligned}$$

but we know that $|w - z| \geq |w| - |z|$ and $|w - \bar{z}| \geq |w| - |\bar{z}|$

$$\begin{aligned} \text{Thus, } \left| \int_{\gamma} \frac{yf(w)}{(w - z)(w - \bar{z})} dw \right| &\leq \int_{\gamma} \frac{|y|M}{(|w| - |z|)(|w| - |\bar{z}|)} dw = \int_{\gamma} \frac{|y|M}{(|w| - |z|)^2} dw \\ &\leq \frac{yM}{(R - |z|)^2} \int_0^{\pi} dt = \frac{yM\pi}{(R - |z|)^2}, \quad \text{since } |w| = R \text{ and } |z| < R, y > 0 \end{aligned}$$

$$\text{when } R \rightarrow \infty, \lim_{R \rightarrow \infty} \left| \int_{\gamma} \frac{yf(w)}{(w-z)(w-\bar{z})} dw \right| = 0, \text{ because } \lim_{R \rightarrow \infty} \frac{yM\pi}{(R-|z|)^2} = 0$$

$$\therefore \frac{y}{\pi} \int_{\gamma} \frac{f(w)}{(w-z)(w-\bar{z})} dw = 0$$

Then,

$$f(z) = \frac{y}{\pi} \int_{-R}^R \frac{f(w)}{(w-z)(w-\bar{z})} dw$$

$$\text{but } f(z) = \phi(x, y) + i\psi(x, y) \text{ and } f(w) = \phi(u, v) + i\psi(u, v)$$

$$\text{Now } \phi(x, y) = \frac{y}{\pi} \int_{-R}^R \frac{\phi(u, v)}{((u-x) + i(v-y))((u-x) + i(v+y))} du$$

Now taking $R \rightarrow \infty$ we have to get

$$\begin{aligned} \phi(x, y) &= \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\phi(u, 0)}{((u-x) + i(-y))((u-x) + i(y))} du \\ &= \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\phi(u, 0)}{((u-x) - iy)((u-x) + iy)} du \\ &= \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\phi(u, 0)}{(u-x)^2 + y^2} du \end{aligned}$$

This is the Poisson integral formula for the upper half plane. Note that the Poisson integral formula gives the value of the harmonic function $\phi(x, y)$ every where in the upper half-plane. Provided we know the values $\phi(u, 0)$ over the entire real axis. ■.

Proposition 2.5: Poisson kernel for the upper half-plane satisfies the following properties:

- a) $p_y(x-t) \geq 0, \int p_z(t) dt = 1$
- b) p_y is even, $p_y(-t) = p_y(t)$
- c) p_y is decreasing in $t > 0$

d) $p_y(x - t) \leq \frac{1}{\pi y}$ for any $\delta > 0$;

e) $\sup_{|t|>\delta} p_y(t) \rightarrow 0$ ($y \rightarrow 0$)

f) $\int_{|t|>\delta} p_y(t) dt \rightarrow 0$ ($y \rightarrow 0$) and if $a \in \partial H$ $\int_{|t-a|>\delta} p_z(t) dt \rightarrow 0$ ($z \rightarrow a$)

Proof:

The proof of the first three properties follows from the definition of Poisson kernel.

d) From $p_y(x - t) = \frac{1}{\pi} \frac{y}{(x-t)^2 + y^2}$, we have that

$$p_y(x - t) = \frac{1}{\pi y} \left[\frac{1}{\left(\frac{x-t}{y}\right)^2 + 1} \right]$$

since $\left[\frac{1}{\left(\frac{x-t}{y}\right)^2 + 1} \right] \leq 1$, $p_y(x - t) \leq \frac{1}{\pi y}$

e) Since p_y is decreasing in $t > 0$, so for $0 < \delta \leq t$, we have $p_y(t) \leq p_y(\delta)$

$$0 \leq \lim_{y \rightarrow 0} p_y(t) \leq \lim_{y \rightarrow 0} p_y(\delta) = \lim_{y \rightarrow 0} \left(\frac{1}{\pi} \frac{y}{\delta^2 + y^2} \right) = 0$$

$$\therefore \sup_{|t|>\delta} p_y(t) \rightarrow 0 \quad (y \rightarrow 0)$$

f). since $p_y \geq 0$, $0 \leq \int_{|t|>\delta} p_y(t) dt$ and from e) we have that,

$$\sup_{|t|>\delta} p_y(t) \rightarrow 0 \quad (y \rightarrow 0)$$

$$\text{Now } 0 \leq \int_{|t|>\delta} p_y(t) dt \leq \int_{|t|>\delta} \sup_{|t|>\delta} p_y(t) dt \leq 0 \quad (y \rightarrow 0).$$

$$\text{Hence } \int_{|t|>\delta} p_y(t) dt \rightarrow 0 \quad (y \rightarrow 0)$$

In general from this directly we have for every $a \in \partial H$ and for any $\delta > 0$;

$$\int_{|t-a|>\delta} p_z(t) dt \rightarrow 0 \quad (z \rightarrow a). \blacksquare.$$

An important tool for studying integrals like $\int p_z(t) f(t) dt$ is the Minkowski inequality for integrals which is stated as follows.

Theorem 2.6: (Minkowski's integral inequality) If μ and ν are δ -finite measures, $1 \leq p < \infty$ and $F(x, t)$ is $\nu \times \mu$ measurable then,

$$\left\| \int F(x, t) d\nu(x) \right\|_{L^p(\mu)} \leq \int \|F(x, t)\|_{L^p(\mu)} d\nu(x)$$

$$\left(\text{or } \left(\int_x \left| \int_t F(x, t) d\mu(t) \right|^p d\nu(x) \right)^{\frac{1}{p}} \leq \int_t \left(\int_x |F(x, t)|^p d\nu(x) \right)^{\frac{1}{p}} d\mu(t) \dots \dots \dots (1) \right)$$

Proof:

Since we know that

$$\left| \int_t F(x, t) d\mu(t) \right| \leq \int_t |F(x, t)| d\mu(t),$$

The case $p = 1$ of the theorem follows by integrating on x with respect to ν and applying the Fubini-Tonelli theorem

$$i.e., \int_x \left| \int_t F(x, t) d\mu(t) \right| \leq \int_x \int_t |F(x, t)| d\mu(t)$$

$$\left\| \int F(x, t) d\nu(x) \right\|_{L^1(\mu)} \leq \int \|F(x, t)\|_{L^1(\mu)} d\nu(x)$$

Also, since

$$\left| \int_t F(x, t) d\mu(t) \right| \leq \int_t \|F(x, t)\|_{\infty} d\mu(t),$$

The case corresponding to $p = \infty$ is obvious.

Suppose that $1 < p < \infty$, we assume that $F(x, t) \geq 0$ and that $F(x, t)$ is a simple function, so that both integrals converge. Set

$$G(t) = \left(\int F(x, t) dv(x) \right)^{p-1} \text{ but we know that } \frac{1}{p} + \frac{1}{q} = 1 \text{ then } q = \frac{p}{(p-1)}$$

$$\|G\|_{L^q(\mu)} = \left\| \int F(x, t) dv(x) \right\|_{L^p(\mu)}^{p-1}$$

And now using Fubini's theorem and Hölder's inequality

$$\begin{aligned} \left\| \int F(x, t) dv(x) \right\|_{L^p(\mu)}^p &= \int \left(\int F(x, t) dv(x) \right)^p d\mu(t) \\ &= \int \left(\int F(x, t) dv(x) \right)^{p-1} \left(\int F(x, t) dv(x) \right) d\mu(t) \\ &= \int G(t) \int F(x, t) dv(x) d\mu(t) \\ &= \int \int G(t) F(x, t) d\mu(t) dv(x) \text{ by Fubini's theorem} \\ &\leq \int \|G\|_{L^q(\mu)} \|F(x, t)\|_{L^p(\mu)} dv(x) \text{ by Hölder's inequality} \\ &= \|G\|_{L^q(\mu)} \int \|F(x, t)\|_{L^p(\mu)} dv(x) \end{aligned}$$

Now cancelling $\|G\|_{L^q(\mu)}$ from each side which gives the Minkowski inequality.

$$\therefore \left\| \int F(x, t) dv(x) \right\|_{L^p(\mu)} \leq \int \|F(x, t)\|_{L^p(\mu)} dv(x) \quad \blacksquare.$$

Using Minkowski's inequality we obtain

$$\left(\int |u(x, y)|^p dx \right)^{\frac{1}{p}} \leq \|f\|_p, \quad 1 \leq p \leq \infty \dots \dots \dots (1)$$

$$\text{if } u(x, y) = p_y * f(x) = \int p_y(x-t) f(t) dt, f \in L^p, \text{ and}$$

$$\int |u(x, y)| dx \leq \int |d\mu| \dots \dots \dots (2) \text{ if } u(x, y) = p_y * \mu = \int p_y(x - t) d\mu(t),$$

Where μ is a finite measure on R . for $p = \infty$ the analog of (1), $Sup_x |u(x, y)| \leq \|f\|_\infty$, is trivial from property (i) of $p_y(t)$.

Theorem 2.7:

a) If $1 \leq p < \infty$ and if $f(x) \in L^p$, then $\|p_y * f - f\|_p \rightarrow 0$ ($y \rightarrow 0$)

b) When $f(x) \in L^\infty$, $p_y * f$ weak-star to $f(x)$

(I.e. for all $g \in L^1$, $\int g(x) (p_y * f)(x) dx \rightarrow \int g(x) f(x) dx$ ($y \rightarrow 0$.)

c) When $f(x)$ is bounded and uniformly continuous on R , $p_y * f(x)$ converges uniformly to $f(x)$.

Proof:

a) let $f \in L^p$, $1 \leq p < \infty$ we note that

$$(p_y * f)(x) - f(x) = \int p_y(t) (f(x - t) - f(x)) dt$$

Then by Minkowski's inequality for integrals we obtain

$$\|p_y * f - f\|_p \leq \int p_y(t) \|f_t(x) - f(x)\|_p dt$$

where $f_t(x) = f(t - x)$, let $\delta > 0$ by suitably chosen, we then write

$$\|p_y * f - f\|_p \leq \int_{|t| \leq \delta} p_y(t) \|f_t - f\|_p dt + \int_{|t| > \delta} p_y(t) \|f_t - f\|_p dt$$

Since $\int p_y(t) dt = 1$, and translation are continuous in L^p , it follows that

$$\|f_t - f\|_p \rightarrow 0, t \rightarrow 0$$

So that for small δ the first integral can be made less than $\epsilon/2$. As for the second integral, we note that for fixed δ

$$\int_{|t| > \delta} p_y(t) \|f_t - f\|_p dt \leq 2\|f\|_p \int_{|t| > \delta} p_y(t) dt \rightarrow 0, (y \rightarrow 0)$$

according to property (f)

$$\therefore \|p_y * f - f\|_p \rightarrow 0 \ (y \rightarrow 0).$$

b) Assume that $g \in L^1$, we then have

$$\begin{aligned} & \left| \int g(x) (p_y * f)(x) dx - \int g(x) f(x) dx \right| \\ &= \left| \int g(x) \int p_y(t) f(x-t) dt dx - \int g(x) f(x) \int p_y(t) dt dx \right| \\ &= \left| \int \int g(x) p_y(t) f(x-t) dt dx - \int \int g(x) f(x) p_y(t) dt dx \right| \\ & \quad \text{by Fubini's theorem} \\ &= \left| \int \int (g(x) p_y(t) f(x-t)) - (g(x) f(x) p_y(t)) dt dx \right| \\ &= \left| \int \int p_y(t) (f(x-t) - f(x)) g(x) dt dx \right| \\ &\leq \int \left| \int p_y(t) (f(x-t) - f(x)) dt \right| |g(x)| dx \end{aligned}$$

But we know $\int p_y(t) (f(x-t) - f(x)) dt = \|p_y * f - f\|_1 \rightarrow 0 \ (y \rightarrow 0)$ by a) then,

$\therefore p_y * f$ Converges weak-star to $f(x)$.

c) Suppose that f is bounded and uniformly continuous on R . As in part a)

$$\|p_y * f - f\|_\infty \leq \int p_y(t) \|f_t - f\|_\infty dt$$

Again we may write the integral on the right as the sum of two integrals on $\{|t| \leq \delta\}$ and $\{|t| > \delta\}$.

$$\left(i.e., \quad \|p_y * f - f\|_\infty \leq \int_{|t| \leq \delta} p_y(t) \|f_t - f\|_\infty dt + \int_{|t| > \delta} p_y(t) \|f_t - f\|_\infty dt \right)$$

By the uniform continuity of f , the first integral can be made small for small δ .

The fact that the second integral approaches zero follows from the estimate

$$\int_{|t| > \delta} p_y(t) \|f_t - f\|_\infty dt \leq 2\|f\|_\infty \int_{|t| > \delta} p_y(t) dt \rightarrow 0, \ (y \rightarrow 0)$$

according to property (e)

$\therefore p_y * f$ Converges uniformly to $f(x)$. ■.

Corollary 2.8: Assume $f(x)$ is bounded and uniformly continuous, and let

$$u(x, y) = \begin{cases} (p_y * f)(x), & y > 0, \\ f(x), & y = 0. \end{cases}$$

Then $u(x, y)$ is harmonic on H and continuous on \bar{H} .

Proof: This corollary follows from (c).

Lemma 2.9: Assume $f(x) \in L^p, 1 \leq p \leq \infty$, and assume f is continuous at x_0 . Let

$$u(x, y) = p_y * f(x) = \int p_y(t) f(x - t) dt. \text{ Then } \lim_{(x,y) \rightarrow x_0} u(x, y) = f(x_0).$$

Proof:

We are going to show that given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|x - x_0| < \delta \implies |u(x, y) - f(x_0)| < \epsilon,$$

Indeed since f is continuous at x_0 there exists $\delta > 0$ such that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon, \text{ if } x - t \in (x_0 - \delta, x_0 + \delta) \text{ then}$$

$$|(x - t) - x_0| < \delta \text{ implies that } |f(x - t) - f(x_0)| < \epsilon, \text{ but}$$

$$|u(x, y) - f(x_0)|$$

$$= \int_{|t| \leq \delta} p_y(t) |f(x - t) - f(x_0)| dt + \int_{|t| > \delta} p_y(t) |f(x - t) - f(x_0)| dt$$

Hence for the fixed δ according to property (e)

$$\int_{|t| > \delta} p_y(t) |f(x - t) - f(x_0)| dt \rightarrow 0 \text{ as } y \rightarrow 0$$

And hence with δ small and $|x - x_0|$ small,

$$\int_{|t| \leq \delta} p_y(t) |f(x - t) - f(x_0)| dt \text{ is small}$$

$$\therefore |u(x, y) - f(x_0)| < \epsilon$$

$$\therefore \lim_{(x,y) \rightarrow x_0} u(x, y) = f(x_0). \text{ (The limit converges to } f(x_0)\text{) } \blacksquare.$$

It is important that the Poisson integrals of L^p functions and measures are characterized by the norm inequalities like (1) and (2). The proof of this in the upper half-plane requires the following lemma.

Lemma 2.10: If $u(z)$ is harmonic on H and bounded and continuous on \bar{H} then

$$u(z) = \int p_y(x - t) u(t) dt.$$

Proof:

If u is continuous at ∞ , then it is a consequence of the definition but u mayn't be continuous at ∞ .

$$\text{Let } U(z) = u(z) - \int p_y(x - t) u(t) dt$$

Claim $U(z) = 0$

Clearly $U(z)$ is harmonic on H and continuous and bounded in \bar{H} , and by the above lemma, $U(z) \equiv 0$ on R . Set

$$V(z) = \begin{cases} U(z), & y \geq 0, \\ -U(\bar{z}), & y < 0. \end{cases}, z = x + iy$$

Then V is a bounded harmonic function on the complex plane, because V has the Mean value property over small discs. Then by Liouville's theorem which says abounded analytic on the complex plane is constant, $V(z) = V(0) = 0$.

Hence $V(z) = 0$ for all z in \mathbb{C} . This implies that

$$u(z) = \int p_y(x - t) u(t) dt \blacksquare.$$

Theorem 2.11: Let $u(z)$ be a harmonic function on the upper half-plane H . Then

- a) If $1 \leq p \leq \infty$, u is the Poisson integral of a function in L^p if and only if

$$\text{Sup}_y \|u(x, y)\|_{L^p(dx)} < \infty$$

- b) $u(z)$ is the Poisson integral of a finite measure on R if and only if

$$\text{Sup}_y \int u(x, y) dx < \infty$$

- c) $u(z)$ is positive if and only if

$$u(z) = cy + \int p_y(x-t) d\mu(t), \text{ where } c \geq 0, \mu \geq 0, \text{ and } \int \frac{d\mu(t)}{1+t^2} < \infty.$$

Proof:-

- a) (\Rightarrow) let u be the Poisson integral of a function f in L^p

$$i. e \ u(z) = \int p_y(x-t) f(t) dt = p_y * f(x)$$

Then by Minikoski inequality for integrals we have

$$\left(\int |u(x, y)|^p dx \right)^{\frac{1}{p}} \leq \|f\|_p, \ 1 \leq p < \infty.$$

This implies that,

$$\|u(x, y)\|_{L^p(dx)} \leq \|f\|_p$$

Now taking the suprumum on both sides we get the requires result

$$i. e, \text{Sup}_y \|u(x, y)\|_{L^p(dx)} \leq \text{Sup}_y \|f\|_p < \infty$$

$$\therefore \text{Sup}_y \|u(x, y)\|_{L^p(dx)} < \infty$$

(\Rightarrow) Conversely

Assume $\text{Sup}_y \|u(x, y)\|_{L^p(dx)} < \infty$, we need to show that

$$u(z) = p_y * f(x) = \int p_y(x-t) f(t) dt$$

To prove this let's prove this inequality,

$$|u(z)| \leq \left(\frac{2}{\pi y} \right)^{\frac{1}{p}} \text{Sup}_{\beta > 0} \|u(x, \beta)\|_{L^p(dx)}, \ y > 0$$

Let $w = \alpha + i\beta$ consider $B(z, y)$

Then by the Mean value property (MVP)

$$u(z) = \frac{1}{2\pi} \int u(z + re^{i\theta}) d\theta.$$

Now integrating from 0 to y

$$i. e, \int_0^y u(z) r dr = \frac{1}{2\pi} \int_0^y \int_0^{2\pi} u(z + re^{i\theta}) r dr d\theta = \frac{1}{2\pi} \iint_{B(z,y)} u(w) d\alpha d\beta$$

From this we get

$$u(z) = \frac{1}{\pi y^2} \iint_{B(z,y)} u(w) d\alpha d\beta \text{ and } |u(z)| = \frac{1}{\pi y^2} \left| \iint_{B(z,y)} u(w) d\alpha d\beta \right|$$

Now applying Hölder's inequality

$$\begin{aligned} |u(z)| &\leq \left(\frac{1}{\pi y^2} \iint_{B(z,y)} |u(w)|^p d\alpha d\beta \right)^{\frac{1}{p}} \\ &\leq \left(\frac{1}{\pi y^2} \int_0^{2y} \int_{-\infty}^{\infty} |u(\alpha + i\beta)|^p d\alpha d\beta \right)^{\frac{1}{p}} \\ &\leq \left(\frac{1}{\pi y^2} \right)^{\frac{1}{p}} \text{Sup}_{\beta > 0} \left(\int |u(\alpha + i\beta)|^p d\alpha \right)^{\frac{1}{p}} < \infty \end{aligned}$$

$u(z)$ is bounded in $y > y_n > 0$. For $y > 0$ we can form a such that $y > y_n > 0$ by the above Lemma 2.10.

$$u(z + iy_n) = \int p_y(x - t) u(t + iy_n) dt, 1 < p \leq \infty$$

Let $f_n(t) = u(t + iy_n)$, $f_n(t)$ is bounded in L^p then by then by the Banach Alague theorem which says the closed unit ball of the dual of a Banach space is compact in the weak star topology, $\{f_n\}$ has a weak star accumulation point $f \in L^p$. Since Poisson kernels are in L^q , $q = \frac{p}{p-1}$

We have

$$\begin{aligned} u(z) &= \lim_n u(z + iy_n) = \lim_n \int p_y(x - t) f_n(t) dt \\ &= \int p_y(x - t) \lim_n f_n(t) dt \end{aligned}$$

$$u(z) = \int p_y(x-t) f(t) dt$$

But $f \in L^p$. Therefore u is the Poisson integral of f .

b) (\Rightarrow) u is the Poisson integral of a finite measure μ on R .

$$u(z) = \int p_y(x-t) d\mu(t) = p_y * \mu$$

Then by Minikoski integral inequality we have

$$\int |u(x+iy)| dx \leq \int |d\mu|, \mu \text{ is a finite measure}$$

Now taking both sides the suprumum we get

$$\text{Sup}_y \int |u(x+iy)| dx \leq \text{Sup}_y \int |d\mu| < \infty$$

$$\therefore \text{Sup}_y \int |u(x+iy)| dx < \infty$$

(\Leftarrow) The proof of the converse is the same as the converse of a) except the measures $u(t+iy_n)dt$ which have the bounded norms, converges weak star to finite measure on R . i. e, let $d\mu_n(t) = u(t+iy_n)dt$, μ_n has a finite measure and a bounded norms and converges to a finite measure $d\mu(t)$ in the weak star topology.

$$\begin{aligned} \text{Hence } u(z) &= \lim_n u(t+iy_n) = \int p_y(x-t) \lim_n u(t+iy_n) \\ &= \int p_y(x-t) d\mu(t) \end{aligned}$$

c) The easiest proof involves mapping H back to D , using the analoge of b) for harmonic functions on the disc and then returning to H . A harmonic function $u(z)$ on D is the Poisson integral of a finite measure ν in ∂D if and only if $\text{Sup}_r |u(re^{i\theta})| d\theta < \infty$. The measure ν is then the limit of the measures $u(re^{i\theta})/2\pi$ in the weak-star topology on measures on ∂D . If $u(z) \geq 0$, then the measures $u(re^{i\theta})d\theta$ are positive and bounded

$$\text{since } \frac{1}{2\pi} \int u(re^{i\theta}) d\theta = u(0),$$

And so the limit ν exists and ν is a positive measure. That proves the disc version of c). Now map D to H by $w \rightarrow z(w) = i(1-w)/(1+w)$. The harmonic

function u on H is positive if and only if the harmonic function $u(z(w))$, which is positive, is the Poisson integral of a positive measure ν on ∂D . Consider first the case when ν is supported on the point $w = -1$, which corresponds to $z = \infty$. Then

$$u(z(w)) = \nu(\{-1\})p_w(-1) = \nu(\{-1\})\frac{1 - |w|^2}{|1 + w|^2} = \nu(\{-1\})\text{Im } z = \nu(\{-1\})y.$$

Now assume $\nu(\{-1\}) = 0$. The map $z(w)$ moves ν on to a finite positive measure $\bar{\nu}$ on R , and for $t = z(e^{i\theta})$

$$p_w(\theta) = \pi(1 + t^2)p_z(t).$$

In this case we have

$$u(z) = \int p_y(x - t)d\mu(t) \text{ where } \mu = \pi(1 + t^2)\bar{\nu}. \blacksquare.$$

Chapter Three

3. THE DIRICHLET PROBLEM ON A DISC AND UPPER HALF PLANE

In this section we construct harmonic functions on D that behave in a prescribed manner near ∂D . The Poisson integral formula shows that if $u(z)$ is harmonic in a disc (upper half plane) and continuous on the closed disc, then its value at any interior point is completely determined by its value on the boundary circle. On the other hand, the Poisson integral is meaning full for every (bounded piecewise) continuous function $u(e^{i\theta})$ on the circle. Here we consider the Dirichlet problem for simple but important case where the domain is disc $D = \{|z - z_0| \leq \rho\}$. These facts suggests the following two questions.

- i) Given a real valued bounded piecewise continuous function $u(e^{i\theta})$ on the unit circle, do we obtain a harmonic function $\tilde{u}(z)$ through the Poisson integral

$$\tilde{u}(z) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) p_r(\theta - \varphi) d\theta, (z = re^{i\varphi})?$$

- ii) If so, do the boundary values of $\tilde{u}(z)$ agree with $u(e^{i\theta})$? The answer is affirmative and the unique solution is given by

$$\tilde{u}(z) = \begin{cases} Pf & \text{if } z \in D(0,1) \\ u(z) & \text{if } z \in \partial D(0,1) \end{cases}$$

The above two questions leads to the problem of finding a function that is harmonic in a region and pre assigned values on the boundary which is known as the Dirichlet problem.

3.1 DIRICHLET PROBLEM ON DISC AND ITS SOLUTION

The Poisson integral formula both reproduces and creates harmonic functions which are analogous to Cauchy integral formula. Cauchy integral formula not only reproduces analytic functions but it is also creates them: If u is any continuous function on $\partial D(0,1)$,

Then

$$U(z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{u(w)}{w-z} dw$$

is analytic for $z \in D(0,1)$. But of course, there is in general no direct connection between u and U ; one can't recover u as the “bounded limit” of the function U . For instance, if

$u(w) = \bar{w} = \frac{1}{w}$ on $\partial D(0,1)$, then $U \equiv 0$ on $D(0,1)$. For harmonic functions the situation is different. But, in contrast to analytic case, there is a simple connection between the continuous function u on ∂D and the created harmonic function \tilde{u} on D . The following theorem states this connection precisely.

Theorem 3.1: (Solution of the Dirichlet problem for the disc)

Let $D := \{z: |z| < 1\}$ and suppose $u: \partial D \rightarrow \mathbb{R}$ is a continuous function. Define

$$\tilde{u}(z) = \begin{cases} Pu & \text{if } z \in D(0,1) \\ u(z) & \text{if } z \in \partial D(0,1) \end{cases}$$

Then $\tilde{u}(z): \bar{D} \rightarrow \mathbb{R}$ is continuous function on $\bar{D}(0,1)$ and harmonic on $D(0,1)$. Moreover \tilde{u} is unique.

Proof: (Part I)

The proof of the continuity involves a good deal of complicated estimation. Here we use property of Poisson kernel

From $\frac{1}{2\pi} \int_0^{2\pi} p_r(\varphi) d\varphi = 1$, we have that

$$\tilde{u}(e^{i\theta}) = u(e^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\varphi}) p_r(\varphi) d\varphi \text{ for } 0 \leq r < 1$$

And from Poisson integral formula

$$\tilde{u}(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\varphi}) p_r(\theta - \varphi) d\varphi = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i(\theta-\varphi)}) p_r(\varphi) d\varphi$$

And from $p_r(\varphi) > 0$ for all φ and $0 \leq r < 1$,

$$|\tilde{u}(re^{i\theta}) - \tilde{u}(e^{i\theta})| = \left| \frac{1}{2\pi} \int_0^{2\pi} [u(e^{i(\theta-\varphi)}) - u(e^{i\varphi})] p_r(\varphi) d\varphi \right|$$

$$|\tilde{u}(re^{i\theta}) - \tilde{u}(e^{i\theta})| \leq 1/2\pi \int_0^{2\pi} |u(e^{i(\theta-\varphi)}) - u(e^{i\varphi})| p_r(\varphi) d\varphi$$

From the hypothesis we have that u is uniformly continuous ∂D , we may select for any given $\varepsilon > 0, \exists \delta > 0$ such that $|u(e^{i(\theta-\varphi)}) - u(e^{i\varphi})| < \varepsilon/2$ provided that $|\varphi| < \delta$

Now we break the integral in to two pieces and make the obvious estimate on each piece.

$$\text{Set } I_1 := 1/2\pi \int_{|\varphi| < \delta} |u(e^{i(\theta-\varphi)}) - u(e^{i\varphi})| p_r(\varphi) d\varphi$$

$$\text{And } I_2 := 1/2\pi \int_{|\varphi| \geq \delta} |u(e^{i(\theta-\varphi)}) - u(e^{i\varphi})| p_r(\varphi) d\varphi$$

$$\text{Then } I_1 \leq \varepsilon/2 \cdot 1/2\pi \int_0^{2\pi} p_r(\varphi) d\varphi$$

$$\therefore I_1 \leq \varepsilon/2$$

$$\text{And } I_2 \leq 1/2\pi \int_{|\varphi| \geq \delta} (|u(e^{i(\theta-\varphi)})| + |u(e^{i\varphi})|) p_r(\varphi) d\varphi$$

From property of Poisson kernel (vi) for each $\delta > 0, \lim_{r \rightarrow 1} p_r(\theta) = 0$ uniformly in θ for $\delta \leq |\theta| \leq \pi$ and $\max_{\delta \leq |\theta| \leq \pi} \{p_r(\theta)\} \rightarrow 0$ as $r \rightarrow 1$ and letting

$$M = \max\{u(e^{i\theta}) : \delta \leq |\theta| \leq \pi\}, \text{ we have that } 0 < p_r(\theta) < \varepsilon/4M$$

$$\begin{aligned} \text{Now } I_2 &\leq 1/2\pi \int_{|\varphi| \geq \delta} |u(e^{i(\theta-\varphi)})| p_r(\varphi) d\varphi + 1/2\pi \int_{|\varphi| \geq \delta} |u(e^{i\varphi})| p_r(\varphi) d\varphi \\ &\leq 1/2\pi \cdot 2M \max_{\delta \leq |\theta| \leq \pi} p_r(\varphi) \int_{\delta \leq |\theta| \leq \pi} d\varphi \end{aligned}$$

$$I_2 < 1/2\pi \cdot 2M \max_{\delta \leq |\theta| \leq \pi} p_r(\varphi) \int_0^{2\pi} d\varphi = 2M \max_{\delta \leq |\theta| \leq \pi} p_r(\varphi) = 2M \frac{\varepsilon}{4M} = \frac{\varepsilon}{2}$$

$$\therefore I_2 < \frac{\varepsilon}{2} \text{ for } r \text{ sufficiently close to } 1.$$

Thus, we have shown that $|\tilde{u}(re^{i\varphi}) - \tilde{u}(e^{i\varphi})| \leq I_1 + I_2 < \varepsilon/2 + \varepsilon/2 = \varepsilon$, for r sufficiently close to 1. Thus, $\tilde{u}(re^{i\varphi}) \rightarrow \tilde{u}(e^{i\varphi})$ as $z \rightarrow e^{i\varphi}$.

Therefore; \tilde{u} is continuous on \bar{D} . ■.

(Part II)

To show \tilde{u} is harmonic in $D(0,1)$, it is sufficient to show \tilde{u} is the real part of analytic function.

From definition of $\tilde{u}(re^{i\theta}) = 1/2\pi \int_0^{2\pi} u(e^{i\varphi}) p_r(\theta - \varphi) d\varphi$ ($z = re^{i\theta}, 0 \leq r < 1$)

We have that,

$$\tilde{u}(re^{i\theta}) = 1/2\pi \int_0^{2\pi} u(e^{i\varphi}) \operatorname{Re} \left(\frac{e^{i\varphi} + re^{i\theta}}{e^{i\varphi} - re^{i\theta}} \right) d\varphi,$$

for $p_r(\theta - \varphi) = \operatorname{Re} \left(\frac{e^{i\varphi} + re^{i\theta}}{e^{i\varphi} - re^{i\theta}} \right)$

$$\tilde{u}(z) = \frac{1}{2\pi} \operatorname{Re} \left\{ \int_{|w|=1} u(w) \left(\frac{w+z}{w-z} \right) \frac{dw}{wi} \right\}, \text{ where } w = e^{i\varphi} \text{ and } z = re^{i\theta}, 0 \leq r < 1$$

$$\tilde{u}(z) = \operatorname{Re} \left\{ \frac{1}{2\pi i} \int_{|w|=1} u(w) \left(\frac{w+z}{w-z} \right) \frac{dw}{w} \right\}$$

Let $g(z) = \frac{1}{2\pi i} \int_{|w|=1} u(w) \left(\frac{w+z}{w-z} \right) \frac{dw}{w}$ and consider $\frac{w+z}{w(w-z)} = \frac{a}{w} + \frac{b}{w-z}$

From this we get $a = -1$, and $b = 2$; hence $\frac{w+z}{w(w-z)} = \frac{-1}{w} + \frac{2}{w-z}$

Now $g(z)$ can be written as $g(z) = \frac{1}{2\pi i} \int_{|w|=1} u(w) \left[\frac{-1}{w} + \frac{2}{w-z} \right] dw$

$$g(z) = \frac{-1}{2\pi i} \int_{|w|=1} \frac{u(w)}{w} dw + \frac{1}{2\pi i} \int_{|w|=1} \frac{2u(w)}{w-z} dw \dots \dots \dots (1)$$

Since $z = re^{i\theta}, 0 \leq r < 1$ and $w = e^{i\varphi}, \frac{z}{w} = re^{i(\theta-\varphi)}$

$$\Rightarrow 0 \leq \left| \frac{z}{w} \right| = r < 1, \text{ hence we have } \frac{1}{w-z} = \frac{1}{w \left(1 - \frac{z}{w} \right)} = \frac{1}{w} \sum_{k=0}^{\infty} \left(\frac{z}{w} \right)^k$$

Now substitute to equation (1) we get that;

$$\begin{aligned}
 g(z) &= \frac{-1}{2\pi i} \int_{|w|=1} \frac{u(w)}{w} dw + \frac{1}{2\pi i} \int_{|w|=1} 2 \sum_{k=0}^{\infty} \left(\frac{z}{w}\right)^k \frac{u(w)}{w} dw \\
 &= \frac{-1}{2\pi i} \int_{|w|=1} \frac{u(w)}{w} dw + 2 \sum_{k=0}^{\infty} z^k \frac{1}{2\pi i} \int_{|w|=1} \frac{u(w)}{w^{k+1}} dw \\
 &= \frac{-1}{2\pi i} \int_{|w|=1} \frac{u(w)}{w} dw + \frac{2}{2\pi i} \int_{|w|=1} \frac{u(w)}{w} dw + 2 \sum_{k=1}^{\infty} z^k \frac{1}{2\pi i} \int_{|w|=1} \frac{u(w)}{w^{k+1}} dw \\
 &= \frac{1}{2\pi i} \int_{|w|=1} \frac{u(w)}{w} dw + 2 \sum_{k=1}^{\infty} z^k \frac{1}{2\pi i} \int_{|w|=1} \frac{u(w)}{w^{k+1}} dw \\
 &= a_0 + 2 \sum_{k=1}^{\infty} a_k z^k, \text{ where } a_k = \frac{1}{2\pi i} \int_{|w|=1} \frac{u(w)}{w^{k+1}} dw
 \end{aligned}$$

The term wise integration is justified by the uniform convergence of the series $w \in D(0,1)$ (with $z \in D$ fixed). Because

$$\begin{aligned}
 |q_k| &= \left| \frac{1}{2\pi i} \int_{|w|=1} \frac{u(w)}{w^{k+1}} dw \right| \\
 &\leq \frac{1}{2\pi} \int_{|w|=1} \left| \frac{u(w)}{w^{k+1}} \right| |dw| = \frac{1}{2\pi} \int_{|w|=1} |u(w)| |dw|
 \end{aligned}$$

Let $M = \max\{|u(w)| : w \in D(0,1)\}$ and since $\int_{|w|=1} |dw|$ is the length of $|w|$ is $2\pi r$, then we have $|q_k| \leq \frac{M}{2\pi} \int_{|w|=1} |dw| = \frac{M}{2\pi} (2\pi) = M$,

The radius of convergence of the power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ is

$$\frac{1}{R} = \lim_{k \rightarrow \infty} \text{Sup} |a_k|^{\frac{1}{k}}$$

Now

$$\frac{1}{R} = \lim_{k \rightarrow \infty} |2a_k|^{\frac{1}{k}} \leq \lim_{k \rightarrow \infty} |2M|^{\frac{1}{k}} = 1 \text{ (assuming } M \neq 0)$$

$$\Rightarrow R \geq 1$$

Thus $g(z)$ has a power series expansion with radius of convergence not smaller than one and, therefore, $g(z)$ is analytic function in $D(0,1)$. Since $\tilde{u}(z)$ is the real part of analytic function, $\tilde{u}(z)$ is harmonic in the unit disc.

(Uniqueness) Suppose that v is a continuous function on \bar{D} which is harmonic in D and $\tilde{u}(z) = v(z)$ for $z \in \partial D$. Then $\tilde{u} - v$ is harmonic in D and $\tilde{u} - v = 0$ on ∂D . Hence by corollary 1.9 $\max_{\bar{D}} \tilde{u} - v = \max_{\partial D} \tilde{u} - v = 0$. Since the value of $\tilde{u} - v$ on ∂D is identically zero, by Poisson integral $\tilde{u} - v = 0$ in D .

Hence $\tilde{u} - v \equiv 0$ in D . This proves the uniqueness. ■.

The preceding result shows that the Dirichlet problem for unit disc D and can easily be adapted to solve Dirichlet problem for an arbitrary disc by translating and rescaling. The following corollary will generalize Dirichlet problem for an arbitrary disc.

Corollary 3.2: Let $z_0 \in \mathbb{C}$ and $\rho > 0$ and suppose u is a continuous real valued function on $D := \{z: |z - z_0| = \rho\}$. Then there is a unique continuous function $\tilde{u}(z): \bar{D} \rightarrow \mathbb{R}$ such that \tilde{u} harmonic on D and $u(z_0 + \rho e^{i\theta}) = \tilde{u}(z_0 + \rho e^{i\theta})$ for all θ on D .

Proof:

Take $u'(e^{i\theta}) = u(z_0 + \rho e^{i\theta})$; then u' is continuous on D . Then by above theorems, there exist a continuous function $\tilde{u}'(z): \bar{D} \rightarrow \mathbb{R}$ such that \tilde{u}' harmonic on D and $\tilde{u}'(e^{i\theta}) = u'(e^{i\theta}) = u(z_0 + \rho e^{i\theta})$. Then we set $\tilde{u}(z) = \tilde{u}'\left(\frac{z-z_0}{\rho}\right)$ is the required function on $D(z_0, \rho)$. This proves Dirichlet problem for arbitrary disc. ■.

In general our result shows that, for any disc, the Poisson integral formula provides the explicit solution of the Dirichlet problem and the uniqueness of the solution is given by maximum principle. Transfer to other region is accomplished using conformal mapping.

3.2 DIRICHLET PROBLEM FOR UPPER HALF PLANE

The Dirichlet problem for the upper half plane H is to find a function $\tilde{u}(x, y)$ that is harmonic in the upper half plane and has the boundary values $\tilde{u}(x, 0) = u(x)$, where $u(x)$, is a real-valued function of the real variable x . An important method for solving this problem is our Poisson integral formula for upper half plane.

Theorem 3.3 (Solution of Dirichlet problem for upper half plane): Let u be a real-valued function that is piecewise continuous and bounded for all real $t \in \partial H$. Define $\tilde{u}(z) = \begin{cases} Pu & \text{if } \text{Im}z > 0 \\ u & \text{if } z \in \partial H \end{cases}$ then $\tilde{u}(z)$ on \bar{H} is continuous function and harmonic in the upper half plane H .

Proof:

We see in chapter (1) the role of conformal mapping: the composition of analytic function with harmonic function is harmonic. Since the equality $\tilde{u}(t) = u\left(\frac{t-i}{t+i}\right)$ holds and the function of t given by the formula $z = \frac{t-i}{t+i}$ is analytic, we know that \tilde{u} is harmonic on half plane H .

Now to show that $\tilde{u}(z)$ continuous on ∂H . Let $a \in \partial H$ and $\delta > 0$. Then using the property $\int p_z(t) dt = 1$, we have that, $u(a) = \int_{\partial H} u(a) p_z(t) dt = \tilde{u}(a)$ and from the Poisson integral for upper half plane $\tilde{u}(z) = \int_{\partial H} \tilde{u}(t) p_z(t) dt$

$$\begin{aligned} |\tilde{u}(z) - \tilde{u}(a)| &= \left| \int_{\partial H} [u(t) - u(a)] p_z(t) dt \right| \\ &\leq \int_{\partial H} |[u(t) - u(a)] p_z(t)| dt \end{aligned}$$

Since $p_z(t) \geq 0$,

$$\begin{aligned} |\tilde{u}(z) - \tilde{u}(a)| &\leq \int_{\partial H} |[u(t) - u(a)]| p_z(t) dt \\ &\leq \int_{|t-a| \leq \delta} |[u(t) - u(a)]| p_z(t) dt + \int_{|t-a| > \delta} |[u(t) - u(a)]| p_z(t) dt \end{aligned}$$

$$\leq \int_{|t-a| \leq \delta} |[u(t) - u(a)]| p_z(t) dt + 2 \max_{|t-a| > \delta} |u(t)| \int_{|t-a| > \delta} p_z(t) dt$$

For all $z \in H$. If δ is very small, the integral over $\{|t - a| \leq \delta\}$ will be small by continuity of u at a and $\int p_z(t) dt = 1$ and the integral over $\{|t - a| > \delta\}$ approaches 0(zero) as $z \rightarrow a$ by property $\int_{|t-a| > \delta} p_z(t) dt \rightarrow 0(z \rightarrow a)$. Hence $\tilde{u}(z)$ is continuous on ∂H . ■.

Example:

The complex temperature is an analytic function that has as its real part the ordinary physical temperature and as its imaginary part the heat flux associated with the temperature distribution. Complex temperature is an artificial constructs that makes temperature and heat transfer problem in two dimensions amenable to our theory. This works because both temperature and heat flux are harmonic functions and curves of constant temperature and curves of constant heat flux are mutually orthogonal families. We can write $\Omega(z) = T(x, y) + iH(x, y)$, where $T(x, y)$ gives the temperature at the point $z = x + iy$, and $H(x, y)$ gives the magnitude of the heat flux.

Suppose the left half of the real axis is maintained at temperature T_1 and the right half at T_2 . We would like to know the temperature at any point in the upper half-plane. We know the boundary temperatures and we know temperature is harmonic

$$\left(i. e, T(x, 0) = \begin{cases} T_1, & x < 0 \\ T_2, & x > 0 \end{cases} \right)$$

Solution:

We use the Poisson integral formula for upper half-plane

$$\begin{aligned} T(s, t) &= \frac{t}{\pi} \int_{-\infty}^{\infty} \frac{T(x, 0)}{t^2 + (x - s)^2} dx \\ &= \frac{t}{\pi} \int_{-\infty}^0 \frac{T_1}{t^2 + (x - s)^2} dx + \frac{t}{\pi} \int_0^{\infty} \frac{T_2}{t^2 + (x - s)^2} dx \end{aligned}$$

Let $w = \frac{(x-s)}{t}$, then $dx = tdw$

And now we have to find

$$\int \frac{tdx}{t^2 + (x-s)^2} = \int \frac{dx}{t \left[\left(\frac{x-s}{t} \right)^2 + 1 \right]} = \int \frac{tdw}{t \left[\left(\frac{x-s}{t} \right)^2 + 1 \right]} = \int \frac{dw}{1+w^2} = \tan^{-1} w$$

Then,

$$\frac{t}{\pi} \int_{-\infty}^0 \frac{T_1}{t^2 + (x-s)^2} dx = \left[\frac{T_1}{\pi} \tan^{-1} \left(\frac{x-s}{t} \right) \right]_{-\infty}^0 = \frac{-T_1}{\pi} \left[\tan^{-1} \left(\frac{S}{t} \right) - \frac{\pi}{2} \right] = \frac{T_1}{\pi} \tan^{-1} \left(\frac{t}{S} \right)$$

Likewise for

$$\frac{t}{\pi} \int_0^{\infty} \frac{T_2}{t^2 + (x-s)^2} dx = \left[\frac{T_2}{\pi} \tan^{-1} \left(\frac{x-s}{t} \right) \right]_0^{\infty} = \frac{T_2}{\pi} \left[\tan^{-1} \left(\frac{S}{t} \right) + \frac{\pi}{2} \right] = \frac{T_2}{\pi} \tan^{-1} \left(\frac{S}{t} \right)$$

So finally

$$T(s, t) = \frac{1}{\pi} \left[T_2 \left(\tan^{-1} \left(\frac{S}{t} \right) + \frac{\pi}{2} \right) - T_1 \left(\tan^{-1} \left(\frac{S}{t} \right) - \frac{\pi}{2} \right) \right]$$

Example:

Solve the following boundary value problem

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, y > 0$$

$$\lim_{y \rightarrow 0^+} \phi(x, y) = G(x) = \begin{cases} T_0, & x < -1 \\ T_1, & -1 < x < 1 \\ T_2, & x > 1 \end{cases}$$

Solution:

We use the Poisson integral formula for upper half-plane

$$\begin{aligned}\phi(x, y) &= \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{G(\eta)}{y^2 + (x - \eta)^2} d\eta \\ &= \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{T_0}{y^2 + (x - \eta)^2} d\eta + \frac{y}{\pi} \int_{-1}^1 \frac{T_1}{y^2 + (x - \eta)^2} d\eta \\ &\quad + \frac{y}{\pi} \int_1^{\infty} \frac{T_2}{y^2 + (x - \eta)^2} d\eta \\ &= \frac{T_0}{\pi} \tan^{-1} \left(\frac{\eta - x}{y} \right)_{-\infty}^{-1} + \frac{T_1}{\pi} \tan^{-1} \left(\frac{\eta - x}{y} \right)_{-1}^1 + \frac{T_2}{\pi} \tan^{-1} \left(\frac{\eta - x}{y} \right)_1^{\infty} \\ &= \frac{T_0 - T_1}{\pi} \tan^{-1} \left(\frac{y}{x + 1} \right) + \frac{T_1 - T_2}{\pi} \tan^{-1} \left(\frac{y}{x - 1} \right) + T_2\end{aligned}$$

REFERENCES

- [1]. John B.Garnett; Bounded Analytic Functions, Academic press, Inc, 1981.
- [2]. Rudin W.; Real and complex analysis, McGraw-Hill, Inc, 1996, 1994.
- [3]. Conway J.B; Functions of complex variables, Springer-Verlag New York, 1978.
- [4]. L.V. Ahlfors; Complex analysis McGraw-Hill, New York, 1979.
- [5]. M.A Evgrafav; Analytic functions, Dover publication, Inc, 1996.
- [6]. Sheldon Axler Paul Bourdon Wade Ramey; Harmonic Function Theory, Springer-Verlag New York, Berlin Heidelberg, 2001, 1992.
- [7]. Robert E. Greene Steven G. Krantz; Function theory of one complex variable, John Wiley & Sons, Inc.1997
- [8]. Moybon W.; Seminar report on harmonic functions, A.A.U. Addis Ababa, 2010.
- [9]. Liang_S.H & Bernard E.; Classical complex analysis Jones and Bartlett, Publisher, Inc, 1996.