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First Order Nonlinear Partial Differential Equation Lagrange Method, Char pit's Method

Submitted in Partial Fulfillment of The Requirement for The Degree of Master of Science in Mathematics

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The undersigned hereby certify that they have read and recommend to the school of graduate studies for acceptance of a project entitled **First Order Nonlinear Partial Differential Equation Lagrange Method, Charpit's Method** by Asfaw Wana in partial fulfillment of the requirements for the degree of Master of Science.

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Abstract

Nonlinear first order equations are typically hard to solve without some conditions placed on the PDE. In this project I used to present the Method of Characteristics and Auxiliary system of equation to solve first order nonlinear Partial Differential Equations using Lagrange's and Charpit's method.

Preface

This Thesis contains the following three chapters each which are briefed below with some definitions, Theorems and examples.

Chapter one_ Deals with Non-linear first order partial differential equation, introduction, Linear equation, quasi linear, Derivation of characteristic equation and compatibility condition, solution of the Cauchy problem and first order non-linear equation in two independent variables.

Chapter two_ Deals with Lagrange method, Lagrange method for partial differential equation with coefficient variable, Lagrange method for first integral The Euler-Lagrange equation and Lagrange method of first integral and Euler-Lagrange equation.

Chapter three_ Char pit's method to find complete integral, Lagrange- Charpit's equation, integrability , integral of Charpit's equation and Char pit's equation for find complete integral of first order non-linear partial differential equation.

Contents

CHAPTERONE

Non Linear First Order Partial DifferentialEquations

1. Introduction

A partial differential equation (DE) is a relation that involves partialderivatives of an unknown function. Let the unknown function be u, and $x, y, z, \ldots$,be independent variables, i.e. $u = u(x, y, z, \ldots)$. Often, one of Variable represents the time. Thus, a partial DE is an equation ofthe form

$$F(x, y, z, u, u_x, u_y, u_z, u_{xy}, u_{xx}, \ldots, u_{xxx}, \ldots) = 0$$

We will always assume that the unknown function *u* is *sufficiently wellbehaved* so that all necessary partial derivatives exist and correspondingmixed partial derivatives are equal

I.e. $u_{xy} = u_{yx}, u_{xxy} = u_{yxx}$, and so on

Definition1: Let $X = (x_1, x_2, x_3, \ldots, x_n) \in \mathbb{R}^n$ and let $u: \overline{\Omega} \subseteq \mathbb{R}^n \to \mathbb{R}$

A first order partial differential equation for $u = u(x)$ is given by $F(x, u, D_u) = 0$ where $F: \mathbb{R}^n \text{x} \mathbb{R} \text{ x} \overline{\Omega} \to \mathbb{R}$ is given function, D_u is a vector of partial derivatives of u.

Notation: we usually denote the entries of F as follows

$F = F(x, z, p)$. Thus $p \in \mathbb{R}^n$, $z \in \mathbb{R}$, $x \in \overline{\Omega}$

Remark1: The partial differential equation $F(D_u, u, x) = 0$ is usually accompanied by a boundary condition of the form $u = g$ on $\partial\Omega$ such a problem is usually called a boundary value problem.

Remark 2: For simplicity, in most of what follows, we restrict n=2, we call the two variables x, y. Thus we reduce to the case

$F(x, y, u, u_x, u_y) = 0$ (1) In this case, the solution $u = u(x, y)$ is surface in $\mathbb{R}^3$

1.1 Linear Equation

The first subclass of nonlinear partial differential, we consider are the linear equations.

Definition 2: A linear 1^{st} order PDE is a PDE which can be put in the form

$a(x,y)u_x + b(x,y)u_y + (x,y)u = f(x,y)$ (2)

is a linear combination of the unknown function $u$and itspartial derivative with coefficients which are given functions of theindependent variables or constants.

Example: Consider the linear equation

$u_x = c_0 u + c_1(x,y)$(3) Where c_0 is constant, and $c_1(x,y)$ is the function of the two variables $x \ and \ y$. since there is no differentiation with respect to$y$, the variable may actually be treated as the parameter, this equation reduces to an ODE. To solve we need some initial condition, for instance

$u(0,y) = y$(4) The solution is now simple

$u(x,y) = e^{c_0 x}(\int_0^x e^{-c_0 \xi} c_1(\xi,y)d\xi + y)$ (5) From this we see that given an initial value along the y-axis $u(0,y_0)$ we are able to extend to a solution $u(.,y_0)$ parallel to x-axis. This is true for any y_0

1.2 Quasi-Linear Equation

1.2.1 **Derivation of Characteristic Equation and Compatibility Condition**It is simple to discuss the theory of a quasi-linear equation

$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$ (6)

Using the theory of the linear equation, we assume that $a, b, c \in C^1(D_2)$

Consider a known solution $u(x,y)$ of (6). Let us substitute the function $u(x,y)$ for u in coefficient a, b, c, then the coefficients$a(x,y,u(x,y))$, b$(x,y,u(x,y))and \ c(x,y,u(x,y))$ become known functions of $x \ and \ y, and \ u$ satisfies a relation in which $a \ and \ b$ are functions of *x* and *y* only. Thus the following ODEs are satisfied

$$\frac{dx}{d\sigma} = a\big(x, y, u(x, y)\big), \frac{dy}{d\sigma} = b\big(x, y, u(x, y)\big),$$

$$\frac{du}{d\sigma} = c\big(x, y, u(x, y)\big) \qquad (7)$$

Now, this is true for every solution $u(x, y)$. Hence it follows that for any arbitrary solution of (6) there are characteristic curves in (x, y)-plane given by

$\frac{dx}{d\sigma} = a(x, y, u), \frac{dy}{d\sigma} = b(x, y, u)$ (8) and along characteristic curves the compatibility condition $\frac{du}{d\sigma} = c(x, y, u)$ (9) must be satisfied.

Remark 3: In the case of linear and Quasi-linear equations, the characteristic equations are independent u.

1.2.2 Solution of the Cauchy problem

Theorem: *Consider the Cauchy problem for PDE (10) with Cauchy data $u_0(\eta)$ prescribed on a curve $\Upsilon: x = x_0(\eta), y = y_0(\eta), \eta \in I$ where I is an open interval say for $0 < \eta < 1$*

Let

i) $x_0(\eta), y_0(\eta), u_0(\eta) \in C^1(I)$

ii) $a(x, y, u), b(x, y, u)$ *and* $c(x, y, u)$ *be* C^1 *functions on Domain* D_2 *of* $\mathbb{R}^3$

iii) *The curve* $\Gamma: x = x_0(\eta), y = y_0(\eta), u = u_0(\eta), for\ \eta \in (0,1)$ *belongs to* D_2

iv) *The transversal condition*

$$\frac{dy_0(\eta)}{d\eta} a\big(x_0(\eta), y_0(\eta), u_0(\eta)\big) - \frac{dx_0(\eta)}{d\eta} b\big((x_0(\eta), y_0(\eta), u_0(\eta)\big) \neq 0 \qquad (10)$$

is satisfied on curve Υ, then there exists a unique solution $u(x, y)$ of the Cauchy problem in a neighborhood D of Υ.

Proof: since a, b, c have continuous partial derivatives with respect to x, y, u the system of ODEs (12) and (13) have a unique continuously differentiable solution with respect to σ of the form

$x = x(\sigma,\eta), y = y(\sigma,\eta), u = u(\sigma,\eta)(11)$

Satisfying the initial condition

$x(0,\eta) = x_0(\eta), y(0,\eta) = y_0(\eta), u(0,\eta) = u_0(\eta), (12)$

for an $\eta \in I$

Consider now a point p_0 on ϒ corresponding to $\eta = \eta_0$. As $x_0(\eta), y_0(\eta)$ and $u_0(\eta)$ are continuously differentiable functions in solution (11) are continuously differentiable functions of two variables σ and η in a Domain of (σ,η)-plane containing $(0,\eta_0)$. In a view our assumption (10) the Jacobian

$$\frac{\partial(x,y)}{\partial(\sigma,\eta)} \equiv \begin{vmatrix} x_\sigma & x_\eta \\ y_\sigma & y_\eta \end{vmatrix}$$

$= ay_\eta - bx_\eta$ (13) Does not vanish at σ=0 for $0 < \eta < 1$ and in particular at the point $(0,\eta_0)$ in $(\sigma,\eta) - plane$

Using the inverse function theorem we can uniquely solve for σ and η from the first two relations in (11) in terms of x and y.

$\sigma = \sigma(x,y), \eta = \eta(x,y)$ (14)

In a neighborhood D_{p_0} of p_0 in (x,y)-plane. More over the functions σ and η in (14) are C^1in D_{p_0} . Substitute (14) in the third relation in (11) to get a function $u(x,y)$ of $x\ andy$ in D_{p_0}

$u(x,y) = u(\sigma(x,y),\eta(x,y))(15)$

Further this function is C^1 on a Domain D_{p_0} in (x,y)-plane. Note that at any point of the intersection of datum curve ϒ and the Domain D_{p_0}

$$u(x_0(\eta),y_0(\eta)) = u(\sigma(x_0,y_0),\eta(x_0,y_0))$$

$$= u(0,\eta)$$

$=u_0(\eta)$

This shows that the initial condition (14) is satisfied. More over

$$au_x + bu_y = a\left(u_\sigma \sigma_x + u_\eta \eta_x\right) + b(u_\sigma \sigma_y + u_\eta \eta_y)\text{using } (11)$$

$$=u_\sigma\left(a\sigma_x + b\sigma_y\right) + u_\eta(a\eta_x + b\eta_y)$$

$$=u_\sigma\left(\sigma_x x_\sigma + \sigma_y y_\sigma\right) + u_\eta(\eta_x x_\sigma + \eta_y y_\sigma)\text{using } (8)$$

$$=u_\sigma \sigma_{\sigma +} u_\eta \eta_\sigma$$

$$=u_\sigma$$

$$=\text{C} \ \ (16)$$

Hence, u satisfies the PDE (10) in D_{p_0}

Let, D= $\{u_p \in \Upsilon D_{p_0}\}$ we define u in D with the help of values of u inD_{p_0}. Now, we have solved the Cauchy problem in a neighborhood D of ϒ.

For uniqueness, note that each step of construction of the function $u(x,y)$ in (13), namely solution of the ODEs (7), inversion of the functions leading to $\sigma(x,y)$ and $\eta(x,y)$ and substitution in $u(\sigma,\eta)$ all lead to unique results. Hence the procedure leading to the construction of the solution (15) gives a unique solution of the Cauchy problem in a neighborhood D_{p_0} of the point $(x_0(\eta_0), y_0(\eta_0))$ on ϒ and hence the Domain D containsϒ. The proof completed.

1.3 First Order Non-Linear Equations in two independent Variables

1.3.1 Derivation of Charpit's Equation

The general form of the first order

$F(x,y,u,p,q) = 0; p = \frac{du}{dx}, q = \frac{du}{dy}(1)$ where we assume that $F \in c^2(D_3)$with D_3as a domain in $\mathbb{R}^5$.With a general expression *F* in the PDE (1), there is no *indication* that we canfind a transport equation for a solution *u* along a curve in (*x, y*)- plane. such a curve was made by Lagrange and Charpit in an attempt to find a complete integral and by Cauchy in a geometric formulation with the help of Monge Cone .

We proceed differently by taking a known c^2solution $u(x,y)$ and differentiate (1) with respect to x. This gives

$F_x + u_x F_u + p_x F_p + q_x F_q = 0.$

Using$q_x = u_{yx} = u_{xy} = p_y$, and rearrange the term we get

$F_p p_x + F_q q_y = -F_x - pF_u$(2)

In which the quantities $F_p, F_q$are now known functions of $x$and$y$.

The first x-derivative of u, namely p is differentiated in the direction (F_p, F_q) in (x, y)-plane. Thus, for the known solution $u(x, y)$, consider a one parameter family of curves$(x, y) -$ plane given by

$\frac{dx}{d\sigma} = F_p, \frac{dy}{d\sigma} = F_q$(3)

Along these curves, (2) gives

$\frac{dp}{d\sigma} = -F_x - pF_u$(4)

Similarly, differentiating (1) with respect to$y$, we find that along the same curves given by (3), we have

$\frac{dq}{d\sigma} = -F_y - qF_u$ (5)

The rate of change of u along these curves is

$$\frac{du}{d\sigma} = u_x \frac{dx}{d\sigma} + u_y \frac{dy}{d\sigma}$$

$= pF_p + qF_q$ (6) Thus, we find set of equations, called *Charpit's equations* consisting of two Characteristic equations (3) and three compatibility conditions (4)-(6).

Given set of values (u_0, p_0, q_0) at any point(x_0, y_0),

so that$(x_0, y_0, u_0, p_0, q_0) \in D_3$, we can find a solution of Charpit's equations.

Since the system is autonomous, the set of solutions of the Charpit's equations form a four parameter family of curves in (x, y)-plane.

Remark 4: Every solution $\big(x(\sigma), y(\sigma), u(\sigma), p(\sigma), q(\sigma)\big)$ of the Charpit's equations satisfies the strip condition

$\frac{du}{d\sigma} = p(\sigma)\frac{dx}{d\sigma} + q(\sigma)\frac{dy}{d\sigma}$(7) on the curve $(x(\sigma), y(\sigma), u(\sigma))$ in (x, y, u)-space

Definition 3: A set of five ordered functions $(x(\sigma), y(\sigma), u(\sigma), p(\sigma), q(\sigma))$ satisfying the Charpit equations (3) - (6) and $F\big(x(\sigma), y(\sigma), u(\sigma), p(\sigma), q(\sigma)\big) = 0$ is called a *Monge strip* of the PDE (1).

The condition $F(x(\sigma), y(\sigma), u(\sigma), p(\sigma), q(\sigma)) = 0$ imposes a relation between the four parameters of the set of all solutions of the Charpit's equations.

Therefore the Monge strips form a three parameter family of strips in (*x, y, u*)-space.

Definition 4: Given a Monge strip $(x(\sigma), y(\sigma), u(\sigma), p(\sigma), q(\sigma))$, the base curve in (*x, y*)-plane given by ($x(\sigma)$, $y(\sigma)$) is called a characteristic curveof the PDE (1).

Remark 5: Characteristic curves of a linear first order PDE form a one parameter family of curves in (x, y)-plane. Those for a quasi-linear equation form a two parameter family of curves. Finally those of a nonlinear equation (1) form a three parameter family of curves.

1.3.2 Solution of Cauchy problem

For the nonlinear PDE (1), we need to solve the Charpit's equations (3)-(6) and for this we need initial values $p_0(\eta)\ and q_0(\eta)$ on a curve ϒ in addition to $x_0(\eta), y_0(\eta)$ and $u_0(\eta)$

Firstly we note that these initial values must satisfy the PDE i.e.

$F(x_0(\eta), y_0(\eta), u_0(\eta), p_0(\eta), q_0(\eta)) = 0$ (8)

Further, differentiating (8) with respect to η we get one more relation

$p_0(\eta)x'_0(\eta) + q_0(\eta)y'_0(\eta) = u'_0(\eta)$ (9) We take two functions $p_0(\eta)$ and $q_0(\eta)$ satisfying (8) and (9) and complete the initial data for (3)-(6) at $\sigma = 0$ as

$x(0,\eta) = x_0(\eta), y(0,\eta) = y_0(\eta), u(0,\eta) = u_0(\eta), p(0,\eta) = p_0(\eta), \quad q(0,\eta) = q_0(\eta)$(10)

Now we solve the Charpit's equations (3)-(6) with initial data (10) and obtain

$x = X(\sigma,\eta), y = Y(\sigma,\eta), u = U(\sigma,\eta),$

$p = P(\sigma,\eta), q = Q(\sigma,\eta)$ (11)

From the first two relations in (11), we solve σ and η as functions of *x* and *y* and substitute in the third relation to get the solution $u(x,y)$ of the Cauchy problem.

Example: Consider a wave front moving into a uniform two-dimensional medium with a constant normal velocity *c*. Let the successive positions of the wave front be denoted by an equation $u(x,y) = ct$, where $t$is the time. Since the velocity of propagation *c* is given by $c = \frac{\phi_t}{((\phi^2{}_x+\phi^2{}_y))^{1/2}}$, where $c := ct - u(x,y) = 0$ the function *u*(*x, y*) satisfies an eikonal equation

$p^2 + q^2 = 1\,; p = u_x, q = u_y$(12)

Let the initial position of the wave front be given by

$\alpha x + \beta y = 0\ , \alpha, \beta = constants, \alpha^2 + \beta^2 = 1$ (13)
We can formulate the problem of finding successive positions of the wave front for $t > 0$ as a Cauchy problem.

Solve the PDE (12) subject to the Cauchy data at

$\sigma = 0, x_0 = \beta\eta, y_0 = -\alpha\eta, u_0 = 0$ (14)

Solution:

First we find out the values of $p_0$and $q_0$from (8) and (9)

i.e.

$p^2{}_0 + q^2{}_0 = 1, \beta p_0 - \alpha q_0 = 0,$

which give two sets of values at $\sigma = 0$

$p_0 = \pm\alpha, q_0 = \pm\beta$(15)

The Charpit's equations of (12) are

$\frac{dx}{d\sigma} = 2p,\ \frac{dy}{d\sigma} = 2q$ (16)

$\frac{du}{dx} = 2(p^2 + q^2)$ (17)

$\frac{dp}{d\sigma} = 0,\ \ \frac{dq}{d\sigma} = 0$ (18)

Solution of (16)-(18) with initial data (14) and (15) at $\sigma = 0$ gives

$x = \pm 2\alpha\sigma + \beta\eta,\ y = \pm 2\beta\sigma - \alpha\eta$ (19)

$u = 2\sigma,\ p = \pm\alpha, q = \pm\beta$ (20)

Solving σin terms of $x$and $y$from (19), we get

$\sigma = \pm\frac{1}{2}(\alpha x + \beta y).$

From (20) we have $u = 2\sigma$and hence the solutions of the Cauchy problem (12) and (14) are

$u = \pm(\alpha x + \beta y$(21)

There are two solutions of this Cauchy problem. Note that this does not violate the uniqueness theorem for a solution of the Cauchy problem for a first order nonlinear PDE. The two solutions correspond to the two sets of determinations of $p_0$and $q_0$in (14) leading actually to two Cauchy problems.

The two solutions have an important physical interpretation in terms of front propagation. The wave fronts (one forward propagating and another backward propagating) starting from the position (13) occupy the positions

$u = ct,$

i.e.

$\alpha x + \beta y = \pm ct$ (22)

Which are at a (normal) distance $\pm ct$ from (14).The equation (12) governs boththe forward and backward moving wave

CHAPTERTWO

LagrangeMethod

2.1 Lagrange method for Partial Differential Equation of first order with variable coefficient The general form of first order linear partial differential equations with variable coefficient is

$P(x,y,)u_x + Q(x,y)u_y + f(x,y)u = R(x,y)$ (1)

We can eliminate the term in u from (1) by substituting $u = ve^{-\zeta(x,y)}$, where $\zeta(x,y)$ satisfies the equation

$$P(x,y)\zeta_x(x,y) + Q(x,y)\zeta_y(x,y) = f(x,y)$$

Hence, equation (1) is reduced to

$P(x,y)u_x + Q(x,y)u_y = R(x,y)$(2)

Where P, Q, R in (2) are not the same as in (1). The following theorem provides a method for solving (2) often called Lagrange's method

Theorem 2.1: *The general solution of linear partial differential of first order*

$Pp + Qq = R$ *(3)* Where$p = \frac{\partial u}{\partial x}, q = \frac{\partial u}{\partial y}$, P, Q and R *are functions of* x *and* y,

$F(\varphi, \psi) = 0$ *(4)* Where F is arbitrary function and $\varphi(x,y,u) = c_1$, $\psi(x,y,u) = c_2$ *form* a solution of the auxiliary system of equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{du}{R}$ (5)

Proof: Let $\varphi(x,y,u) = c_1$ *and* $\psi(x,y,u) = c_2$ satisfy (5), then equations $\varphi_x dx + \varphi_y dy + \varphi_u du = 0$ and $\frac{dx}{P} = \frac{dy}{Q} = \frac{du}{R}$, must be compatible, that is, we have

$$P\varphi_x + Q\varphi_y + R\varphi_u = 0,$$

Similarly, we have,

$P\psi_x + Q\psi_y + R\psi_u = 0 \begin{cases} P\varphi_x + Q\varphi_y + R\varphi_u = 0 \\ P\varphi_x + Q\varphi_y + R\varphi_u = 0 \end{cases}$ solving these equations for P, Q *and* R by cross multiplication we have

$$\frac{P}{\partial(\varphi,\psi)/\partial(y,u)} = \frac{Q}{\partial(\varphi,\psi)/\partial(x,u)} = \frac{R}{\partial(\varphi,\psi)/\partial(x,y)} \quad (6)$$

Where $\frac{\partial(\varphi,\psi)}{\partial(y,u)} = \varphi_y\psi_u - \varphi_u\psi_y \neq 0$ denotes Jacobean

Let $F(\varphi,\psi) = 0$, then by differentiating this equation with respect to x and y respectively,we obtain the equations

$$\frac{\partial F}{\partial \varphi}\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial u}p\right) + \frac{\partial F}{\partial \psi}\left(\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial u}p\right) = 0$$

$$\frac{\partial F}{\partial \varphi}\left(\frac{\partial \varphi}{\partial y} + \frac{\partial \varphi}{\partial u}q\right) + \frac{\partial F}{\partial \psi}\left(\frac{\partial \psi}{\partial y} + \frac{\partial \psi}{\partial u}q\right) = 0$$

And if we now eliminate $\frac{\partial F}{\partial \varphi}$ and $\frac{\partial F}{\partial \psi}$ for these equations, we have

$$\begin{vmatrix} \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial u}p & \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial u}p \\ \frac{\partial \varphi}{\partial y} + \frac{\partial \varphi}{\partial u}q & \frac{\partial \psi}{\partial y} + \frac{\partial \psi}{\partial u}q \end{vmatrix} = 0$$ And simplified as

$$p\frac{\partial(\varphi,\psi)}{\partial(y,u)} + q\frac{\partial(\varphi,\psi)}{\partial(x,u)} = \frac{\partial(\varphi,\psi)}{\partial(x,y)} \quad (7)$$

Substituting from equation (6) into equation (7) we see that $F(\varphi,\psi) = 0$ is general solution of (3). The solution can also be written as

$$\varphi = g(\psi) \;\; or \psi = h(\varphi)$$

Example: Find the general solution of the partial differential equation

$$y^2up + x^2uq = y^2x$$

Solution: the auxiliary system of equation

$$\frac{dx}{y^2u} = \frac{dy}{x^2u} = \frac{du}{xy^2}$$

From the first two equations we have,

$$x^2dx = y^2dy$$

By integrating

$x^3 - y^3 = c_1$

Similarly, from the first and the third, we have

$xdx = udu$ By integration $x^2 - u^2 = c_2$

Hence, the general solution is given by

$$F(c_1, c_2) = F(x^3 - y^3,\, x^2 - u^2) = 0$$

2.2 Lagrange Method of First Integral

In a linear case, we can construct a general solution, depending on an arbitrary function from the knowledge of a first integral for reduced characteristic system. The method can be extended to the equations of the form

$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$ (1)

We can say that two first integrals $\varphi = \varphi(x,y,u)$ and $\psi = \psi(x,y,u)$ are independent if $\nabla\varphi$ and $\nabla\psi$ are nowhere collinear.

Theorem 2.2: Let $\varphi = \varphi(x,y,u)$ and $\psi = \psi(x,y,u)$ be two independent first integrals of characteristic system and $F = F(h,k)$ be a $c^1 -$ function.

If $F_h\varphi_u + F_k\psi_u \neq 0$, the equation

$F(\varphi(x,y,u)\,,\,\psi(x,y,u)) = 0$ defined the general solution of (1)in implicit form.

Proof: It is based on two observations.

First, the function

$w = F(\varphi(x,y,u)\,,\,\psi(x,y,u))$(2) is first integral. In fact

$\nabla w = F_h\nabla\varphi + F_k\nabla\psi$

So that,

$\nabla w.(a,b,c) = F_h\nabla\varphi(a,b,c) + F_k\nabla\psi(a,b,c) \equiv 0$,since φ and ψ are first integrals.

Moreover, by hypothesis

$$w_u = F_h\varphi_u + F_k\psi_u \neq 0$$

Second, if w is first integral, and $w_u \neq 0$, then the equation

$w(x, y, u) = 0$(3) defines implicitly an integral surface $u(x, y)$ of (1). In fact, since w is first integral it satisfies the equation

$a(x, y, u)w_x + b(x, y, u)w_y + c(x, y, u)w_u = 0$ (4)
Moreover, from the implicit function theorem, we have,

$u_x = -\frac{w_x}{w_u}, \qquad u_y = -\frac{w_y}{w_u}$ and from (4) we get easily

2.3 The Euler-Lagrange Equation

The basic minimization problem is to determine a suitable function $y = u(x) \in c^1\,[a, b]$ that minimizes the objective functional

$J[u] = \int_a^b L(x, u, u')dx$ (1)The integrand is known as the Lagrange for variation problem.

In order to uniquely specify a minimizing function, we must impose suitable boundary conditions, Either Dirichlet (fixed) or Neumann (free) or mixed or periodic boundary. In this case we shall concentrate on Dirichlet boundary condition.

$u(a) = \alpha, \quad u(b) = \beta$ (2) The gradient $\nabla J[u]$ of the functional (1) will then be defined by the same basic directional derivative formula

$<\nabla J[u]; v> = \frac{d}{d\varepsilon}J[u + \varepsilon v]$ (3)

Here $v(x)$ is the function that prescribes direction of the derivative is computed. The inner product used in (3) is usually taken to be standard L^2 inner product

$< f; g > = \int_a^b f(x)g(x)dx$ (4) On function space.

Now starting with (1) for each fixed u and$v$, we must compute the derivative of the function

$h(\varepsilon) = J[u + \varepsilon v] = \int_a^b L(x, u + \varepsilon v, u' + \varepsilon v')$ (5) Assuming sufficient smoothness of $L(x, u + \varepsilon v, u' + \varepsilon v')$ allows us to bring the derivative inside the integral and by chain rule $h'(\varepsilon) = \frac{d}{d\varepsilon} J[u + \varepsilon v] = \int_a^b \frac{d}{d\varepsilon} L(x, u + \varepsilon v, u' + \varepsilon v') dx$

$= \int_a^b [v \frac{\partial L}{\partial \varepsilon}(x, u + \varepsilon v, u' + \varepsilon v') + v' (\frac{\partial L}{\partial u}(x, u + \varepsilon v, u' + \varepsilon v'] \, dx$ Therefore, setting $\varepsilon = 0$ in order to evaluate (3), we find $< \nabla J[u]; v >= \int_a^b [v (\frac{\partial L}{\partial \varepsilon}(x, u, u') + v' \frac{\partial L}{\partial p}(x, u, u'] \, dx$(6)

The resulting integral often, referred to as the first variation of the function $J[u]$.To obtain an explicit formula for $\nabla J[u]$, the right hand side of (6) needs to be written as

$$< \nabla J[u]; v >= \int_a^b \nabla J(u) v dx = \int_a^b h v dx$$

If we set $\Upsilon(x) = \frac{\partial L}{\partial p}\left(x, u(x), u'(x)\right)$, we can rewrite the offending term as

$\int_a^b \Upsilon(x) v'(x) dx = [\Upsilon(b) v(b) - \Upsilon(a) v(a)] - \int_a^b \Upsilon'(x) v(x) dx$ (7) Where again by chain rule

$$\Upsilon'(x) = \frac{d}{dx}(\frac{\partial L}{\partial p}(x, u, u'))$$

$= \frac{\partial^2 L}{\partial x \partial p}(x, u, u') + u' \frac{\partial^2 L}{\partial u \partial p}(x, u, u') + u'' \frac{\partial^2 L}{\partial p^2}(x, u, u')$ (8)

With boundary condition $u(a) = \alpha, \quad u(b) = \beta$. Therefore, we must make sure that the varied v, function $\hat{u}(x) = u(x) + \varepsilon v(x)$ and so

$$\hat{u}(a) = u(a) + \varepsilon v(a) = \alpha$$

$\hat{u}(b) = u(b) + \varepsilon v(b) = \beta$, the variation $v(x)$ must satisfy the corresponding homogeneous boundary condition

$v(a) = 0 \; and \; v(b) = 0$ (9)We can write (6) as $<\nabla J[u]; v> = \int_a^b \nabla J(u) v dx = \int_a^b v[\frac{\partial L}{\partial u}(x,u,u') - \frac{d}{dx}(\frac{\partial L}{\partial p}(x,u,u')]dx$

Since, this holds for all variations $v(x)$ we conclude that $\nabla J[u] = \frac{\partial L}{\partial u}(x,u,u') - \frac{d}{dx}(\frac{\partial L}{\partial p}(x,u,u')$ (10)This is our explicit formula for the functional gradient or variation derivative of the function (1) with Lagrange $L(x,u,p)$ where $p = u'$

The critical functions $u(x)$ are those for which the functional gradient vanishessatisfy$\nabla J[u] = \frac{\partial L}{\partial u}(x,u,u') = \frac{d}{dx}(\frac{\partial L}{\partial p}(x,u,u') = 0$ (11)

In view of (8) the critical equation (11) is the second order Ordinary differential equation

$$E(x,u,u',u'') = \frac{\partial y}{\partial x}(x,u,u') - \frac{\partial^2 L}{\partial x \partial p}(x,u,u') -$$

$u' \frac{\partial^2 L}{\partial u \partial p}(x,u,u') - u'' \frac{\partial^2 L}{\partial p^2}(x,u,u') = 0$ (12)

is known as Euler-Lagrange equation

CHAPTER THREE

Charpit's Method to Find the Complete Integral

3.1 Lagrange _ Charpit's Equation

Consider a first order partial differential equation with two independent variables

$F(x, y, u, p, q) = 0$ (1)

Where $p = \frac{\partial u}{\partial x}$ and $q = \frac{\partial u}{\partial y}$ and we assume that $F_p^2 + F_q^2 \neq 0$.The equationfor the characteristic strips for this equation are

$\frac{dx}{d\sigma} = F_p, \frac{dy}{d\sigma} = F_q, \frac{dp}{d\sigma} = -F_x - pF_u, \frac{dy}{d\sigma} = -F_y - qF_u$ and $\frac{du}{d\sigma} = pF_p + qF_q$

By eliminating parameter σ from these equations one often writes as

$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{du}{pF_p + qF_q} = \frac{dp}{-F_x - pF_u} = \frac{dq}{-F_y - qF_u}$ (2)

These equations are called Lagrange –Charpit's equation. In interpretation these equations, it is convenient to allow zero denominators. For example, if $F_p = 0$, these equations require that $dx = 0$, that is the denominator is zero just *means* that the numerator is also zero. *Assume* that from eq(1) and eq(2) we can derive a new equation

$\phi(x, y, u, p, q) = a$ (3)

$\mathrm{J} = det \frac{d(F,\phi)}{d(p,q)} = \begin{vmatrix} F_p & F_q \\ \phi_p & \phi_p \end{vmatrix} = F_p \phi_q - \phi_p F_q \neq 0$(4) this assumption implies that equation (1) and equation (3) can be solved locally for $p \, and \, q$. Let this solution be

$p = p(x, y, u, a)$and$q = q(x, y, u, a)$(5)

To simplify the notation, consider a fixed value of a and we write $p(x, y, u, a) = p(x, y, u)$and $q(x, y, u, a) = q(x, y, u)$ for now, then to get $u = u(x, y)$, we need to solve the system

$u_x = p(x, y, u)$, $u_y = q(x, y, u)$ (6) of partial differential equations.

By differentiating the first equation of (6) with respect to y we have

$$u_{xy} = p_y + p_u u_y = p_y + p_u q$$

Similarly, by differentiating the second equation of (6) with respect to x, we have

$u_{yx} = q_x + q_u u_x = q_x + q_u p$
The equality of these mixed derivatives lead to the requirement

$$p_y + p_u q = q_x + q_u p \quad (7)$$

Therefore, to make the above approach viable, we need to do two things: we need to show that condition (7) is satisfied and then to show that this condition is sufficient to solve the system of equation given in (6). Solving the above system is described as integrating the system and condition (7) is called integrality condition, and a system satisfying this condition is an integral system. The system of equations given in (6) can also be written as

$du = p(x, y, u)dx + q(x, y, u)dy$ (8) Here, du denotes the total differential of $u = u(x, y)$

Assuming that condition (7) is satisfied and a function $u = u(x, y)$ satisfy (6) exists (locally, that means on an open set containing the point (x_0, y_0)), given the value of u at a point (x_0, y_0) such a u can be determined as follows: Let $x = x(t), y = y(t)$ be differentiable functions such that $x(0) = x_0$ and $y(0) = y_0$ and write $u(t) = u\big(x(t), y(t)\big)$, then it follows from (6) (or equivalently, from (8)), with aid of chain rule that

$$u'(t) = p\big(x(t), y(t), u(t)\big)x'(t) + q\big(x(t), y(t), u(t)\big)y'(t) \qquad (9)$$

Solving this differential equation for u with initial condition $U(0) = u(x_0, y_0)$ we can, determine $u\big(x(t), y(t)\big) = u(t)$

Example: Find the solution of
$p^2u + q^2 - 4 = 0$ (10)
Solution: the Lagrange _ Charpit equation (2) for the above equation can be written as

$$\frac{dx}{2pu} = \frac{dy}{2q} = \frac{du}{2p^2u + 2q^2} = \frac{dp}{-p^3} = \frac{dq}{-p^2q}$$

The last (4th) equation can be written as

$$\frac{dp}{p} = \frac{dq}{q} \Rightarrow ln|p| = ln|q| + ln|c|$$

That is $q = ap$ with $a = \pm e^{-c}$, substituting this into equation (10) we obtain

$$p^2u + a^2p^2 - 4 = 0$$

$\Rightarrow \; p = \pm \frac{2}{\sqrt{u+a^2}}$

For the sake of simplicity, we will consider the + sign in the $\pm$ here with this equation together with the equation $q = ap$ obtained just before equation (8) can be written as

$$du = \frac{2}{\sqrt{u + a^2}}dx + \frac{2a}{\sqrt{u + a^2}}dy$$

Taking $x = \sigma$ and $y = \lambda\sigma$ with λ an arbitrary constant, equation (9) becomes

$$du = \frac{2}{\sqrt{u + a^2}}d\sigma + \frac{2a\lambda}{\sqrt{u + a^2}}d\sigma$$

$$\Rightarrow (u + a^2)^{\frac{1}{2}}du = 2(1 + a\lambda)d\sigma$$

Integrating both sides, we obtain

$$\frac{2}{3}(u + a^2)^{\frac{3}{2}} = 2(1 + a\lambda)\sigma + c'$$

$= 2\sigma + 2a\lambda\sigma + c'$ $=2x + 2ay + c'$
where c' is arbitrary constant, $= \sigma, \;\; y = \lambda\sigma$, Solving this for u we obtain

$u = (3x + 3ay + b)^{\frac{2}{3}} - a^2$, Where a and b are arbitrary constant ($b = \frac{3}{2}c'$)

3.2 Integrability

Condition (7) ensures that equation (6) or equivalently (8) are solvable. This is expressed by the following theorem

Theorem3.1: Let $(x_0, y_0, u_0) \in \mathcal{R}^3$ be a point, and assume that the functions p and q have continuous partial derivatives in a neighborhood of(x_0, y, u_0),assume that

$p_y + p_u q = q_x + q_u p$(11) in this neighborhood of (x_0, y_0, u_0), then there is unique function u in a neighborhood of (x_0, y_0) in $\mathcal{R}^2$ such that $u(x_0, y_0) = u_0$ and

$du = p(x, y, u)dx + q(x, y, u)dy$ (12) In a neighborhood of (x_0, y_0)

Proof: Consider the differential equation $u_x(x, y_0) = p(x, y_0, u(x, y_0)),\ u_y(x, y) = q\big(x, y, u(x, y)\big),$

$u_y(x_0, y_0) = u_0$(13)

It is clear that any function u satisfying eq(12) must also satisfy these equations. These later equations can be thought of ordinary differential equation. The first equation together with the initial condition given by the third equation is to describe $u(x, x_0)$ and the second equation for a fixed value of x given by the value of $u(x, y_0)$ determined by the first and the third equations as initial condition is to determine $u(x, y)$ for arbitrary (x, y).It follows from theory of ordinary differential equations that these equations uniquely determine $u(x, y)$ in a neighborhood of(x_0, y_0).

We need to show that the function so determined satisfies the requirements of the theorem The Picard-Lindelőf theorem states that the following:
Let (x_0, y_0) be a point and let f be a continuous function on a set $S = \{(x, y) \in \mathcal{R}^2 : |x - x_0| \le a, |y - y_0| \le b\}$ Where a and b are positive real numbers. Assume that there is a real number L such that$|f(x, y_1) -$

$f(x, y_2)| \leq L|y_2 - y_1|$(Such condition is called Lipschitz condition) for all points (x, y_1) and (x, y_2) in S.

Let $M = max\{|f(x, y)|: (x, y) \in \mathcal{R}^2\}$, then the differential equation $\frac{dg(x)}{dx} = f\big(x, g(x)\big)$, with initial condition $g(x_0) = y_0$ has a unique solution for g in the interval $(x_0 - k, x_0 + k)$ where $k = \min\,(a, \frac{b}{m})$

This theorem ensures the uniqueness and the existence of a function $u(x, y)$ satisfying eq(13) in a neighborhood of a point (x_0, y_0)

To complete the proof of the theorem we need to show that the function $u(x, y)$ so defined satisfies the first equation in (6); the second equation there is satisfied, since this equation is required in (13) used to define $u(x, y)$. Integrating equation (13) , we obtain

$$u(x,y) = u(x_0, y_0) + \int_{x_0}^{x} p(\xi, y_0, u(\xi, x_0))d\xi + \int_{y_0}^{y} q(x, \eta, u(x, \eta))d\eta$$

Differentiating this with respect to x, we obtain, by Fundamental theorem of calculus and interchangeability of differentiation and integration that

$$\begin{aligned} u_x(x,y) &= p\big(x, y_0, u(x, y_0)\big) + \int_{y_0}^{y} (q_x(x, \eta, u(x, \eta)) + u_x(x, \eta) q_u(x, \eta, u(x, \eta))d\eta \\ &= p\big(x, y_0, u(x, y_0)\big) + \int_{y_0}^{y} (q_x(x, \eta, u(x, \eta)) \\ &\quad + p(x, \eta, u(x, \eta)) q_u(x, \eta, u(x, \eta))d\eta \ + \int_{y_0}^{y} (u_x(x, \ \eta) \\ &\quad - p(x, \eta, u(x, \eta)) q_u(x, \eta, u(x, \eta))d\eta \end{aligned}$$

$$= \quad p\big(x, y_0, u(x, y_0)\big) + \textstyle\int_{y_0}^{y} (p_y(x, \eta, u(x, \eta)) + q(x, \eta, u(x, \eta)) p_u(x, \eta, u(x, \eta))d\eta$$

$$+ \ \int_{y_0}^{y} (u_x(x, \ \eta) - p(x, \eta, u(x, \eta)) q_u(x, \eta, u(x, \eta))d\eta$$

Where the last equation was obtained by (11) using the second equation in (13), we can write

$$u_x(x,y) = p\big(x, y_0, u(x,y_0)\big) + \int_{y_0}^{y} (p_y(x,\eta,u(x,\eta)) + u_y(x,\eta)\, p_u(x,\eta,u(x,\eta))d\eta$$

$$+ \int_{y_0}^{y} (u_x(x,\ \eta) - p(x,\eta,u(x,\eta))q_u(x,\eta,u(x,\eta))d\eta$$

$$= \ p\big(x_0, y_0, u(x,y_0)\big) + p\big(x,y,u(x,y)\big) - p(x,y_0,u(x,y_0)) \qquad +$$
$\int_{y_0}^{y}(p_y(x,\eta,u(x,\eta)) + u_y(x,\eta)\ p_u(x,\eta,u(x,\eta))d\eta$

$$= \ p\big(x,y,u(x,y)\big) + \int_{y_0}^{y}(u_x(x,\ \eta) - p(x,\eta,u(x,\eta))q_u(x,\eta,u(x,\eta))d\eta$$

Where, the second equation used the Newton-Leibniz formula to evaluate the definite integral. Therefore, we have$\left|u_x(x,y) - p\big(x,y,u(x,y)\big)\right| \le$

$\int_{y_0}^{y}\left|u_x(x,\eta) - p\big(x,\eta,u(x,\eta)\big)q_u\big(x,\eta,u(x,\eta)\big)\right| d\eta.$(14)Let u be a neighborhood of (x_0, y_0) in which the function $u(x,y)$ described in (13) is defined and in which the functions $u_x(x,y), p\big(x,y,u(x,y)\big), and\ p_u\big(x,y,u(x,y)\big)$ are bounded. Let M> 0 be such that

$M \ge |p_u(x,y,u(x,y))|$ For $(x,y) \subset U$

Let α with $0 < \alpha \le \frac{1}{2M}$ such that

$\{(x,y): |x - x_0| < \alpha\ , |y - y_0| < \alpha\} \subset U$ Let$B =$
$sup\{\left|u_x(x,y) - p\big(x,y,u(x,y)\big)\right|:$

$|x - x_0| < \alpha, |y - y_0| < \alpha\}$ (15)

Using (14), for (x,y) with $|x - x_0| < \alpha$ and $|y - y_0| < \alpha$ we obtain

$$\left|u_x(x,y) - p\big(x,y,u(x,y)\big)\right| \le BM|y - y_0| \le BM\delta \le B/2$$

This implies that

$$sup\{\left|u_x(x,y) - p\big(x,y,u(x,y)\big)\right|: |x - x_0| < \alpha, |y - y_0| < \alpha\} \le B/2$$

Contradicting (15) unless B=0. This shows the first equation in (6) is satisfied in a neighborhood of(x_0, y_0), the proof completed Type equation here.

3.3 Integrability of Charpit's Equations

In order to show that the integrability condition (7) to solve equation (6) (or equivalently (8)) is satisfied we need the following lemma.

Lemma: The function ϕ given in equation (3) satisfies the equation

$$F_p\phi_x + F_q\phi_y + (pF_p + qF_q)\phi_u - (F_x + pF_u)\phi_p -$$

$$(F_y + qF_u)\phi_q = 0 \qquad (16)$$

The above equation can also be written as

$$(F_p\phi_x - \phi_p F_x) + (F_q\phi_y - \phi_q F_y) + p(F_P\phi_u - \phi_p F_u) +$$

$$q(F_q\phi_u - \phi_q F_u)=0 \qquad (17)$$

So it clear that this equation is satisfied if F and ϕ are the same function. On the other hand equation(4)ensures that F and ϕ are not the same.

Proof: This equation means that the gradient vector $(\phi_x, \phi_u, \phi_p, \phi_q)$ is orthogonal to the tangent vector$(F_p, F_q, pF_p + qF_q, -F_x - pF_u, -F_u - qF_u)$ of the characteristic curves described by equation (2). As equation (3) is the consequence of equations (2) and (1), it must be compatible with characteristic curves. The gradient of the surface $\phi(x, y, u, p, q) = a$ must be orthogonal to the characteristic curves contained in it.

Now consider equations (1) and (3) with p and q as in (5). That is, suppressing the parameter a, we have

$$F(x, y, u, p(x, y, u), q(x, y, u)) = 0, \phi(x, y, u, p(x, y, u), q(x, y, u)) = 0$$

Differentiating these with respect to x, we obtain

$$F_x + F_p p_x + F_q q_x = 0,\ \phi_x + \phi_p p_x + \phi_q q_x = 0$$

Using the notation $\mathrm{J} = F_p\phi_q - F_q\phi_p$ *equation* (4)multiplying the first equation byϕ_p, the second by ϕ_q and subtracting the equation, *we obtain*

$$F_x\phi_p - \phi_x F_p - \mathrm{J}q_x = 0(18)$$

Similarly, differentiating the above equations with respect to y, we obtain

$F_y + F_p P_y + F_q q_y = 0, \phi_y + \phi_p P_y + \phi_q q_y = 0$

Multiply, the first equation by ϕ_q and the second by F_q, we obtain

$F_y \phi_q - \phi_y F_q + \mathrm{J} p_y = 0$ (19)

Next, differentiating the above equations with respect to u, we obtain

$F_u + F_p p_u + F_q q_u = 0, \phi_u + \phi_p p_u + \phi_q q_u = 0$ Multiply, the first equation by ϕ_p and the second equation by F_p, we obtain

$F_u \phi_p - \phi_u F_p - \mathrm{J} q_u = 0$(20)

Similarly, multiply the first equation by ϕ_q and the second one by F_q , we obtain

$F_u \phi_q - \phi_u F_q + \mathrm{J} p_u = 0$(21)

Adding equation (18) , (19), p times(20) and q times(21) to equation (17), we obtain

$(-q_x + p_y - p q_u + q p_u)\mathrm{J} = 0$

Noting that $\mathrm{J} \neq 0$ according to (4), equation (7) follows.

3.4 Charpit's Equation for Finding Complete Integral of the First Order Non-Linear Partial Differential Equation

Definition: A solution of anon-linear partial differential equation of the form

$f(x, y, u, p, q) = 0$ (1)

Where $p = \frac{\partial u}{\partial x}$, $q = \frac{\partial u}{\partial y}$ involving two arbitrary constants is called a complete integral, and it is the form

$F(x, y, u, a, b) = 0$ (2) A general method of finding such a complete integral involves a solution of Charpit's equation

$$\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{du}{pf_p+qf_q}=\frac{-dp}{f_x+pf_u}=\frac{-dq}{f_y+qf_u} \quad (3)$$

A solution of the form

$g(x,y,p,q,a)=0$ (4) Involving only one arbitrary constant a can be combined with the given non-linear partial differential equation (1) to obtain p and q in the form

$p=p(x,y,u,a), \quad q=q(x,y,u,a).$
Using these and the relation
$du=pdx+qdy$ leads to obtain a complete integral of the form
$F(x,y,u,a,b)=0$.this technique can be easily used in obtaining the complete integral for a non-linear partial differential equation of (1)

$u=px+qy+f(p,q)$
Consider a non-linear partial differential equation of the form

$u=px+qy+f(p,q)$ (5) Charpit's equation gives us

$$\frac{dx}{x+f_p}=\frac{dy}{y+f_q}=\frac{du}{px+qy+pf_p+qf_q}=\frac{dp}{0}=\frac{dq}{0}$$

Clearly, $p=\text{constant}=a$ is a solution part, this combined with (5) gives us
$q=g(u-ax,y,a), du=adx+g(u-ax,y,a)dy$ (6)

Now consider the function

$u=ax+by+f(a,b)$ (7)

Solving for b, we get

$b=g(u-ax,y,a)$ (8) Hence, from (7) and (8), we have $du=adx+bdy=adx+g(u-ax,y,a)dy$ this shows that (7) is the solution of (5). Further $q=a$ is also an integral of (5). By eliminating p and q we obtain $u=ax+by+f(a,b)$ which is the complete integral of the given non-linear partial differential equation. To get the complete integral of the given non-linear partial differential equation we should try to obtain a second solution, say $h(x,y,z,p,q,a,b)$of the Charpit's equations, then the elimination of p and q will give us the complete integral.

Theorem 3.2: *Let*

$f(x,y,u,p,q) = 0$ *(9)* *Be a first order partial differential equation, then there exist two solutions* $g(x,y,u,p,q,a) = 0$ *(10)* $h(x,y,u,p,q,b) = 0$*(11)* *of Charpit's equations, such that the elimination*

$F(x,y,z,a,b) = 0$*(12)* *of* p *and* q *from equations (9), (10) and (11) is the complete integral of equation(12).*

Proof: Let $g(x,y,p,q,a) = 0$ be an arbitrary solution of Charpit's equation, then from equation (9) and (10), we get
$p = p_1(x,y,u,a), q = q_1(x,y,u,a)$and $du = p_1 dx + q_1 dy$ is integrable with integral

$F_1(x,y,u,a) = 0$ (13)

Solving equation (13) for *a*, we get
$F_2(x,y,u,b) = a$ (14) then
$dF_2 = \frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy + \frac{\partial F_2}{\partial z} du = 0$ or $du = p_2(x,y,u,b)dx + q_2(x,y,u,b)dy$,
with $p_2 = \frac{\partial F_2}{\partial x} / \frac{\partial F_2}{\partial u}, q_2 = \frac{\partial F_2}{\partial y} / \frac{\partial F_2}{\partial u}$
The equation $f(x,y,u,p_2,q_2) = 0$ must be satisfied

The equation (14) is a complete integral of equation (9). Thus for arbitrary function ϕ,

$\phi(p - p_2, q - q_2) = 0 = h(x,y,p,q,b)$(15) is an equation of the required form (12). This follows from the fact that

$p = p_1(x,y,u,a) = p_2(x,y,u,b), q = q_1(x,y,u,a) = q_2(x,y,u,b)$

Satisfy this equation and also (9) and (10). This completes the proof.

Example: Solve the equation

$2py^2 - q^2 u = 0$ (16)

Solution: Charpit's equations for (16) are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{du}{pf_p + qf_q} = \frac{-dp}{f_x + pf_u} = \frac{-dq}{f_y + qf_u}$$

$\Rightarrow \frac{dx}{2y^2} = \frac{dy}{-2qu} = \frac{du}{2py^2-2q^2u} = \frac{-dp}{-pq^2} = \frac{-dq}{4py-q^3}$,

from which we obtain

$$\frac{du}{2py^2 - 2q^2u} = \frac{du}{-q^2u} = \frac{dp}{pq^2}$$

$2p = \text{constant} = \frac{1}{2}a^2$ (17) solving for q we get

$q = \frac{ay}{u}$, and so, $udu = \frac{1}{2}a^2dx + aydy.$

thus the complete integral is

$u^2 = a^2x + ay^2 + b$ (18) by solving the equation for a, we get

$a = \frac{-y^2+(y^4+4u^2x-4xb)^{1/2}}{2x}.$

Differentiating partially with respect to y, we get

$q = \frac{y(y^4+4u^2x-4xb)^{1/2}-y^3}{2xu}.$

Eliminating p and q from equations (16), (17) and (18) we get

$u^2 = a^2x + ay^2 + b$as a complete integral of (16)

Summary

A partial differential equation (DE) is a relation that involves partial derivative of an unknown function. Let an unknown function be u and independent variable-s $x \; and \; y$, thus, a partial DE is an equation of the form

$$F(x, y, u, p, q) = 0, p = \frac{du}{dx}, q = \frac{du}{dy}$$

Where, we assume that $F \in c^2(D_3)$with D_3as a domain in$\mathbb{R}^5$. With a general expression *F* in the PDE above, we canfind a solution *u* along a curve in (x, y)plane. such a curve was made by Lagrange and Charpit in an attempt to find a complete integral and by Cauchy in a geometric formulation with the help of Monge Cone.

Characteristic curves of a linear first order PDE form a one parameter family of curves in (x, y)-plane. Those for a quasi-linear equation form a two parameter family of curves. Finally those of a nonlinear form a three parameter family of curves.

For the nonlinear PDE, we need to solve the Charpit's equations$\frac{dx}{d\sigma} = F_p$, $\frac{dp}{d\sigma} = -F_x - pF_u$, $\frac{dq}{d\sigma} = -F_y - qF_u$, $\frac{du}{d\sigma} = pF_p + qF_q$ and for these we need initial values $p_0(\eta) \; and q_0(\eta)$ on a curve ϒ in addition to $x_0(\eta), y_0(\eta)$ and $u_0(\eta)$
Every solution $\big(x(\sigma), y(\sigma), u(\sigma), p(\sigma), q(\sigma)\big)$ of the Charpit's equations satisfies the strip condition$\frac{du}{d\sigma} = p(\sigma)\frac{dx}{d\sigma} + q(\sigma)\frac{dy}{d\sigma}$ on the curve $(x(\sigma), y(\sigma), u(\sigma))$ in (x, y, u)-space.

Theorem 2.1: The general solution of linear partial differential of first order $\mathrm{Pp} + \mathrm{Qq} = \mathrm{R}$, Where $\mathrm{p} = \frac{\partial \mathrm{u}}{\partial \mathrm{x}}, \mathrm{q} = \frac{\partial \mathrm{u}}{\partial \mathrm{y}}$, P, Q and R are functions of$\mathrm{x}$ and y, $\mathrm{F}(\varphi, \psi) = 0$, Where F is arbitrary function and $\varphi(\mathrm{x}, \mathrm{y}, \mathrm{u}) = \mathrm{c}_1$, $\psi(\mathrm{x}, \mathrm{y}, \mathrm{u}) = \mathrm{c}_2$ form a solution of the auxiliary system of equations
$\frac{\mathrm{dx}}{\mathrm{P}} = \frac{\mathrm{dy}}{\mathrm{Q}} = \frac{\mathrm{du}}{\mathrm{R}}$.

For a first order partial differential equation with two independent variables $F(x, y, u, p, q) = 0$, the characteristic strips for this equation are

$\frac{dx}{d\sigma} = F_p, \frac{dy}{d\sigma} = F_q, \frac{dp}{d\sigma} = -F_x - pF_u$, $\frac{dy}{d\sigma} = -F_y - qF_u$ and $\frac{du}{d\sigma} = pF_p + qF_q$
By eliminating parameter σ from these equations one often writes as

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{du}{pF_p + qF_q} = \frac{dp}{-F_x - pF_u} = \frac{dq}{-F_y - qF_u}$$

These equations are called Lagrange –Charpit's equation.

Theorem 3.1: Let $(x_0, y_0, u_0) \in \mathcal{R}^3$ be a point, and assume that the functions p and q have continuous partial derivatives in a neighborhood of(x_0, y, u_0), assume that

$p_y + p_u q = q_x + q_u p$,In this neighborhood of$(x_0, y_0, u_0)$, then there is unique function u in a neighborhood of (x_0, y_0) in $\mathcal{R}^2$ such that $u(x_0, y_0) = u_0$ and$du = p(x, y, u)dx + q(x, y, u)dy$, in a neighborhood of (x_0, y_0)

A solution of anon-linear partial differential equation of the form $f(x, y, u, p, q) = 0$, Where $p = \frac{\partial u}{\partial x}$, $q = \frac{\partial u}{\partial y}$involving two arbitrary constants is called a complete integral, and it is the form $F(x, y, u, a, b) = 0$

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