Addis Ababa University
College of Natural and Computational Sciences

ANALYSIS OF TWO-OPERATOR BOUNDARY-DOMAIN INTEGRAL EQUATIONS FOR VARIABLE COEFFICIENT BOUNDARY VALUE PROBLEMS IN 2D

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A thesis submitted to the Department of Mathematics

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This is to certify that the thesis prepared by Solomon Tesfaye Bekele entitled: **"*Analysis of two-operator Boundary-Domain Integral Equations for variable-coefficient boundary value problems in 2D*"** submitted in fulfilment of the requirements for the Degree of Doctor of Philosophy in Mathematics complies with the regulations of the University and meets the accepted standards with respect to originality and quality.

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to Elfie
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and Soli

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Abstract

In this dissertation, the boundary value problems (BVPs) for the second order elliptic partial differential equation with variable coefficient in two-dimensional bounded domain is considered. Using an appropriate parametrix (Levi function) and applying the two-operator approach (where the one operator approach fails), the problems are reduced to some systems of boundary-domain integral equations (BDIEs). The two-operator BDIEs in 2D need a special consideration due to their different equivalence properties as compared to higher dimensional case due to the logarithmic term in the parametrix for the associated partial differential equation. Consequently, we need to set conditions on the domain or function spaces to insure the invertibility of the corresponding layer potentials, and hence the unique solvability of BDIEs. Equivalence of the two-operator BDIE systems to the original BVPs, BDIEs solvability, uniqueness/non uniqueness of the solution, as well as Fredholm property and invertibility of the BDIEs are analysed. Moreover, the two-operator boundary domain integral operators for the Neumann BVP are not invertible, and appropriate finite-dimensional perturbations are constructed leading to invertibility of the perturbed operators.

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Notation and Abbreviations

$\Omega^+ = \Omega$	A bounded two-dimensional open set of $\mathbb{R}^2$.
$\partial\Omega$	The boundary of Ω.
$C^k(\Omega)$	The space of real valued functions having continuous derivatives up to order k on Ω
$C^\infty(\Omega)$	The set of all infinitely differentiable functions on Ω
α	Multi-index in $\mathbb{N}_0^2, \alpha = (\alpha_1, \alpha_2)$
$\lvert\alpha\rvert$	The length of multi-index α, $\lvert\alpha\rvert = \alpha_1 + \alpha_2$
x^α	The multi-index notation for the power of x, $x = x_1^{\alpha_1} x_2^{\alpha_2}$
$\overline{\Omega}$	$\Omega \cup \partial\Omega$
$\mathscr{D}(\Omega)$	The space of test functions on Ω.
$\mathscr{D}^{'}(\Omega)$	The space of all continuous linear functionals over $\mathscr{D}(\Omega)$.
$\mathscr{S}(\mathbb{R}^2)$	The space of rapidly decreasing test functions.
$\mathscr{S}^{'}(\mathbb{R}^2)$	The space of linear continuous functionals on $\mathscr{S}(\mathbb{R}^2)$.
r_{S_1}	restriction operator on a set S_1.
T_a^+, T_b^+	Co-normal derivative operator for the differential operators A, B.
n^+	A unit outward normal vector to Ω^+.
$\gamma^\pm$	The trace operator.
$W_p^k(\Omega)$	Sobolev space of order k
$H^s(\Omega)$	Bessel potential space or Sobolev-Slobodetskii space for $s \in \mathbb{R}$
$supp(\cdot)$	Support of a function
Δ	The Laplace's operator
∇	The gradient operator
δ	The Dirac delta distribution
$\mathscr{F}$	The Fourier transform operator
$\mathscr{F}(\mathbb{E}, \mathbb{F})$	The space of all Fredholm operators from Banach spaces $\mathbb{E}$ to $\mathbb{F}$.
$\mathscr{L}(\mathbb{E}, \mathbb{F})$	The space of bounded linear operators from $\mathbb{E}$ to $\mathbb{F}$.
$\mathscr{K}(\mathbb{E}, \mathbb{F})$	The space of all Compact operators from $\mathbb{E}$ to $\mathbb{F}$.
PDEs	Partial Differential Equations.
BVPs	Boundary Value Problems.

BIE	Boundary Integral Equation.
BDIEs	Boundary Domain Integral Equations.
BDIDEs	Boundary Domain Integro-differential Equations.

Chapter 1
Introduction and Background

Any phenomena in physical science and engineering can be modeled using mathematics. In some situations, this modeling leads to linear or non-linear differential, or integral equation or even integro-differential equations. These equations play a considerable role in understanding problems related to first Fluid mechanics that enables us to understand how a circulatory system works, how rockets and planes fly and even to some extent how the weather behaves. Secondly, heat and mass transfer used to understand how drug delivery devices work, how kidney dialysis works, and the process of controlling heat for temperature sensetive things. The other point related with elctromagnetism that is used for all electricity out there and everything that involves light at all form X-rays to pulse Oximetry and laser pointers.

Particularly, Partial Differential Equations (PDEs) with variable coefficients often arise in mathematical modeling of inhomogeneous media in solid mechanics, electro-magnetics, thermo conductivity, fluid flow through porous media, and other areas of physics and engineering. Through centuries, different methods of solving such problems have been developed. In any method, it is crucial to investigate the existence and uniqueness of solution, and the well-posedness of the problem or whether the solution depends continuously on the given data. In most engineering and science problems, it is impossible to find the analytical solution for a BVP. In a narrow scope, one can find the analytical solution for a given boundary value problem (BVP). In this case, one has to implement a suitable numerical method to obtain the approximate solution. Basically, there are different kinds of transformations that are efficient in analyzing and obtaining solution. One of a classical method to investigate the existence of solution of various BVPs comprises of reducing the solution of this problem to an integral equation or systems of integral equations. This method of solving a BVP by transforming to an integral equation or systems of integral equations is known as Boundary Integral Equation (BIE) method. Furthermore, the importance of integral equation methods in the solution, both theoretical and practical, of certain types of BVPs is universally recognized. The essential feature and the main advantage of such methods is that these methods often allow the problem to be reduced from one involving the whole of the domain of interest to one involving only its boundary so that the dimension of the problem is reduced by one.

The central idea of eliminating the field equations in the domain is to reduce BVPs to equivalent equations only on the boundary which requires the knowledge of corresponding fundamental solutions. This idea has a long history dating back to the work of Green and Gauss. Recently, the theory and the application of BIE is well studied, (see e.g., [Con00], [Ste07], [Cos88]). The main ingridient necessary for the reduction of the BVP to a BIE is a fundamental solution to the original PDE. Employing the fundamental solution in the corresponding Green formula one can reduce the problem to a BIE. This method, i.e., boundary integral equation method, is among the most powerful and elegant techniques developed by analysts for solving elliptic BVPs. Applied to a wide variety of situations in physical science and engineering, these procedures have great advantages in delivering the solution in closed form which is very helpful for numerical computation. The well known numerical methods for the approximation of BIE is the Boundary Element Method (BEM). This method requires discritizing only the boundary of the domain that enables one to consider unbounded or exterior problems.

Currently, the resulting of BIEs still serve as a major tool for the analysis and construction of solutions to BVPs. Eventhough the BIE method has many advantages in finding the solution of linear PDE, it also has limitations. The reduction of the BVP with an arbitrary variable coefficient to BIE is not usually possible since the fundamental solution necessary for such reduction is generally not available in analytical form except for some special dependence of the coefficients on coordinates. However, using a parametrix (Levi function) as a substitute of the fundamental solution in the Green's formula is possible to reduce such a BVP to Boundary-Domain Integral Equation (BDIE) or Boundary Domain Integro- Differential Equation (BDIDE), see e.g., [Mik02], [CMN09a],[DM15] and the references therein for the corresponding two and three-dimensional BVPs.

Recently, the theory of BDIE system in three-dimensions is well studied. For instance, in [Mik02],[Mik05b] localized parametrixes were implemented to reduce a BVP with variable coefficient to a localized BDIE or BDIDE. In [Mik06] analysis of united boundary-domain integral and integro-differential equations for a mixed BVPs are analyzed. In [CMN09a], the parametrix-based direct boundary-domain integral equations for mixed BVPs with variable coefficients and many more have been investigated. Besides, one-operator approach does not work when the fundamental solution of the frozen-coefficient PDE is not known explicitly (as e.g., in the Lam*é* system of anisotropic elasticity). To overcome this difficulty, one can apply the two-operator approach which is formulated in [AM10], [AM11] and also in [Mik05b] two-operator direct integro-differential formulated for a certain non-linear problem that employs a parametrix of another (second) PDE, not related with the PDE in question to reduce the BVP to a BDIE system. Since the second PDE is rather arbitrary, one can always chose it in such a way that its parametrix is known explicitly. The simplest choice for the second PDE is the one with an explicit fundamental solution.

Furthermore, in the works of [DM15], [ADM17] BDIEs for variable cofficient BVP in two dimensional domain is researched. In these works, the equivalence of the BDIEs to the respective original BVP has been proved and the invertibility of the corresponding operators has been shown in appropriate Sobolev spaces.

In connection with the above research findings, one may ask like the problem raised in [AM10], [AM11] which are studied for three dimensional case where the one operator approach fails to solve and the two-operator approach is efficient in handling them. As the authors in [DM15] breifly indicated and in [Con00, preface] remarked, the importance of studying the construction of the BIE for 2D is independent of the higher dimensions. In the two dimensional case, the fundamental solution for the formulation of BIE or the parametrix (Levi function) in the formulation of the BDIEs contains Logarithmic function. As a result of this, the corresponding potentials have singularities in the domain as well as in the far field. Therefore, the two dimensional specially for the cases where the fundamental solution of the frozen-coefficient PDE is not known explicitly and where the one operator approach fails to work, the problem should be considered independently. Generally, real materials are never perfectly isotropic. There are many materials which can be well modelled using the linear elastic model, that may not be nearly isotropic like wood, composite materials and many biological and other physical materials. The mechanical properties of these materials differ in different directions. Materials with this direction dependence are called Anisotropic. Due to this facts, non-linear PDEs arise naturally in mathematical modeling of non-linear physical processes, e.g., non-linear heat transfer in material with the thermo-conductivity coefficient depending on the point temperature and coordinate, materials with damaged induced inhomogeneneity, elasto-plastic matarials, non-linear equation of stationary potential compressible flow, non-linear flow through porous media, non-linear electromagnetics and other areas of physics and engineering.

Among variety of the above physical problems that can be solved using the method of integral equation, the elasticity problem is typical. By using reasonable assumptions, we can bring down the complexity in the problem in 2D. From the 2D theory of elasticity, the field variables, here is $\phi = \phi(x, y, t)$. Usually, the mathematical model of elasticity can be divided into two parts. One is the basic equations for linear anisotropic elasticity that are the strain-displacement equation, the stress-strain equation and the equation of equilibrium, which can be expressed in a fixed rectangular system $x_i, i = 1, 2$. The other is the boundary condition which can be distinguished into traction-boundary (Neumann boundary), displacement-boundary (Dirichlet boundary) and mixed-boondary value problems. Once a problem is formulated based upon the basic equation, its solvability is usually dependent on the boundary condition. Where the field equations can be catagorized as:

i. The strain-displacement equation

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{1.1}$$

ii. Compatibility equations

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0 \tag{1.2}$$

The constitutive law is $\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{ij}\delta_{ij}$ we derive an equivalent expression for the compatibility equation (1.2) by eliminating the two displacement in the three strain-displacement relation to obtain

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

by using the constitutive law, i.e.,

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu}{E}(\sigma_{yy} + \sigma_{zz}) \quad \varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{zz}) \quad \varepsilon_{zz} = 0 = \frac{\sigma_{zz}}{E} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{yy})$$

Thus, $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$ which implies

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}$$

where $E > 0$ is the Young Modulus and $\nu \in (0, \frac{1}{2})$ describes the Poisson ratio.

iii. Equilibrium equations

$$\sigma_{ij,j} + B_i = 0 \tag{1.3}$$

From the constitutive law

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

differentiating it with respect to x_j

$$\sigma_{ij,j} = \lambda \varepsilon_{kk,j} \delta_{ij} + 2\mu \varepsilon_{ij,j}$$

putting this in equation (1.3)

$$\sigma_{ij,j} + B_i = \lambda \varepsilon_{kk,j} \delta_{ij} + 2\mu \varepsilon_{ij,j} + B_i = 0 \tag{1.4}$$

express all strain components in (1.4) in terms of displacements and write the equivalent form of (1.3) as

$$(\lambda + \mu) u_{j,jj} + \mu u_{i,jj} + B_i = 0$$

where λ, μ are Lam$\acute{e}$'s coefficients (or λ-volume viscosity and μ-shear viscosity coefficients).

where u_i, σ_{ij} and ε_{ij} are respectively the displacement, stress and strain; the repeated indices imply summation, a comma stands for differentiation. In 3D domain, there are 15 equations , i.e., three equilibrium (1.3), six strain displacement (1.1) and six constitutive laws (1.2), where there are 3 displacement, 6 strain components and 6 stress components are unknown, where as in 2D elasticity problem there are 8 equations and 8 unknowns. Here, mathematically there are two convinient ways to reformulate the elasticity problems; the displacement based or the stress based formulation. In the displacement based formulation, we express the field equation in terms of displacement, u_i which is the Navier equation. In this formulation the primary variable is displacement and the solution is obtained for displacement; stress and strain are secondary / derived variables, thus we can solve for strain by derivation of displacement and for stress by using the constitutive law. In the stress based formulation, we express the field equation in terms of stress, σ_{ij}. In this formulation the primary variable is stress and the solution is obtained for stress; strain and displacement are secondary / derived variables, thus we can solve for strain using the compatibility equation and the governing eqution is expressed in displacement. We can cast the BVP as

$$\tau \left(\underbrace{u_i, \varepsilon_{ij}, \sigma_{ij},}_{\text{unknowns}} \quad \underbrace{\lambda, \mu}_{Lame's\ constants} \quad , \quad \underbrace{B_i}_{Body\ load} \right) = 0$$

According to [Sol13], the stress tensor

$$\sigma = \begin{pmatrix} \sigma_1 & \sigma_3 \\ \sigma_3 & \sigma_2 \end{pmatrix}$$

of plane elasticity medium is connected with a displacement vector $u = (u_1, u_2)^t$ by Hooke's law

$$\begin{pmatrix} \sigma_1 \\ \sigma_3 \end{pmatrix} = a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y} \qquad \begin{pmatrix} \sigma_3 \\ \sigma_2 \end{pmatrix} = a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y} \tag{1.5}$$

The coefficient $a_{ij} \in \mathbb{R}^{2\times 2}$ are defined by

$$a_{11} = \begin{pmatrix} \alpha_1 & \alpha_6 \\ \alpha_6 & \alpha_3 \end{pmatrix} \qquad a_{12} = \begin{pmatrix} \alpha_6 & \alpha_4 \\ \alpha_3 & \alpha_5 \end{pmatrix}$$
$$a_{21} = \begin{pmatrix} \alpha_6 & \alpha_3 \\ \alpha_4 & \alpha_5 \end{pmatrix} \qquad a_{22} = \begin{pmatrix} \alpha_3 & \alpha_5 \\ \alpha_5 & \alpha_2 \end{pmatrix}$$

where modulus elasticity α_j from positively defined matrix

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_4 & \alpha_6 \\ \alpha_4 & \alpha_2 & \alpha_5 \\ \alpha_6 & \alpha_5 & \alpha_3 \end{pmatrix}$$

by Silvester criterium we have

$$\alpha_j > 0, j = 1,2,3 \qquad \alpha_1\alpha_2 > \alpha_4^2$$
$$\alpha_1\alpha_2\alpha_3 + 2\alpha_4\alpha_5\alpha_6 > \alpha_1\alpha_5^2 + \alpha_2\alpha_6^2 + \alpha_3\alpha_4^2$$

The elastic medium is orthotropic if $\alpha_5 = \alpha_6 = 0$ and isotropic if $\alpha_5 = \alpha_6 = 0, \alpha_1 = \alpha_2 = 2\alpha_3 + \alpha_4$.
The stress tensor satisfies the equilibrium equation

$$\frac{\partial}{\partial x}\begin{pmatrix} \sigma_1 \\ \sigma_3 \end{pmatrix} + \frac{\partial}{\partial y}\begin{pmatrix} \sigma_3 \\ \sigma_2 \end{pmatrix} = 0$$

together with (1.5) it yields the Lamé system

$$a_{11}\frac{\partial^2 u}{\partial x^2} + (a_{12} + a_{21})\frac{\partial^2 u}{\partial y^2} + a_{22}\frac{\partial^2 u}{\partial y^2} = 0$$

for the displacement vector $u = (u_1, u_2)^t$

Assume that $\mathscr{B}$ is anisotropic elastic body and the reference configuration of $\mathscr{B}$ is Ω, a bounded open connected set in $\mathbb{R}^2$. Let $C(x) = \big(C_{ijkl}(x)\big)$ be the elastic tensor where $i,j,k,l = 1,2$. However, for simplicity of the expression we assumed that the elastic tensor C satisfies the full symmetric properties.

$$C_{ijkl} = C_{jilk} = C_{klij}, \qquad \forall i,j,k,l$$

The displacement equation of the equilibrium where $\mathscr{B}$ is body force is given by

$$\mathscr{L}_C u = \nabla \cdot (C(x)\nabla u) + \mathscr{B} = 0 \quad \text{in} \quad \Omega$$

i.e.,

$$\mathscr{L}_C u = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(C\frac{\partial u}{\partial x_i}\right) + \mathscr{B} = 0 \quad \text{in} \quad \Omega \tag{1.6}$$

where $(\nabla u)_{kl} = \partial_l u_k$ and $(\nabla \cdot G)_i = \sum_{j=1}^{2} \partial_j g_{ij}$ for any matrix function $G = (g_{ij})$. Also we have used the convention notation

$$(CH)_{ij} = \sum_{kl} C_{ijkl} h_{ij}$$

where $H = (h_{ij})$ is a 2×2 matrix. Here $u = (u_1, u_2)^t$ is the displacement vector.

Here is an outline of the content.

In Chapter 2, some concepts that are intended to make the reader familiar with some terminologies and analytical tools to implement the integral equation method in variational setting are focused. In this chapter, definitions of important notions, examples, Theorems, Lemmas and Corollaries (some with their proofs) are incorporated. To mention few, the distribution theory, infinitely differentiable functions with compact support $C_0^\infty = \mathscr{D}$, the space of linear continous functionals on $\mathscr{D}$, $\mathscr{D}'$, the space of rapidly decreasing functions $\mathscr{S}$, and the space of linear continous functionals on $\mathscr{S}$, $\mathscr{S}'$, Fourier transform and convolution. Moreover, the Sobolev space $W_p^s(\Omega)$ based on $L_p(\Omega)$ for $1 \le p \le \infty$ is defined, but soon focus almost exclusively on the case $p = 2$. The space $H^s(\mathbb{R}^2)$, which concides with $W_2^s(\mathbb{R}^2) = W^s(\mathbb{R}^2)$, is then defined via the Fourier transform in the usual way, and after that the spaces $H^s(\Omega)$ and $\tilde{H}^s(\Omega)$ for a general open set $\Omega \subset \mathbb{R}^2$ are introduced. In addition, the construction of fundamental solution and parametrix (Levi function) for the considered differential equation is presented. Furthermore, the theory of compact operators, Fredholm operator and perturbation theory for Fredholm operator are also discussed.

In chapter 3, the PDE of the form (1.6) in bounded 2D domain where the right hand side is from $L_2(\Omega)$ is considered. The second section, 3.2, is given for the Dirichlet BVP, where the displacement boundary is prescribed, i.e., $\gamma^+ u = \varphi_0$ on

$\partial\Omega$. On the third section, 3.3, the Neumann BVP where the traction boundary is known, i.e., $T_a^+ u = \psi_0$ on $\partial\Omega$ is investigated. Finally, on the fourth section, 3.4, the mixed (Dirichlet-Neumann) BVP where the displacement boundary is given on one part and the traction boundary is on the other; that is $\gamma^+ u = \varphi_0$ on $\partial_D\Omega$ and $T_a^+ u = \psi_0$ on $\partial_N\Omega$ is presented. Each BVP is transformed to different systems of two-operator BDIEs in appropraite Sobolev space and analyzed.

The single layer potential operator mapping from $H^{-\frac{1}{2}}(\partial\Omega)$ to $H^{\frac{1}{2}}(\partial\Omega)$ is invertible in three dimensional [CMN09a]; whereas in the two dimensional case, the homogenuous integral equation has non trivial solution which imply it is not invertible. But it is Fredholm operator of index zero [McL00]. Hence, to obtain invertibility, we have considered a subspace of $H^{-\frac{1}{2}}(\partial\Omega)$ whose element g satisfies $\langle g, 1\rangle_{\partial\Omega} = 0$. Alternatively, using the Logarithmic capacity idea, we have restricted the diameter of the domain to be less than r_0, where $r_0 > 0$.

The Hypersingular operator, i.e.,the co-normal derivative of the double layer operator, whose kernel consists a non-zero function for which we have considered appropriate space to obtain its invetibility. Moreover, for the Neumann BVP, the BDIE systems are neither uniquely nor unconditionally solvable. In similar fashion with [Mik17] and [ADM17], in which the right hand side function is from $\tilde{H}^{-1}(\Omega)$, the range of these operators are described or the bases of the co-kernel of the operators are given, and following [Mik99] appropriate finite-dimensional perturbation are constructed leading to invertability of the operators.

Chapter 2
Preliminaries

In the following chapter we cover the basic definition and properties of some concepts of which we use in the dissertation following references, e.g.,[Vlad79], [AF03], [McL00], [Mir70], [Tri08]. Our work is in two dimensional space, thus we focus on the theories in $\mathbb{R}^2$.

Let Ω^+ be a bounded open two-dimensional region of $\mathbb{R}^2$ and $\Omega^- = \mathbb{R}^2 \backslash \overline{\Omega^+}$. For simplicity, we assume that the boundary $\partial\Omega = \partial\Omega^+$ is a closed, infinitely smooth curve. Moreover, $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega$ where $\partial_D\Omega$ and $\partial_N\Omega$ are open, non-empty, non-interesting curve of $\partial\Omega$. Let also $\partial_j = \partial_{x_j} := \frac{\partial}{\partial x_j}$ for $j = 1,2$. A vector of the form $\alpha = (\alpha_1, \alpha_2)$, where each component α_i is a non-negative integer, is called a multi-index of order

$$|\alpha| = \alpha_1 + \alpha_2$$

Given a multi-index α, we define

$$\partial^\alpha u = D^\alpha u(x) = \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} u(x), \qquad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \quad \text{where} \quad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$$

2.1 The space of test functions $\mathscr{D}(\Omega)$

For an open bounded domain $\Omega \subset \mathbb{R}^2$ and $k \in \mathbb{N}_0$ we denote the space $C^k(\Omega)$, the space of $k-$ times continuously differentiable functions equipped with the norm

$$\|f\|_{C^k(\Omega)} = \sum_{|\alpha| \le k} \sup_{x \in \overline{\Omega}} |\partial^\alpha f(x)|$$

$C^\infty(\Omega)$ is the collection of all functions infinitely differentiable in Ω . That is: for any integer $k \geq 0$, we let

$$C^k(\Omega) = \{u : \partial^\alpha u \quad \text{exists and is continuous on} \quad \Omega \quad \text{for all } |\alpha| \leq k\},$$

and we put $C^\infty(\Omega) = \bigcap_{k\geq 0} C^k(\Omega)$

which is the collection of all functions infinitely differentiable in Ω.

The support of a function u continuous in Ω, denoted by *supp u*, is the closure of the set where u does not vanish; $\overline{\{x \in \Omega : u(x) \neq 0\}}$.

$C_0^\infty(\Omega)$ the space of infinite times continuously differentiable functions with compact support.

$$C_0^\infty(\Omega) = \{u \in C^\infty(\Omega) : supp\ u \subset \Omega\}$$

Definition 2.1. The set of test functions $\mathscr{D}(\Omega)$ is the collection of all infinitely differentiable functions in Ω with compact support. The set $\mathscr{D}(\Omega)$ is also denoted by $C_0^\infty(\Omega)$

$$\mathscr{D}(\Omega) = \{\varphi \in C^\infty(\Omega) : supp(\varphi) \text{is} \quad \text{compact in} \quad \Omega.\}$$

Example 2.1. For $x \in \mathbb{R}^2$, let

$$\varphi(x) = \begin{cases} e^{\frac{-1}{1-|x|^2}} & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$$

We claim that, for any multi-indix $\alpha, \varphi^\alpha(x) = \frac{P_\alpha(x)\varphi(x)}{(1-|x|^2)^{|\alpha|}}$ for some polynomial P_α. For $|x| < 1$, we can differentiate and determine inductively in α, that $\varphi^\alpha(x) = P_\alpha(x)e^{-t}t^{|\alpha|}$ for some polynomial P_α, where $t = \frac{1}{(1-|x|^2)^{|\alpha|}}$. Further, $\varphi^\alpha(x) = 0$ for $|x| > 1$. Thus, the formula above for φ^α is verfied in the case $|x| \neq 1$. Since the exponential increases faster than any finite power, $\frac{P_\alpha(x)\varphi(x)}{(1-|x|^2)^k} = \frac{P_\alpha(x)t^{|\alpha|+k}}{e^t} \to 0$ as $|x| \to 1$ (i.e., as $t \to \infty$). Apply (inductively) this facts with $k = 0$ shows φ^α is continuous at $|x| = 1$, and using $k = 1$ shows it is also differentiable there, and has derivative zero. Thus, the claimed formula holds for all x. Moreover, we also see from the argument that φ^α is bounded and continuous for all α. Thus, $\varphi \in \mathscr{D}(\Omega)$ for any open set Ω containing the closed unit disc. Consequently φ is infinitely differentiable and one can finds that its n^{th} derivative has the form $\varphi^{(n)}(x) = p_n\left(x, \frac{1}{1-x^2}\right)\varphi(x)$ for some polynomial $p_n(s,t)$ and therefore $\varphi \in C_0^\infty(\mathbb{R}^2)$ with compact support. The support of φ is a closed unit disc, i.e.,
$supp\ \varphi = \{(x,y) : x^2 + y^2 \leq 1\}$ and it is infinitely differentiable.

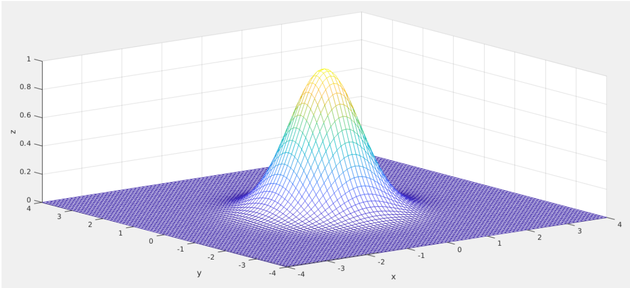

Fig. 2.1: The graph of the test function $\varphi(x)$.

Remark 2.1.1 *i. If $\varphi_1(x)$ and $\varphi_2(x)$ are test functions on Ω, so is $C_1\varphi_1(x)+C_2\varphi_2(x)$ for any $C_1, C_2 \in \mathbb{R}$. Hence, the space $\mathscr{D}(\Omega)$ is a real linear space.*
ii. If $a \in C^\infty(\Omega)$ and $\varphi \in \mathscr{D}(\Omega)$, then $a\varphi \in \mathscr{D}(\Omega)$

Convergence in $\mathscr{D}(\Omega)$: A sequence $\{\varphi_n\}_{n=1}^{\infty}$ in $\mathscr{D}(\Omega)$ is said to be convergent in $\mathscr{D}(\Omega)$ to φ in $\mathscr{D}(\Omega)$, if there is a compact subset G of Ω with *supp* $(\varphi_n) \subset G, \quad n \in \mathbb{N}$ and $\partial^\alpha \varphi_n \to \partial^\alpha \varphi$ uniformly on G, $\forall \alpha \in \mathbb{N}_0^2$ i.e., uniform convergence for all derivatives, $\|\varphi_n - \varphi\|_{C^m(\Omega)} \to 0$ if $n \to \infty, \quad \forall m \in \mathbb{N}_0$.

Lipchitz continuous functions and Lipschitz domain

For $k \in \mathbb{N}_0; \kappa \in (0,1]$ we define the space $C^{k,\kappa}(\Omega)$ of Hölder continuous functions equipped with the norm

$$\|u\|_{C^{k,\kappa}(\Omega)} = \|u\|_{C^k(\Omega)} + \sum_{|\alpha|=k} \sup_{x,y\in\Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\kappa}$$

In particular, for $k=0,$ and $\kappa = 1$ we obtain the space $C^{0,1}(\Omega)$ of Lipchitz continuous functions.

We assume that the boundary $\partial\Omega$ can be represented by a certain decomposition

$$\partial\Omega = \bigcup_{i=1}^{p} S_i \tag{2.1}$$

where each boundary segment S_i is described via a local parametrization,

$$S_i = \left\{x \in \mathbb{R}^2 : x = \chi_i(\xi) \ \text{ for } \ \xi \in \mathscr{T}_i \subset \mathbb{R}\right\} \tag{2.2}$$

with respect to some parameter.

A domain Ω is said to be a Lipschitz domain, when all functions χ_i in (2.2) are Lipschitz continuous for any arbitrary decomposition (2.1).

2.2 The space of distributions $\mathscr{D}'(\Omega)$

Definition 2.2. Let Ω be a domain in $\mathbb{R}^2$ and $\mathscr{D}(\Omega)$ be the space of test functions. The set of distributions(or generalized functions) is a collection of all complex-valued linear continuous functionals u over $\mathscr{D}(\Omega)$.

A distribution specified on an open set Ω is any continuous linear functional on the space of test functions $\mathscr{D}(\Omega)$. We can interpret the definition of the distribution f as follows:

1. A distribution f is a functional over $\mathscr{D}(\Omega)$, that is, with each $\varphi \in \mathscr{D}(\Omega)$ there is associated (complex-valued) number
$$\langle f, \varphi \rangle = f(\varphi)$$
2. A distribution f is a linear functional over $\mathscr{D}(\Omega)$, that is, if $\psi, \varphi \in \mathscr{D}(\Omega)$ and λ, μ are complex numbers, then
$$\langle f, \lambda\varphi + \mu\psi \rangle = \lambda \langle f, \varphi \rangle + \mu \langle f, \psi \rangle$$
3. A distribution f is a continuous functional over $\mathscr{D}(\Omega)$, that is, if $\varphi_k \to \varphi$ as $k \to \infty$ in $\mathscr{D}(\Omega)$, then
$$\langle f, \varphi_k \rangle \to \langle f, \varphi \rangle, \quad k \to \infty$$

We denote by $\mathscr{D}'(\Omega)$ the set of all distribution specified in Ω.

Convergence in $\mathscr{D}'(\Omega)$: A sequence of distributions $\{u_n\}$ is said to be convergent to u in $\mathscr{D}'(\Omega)$, written as $u_n \to u$ in $\mathscr{D}'(\Omega)$, if $\langle u_n, \varphi \rangle \to \langle u, \varphi \rangle$ in $\mathbb{C}$ as $n \to \infty, \quad \forall \varphi \in \mathscr{D}(\Omega)$.

Distributions, which are definable in terms of functions locally integrable are said to be regular distributions. The remaining generalized functions are called singular distributions.

For example, the Dirac delta distribution δ, which is defined by $\langle \delta, \varphi \rangle = \varphi(0), \quad \forall \varphi \in \mathscr{D}(\Omega)$ is singular distribution. The Dirac delta distribution cannot be evaluated at points, it makes sense to say that it vanishes except at origin. Thus, $supp(\delta) = \{0\}$.

Remark 2.2.1 • *Locally integrable functions are a subset of distributions, $L^1_{loc}(\Omega) \subset \mathscr{D}'(\Omega)$ and hence $\mathscr{D}(\Omega) \subset \mathscr{D}'(\Omega)$ since every test function is integrable.*

- *For a distribution f, a multiplication by $a \in C^\infty(\Omega)$ is defined by $\langle af, \varphi\rangle = \langle f, a\varphi\rangle$ for all $\varphi \in \mathscr{D}(\Omega)$.*
- *Multiplication of two arbitrary distributions is not possible. For instance,*

$$0 = 0 \cdot \mathscr{P}\frac{1}{x} = (x\delta(x))\mathscr{P}\frac{1}{x} = (\delta(x)x)\mathscr{P}\frac{1}{x} = \delta(x)\left(x\mathscr{P}\frac{1}{x}\right) = \delta(x)$$

Definition 2.3. We say that a distribution $u \in \mathscr{D}'(\Omega)$ vanishes on an open set $G \subset \Omega$ if $\langle u, \varphi\rangle = 0$ for all $\varphi \in \mathscr{D}(G)$. Two distributions are equal on G if their difference vanishes on G. The complement of a largest open set on which u vanishes in Ω is called the support of u.

Definition 2.4. Let $u \in C^k(\Omega)$, where k is a positive integer. Then for multi-index α, $|\alpha| \le k$ and $\varphi \in \mathscr{D}(\Omega)$, we have the following integration by parts formula:

$$\langle D^\alpha u, \varphi\rangle = \int D^\alpha u(x)\varphi(x)dx = (-1)^{|\alpha|}\int u(x)D^\alpha\varphi(x)dx = (-1)^{|\alpha|}\langle u, D^\alpha\varphi\rangle$$

For example, let $f \in \mathscr{D}'(\Omega)$. Then the derivative of f with respect to x_j, $j = 1,2$ is defined as:

$$\langle\frac{\partial f}{\partial x_j}, \varphi\rangle = -\langle f, \frac{\partial \varphi}{\partial x_j}\rangle \qquad \text{for all} \quad \mathscr{D}(\Omega)$$

This motivates the definition of the derivative $D^\alpha u$ of the distribution $u \in \mathscr{D}'(\Omega)$:

$$\langle D^\alpha u, \varphi\rangle = (-1)^{|\alpha|}\langle u, D^\alpha\varphi\rangle \tag{2.3}$$

Since $u \in \mathscr{D}'(\Omega)$ the functional $D^\alpha u$, defined on the right hand side of equation (2.3), is linear and continuous :

$$\langle D^\alpha u, \varphi_k\rangle = (-1)^{|\alpha|}\langle u, D^\alpha\varphi_k\rangle \to (-1)^{|\alpha|}\langle u, D^\alpha\varphi\rangle = \langle D^\alpha u, \varphi\rangle$$

for, if $\varphi_k \to \varphi$ as $k \to \infty$ in $\mathscr{D}$, then also $\partial^\alpha\varphi_k \to \partial^\alpha\varphi$ as $k \to \infty$ in $\mathscr{D}$. Therefore, $D^\alpha u \in \mathscr{D}'$.

Example 2.2. If $0 \in \Omega$ and $\delta \in \mathscr{D}'(\Omega)$ is the Dirac distribution, then $D^\alpha\delta$ given by

$$\langle D^\alpha\delta, \varphi\rangle = (-1)^{|\alpha|}\langle\delta, D^\alpha\varphi\rangle = D^\alpha\varphi(0).$$

Furthermore, let T be a distribution. Let G be an open subset of $\mathbb{R}^2$ such that $\langle T, \phi\rangle = 0 \quad \forall\phi \in \mathscr{D}$ with $Supp\phi \subset G$. Then we say that T vanishes on G. Let U

be the union of such open sets G. Then U is again open, and T vanishes on U. Thus U is the largest open set on which T vanishes. Therefore we can now give the following definition.

Definition 2.5. The support of a distribution T on $\mathbb{R}^2$, denoted by $SuppT$, is the complement of the largest open set in $\mathbb{R}^2$ on which T vanishes.

Clearly, $SuppT$ is a closed subset of $\mathbb{R}^2$.

2.3 The space $\mathscr{S}(\mathbb{R}^2)$ and $\mathscr{S}'(\mathbb{R}^2)$

One of the most effective means of solving problems in science and engineering is the transform method. Among variety of transform methods the Fourier transform is well defined on the spaces $\mathscr{S}(\mathbb{R}^2), \mathscr{S}'(\mathbb{R}^2)$ and also, the result belongs to these spaces, which is not practical on the space of test functions $\mathscr{D}(\Omega)$. The investigation on the BDIEs in appropraite Sobolev spaces and one way of defining the Sobolev space, $H^s(\Omega)$, is based on Fourier transform of distributions. Hence, first we define the following spaces.

Definition 2.6. The space of basic functions $\mathscr{S}(\mathbb{R}^2)$ is the collection of all functions infinitely differentiable in $\mathbb{R}^2$ that decrease together with all their derivatives, as $|x| \to \infty$, faster than any power of $|x|^{-1}$.

Or equivalently, the space $\mathscr{S}(\mathbb{R}^2)$ is defined by:

$$\mathscr{S}(\mathbb{R}^2) = \left\{ \varphi \in C^\infty(\mathbb{R}^2) : \sup_{x \in \mathbb{R}^2} |x^\alpha \partial^\beta \varphi(x)| < \infty \quad \text{for all} \quad \text{multi-indices} \quad \alpha \ \& \ \beta. \right\}$$

We introduce the norm in $\mathscr{S}(\mathbb{R}^2)$ via the formula

$$\|\varphi\|_p = \sup_{|\alpha| \le p} \left(1 + |x|^2\right)^{\frac{p}{2}} |D^\alpha \varphi(x)|, \quad \varphi \in \mathscr{S}(\mathbb{R}^2), \quad p = 0, 1, 2, \cdots$$

Clearly, $\|\varphi\|_0 \le \|\varphi\|_1 \le \|\varphi\|_2 \le \cdots$
The space $\mathscr{S}(\mathbb{R}^2)$ is usually called the Schwartz space of all rapidly decreasing infinitely differentiable functions. The space $\mathscr{S}(\mathbb{R}^2)$ is a larger class of functions than the space $\mathscr{D}(\mathbb{R}^2)$, i.e., $\mathscr{D}(\mathbb{R}^2) \subset \mathscr{S}(\mathbb{R}^2)$ ($\mathscr{S}(\mathbb{R}^2)$ does not coincides with $\mathscr{D}(\mathbb{R}^2)$). A typical example for this is, the function $\phi(x) = e^{-|x|^2}$ does not have compact support so it is not in $\mathscr{D}(\mathbb{R}^2)$ but belongs $\mathscr{S}(\mathbb{R}^2)$.

Convergence in $\mathscr{S}(\mathbb{R}^2)$: A sequence of basic functions $\{\phi_n\}$ in $\mathscr{S}(\mathbb{R}^2)$ is said to converge to zero in $\mathscr{S}(\mathbb{R}^2)$ if for all multi-indices α, β

$$x^\alpha \partial^\beta \phi_n(x) \to 0 \quad \text{uniformly for} \quad x \in \mathbb{R}^2.$$

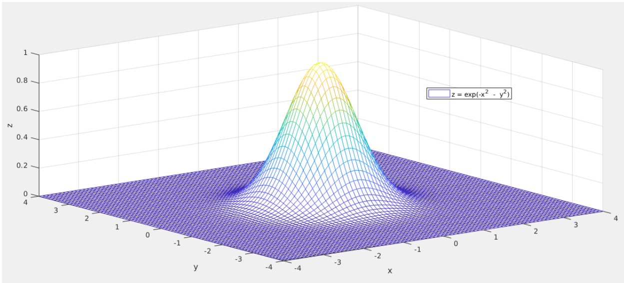

Fig. 2.2: The graph of the function $\phi(x) = e^{-|x|^2}$.

Generally, a sequence functions $\{\phi_n\}$ in $\mathscr{S}(\mathbb{R}^2)$ is said to converge to ϕ in $\mathscr{S}(\mathbb{R}^2)$ if $\phi_n - \phi$ converges to 0 in $\mathscr{S}(\mathbb{R}^2)$. Since the basic functions from $\mathscr{S}$ are locally summable in $\mathbb{R}^2$, the operation of Fourier Transform $\mathscr{F}(\varphi)$ and $\mathscr{F}^{-1}(\varphi)$ are defined in $\mathscr{S}(\mathbb{R}^2)$.

Definition 2.7. Let $\varphi(x) \in \mathscr{S}(\mathbb{R}^2)$. Then the integral transformation

$$\mathscr{F}[\varphi](\xi) = \mathscr{F}_{x\to\xi}[\varphi(x)] := \int_{\mathbb{R}^2} \varphi(x) e^{-i2\pi x\cdot\xi} dx, \qquad \xi \in \mathbb{R}^2$$

is called the Fourier transform of φ and

$$\varphi(x) = \mathscr{F}^{-1}_{\xi\to x}[\hat{\varphi}(\xi)] := \int_{\mathbb{R}^2} \hat{\varphi}(\xi) e^{i2\pi x\cdot\xi} d\xi, \qquad \xi \in \mathbb{R}^2.$$

is the inverse Fourier transform of φ

The test function $\varphi(x)$ decrease at infinity faster than any power of $|x|^{-1}$. Therefore, its Fourier transform may be differentiated under the integral sign any number of times:

$$\partial^\alpha \mathscr{F}[\varphi](\xi) = \int_{\mathbb{R}^2} (-i2\pi x)^\alpha \varphi(x) e^{-i2\pi\xi\cdot x} dx = \mathscr{F}[(-i2\pi x)^\alpha \varphi](\xi) \tag{2.4}$$

Hence it follows that $\mathscr{F}[\varphi] \in C^\infty(\mathbb{R}^2)$.
Furthermore, every derivative $\partial^\alpha \varphi(x)$ has the same property and so

$$\mathscr{F}[\partial^\alpha \varphi](\xi) = \int_{\mathbb{R}^2} \partial^\alpha \varphi(x) e^{-i2\pi\xi\cdot x} dx = (i2\pi\xi)^\alpha \mathscr{F}[\varphi](\xi) \tag{2.5}$$

From equations (2.4) and (2.5), we obtain

$$\xi^\beta D^\alpha \mathscr{F}[\varphi](\xi) = \xi^\beta \mathscr{F}[(ix)^\alpha \varphi](\xi) = i^{|\alpha|+|\beta|} \mathscr{F}[D^\beta (x^\alpha \varphi)](\xi). \tag{2.6}$$

It follows from equation (2.6) that α and β the magnitudes $\xi^\beta D^\alpha \mathscr{F}[\varphi](\xi)$ are uniformly bounded with respect to $\xi \in \mathbb{R}^2$

$$|\xi^\beta D^\alpha \mathscr{F}[\varphi](\xi)| \leq \int |D^\beta(x^\alpha \varphi)|dx.$$

This means that $\mathscr{F}[\varphi] \in \mathscr{S}$. So the Fourier transform maps the space $\mathscr{S}$ into itself.

Note that if $\varphi \in \mathscr{S}(\mathbb{R}^2)$, then $\mathscr{F}\varphi \in \mathscr{S}(\mathbb{R}^2), \mathscr{F}^{-1}\mathscr{F}\varphi = \varphi \quad \text{and} \quad \mathscr{F}\mathscr{F}^{-1}\varphi = \varphi.$ where

$$\begin{aligned}\mathscr{F}^{-1}[\psi](x) &= \int_{\mathbb{R}^2} \psi(\xi) e^{i2\pi\xi\cdot x} d\xi \\ &= \int_{\mathbb{R}^2} \psi(\xi) e^{i2\pi\xi\cdot(-x)} d\xi = \mathscr{F}[\psi](-x).\end{aligned}$$

Let $u \in \mathscr{S}(\mathbb{R}^2)$. We define the transformation $\mathscr{F} : u \to \hat{u}$. Then $\mathscr{F}$ is called the Fourier transformation. It is known that $\hat{u} \in \mathscr{S}(\mathbb{R}^2)$, if $u \in \mathscr{S}(\mathbb{R}^2)$.
So, $\mathscr{F} : \mathscr{S}(\mathbb{R}^2) \to \mathscr{S}(\mathbb{R}^2)$ is a linear operator.

Theorem 2.1. (Parseval's formula). *For any $u, v \in \mathscr{S}(\mathbb{R}^2)$*

$$\int_{\mathbb{R}^2} \overline{\hat{u}(x)}\hat{v}(x)dx = \int_{\mathbb{R}^2} \overline{u(x)}v(x)dx$$

*In particular, $\|\hat{u}\|_{L^2} = \|u\|_{L^2}$ (**Plancheral's formula**).*

Proof. We can show that $\int_{\mathbb{R}^2} v(\xi)\hat{u}(\xi)e^{i2\pi y\xi}dx = \int_{\mathbb{R}^2} \hat{v}(x)u(y+x)dx$ that is

$$\begin{aligned}\int_{\mathbb{R}^2} v(\xi)\hat{u}(\xi)e^{i2\pi y\xi}d\xi &= \int_{\mathbb{R}^2} v(\xi)\left(\int_{\mathbb{R}^2} u(x)e^{-i2\pi\xi\cdot x}dx\right)e^{i2\pi\xi y}d\xi \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} v(\xi)u(x)e^{-i2\pi\xi(x-y)}dxd\xi \\ &= \int_{\mathbb{R}^2} u(x)\int_{\mathbb{R}^2} v(\xi)e^{-i2\pi\xi(x-y)}d\xi dx \\ &= \int_{\mathbb{R}^2} u(x)\hat{v}(x-y)dx = \int_{\mathbb{R}^2} \hat{v}(x)u(x+y)dx.\end{aligned}$$

Setting $y = 0$, we get $\int_{\mathbb{R}^2} v(\xi)\hat{u}(\xi)dx = \int_{\mathbb{R}^2} \hat{v}(x)u(x)dx$. Replacing u by its inverse Fourier transform $\mathscr{F}^{-1}u$ and using the identity $\mathscr{F}^{-1}u(\xi) = \hat{u}(-\xi)$, we obtain

$$\int_{\mathbb{R}^2} v(x)u(x)dx = \int_{\mathbb{R}^2} \hat{v}(x)\hat{u}(-x)dx.$$

However, $\overline{\hat{u}(x)} = \hat{\bar{u}}(-x)$, so putting $h = \bar{u}$. We see that $\hat{u}(-x) = \hat{\hat{h}}(-x) = \hat{h}(x)$ hence,

$$\int_{\mathbb{R}^2} v(x)\bar{h}(x)dx = \int_{\mathbb{R}^2} \hat{v}(x)\overline{\hat{h}(x)}dx$$

as required.

Definition 2.8. A distribution of slow growth, a tempered distribution, is any continuous linear functional on the space $\mathscr{S}(\mathbb{R}^2)$ of basic functions; and is denoted by $\mathscr{S}'(\mathbb{R}^2)$.

Equivalently, we can define the space $\mathscr{S}'(\mathbb{R}^2)$ to be the collection of all complex-valued linear functionals over $\mathscr{S}(\mathbb{R}^2)$.

Note that, if $f \in \mathscr{S}'(\mathbb{R}^2)$, then $\langle f, \varphi \rangle \in \mathbb{R}$ or $\mathbb{C}$ for all $\varphi \in \mathscr{S}(\mathbb{R}^2)$. Since $\mathscr{D}(\mathbb{R}^2) \subset \mathscr{S}(\mathbb{R}^2)$, we can observe that $f \in \mathscr{D}'(\mathbb{R}^2)$. Hence, $\mathscr{S}'(\mathbb{R}^2) \subset \mathscr{D}'(\mathbb{R}^2)$

The Fourier transformation is extended to the class $\mathscr{S}'(\mathbb{R}^2)$. Let $u \in \mathscr{S}'(\mathbb{R}^2)$, a functional $\mathscr{F}[u] \in \mathscr{S}'(\mathbb{R}^2)$ is called the Fourier image of u , if

$$\langle \mathscr{F}[u], \varphi \rangle = \langle u, \mathscr{F}[\varphi] \rangle, \ \forall \varphi \in \mathscr{S}(\mathbb{R}^2).$$

Definition 2.9. Let $u \in \mathscr{S}'(\mathbb{R}^2)$. Then the Fourier transform $\mathscr{F}[u]$ and the inverse Fourier transform $\mathscr{F}^{-1}[u]$ are given by

$$\langle \mathscr{F}[u], \varphi \rangle = \langle u, \mathscr{F}[\varphi] \rangle \qquad \text{and} \quad \langle \mathscr{F}^{-1}[u], \varphi \rangle = \langle u, \mathscr{F}^{-1}[\varphi] \rangle \qquad \forall \varphi \in \mathscr{S}(\mathbb{R}^2)$$

Example 2.3.1 *Let $\varphi \in \mathscr{S}(\mathbb{R}^2)$. Then*

$$\langle \mathscr{F}[\delta], \varphi \rangle = \langle \delta, \mathscr{F}[\varphi] \rangle = \mathscr{F}[\varphi](0)$$

But by the definition of the Fourier transform

$$\mathscr{F}[\varphi](0) = \int_{\mathbb{R}^2} \varphi(x)dx = \langle 1, \varphi \rangle$$

which implies $\mathscr{F}\delta = 1$

Note that if $f \in \mathscr{S}'(\mathbb{R}^2)$, then $\mathscr{F}f \in \mathscr{S}'(\mathbb{R}^2), \mathscr{F}^{-1}\mathscr{F}f = f$ and $\mathscr{F}\mathscr{F}^{-1}f = f$.

Theorem 2.2. *$\mathscr{F}$ and $\mathscr{F}^{-1}$ are continuous on $\mathscr{S}'(\mathbb{R}^2)$.*

Proof. Suppose that $f_n \to f$ in $\mathscr{S}'(\mathbb{R}^2)$. Then $\langle f_n, \varphi \rangle \to \langle f, \varphi \rangle$ for every $\varphi \in \mathscr{S}(\mathbb{R}^2)$. Hence

$$\langle \hat{f}_n, \varphi \rangle = \langle f_n, \hat{\varphi} \rangle \to \langle f, \hat{\varphi} \rangle = \langle \hat{f}, \varphi \rangle$$

so, $\hat{f}_n \to \hat{f}$ in $\mathscr{S}'(\mathbb{R}^2)$.
A similar argument holds for the inverse Fourier transform.

Theorem 2.3. *For any $u \in \mathscr{S}'(\mathbb{R}^2)$ and multi-indices $\alpha, \beta \in \mathbb{Z}^2$*

$$(ix)^\alpha \hat{u} = (\widehat{D^\alpha u}) \qquad and \quad D^\beta \hat{u} = ((\widehat{-ix)^\beta u}).$$

In general,

$$(ix)^\alpha D^\beta \hat{u} = \left(D^\alpha \widehat{((-ix)}^\beta u)\right).$$

Proof. Let $\varphi \in \mathscr{S}(\mathbb{R}^2)$. Then

$$\begin{aligned}(ix)^\alpha \hat{u}(\varphi) &= \hat{u}((ix)^\alpha \varphi) = u\left(\widehat{(ix)^\alpha \varphi}\right) = u((-D)^\alpha \hat{\varphi}) \\ &= D^\alpha u(\hat{\varphi}) = \left(\widehat{D^\alpha u}\right)(\varphi).\end{aligned}$$

Similarly,

$$\begin{aligned}D^\beta \hat{u}(\varphi) &= \hat{u}\left((-D)^\beta \varphi\right) = u\left((\widehat{-D)^\beta} \varphi)\right) = u\left(-ix)^\beta \hat{\varphi}\right) = (-ix)^\beta u(\hat{\varphi}) \\ &= \left((\widehat{-ix)^\beta} u\right)(\varphi).\end{aligned}$$

Let $f(x)$ and $g(x)$ be locally integrable functions in the space $\mathbb{R}^2$. The function $f(x)g(x)$ also will be locally integrable in $\mathbb{R}^4$. It defines the regular distribution, acting on the test functions $\varphi(x,y) \in \mathscr{D}$, according to the formulae

$$\begin{aligned}\langle f(x)g(y), \varphi\rangle &= \int f(x) \int g(y)\varphi(x,y)dydx = \langle f(x), \langle g(y), \varphi(x,y)\rangle\rangle, \\ \langle g(y)f(x), \varphi\rangle &= \int g(y) \int f(x)\varphi(x,y)dxdy = \langle g(y), \langle f(x), \varphi(x,y)\rangle\rangle.\end{aligned}$$

These equations expresses Fubini's Theorem concerning the equality of a repeated and a multiple integral.

Definition 2.10. Let $f, g \in \mathscr{D}'(\mathbb{R}^2)$. Then the direct product $f(x)g(x)$ is the distribution on $\mathbb{R}^4$ given by:

$$\langle f(x)g(y), \varphi(x,y)\rangle = \langle f(x), \langle g(x), \varphi(x,y)\rangle\rangle.$$

For instance, we can take as a direct product, $\delta(x,y) = \delta(x)\delta(y)$.

Definition 2.11. Let φ and ψ be functions in $\mathscr{S}(\mathbb{R}^2)$, the convolution $\varphi * \psi$ is defined by

$$(\varphi * \psi)(x) = \int_{\mathbb{R}^2} \varphi(x-y)\psi(y)dy$$

Remark 2.3.1 *For φ and ψ be functions in $\mathscr{S}(\mathbb{R}^2)$, then it is well known that, $\mathscr{F}(\varphi * \psi) = \mathscr{F}\varphi \mathscr{F}\psi$. As the product of two distributions are not generally defined, the* *convolution of tempered distributions is impossible. However if one factor is in $\mathscr{S}(\mathbb{R}^2)$, these problem will be solved. If $f \in \mathscr{S}'(\mathbb{R}^2)$ and $\psi \in \mathscr{S}(\mathbb{R}^2)$, then $f * \psi$ is defined by*

$$\langle f * \psi, \varphi \rangle = \langle f, \tilde{\psi} * \varphi \rangle \qquad \textit{where} \quad \tilde{\psi}(x) = \psi(-x)$$

In this case, $\mathscr{F}(\psi * f) = \hat{\psi} \cdot \hat{f}$

Definition 2.12. Let $f, g \in \mathscr{D}'(\mathbb{R}^2)$. Then the convolution of $f(x)$ and $g(x)$ is defined by:

$$\langle (f * g), \varphi \rangle = \langle f(x)g(y), \varphi(x+y) \rangle$$

Note that this definition is meaningful under the conditions: either f or g has compact support or in one dimension, the support of f and g are bounded from the same side.

Remark 2.3.2 *If the convolution $f * g$ exists, then there is also a convolution $g * f$ and they are equal. i.e., the operator convolution is commutative. The assertion follows from the commutativity of the direct product and the definition of convolution.*

Let us consider some special properties of convolution. The convolution of any distribution f with the δ function exists and is equal to f

$$f * \delta = \delta * f = f. \tag{2.7}$$

Indeed, for all $\varphi \in \mathscr{D}(\mathbb{R}^2)$,

$$\begin{aligned} \langle \delta * f, \varphi \rangle &= \langle \delta(x) f(y), \varphi(x+y) \rangle = \langle f(y), \langle \delta(x), \varphi(x+y) \rangle \rangle \\ &= \langle f(y), \varphi(y) \rangle = \langle f, \varphi \rangle. \end{aligned}$$

Hence, $\delta * f = f$.

Let us consider $f * \psi$, where $\psi \in \mathscr{D}(\mathbb{R}^2)$. Then we have

$$\begin{aligned} \langle f * \psi, \varphi \rangle &= \langle f(x)\psi(y), \varphi(x+y) \rangle = \langle f(x), \int_{\mathbb{R}^2} \psi(y)\varphi(x+y)dy \rangle \\ &= \langle f(x), \int_{\mathbb{R}^2} \psi(y-x)\varphi(y)dy \rangle = \int_{\mathbb{R}^2} \langle f(x), \psi(y-x)\varphi(y) \rangle dy. \end{aligned}$$

Therefore, $f * \psi(y) = \langle f(x), \psi(y-x) \rangle$. This function is a C^∞, and if f has compact support, it is a test function.

If the convolution $f * g$ exist, then the convolution $D^\alpha f * g$ and $f * D^\alpha g$ exists and

$$D^\alpha f * g = D^\alpha (f * g) = f * D^\alpha g. \tag{2.8}$$

By the definition of distributional derivative

$$\begin{aligned}\langle D^\alpha(f*g),\varphi\rangle &= (-1)^{|\alpha|}\langle f*g, D^\alpha\varphi\rangle\\ &= (-1)^{|\alpha|}\langle g(y),\langle f(x), D^\alpha\varphi(x+y)\rangle\rangle\\ &= \langle g(y),\langle D^\alpha f(x),\varphi(x+y)\rangle\rangle = \langle D^\alpha f*g,\varphi\rangle.\end{aligned}$$

Thus, $D^\alpha f*g = D^\alpha(f*g)$ and using commutativity, we also find $f*D^\alpha g = D^\alpha(f*g)$.

From this relation and using the existence of the convolution $f*g$, we obtain the following sequence of equations.

Let $\psi\in\mathscr{D}(\mathbb{R}^2)$ satisfies $\psi\geq 0$ on $\mathbb{R}^2$,

$$\psi = 0 \quad \text{for} \quad |x|>1, \quad \text{and} \quad \int_{\mathbb{R}^2}\psi(x)dx = 1, \tag{2.9}$$

we define

$$\psi_\varepsilon(x) = \varepsilon^{-2}\psi(\varepsilon^{-1}x) \qquad \text{for} \quad x\in\mathbb{R}^2 \quad \text{and} \quad \varepsilon>0.$$

For example, we may take for $x\in\mathbb{R}^2$, let

$$\psi(x) = \begin{cases} Ce^{\frac{-1}{1-|x|^2}} & |x|<1\\ 0 & |x|\geq 1\end{cases}$$

where constant C is chosen so that condition (2.9) is satisfied.
For $\varepsilon>0$, we put $\psi_\varepsilon(x) = \varepsilon^{-2}\psi(\varepsilon^{-1}x)$ for $x\in\mathbb{R}^2$, then $\psi_\varepsilon\in\mathscr{D}(\mathbb{R}^2)$, $\psi_\varepsilon(x)\geq 0$. And $\psi_\varepsilon(x)=0$ if $|x|\geq\varepsilon$ which implies $\int_{\mathbb{R}^2}\psi_\varepsilon(x)dx = 1$.

This properties of ψ_ε mean that $(\psi_\varepsilon * u)$ is a kind of local average of u around x, and for this reason $\psi_\varepsilon * u\approx u$ when ε is small.

2.4 Fundamental solutions and Parametrix

In this section, using [Vlad79], we discuss the theory of distribution applied to the solution of the Cauchy problem, particularly give the definition of a fundamental solution for a second order partial differential equation with constant coefficient in two dimension and describe important concepts related to it. In this connection, the Cauchy problem is considered in a generalized context which allows one to include boundary conditions. The problem is solved by a standard method of summing the perturbations produced by each point of the distributed source so that its solution appears in the form of a convolution of the fundamental solution together with the right-hand side. We also discuss how to construct a kind of an approximation of fundamental solution, parametrix, using the method of “frozen" coefficient.

Fundamental solution

Consider the following linear differential operator L with constant coefficients $a_\alpha(x) = a_\alpha$:

$$L(D) = \sum_{|\alpha|=0}^{2} a_\alpha D^\alpha \tag{2.10}$$

A distribution $\mathscr{E} \in \mathscr{D}'(\mathbb{R}^2)$ which satisfies the relation

$$L(D)\mathscr{E} = \delta(x) \qquad \text{in} \quad \mathbb{R}^2 \tag{2.11}$$

is called the fundamental solution (or the function of influence) of the differential operator $L(D)$.

The fundamental solution $\mathscr{E}(x)$ of the operator $L(D)$, generally speaking, is not unique, it is defined accurately as far as the term $\mathscr{E}_0(x)$, which is an arbitrary solution of the homogeneous equation $L(D)\mathscr{E}_0(x) = 0$. In fact, the distribution $\mathscr{E}(x) + \mathscr{E}_0(x)$ is also a fundamental solution of the operator $L(D)$.

$$L(D)\,(\mathscr{E} + \mathscr{E}_0) = L(D)\mathscr{E} + L(D)\mathscr{E}_0 = \delta(x)$$

Fundamental solution of a differential equation plays an important role in the implementation of BIE method. In addition, it has significance in constructing a solution of the equation:

$$Lu = f \tag{2.12}$$

where $f \in \mathscr{D}'(\mathbb{R}^2)$ and $supp(f)$ is compact. The following Theorem addresses this essential idea.

Theorem 2.4. *Let $f \in \mathscr{D}'(\mathbb{R}^2)$ be such that the convolution $\mathscr{E} * f$ exists in $\mathscr{D}'(\mathbb{R}^2)$. Then a solution of equation (2.12) exists in $\mathscr{D}'(\mathbb{R}^2)$ and is given by the formula*

$$u = \mathscr{E} * f.$$

This solution is unique in the class of distributions belonging to $\mathscr{D}'(\mathbb{R}^2)$ for which a convolution with $\mathscr{E}$ exists.

Proof. To show that the formula $u = \mathscr{E} * f$ is a solution of (2.12), we insert it into the equation and arrive at the identity. Here, we implement the properties of convolution stated in equations (2.8) and (2.7).

$$L(D)(\mathscr{E}*f)=\sum_{|\alpha|=0}^{2}a_\alpha D^\alpha(\mathscr{E}*f)=\left(\sum_{|\alpha|=0}^{2}a_\alpha D^\alpha\mathscr{E}\right)*f=L(D)\mathscr{E}*f=\delta*f=f.$$

Therefore, $u=\mathscr{E}*f$ is a solution of the equation (2.12).
We shall prove the uniqueness of the solution of (2.12) in the class of distribution belonging to $\mathscr{D}'$ for which a convolution with $\mathscr{E}$ exists in $\mathscr{D}'$.

For this it is suficient to establish that the corresponding homogeneouos equation

$$L(D)u=0$$

has only a zero in this class. But this is in fact so, by virtue of

$$u=u*\delta=u*L(D)\mathscr{E}=L(D)u*\mathscr{E}=0.$$

Hence, the solution is unique. Or equivalently, we can check the uniqueness of the solution by letting u_1 and u_2 be two solutions of equation (2.12) in $\mathscr{D}'$, i.e., $L(D)u_1=f(x)$ and $L(D)u_2=f(x)$. $u_1-u_2=(u_1-u_2)*\delta=(u_1-u_2)*L(D)\mathscr{E}=L(D)(u_1-u_2)*\mathscr{E}=0*\mathscr{E}=0$. Hence, $u_1=u_2$.

The physical sense of the solution $u=\mathscr{E}*f$ is described as follows. Let us represent a source $f(x)$ in the form of a “sum" of the point sources $f(\xi)\delta(x-\xi)$,

$$f(x)=\delta*f=\int f(\xi)\delta(x-\xi)d\xi.$$

Since, $L(D)\mathscr{E}=\delta(x)$, each point source $f(\xi)\delta(x-\xi)$ defines the influence $f(\xi)\mathscr{E}(x-\xi)$. Therefore, the solution

$$u(x)=\mathscr{E}*f=\int f(\xi)\mathscr{E}(x-\xi)d\xi$$

is the superposition of these influences.

As an example, let us see the fundamental solution for Laplace operator in 2D. Fundamental solution of the Laplace operator in $\mathbb{R}^2$

$$\Delta\mathscr{E}=\delta(x) \tag{2.13}$$

we shall calculate these fundamental solution by means of Fourier transform. Applying the Fourier transform to (2.13) we obtain:

$$-|\xi|^2\mathscr{F}(\mathscr{E})=(2\pi)^{-1}\quad\text{which implies}\quad\mathscr{F}(\mathscr{E})=-\frac{1}{|\xi|^2}(2\pi)^{-1}.$$

In order to determine the fundamental solution of Laplace operator we need the following Lemma.

Lemma 2.4.1 *([Ste07, Lemma 2.13]) The Fourier transform maintains rotational symmetries. i.e., For* $u \in \mathscr{S}(\mathbb{R}^2)$ *we have* $\hat{u}(\xi) = \hat{u}(|\xi|)$ *for all* $\xi \in \mathbb{R}^2$ *if and only if* $u(x) = u(|x|)$ *for all* $x \in \mathbb{R}^2$.

For the two dimensional case the inverse Fourier transform of the fundamental solution has to be regularized in some appropriate way. By

$$\langle \mathscr{P}\frac{1}{|x|^2}, \varphi\rangle_{L_2(\mathbb{R}^2)} = \int_{x\in\mathbb{R}^2;|x|\leq 1} \frac{\varphi(x)-\varphi(0)}{|x|^2}dx + \int_{x\in\mathbb{R}^2;|x|>1} \frac{\varphi(x)}{|x|^2}dx$$

we first define the tempered distribution $\mathscr{P}\frac{1}{|x|^2} \in \mathscr{S}'(\mathbb{R}^2)$. Then

$$\begin{aligned} 2\pi\langle v, \hat{\varphi}\rangle_{L_2(\mathbb{R}^2)} &= \langle \mathscr{P}\frac{1}{|\xi|^2}, \varphi\rangle_{L_2(\mathbb{R}^2)} \\ &= \int_{\xi\in\mathbb{R}^2;|\xi|\leq 1} \frac{\varphi(\xi)-\varphi(0)}{|\xi|^2}d\xi + \int_{\xi\in\mathbb{R}^2;|\xi|>1} \frac{\varphi(\xi)}{|\xi|^2}d\xi, \qquad \forall\varphi \in \mathscr{S}(\mathbb{R}^2) \end{aligned}$$

with

$$\varphi(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i(z.\xi)}\hat{\varphi}(z)dz, \qquad \varphi(0) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\hat{\varphi}(z)dz$$

we then obtain

$$\begin{aligned} (2\pi)^2\langle v, \hat{\varphi}\rangle_{L_2(\mathbb{R}^2)} = &\int_{\xi\in\mathbb{R}^2;|\xi|\leq 1} \frac{1}{|\xi|^2}\int_{\mathbb{R}^2}\left(e^{i(z.\xi)}-1\right)\hat{\varphi}(z)dzd\xi \\ &+ \int_{\xi\in\mathbb{R}^2;|\xi|>1} \frac{1}{|\xi|^2}\int_{\mathbb{R}^2} e^{i(z.\xi)}\hat{\varphi}(z)dzd\xi. \end{aligned}$$

We can not exchange the order of integration in the second term. However, we can write

$$(2\pi)^2\langle v, \hat{\varphi}\rangle_{L_2(\mathbb{R}^2)} = \int_{\mathbb{R}^2}\hat{\varphi}(z)\left\{\int_{\xi\in\mathbb{R}^2;|\xi|\leq 1}\frac{e^{i(z.\xi)}-1}{|\xi|^2}d\xi + \int_{\xi\in\mathbb{R}^2;|\xi|>1}\frac{e^{i(z.\xi)}}{|\xi|^2}d\xi\right\}dz.$$

With Lemma 2.4.1([Ste07, Lemma 2.13]), we further have

$$v(z) = v(|z|) = \frac{1}{(2\pi)^2}\int_{\xi\in\mathbb{R}^2;|\xi|\leq 1}\frac{e^{i(z.\xi)}-1}{|\xi|^2}d\xi + \int_{\xi\in\mathbb{R}^2;|\xi|>1}\frac{e^{i(z.\xi)}}{|\xi|^2}d\xi$$

and using polar coordinate, we obtain

$$v(z) = \frac{1}{(2\pi)^2}\int_0^1\int_0^{2\pi}\frac{1}{r}\left[e^{ir|z|cos\theta}-1\right]d\theta dr + \frac{1}{(2\pi)^2}\int_1^\infty\int_0^{2\pi}\frac{1}{r}e^{ir|z|cos\theta}d\theta dr$$
$$= \frac{1}{2\pi}\int_0^1\frac{1}{r}\left[J_0(r|z|)-1\right]dr + \frac{1}{2\pi}\int_1^\infty\frac{1}{r}J_0(r|z|)dr$$

with the first order Bessel function $J_0(s) = \frac{1}{2\pi}\int_0^{2\pi} e^{iscos\theta}d\theta$.

Substituting $r = \frac{s}{\rho}$, we compute

$$v(z) = \frac{1}{2\pi}\int_0^\rho \frac{J_0(s)-1}{s}ds + \frac{1}{2\pi}\int_\rho^\infty\frac{J_0(s)}{s}ds$$
$$= \frac{1}{2\pi}\int_0^1 \frac{J_0(s)-1}{s}ds + \frac{1}{2\pi}\int_1^\infty\frac{J_0(s)}{s}ds - \frac{1}{2\pi}\int_\rho^1\frac{J_0(s)}{s}ds$$
$$= \frac{1}{2\pi}\log|z| - \frac{\log C_0}{2\pi} = \frac{1}{2\pi}\log\frac{|z|}{C_0}$$

where $C_0 > 0$ with constant $\frac{1}{2\pi}\int_0^1\frac{J_0(s)-1}{s}ds - \frac{1}{2\pi}\int_1^\infty\frac{J_0(s)}{s}ds$.
Hence, the fundamental solution of the Laplace operator in 2D is

$$\mathscr{E}(z) = \frac{1}{2\pi}\log\frac{|z|}{r}, \quad r > 0.$$

Note that if we shift the origin to a new point y, the PDE $\Delta u = 0$ is unchanged (Laplace's equation is translation invariant). If $u(x)$ is harmonic, then $u(x-y)$ is also harmonic for x different from y. Therefore,

$$\mathscr{E}(x,y) = \frac{1}{2\pi}\log\frac{|x-y|}{r}, \quad r > 0 \tag{2.14}$$

is a fundamental solution for Laplace equation. That is, $\Delta\mathscr{E}(x,y) = \delta(x-y)$. From the discussion in the computation of the fundamental solution for Laplace's operator, we extract the following important remark.

Lemma 2.4.2 *[Vlad79] Let Ω be a bounded open set in $\mathbb{R}^2$ with a piecewise C^1 -boundary. If u is in $C^1(\overline{\Omega})\cap C^2(\Omega)$ and $\mathscr{E}(x,y)$ is the fundamental solution given by (2.14), then*

$$u(x) = \int_{\partial\Omega}\left(u(y)\frac{\partial\mathscr{E}(x,y)}{\partial n_y} - \mathscr{E}(x,y)\frac{\partial u(y)}{\partial n}\right)ds + \int_\Omega \mathscr{E}(x,y)\Delta u(y)dy, \qquad \text{for} \quad x\in\Omega.$$

The following fundamental identities are well known, see e.g., [BJS64].

Corollary 2.4.1 *If $\phi(x)\in\mathscr{D}(\mathbb{R}^2)$, then for $r > 0$*

$$\phi(x) = \int_{\mathbb{R}^2}\Delta\phi(x)\frac{1}{2\pi}\log\frac{|x-y|}{r}dy, \quad \& \quad \phi(x) = \Delta\int_{\mathbb{R}^2}\phi(x)\frac{1}{2\pi}\log\frac{|x-y|}{r}dy.$$

Using the convolution notation, $2\pi\phi(x) = \Delta\phi(x) * \log(x) = \Delta(\phi(x) * \log(x))$.

Proof. Let $\phi(x) \in \mathscr{D}(\mathbb{R}^2)$, then Lemma 2.4.2 implies that,

$$2\pi\phi(x) = \int_{\Omega} \Delta\phi(y) \log\frac{|x-y|}{r} dy + \int_{\partial\Omega} \left(\phi(y)\frac{\partial}{\partial n}\log\frac{|x-y|}{r} - \log\frac{|x-y|}{r}\frac{\partial\phi(y)}{\partial n}\right) ds.$$

Here Ω is a domain with sufficiently smooth boundary $\partial\Omega$ and x is a point of Ω. Let us take Ω sufficiently large such that $supp(\phi) \subset \Omega$, in this case, ϕ vanishes on $\partial\Omega$. Hence, equation (2.11) gives us the identity (2.9)

$$2\pi\phi(x) = \int_{\mathbb{R}^2} \Delta\phi(y) \log\frac{|x-y|}{r} dy.$$

From the distribution theory, the identity can be rewritten as

$$\Delta\left(\frac{\log\frac{|x-y|}{r}}{2\pi}\right) = \delta(x-y).$$

Now the identity (2.10) follows from (2.9) (cf., equation 2.8).

$$\Delta_x \int_{\mathbb{R}^2} \phi(y) \log\frac{|x-y|}{r} dy = \Delta_x \int_{\mathbb{R}^2} \phi(x-z) \log\frac{|z|}{r} dz = \int_{\mathbb{R}^2} \Delta_x\phi(x-z) \log\frac{|z|}{r} dz$$
$$= \int_{\mathbb{R}^2} \Delta_y\phi(y) \log\frac{|x-y|}{r} dy.$$

Parametrix

Consider the following elliptic second order partial differential operator with variable coefficients $a_\alpha(x)$:

$$L(x,D) = \sum_{|\alpha|=0}^{2} a_\alpha(x) D^\alpha \tag{2.15}$$

Definition 2.13. A function $P(x,y)$ is a parametrix (Levi function) for the operator L if

$$L_x P(x,y) = \delta(x-y) + R(x,y)$$

where δ is the Dirac-delta distribution, while $R(x,y)$ is a remainder possessing at most a weak singularity at $x = y$.

On the operator (2.15), to every point x_0 we associate the coefficient $a_\alpha(x)$ “frozen" coefficient and obtain the operator

$$L_{x_0}(D) = \sum_{|\alpha|=0}^{2} a_\alpha(x_0) D^\alpha.$$

Denote by $\mathscr{E}_{x_0}(x)$ the fundamental solution of the operator L_{x_0}. Then, $\mathscr{E}_{x_0}(x)$ is a parametrix for the operator L with singularity at x_0. In the thesis, we consider the differential operator given by

$$A(x,\partial_x) := \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left[a(x)\frac{\partial}{\partial x_i}\right], \tag{2.16}$$

Then at a point x_0, the operator with "frozen" coefficient is $A_{x_0}(\partial_x) = a_{x_0}\Delta$ whose fundamental solution is given by $\mathscr{E}_{x_0}(x,y) = \frac{1}{2\pi a(x_0)} \log \frac{|x-y|}{r}$, which can be easily obtained form the fundamental solution for Laplace operator, (2.14). Therefore, the parametrix for the operator in equation (2.16) is given by

$$P(x,y) = \mathscr{E}_y = \frac{1}{2\pi a(y)} \log \frac{|x-y|}{r}.$$

2.5 The $L^p(\Omega)$ space

The analysis of BVPs naturally involves function spaces that are not only defined in terms of the properties of the function itself, but also in terms of the properties of its derivatives. Hence, in the following we discuss on a wider space.

Definition 2.14. Let Ω be a domain in $\mathbb{R}^2$ and let p be a positive real number. Denote by $L^p(\Omega)$ for $1 \le p < \infty$, the class of all measurable function u defined on Ω for which

$$\int_\Omega |u(x)|^p dx < \infty$$

and we define the **norm** of $u \in L^p(\Omega)$ as follows

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p dx\right)^{\frac{1}{p}}.$$

A function u that is measurable on Ω is said to be essentially bounded on Ω if there is a constant C such that $|u(x)| \le C$ a.e on Ω . The greatest lower bound of such constant C is called the essential supremum of $|u|$ on Ω and is denoted by $\operatorname{ess\,sup}_{x\in\Omega} |u(x)|$. We denote $L^\infty(\Omega)$ the space of all functions u that are essentially bounded on Ω, that is

$$L^\infty(\Omega) = \left\{u : \Omega \to \mathbb{R} : \ u \text{ is measurable and } \|u\|_{L^\infty(\Omega)} < \infty \text{ in } \Omega\right\}.$$

The norm in $L^\infty(\Omega)$ is defined by

$$\|u\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)|.$$

Definition 2.15. The $L^p_{loc}(\Omega)$ consists of all measurable functions in Ω (that are p-locally integrable) such that

$$\int_{\Omega'} |u(x)|^p dx < \infty$$

for any bounded strictly interior sub-domain $\Omega' \subset\subset \Omega$.

$$L^p_{loc}(\Omega) = \{u : u \in L^p(\Omega') \quad \forall \Omega' \subset\subset \Omega, \quad \Omega' \text{ compact}\}.$$

Definition 2.16. Let $u \in L^1_{loc}(\Omega)$ and α be a multi-index. A function $w \in L^1_{loc}(\Omega)$ is called the α^{th}-weak derivative of u, if it satisfies

$$\int_\Omega u(x) D^\alpha \varphi(x) dx = (-1)^{|\alpha|} \int_\Omega w(x)\varphi(x) dx \qquad \forall \varphi \in C_0^\infty(\Omega).$$

Remark 2.1. If $u(x)$ is sufficiently smooth to have continuous derivative $D^\alpha u$, then we can integrate by part

$$\int_\Omega u(x) D^\alpha \varphi(x) dx = (-1)^{|\alpha|} \int_\Omega D^\alpha u(x)\varphi(x) dx \qquad \forall \varphi \in C_0^\infty(\Omega).$$

Hence, the classical derivatives is also a weak derivatives. To define the weak derivative of $D^\alpha u$, we don't need the existence of the smaller order derivative (like classical derivative).

Lemma 2.5.1 *(Generalized Variational Lemma) Let $v \in L^1_{loc}(\Omega)$ with Ω a nonempty open set in $\mathbb{R}^2$. If*

$$\int_\Omega v(x)\phi(x) dx = 0 \qquad \textit{for all} \quad \phi \in \mathscr{D}(\Omega)$$

then $v = 0$ a.e. on Ω .

For the proof, see e.g., [Tri08, Proposition 2.7(ii)].

2.6 Sobolev space

We will use distributions to define an important class of function spaces for PDEs, the Sobolev space. As we have seen, every function in $L^p(\Omega)$ is actually a distribution, therefore it has a distributional derivative. Sobolev spaces are useful tools in the analysis of boundary value problem.

Integer order Sobolev spaces

We always interprete $f \in L^p$ with $1 \leq p \leq \infty$ as tempered distribution. In particular, $D^\alpha f \in \mathscr{S}'$ make sense for all $\alpha \in \mathbb{N}_0^2$.

Definition 2.17. Let k be a non negative integer and $1 \leq p < \infty$ and let Ω be non-empty open subset of $\mathbb{R}^2$. Then we define Sobolev space $W_p^k(\Omega)$ of order k to be the set of all distribution $u \in L^p(\Omega)$ such that $D^\alpha u \in L^p(\Omega)$ for $|\alpha| \leq k$. That is

$$W_p^k(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), \ \ \forall \alpha \in \mathbb{N}_0^2, \ \ |\alpha| \leq k\}$$

Remark 2.2. Here, of course, $D^\alpha u$ is viewed as a distributional derivative of u on Ω.

In $W_p^k(\Omega)$, we define a norm by

$$\|u\|_{W_p^k(\Omega)} = \begin{cases} \left(\int_\Omega \sum\limits_{|\alpha| \leq k} |D^\alpha u|^p dx \right)^{\frac{1}{p}} & 1 \leq p < \infty \\ \sum\limits_{|\alpha| \leq k} \operatorname{ess\,sup}_\Omega |D^\alpha u| & p = \infty \end{cases}$$

Example 2.3. Let $1 \leq p < \infty$, we have

$$\|u\|^p_{W_p^1(\Omega)} = \int_\Omega \left(|u|^p + \sum_{i=1}^m |\frac{\partial u}{\partial x_i}|^p \right) dx$$

$$\|u\|^p_{W_p^2(\Omega)} = \int_\Omega \left(|u|^p + \sum_{i=1}^m |\frac{\partial u}{\partial x_i}|^p + \sum_{i,j=1}^m |\frac{\partial^2 u}{\partial x_i \partial x_j}|^p \right) dx$$

Fractional order Sobolev spaces

To define the Sobolev space of fractional order, we denote the Slobodeckii seminorm by

$$|u|^p_{\lambda,p,\Omega} = \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+p\lambda}} dxdy \text{ for } 0<\lambda<1$$

.

Definition 2.18. For $s=\lambda+r$ with a real number $\lambda \in (0,1)$ and an integer $r \geq 0$, we define

$$W^s_p(\Omega) = \{u \in W^r_p(\Omega) : |\partial^\alpha u|_{\lambda,p,\Omega} < \infty \text{ for } |\alpha| = r\}$$

In $W^s_p(\Omega)$ we define the norm

$$\|u\|_{W^s_p(\Omega)} = \left(\|u\|_{W^r_p(\Omega)} + \sum_{|\alpha|=r} |\partial^\alpha u|^p_{\lambda,p,\Omega} \right)^{\frac{1}{p}}$$

Sobolev space-second definition

Definition 2.19. For $s \in \mathbb{R}$, we define a continuous linear operator $\mathscr{J}^s : \mathscr{S}(\mathbb{R}^2) \to \mathscr{S}(\mathbb{R}^2)$ called the *Bessel potential* of order s, by

$$\mathscr{J}^s u(x) = \int_{\mathbb{R}^2} \left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u}(\xi) e^{i2\pi\xi\cdot x} d\xi \qquad x \in \mathbb{R}^2 \tag{2.17}$$

In this way, we have

$$\mathscr{F}_{x\to\xi}\{\mathscr{J}^s u(x)\} = \left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u}(\xi)$$

Indeed

$$\begin{aligned}
\langle \mathscr{F}_{x\to\xi}\left(\mathscr{J}^s u(x)\right), \varphi\rangle &= \langle \mathscr{J}^s u(x), \mathscr{F}_{\xi\to x}\left(\varphi(x)\right)\rangle = \int_{\mathbb{R}^2} \mathscr{J}^s u(x) \mathscr{F}_{\xi\to x}[\varphi(x)]dx \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u}(\xi) e^{i2\pi\xi . x} d\xi\, \hat{\varphi}(x)dx \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u}(\xi)\hat{\varphi}(x) e^{i2\pi\xi . x} dxd\xi \\
&= \int_{\mathbb{R}^2} \left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u}(\xi)\varphi(\xi)d\xi = \left\langle \left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u}(\xi), \varphi(\xi) \right\rangle
\end{aligned}$$

Therefore,

$$\mathscr{F}_{x\to\xi}\{\mathscr{J}^s u(x)\} = \left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u}(\xi)$$

It follow from (2.17) that $\langle \mathscr{J}^s u, v\rangle = \langle u, \mathscr{J}^s v\rangle$ for all $u, v \in \mathscr{S}(\mathbb{R}^2)$. Since,

$$\begin{aligned}
\langle \mathscr{J}^s u, v\rangle &= \int_{\mathbb{R}^2} \mathscr{J}^s u(x)v(x)dx = \int_{\mathbb{R}^2}\left[\int_{\mathbb{R}^2} \left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u}(\xi)e^{i2\pi\xi.x}d\xi\right] v(x)dx \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u}(\xi)e^{i2\pi\xi.x}v(x)dxd\xi \\
&= \int_{\mathbb{R}^2} \left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u}(\xi)e^{i4\pi\xi.x}\hat{v}(\xi)d\xi \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left(1+|\xi|^2\right)^{\frac{s}{2}} u(x)e^{i2\pi\xi.x}\hat{v}(\xi)d\xi dx \\
&= \int_{\mathbb{R}^2} u(x)\mathscr{J}^s v(x)dx = \langle u, \mathscr{J}^s v\rangle
\end{aligned}$$

Therefore, $\langle \mathscr{J}^s u, v\rangle = \langle u, \mathscr{J}^s v\rangle$.

Note that for all $s, t \in \mathbb{R}$, we have the followings:

$$\mathscr{J}^{s+t} = \mathscr{J}^s\mathscr{J}^t, \qquad (\mathscr{J}^s)^{-1} = \mathscr{J}^{-s}, \qquad \mathscr{J}^0 = \text{identity operator}$$

Using the definition of Bessel potential we have

$$\begin{aligned}
\left\langle \mathscr{J}^s\mathscr{J}^t u, v\right\rangle &= \left\langle \mathscr{J}^t u, \mathscr{J}^s v\right\rangle \\
&= \int_{\mathbb{R}^2} \mathscr{J}^t u(x)\mathscr{J}^s v(x)dx \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u}(\xi)e^{i2\pi\xi.x}d\xi \int_{\mathbb{R}^2} \left(1+|\xi|^2\right)^{\frac{t}{2}} \hat{v}(\xi)e^{i2\pi\xi.x}d\xi dx \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u}(\xi)e^{i2\pi\xi.x}d\xi \left(1+|\xi|^2\right)^{\frac{t}{2}} \left(\int_{\mathbb{R}^2} \hat{v}(\xi)e^{i2\pi\xi.x}d\xi\right) dx \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left(1+|\xi|^2\right)^{\frac{s}{2}} \left(1+|\xi|^2\right)^{\frac{t}{2}} \hat{u}(\xi)e^{i2\pi\xi.x}v(x)d\xi dx \\
&= \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2} \left(1+|\xi|^2\right)^{\frac{s+t}{2}} \hat{u}(\xi)e^{i2\pi\xi.x}d\xi\right) v(x)dx = \int_{\mathbb{R}^2} \mathscr{J}^{s+t}u(x)v(x)dx \\
&= \left\langle \mathscr{J}^{s+t}u, v\right\rangle
\end{aligned}$$

which implies that $\mathscr{J}^{s+t} = \mathscr{J}^s\mathscr{J}^t$.
And also,

$$\mathscr{J}^0 u(x) = \int_{\mathbb{R}^2} \left(1+|\xi|^2\right)^{\frac{0}{2}} \hat{u}(\xi)e^{i2\pi\xi.x}d\xi = \int_{\mathbb{R}^2} \hat{u}(\xi)e^{i2\pi\xi.x}d\xi = u(x)$$

which follows $\mathscr{J}^0 = \text{identity operator}$.
Finally, let $(\mathscr{J}^s)^{-1}$ be the inverse of operator $\mathscr{J}^s$, then

$$\langle u, v\rangle = \left\langle \mathscr{J}^s(\mathscr{J}^s)^{-1}u, v\right\rangle = \left\langle (\mathscr{J}^s)^{-1}u, \mathscr{J}^s v\right\rangle \tag{2.18}$$

Using the identity operator $\mathscr{J}^0$

$$\langle u,v\rangle = \left\langle \mathscr{J}^0 u,v\right\rangle = \left\langle \mathscr{J}^{s+(-s)}u,v\right\rangle = \left\langle \mathscr{J}^s \mathscr{J}^{-s}u,v\right\rangle = \left\langle \mathscr{J}^{-s}u, \mathscr{J}^s v\right\rangle \qquad (2.19)$$

From (2.18) and (2.19) follows that

$$\left\langle (\mathscr{J}^s)^{-1} u, \mathscr{J}^s v\right\rangle = \left\langle \mathscr{J}^{-s}u, \mathscr{J}^s v\right\rangle$$

which implies $(\mathscr{J}^s)^{-1} = \mathscr{J}^{-s}$.

The Space $\mathrm{H}^s(\mathbb{R}^2)$

For any $s\in\mathbb{R}$, we define the Sobolev space $H^s(\mathbb{R}^2)$ of order s as follows:

Definition 2.20. For $s\in\mathbb{R}$, we denote by $H^s(\mathbb{R}^2)$ the space of distributions $u\in\mathscr{S}'(\mathbb{R}^2)$ such that $\left(1+|\xi|^2\right)^{\frac{s}{2}}\hat{u}(\xi)\in L_2(\mathbb{R}^2)$.

$$\begin{aligned} H^s(\mathbb{R}^2) &= \{u\in\mathscr{S}'(\mathbb{R}^2) : \mathscr{F}_{x\to\xi}\{\mathscr{J}^s u(x)\}\in L_2(\mathbb{R}^2)\} \\ &= \{u\in\mathscr{S}'(\mathbb{R}^2) : \left(1+|\xi|^2\right)^{\frac{s}{2}}\hat{u}(\xi)\in L_2(\mathbb{R}^2)\} \end{aligned}$$

We equip this space with the inner product

$$\langle u,v\rangle_{H^s(\mathbb{R}^2)} = \langle \mathscr{J}^s u, \mathscr{J}^s v\rangle = \int_{\mathbb{R}^2}\left(1+|\xi|^2\right)^s \hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi$$

and with the associated norm $\|.\|_{H^s(\mathbb{R}^2)}$ defined by

$$\|u\|_{H^s(\mathbb{R}^2)} = \sqrt{\langle u,u\rangle_{H^s(\mathbb{R}^2)}} = \|\mathscr{J}^s u\|_{L_2(\mathbb{R}^2)}$$

Plancherel's theorem and equation (2.17), imply that

$$\|u\|^2_{\mathrm{H}^s(\mathbb{R}^2)} = \int_{\mathbb{R}^2}\left(1+|\xi|^2\right)^s|\hat{u}(\xi)|^2 d\xi$$

So, if $s\le t$, then $\|u\|_{\mathrm{H}^s(\mathbb{R}^2)}\le\|u\|_{\mathrm{H}^t(\mathbb{R}^2)}$.

Notice that the Bessel potential $\mathscr{J}^s : \mathrm{H}^s(\mathbb{R}^2)\to L_2(\mathbb{R}^2)$ is a unitary isomorphic. In particular, since $\mathscr{J}^0 =$ identity operator so that $\mathscr{J}^0 u = u$, and

$$\mathrm{H}^0(\mathbb{R}^2) = L_2(\mathbb{R}^2)$$

Remark 2.3. 1. $W_p^0(\Omega) = L^p(\Omega)$.
2. The norm $\sum_{|\alpha|\leq k} \|D^\alpha u\|_{L^p}^p$ is equivalent to the standard norm.
3. In particular $p = 2$, we have $W_2^k(\Omega) = W^k(\Omega)$.

It turns out that the norm topology on $W_2^k(\Omega)$ is complete (see e.g.,[AF03]). So that the Sobolev spaces are Banach space. The case $p = 2$, which is denoted by $W_2^k(\Omega) = W^k(\Omega) = H^k(\Omega)$ is of special importance because these are even Hilbert spaces with the inner products given by

$$\langle u, v\rangle_{H^k(\Omega)} = \sum_{|\alpha|\leq k} \int_\Omega D^\alpha u(x)\overline{D^\alpha v(x)}dx.$$

In particular, for $k = 1$, the space $H^1(\Omega)$, expressed as follows,

$$H^1(\Omega) = \left\{u : u \in L_2(\Omega) \quad \text{and} \quad \frac{\partial u_i}{\partial x_i} \in L_2(\Omega), i = 1,2\right\}.$$

And the inner product in $H^1(\Omega)$ is given by

$$\begin{aligned}\langle u, v\rangle_{H^k(\Omega)} &= \sum_{|\alpha|\leq 1} \int_\Omega \overline{D^\alpha u(x)} D^\alpha v(x)dx = \int_\Omega \overline{u(x)}v(x)dx + \sum_{i=1}^{2} \int_\Omega \overline{\frac{\partial u}{\partial x_i}} \frac{\partial v}{\partial x_i}dx \\ &= \int_\Omega \overline{u(x)}v(x)dx + \int_\Omega \overline{\nabla u} \cdot \nabla v dx,\end{aligned}$$

while the norm is given by

$$\|u\|_{H^1(\Omega)}^2 = \|u\|_{L_2(\Omega)}^2 + \|\nabla u\|_{L_2(\Omega)}^2.$$

If $t \geq s > 0$, then we can see that, $u \in H^t(\Omega)$ implies that $u \in H^s(\Omega)$. We have the inclusions $u \in H^t(\Omega) \subset H^s(\Omega) \subset H^0(\Omega) = L_2(\Omega)$, with continuous injection, i.e., $\|u\|_{H^s(\Omega)} \leq \|u\|_{H^t(\Omega)}$. Hence, for all $k \in \mathbb{N}$ we have the following inclusions.

$$\mathscr{D}(\Omega) \subset H^k(\Omega) \subset H^{k-1}(\Omega) \subset \cdots \subset H^1(\Omega) \subset L_2(\Omega) \subset \mathscr{D}'(\Omega)$$

Definition 2.21. Let G be a closed subset of $\mathbb{R}^2$; we denote by $\mathrm{H}_G^s(\mathbb{R}^2)$ the subspace of $\mathrm{H}^s(\mathbb{R}^2)$ formed by the elements with support in G. that is

$$\mathrm{H}_G^s(\mathbb{R}^2) = \{u \in \mathrm{H}^s(\mathbb{R}^2) : supp\ u \subset G\}$$

Since $H_G^s(\mathbb{R}^2)$ is a closed subspace of $H^s(\mathbb{R}^2)$ and is therefore a Hilbert space when equipped with the restriction of the inner product of $H^s(\mathbb{R}^2)$.

Theorem 2.5. *[DL88, page 98]* $\mathrm{H}^{-s}(\mathbb{R}^2)$ *is the dual space of* $\mathrm{H}^s(\mathbb{R}^2)$ *, i.e.,* $\mathrm{H}^{-s}(\mathbb{R}^2)$ *is the space of all linear functionals on* $\mathrm{H}^s(\mathbb{R}^2)$.

Since $\mathscr{D}(\mathbb{R}^2)$ is dense subset of $\mathrm{H}^s(\mathbb{R}^2)$, the dual space of $\mathrm{H}^s(\mathbb{R}^2)$ is space of distributions, i.e, $\mathrm{H}^{-s}(\mathbb{R}^2) \subset \mathscr{D}'(\mathbb{R}^2)$. For any non-empty open set, $\Omega \subset \mathbb{R}^2$, the space $\mathrm{H}^s(\Omega)$ consists of restriction on Ω of distributions from $\mathrm{H}^s(\mathbb{R}^2)$, i.e.,

$$\mathrm{H}^s(\Omega) = \left\{ u : u = v|_\Omega \text{for} \quad \text{some} \quad v \in \mathrm{H}^s(\mathbb{R}^2) \right\}$$

equipped with norm

$$\|u\|_{H^s(\Omega)} := \inf_{v \in H^s(\mathbb{R}^2), u = v|_\Omega} \|u\|_{H^s(\mathbb{R}^2)}$$

We denote by $\tilde{H}^s(\Omega)$ the closure of $\mathscr{D}(\Omega)$ in $\mathrm{H}^s(\mathbb{R}^2)$, which can be characterized as

$$\tilde{H}^s(\Omega) = \{ g : g \in \mathrm{H}^s(\mathbb{R}^2), supp(g) \subset \overline{\Omega} \}$$

(see e.g., [McL00, Theorem 3.29]. The space $\mathrm{H}^s(\Omega)$ consists of restrictions on Ω of distributions from $\mathrm{H}^s(\mathbb{R}^2)$,

$$H^s(\Omega) = \{ g|_\Omega : g \in \mathrm{H}^s(\mathbb{R}^2) \}$$

and $H_0^s(\Omega)$ the closure of $\mathscr{D}(\Omega)$ in $\mathrm{H}^s(\Omega)$. We denote

$$H^s_{loc}(\Omega) = \{ g : \varphi g \in \mathrm{H}^s(\Omega), \forall \varphi \in \mathscr{D}(\Omega) \}.$$

For infinite (unbounded) domains Ω we will use also the notation

$$H^s_{loc}(\overline{\Omega}) = \{ g : \varphi g \in \mathrm{H}^s(\Omega), \forall \varphi \in \mathscr{D}(\overline{\Omega}) \}.$$

for bounded domains $H^s_{loc}(\overline{\Omega}) = \mathrm{H}^s(\Omega)$.
Note that distributions from $\mathrm{H}^s(\Omega)$ and $\mathrm{H}^s_0(\Omega)$ are defined only in Ω, while distributions from $\tilde{H}^s(\Omega)$ are defined in $\mathbb{R}^2$ and particularly on the boundary $\partial\Omega$. For $s \geq 0$, we can identify $\tilde{H}^s(\Omega)$ with the subset of functions from $H^s(\Omega)$, whose extensions by zero outside Ω belong to $\tilde{H}^s(\mathbb{R}^2)$, i.e., identify functions $u \in \tilde{H}^s(\Omega)$ with their restrictions, $u|_\Omega \in H^s(\Omega)$(see, [Mik11], [AF03], [HCW08]).

Definition 2.22. (Symbol). Let $m \in \mathbb{R}$. Then we define S^m to be the set of all functions $\sigma(x,\xi)$ in $C^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ such that for any two multi-indices α and β , there are positive constants $C_{\alpha,\beta}$, depending on α and β only, such that

$$|\left(D_x^\alpha D_\xi^\beta \sigma(x,\xi) \right)| \leq C_{\alpha,\beta} \left(1 + |\xi|\right)^{m-|\beta|}, \qquad x, \xi \in \mathbb{R}^2$$

We call any function σ in $\cup_{m \in \mathbb{R}} S^m$ a symbol.

A symbol, σ in ξ, of degree at most m, satisfies a bound

$$|\sigma(\xi)| \leq C(1+|\xi|)^m \quad \forall \xi \in \mathbb{R}^2.$$

Definition 2.23. (Pseudo Differential Operator). Let σ be a symbol. Then the pseudo-differential operator T_σ associated with σ is defined by

$$(T_\sigma)(x) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^2} e^{ix.\xi} \sigma(x,\xi)\hat{\varphi}(\xi)d\xi = \mathscr{F}^{-1}_{\xi\to x}\sigma(x,\xi)\hat{\varphi}(\xi), \qquad \varphi \in S$$

with a symbol $\sigma(x,\xi) \in S^m$.

In studying boundary value problem, we shall need to make sense of the restriction $u|_{\partial\Omega}$ as an element of a Sobolev space on $\partial\Omega$, when u belongs to a Sobolev space on Ω.

Definition 2.24. An operator $\gamma^{\pm} : H^s(\Omega^{\pm}) \to H^{\sigma}(\partial\Omega)$ is a trace operator if for each $u \in H^s(\Omega)$ and for any sequence $\varphi_k \in \mathscr{D}(\overline{\Omega})$ converging to u in $H^s(\Omega)$, the sequence of the boundary values $\varphi_k|_{\partial\Omega}$ converges to $\gamma^{\pm}u$ in $H^{\sigma}(\partial\Omega)$. If $\gamma^+u = \gamma^-u$ we denote them as γu.

Lemma 2.1. *Define the trace operator $\gamma : \mathscr{D}(\mathbb{R}^n) \to \mathscr{D}(\mathbb{R}^{n-1})$ by*

$$\gamma u(x') = u(x',0) \quad for \ x' \in \mathbb{R}^{n-1}.$$

If $s > \frac{1}{2}$, then γ has a unique extension to a bounded linear operator

$$\gamma : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}).$$

Proof. For $u \in \mathscr{D}(\mathbb{R}^n)$, the Fourier inversion formula gives

$$\gamma u(x') = \int_{\mathbb{R}^n} \hat{u}(\xi)e^{i2\pi\xi'.x'}d\xi = \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{\infty} \hat{u}(\xi',\xi_n)d\xi_n\right) e^{i2\pi\xi'.x'}d\xi'$$

$$\text{and so} \quad \widehat{\gamma u}(x') = \int_{-\infty}^{\infty} \hat{u}(\xi',\xi_n)d\xi_n = \int_{-\infty}^{\infty} \left(1+|\xi|^2\right)^{-\frac{s}{2}} \left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u}(\xi',\xi_n)d\xi_n.$$

Applying the Cauchy-Schwarz inequality, we obtain the bound

$$|\widehat{\gamma u}(x')|^2 \leq M_s(\xi') \int_{-\infty}^{\infty} \left(1+|\xi|^2\right)^s |\hat{u}(\xi)|^2 d\xi_n$$

where, the substitution $\xi_n = (1+|\xi'|^2)^{\frac{1}{2}}t$,

$$M_s(\xi') = \int_{-\infty}^{\infty} \frac{d\xi_n}{\left(1+|\xi'|^2+|\xi_n|^2\right)^s} = \frac{1}{\left(1+|\xi'|^2\right)^{s-\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{dt}{\left(1+t^2\right)^s}.$$

The integral with respect to t converges because $s > \frac{1}{2}$, so if we write $M_s = M_s(0)$, then

$$\left(1+|\xi'|^2\right)^{s-\frac{1}{2}} |\widehat{\gamma u}(\xi')|^2 \le M_s \int\limits_{-\infty}^{\infty} \left(1+|\xi|^2\right)^s |\hat{u}(\xi)|^2 d\xi_n.$$

Integrating over $\xi' \in \mathbb{R}^{n-1}$ gives $\|\gamma u\|_{\mathrm{H}^{s-1/2}(\mathbb{R}^{n-1})} \le M_s ||u||_{\mathrm{H}^s(\mathbb{R}^n)}$ and since $\mathscr{D}(\mathbb{R}^n)$ is dense in $\mathrm{H}^s(\mathbb{R}^n)$, we obtain a unique continuous extension for γ.

The above Lemma is sharp in the sense that

$$H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}) = \{\gamma u : u \in H^s(\mathbb{R}^n)\} \qquad \textbf{for} \qquad s > \frac{1}{2}$$

because γ has a continuous right inverse.

Theorem 2.6. *[McL00, The Trace Theorem], Define a trace operator*

$$\gamma : \mathscr{D}(\overline{\Omega}) \to \mathscr{D}(\partial\Omega)$$

by $\gamma u = u|_{\partial\Omega}$. If Ω is a $C^{k-1,1}$ domain and if $\frac{1}{2} < s < k$, then γ has a unique extension to a bounded linear operator.

$$\gamma : H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega)$$

and this extension has a continuous right inverse.

Proof. Since $H^s(\mathbb{R}^2)$ is invariant under $C^{k-1,1}$ change of coordinates if $1-k \le s \le k$, one sees, via partition of unity and local flattening of the boundary, that if $\frac{1}{2} < s \le k$, then

$$\|\gamma u\|_{H^{s-\frac{1}{2}}(\partial\Omega)} \le C\|U\|_{H^s(\mathbb{R}^2)}$$

if $u = U|_\Omega$ for $U \in \mathscr{D}(\mathbb{R}^2)$. Hence $\|\gamma u\|_{H^{s-\frac{1}{2}}(\partial\Omega)} \le C\|u\|_{H^s(\Omega)}$ for all $u \in \mathscr{D}(\overline{\Omega})$ and we obtain a unique extension because $\mathscr{D}(\overline{\Omega})$ is dense in $H^s(\Omega)$. A right inverse for γ can be pieced together using the same partition of unity.

Theorem 2.7. *[Tri08, Theorem 4.24], Let Ω be a bounded C^∞ domain in $\mathbb{R}^2$ and let $\partial\Omega$ be its boundary.*

i. *Let $s > \frac{1}{2}$. Then the trace operator γ is a linear and bounded map of $H^s(\Omega)$ onto $H^{s-\frac{1}{2}}(\partial\Omega)$, i.e., $\gamma(H^s(\Omega)) = H^{s-\frac{1}{2}}(\partial\Omega)$.*
ii. *Let $s > \frac{3}{2}$. Then the normal derivative operator $\gamma_1 = \gamma(\frac{\partial}{\partial n})$ is a linear and bounded map of $H^s(\Omega)$ onto $H^{s-\frac{3}{2}}(\partial\Omega)$, i.e., $\gamma_1\left(H^s(\Omega)\right) = H^{s-\frac{3}{2}}(\partial\Omega)$*

2.7 Variational Method

The weak formulation of boundary value problems leads to variational problems and associated operator equations. In particular, the representation of solutions of partial differential equations by using surface and volume potentials requires the solution of boundary integral operator equations to find the complete Cauchy data.

Definition 2.25. Let X,Y be Hilbert spaces with norms $\|\cdot\|_X, \|\cdot\|_Y$ over $\mathbb{K}$ - the set of real or complex numbers. A mapping $\mathscr{A}(\cdot,\cdot) : X \times Y \to \mathbb{K}$ is called a sesquilinear form if for all $u_1, u_2 \in X, v_1, v_2 \in Y$ and $\lambda \in \mathbb{K}$, the following hold:

$$\mathscr{A}(u_1 + \lambda u_2, v_1) = \mathscr{A}(u_1, v_1) + \lambda \mathscr{A}(u_2, v_1)$$
$$\mathscr{A}(u_1, v_1 + \lambda v_2,) = \mathscr{A}(u_1, v_1) + \overline{\lambda} \mathscr{A}(u_1, v_2)$$

If $\mathbb{K} = \mathbb{R}$ we speak of a bilinear form . For a sesquilinear form $\mathscr{A} : H \times H \to \mathbb{C}$ the adjoint sesquilinear form $\mathscr{A}^* : H \times H \to \mathbb{C}$ is defined by

$$\mathscr{A}^*(u,v) = \overline{\mathscr{A}(v,u)}, \qquad \forall \quad u,v \in H$$

If $\mathscr{A} = \mathscr{A}^*$, then it is called Hermitian . The bilinear form $\mathscr{A} : H \times H \to \mathbb{R}$ is called symmetric if $\mathscr{A}(u,v) = \mathscr{A}(v,u)$, for all $u,v \in H$.

A sesquilinear form $\mathscr{A}(\cdot,\cdot) : X \times Y \to \mathbb{K}$ is continuous (or bounded) if there exist a finite constant C, such that,

$$|\mathscr{A}(u,v)| \leq C\|u\|_X \|v\|_Y, \qquad \forall \quad u \in X, v \in Y \tag{2.20}$$

The infimum of the set of C in (2.20) is the norm of $\mathscr{A}(\cdot,\cdot)$ and we write

$$\|\mathscr{A}\| := \sup_{u \in X\setminus\{0\}} \sup_{v \in Y\setminus\{0\}} = \frac{|\mathscr{A}(u,v)|}{\|u\|_X \|v\|_Y}.$$

Let $A : H \to H'$ be a self-adjoint bounded linear operator. Consider the problem: given $f \in H'$, find $u \in H$ such that

$$Au = f \tag{2.21}$$

Instead of the operator equation (2.21) we may consider the variational problem, given $f \in H'$ find $u \in H$ such that

$$\langle Au, v \rangle = \langle f, v \rangle \qquad \forall \quad v \in H \tag{2.22}$$

Obviously, any solution $u \in H$ of the operator equation (2.21) is also a solution of the variational problem (2.22). To show the reverse direction, we now consider $u \in H$ solves the variational problem (2.22). Then using the norm

$$\|Au - f\|_{H'} = \sup_{0 \neq v \in H} \frac{|\langle Au - f, v\rangle|}{\|v\|_H} = 0$$

and therefore $0 = Au - f \in H'$, i.e., $u \in H$ is a solution of the operator equation (2.21).

The operator $A : H \to H'$ induces a bilinear form $\mathscr{A}(u,v) = \langle Au, v\rangle$ for all $u, v \in H$ with the mapping property

$$\mathscr{A}(\cdot,\cdot) : H \times H \to \mathbb{R}.$$

Let $\mathscr{A}(u,v) : H \times H \to \mathbb{R}$. be a continuous bilinear form, given $f \in H'$; we find $u \in H$, solution of

$$\mathscr{A}(u,v) = \langle f, v\rangle \qquad \forall \quad v \in H \tag{2.23}$$

where $\mathscr{A}(u,v)$ may be represented in the form $\mathscr{A}(u,v) = \langle Au, v\rangle$ where A is a bounded linear operator $A : H \to H'$, see e.g., [Ste07, Lemma 3.1]. Then the variational problem (2.23) is equivalent to the linear equation (2.21).

To ensure the unique solvability of the operator equation (2.21) and of the variational problem (2.22) we need to have a further assumption for the operator A and for the bilinear form $\mathscr{A}(\cdot,\cdot)$, respectively. The operator $A : H \to H'$ is called H-elliptic if

$$\langle Au, u\rangle \geq C_1 \|u\|_H^2 \qquad \forall \quad u \in H$$

is satisfied with some positive constant C_1. And also, the sesquilinear form $\mathscr{A}(u,v)$ is said to be H - elliptic if

$$|\mathscr{A}(u,u)| \geq C\|u\|_H^2, \qquad \forall u \in H,\ C > 0$$

Theorem 2.8. *(Lax-Milgramn Lemma, [Ste07, Theorem 3.4]) Let the operator $A : X \to X'$ be bounded and X- elliptic. For any $f \in X'$ there exists a unique solution of the operator equation (2.21) satisfying the estimate*

$$\|u\|_X \leq \frac{1}{C_1^A}\|f\|_{X'}$$

Then, we have the following well known theorem due to Lax-Milgram (see e.g.,[LM72, Theorem 9.1],). The theorem ensures the unique solvability of the operator equation (2.21) and of the variational problem (2.23).

Theorem 2.9. *Let the sesquilinear form $\mathscr{A}(u,v)$ be bounded and H - elliptic, then the variational problem (2.23) admits a unique solution, the mapping $f \to u$ being continuous from $H \to H'$ (or: A is an isomorphism of H onto H').*

2.8 Equivalence of the norms

Lemma 2.2. *If $0 < \lambda < 1$, then $|u|^2_{\lambda,\mathbb{R}^2} = a_\lambda \int_{\mathbb{R}^2} |\xi|^{2\lambda} |\hat{u}(\xi)|^2 d\xi$ where*

$$a_\lambda = \int_{t>0} t^{-2\lambda-1} \int_{|\omega|=1} |e^{2i\pi\omega_1 t} - 1|^2 d\omega dt.$$

Proof. First, we define the forward difference operator δ_h by

$$\delta_h u(x) = u(x+h) - u(x),$$

Then applying the Fourier transform to both sides, we obtain

$$\begin{aligned}\mathscr{F}_{x\to\xi}\delta_h u(x) &= \mathscr{F}_{x\to\xi}\left(u(x+h)-u(x)\right) = \mathscr{F}_{x\to\xi}[u(x+h)] - \mathscr{F}_{x\to\xi}[u(x)]\\ &= \int_{\mathbb{R}^2} u(x+h)e^{-i2\pi\xi.x}dx - \hat{u}(\xi) = \int_{\mathbb{R}^2} u(y)e^{-i2\pi\xi.(y-h)}dy - \hat{u}(\xi)\end{aligned}$$

$$= e^{i2\pi\xi.h}\int_{\mathbb{R}^2} u(y)e^{-i2\pi\xi.y}dy - \hat{u}(\xi) = e^{i2\pi\xi.h}\hat{u}(\xi) - \hat{u}(\xi) = \left(e^{i2\pi\xi.h} - 1\right)\hat{u}(\xi)$$

Since, the Slobodeckii semi-norm is

$$|u|^2_{\lambda,\mathbb{R}^2} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|u(x)-u(y)|^2}{|x-y|^{2\lambda+n}} dxdy$$

By taking substitution $h = y - x$, applying Plancherel's theorem and then reversing the order of integration,

$$
\begin{aligned}
|u|^2_{\lambda,\mathbb{R}^2} &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|u(x)-u(x+h)|^2}{|h|^{2\lambda+n}} dhdx = \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2} \frac{|\delta_h u|^2}{|h|^{2\lambda+n}} dx\right) dh \\
&= \int_{\mathbb{R}^2} \frac{||\delta_h u||^2_{L^2\mathbb{R}^2}}{|h|^{2\lambda+n}} dh = \int_{\mathbb{R}^2} \frac{||\mathscr{F}_{x\to\xi}\{\delta_h u\}||^2_{L^2\mathbb{R}^2}}{|h|^{2\lambda+n}} dh \\
&= \int_{\mathbb{R}^2} \frac{||\left(e^{i2\pi\xi.h}-1\right)\hat{u}(\xi)||^2_{L^2\mathbb{R}^2}}{|h|^{2\lambda+n}} dh \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|\left(e^{i2\pi\xi.h}-1\right)\hat{u}(\xi)|^2}{|h|^{2\lambda+n}} dhd\xi \\
&= \int_{\mathbb{R}^2} |\hat{u}(\xi)|^2 \int_{\mathbb{R}^2} \frac{|\left(e^{i2\pi\xi.h}-1\right)|^2}{|h|^{2\lambda+n}} dhd\xi
\end{aligned}
$$

Now, transform the inner integral to polar coordinate. Letting $h=\rho\omega$, where $\rho=|h|$ and $\omega=\frac{h}{|h|}$. Then $dh=\rho^{n-1}d\rho d\omega$. Therefore,

$$
\begin{aligned}
\int_{\mathbb{R}^2} \frac{|\left(e^{i2\pi\xi.h}-1\right)|^2}{|h|^{2\lambda+n}} dh &= \int_{\rho>0}\int_{|\omega|=1} \frac{|\left(e^{i2\pi\xi.\rho\omega}-1\right)|^2}{\rho^{2\lambda+n}} \rho^{n-1} d\rho d\omega \\
&= \int_{\rho>0} \rho^{-2\lambda-1} \int_{|\omega|=1} |\left(e^{i2\pi\xi.\rho\omega}-1\right)|^2 d\omega d\rho
\end{aligned}
$$

and letting $\rho=|\xi|^{-1}t$,then we obtain

$$
\begin{aligned}
&= \int_{\rho>0} t^{-2\lambda-1}|\xi|^{2\lambda+1} \int_{|\omega|=1} |e^{i2\pi\xi.|\xi|^{-1}t\omega}-1|^2|\xi|^{-1} d\omega dt \\
&= |\xi|^{2\lambda} \underbrace{\int_{\rho>0} t^{-2\lambda-1} \int_{|\omega|=1} |e^{i2\pi\omega t}-1|^2|\xi|^{-1} d\omega dt}_{a_\lambda}
\end{aligned}
$$

therefore,

$$
|u|^2_{\lambda,\mathbb{R}^2} = a_\lambda \int_{\mathbb{R}^2} |\xi|^{2\lambda}|\hat{u}(\xi)|^2 d\xi.
$$

Remark 2.4. For $u \in H^r(\mathbb{R}^2)$, by Fourier transform derivative and Plancherel's theorem, we deduces that

$$
||\partial^\alpha u||_{0,\mathbb{R}^2} = ||\xi^\alpha \hat{u}||_{0,\mathbb{R}^2}
$$

Thus, by definition

$$||u||^2_{H^r(\mathbb{R}^2)} = \sum_{|\alpha|\le r} \int_{\mathbb{R}^2} |\partial^\alpha u|^2 dx = \int_{\mathbb{R}^2} \left(\sum_{|\alpha|\le r} |\xi^\alpha|^2 \right) |\hat{u}(\xi)|^2 d\xi,$$

since,

$$\sum_{|\alpha|\le r} |\xi^\alpha|^2 \sim \left(1+|\xi|^2\right)^r.$$

Then for any arbitrary constants c_1 and c_2, we have

$$c_1 \left(1+|\xi|^2\right)^r \le \sum_{|\alpha|\le r} |\xi^\alpha|^2 \le c_2 \left(1+|\xi|^2\right)^r.$$

Then by using the elementary inequalities

$$\left(1+|\xi|^2\right)^r \lesssim \sum_{|\alpha|\le r} |\xi^\alpha|^2 \lesssim \left(1+|\xi|^2\right)^r$$

we conclude that

$$||u||_{r,\mathbb{R}^2} \eqsim ||(1+|.|^2)^{\frac{r}{2}}||_{0,\mathbb{R}^2}.$$

Theorem 2.10. *If $s \ge 0$, then $W^s(\mathbb{R}^2) = \mathrm{H}^s(\mathbb{R}^2)$ with equivalent norms.*

Proof. Let r be a non-negative integer, and let $0 < \lambda < 1$. Since, the norm in $W^r(\mathbb{R}^2)$ is given by

$$\|u\|_{W^r(\mathbb{R}^2)} = \left(\int_\Omega \sum_{|\alpha|\le r} |D^\alpha u|^2 dx \right)^{\frac{1}{2}}$$

follows that

$$\|u\|^2_{W^r(\mathbb{R}^2)} = \int_\Omega \sum_{|\alpha|\le r} |D^\alpha u|^2 dx = \sum_{|\alpha|\le r} \int_\Omega |D^\alpha u|^2 dx = \sum_{|\alpha|\le r} \|D^\alpha u\|^2_{L_2(\mathbb{R}^2)}$$

thus $\quad \|u\|^2_{W^r(\mathbb{R}^2)} = \sum_{|\alpha|\le r} \|D^\alpha u\|^2_{L_2(\mathbb{R}^2)}.$

By Plancherel's theorem, we have

$$\|u\|^2_{W^r(\mathbb{R}^2)} = \sum_{|\alpha|\le r} \|D^\alpha u\|^2_{L_2(\mathbb{R}^2)} = \sum_{|\alpha|\le r} \|\mathscr{F}_{x\to\xi}\{D^\alpha u\}\|^2_{L_2(\mathbb{R}^2)}.$$

But,

$$\mathscr{F}_{x\to\xi}\{D^\alpha u\} = \int D^\alpha u(x) e^{-i2\pi\xi.x} dx = \int (-1)^{|\alpha|} u(x) D^\alpha e^{-i2\pi\xi.x} dx$$
$$= (-1)^{|\alpha|} \int u(x)(-i2\pi\xi)^\alpha e^{-i2\pi\xi.x} dx = (i2\pi\xi)^\alpha \hat{u}(\xi)$$

Therefore,

$$\begin{aligned}\|u\|^2_{W^r(\mathbb{R}^2)} &= \sum_{|\alpha|\leq r} \|(i2\pi\xi)^\alpha \hat{u}(\xi)\|_{L^2(\mathbb{R}^2)} = \sum_{|\alpha|\leq r} \int_{\mathbb{R}^2} |(i2\pi\xi)^\alpha \hat{u}(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^2} \sum_{|\alpha|\leq r} (i2\pi\xi)^{2\alpha} |\hat{u}(\xi)|^2 d\xi \sim \int_{\mathbb{R}^2} (1+|\xi|^2)^r |\hat{u}(\xi)|^2 d\xi\end{aligned}$$

proving the result if $s = r$ By the Lemma 2.2, if $s = r + \lambda$, we have

$$\|u\|^2_{W^s(\mathbb{R}^2)} = \|u\|^2_{W^r(\mathbb{R}^2)} + \sum_{|\alpha|=r} |D^\alpha u|^2_{\lambda,\mathbb{R}^2}$$

But,

$$\begin{aligned}\sum_{|\alpha|=r} |D^\alpha u|^2_{\lambda,\mathbb{R}^2} &= \sum_{|\alpha|=r} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|D^\alpha\left(u(x)-u(y)\right)|^2}{|x-y|^{2\lambda+n}} dxdy \\ &= \sum_{|\alpha|=r} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|D^\alpha\left(u(x)-u(x+h)\right)|^2}{|h|^{2\lambda+n}} dhdx \\ &= \sum_{|\alpha|=r} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|D^\alpha\left(\delta_h u(x)\right)|^2}{|h|^{2\lambda+n}} dxdh \\ &= \sum_{|\alpha|=r} \int_{\mathbb{R}^2} \frac{\|D^\alpha \delta_h u\|^2_{L_2\mathbb{R}^2}}{|h|^{2\lambda+n}} dxdh \\ &= \sum_{|\alpha|=r} \int_{\mathbb{R}^2} \frac{\|\mathscr{F}_{x\to\xi}\{D^\alpha \delta_h u\}\|^2_{L_2\mathbb{R}^2}}{|h|^{2\lambda+n}} dxdh \\ &= \sum_{|\alpha|=r} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|(i2\pi\xi)^\alpha \left(e^{i2\pi\xi.h}-1\right)\hat{u}(\xi)|^2}{|h|^{2\lambda+n}} dhd\xi \\ &= \sum_{|\alpha|=r} \int_{\mathbb{R}^2} |\hat{u}(\xi)|^2 (i2\pi|\xi|)^\alpha \int_{\mathbb{R}^2} \frac{|\left(e^{i2\pi\xi.h}-1\right)|^2}{|h|^{2\lambda+n}} dhd\xi \\ &= \sum_{|\alpha|=r} \int_{\mathbb{R}^2} a_\lambda |\xi|^{2\lambda} (2\pi|\xi|)^{2\alpha} |\hat{u}(\xi)|^2 d\xi\end{aligned}$$

Therefore,

$$\begin{aligned}
\|u\|^2_{W^s(\mathbb{R}^2)} &= \|u\|^2_{W^r(\mathbb{R}^2)} + \sum_{|\alpha|=r} |D^\alpha u|^2_{\lambda,\mathbb{R}^2} \\
&= \sum_{|\alpha|\le r} \int_{\mathbb{R}^2} (i2\pi|\xi|)^{2\alpha} |\hat{u}(\xi)|^2 d\xi + \sum_{|\alpha|=r} \int_{\mathbb{R}^2} a_\lambda |\xi|^{2\lambda} (2\pi|\xi|)^{2\alpha} |\hat{u}(\xi)|^2 d\xi \\
&= \sum_{|\alpha|\le r} \int_{\mathbb{R}^2} \left(1 + a_\lambda |\xi|^{2\lambda}\right) (2\pi|\xi|)^{2\alpha} |\hat{u}(\xi)|^2 d\xi \\
&\sim \int_{\mathbb{R}^n} (1+|\xi|^2)^{r+\lambda} |\hat{u}(\xi)|^2 d\xi = \|u\|_{\mathrm{H}^{r+\lambda}(\mathbb{R}^2)}
\end{aligned}$$

Corollary 2.8.1 *For any non-empty open set $\Omega \subseteq \mathbb{R}^2$ their is a continuous inclusion $H^s(\Omega) \subseteq W_2^s(\Omega)$ for $s \geq 0$.*

Proof. Given $u \in H^s(\Omega)$, we can find $U \in H^s(\mathbb{R}^2)$ such that $U|_\Omega = u$ and $\|u\|_{H^s(\Omega)} = \|U\|_{H^s(\mathbb{R}^2)}$. By the above theorem, $U \in W_2^s(\mathbb{R}^2)$, so $u \in W_2^s(\Omega)$ and

$$\|u\|_{W_2^s(\Omega)} \leq \|U\|_{W_2^s(\mathbb{R}^2)} \sim \|U\|_{H^s(\mathbb{R}^2)} = \|u\|_{H^s(\Omega)}.$$

Theorem 2.11. *([McL00, Theorem 3.18]) For any non-empty open set $\Omega \subseteq \mathbb{R}^2$ and any real $s \geq 0$, if there exists a continuous linear operator $E : W_2^s(\Omega) \to W_2^s(\mathbb{R}^2)$ such that $Eu|_\Omega = u$ for all $u \in W_2^s(\Omega)$, then $H^s(\Omega) = W_2^s(\Omega)$ with equivalent norm.*

Proof. If $u \in W^s(\Omega)$, then $U|_\Omega = u$ for $U = Eu \in W_2^s(\mathbb{R}^2) = H^s(\mathbb{R}^2)$, so $u \in H^s(\Omega)$ and

$$\|u\|_{H^s(\Omega)} \leq \|U\|_{H^s(\mathbb{R}^2)} \sim \|Eu\|_{W_2^s(\mathbb{R}^2)} \leq C\|u\|_{W_{2^s(\Omega)}}$$

giving a continuous inclusion $W_2^s(\Omega) \subseteq H^s(\Omega)$.

2.9 Fredholm Operators: basic properties

Let $\mathbb{E}$ and $\mathbb{F}$ be two Banach spaces. We denote by $\mathscr{L}(\mathbb{E},\mathbb{F})$ the space of bounded linear operators from $\mathbb{E}$ to $\mathbb{F}$. Let a linear (not necessarily continuous) map $T : \mathbb{E} \to \mathbb{F}$ be given. Recall that

i. The kernel $Ker(T)$ of $T : \mathbb{E} \to \mathbb{F}$ is $Ker(T) = \{x \in \mathbb{E} : Tx = 0 \in \mathbb{F}\}$
ii. The image $Im(T)$ of $T : \mathbb{E} \to \mathbb{F}$ is $Im(T) = \{Tx : x \in \mathbb{E}\}$
iii. The cokernel $Coker(T)$ of $T : \mathbb{E} \to \mathbb{F}$ is $Coker(T) = \mathbb{F}/Ker(T)$

Definition 2.26. A bounded linear operator $T : \mathbb{E} \to \mathbb{F}$ is said to be Fredholm if the subspaces $Ker(T)$ and $Coker(T)$ are finite dimensional and the subspace ImT is closed in $\mathbb{F}$. We denote by $\mathscr{F}(\mathbb{E},\mathbb{F})$ the space of all Fredholm operators from $\mathbb{E}$ to $\mathbb{F}$. The index of a Fredholm operator T is defined by

$$Index(T) = dim(Ker(T)) - dim(Coker(T))$$

The $dim(\mathbb{F} \setminus Im(T))$ also denoted by $dim(coker(T))$.

Note that a consequence of the Fredholmness is the fact that $R(T) = Im(T)$ is closed. Here are the first properties of Fredholm operators.

Theorem 2.12. *If $T : \mathbb{E} \to \mathbb{F}$ is Fredholm operator, and if $K : \mathbb{E} \to \mathbb{F}$ is compact operator, then their sum $T + K : \mathbb{E} \to \mathbb{F}$ is Fredholm, and $index(T + K) = index(T)$.*

For the proof see ([McL00, Theorem 2.26]).

Theorem 2.13. *Assume $T : \mathbb{E} \to \mathbb{F}$ is Fredholm with $index(T) = 0$ and T is injective, then for each $f \in \mathbb{F}$, the inhomogeneous equation $Tu = f$ has a unique solution $u \in \mathbb{E}$.*

For the proof see ([McL00, Theorem 2.27]).

Let X and Y be Banach space, X^* and Y^* be adjoined (dual) spaces of bounded linear functionals defined on X and Y. Consider a linear map $A : X \to Y$, then we define the adjoint $A^* : Y^* \to X^*$ by

$$(A^*u, v) = (u, Av) \qquad \text{for} \quad v \in X, u \in Y^*.$$

Theorem 2.14. *(Fredholm Alternative)[McL00, Theorem 2.27] Assume that $A : X \to Y$ is Fredholm with $index(A) = 0$. There are two, mutually exclusive possibilities:*

i. *The homogeneous equation $Au = 0$ has only the trivial solution $u = 0$. In this case,*
   a. *for each $f \in Y$, the inhomogeneous equation $Au = f$ has a unique solution $u \in X$;*
   b. *for each $g \in X^*$, the adjoint equation $A^*v = g$ has a unique solution $v \in Y^*$.*

ii. *The homogeneous equation $Au = 0$ has exactly p linearly independent solutions $u_1, \cdots, u_p$ for some finite $p \geq 1$. In this case,*
   a. *the homogeneous adjoint equation $A^*v = 0$ has exactly p linearly independent solutions $v_1, \cdots, v_p$;*
   b. *the inhomogeneous equation $Au = f$ is solvable if and only if the right-hand side f satisfies $(v_j, f) = 0$ for $j = 1, \cdots, p$;*
   c. *the inhomogeneous adjoint equation $A^*v = g$ is solvable if and only if the right-hand side g satisfies $(g, u_j) = 0$ for $j = 1, \cdots, p$.*

Theorem 2.15. *Compact perturbations do not change Fredholmness and do not change the index, and zero index is achieved only by compact perturbations of invertible operators. More precisely:*

(i) *If $K \in \mathscr{K}(\mathbb{E}, \mathbb{F})$ and $A \in \mathscr{F}(\mathbb{E}, \mathbb{F})$, then $A + K \in \mathscr{F}(\mathbb{E}, \mathbb{F})$ and $Index(A + K) = Index(A)$.*

(ii) *If* $A \in \mathscr{F}(\mathbb{E},\mathbb{F})$*, then* $Index(A) = 0$ *if and only if* $A = A_0 + K$ *for some invertible operator* A_0 *and some compact operator* K*.*

There is yet another relation between Fredholm and compact operators, know as the Atkinson characterization of Fredholm operators:

Theorem 2.16. *Fredholmness = invertible modulo compact operators. More precisely, given a bounded operator* $A : \mathbb{E} \to \mathbb{F}$*, the following are equivalent:*

(i) *A is Fredholm.*
(ii) *A is invertible modulo compact operators, i.e. there exists an operator* $B \in \mathscr{L}(\mathbb{F},\mathbb{E})$ *and compact operators* K_1 *and* K_2 *such that*

$$BA = I + K_1, \quad AB = I + K_2$$

Theorem 2.17. *([McL00, Theorem 2.33]) If* $A = A_0 + K$*, where* $A_0 : X \to X^*$ *is positive and bounded below, and* $K : X \to X^*$ *is compact, then* $A : X \to X^*$ *is Fredholm with zero index, and hence Theorem 2.14 holds for the equation* $Au = f$*.*

Theorem 2.18. *(Perturbations for Fredholm operator [Mik99]). Let* B_1 *and* B_2 *be two Banach spaces. Let* $\underline{A} : B_1 \to B_2$ *be a linear continuous Fredholm operator with zero index,* $\underline{A}^* : B_2^* \to B_1^*$ *be the operator adjoined to it, and* $dimker\underline{A} = dimker\underline{A}^* = n < \infty$*, where* $ker\underline{A} = span\{\mathring{x}_i\}_{i=1}^n \subset B_1, ker\underline{A}^* = span\{\mathring{x}_i^*\}_{i=1}^n \subset B_2^*$*. Let*

$$\underline{A}_1 x = \sum_{i=1}^{k} h_i h_i^*(x),$$

where $h_i^*, h_i (i = 1, \cdots n)$ *are elements from* B_1^* *and* B_2 *, respectively, such that*

$$det\left[h_i^*(\mathring{x}_j)\right] \neq 0, \qquad det\left[\mathring{x}_i^*(h_j)\right] \neq 0 \quad i,j,= 1,\cdots,n.$$

Then:

i. *the operator* $\underline{A} - \underline{A}_1 : B_1 \to B_2$ *is continuous and continuously invertable;*
ii. *if* $y \in B_2$ *satisfies the solvability conditions,*

$$\mathring{x}_i^*(y) = 0, \qquad i = 1,\cdots,n \tag{2.24}$$

of equation

$$\underline{A}x = y, \tag{2.25}$$

then the unique solution x of equation

$$(\underline{A} - \underline{A}_1)x = y \tag{2.26}$$

is a solution of equation (2.25) such that

$$h_i^*(x) = 0, \qquad i = 1,\cdots,k \tag{2.27}$$

iii. *Vice versa, if x is a solution of equation (2.26) satisfying condition (2.27), then conditions (2.24) are satisfied for the right-hand side y of equation (2.26) and x is a solution of equation (2.25) with the same right-hand side y.*

Chapter 3
Analysis of Two-operator Boundary- Domain integral Equations for variable-coefficient BVPs in 2D

In this chapter, the Dirichlet, Neumann and Mixed boundary value problems for the second order elliptic partial differential equation with variable coefficient in two dimensional bounded open domain are considered. These BVPs are transformed to some direct two-operator BDIEs or BDIDEs based on a specially constructed parametrix. The two-operator BDIEs contain potential operators defined on the domain under consideration and acting on the unknown solution as well as integral operators defined on the boundary and acting on the trace and/or conormal derivative of the unknown solution or on an auxiliary function. The properties of corresponding potential operators are investigated. Solvability, solution uniqueness, and equivalence of the BDIEs to the original BVP and the invertibility of the boundary domain integral operators are investigated in appropriate Sobolev spaces.

3.1 Green's identities and integral relations

Let Ω be a domain in $\mathbb{R}^2$ bounded by a closed infinitely differentiable curve $\partial\Omega$, and $n(x)$ be the exterior unit normal vector, which is defined for all x in $\partial\Omega$. We shall consider the following PDE with scalar variable coefficient

$$Au(x) = A(x,\partial_x)u(x) := \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left[a(x)\frac{\partial u(x)}{\partial x_i}\right] = f(x), \quad x \in \Omega, \tag{3.1}$$

where u is an unknown function and f is a given function in Ω. Also assume $a \in C^\infty(\mathbb{R}^2)$, where $0 < a_{min} \le a(x) \le a_{max} < \infty, \quad \forall x \in \mathbb{R}^2$.

For $\phi,\psi \in \mathscr{D}(\Omega)$, we can easily see that $(A\phi,\psi)_{L_2(\Omega)} = (\phi, A\psi)_{L_2(\Omega)}$ which implies that the operator A in (3.1) is formally self-adjoint. Also we can find the ellipticity constant C such that

$$\sum_{i=1}^{2} a(x)\xi_i^2 \geq C|\xi|^2$$

holds for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^2$. In particular we can choose $C = \min_{x\in\overline{\Omega}} a(x)$.

3.1.1 The first Green identity

We can derive the first Green identity from the well known Theorem of Gauss Ostrogradski, if $h \in C^1(\overline{\Omega})$, then (here we adapt the derivation technique from [DM15])

$$\int_{\Omega} \frac{\partial}{\partial x_i} h(x)dx = \int_{\partial\Omega} \gamma^+ h(x) n_i(x) dS_x, \qquad i = 1,2, \tag{3.2}$$

where $\gamma^+ h(x) = \lim\limits_{\Omega\ni y\to x\in\partial\Omega} h(y)$ for $x \in \Omega$ is the interior boundary trace of $h(x)$.

Using the results for density and trace theorem (see e.g. [McL00, Theorem 3.29, Theorem 3.38]), we can show that, the integral relation (3.2) implies

$$\int_{\Omega} \frac{\partial}{\partial x_i} h_n(x)dx = \int_{\partial\Omega} \gamma^+ h_n n_i(x) dS_x, \qquad i = 1,2$$

Taking the limit as $n \to \infty$ in the above equation

$$|\int_{\Omega} \frac{\partial h_n}{\partial x_i}dx - \int_{\Omega} \frac{\partial h}{\partial x_i}dx| \leq \int_{\Omega} |\frac{\partial h_n}{\partial x_i}dx - \frac{\partial h}{\partial x_i}|dx \leq \|h_n - h\|_{H^1(\Omega)} \to 0$$

Hence $\int_{\Omega} \frac{\partial h_n}{\partial x_i}dx - \int_{\Omega} \frac{\partial h}{\partial x_i}dx \to 0$ as $n \to \infty$. For the right hand side

$$|\int_{\partial\Omega} \gamma^+ h_n n_i dS_x - \int_{\partial\Omega} \gamma^+ h n_i dS_x| \leq \int_{\partial\Omega} |\gamma^+ (h_n - h) n_i| dS_x \leq \|\gamma^+(h_n - h)\|_{L_2(\partial\Omega)}$$

implies $\|\gamma^+(h_n - h)\|_{L_2(\partial\Omega)} \leq C\|h_n - h\|_{H^1(\Omega)} \to 0$ as $n \to \infty$
Hence, $\int_{\partial\Omega} \gamma^+ h_n n_i dS_x - \int_{\partial\Omega} \gamma^+ h n_i dS_x \to 0$ as $n \to \infty$.

Now for $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$ if we put $h(x) = a(x)\frac{\partial u(x)}{\partial x_i} v(x)$ and apply the Gauss-Ostrogradski theorem, we obtain

$$\int_{\Omega} \frac{\partial}{\partial x_j}\left(a(x)\frac{\partial u(x)}{\partial x_j} v(x)\right) dx = \int_{\partial\Omega} n_j \gamma^+ \left(a(x)\frac{\partial u(x)}{\partial x_j} v(x)\right) dS_x.$$

Summing over j and noting the relation

$$\frac{\partial}{\partial x_j}\left[a(x)\frac{\partial u(x)}{\partial x_j} v(x)\right] = \frac{\partial}{\partial x_j}\left[a(x)\frac{\partial u(x)}{\partial x_j}\right] v(x) + a(x)\frac{\partial u(x)}{\partial x_j}\frac{\partial v(x)}{\partial x_j}$$

We obtain the following Green's first identity

$$\mathscr{E}_a(u,v) = -\int_\Omega (Au)(x)v(x)dx + \int_{\partial\Omega} T_a^+ u(x)\gamma^+ v(x)dS_x,$$

where $\mathscr{E}_a(u,v) = \sum_{i=1}^2 \int_\Omega a(x)\frac{\partial u(x)}{\partial x_j}\frac{\partial v(x)}{\partial x_j}$ is the symmetric bilinear form and

$$T_a^+ u(x) = \lim_{\Omega\ni y\to x\in\partial\Omega}\left[\sum_{i=1}^2 n_i(x)a(y)\frac{\partial}{\partial y_i}u(y)\right] \quad \text{for } x\in\partial\Omega \tag{3.3}$$

where $n(x)$ is the exterior (to Ω)unit normal at the point $x\in\Omega$, is the interior co-normal derivative.

From the above proved relation we have verified the following Lemma.

Lemma 3.1.1 *For $u\in H^2(\Omega)$ and $v\in H^1(\Omega)$,*

$$\mathscr{E}_a(u,v) = -(Au,v)_\Omega + (T_a^+ u, \gamma^+ v)_{\partial\Omega}$$

Remark 3.1.1 *For $\mathscr{D}(\Omega), \gamma^+ v = 0$. If $u\in H^1(\Omega)$, then we can find Au as a distribution on Ω by*

$$(Au,v)_\Omega = -\mathscr{E}_a(u,v) \quad \textit{for} \quad v\in\mathscr{D}(\Omega)$$

For a linear operator A, we introduce the following subspace of $H^1(\Omega)$, [Gri85], [Cos88]

$$H^{1,0}(\Omega;A) = \{g\in H^1(\Omega) : Ag\in L_2(\Omega)\}$$

endowed with the norm

$$\|g\|^2_{H^{1,0}(\Omega;A)} := \|g\|^2_{H^1(\Omega)} + \|Ag\|^2_{L_2(\Omega)}.$$

For $u\in H^1(\Omega^+)$ the co-normal differentiation operators on $\partial\Omega$ do not generally exist in the trace sense (3.3). However, if $u\in H^{1,0}(\Omega^+;A)$, one can define the generalized (canonical) co-normal derivative $T_a^+ u = [T_a u]^+ \in H^{-\frac{1}{2}}(\partial\Omega)$ with the help of the Green's first identity

$$\left\langle T_a^+ u, w\right\rangle_{\partial\Omega} := \int_\Omega [(\gamma_{-1}^+ w)Au + E_a(u,\gamma_{-1}^+ w)]dx \qquad \forall w\in H^{\frac{1}{2}}(\partial\Omega), \tag{3.4}$$

where

$$E_a(u,v) := \sum_{i=1}^2 a(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_i}, \quad \forall u,v\in H^(\Omega)$$

and $\gamma_{-1}^+ : H^{\frac{1}{2}}(\partial\Omega)\to H^1(\Omega)$ is a continuous right inverse of the continuous interior trace operator $\gamma^+ : H^1(\Omega)\to H^{\frac{1}{2}}(\partial\Omega)$ while $\langle .,.\rangle$ denotes the duality brackets

between the spaces $H^{\frac{1}{2}}(\partial\Omega)$ and $H^{-\frac{1}{2}}(\partial\Omega)$, extending the usual $L_2(\partial\Omega)$ scalar product.
The operator $T_a^+ : H^{1,0}(\Omega,A) \to H^{-\frac{1}{2}}(\partial\Omega)$ is continuous and gives a continuous extension in $H^{1,0}(\Omega;A)$ of the classical co-normal derivative operator.

For $u \in H^s(\Omega), \quad s > \frac{3}{2}$, the canonical co-normal derivative operator $T_a^+ u$ defined by (3.4) reduces to its strong(classical) canonical derivative form (3.3)and is well defined on $\partial\Omega$ in the trace sense. i.e.,

$$T_a^+ u(x) := T_a^+ (x, n(x), \partial_x) u(x) = \sum_{i=1}^{2} a(x) n_i(x) \gamma^+ \left(\frac{\partial u(x)}{\partial x_i} \right) = a(x)\gamma^+ \left(\frac{\partial u(x)}{\partial n(x)} \right)$$

where $n(x)$ is the exterior (to Ω) unit normal at the point $x \in \partial\Omega$.

Then for $u \in H^{1,0}(\Omega;\Delta), \quad v \in H^1(\Omega)$ the first Green identity holds, [Mik11] [Cos88, Lemma 3.4],

$$\int_\Omega v(x)Au(x)dx = \int_{\partial\Omega} \gamma^+ v(x) T_a^+ u(x) dS(x) - \int_\Omega E_a(u,v)dx. \tag{3.5}$$

3.1.2 The Two-operator second Green identity

In the Green's first formula, by interchanging the role of u and v and subtracting the result, we obtain Green's second formula. For $u, v \in H^{1,0}(\Omega, A)$, the Green's second formula is,

$$\int_\Omega (v(x)Au(x) - u(x)Av(x))\, dx = \int_{\partial\Omega} \gamma^+ v(x) T_a^+ u(x) dS(x) - \int_{\partial\Omega} \gamma^+ u(x) T_a^+ v(x) dS(x). \tag{3.6}$$

i.e.,

$$\int_\Omega (v(x)Au(x) - u(x)Av(x))\, dx = \langle T_a^+ u, \gamma^+ v \rangle_{\partial\Omega} - \langle T_a^+ v, \gamma^+ u \rangle_{\partial\Omega}.$$

As real materials are generally never perfectly isotropic their behaviours are modelled by some non-linear and linear PDEs for which the fundamental solution of the “frozen"-coefficient PDE is not available explicitly (as e.g. in the Lamè system of anisotropic elasticity). For such type of problems, as the authors of [AM11] and [Mik05b] showed the need for two-operator approach in 3D; which is efficient in handling these problems where the one operator approach fails to work. Hence, we need to define a second operator in order to overcome this problem in 2D. (The one operator approach is investigated in [DM15]).

Let us consider the auxiliary linear elliptic partial differential operator B defined by

$$Bu(x) := B(x,\partial_x)u(x) := \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left[b(x) \frac{\partial u(x)}{\partial x_i} \right], \tag{3.7}$$

where $b \in C^\infty(\mathbb{R}^2)$, $\quad 0 < b_{min} \leq b(x) \leq b_{max} < \infty, \quad \forall x \in \mathbb{R}^2$.
Then for $u \in H^{1,0}(\Omega,\Delta) = H^{1,0}(\Omega,B)$ the associate co-normal derivative operator T_b^+ is defined by (3.4) (and for $u \in H^2(\Omega)$ by (3.3)) with a replaced by b.
If $v \in H^{1,0}(\Omega,\Delta), u \in H^1(\Omega)$, then for the operator B in (3.7) holds the Green's first identity,

$$\int_\Omega u(x)Bv(x)dx = \int_{\partial\Omega} \gamma^+ u(x) T_b^+ v(x) dS(x) - \int_\Omega E_b(u,v)dx. \tag{3.8}$$

If $u,v \in H^{1,0}(\Omega,\Delta)$, then subtracting (3.5) from (3.8), we obtain the two-operator second Green identity,

$$\int_\Omega (u(x)Bv(x) - v(x)Au(x))\, dx =$$
$$\int_{\partial\Omega} \left[\gamma^+ u(x) T_b^+ v(x) - \gamma^+ v(x) T_a^+ u(x)\right] dS(x) - \int_\Omega [a(x) - b(x)] \nabla v(x) \cdot \nabla u(x) dx. \tag{3.9}$$

Note that if $a = b$, then the last domain integral disappears, and the two-operator second Green identity reduces to the classical second Green identity, equation (3.6).

3.1.3 Parametrix and potential type operators

Before we give our solution procedure let us recall the idea of surface layer and Logarithmic potential by introducing the parametrix.

Definition 3.1. A function $P_b(x,y)$ of two variables $x, y \in \mathbb{R}^2$ is called a parametrix (Levi function) for the operator $B(x,\partial_x)$ in $\mathbb{R}^2$ if

$$B(x,\partial_x)P_b(x,y) = \delta(x-y) + R_b(x,y),$$

where $\delta(.)$ is a Dirac-delta distribution and $R_b(x,y)$ is a remainder possessing at most a weak (integrable) singularity at $x = y$. i.e,

$$R_b(x,y) = \mathcal{O}(|x-y|^{-\varkappa}) \text{ with } \varkappa < 3.$$

For the operator B, to every point y associate "frozen" coefficient $b(x) = b(y)$ and obtain $\quad B_y = \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left[b(x) \frac{\partial}{\partial x_i} \right] = b(y)\Delta$.

Using equation (2.11), the operator B_y is a constant coefficient and its fundamental solution is given by

$$P_b(x,y) = \frac{\log\left(\frac{|x-y|}{r_0}\right)}{2\pi b(y)} \qquad \text{where} \quad r_0 > 0. \tag{3.10}$$

Now let us show that the function given by (3.10) is the parametrix for the operator $B(x,\partial_x)$. Indeed,

$$\begin{aligned}
B(x,\partial_x)P_b(x,y) &= \sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left[b(x)\frac{\partial P_b(x,y)}{\partial x_i}\right] = \sum_{i=1}^{2}\left[\frac{\partial b(x)}{\partial x_i}\frac{\partial P_b(x,y)}{\partial x_i} + b(x)\frac{\partial^2 P_b(x,y)}{\partial x_i^2}\right] \\
&= \frac{b(x)}{b(y)}\sum_{i=1}^{2}\frac{\partial^2}{\partial x_i^2}\frac{\log\left(\frac{|x-y|}{r_0}\right)}{2\pi b(y)} + \sum_{i=1}^{2}\frac{\partial}{\partial x_i}\frac{\log\left(\frac{|x-y|}{r_0}\right)}{2\pi b(y)}\frac{\partial b(x)}{\partial x_i} \\
&= \frac{b(x)}{b(y)}\Delta\left(\frac{\log\left(\frac{|x-y|}{r_0}\right)}{2\pi}\right) + \frac{1}{2\pi b(y)}\sum_{i=1}^{2}\left(\frac{x_i-y_i}{|x-y|^2}\right)\frac{\partial b(x)}{\partial x_i} \\
&= \frac{b(x)}{b(y)}\delta(x-y) + \sum_{i=1}^{2}\frac{x_i-y_i}{2\pi b(y)|x-y|^2}\frac{\partial b(x)}{\partial x_i} \\
&= \delta(x-y) + \sum_{i=1}^{2}\frac{x_i-y_i}{2\pi b(y)|x-y|^2}\frac{\partial b(x)}{\partial x_i} = \delta(x-y) + R_b(x,y)
\end{aligned}$$

that is

$$B(x,\partial_x)P_b(x,y) = \delta(x-y) + R_b(x,y)$$

Hence $P_b(x,y)$ is a parametrix for the operator $B(x,\partial_x)$ with singularity at $x=y$ (see e.g., [Con00], [Mik02], [CMN09a]) while the corresponding remainder is given by

$$R_b(x,y) = \sum_{i=1}^{2}\frac{x_i-y_i}{2\pi b(y)|x-y|^2}\frac{\partial b(x)}{\partial x_i}, \quad x,y\in\mathbb{R}^2.$$

and $\frac{\partial P_b}{\partial x_i} = \frac{x_i-y_i}{2\pi b(y)|x-y|^2}$, $\quad i=1,2$ and satisfies the above estimate with $\varkappa = 2$ due to the smoothness of the function $b(x)$. From the definition of the fundamental solution of the Laplace operator, we obtain that

$$\sum_{i=1}^{2}\frac{\partial^2}{\partial x_i^2}\left(\frac{1}{2\pi}\log\left(\frac{|x-y|}{r_0}\right)\right) = \Delta_x\left(\frac{1}{2\pi}\log\left(\frac{|x-y|}{r_0}\right)\right) = \delta(x-y).$$

Also, the multiplication of a smooth function $b(y)$ by δ, gives $b(y)\delta(x-y) = b(x)\delta(x-y)$.

Observe that,

$$BP_b(x,y) = \delta(x-y) + \nabla b(x) \cdot \nabla P_b(x,y)$$

and we can write the remainder as $R_b(x,y) = \nabla b(x) \cdot \nabla P_b(x,y)$. If $b(x) = 1$, then the operator $B(x, \partial_x)$ becomes the Laplace operator, Δ, and the parametrix $P_b(x,y)$ becomes its fundamental solution, and $R_b(x,y) \equiv 0$.

3.1.4 Logarithmic and Remainder Potentials

Similar to [CMN09a], [AM11], [DM15], we define the parametrix-based logarithmic and remainder potential operators as:

$$\mathscr{P}_b g(y) = \int_\Omega P_b(x,y) g(x) dx, \tag{3.11}$$

$$\mathscr{R}_b g(y) = \int_\Omega R_b(x,y) g(x) dx \tag{3.12}$$

We deduce representations of the parametrix-based surface potential boundary operators in terms of their counterparts for $b = 1$, that is, associated with the fundamental solution $P_\Delta = \frac{1}{2\pi} \log\left(\frac{|x-y|}{r_0}\right)$ of the Laplace operator Δ. Then

$$\mathscr{P}_b g = \frac{1}{b} \mathscr{P}_\Delta g, \qquad \mathscr{R}_b g = \frac{1}{b} \sum_{i=1}^{2} \partial_i \mathscr{P}_\Delta \left(g \partial_i b\right),$$

We will verify the above formulae as follows:

$$\mathscr{P}_b g(y) = \int_\Omega P_b(x,y) g(x) dx = \int_\Omega \frac{P_\Delta(x,y)}{b(y)} g(x) dx = \frac{1}{b(y)} \int_\Omega P_\Delta(x,y) g(x) dx = \frac{1}{b} \mathscr{P}_\Delta g(y).$$

$$\begin{aligned}
\mathscr{R}_b g(y) &= \int_\Omega R_b(x,y) g(x) dx = \int_\Omega \sum_{i=1}^{2} \frac{x_i - y_i}{2\pi b(y) |x-y|^2} \frac{\partial b(x)}{\partial x_i} g(x) dx \\
&= \frac{1}{b(y)} \int_\Omega \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(\frac{1}{2\pi} \log\left(\frac{|x-y|}{r_0} \right) \right) \frac{\partial b(x)}{\partial x_i} g(x) dx \\
&= \frac{1}{b(y)} \int_\Omega \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(P_\Delta\right) \frac{\partial b(x)}{\partial x_i} g(x) dx = \frac{1}{b} \sum_{i=1}^{2} \partial_i \mathscr{P}_\Delta \left(g \partial_i b\right)(y).
\end{aligned}$$

Remark 3.1.2 *The logarithmic potential*

$$\mathscr{P}_\Delta u(x) = \int_{\mathbb{R}^2} P_\Delta(x,y) u(y) dy \tag{3.13}$$

is a pseudo-differential operator of degree -2

We can verify this by using $P_\Delta(x,y)$ since it is the fundamental solution of the Laplace operator, i.e., $\Delta P_\Delta(x,y) = \delta(x-y)$. Applying the Fourier transform in x we get,

$$-|\xi|^2 \mathscr{F} P_\Delta(\cdot,y) = \mathscr{F}\delta(\cdot - y) = (2\pi)^{-1} e^{-iy\xi}$$

which implies for $\xi \neq 0, P_\Delta(x,y) = -(2\pi)^{-1}\mathscr{F}^{-1}(e^{-iy\xi}\frac{1}{|\xi|^2})$ substituting this into equation (3.13), we get

$$\begin{aligned}
\mathscr{P}_\Delta u(x) &= \int_{\mathbb{R}^2} P_\Delta(x-y)u(y)dy = -\int_{\mathbb{R}^2}(2\pi)^{-1}\mathscr{F}^{-1}(e^{-iy\xi}\frac{1}{|\xi|^2})u(y)dy \\
&= -\int_{\mathbb{R}^2}(2\pi)^{-2}\left[\int_\Omega e^{ix\xi}e^{-iy\xi}\frac{1}{|\xi|^2}d\xi\right]u(y)dy \\
&= -\int_{\mathbb{R}^2}(2\pi)^{-2}\left[\int_\Omega e^{-iy\xi}u(y)dy\right]e^{ix\xi}\frac{1}{|\xi|^2}d\xi = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{ix\xi}\frac{-1}{|\xi|^2}\hat{u}(\xi)d\xi
\end{aligned}$$

$$\text{Therefore,} \quad \mathscr{P}_\Delta u(x) = \mathscr{F}^{-1}_{\xi\to x}(-\frac{1}{|\xi|^2}\hat{u}(\xi))$$

Hence, the logarithmic potential operator is pseudo-differential operator with homogeneous symbol $\sigma(\xi) = -\frac{1}{|\xi|^2}$.

3.1.5 Layer potential

In this section, we discuss properties of layer potentials. The reason why we are interested in layer potentials is that they are good candidates for being solutions to the boundary value problems for Laplace's equation, they are harmonic and also obey certain jump relations on the boundary. Let us introduce the single and double layer potential operators based on the parametrix $P_b(x,y) = \frac{1}{2\pi b(y)}\log\left(\frac{|x-y|}{r_0}\right)$.

Definition 3.2. The single and double layer potential operators corresponding to the parametrix $P_b(x,y)$ are defined

I. Single layer potential operator

$$V_b g(y) := -\int_{\partial\Omega} P_b(x,y)g(x)dS_x \quad \text{for} \quad y \notin \partial\Omega$$

II. Double layer potential operator

$$W_b g(y) := -\int_{\partial\Omega} T^+_{b_x} P_b(x,y) g(x) dS_x \ \text{ for } \ y \notin \partial\Omega$$

where g is some scalar function and the integrals are understood in the distributional sense if g is not integrable.

The corresponding boundary integral (pseudo-differential) operators of direct surface values of the single layer potential $\mathscr{V}_b$ and of the double layer potential $\mathscr{W}_b$ for $y \in \partial\Omega$, are

$$\mathscr{V}_b g(y) := -\int_{\partial\Omega} P_b(x,y) g(x) dS_x,$$
$$\mathscr{W}_b g(y) := -\int_{\partial\Omega} [T^+_b(x, n(x), \partial_x) P_b(x,y)] g(x) dS_x.$$

For $y \in \partial\Omega$, the co-normal derivatives associated with the operator A of the single layer potential and of the double layer potential are given by:

$$T^+_a V_b g(y) := \frac{a(y)}{b(y)} T^+_b V_b g(y), \tag{3.14}$$

$$\mathscr{L}^+_{ab} g(y) := T^+_a W_b g(y) = \frac{a(y)}{b(y)} T^+_b W_b g(y) = \frac{a(y)}{b(y)} \mathscr{L}^+_b g(y), \tag{3.15}$$

The direct value operators associated with (3.14) are

$$\mathscr{W}'_{ab} g(y) := -\int_{\partial\Omega} [T^+_a(y, n(y), \partial_y) P_b(x,y)] g(x) dS_x = \frac{a(y)}{b(y)} \mathscr{W}'_b g(y),$$
$$\mathscr{W}'_b g(y) := -\int_{\partial\Omega} [T^+_b(y, n(y), \partial_y) P_b(x,y)] g(x) dS_x.$$

The verification of the formula (3.14) is given below. Let us first find the value of $T^+_b V_b g(y)$.

$$\begin{aligned}
T^+_b V_b g(y) &= T^+_b(y, n(y), \partial_y) V_b g(y) = \sum_{i=1}^{2} b(y) n_i(y) \frac{\partial}{\partial y_i} V_b g(y) \\
&= \sum_{i=1}^{2} b(y) n_i(y) \frac{\partial}{\partial y_i} \left[-\int_{\partial\Omega} P_b(x,y) g(x) dS_x \right] = -\sum_{i=1}^{2} b(y) n_i(y) \frac{\partial}{\partial y_i} \int_{\partial\Omega} \frac{P_\Delta(x,y)}{b(y)} g(x) dS_x \\
&= -\sum_{i=1}^{2} b(y) n_i(y) \int_{\partial\Omega} \left(\frac{b(y) \frac{\partial}{\partial y_i} P_\Delta(x,y) - P_\Delta \frac{\partial b}{\partial y_i}}{(b(y))^2} \right) g(x) dS_x \\
&= -\sum_{i=1}^{2} b(y) n_i(y) \int_{\partial\Omega} \left(\frac{1}{b(y)} \frac{\partial}{\partial y_i} P_\Delta(x,y) - \frac{P_\Delta}{(b(y))^2} \frac{\partial b}{\partial y_i} \right) g(x) dS_x
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{i=1}^{2} n_i(y)\int_{\partial\Omega}\frac{\partial}{\partial y_i}P_\Delta(x,y)g(x)dS_x + \sum_{i=1}^{2} n_i(y)\frac{1}{b(y)}\frac{\partial b}{\partial y_i}\int_{\partial\Omega}P_\Delta g(x)dS_x\\
&= -\sum_{i=1}^{2} n_i(y)\frac{\partial}{\partial y_i}\int_{\partial\Omega}P_\Delta(x,y)g(x)dS_x + \sum_{i=1}^{2} n_i(y)\frac{1}{b(y)}\frac{\partial b}{\partial y_i}\int_{\partial\Omega}P_\Delta g(x)dS_x\\
&= -\sum_{i=1}^{2} n_i(y)\frac{\partial}{\partial y_i}\left(-V_\Delta g\right)(y) + \sum_{i=1}^{2}\left(b\frac{\partial}{\partial n}[\frac{1}{b}]\right)(y)\left(-V_\Delta g\right)(y)\\
&= \sum_{i=1}^{2} n_i(y)\frac{\partial}{\partial y_i}\left(V_\Delta g\right)(y) - \sum_{i=1}^{2}\left(b\frac{\partial}{\partial n}[\frac{1}{b}]\right)(y)\left(V_\Delta g\right)(y).
\end{aligned}
$$

which implies that

$$
\begin{aligned}
T_a^+V_b g(y) &= T_a(y,n(y),\partial_y)V_b g(y) = \sum_{i=1}^{2} a(y)n_i(y)\frac{\partial}{\partial y_i}V_b g(y)\\
&= \sum_{i=1}^{2} a(y)n_i(y)\frac{\partial}{\partial y_i}\left[-\int_{\partial\Omega}P_b(x,y)g(x)dS_x\right]\\
&= -\sum_{i=1}^{2} a(y)n_i(y)\frac{\partial}{\partial y_i}\int_{\partial\Omega}\frac{P_\Delta(x,y)}{b(y)}g(x)dS_x\\
&= -\sum_{i=1}^{2} a(y)n_i(y)\int_{\partial\Omega}\left(\frac{b(y)\frac{\partial}{\partial y_i}P_\Delta(x,y) - P_\Delta\frac{\partial b}{\partial y_i}}{(b(y))^2}\right)g(x)dS_x\\
&= -\sum_{i=1}^{2} a(y)n_i(y)\int_{\partial\Omega}\left(\frac{1}{b(y)}\frac{\partial}{\partial y_i}P_\Delta(x,y) - \frac{P_\Delta}{(b(y))^2}\frac{\partial b}{\partial y_i}\right)g(x)dS_x\\
&= -\sum_{i=1}^{2} a(y)n_i(y)\int_{\partial\Omega}\frac{1}{b(y)}\frac{\partial}{\partial y_i}P_\Delta(x,y)g(x)dS_x + \sum_{i=1}^{2} a(y)n_i(y)\int_{\partial\Omega}\frac{1}{[b(y)]^2}\frac{\partial b}{\partial y_i}P_\Delta g(x)dS_x\\
&= -\frac{a(y)}{b(y)}\left(\sum_{i=1}^{2} n_i(y)\int_{\partial\Omega}\frac{\partial}{\partial y_i}P_\Delta(x,y)g(x)dS_x - \sum_{i=1}^{2} n_i(y)\frac{1}{b(y)}\frac{\partial b}{\partial y_i}\int_{\partial\Omega}P_\Delta g(x)dS_x\right)\\
&= -\frac{a(y)}{b(y)}\left(\sum_{i=1}^{2} n_i(y)\frac{\partial}{\partial y_i}\left(-V_\Delta g\right)(y) - \sum_{i=1}^{2}\left(b\frac{\partial}{\partial n}[\frac{1}{b}]\right)(y)\left(-V_\Delta g\right)(y)\right)\\
&= -\frac{a(y)}{b(y)}\left(-\sum_{i=1}^{2} n_i(y)\frac{\partial}{\partial y_i}\left(V_\Delta g\right)(y) + \sum_{i=1}^{2}\left(b\frac{\partial}{\partial n}[\frac{1}{b}]\right)(y)\left(V_\Delta g\right)(y)\right).
\end{aligned}
$$

Putting the last row of this equation to the above will give the required relation.

The verification of the formula (3.15) is given below as follows,

$$
\begin{aligned}
T_a^+ W_b g(y) &= T_a(y,n(y),\partial_y) W_b g(y) = \sum_{i=1}^{2} a(y) n_i(y) \frac{\partial}{\partial y_i} W_b g(y) \\
&= \sum_{i=1}^{2} a(y) n_i(y) \frac{\partial}{\partial y_i} \left[-\int_{\partial\Omega} [T_b(x,n(x),\partial_x) P_b(x,y)] g(x) dS_x \right] \\
&= -\sum_{i=1}^{2} a(y) n_i(y) \frac{\partial}{\partial y_i} \int_{\partial\Omega} \sum_{j=1}^{2} b(x) n_j(x) \frac{\partial}{\partial x_j} P_b(x,y) g(x) dS_x \\
&= -\sum_{i=1}^{2} a(y) n_i(y) \frac{\partial}{\partial y_i} \int_{\partial\Omega} \sum_{j=1}^{2} b(x) n_j(x) \frac{\partial}{\partial x_j} \frac{P_\Delta(x,y)}{b(y)} g(x) dS_x \\
&= -\sum_{i=1}^{2} a(y) n_i(y) \int_{\partial\Omega} \sum_{j=1}^{2} b(x) n_j(x) \frac{\partial}{\partial y_i} \left(\frac{1}{b(y)} \frac{\partial P_\Delta(x,y)}{\partial x_j} \right) g(x) dS_x \\
&= -\sum_{i=1}^{2} a(y) n_i(y) \int_{\partial\Omega} \sum_{j=1}^{2} n_j(x) \left(\frac{b(y) \frac{\partial}{\partial y_i} \frac{\partial P_\Delta(x,y)}{\partial x_j} - \frac{\partial P_\Delta}{\partial x_j} \frac{\partial b}{\partial y_i}}{(b(y))^2} \right) (bg)(x) dS_x \\
&= -\sum_{i=1}^{2} a(y) n_i(y) \int_{\partial\Omega} \sum_{j=1}^{2} n_j(x) \frac{1}{b(y)} \frac{\partial}{\partial y_i} \frac{\partial P_\Delta(x,y)}{\partial x_j} (bg)(x) dS_x \\
&+ \sum_{i=1}^{2} a(y) n_i(y) \int_{\partial\Omega} \sum_{j=1}^{2} n_j(x) \frac{\partial P_\Delta}{\partial x_j} \frac{\partial b}{\partial y_i} \frac{1}{(b(y))^2} (bg)(x) dS_x \\
&= -\frac{a(y)}{b(y)} \sum_{i=1}^{2} n_i(y) \frac{\partial}{\partial y_i} \int_{\partial\Omega} n_i(x) \frac{\partial}{\partial x_i} P_\Delta(x,y) (bg)(x) dS_x \\
&+ \frac{a(y)}{b(y)} \sum_{i=1}^{2} n_i(y) \frac{1}{b(y)} \frac{\partial b}{\partial y_i} \int_{\partial\Omega} \sum_{i=1}^{2} n_i(x) \frac{\partial}{\partial x_i} P_\Delta g(x) dS_x \\
&= -\frac{a(y)}{b(y)} \sum_{i=1}^{2} n_i(y) \frac{\partial}{\partial y_i} \left(-W_\Delta(bg)(y)\right) + \frac{a(y)}{b(y)} \sum_{i=1}^{2} n_i(x) \frac{1}{b} \frac{\partial b(y)}{\partial y_i} \left(-W_\Delta(bg)(y)\right) \\
&= \frac{a(y)}{b(y)} \left[\sum_{i=1}^{2} n_i(y) \frac{\partial}{\partial y_i} \left(W_\Delta(bg)(y)\right) + \left([b \frac{\partial}{\partial n} (\frac{1}{b})(y)] \right) \left(-W_\Delta(bg)(y)\right) \right] \\
&= \frac{a(y)}{b(y)} \left(\mathscr{L}_\Delta(bg)(y) + \left([b \frac{\partial}{\partial n} (\frac{1}{b})(y)] \right) \left(W_\Delta(bg)(y)\right) \right) \\
&= \frac{a(y)}{b(y)} T_b^+ W_b g(y) = \frac{a(y)}{b(y)} \mathscr{L}_b g(y).
\end{aligned}
$$

The direct value operators associated with $T_a^+ V_b g(y) = \frac{a(y)}{b(y)} T_b^+ V_b g(y)$ are

$$\mathscr{W}'_{ab}g(y) = -\int_{\partial\Omega}[T_a(y,n(y),\partial_y)P_b(x,y)]g(x)dS_x = \frac{a(y)}{b(y)}\mathscr{W}'_b g(y), \tag{3.16}$$

where

$$\mathscr{W}'_b g(y) = -\int_{\partial\Omega}[T_b(y,n(y),\partial_y)P_b(x,y)]g(x)dS_x. \tag{3.17}$$

Remark 3.1.3 *The direct value of single layer operator $\mathscr{V}_\Delta$, double layer operators $\mathscr{W}_\Delta$ and $\mathscr{W}'_\Delta$, and the hypersingular operator $\mathscr{L}_\Delta$ are pseudo-differential operators.*

Referring to the book [HCW08, pages 550-556], on the subsection 10.1 **Representation of the basic integral operators for the 2D- Laplacian in terms of Fourier series**; The authors gave general formulations concerning the pseudo-differential operators on the boundary $\partial\Omega$ in terms of the Fourier series expansion by analyzing the four basic boundary integral operators $\mathscr{V}_\Delta, \mathscr{W}_\Delta, \mathscr{W}'_\Delta$, and $\mathscr{L}_\Delta$ of the two dimensional Laplacian.

i. The single layer potential operator $\mathscr{V}_\Delta$ on the closed curve $\partial\Omega$ is pseudo-differential operator with homogeneuos symbol $\sigma(x,\xi) = \frac{1}{2|\xi|}$ i.e., pseudo-differential operator of order -1.
ii. The double layer potential operators $\mathscr{W}_\Delta$ and $\mathscr{W}'_\Delta$ are pseudo-differential operators of order zero.
iii. The hypersingular operator $\mathscr{L}_\Delta$ of Laplacian is a pseudo-differential operator of order $+1$ with homogeneuos symbol $\sigma(x,\xi) = \frac{1}{2}|\xi|$.

From equations (3.14)-(3.17) we deduce representations of the parametrix-based surface potential boundary operators in terms of their counterparts for Laplacian, that is, associated with the fundamental solution $P_\Delta = \frac{1}{2\pi}\log\left(\frac{|x-y|}{r_0}\right)$ (see e.g., [DM15] for 2D and [CMN09a] and [AM11] for 3D case).

$$\frac{a}{b}V_a g = V_b g = \frac{1}{b}V_\Delta g, \quad \frac{a}{b}W_a\left(\frac{bg}{a}\right) = W_b g = \frac{1}{b}W_\Delta(bg), \tag{3.18}$$

$$\frac{a}{b}\mathscr{V}_a g = \mathscr{V}_b g = \frac{1}{b}\mathscr{V}_\Delta g, \quad \frac{a}{b}\mathscr{W}_a\left(\frac{bg}{a}\right) = \mathscr{W}_b g = \frac{1}{b}\mathscr{W}_\Delta(bg), \tag{3.19}$$

$$\mathscr{W}'_{ab}g = \frac{a}{b}\mathscr{W}'_b g = \frac{a}{b}\left\{\mathscr{W}'_\Delta(g) + \left[b\frac{\partial}{\partial n}\left(\frac{1}{b}\right)\right]\mathscr{V}_\Delta g\right\}, \tag{3.20}$$

$$\mathscr{L}^+_{ab}g = \frac{a}{b}\mathscr{L}^+_b g = \frac{a}{b}\left\{\mathscr{L}^+_\Delta(bg) + \left[b\frac{\partial}{\partial n}\left(\frac{1}{b}\right)\right]\gamma^+\mathscr{W}_\Delta(bg)\right\}. \tag{3.21}$$

We will verify the above relations as follows:

$$\frac{a}{b}V_a g(y) = -\frac{a}{b}\int_{\partial\Omega} P_a(x,y)g(x)dSx = -\frac{a(y)}{b(y)}\int_{\partial\Omega}\frac{P_\Delta(x,y)}{a(y)}g(x)dSx$$
$$= -\frac{1}{b(y)}\int_{\partial\Omega} P_\Delta(x,y)g(x)dSx = \frac{1}{b}V_\Delta g(y) = -\int_{\partial\Omega} P_b(x,y)g(x)dSx = V_b g(y).$$

$$\frac{a}{b}W_a\left(\frac{bg}{a}\right)(y) = \frac{a}{b}\left\{-\int_{\partial\Omega}\left[T_a^+(x,n(x),\partial_x)P_a(x,y)\right]\left(\frac{bg}{a}\right)(x)dSx\right\}$$
$$= -\frac{a}{b}\int_{\partial\Omega}\sum_{i=1}^{2} a(x)n_i(x)\frac{\partial}{\partial x_i}P_a(x,y)\left(\frac{bg}{a}\right)(x)dSx$$
$$= -\frac{a}{b}\int_{\partial\Omega}\sum_{i=1}^{2} n_i(x)\frac{\partial}{\partial x_i}\frac{P_\Delta(x,y}{a(y)}(bg)(x)dSx$$
$$= -\frac{1}{b}\int_{\partial\Omega}\sum_{i=1}^{2} n_i(x)\frac{\partial}{\partial x_i}P_\Delta(x,y\,(bg)(x)dSx = \frac{1}{b}W_\Delta(bg) = W_b g(y).$$
$$\mathscr{W}'_{ab}g(y) = \frac{a}{b}\mathscr{W}'_b g(y) = \frac{a}{b}\left\{\mathscr{W}'_\Delta(g) + \left[b\frac{\partial}{\partial n}\left(\frac{1}{b}\right)\right]\mathscr{V}_\Delta g\right\}(y)$$

$$\frac{a}{b}\mathscr{W}'_b g(y) = \frac{a}{b}\int_{\partial\Omega}\left[T_b^+(y,n(y),\partial_y)P_b(x,y)\right]g(x)dSx$$
$$= -\frac{a}{b}\int_{\partial\Omega}\sum_{i=1}^{2} b(y)n_i(y)\frac{\partial}{\partial y_i}P_b(x,y)g(x)dSx$$
$$= -\frac{a}{b}\int_{\partial\Omega}\sum_{i=1}^{2} b(y)n_i(y)\frac{\partial}{\partial y_i}\frac{P_\Delta(x,y}{b(y)}g(x)dSx$$
$$= -\frac{a}{b}\int_{\partial\Omega}\sum_{i=1}^{2} b(y)n_i(y)\left[\frac{b(y)\frac{\partial}{\partial y_i}P_\Delta - \frac{\partial b}{\partial y_i}P_\Delta}{[b(y)]^2}\right]g(x)dSx$$
$$= -\frac{a}{b}\int_{\partial\Omega}\sum_{i=1}^{2} n_i(y)\frac{\partial}{\partial y_i}P_\Delta g(x)dSx + \frac{a}{b}\int_{\partial\Omega}\sum_{i=1}^{2} n_i(y)P_\Delta\frac{\partial}{\partial y_i}b(y)g(x)dSx$$
$$= \frac{a}{b}\mathscr{W}'_\Delta g + \frac{a}{b}\sum_{i=1}^{2} n_i(y)\frac{1}{b(y)}\frac{\partial b(y)}{\partial y_i}\int_{\partial\Omega} P_\Delta(x,y)g(x)dSx$$
$$= \frac{a}{b}\mathscr{W}'_\Delta g(y) + \frac{a}{b}\sum_{i=1}^{2} n_i(y)\frac{1}{b(y)}\frac{\partial b(y)}{\partial y_i}(-\mathscr{V}_\Delta g)(y)$$
$$= \frac{a}{b}\left\{\mathscr{W}'_\Delta g(y) + \left[b\frac{\partial}{\partial n}\left(\frac{1}{b(y)}\right)\right]\mathscr{V}_\Delta g(y)\right\}.$$

It is taken into account that b and its derivatives are continuous in $\mathbb{R}^2$ and $\mathscr{L}_\Delta(bg) := \mathscr{L}_\Delta^+(bg) = \mathscr{L}_\Delta^-(bg)$ by the Lyapunov-Tauber theorem.

3.1.6 Representation formula

The construction of appropriate parametrix, $P_b(x,y)$, is essential to derive the representation formula, and therefore to formulate appropriate BDIEs to find the complete Cauchy data $[\gamma^+ u(y), T_a^+ u(y)]$ for $y \in \partial\Omega$. If $u \in H^{1,0}(\Omega;A)$, then from the Green's second identity (3.9), we have the following parametrix-based third Green identity for $y \in \Omega$, [Mik02], cf. also [CMN09a] in 3D and [DM15, Equation 15.6]

$$u(y) = -\int_\Omega u(x)R_b(x,y)dx + \int_{\partial\Omega}\left(\gamma^+ u(x)T_b^+ P_b(x,y) - \gamma^+ P_b(x,y)T_a^+ u(x)\right)dS(x) + \int_\Omega (a(x)-b(x))\,\nabla P_b(x,y)\cdot\nabla u(x)dx + \int_\Omega P_b(x,y)Au(x)dx. \quad (3.22)$$

The integral representation (3.22) does not only contain the usual integral over the boundary as in the case when the parametrix is a fundamental solution, but also integrals over the entire domain with the unknown function and its derivative in the integrand.

Before proving the ellipticity of the single layer potential V_b and of the hypersingular boundary integral operator $\mathscr{L}_{ab}$ we will derive some basic relations of boundary integral operators and study the properties of logarithmic potentials, single and double layer potentials that appears in the representation formula (3.22).

The mapping and jump properties of the parametrix-based Logarithmic and surface potentials follow from [CMN09a], [DM15], [AM11].

Theorem 3.1. *The following operators are continuous*

$$V_b : H^s(\partial\Omega) \to H^{s+\frac{3}{2}}(\Omega), \qquad s \in \mathbb{R}$$
$$W_b : H^s(\partial\Omega) \to H^{s+\frac{1}{2}}(\Omega) \qquad s \in \mathbb{R}.$$

Proof. The continuity of the operator V_a and W_a are proved in [DM15, Theorem 1], using the constant coefficient opertors (see e.g., [McL00], [Cos88], [Ste07]) and using the relation (3.18) as $b(y) > 0$ follows the required result.

Theorem 3.2. *Let $s \in \mathbb{R}$. The following pseudo-differential operators are continuous,*

$$\begin{aligned} \mathscr{V}_b &: H^s(\partial\Omega) \to H^{s+1}(\partial\Omega), \\ \mathscr{W}_b &: H^s(\partial\Omega) \to H^{s+1}(\partial\Omega), \\ \mathscr{W}'_{ab} &: H^s(\partial\Omega) \to H^{s+1}(\partial\Omega), \\ \mathscr{L}^+_{ab} &: H^s(\partial\Omega) \to H^{s-1}(\partial\Omega). \end{aligned}$$

Proof. The proof follows annalogously with (see, e.g., [DM15], Theorem 1). We have the corresponding mappings for the corresponding constant- coefficient operators. Then (3.18)-(3.20) imply the desired result of the theorem claim.

Due to the Rellich compact embedding theorem,(see, e.g., [McL00, Theorem 3.27]) and Theorem 3.2 implies the following assertion.

Theorem 3.3. *Let $s \in \mathbb{R}$. Then the following operators are compact.*

$$\begin{aligned} \mathscr{V}_b &: \tilde{H}^s(\partial\Omega) \to H^s(\partial\Omega), \\ \mathscr{W}_b &: \tilde{H}^s(\partial\Omega) \to H^s(\partial\Omega), \\ \mathscr{W}'_{ab} &: \tilde{H}^s(\partial\Omega) \to H^s(\partial\Omega). \end{aligned}$$

Proof. By the Rellich compact embedding theorem (see, e.g., [McL00, Theorem 3.7], [DM15, Corollary 1]) and the relation $\mathscr{V}_b = \frac{1}{b}\mathscr{V}_\Delta$ and $\mathscr{W}_b = \frac{1}{b}\mathscr{W}_\Delta$ the first two holds true. Next

$$\mathscr{W}'_{ab}g = \frac{a}{b}\left\{\mathscr{W}'_\Delta(g) + \left[b\frac{\partial}{\partial n}\left(\frac{1}{b}\right)\right]\mathscr{V}_\Delta g\right\},$$

$$\mathscr{W}'_{ab}g - \frac{a}{b}\mathscr{W}'_\Delta(bg) = \left[b\frac{\partial}{\partial n}\left(\frac{1}{b}\right)\right]\mathscr{V}_\Delta g.$$

The operator $\mathscr{V}_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is continuous for $g \in H^{-\frac{1}{2}}(\partial\Omega)$ and $a, b \in L_\infty(\Omega)$

$$\|\mathscr{V}_\Delta g\|_{H^{\frac{1}{2}}(\partial\Omega)} \le C_1 \|g\|_{H^{-\frac{1}{2}}(\partial\Omega)}$$

which implies,

$$\begin{aligned} \|\mathscr{W}'_{ab}g - \frac{a}{b}\mathscr{W}'_\Delta(bg)\|_{H^{-\frac{1}{2}}(\partial\Omega)} &\le \max\left|\left[b\frac{\partial}{\partial n}\left(\frac{1}{b}\right)\right]\right| \|\mathscr{V}_\Delta g\|_{H^{\frac{1}{2}}(\partial\Omega)} \\ &\le C\|\mathscr{V}_\Delta g\|_{H^{\frac{1}{2}}(\partial\Omega)} \\ &\le \tilde{C}\|g\|_{H^{-\frac{1}{2}}(\partial\Omega)}, \end{aligned}$$

which implies boundedness of the operator and the embedding $H^{\frac{1}{2}}(\partial\Omega) \subset H^{-\frac{1}{2}}(\partial\Omega)$ is compact, and due to the Rellich compact embedding theorem

$\mathscr{W}'_{ab}g - \frac{a}{b}\mathscr{W}'_{\Delta}(bg)$ is compact.
In addition $\mathscr{W}'_{\Delta} : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ by [DM15, Corollary 1] implying that $\frac{a}{b}\mathscr{W}'_{\Delta} : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is compact. Therefore, $\mathscr{W}'_{ab}g - \frac{a}{b}\mathscr{W}'_{\Delta}(bg) + \frac{a}{b}\mathscr{W}'_{\Delta}(bg) = \mathscr{W}'_{ab}g$ is compact.

The following well-known jump relations might be useful for further discussions.

Theorem 3.4. *Let $g_1 \in H^{-\frac{1}{2}}(\partial\Omega)$ and $g_2 \in H^{\frac{1}{2}}(\partial\Omega)$. Then there holds the following jump relation on $\partial\Omega$.*

$$\gamma^{\pm}V_b g_1 = \mathscr{V}_b g_1, \tag{3.23}$$

$$\gamma^{\pm}W_b g_2 = \mp\frac{1}{2}g_2 + \mathscr{W}_b g_2, \tag{3.24}$$

$$T_a^{\pm}V_b g_1 = \pm\frac{1}{2}\frac{a}{b}g_1 + \mathscr{W}'_{ab}g_1 = \frac{a}{b}\left(\pm\frac{1}{2}g_1 + \mathscr{W}'_b g_1\right). \tag{3.25}$$

$$T_a^{\pm}W_b g_2 = \mathscr{L}_{ab}^{\pm}g_2. \tag{3.26}$$

Theorem 3.5. *Let Ω be a bounded open region in $\mathbb{R}^2$ with closed, infinitely smooth boundary $\partial\Omega$. The operators:*

$$\mathscr{P}_b : \tilde{H}^s(\Omega) \to H^{s+2}(\Omega), \quad s \in \mathbb{R} \tag{3.27}$$

$$: H^s(\Omega) \to H^{s+2}(\Omega), \quad s > -\frac{1}{2} \tag{3.28}$$

$$\mathscr{R}_b : \tilde{H}^s(\Omega) \to H^{s+1}(\Omega), \quad s \in \mathbb{R} \tag{3.29}$$

$$: H^s(\Omega) \to H^{s+1}(\Omega), \quad s > -\frac{1}{2} \tag{3.30}$$

$$\gamma^{+}\mathscr{P}_b : \tilde{H}^s(\Omega) \to H^{s+\frac{3}{2}}(\partial\Omega), \quad s > -\frac{3}{2} \tag{3.31}$$

$$: H^s(\Omega) \to H^{s+\frac{3}{2}}(\partial\Omega), \quad s > -\frac{1}{2} \tag{3.32}$$

$$\gamma^{+}\mathscr{R}_b : \tilde{H}^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \quad s > -\frac{1}{2} \tag{3.33}$$

$$: H^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \quad s > -\frac{1}{2} \tag{3.34}$$

$$T_a^{+}\mathscr{P}_b : \tilde{H}^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \quad s > -\frac{1}{2} \tag{3.35}$$

$$: H^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \quad s > -\frac{1}{2} \tag{3.36}$$

$$T_a^{+}\mathscr{R}_b : \tilde{H}^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad s > \frac{1}{2} \tag{3.37}$$

$$: H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad s > \frac{1}{2} \tag{3.38}$$

are continuous and the operators

$$\mathscr{R}_b : H^s(\Omega) \to H^s(\Omega), \quad s > -\frac{1}{2} \tag{3.39}$$

$$: H^s(\Omega) \to H^{s,0}(\Omega, A), \quad s > 1 \tag{3.40}$$

$$\gamma^+\mathscr{R}_b : H^s(\Omega) \to H^{s-\frac{1}{2}}(S_1), \quad s > -\frac{1}{2} \tag{3.41}$$

$$T_a^+\mathscr{R}_b : H^s(\Omega) \to H^{s-\frac{3}{2}}(S_1), \quad s > \frac{1}{2} \tag{3.42}$$

are compact.

Proof. Since $\partial\Omega \in C^\infty$, the Logarithmic potential operator $\mathscr{P}_\Delta$ is a pseudo-differential operator of order -2 see Remark 3.1.2 on $\mathbb{R}^2$. Hence, the continuity of the operators (3.28), (3.30), (3.32), (3.34), (3.36) and (3.38) follows from the mapping properties of pseudo-differential operator on $\mathbb{R}^2$, thus for any $s \in \mathbb{R}$, the mapping $\mathscr{P}_\Delta : H^s_{comp}(\mathbb{R}^2) \to H^{s+2}_{loc}(\mathbb{R}^2)$ is continuous and application of trace theorem along with the formulae

$$\mathscr{P}_b g = \frac{1}{b}\mathscr{P}_\Delta,$$

$$\mathscr{R}_b g = -\frac{1}{b}\sum_{j=1}^{2} \partial_j \left[\mathscr{P}_\Delta(g\partial_j b)\right]$$

$$\text{and} \quad \mathscr{P}_\Delta g := \int_\Omega \frac{1}{2\pi} \log\left(\frac{|x-y|}{r_0}\right) g(x)dx.$$

To prove the remaining items of the theorem, we first assume that $s \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, where $\tilde{H}^s(\Omega) = H^s(\Omega)$,

To prove the case, suppose $g \in H^s(\Omega)$, with $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$. Clearly, $\partial_j g \in H^{s-1}(\Omega)$ and $\gamma^+ g \in H^{s-\frac{1}{2}}(\partial\Omega)$, due to the continuity of the operator $\partial_j : H^s(\Omega) \to H^{s-1}(\Omega)$ and the trace theorem. Then integrating by parts we have the representation,

$$\partial_j \mathscr{P}_\Delta g(y) = \mathscr{P}_\Delta(\partial_j g)(y) + V_\Delta(n_j\gamma^+ g)(y) \quad y \in \Omega \tag{3.43}$$

where $n_j (j = 1, 2)$ are the components of the outward unit normal vector to $\partial\Omega$. Due to (3.43) and the mapping properties of the single layer potential, we conclude that $\partial_j \mathscr{P}_\Delta : H^s(\Omega) \to H^{s+1}(\Omega)$ is continuous for $j = 1, 2$ *i.e.*, $\nabla\mathscr{P}_\Delta : H^s(\Omega) \to H^{s+1}(\Omega)$ is continuous. This implies that $\mathscr{P}_\Delta : H^s(\Omega) \to H^{s+2}(\Omega)$ is continuous.

Further, with the help of these result and the relation (3.43), we can verify by induction that the operator (3.28) is continuous for $s \in \left(k-\frac{1}{2}, k+\frac{1}{2}\right)$, where k is an arbitrary non-negative integer. For the values $s = k+\frac{1}{2}$ the continuity of the operator (3.28) then follows due to the complex interpolation properties of Bessel potential spaces.

$$\mathscr{P}_b : H^s(\Omega) \to H^{s+2,0}(\Omega;A) \;\; s \geq 0$$

For $s = 0$, let $g \in H^0(\Omega) = L_2(\Omega)$, we have $\mathscr{P}_b g \in H^2(\Omega)$, from (3.27) and (3.28), then

$$\begin{aligned}\Delta \mathscr{P}_b g &= \Delta \left[\frac{1}{b}\mathscr{P}_\Delta g\right] \\ &= \frac{1}{b}g + 2\sum_{j=1}^{2} \partial_j \left[\frac{1}{b}\right] \partial_j \left[\mathscr{P}_\Delta g\right] + \left[\frac{1}{b}\right] \mathscr{P}_\Delta g \quad \text{in} \quad \mathbb{R}^2\end{aligned}$$

where $\mathscr{P}_\Delta := \mathscr{P}|_{b=1}$ and we take account into $\mathscr{P}_\Delta g = g$, then the first term of the second line of the above equation belongs to $L_2(\Omega)$ while since $b \in C^\infty(\overline{\Omega}), b > 0$, the sum of the second and the third terms belongs to $H^1(\Omega)$ and it can be extended by zero to $\tilde{H}^0(\Omega)$, which completes the proof of continuity for the operator $\mathscr{P}_b$ for $s = 0$. For $s > 0$, the continuity of (3.27) and (3.28) implies continuity of operator (3.29) for $s \geq 0$

Evidently, the proof of (3.33) and (3.37) are direct consequences of the trace theorem. We can follow the same procedure to prove the claim of of the theorem concerning the operator $\mathscr{R}_b$. The continuity of the operator (3.36)-(3.39) follows if we remark that for the chosen s the canonical derivative can be understood in the classical sense. In addition, for the case $a \neq b$, follows by taking into account the relation $T_a^+ = \frac{a}{b}T_b^+$ for (3.36)-(3.39) and (3.32)

3.1.7 Invertibility of the single layer potential in 2D

The boundary integral operator $\mathscr{V}_\Delta$ is a Fredholm operator of index zero [McL00, Theorem 7.6]. Thus the relation (3.18) [DM15, Equation 15.9] lead to the same result for the single layer potential $\mathscr{V}_b$. For the 3D case the following holds [CMN09a, Lemma 4.2(i)].

$$\text{For} \quad \Psi^* \in H^{-\frac{1}{2}}(\partial\Omega), \;\; \text{if} \;\; V_b\Psi^*(y) = 0, y \in \Omega, \text{then} \quad \Psi^* = 0,$$

which implies the invertibilty of the single layer potential operator mapping from $H^{-\frac{1}{2}}(\partial\Omega)$ to $H^{\frac{1}{2}}(\partial\Omega)$. But this is not true for 2D case. It is well known (see, e.g.[Con00, Remark 1.42(ii)], [Ste07, Proof of Theorem 6.22]) that for some 2D domains the Kernel of the operator $\mathscr{V}_\Delta$ is nontrivial, which by the first relation (3.18) is also nontrivial as well for some domains. The following example will illustrate this fact.

Example 3.1.1 *Take the density function $\phi \equiv 1$ and $\Omega = B(0,R)$ to be a disk of radius R centered at the origin and $\partial\Omega = \partial B(0,R)$ be the circular boundary of the disk. We can show that*

$$b(y)V_b\phi(y) = V_\Delta \phi(y) = \begin{cases} R\log\frac{r_0}{|y|}, & \text{for } |y| > R \\ R\log\frac{r_0}{R}, & \text{for } |y| \leq R. \end{cases}$$

Proof. Let $\phi \equiv 1$, then

$$(V_\Delta \phi)(y) = -\frac{1}{2\pi}\int_{|x|=R} \log\frac{|y-x|}{r_0} dS_x.$$

If $|y| > R$, then the function $g(z) = \log\frac{|y-z|}{r_0}$ is harmonic in the disk $B(0,R)$. Then $g(z)$ has the mean value property (cf., [Hac92], Lemma 2.3.5),

$$\log|y| = g(0) = \frac{1}{2\pi R}\int_{|x|=R} g(x)dS_x.$$

Therefore,

$$R\log\frac{|y|}{r_0} = \frac{1}{2\pi}\int_{|x|=R} g(x)dS_x = \frac{1}{2\pi}\int_{|x|=R} \log\frac{|y-x|}{r_0} dS_x, \quad for \ \ |y| > R. \tag{3.44}$$

Which implies,

$$R\log\frac{r_0}{|y|} = -\frac{1}{2\pi}\int_{|x|=R} \log\frac{|y-x|}{r_0} dS_x, \quad for \ \ |y| > R. \tag{3.45}$$

For $|y| \leq R$ in paricular take $y = 0$

$$(V_\Delta \phi)(0) = -\frac{1}{2\pi}\int_{|x|=R} \log\frac{|x|}{r_0} dS_x = -R\log\frac{R}{r_0} = R\log\frac{r_0}{R}.$$

The relation (3.45) implies that, the limit of the value of the potential when $|y|$ approach the boundary from exterior is given by:

$$\lim_{|y|\to R^+} (V_\Delta \phi)(y) = R\log\frac{r_0}{R}$$

Furthermore, since the single layer potential is continuous on $\mathbb{R}^2$ we have

$$(V_\Delta \phi)(y) = R \log \frac{r_0}{R} \quad for \;\; |y| = R.$$

To determine the value of the potential inside the disk for $y \neq 0$, we use the maximum/ minimum principle. Since the single layer potential is harmonic on Ω it has neither maximum nor minimum in the disk.
Let

$$C_0 = (V_\Delta \phi)(y_0) \quad \text{for} \quad 0 < |y_0| < R.$$

If we assume $C_0 \neq R \log \frac{r_0}{R}$, i.e., C_0 is different from the value the potential on the boundary, we will arrive at contradiction of the maximum principle. Thus $(V_\Delta \phi)(y)$ is continuous on $\overline{\Omega}$. Therefore, $(V_\Delta \phi)(y) = R \log \frac{r_0}{R}, \quad \text{for} \quad |y| \leq R.$

Remark 3.1.4 *In the above example, if we take the value of $R = r_0$, and since $b(y) \neq 0$, then $V_b \phi(y) = 0$ in $\overline{\Omega}$; and the kernel of single layer V_b is nontrivial and therefore V_b is not invertible.*

In order to have the invertibility for the single layer potential operator in 2D, we define the following subspace of the space $H^{-\frac{1}{2}}(\partial\Omega)$ based on (see e.g. [Ste07, Equation (6.30)], and [McL00, Corollary 8.11])

$$H_{**}^{-\frac{1}{2}}(\partial\Omega) := \left\{ \phi \in H^{-\frac{1}{2}}(\partial\Omega) : \langle \phi, 1 \rangle = 0 \right\}, \tag{3.46}$$

where the norm in $H_{**}^{-\frac{1}{2}}(\partial\Omega)$ is the induced norm in $H^{-\frac{1}{2}}(\partial\Omega)$. The following Theorem is stated and proved for $\mathscr{V}_a$ in [DM15, Theorem 4]

Theorem 3.6. *If $\Psi \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$ satisfies $\mathscr{V}_b \Psi = 0$ on $\partial\Omega$, then $\Psi = 0$*

Proof. The theorem holds for the operator $\mathscr{V}_\Delta$ (see, e.g., [McL00, Corollay 8.11(ii)]), which by the relation $\mathscr{V}_b = \frac{1}{b}\mathscr{V}_\Delta$ as $b(y) \neq 0$ implies it for the operator $\mathscr{V}_b$ as well.

The following Lemma gives information about the behaviour of the single layer potential and its gradient at the far field and it helps to give an alternative proof of Theorem 3.6 (see, e.g., [Ste07, Lemma 6.21]).

Lemma 3.1. *For $x_0 \in \Omega$ and $y \in \mathbb{R}^2$, assume $|y - x_0| > max\{1, 2diam(\Omega)\}$ to be satisfied, and the coefficient $b(y), \;\; \nabla b(y) \in L_\infty(\mathbb{R}^2)$. Let $\Psi \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$, then the single layer potential operator $V_b \Psi$ has the bounds*

$$|(V_b \Psi)(y)| \leq C_1(\Psi) \frac{1}{|y - x_0|}, \quad and \quad |\nabla (V_b \Psi)(y)| \leq C_2(\Psi) \frac{1}{|y - x_0|^2}.$$

The following is an alternative proof of Theorem 3.6.

Proof. For $x \in \Omega$ and $diam(\Omega) < \frac{R}{2}$ define

$$B(x,R) = \{y \in \mathbb{R}^2 : |y-x| < R\}.$$

Then $u(y) = b(y)V_b\Psi(y) = V_\Delta\Psi(y)$ satisfies $\Delta u(y) = 0$ for $y \in B(x,R)\backslash\overline{\Omega} := G$ and $\gamma u = 0$ on $\partial\Omega$. Applying Green's first identity for the Laplace's operator Δ, with respect to the bounded domain G,we have

$$\mathscr{E}_\Delta(u,u)_G = -\left\langle T^-_{a_\Delta}u, \gamma^- u\right\rangle_{\partial\Omega} + \left\langle T^+_{a_\Delta}u, \gamma^+ u\right\rangle_{\partial B(x,R)}, \tag{3.47}$$

where we have used the opposite direction of the exterior normal vector along the boundary $\partial\Omega$. The first integral in the right side of equation (3.47) vanishes and Lemma 3.1 we obtain the following bound,

$$|\left\langle T^+_\Delta u, \gamma^+ u\right\rangle_{\partial B(x,R)}| \leq C_1(\Psi)C_2(\Psi)\int_{|y-x|=R} |y-x|^{-3} dS_x \leq CR^{-2}.$$

If we let R to be very large, we get $\mathscr{E}_\Delta(u,u)_{\Omega^c} = 0$ which implies the gradient $\nabla u = 0$ on Ω^c and thus u is constant on each component of Ω^c. Hence, u is identically zero, and also $V_b\Psi = \frac{1}{b(y)}u(y) = 0$. From the jump properties of the single layer potential, we have $\Psi = T^+_a V_b\Psi - T^-_a V_b\Psi = 0$ which implies $\Psi = 0$.

3.1.8 Logarthmic capacity

The unique solvability of the variational problem follows from the $H^{-\frac{1}{2}}_{**}(\partial\Omega)$ - ellipticity of the single layer potential V_b , (see [Ste07, Theorem 6.22]). The resulting solution w_{eq} is denoted as the natural density . We can compute the Lagrange parameter by using

$$\lambda = \langle V w_{eq}, w_{eq}, \rangle_{\partial\Omega}.$$

In the three–dimensional case it follows that $\lambda > 0$ [Ste07, Theorem 6.22]. In this case the Lagrange parameter λ is called the capacity of $\partial\Omega$. In the two–dimensional case we define the logarithmic capacity by

$$Cap_{\partial\Omega} = e^{-2\pi\lambda}.$$

For a positive number $r \in \mathbb{R}$ we may define the parameter dependent fundamental solution

$$\mathscr{E}^*_r = \frac{1}{2\pi}\log|x-y| - \frac{1}{2\pi}logr$$

which induces an associated boundary integral operator

$$(V_r w)(x) = -\int_{\partial\Omega} \mathscr{E}_r^*(x,y)w(y)dS_y \quad \text{for} \quad x \in \partial\Omega$$

satisfying

$$(V_r w_{eq})(x) = \frac{1}{2\pi}\log r + \lambda = \frac{1}{2\pi}\log\frac{r}{Cap_{\partial\Omega}}$$

If the logarithmic capacity $Cap_{\partial\Omega} < r$, then we conclude $\lambda > 0$. To ensure $Cap_{\partial\Omega} < r$ a sufficient criteria is to assume $diam(\Omega) < r$. This assumption can be always guaranteed when considering a suitable scaling of the domain $\Omega \subset \mathbb{R}^2$.

Theorem 3.7. *i. The operator $\mathscr{V}_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$, is $H^{-\frac{1}{2}}(\partial\Omega)$- elliptic. i.e., $\langle \mathscr{V}_\Delta \Psi, \Psi\rangle_{\partial\Omega} \geq c\|\Psi\|^2_{H^{-\frac{1}{2}}(\partial\Omega)}$ for all $\Psi \in H^{-\frac{1}{2}}(\partial\Omega)$ if and only if $Cap_{\partial\Omega} < r_0$.*

ii. The operator $\mathscr{V}_b : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$, has a bounded inverse if and only if $Cap_{\partial\Omega} \neq r_0$.

Proof. The proof is similar to [McL00, Theorem 8.16], and [Ste07, Theorem 6.23] for part (i) if $r_0 = 1$. To show the ellipticity of the single layer potential $\mathscr{V}_\Delta$, let $\Psi_{eq} \in H^{-\frac{1}{2}}(\partial\Omega)$ be the equilibrium density and put

$$\lambda = \mathscr{V}_\Delta \Psi_{eq} = \frac{1}{2\pi}\log\left(\frac{r_0}{Cap_{\partial\Omega}}\right).$$

Then for arbitrary $\Psi \in H^{-\frac{1}{2}}(\partial\Omega)$, let $\langle 1, \Psi\rangle_{\partial\Omega} = \alpha \in \mathbb{R}$ and define $\Psi_0 = \Psi - \alpha\Psi_{eq}$. Observe that $\langle 1, \Psi_0\rangle_{\partial\Omega} = 0$ i.e. $\Psi_0 \in H_{**}^{-\frac{1}{2}}(\partial\Omega), \Psi = \Psi_0 + \alpha\Psi_{eq}$ and $\mathscr{V}_\Delta\Psi = \mathscr{V}_\Delta\Psi_0 + \lambda\alpha$
since $\langle \mathscr{V}_\Delta\Psi_0, \Psi_{eq}\rangle_{\partial\Omega} = \langle \Psi_0, \mathscr{V}_\Delta\Psi_{eq}\rangle_{\partial\Omega} = 0$, we have

$$\langle \mathscr{V}_\Delta\Psi, \Psi\rangle_{\partial\Omega} = \langle \mathscr{V}_\Delta\Psi_0, \Psi_0\rangle_{\partial\Omega} + \lambda\alpha^2. \tag{3.48}$$

If $Cap_{\partial\Omega} \geq r_0$, then $\langle \mathscr{V}_\Delta\Psi_{eq}, \Psi_{eq}\rangle_{\partial\Omega} = \lambda \leq 0$. But the boundary operator $\mathscr{V}_\Delta$ is strictly positive-definite, $\langle \mathscr{V}_\Delta\Psi, \Psi\rangle_{\partial\Omega} > 0$ for all $\psi \in \mathrm{H}^{-\frac{1}{2}}(\Omega)$ (see, e.g., [McL00, Theorem 8.12]). If $Cap_{\partial\Omega} < r_0$, then $\lambda > 0$, in addition $\langle \mathscr{V}_\Delta\Psi_0, \Psi_0\rangle_{\partial\Omega} > 0$. Hence both terms on the right hand side of (3.48) are non negative, and by Theorem 3.6 the first is zero iff $\Psi_0 = 0$. Thus $\langle \mathscr{V}_\Delta\Psi, \Psi\rangle_{\partial\Omega} \geq 0$, with equality iff Ψ_0 and $\alpha = 0$ i.e., if and only if $\Psi = 0$. Hence, $\mathscr{V}_\Delta$ is strictly positive definite on the whole of $H^{-\frac{1}{2}}(\partial\Omega)$.
Now since $\mathscr{V}_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$, is Fredholm operator of zero index, and $\mathscr{V}_\Delta$ is strictly positive definite, *ker* $\mathscr{V}_\Delta = \{0\}$ we conclude that $\mathscr{V}_\Delta$ is $H^{-\frac{1}{2}}(\partial\Omega)$-elliptic.

To prove (ii) we note that if $r_0 = Cap_{\partial\Omega}$, then $\mathscr{V}_\Delta$ cannot be invertible, because $\mathscr{V}_\Delta \Psi_{eq} = 0$ Thus suppose that $Cap_{\partial\Omega} \neq r_0$ and $\mathscr{V}_\Delta \Psi = 0$. we have $\mathscr{V}_\Delta \Psi_0 = -\lambda\alpha$ hence $\langle \mathscr{V}_\Delta \Psi_0, \Psi_0 \rangle_{\partial\Omega} = 0$ and therefore, $\Psi_0 = 0$ and by Theorem 3.6. in turn $\alpha = 0$ because $\lambda \neq 0$ giving $\Psi = 0$. Thus the homogeneuos equation has only the trivial solution, and $\mathscr{V}_\Delta$ is invertible. Therefore the invertibility of $V_b : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ follows.

There is a connection between the logarithmic capacity and the Euclidean diameter of Ω(see, e.g., [YaS88]). In particular, $Cap_{\partial\Omega} < diam(\Omega)$. Therefore to ensure $Cap_{\partial\Omega} < r_0$ a sufficient criteria is to assume $diam(\Omega) < r_0$(see, e.g.,[McL00, Theorem 8.12, Theorem 8.16], [Ste07, Theorem 6.23]).

Theorem 3.8. *Let $\Omega \subset \mathbb{R}^2$ have diameter $diam(\Omega) < r_0$. Then the single layer potential operator $\mathscr{V}_b : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is invertible.*

Proof. For $diam(\Omega) < r_0$ the operator $\mathscr{V}_b : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is $H^{-\frac{1}{2}}(\partial\Omega)-$ elliptic by Theorem 3.7 and also $\mathscr{V}_b$ is invertible (see, e.g., [DM15, Equation 15.8]). Then by the first relation in (3.18) the invertibility of the operator $\mathscr{V}_b : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ follows.

3.1.9 The Two-operator third Green identity

Let $u, v \in H^{1,0}(\Omega;\Delta)$. Then by using the two-operator second Green identity (3.9); and replacing $v(x) = P_b(x,y)$ on the equation (3.9), then by the standard limiting procedures we have the following parametrix-based two-operator third Green identity for $y \in \Omega$. Note that the direct substitution of $v(x)$ by $P_b(x,y)$ in the second Green identity is not possible as it has singularity at $x = y$. Thus we can avoid this difficulty by replacing Ω by $\Omega \backslash D(y,\varepsilon)$, where $D(y,\varepsilon)$ is a disk of radius ε centred at y;

$$\int_{\Omega_\varepsilon} \{u(x)BP_b(x,y) - P_b(x,y)Au(x)\}\, dx =$$
$$\int_{\partial\Omega_\varepsilon} \left[\gamma^+ u(x) T_b^+ P_b(x,y) - \gamma^+ P_b(x,y) T_a^+ u(x)\right] dS(x) - \int_{\Omega_\varepsilon} (a(x) - b(x))\, \nabla P_b(x,y) \cdot \nabla u(x) dx.$$

where $\Omega_\varepsilon = \Omega \setminus D_\varepsilon$ and taking the limit $\varepsilon \to 0$ we then arrive at the required two operator BDIE,

$$u + \mathscr{Z}_b u + \mathscr{R}_b u - V_b T_a^+ u + W_b \gamma^+ u = \mathscr{P}_b Au \quad \text{in} \quad \Omega, \tag{3.49}$$

where

$$\mathscr{Z}_b u(y) = -\int_\Omega (a(x)-b(x))\,\nabla_x P_b(x,y)\cdot\nabla u(x)dx = \frac{1}{b(y)}\sum_{i=1}^{2}\partial_i \mathscr{P}_\Delta\left[(a-b)(\partial_i u)\right](y).$$

Remark 3.1.5 *The equation (3.49) is Boundary Domain Integro-Differential representation.*

To verify this result, let $u \in D(\Omega)$, let $y \in \Omega$ be an arbitrarily fixed interior point in Ω. Denote the disk by $D(y,\varepsilon) \subset \Omega$ be a disk centred at a point y with sufficiently small $\varepsilon > 0$ and $\Omega_\varepsilon := \Omega \backslash \overline{D(y,\varepsilon)}$. For a fixed y, evidently $P_b(.,y) \in D(\overline{\Omega}) \subset H^{1,0}(\Omega_\varepsilon; B)$ and has coinciding classical and canonical co-normal derivatives on $\partial\Omega_\varepsilon$. Then from the definition of parametrix and Green's second identity employed for Ω_ε with $P_b(.,y)$ we obtain

$$\begin{aligned}\int_{\Omega_\varepsilon} \{u(x)BP_b(x,y) - P_b(x,y)Au(x)\}\,dx = \\ \int_{\partial\Omega_\varepsilon}\left[\gamma^+ u(x)T_b^+ P_b(x,y) - \gamma^+ P_b(x,y)T_a^+ u(x)\right]dS(x) \\ -\int_{\Omega_\varepsilon}(a(x)-b(x))\,\nabla P_b(x,y)\cdot\nabla u(x)dx.\end{aligned}$$

Here $\partial\Omega_\varepsilon = \partial\Omega \cup \partial D_\varepsilon$, then follows.

$$\begin{aligned}\int_{\Omega_\varepsilon} u(x)BP_b(x,y)dx - \int_{\Omega_\varepsilon} P_b(x,y)Au(x)dx = \\ \int_{\partial\Omega}\left[\gamma^+ u(x)T_b^+ P_b(x,y) - \gamma^+ P_b(x,y)T_a^+ u(x)\right]dS(x) \\ +\int_{\partial D_\varepsilon}\left[\gamma^+ u(x)T_b^+ P_b(x,y) - \gamma^+ P_b(x,y)T_a^+ u(x)\right]dS(x) \\ -\int_{\Omega_\varepsilon}(a(x)-b(x))\,\nabla P_b(x,y)\cdot\nabla u(x)dx,\end{aligned}$$

and $P_b(x,y)$ is parametrix for $B(x,\partial_x)$

$$\begin{aligned}\int_{\Omega_\varepsilon} u(x)\left(\delta(x-y) + R_b(x,y)\right)dx - \int_{\Omega_\varepsilon} P_b(x,y)Au(x)dx = \\ \int_{\partial\Omega}\left[\gamma^+ u(x)T_b^+ P_b(x,y) - \gamma^+ P_b(x,y)T_a^+ u(x)\right]dS(x) \\ +\int_{\partial D_\varepsilon}\left[\gamma^+ u(x)T_b^+ P_b(x,y) - \gamma^+ P_b(x,y)T_a^+ u(x)\right]dS(x) \\ -\int_{\Omega_\varepsilon}(a(x)-b(x))\,\nabla P_b(x,y)\cdot\nabla u(x)dx,\end{aligned}$$

$\int_{\Omega_\varepsilon} u(x)\delta(x-y)dx = 0$ since $y \notin \Omega_\varepsilon$,

$$\int_{\Omega_\varepsilon} u(x)R_b(x,y)dx - \int_{\Omega_\varepsilon} P_b(x,y)Au(x)dx =$$
$$\int_{\partial\Omega} \left[\gamma^+ u(x) T_b^+ P_b(x,y) - \gamma^+ P_b(x,y) T_a^+ u(x)\right] dS(x)$$
$$+ \int_{\partial D_\varepsilon} \left[\gamma^+ u(x) T_b^+ P_b(x,y) - \gamma^+ P_b(x,y) T_a^+ u(x)\right] dS(x)$$
$$- \int_{\Omega_\varepsilon} (a(x) - b(x)) \nabla P_b(x,y) \cdot \nabla u(x) dx.$$

Letting the limit as $\varepsilon \to 0$ we have

$$\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} u(x)R_b(x,y)dx - \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} P_b(x,y)Au(x)dx =$$
$$\lim_{\varepsilon\to 0} \int_{\partial\Omega} \left[\gamma^+ u(x) T_b^+ P_b(x,y) - \gamma^+ P_b(x,y) T_a^+ u(x)\right] dS(x)$$
$$\lim_{\varepsilon\to 0} \int_{\partial D_\varepsilon} \left[\gamma^+ u(x) T_b^+ P_b(x,y) - \gamma^+ P_b(x,y) T_a^+ u(x)\right] dS(x)$$
$$- \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} (a(x) - b(x)) \nabla P_b(x,y) \cdot \nabla u(x) dx.$$

In the above equation from the right hand side of the third line since $T_a^+ u(x)$ is continuous, then it is bounded, say by $M_a > 0$ then

$$\begin{aligned}
\Big| \int_{\partial D_\varepsilon(y)} T_a^+ u(x) P_b(x,y) dS_x \Big| &\leqslant M_a \int_{\partial D_\varepsilon(y)} |P_b(x,y)| dS_x \\
&= M_a \int_{\partial D_\varepsilon(y)} \frac{1}{2\pi b(y)} \log\left(\frac{|x-y|}{r_0}\right) dS_x \\
&= \frac{M_a}{2\pi b(y)} \int_{|x-y|=\varepsilon} \log \frac{|x-y|}{r_0} dS_x.
\end{aligned}$$

using the polar coordinate,

$$\begin{aligned}
&= \frac{M_a}{2\pi b(y)} \int_0^{2\pi} \int_0^{\varepsilon} |\log r| r dr d\varphi \\
&= \frac{M_a}{2\pi b(y)} \int_0^{2\pi} \left(\frac{\varepsilon^2}{2}\log\varepsilon - \frac{\varepsilon^2}{4}\right) d\varphi \\
&= \frac{M_a}{2\pi b(y)} \left(\frac{\varepsilon^2}{2}\log\varepsilon - \frac{\varepsilon^2}{4}\right) 2\pi \\
&= \frac{M_a}{b(y)} \frac{\varepsilon^2}{2} \left(\log\varepsilon - \frac{1}{2}\right) \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0.
\end{aligned}$$

Therefore,

$$\lim_{\varepsilon\to 0}\int_{\partial D_\varepsilon(y)} T_a^+ u(x) P_b(x,y) dS_x = 0.$$

And for the remaining part of the line of the equation (third line)

$$\lim_{\varepsilon\to 0}\int_{\partial D_\varepsilon(y)} T_b^+ P_b(x,y)\gamma^+ u(x) dS_x,$$

the co-normal derivative is given by

$$\int_{\partial D_\varepsilon(y)} \gamma^+ u(x) T_b^+ P_b(x,y) dS_x = \int_{\partial D_\varepsilon(y)} \left[\gamma^+ u(x) - \gamma^+ u(y)\right] T_b^+ P_b(x,y) dS_x + \int_{\partial D_\varepsilon(y)} \gamma^+ u(y) T_b^+ P_b(x,y) dS_x.$$

$$\begin{aligned}\Big|\int_{\partial D_\varepsilon(y)} &\left[\gamma^+ u(x) - \gamma^+ u(y)\right] T_b^+ P_b(x,y) dS_x\Big| \\ &\le \max_{|x-y|=\varepsilon}\Big|u(x)-u(y)\Big| \int_{\partial D_\varepsilon(y)} \Big|T_b^+ P_b(x,y)\Big| dS_x \longrightarrow 0 \ \text{ as } \ \varepsilon\to 0,\end{aligned}$$

since $\int_{\partial D_\varepsilon(y)} T_b^+ P_b(x,y) dS_x = -1$.
Hence,

$$\int_{\partial D_\varepsilon(y)} \gamma^+ u(y) T_b^+ P_b(x,y) dS_x = u(y)\int_{\partial D_\varepsilon(y)} T_b^+ P_b(x,y) dS_x = -u(y).$$

Taking into account the density of $D(\overline{\Omega})$ in $H^s(\Omega)$ and mapping properties of the integral potentials, we obtain that (3.49) hold true for any $u\in H^1(\Omega)$.

$$\begin{aligned}\int_{\Omega_\varepsilon} u(x) R_b(x,y) dx - \int_{\Omega_\varepsilon} P_b(x,y) A u(x) dx = \\ \int_{\partial\Omega}\left[\gamma^+ u(x) T_b^+ P_b(x,y) - \gamma^+ P_b(x,y) T_a^+ u(x)\right] dS(x) \\ - u(y) - \int_{\Omega_\varepsilon} (a(x)-b(x))\nabla P_b(x,y)\cdot\nabla u(x) dx.\end{aligned}$$

Substituting the Newtonian remainder potential, we get

$$\begin{aligned}u(y) + \mathscr{R}_b u - \int_{\partial\Omega}\gamma^+ u(x) T_b^+ P_b(x,y) dS_x + \int_{\partial\Omega}\gamma^+ P_b(x,y) T_a^+ u(x) dS(x) \\ - \int_{\Omega_\varepsilon} (a(x)-b(x))\nabla P_b(x,y)\cdot\nabla u(x) dx = \int_{\Omega_\varepsilon} P_b(x,y) A u(x) dx.\end{aligned}$$

Substituting the single-layer and double layer potential, we have

$$u(y) + \mathscr{R}_b u - V_b T_a^+ u - \int_{\partial\Omega} \gamma^+ u(x) b(x) n(x) \nabla_x P_b(x,y) dS_x$$
$$- \int_{\Omega_\varepsilon} (a(x) - b(x)) \nabla P_b(x,y) \cdot \nabla u(x) dx = \int_{\Omega_\varepsilon} P_b(x,y) Au(x) dx.$$

$$u(y) + \mathscr{R}_b u - V_b T_a^+ u + W_b \gamma^+ u - \int_{\Omega_\varepsilon} (a(x) - b(x)) \nabla P_b(x,y) \cdot \nabla u(x) dx = \int_{\Omega_\varepsilon} P_b(x,y) Au(x) dx.$$

Finally, we have, the required formula, the two-operator third Green identity which is a boundary domain integro-differential equation due to the term $\mathscr{Z}_b u$ that contains the derivative of the unknown function u

$$u + \mathscr{Z}_b u + \mathscr{R}_b u - V_b T_a^+ u + W_b \gamma^+ u = \mathscr{P}_b A u \quad \text{in} \quad \Omega,$$

$$\text{where} \quad \mathscr{Z}_b u(y) = - \int_{\Omega} (a(x) - b(x)) \nabla_x P_b(x,y) \cdot \nabla u(x) dx$$
$$= \frac{1}{b(y)} \sum_{i=1}^{2} \partial_i \mathscr{P}_\Delta \left[(a-b)(\partial_i u) \right](y).$$

By using the Gauss Divergence Theorem, we can convert the BDIDE (3.49) to the BDIE.
By rewriting $\mathscr{Z}_b u$ in the form that does not involve derivative of u. That is,

$$\mathscr{Z}_b u = \left[\frac{a(y)}{b(y)} - 1 \right] u(y) + \widehat{\mathscr{Z}}_b u(y), \tag{3.50}$$

where

$$\widehat{\mathscr{Z}}_b u(y) = \frac{a(y)}{b(y)} W_a \gamma^+ u(y) - W_b \gamma^+ u(y) + \frac{a(y)}{b(y)} \mathscr{R}_a u(y) - \mathscr{R}_b u(y), \tag{3.51}$$

which allows to call $\mathscr{Z}_b$ integral operator inspite of its Integro-Differential representation.

3.1.10 The Gauss Divergence Theorem

We state the divergence (Gauss-Green) theorem.

Theorem 3.9. *Let $X : \overline{\Omega} \to \mathbb{R}^n$ be a $\mathbf{C}^1(\overline{\Omega})$- vector field, and $\Omega \subset \mathbb{R}^n$ a bounded set with $\mathbf{C}^1$ boundary $\partial\Omega$. Then*

$$\int_{\Omega} divX dx = \int_{\partial\Omega} X \cdot \nu dS.$$

In particular, if $u, v \in \mathbf{C}^1(\overline{\Omega})$, then an application of the divergence theorem to the vector field $X = (0, 0, \ldots, uv, \ldots, 0)$, with i^{th} component uv gives the integration by parts formula

$$\int_{\Omega} u \frac{\partial v}{\partial x_i} dx = -\int_{\Omega} v \frac{\partial u}{\partial x_i} dx + \int_{\partial\Omega} uv\nu_i dS.$$

Let $u \in \mathbf{C}^1(\overline{\Omega})$ and $w \in \mathbf{C}^2(\overline{\Omega})$ and using the poduct rule of derivative

$$\nabla(u\nabla w) = u\Delta w + \nabla u \cdot \nabla w,$$

then applying the Gauss divergence theorem to the above equation we obtain

$$\int_{\Omega} \frac{\partial}{\partial x_i} u(x) dx = \int_{\partial\Omega} u n_i dS_x,$$

$$\int_{\Omega} \frac{\partial}{\partial x_i} (u\nabla w) dx = \int_{\partial\Omega} u\nabla w n_i dS_x,$$

$$\int_{\Omega} (u\Delta w + \nabla u \cdot \nabla w) dx = \int_{\partial\Omega} u\nabla w \cdot n dS_x = \int_{\partial\Omega} u \frac{\partial w}{\partial x_i} n_i dS_x = \int_{\partial\Omega} u \frac{\partial w}{\partial n} dS_x,$$

this implies that

$$\int_{\Omega} u\Delta w dx = -\int_{\Omega} \nabla u \cdot \nabla w dx + \int_{\partial\Omega} u \frac{\partial w}{\partial n} dS_x.$$

Next consider the following let $a, b \in \mathbf{C}^{\infty}(\mathbb{R}^2)$

$$\nabla\left[(a-b)u\nabla w\right] = \nabla(a-b)u\nabla w + (a-b)\nabla u \cdot \nabla w + (a-b)u\Delta w,$$

using the Gauss divergence theorem,

$$\int_{\Omega} \frac{\partial}{\partial x_i} \left[(a-b)u \frac{\partial w}{\partial x_i}\right] dx = \int_{\partial\Omega} (a-b)u \frac{\partial w}{\partial x_i} n_i dS_x,$$

which implies

$$\int_{\Omega} \left[\frac{\partial}{\partial x_i}(a-b)u\frac{\partial w}{\partial x_i} + (a-b)\frac{\partial u}{\partial x_i}\frac{\partial w}{\partial x_i} + (a-b)u\frac{\partial^2 w}{\partial x_i^2}\right] dx = \int_{\partial\Omega} (a-b)u\frac{\partial w}{\partial x_i} n_i dS_x.$$

Putting $P_b(x, y) = w(x)$, clearly it is well known that direct substitution of $w(x)$ by $P_b(x, y)$ is impossible as it has singularity at $x = y$. We can avoid this difficulty by replacing Ω by $\Omega_\varepsilon := \Omega \backslash \overline{D(y, \varepsilon)}$ a disc of radius ε centered at y.

$$\int_{\Omega_\varepsilon}\left[\frac{\partial}{\partial x_i}(a-b)u\frac{\partial P_b}{\partial x_i}+(a-b)\frac{\partial u}{\partial x_i}\frac{\partial P_b}{\partial x_i}+(a-b)u\frac{\partial^2 P_b}{\partial x_i^2}\right]dx=\int_{\partial\Omega_\varepsilon}(a-b)u\frac{\partial P_b}{\partial x_i}n_i dS_x,$$

which imply

$$\int_{\Omega_\varepsilon}(a-b)\frac{\partial u}{\partial x_i}\frac{\partial P_b}{\partial x_i}dx=-\int_{\Omega_\varepsilon}(a-b)u\frac{\partial^2 P_b}{\partial x_i^2}dx-\int_{\Omega_\varepsilon}\left[\frac{\partial}{\partial x_i}(a-b)u\frac{\partial P_b}{\partial x_i}dx+\int_{\partial\Omega_\varepsilon}(a-b)u\frac{\partial P_b}{\partial x_i}n_i\right]dS_x.$$

$$\begin{aligned}\int_{\Omega_\varepsilon}(a-b)\nabla u\cdot\nabla_x P_b dx &= -\int_{\Omega_\varepsilon}(a-b)u\Delta_x P_b dx-\int_{\Omega_\varepsilon}\nabla(a-b)u\nabla_x P_b dx+\int_{\partial\Omega_\varepsilon}(a-b)u\nabla_x P_b\cdot n dS_x\\ &= -\frac{1}{b(y)}\int_{\Omega_\varepsilon}(a-b)u\Delta_x P_\Delta dx-\int_{\Omega_\varepsilon}\nabla(a-b)u\nabla_x P_b dx\\ &+\int_{\partial\Omega}(a-b)u\nabla_x P_b\cdot n dSx+\int_{\partial D_\varepsilon}(a-b)u\nabla_x P_b\cdot n dS_x,\\ &= -\frac{1}{b(y)}\int_{\Omega_\varepsilon}(a-b)u\delta(x-y)dx-\int_{\Omega_\varepsilon}\nabla(a-b)u\nabla_x P_b dx\\ &+\int_{\partial\Omega}(a-b)u\nabla_x P_b\cdot n dSx+\int_{\partial D_\varepsilon}(a-b)u\nabla_x P_b\cdot n dS_x,\end{aligned}$$

$-\frac{1}{b(y)}\int_{\Omega_\varepsilon}(a-b)u\delta(x-y)dx=0$ since $y\notin\Omega_\varepsilon$.

Thus,

$$\begin{aligned}\int_{\Omega_\varepsilon}(a-b)\nabla u\cdot\nabla_x P_b dx &= -\int_{\Omega_\varepsilon}\nabla(a-b)u\nabla_x P_b dx\\ &+\int_{\partial\Omega}(a-b)u\nabla_x P_b\cdot n dS_x+\frac{1}{b(y)}\int_{\partial D_\varepsilon}(a-b)u\nabla_x P_\Delta\cdot n dS_x,\end{aligned}$$

Next let us evaluate

$$\lim_{\varepsilon\to 0}\int_{\partial D_\varepsilon(y)}(a-b)u(x)\nabla_x P_\Delta\cdot n dS_x,$$

$$\int_{\partial D_\varepsilon(y)}(a-b)u(x)\nabla_x P_\Delta\cdot n dS_x=\int_{\partial D_\varepsilon(y)}(a-b)\left[\gamma^+u(x)-\gamma^+u(y)\right]\nabla_x P_\Delta dS_x+\int_{\partial D_\varepsilon(y)}\gamma^+u(y)\nabla_x P_\Delta\cdot n dS_x.$$

$$\begin{aligned}&\Big|\int_{\partial D_\varepsilon(y)}(a-b)\left[\gamma^+u(x)-\gamma^+u(y)\right]\nabla_x P_\Delta\cdot n dSx\Big|\\ &\qquad\le\max_{|x-y|=\varepsilon}\Big|(a-b)(u(x)-u(y))\Big|\int_{\partial D_\varepsilon(y)}\Big|\nabla_x P_\Delta\Big|dS_x\longrightarrow 0 \text{ as } \varepsilon\to 0,\end{aligned}$$

since $\int_{\partial D_\varepsilon(y)} \nabla_x P_\Delta dS_x = -1$.
Hence,

$$\int_{\partial D_\varepsilon(y)} (a-b)u(y)\nabla_x P_\Delta dS_x = (a-b)u(y)\int_{\partial D_\varepsilon(y)} \nabla_x P_\Delta dS_x = -(a-b)u(y). \quad (3.52)$$

Putting the double layer potential and the Newtonian remainder and equation (3.52) below, will give us required result.

$$\int_{\Omega_\varepsilon} (a-b)\nabla u\cdot\nabla_x P_b dx = -\int_{\Omega_\varepsilon} \nabla a u \nabla_x P_b dx + \int_{\Omega_\varepsilon} \nabla b u \nabla_x P_b dx + \int_{\partial\Omega} a u \nabla_x P_b \cdot n dS_x - \int_{\partial\Omega} b u \nabla_x P_b \cdot n dS_x + \frac{1}{b(y)}\int_{\partial D_\varepsilon} (a-b)u\nabla_x P_\Delta\cdot n dS_x,$$

Hence, the desired result:

$$\int_{\Omega} (a-b)\nabla u\cdot\nabla_x P_b dx = \left(1-\frac{a(y)}{b(y)}\right)u(y) - \frac{a(y)}{b(y)}\mathscr{R}_a(x,y)u(y) + \mathscr{R}_b(x,y)u(y) - \frac{a(y)}{b(y)}(W_a u)(y) + W_b u(y).$$

$$\mathscr{Z}_b u = -\int_{\Omega} (a-b)\nabla u\cdot\nabla_x P_b dx$$
$$= -\left(1-\frac{a(y)}{b(y)}\right)u(y) + \frac{a(y)}{b(y)}\mathscr{R}_a(x,y)u(y) - \mathscr{R}_b(x,y)u(y) + \frac{a(y)}{b(y)}W_a\gamma^+u(y) - W_b\gamma^+u(y)$$
$$\mathscr{Z}_b u = \left(\frac{a(y)}{b(y)}-1\right)u(y) + \widehat{\mathscr{Z}_b}u(y).$$

where

$$\widehat{\mathscr{Z}_b}u = \frac{a(y)}{b(y)}W_a\gamma^+u - W_b\gamma^+u + \frac{a(y)}{b(y)}\mathscr{R}_a(x,y)u - \mathscr{R}_b(x,y)u.$$

Note that substituting (3.50)-(3.51) in (3.49) and multiplying by $\frac{b(y)}{a(y)}$ one reduces (3.49) to the one-operator parametrix-based third Green identity obtained in [CMN09a],

$$u + \mathscr{R}_a u - V_a T_a^+ u + W_a\gamma^+ u = \mathscr{P}_a A u \qquad \text{in} \quad \Omega.$$

Relations (3.50)-(3.51) and the mapping properties of $\mathscr{P}_a, \mathscr{R}_a, \mathscr{R}_b, W_a$ and W_b imply the following assertion.

Theorem 3.10. *The operators,*

$$\mathscr{Z}_b : H^s(\Omega) \to H^s(\Omega), \quad s > \frac{1}{2},$$

$$\widehat{\mathscr{Z}_b} : H^s(\Omega) \to H^{s,0}(\Omega, \Delta), \quad s \geq 1$$

are continuous

Proof. Let $u \in H^s(\Omega)$, then using the continuity of the double layer potential operators W_a & W_b, the Remainder potentials $\mathscr{R}_a$ & $\mathscr{R}_b$ and the trace operator γ^+, and the space $H^{s+1}(\Omega)$ is continuously embedded in $H^s(\Omega)$ follows that

$$\begin{aligned}
\|\mathscr{Z}_b u\|_{H^s(\Omega)} &= \| \left(\frac{a(y)}{b(y)} - 1\right) u(y) + \frac{a(y)}{b(y)} \mathscr{R}_a(x,y)u - \mathscr{R}_b(x,y)u + \frac{a(y)}{b(y)} W_a \gamma^+ u - W_b \gamma^+ u \|_{H^s(\Omega)} \\
&\leq \| \left(\frac{a(y)}{b(y)} - 1\right) u(y) \|_{H^s(\Omega)} + \| \frac{a(y)}{b(y)} \mathscr{R}_a(x,y)u \|_{H^s(\Omega)} + \| \mathscr{R}_b(x,y)u \|_{H^s(\Omega)} + \| \frac{a(y)}{b(y)} W_a \gamma^+ u \|_{H^s(\Omega)} \\
&+ \|W_b \gamma^+ u\|_{H^s(\Omega)} \\
&\leq C_1 \|u\|_{H^s(\Omega)} + C_2 \|\mathscr{R}_a(x,y)u\|_{H^s(\Omega)} + \|\mathscr{R}_b(x,y)u\|_{H^s(\Omega)} + C_3 \|W_a \gamma^+ u\|_{H^s(\Omega)} + \|W_b \gamma^+ u\|_{H^s(\Omega)} \\
&\leq C_1 \|u\|_{H^s(\Omega)} + C_2 \|\mathscr{R}_a(x,y)u\|_{H^{s+1}(\Omega)} + \|\mathscr{R}_b(x,y)u\|_{H^{s+1}(\Omega)} + \tilde{C}_3 \|\gamma^+ u\|_{H^{s-\frac{1}{2}}(\partial\Omega)} + \tilde{C}_4 \|\gamma^+ u\|_{H^{s-\frac{1}{2}}(\partial\Omega)} \\
&\leq C_1 \|u\|_{H^s(\Omega)} + C_2 \|u\|_{H^s(\Omega)} + \|u\|_{H^s(\Omega)} + \tilde{C}_3' \|u\|_{H^s(\Omega)} + \tilde{C}_4' \|u\|_{H^s(\Omega)} \\
&= \tilde{C} \|u\|_{H^s(\Omega)} \quad \text{for} \quad \mathrm{s} > \frac{1}{2}.
\end{aligned}$$

Therefore,

$$\|\mathscr{Z}_b u\|_{H^s(\Omega)} \leq \tilde{C} \|u\|_{H^s(\Omega)} \quad \text{for} \quad \mathrm{s} > \frac{1}{2}.$$

Hence, the operator,

$$\mathscr{Z}_b : H^s(\Omega) \to H^s(\Omega) \quad \text{for} \quad \mathrm{s} > \frac{1}{2}.$$

is continuous.

If $u \in H^{1,0}(\Omega; \Delta)$ is a solution of equation (3.1), then (3.49) gives

$$u + \mathscr{Z}_b u + \mathscr{R}_b u - V_b T_a^+ u + W_b \gamma^+ u = \mathscr{P}_b f \quad \text{in} \quad \Omega.$$

Taking the Trace of the above equation

$$\gamma^+ u + \gamma^+ \mathscr{Z}_b u + \gamma^+ \mathscr{R}_b u - \gamma^+ V_b T_a^+ u + \gamma^+ W_b \gamma^+ u = \gamma^+ \mathscr{P}_b f \quad \text{on} \quad \partial\Omega,$$

and applying the jump relation to the above equation gives us:

$$\frac{1}{2} \gamma^+ u + \gamma^+ \mathscr{Z}_b u + \gamma^+ \mathscr{R}_b u - \mathscr{V}_b T_a^+ u + \mathscr{W}_b \gamma^+ u = \gamma^+ \mathscr{P}_b f \quad \text{on} \quad \partial\Omega.$$

Taking the canonical co-normal derivative yields

$$T_a^+ u + T_a^+ \mathscr{Z}_b u + T_a^+ \mathscr{R}_b u - T_a^+ V_b T_a^+ u + T_a^+ W_b \gamma^+ u = T_a^+ \mathscr{P}_b f \quad \text{on} \quad \partial\Omega,$$

and applying the jump relation on the above equation, we get

$$\left(1 - \frac{a}{2b}\right) T_a^+ u + T_a^+ \mathscr{Z}_b u + T_a^+ \mathscr{R}_b u - \mathscr{W}'_{ab} T_a^+ u + \mathscr{L}_{ab} \gamma^+ u = T_a^+ \mathscr{P}_b f \quad \text{on} \quad \partial\Omega.$$

Combining these set of equations, we get

$$u + \mathscr{Z}_b u + \mathscr{R}_b u - V_b T_a^+ u + W_b \gamma^+ u = \mathscr{P}_b f \quad \text{in} \quad \Omega, \tag{3.53}$$

$$\frac{1}{2}\gamma^+ u + \gamma^+ \mathscr{Z}_b u + \gamma^+ \mathscr{R}_b u - \mathscr{V}_b T_a^+ u + \mathscr{W}_b \gamma^+ u = \gamma^+ \mathscr{P}_b f \quad \text{on} \quad \partial\Omega, \tag{3.54}$$

$$\left(1 - \frac{a}{2b}\right) T_a^+ u + T_a^+ \mathscr{Z}_b u + T_a^+ \mathscr{R}_b u - \mathscr{W}'_{ab} T_a^+ u + \mathscr{L}_{ab}^+ \gamma^+ u = T_a^+ \mathscr{P}_b f \quad \text{on} \quad \partial\Omega. \tag{3.55}$$

Remark 3.1.6 *Note that if P_b is not only the parametrix but also the fundamental solution of the operator B, then the remainder operator R_b vanishes in (3.53)-(3.55) (and everywhere in the paper), while the operator $\mathscr{Z}_b$ does not unless $A = B$.*

For some functions f, Ψ and Φ let us consider a more general 'indirect' integral relation associated with equation (3.53).

$$u + \mathscr{Z}_b u + \mathscr{R}_b u - V_b \Psi + W_b \Phi = \mathscr{P}_b f \quad \text{in} \quad \Omega. \tag{3.56}$$

Lemma 3.2. *Let $f \in L_2(\Omega), \Psi \in H^{-\frac{1}{2}}(\partial\Omega), \Phi \in H^{\frac{1}{2}}(\partial\Omega)$ and $u \in H^1(\Omega)$, satisfy equation (3.56). Then $u \in H^{1,0}(\Omega;\Delta)$ and is a solution of PDE*

$$Au = f \quad in \quad \Omega, \tag{3.57}$$

and

$$V_b(\Psi - T_a^+ u) - W_b(\Phi - \gamma^+ u) = 0, \quad y \in \Omega. \tag{3.58}$$

Proof. In order to prove $u \in H^{1,0}(\Omega;A)$ we use the following steps: First we will show the following relation

$$\Delta(au) - \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(u(x)\frac{\partial a(x)}{\partial x_i}\right) = \sum_{i=1}^{2}\left[u(x)\frac{\partial^2 a(x)}{\partial x_i^2} + 2\frac{\partial u(x)}{\partial x_i}\frac{\partial a(x)}{\partial x_i} + a(x)\frac{\partial^2 u(x)}{\partial x_i^2}\right]$$

$$- \sum_{i=1}^{2}\left(u(x)\frac{\partial^2 a(x)}{\partial x_i^2} + \frac{\partial u(x)}{\partial x_i}\frac{\partial a(x)}{\partial x_i}\right) = \sum_{i=1}^{2} a(x)\frac{\partial^2 u(x)}{\partial x_i^2} + \sum_{i=1}^{2}\frac{\partial u(x)}{\partial x_i}\frac{\partial a(x)}{\partial x_i}$$

$$= \sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(a(x)\frac{\partial u(x)}{\partial x_i}\right) = A(x, \partial_x)u(x).$$

Therefore,

$$A(x,\partial_x)u(x) = \Delta(au) - \sum_{i=1}^{2} \partial_i(u\partial_i a).$$

Since, $u \in H^1(\Omega), \partial_i u \in L_2(\Omega)$ and $a(x) \in C^\infty(\Omega)$, and hence the term $\sum_{i=1}^{2} \partial_i(u\partial_i a)$ belongs to $L_2(\Omega)$. Next, we need to show that $\Delta(au) \in L_2(\Omega)$. (Here the derivative is understood in the distribution sense). In (3.53), by multiplying both sides by b and substituting $\mathscr{Z}_b$ then simplifying it will give us:

$$au = \mathscr{P}_\Delta f - b\widehat{\mathscr{Z}_b}u - b\mathscr{R}_b u + V_\Delta \Psi - W_\Delta \Phi \quad \text{in} \quad \Omega.$$

Apply the Laplace operator, and since the last two terms of the right hand side,i.e., $V_\Delta \Psi$ and $W_\Delta(\Phi)$ are harmonic functions, so we obtain

$$\Delta(au) = \Delta(\mathscr{P}_\Delta f) - \Delta(b\widehat{\mathscr{Z}_b}u) - \Delta(b\mathscr{R}_b u) \;\text{ in }\; \Omega.$$

Since $u \in H^1(\Omega)$, then from the mapping property of $\mathscr{R}_b$ we have that $\mathscr{R}_b u \in H^2(\Omega)$ and hence $b\mathscr{R}_b u \in H^2(\Omega)$. By the definition of the space $H^2(\Omega)$ we have $\frac{\partial^2(b\mathscr{R}_b u)}{\partial x_i^2} \in L_2(\Omega)$, then $\Delta(b\mathscr{R}_b u) \in L_2(\Omega)$ and it is clear from Theorem 3.10 that $\widehat{\mathscr{Z}_b}u \in H^{1,0}(\Omega;B)$, then

$$\begin{aligned}\Delta(\mathscr{P}_\Delta f) &= \Delta_x\left(\int_\Omega \mathscr{P}_\Delta(x,y)f(y)dy\right) = \int_\Omega \Delta_x \mathscr{P}_\Delta(x,y)f(y)dy \\ &= \int_\Omega \Delta_x\left[\frac{1}{2\pi}\log\frac{|x-y|}{r_0}\right]f(y)dy = \int_{\mathbb{R}^2} \delta(x-y)\tilde{f(y)}dy \\ &= \delta * \tilde{f} = f \in L_2(\Omega).\end{aligned}$$

where $\tilde{f}$ is the extension of f from the domain Ω in to $\mathbb{R}^2$ by zero. Therefore,

$$\Delta(au) \in L_2(\Omega) \quad \text{which implies} \quad Au \in L_2(\Omega),$$

$$u \in H^{1,0}(\Omega;A) = \{u \in H^1(\Omega) : Au \in L_2(\Omega)\}.$$

So, we can write the third green identity (3.53) for the function u and subtracting (3.56) from the identity (3.49),

$$V_b(\Psi - T_a^+ u)(y) - W_b(\Phi - \gamma^+ u)(y) = \mathscr{P}_b(Au - f), \quad y \in \Omega.$$

Setting $\Psi^* = \Psi - T_a^+ u$ and $\Phi^* = \Phi - \gamma^+ u$ and multiplying by b, we get

$$V_\Delta \Psi^* - W_\Delta \Phi^* = \mathscr{P}_\Delta(Au - f) \quad \text{in} \quad \Omega.$$

Applying the Laplace operator, Δ to the above equation and taking into account the left hand side functions are both harmonic surface potentials while the right hand side function is classical Logarithmic Potential, we obtain

$$\Delta\left(\mathscr{P}_\Delta(Au-f)\right)=0$$

which implies $\Delta_x\left(\int_\Omega P_\Delta(x,y)(Au-f)(y)dy\right)=0$ as the integral and the differentiation are in different variable, we can commute them.

$\int_\Omega \Delta\left(\frac{1}{2\pi}\log\frac{|x-y|}{r_0}\right)(Au-f)(y)dy=0$ extending $(Au-f)(y)$ mapped in to the space $\mathbb{R}^2$ by zero and using convolution of $\int_{\mathbb{R}^2}\delta(x-y)(Au-\tilde{f})(y)dy=0$ will give us (3.57), then the required result (3.58) follows.

Lemma 3.3. *i) Let $\Psi^*\in H^{-\frac{1}{2}}(\partial\Omega)$ and $diam(\Omega)<r_0$ or $\Psi^*\in H_{**}^{-\frac{1}{2}}(\partial\Omega)$. If $V_b\Psi^*=0$ in Ω, then $\Psi^*=0$.*
ii) Let $\Phi^*\in H^{\frac{1}{2}}(\partial\Omega)$. If $W_b\Phi^*=0$ in Ω, then $\Phi^*=0$.

Proof. (i.) Let us take the trace of equation $V_b\Psi^*=0$ on $\partial\Omega$, by the jump relation we have

$$\gamma^+V_b\Psi^*=\mathscr{V}_b\Psi^*=0 \qquad \text{on} \quad \partial\Omega.$$

If $\Psi^*\in H^{-\frac{1}{2}}(\partial\Omega)$, and $diam(\Omega)<r_0$, then the result follows from the invertibility of the single layer potential implies $\Psi^*=0$, by Theorem 3.8

On the other hand, if $\Psi^*\in H_{**}^{-\frac{1}{2}}(\partial\Omega)$, then the result is implied by Theorem 3.6

(ii.) Let us take the trace of equation $W_b\Phi^*=0$ on $\partial\Omega$, and use the jump relation to obtain,

$$-\frac{1}{2}\Phi^*+\mathscr{W}_b\Phi^*=0 \quad \text{on} \quad \partial\Omega.$$

Multiplying this equation by $b(y)$, denoting $\hat{\Phi}^*=b\Phi^*$ and we obtain equation

$$-\frac{1}{2}\hat{\Phi}^*+\mathscr{W}_\Delta\hat{\Phi}^*=0 \qquad \text{on} \quad \partial\Omega,$$

follows that

$$\hat{\Phi}^*-\left(\frac{1}{2}I+\mathscr{W}_\Delta\right)\hat{\Phi}^*=0 \qquad \text{on} \quad \partial\Omega.$$

This equation has only the trivial solution. It is due to the contraction property of the operator $\frac{1}{2}I+\mathscr{W}_\Delta$ (see, e.g., [SW01, Theorem 3.1]). Since $b(y)\neq 0$, the result follows.

3.2 Dirichlet Problem

Let Ω be a domain in $\mathbb{R}^2$ bounded by simple closed infinitely differentiable curve $\partial\Omega$. Consider the following second order elliptic PDE with scalar variable coefficient in two-dimensional bounded domain Ω defined as:

$$Au(x) := \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left[a(x)\frac{\partial u(x)}{\partial x_i}\right] = f(x), \quad x \in \Omega,$$

where u is an unknown function and f is a given function in Ω and assume $a \in C^\infty(\mathbb{R}^2)$, where $0 < a_{min} \leq a(x) \leq a_{max} < \infty, \quad \forall x \in \mathbb{R}^2$ and $n(x)$ be the exterior unit normal vector, which is defined for all x in $\partial\Omega$.

In this subsection, we shall derive and investigate the two-operator boundary-domain integral equation systems for the following Dirichlet boundary value problem.
Find a function $u \in H^1(\Omega)$ subject to the Dirichlet boundary condition:

$$Au = f \quad \text{in} \quad \Omega, \tag{3.59}$$

$$\gamma^+ u = \varphi_0 \quad \text{on} \quad \partial\Omega, \tag{3.60}$$

where $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $f \in L_2(\Omega)$.

Remark 3.2.1 *Differential equation (3.59) is understood in the distributional sense and condition (3.60) in the trace sense.*

Here the BVP defined (3.59) - (3.60) has important applications in engineering. As an example, it may describe a steady-state temperature distribution in plane body Ω, which is thermally anisotropic and inhomogeneous. Where $u(x)$ is an unknown temperature, $a(x)$ is a known variable thermo-conductivity coefficient, $f(x)$ is a known distributed heat source, $\varphi_0(x)$ is the known heat on the boundary.

3.2.1 Two-operator BDIEs for Dirichlet BVP

A way of reducing the Dirichlet BVP (3.59) -(3.60) to a direct segregated BDIE system is to substitute the Dirichlet boundary condition (3.59) -(3.60) in to the Green's third identity (3.53) and either into its trace (3.54) or into its co-normal derivative (3.55) on $\partial\Omega$. Assuming that the function u satisfies PDE $Au = f$, by denoting the unknown conormal derivative as $\psi = T^+u \in H^{-\frac{1}{2}}(\partial\Omega)$ and will further will be considered ψ as formally independent on u. We can reduce the BVP (3.59)-(3.60) to two different systems of two-operator Boundary Domain-Integral Equations for the unknown function $u \in H^1(\Omega)$ and $\psi = T^+u \in H^{-\frac{1}{2}}(\partial\Omega)$.

Boundary-Domain Integral Equation system D1

To obtain a system, we use equation (3.53) in Ω, and equation (3.54) on the whole boundary $\partial\Omega$,

$$u + \mathscr{Z}_b u + \mathscr{R}_b u - V_b T_a^+ u + W_b \gamma^+ u = \mathscr{P}_b f \quad \text{in} \quad \Omega.$$

Putting $\gamma^+ u := \varphi_0, \quad T_a^+ u := \psi$ and taking the trace of the above equation. Then, we arrive at the following two-operator segregated system of BDIE system D1:

$$u + \mathscr{Z}_b u + \mathscr{R}_b u - V_b \psi = F_0 \qquad \text{in} \quad \Omega, \tag{3.61}$$

$$\gamma^+ \mathscr{Z}_b u + \gamma^+ \mathscr{R}_b u - \mathscr{V}_b \psi = \gamma^+ F_0 - \varphi_0 \quad \text{on} \quad \partial\Omega. \tag{3.62}$$

where $F_0 = \mathscr{P}_b f - W_b \varphi_0$

System (3.61)- (3.62) can be rewritten in the form

$$\mathscr{D}^1 \mathscr{U} = \mathscr{F}^1,$$

where

$$\mathscr{D}^1 := \begin{bmatrix} I + \mathscr{Z}_b + \mathscr{R}_b & -V_b \\ \gamma^+[\mathscr{Z}_b + \mathscr{R}_b] & -\mathscr{V}_b \end{bmatrix}, \qquad \mathscr{F}^1 := \left[F_0, \gamma^+ F_0 - \varphi_0\right]^T,$$

Remark 3.2.2 *: $\mathscr{F}^1 = 0$ if and only if $(f, \varphi_0) = 0$*

$$\mathscr{F}^1 = 0 \quad \text{implies} \quad \mathrm{F}_0 = 0 \quad \text{and} \quad \gamma^+ \mathrm{F}_0 - \varphi_0 = 0$$

which implies $\varphi_0 = 0$. Thus, as a result we get $\mathscr{P}_b f = 0$ multiplying $\mathscr{P}_b f = 0$ by $b(y)$, and applying Laplace operator, we get $f = 0$. Therefore, the desired result follows.

Boundary-Domain Integral Equation system D2

To obtain another system, we use equation (3.53) in Ω, and equation (3.55) on the whole boundary $\partial\Omega$

$$u + \mathscr{Z}_b u + \mathscr{R}_b u - V_b T_a^+ u + W_b \gamma^+ u = \mathscr{P}_b f \ \text{ in } \ \Omega.$$

Putting $T_a^+ u := \psi$ and taking the conormal derivative of the above equation. Then we arrive at the following two-operator segregated system of BDIE system D2:

$$u + \mathscr{Z}_b u + \mathscr{R}_b u - V_b \psi = F_0 \quad \text{in} \quad \Omega, \tag{3.63}$$

$$\left(1 - \frac{a}{2b}\right) \psi + T_a^+ \mathscr{Z}_b u + T_a^+ \mathscr{R}_b u - \mathscr{W}'_{ab} \psi = T_a^+ F_0 \quad \text{on} \quad \partial\Omega. \tag{3.64}$$

System (3.63)-(3.64) can be rewritten in the form

$$\mathscr{D}^2 \mathscr{U} = \mathscr{F}^2,$$

where

$$\mathscr{D}^2 := \begin{bmatrix} I + \mathscr{Z}_b + \mathscr{R}_b & -V_b \\ T_a^+[\mathscr{Z}_b + \mathscr{R}_b] & \left(1 - \frac{a}{2b}\right) I - \mathscr{W}'_{ab} \end{bmatrix}, \qquad \mathscr{F}^2 := \left[F_0, T_a^+ F_0\right]^T,$$

Remark 3.2.3 *: $\mathscr{F}^2 = 0$ if and only if $(f, \varphi_0) = 0$.*

$$\mathscr{F}^2 = 0 \Rightarrow F_0 = 0 \qquad \text{and} \qquad T^+ F_0 = 0$$
$$\Rightarrow F_0 = 0 \Rightarrow \mathscr{P}_b f - W_b \varphi_0 = 0$$

and multiplying by $b(y)$ and applying Laplace operator, we get $f = 0$ which implies $\varphi_0 = 0$
Hence,

$$\mathscr{F}^2 = 0 \qquad \text{if and only if} \qquad (f, \varphi_0) = 0$$

Now consider the original Dirichlet BVP for $u \in H^1(\Omega)$ given on (3.59) -(3.60). We can rewrite the given Dirichlet BVP in a matrix form

$$A^D u = F^D$$

$$\text{where} \qquad A^D := \begin{bmatrix} A \\ \gamma^+ \end{bmatrix}. \qquad F^D := \begin{bmatrix} f \\ \varphi_0 \end{bmatrix}.$$

The following assertion is well-known and can be proved e.g. using variational setting and the Lax-Milgram Lemma (see, e.g.,[Mik17])

Theorem 3.11. *The Dirichlet problem (3.59) - (3.60) is uniquely solvable in $H^1(\Omega)$. The solution is $u = (A^D)^{-1}(f, \varphi_0)$, where the inverse operator $(A^D)^{-1} : L_2(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$, to the left hand side operator, $A^D : H^1(\Omega) \to L_2(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, of the Dirichlet problem (3.59) - (3.60), is continuous.*

In order to prove the invertibility of BDIE operators first let us prove the following representation Lemma (see, e.g., [Mik05b, Lemma 3] and reference therein). We adapt here its proof scheme.

Lemma 3.4. *For any function* $\mathscr{F}_{\Phi_*} \in H^{1,0}(\Omega;A)$ *there exists a unique couple* $(f_*, \Phi_*) = C_\Phi \mathscr{F}_{\Phi_*} \in L_2(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ *such that*

$$\mathscr{F}_{\Phi_*} = \mathscr{P}_b f_* - W_b \Phi_* \qquad \text{in} \quad \Omega \tag{3.65}$$

and $C_\Phi : H^{1,0}(\Omega;A) \to L_2(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ *is a bounded linear operator.*

Proof. Suppose first there exist some functions $f_*(y), \Phi_*(y)$ satisfying (3.65) and find their expressions in terms of $\mathscr{F}_{\Phi_*}(y)$. Taking into account definitions for the volume and the double layer potentials, ansatz (3.65) can be rewritten as :

$$b(y)\mathscr{F}_{\Phi_*}(y) = \mathscr{P}_\Delta f_* - W_\Delta(b\Phi_*)(y) \qquad y \in \Omega \tag{3.66}$$

Applying the Laplace operator to (3.66) we obtain the

$$\Delta(b\mathscr{F}_{\Phi_*})(y) = f_*(y) \tag{3.67}$$

Then (3.66) can be rewritten as:

$$W_\Delta(b\Phi_*)(y) = Q(y) \qquad y \in \Omega \tag{3.68}$$

$$\text{where} \quad Q(y) = \mathscr{P}_\Delta f_* - b\mathscr{F}_{\Phi_*}(y) \qquad y \in \Omega \tag{3.69}$$

It is easy to check that Q is harmonic functions in Ω as well as (3.67). Then Trace of (3.68) on the boundary gives

$$\gamma^+ W_\Delta(b\Phi_*)(y) = \gamma^+ Q(y) \quad \text{implies} \quad \left[-\frac{1}{2}\mathrm{I} + \mathscr{W}_\Delta\right](\mathrm{b}\Phi_*)(\mathrm{y}) = \gamma^+\mathrm{Q}(\mathrm{y}) \tag{3.70}$$

Since $\left[-\frac{1}{2}I + \mathscr{W}_\Delta\right]$ is an isomorphism, (see, e.g., [DL90], chap XI, Part B, sec. 2, Remark 8) and $b(y) \neq 0$ we obtain the following expression of Φ_*

$$\Phi_*(y) = \frac{1}{b(y)}\left[-\frac{1}{2}I + \mathscr{W}_\Delta\right]^{-1} \gamma^+ Q(y) \quad y \in \partial\Omega \tag{3.71}$$

Now we have to prove that $f_*(y), \Phi_*(y)$ given by (3.67) and (3.71) do satisfy (3.65). Indeed, the potential $W_\Delta(b\Phi_*)(y)$ with $\Phi_*(y)$ given by (3.71) is harmonic function, and one can check that Q given by (3.69) is also harmonic. Since (3.70) implies that they coincide on the boundary, the two harmonic functions should also coincide in the domain, i.e, (3.68) holds true, which implies (3.65). Thus we constructed a bounded operator $C_\Phi : H^{1,0}(\Omega;A) \to L_2(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ given by (3.67), (3.71) , (3.69)

Lemma 3.5. *For any couple* $(\mathscr{F}_1, \mathscr{F}_2) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ *there exists a unique couple* $(f_{**}, \Phi_*) \in L_2(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ *such that*

$$\mathscr{F}_1 = \mathscr{P}_b f_{**} - W_b \Phi_* \quad \text{in} \quad \Omega \tag{3.72}$$

$$\mathscr{F}_2 = T_a^+ \left(\mathscr{P}_b f_{**} - W_b \Phi_*\right) \quad \text{on} \quad \partial\Omega, \tag{3.73}$$

Moreover, $(f_{**}, \Phi_*) = C_\Phi(\mathscr{F}_1, \mathscr{F}_2)$ *and* $C_\Phi : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to L_2(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ *is a linear continuous operator given by*

$$f_{**} = \Delta(b\mathscr{F}_1) \tag{3.74}$$

$$\Phi_*(y) = \frac{1}{b(y)} \left[-\frac{1}{2}I + \mathscr{W}_\Delta\right]^{-1} \gamma^+ \left(b(y)\mathscr{F}_1\right) \quad y \in \partial\Omega \tag{3.75}$$

Proof. First let us assume a couple $(f_{**}, \Phi_*) \in L_2(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ satisfying equations (3.72) and (3.73) and find their expressions in terms of $\mathscr{F}_1$ and $\mathscr{F}_2$. Let us re-write (3.72) as:

$$\mathscr{P}_b f_{**} - \mathscr{F}_1 = W_b \Phi_* \quad \text{in} \quad \Omega \tag{3.76}$$

Multiply equation (3.76) by b and applying the Laplace operator to it, we get

$$\Delta\left(\mathscr{P}_\Delta f_{**} - b\mathscr{F}_1\right) = \Delta\left(W_\Delta(b\Phi_*)\right) = 0, \quad \text{in} \quad \Omega \tag{3.77}$$

Which means

$$f_{**} = \Delta\left(b\mathscr{F}_1\right) \quad \text{in} \quad \Omega \tag{3.78}$$

and $b\mathscr{F}_1 - \mathscr{P}_\Delta f_{**} \in H^{1,0}(\Omega, \Delta)$ and hence $\mathscr{F}_1 - \mathscr{P}_b f_{**} \in H^{1,0}(\Omega, A)$. Hence, the canonical co-normal derivative $T_a^+(\mathscr{F}_1 - \mathscr{P}_b f_{**})$ is well-defined.
Now (3.76) can be written in the form

$$Q = W_\Delta(b\Phi_*), \tag{3.79}$$

where $Q = \mathscr{P}_\Delta f_{**} - b\mathscr{F}_1$.

Taking the trace of (3.79) and equations (3.70) and (3.71) follows that

$$\Phi_*(y) = \frac{1}{b(y)} \left[-\frac{1}{2}I + \mathscr{W}_\Delta\right]^{-1} \gamma^+ \left(\mathscr{P}_\Delta f_* - W_\Delta(b\Phi_*)(y)\right) \quad y \in \partial\Omega$$

Thus,

$$\Phi_*(y) = \frac{1}{b(y)} \left[-\frac{1}{2}I + \mathscr{W}_\Delta\right]^{-1} \gamma^+ \left(b(y)\mathscr{F}_1\right) \quad y \in \partial\Omega \tag{3.80}$$

The relation (3.78) and (3.80) can be written as $(f_{**}, \Phi_*) = C_\Phi(\mathscr{F}_1, \mathscr{F}_2)$, where

$$C_\Phi : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$$

is linear and continuous operator. And also relation (3.78) and (3.80) satisfy (3.72) and (3.73).

To prove the uniqueness of the operator C_Φ, let a couple $(f_{**}, \Phi_*) \in L_2(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ be a solution of linear system (3.72) and (3.73) with $\mathscr{F}_1 = 0$ and $\mathscr{F}_2 = 0$. Then (3.78) implies $f_{**} = 0$ in Ω and taking equation (3.72) due to Lemma 3.3 (ii) $\Phi_* = 0$.

Hence, any solution of non-homogeneous linear system (3.72) and (3.73) has only one solution, which implies C_Φ is unique.

3.2.2 Equivalence and Invertibility

In the following theorem we will check the equivalence of the newly obtained BDIE systems (D1) and (D2) with the original Dirichlet BVP (3.59)-(3.60).

Theorem 3.12. *Let $f \in L_2(\Omega)$ and $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$*

i. If some $u \in H^1(\Omega)$ solves the Dirichlet BVP (3.59)-(3.60) in Ω, then the pair (u, ψ) where

$$\psi = T_a^+ u \in H^{-\frac{1}{2}}(\partial\Omega) \tag{3.81}$$

solves BDIEs (D1) and (D2).

ii. If a pair $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves BDIE system (D1) and diam$(\Omega) < r_0$, then u solves BDIEs (D2) and the BVP (3.59)-(3.60), this solution is unique, and ψ satisfies (3.81).

iii. If a pair $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves BDIE system (D2), then u solves BDIEs (D1) and the BVP (3.59)-(3.60), this solution is unique,and ψ satisfies (3.81).

Proof. Let $u \in H^1(\Omega)$ be a solution of the BVP (3.59)-(3.60). Then since $f \in L_2(\Omega)$, we have that $u \in H^{1,0}(\Omega; A)$. Setting ψ and recalling how the BDIE systems (D1) and (D2) were constructed, we obtain that (u, ψ) solves them.

Let now a pair $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves system (D1) or (D2). Due to the first equations in the BDIE systems, the hypotheses of Lemma (3.2) are satisfied implying that u belongs to $H^{1,0}(\Omega; A)$ and solves PDE (3.59) in Ω, while the following equation also holds,

$$V_b(\psi - T_a^+ u)(y) - W_b(\varphi_0 - \gamma^+ u)(y) = 0 \qquad y \in \Omega.$$

(ii) Let $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solve system (D1). Taking the trace of the first equation in (D1) and subtracting the second equation from it, we get $\gamma^+ u =$

φ_0 on $\partial\Omega$. Thus, the Dirichlet boundary condition is satisfied, and using it in (3.58), we have $V_b(\psi - T^+u)(y) = 0$ for $y \in \Omega$. Lemma 3.3 then implies $\psi = T_a^+ u$

(iii) Let now $(u,\psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solve system (D2). Taking the conormal derivative of the first equation in (D2) and subtracting the second equation from it, we get $\psi = T_a^+ u$ on $\partial\Omega$. Then inserting this in (3.58) gives $W_b(\varphi_0 - \gamma^+ u)(y) = 0 \quad y \in \Omega$ and Lemma 3.3 implies $\varphi_0 = \gamma^+ u$ on $\partial\Omega$.

The uniqueness of the BDIE system solutions follows form the fact that the corresponding homogeneous BDIE systems can be associated with the homogeneous Dirichlet problem, which has only the trivial solution. Then paragraphs (ii) and (iii) above imply that the homogeneous BDIE systems also have only the trivial solutions.

To prove the invertibility of BDIE operators we need the representation Lemma (see, e.g., [AM11, Theorem B.2 and Lemma B.3] and reference therein)

Theorem 3.13. *If diam$(\Omega) < r_0$, then the following operators are invertible,*

$$\mathscr{D}^1 : H^{1,0}(\Omega, A) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^{1,0}(\Omega, A) \times H^{\frac{1}{2}}(\partial\Omega). \tag{3.82}$$

$$\mathscr{D}^2 : H^{1,0}(\Omega, A) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^{1,0}(\Omega, A) \times H^{-\frac{1}{2}}(\partial\Omega). \tag{3.83}$$

Proof. To prove the invertibility of operator (3.82). Consider the BDIE system D1 with an arbitrary right hand side $\mathscr{F}^{D1} = (\mathscr{F}_1^{D1}, \mathscr{F}_2^{D1}) \in H^{1,0}(\Omega, A) \times H^{\frac{1}{2}}(\partial\Omega)$ such that

$$\mathscr{F}_1^{D1} = \mathscr{P}_b f_* - W_b \Phi_* \quad \text{in} \quad \Omega \tag{3.84}$$

$$\mathscr{F}_2^{D1} = \gamma^+ \mathscr{F}_1^{D1} - \Phi_* \quad \text{on} \quad \partial\Omega \tag{3.85}$$

First assume $(f_*, \Phi_*) \in L_2(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ satisfying equations (3.84) and (3.85) and find their expressions in terms of $\mathscr{F}_1^{D1}$ and $\mathscr{F}_2^{D1}$. Let us re-write (3.84) as:

$$\mathscr{F}_1^{D1} - \mathscr{P}_b f_* = -W_b \Phi_* \quad \text{in} \quad \Omega \tag{3.86}$$

Multiply equation (3.86) by b and applying the Laplace operator to it, we get

$$\Delta\left(b\mathscr{F}_1^{D1} - \mathscr{P}_\Delta f_*\right) = -\Delta\left(W_\Delta(b\Phi_*)\right) \quad \text{in} \quad \Omega$$

Which means

$$f_* = \Delta\left(b\mathscr{F}_1^{D1}\right) \quad \text{in} \quad \Omega \tag{3.87}$$

and $b\mathscr{F}_1^{D1} - \mathscr{P}_\Delta f_* \in H^{1,0}(\Omega, \Delta)$ and hence $\mathscr{F}_1^{D1} - \mathscr{P}_b f_* \in H^{1,0}(\Omega, A)$.

Taking the trace of the first equation of BDIE system D1 (3.61), we obtain

$$\gamma^+ \mathscr{Z}_b u + \gamma^+ \mathscr{R}_b u - \mathscr{V}_b \psi = \gamma^+ \mathscr{F}_1^{D1} - \Phi_* \ \text{ on } \ \partial\Omega.$$

That is, $\mathscr{F}_2^{D1} = \gamma^+ \mathscr{F}_1^{D1} - \Phi_*$ which implies

$$\Phi_* = \gamma^+ \mathscr{F}_1^{D1} - \mathscr{F}_2^{D1}. \tag{3.88}$$

Relation (3.87) and (3.88) can be written as $(f_*, \Phi_*) = C_\Phi(\mathscr{F}_1^{D1}, \mathscr{F}_2^{D1})$, where

$$C_\Phi : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to L_2(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

is linear and continuous operator. And also relation (3.87) and (3.88) satisfy (3.84) and (3.85).

To prove the uniqueness of the operator C_Φ, let a couple $(f_*, \Phi_*) \in L_2(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ be a solution of linear system (3.84) and (3.85) with $\mathscr{F}_1^{D1} = 0$ and $\mathscr{F}_2^{D1} = 0$. Then (3.87) implies $f_* = 0$ in Ω and $\gamma^+ \mathscr{F}_1^{D1} - \Phi_* = 0$ which implies $\Phi_* = 0$.

Hence, any solution of non-homogeneous linear system (3.84) and (3.85) has only one solution, which implies C_Φ is unique.

If $u \in H^{1,0}(\Omega, A)$, then the third Green's identity implies $\mathscr{D}^1 u = (\mathscr{P}_b A u, \gamma^+ u)^T$, i.e., the operator $\mathscr{D}^1$ is continuous. On the other hand, if $(\mathscr{F}_1^{D1}, \mathscr{F}_2^{D1}) \in H^{1,0}(\Omega, A) \times H^{\frac{1}{2}}(\partial\Omega)$, then $\mathscr{F}_1^{D1} = \mathscr{P}_b f_* - W_b \Phi_*$ due to Lemma 3.4, $f_* = C_{\Phi 1} \mathscr{F}_1^{D1}$ where $C_{\Phi 1} : H^{1,0}(\Omega; A) \to L_2(\Omega)$ is a linear and bounded operator. Then the equivalence theorem, Theorem 3.12 and invertibility of the BVP operator given by Theorem 3.11 imply that $\mathscr{D}^1 u = \mathscr{F}^{D1}$ has a unique solution

$$u = \left(A^D\right)^{-1} (f_*, \mathscr{F}_2^{D1})^T = \left(A^D\right)^{-1} diag(C_{\Phi 1}, I) \mathscr{F}^{D1},$$

Here, $\left(A^D\right)^{-1} : L_2(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^{1,0}(\Omega; A)$ is a bounded inverse of the operator A^D of the Dirichlet BVP. Thus, $\left(A^D\right)^{-1} diag(C_{\Phi 1}, I)$ is a bounded inverse of $\mathscr{D}^1$. Which shows $\mathscr{D}^1$ is invertible.

To prove the invertibility of operator (3.83), consider the BDIE system D2 with arbitrary right hand side, i.e,

$$u + \mathscr{Z}_b u + \mathscr{R}_b u - V_b \psi = \mathscr{F}_1^{D2} \quad \text{in} \quad \Omega, \tag{3.89}$$

$$\left(1 - \frac{a}{2b}\right) \psi + T_a^+ \mathscr{Z}_b u + T_a^+ \mathscr{R}_b u - \mathscr{W}'_{ab} \psi = \mathscr{F}_2^{D2} \quad \text{on} \quad \partial\Omega. \tag{3.90}$$

where $\mathscr{F}^{D2} = (\mathscr{F}_1^{D2}, \mathscr{F}_2^{D2}) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ for $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$
Let us introduce a new variable

$$\Psi' = \psi - (\mathscr{F}_2^{D2} - T_a^+ \mathscr{F}_1^{D1}) \in H^{-\frac{1}{2}}(\partial\Omega), \tag{3.91}$$

then $\psi = \Psi' + \left(\mathscr{F}_2^{D2} - T_a^+ \mathscr{F}_1^{D1}\right)$ and putting this value in to equation (3.89) yields

$$u + \mathscr{Z}_b u + \mathscr{R}_b u - V_b \left(\Psi' + \left(\mathscr{F}_2^{D2} - T_a^+ \mathscr{F}_1^{D1}\right)\right) = \mathscr{F}_1^{D2}$$

which implies

$$u+\mathscr{Z}_b u+\mathscr{R}_b u-V_b\Psi' = \mathscr{F}_1^{D2}+V_b\left(\mathscr{F}_2^{D2}-T_a^+\mathscr{F}_1^{D1}\right) \quad \text{in} \quad \Omega$$

Let $\mathscr{F}_{1*}^{D2}=\mathscr{F}_1^{D2}+V_b\left(\mathscr{F}_2^{D2}-T_a^+\mathscr{F}_1^{D1}\right)$; then putting this value above equation, we obtain,

$$u+\mathscr{Z}_b u+\mathscr{R}_b u-V_b\Psi' = \mathscr{F}_{1*}^{D2} \quad \text{in} \quad \Omega,$$

Similarly, substituting equation (3.91) in (3.90)

$$\left(1-\frac{a}{2b}\right)(\Psi'+(\mathscr{F}_2^{D2}-T_a^+\mathscr{F}_1^{D1})) + T_a^+\mathscr{Z}_b u+T_a^+\mathscr{R}_b u-\mathscr{W}'_{ab}(\Psi'+(\mathscr{F}_2^{D2}-T_a^+\mathscr{F}_1^{D1})) = \mathscr{F}_2^{D2}$$

$$\begin{aligned}\left(1-\frac{a}{2b}\right)\Psi'+T_a^+\mathscr{Z}_b u+T_a^+\mathscr{R}_b u-\mathscr{W}'_{ab}\Psi' &= \mathscr{F}_2^{D2}-\left(1-\frac{a}{2b}\right)\left(\mathscr{F}_2^{D2}-T_a^+\mathscr{F}_1^{D1}\right)+-\mathscr{W}'_{ab}\left(\mathscr{F}_2^{D2}-T_a^+\mathscr{F}_1^{D1}\right)\\
&= T_a^+\mathscr{F}_1^{D1}+\left[\frac{a}{2b}\left(\mathscr{F}_2^{D2}-T_a^+\mathscr{F}_1^{D1}\right)+\mathscr{W}'_{ab}\left(\mathscr{F}_2^{D2}-T_a^+\mathscr{F}_1^{D1}\right)\right]\\
&= T_a^+\mathscr{F}_1^{D1}+T_a^+V_b\left(\mathscr{F}_2^{D2}-T_a^+\mathscr{F}_1^{D1}\right)\\
&= T_a^+\left[\mathscr{F}_1^{D1}+V_b\left(\mathscr{F}_2^{D2}-T_a^+\mathscr{F}_1^{D1}\right)\right]=T_a^+\mathscr{F}_{1*}^{D2}\end{aligned}$$

Then the system in the new variable becomes

$$u+\mathscr{Z}_b u+\mathscr{R}_b u-V_b\Psi' = \mathscr{F}_{1*}^{D2} \quad \text{in} \quad \Omega,$$
$$\left(1-\frac{a}{2b}\right)\Psi'+T_a^+\mathscr{Z}_b u+T_a^+\mathscr{R}_b u-\mathscr{W}'_{ab}\Psi' = T_a^+\mathscr{F}_{1*}^{D2} \quad \text{on} \quad \partial\Omega.$$

identical with the given BDIE system D2, where

$$\mathscr{F}_{1*}^{D2}=\mathscr{P}_b f_*-W_b\Phi_* \quad \text{in} \quad \Omega \tag{3.92}$$
$$\mathscr{F}_2^{D2}=T_a^+\mathscr{F}_{1*}^{D2} \quad \text{on} \quad \partial\Omega \tag{3.93}$$

Indeed, let us re-write (3.92) as:

$$\mathscr{F}_{1*}^{D2}-\mathscr{P}_b f_* = -W_b\Phi_* \quad \text{in} \quad \Omega \tag{3.94}$$

Multiply equation (3.94) by b and applying the Laplace operator to it, we get

$$\Delta\left(b\mathscr{F}_{1*}^{D2}-\mathscr{P}_\Delta f_*\right) = -\Delta\left(W_\Delta(b\Phi_*)\right)=0, \quad \text{in} \quad \Omega$$

Which means

$$f_*=\Delta\left(b\mathscr{F}_{1*}^{D2}\right) \quad \text{in} \quad \Omega \tag{3.95}$$

and $b\mathscr{F}_{1*}^{D2}-\mathscr{P}_\Delta f_*\in H^{1,0}(\Omega,\Delta)$ and hence $\mathscr{F}_{1*}^{D2}-\mathscr{P}_b f_*\in H^{1,0}(\Omega,A)$. Hence, the canonical co-normal derivative $T_a^+(\mathscr{F}_{1*}^{D2}-\mathscr{P}_b f_*)$ is well-defined.
Now (3.94) can be written in the form

$$Q=-W_\Delta(b\Phi_*), \tag{3.96}$$

where $Q = b\mathscr{F}_{1*}^{D2} - \mathscr{P}_{\Delta} f_*$.

Taking the co-normal derivative of (3.96)

$$T_a^+ Q = -T_a^+ \left(W_\Delta(b\Phi_*)\right), \quad \text{that} \quad \text{implies} \quad T_a^+ Q = -\mathscr{L}_\Delta^+(b\Phi_*)$$

Where the pseudo-differential operator $\mathscr{L}_\Delta^+ : H_{**}^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is invertible (see, e.g., [ADM17, Theorem 2]). Then $b\Phi_* = (\mathscr{L}_\Delta^+)^{-1}(-T_a^+ Q)$ which implies

$$\begin{aligned}
\Phi_* &= \frac{1}{b}(\mathscr{L}_\Delta^+)^{-1}\left(-T_a^+ Q\right) \\
&= \frac{1}{b}(\mathscr{L}_\Delta^+)^{-1}\left(-T_a^+[b\mathscr{F}_{1*}^{D2} - \mathscr{P}_\Delta(\Delta(b\mathscr{F}_{1*}^{D2}))]\right) \\
&= -\frac{1}{b}(\mathscr{L}_\Delta^+)^{-1}\left(T_a^+[b\mathscr{F}_{1*}^{D2} - \mathscr{P}_\Delta(\Delta(b\mathscr{F}_{1*}^{D2}))]\right) \\
&= -\frac{1}{b}(\mathscr{L}_\Delta^+)^{-1}\left(T_a^+ b\mathscr{F}_{1*}^{D2} - T_a^+\mathscr{P}_\Delta(\Delta(b\mathscr{F}_{1*}^{D2}))\right)
\end{aligned}$$

Thus,

$$\Phi_* = -\frac{1}{b}(\mathscr{L}_\Delta^+)^{-1}\left\{T_a^+(b\mathscr{F}_{1*}^{D2}) - T_a^+\mathscr{P}_\Delta(\Delta(b\mathscr{F}_{1*}^{D2}))\right\}. \qquad (3.97)$$

The relation (3.95) and (3.97) can be written as
$(f_*, \Phi_*) = \hat{C}_\Phi(\mathscr{F}_1^{D2}, \mathscr{F}_2^{D2})$, where

$$\hat{C}_\Phi : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$$

is linear and continuous operator. And also relation (3.95) and (3.97) satisfy (3.89) and (3.90).

To prove the uniqueness of the operator $\hat{C}_\Phi$, let a couple $(f_*, \Phi_*) \in L_2(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ be a solution of linear system (3.89) and (3.90) with $\mathscr{F}_1^{D2} = 0$ and $\mathscr{F}_2^{D2} = 0$. Then (3.95) implies $f_* = 0$ in Ω and taking equation (3.92) due to Lemma 3.3 (ii) $\Phi_* = 0$.

Hence, any solution of non-homogeneous linear system (3.89) and (3.90) has only one solution, which implies $\hat{C}_\Phi$ is unique.

If $u \in H^{1,0}(\Omega, A)$, then the third Green's identity implies $\mathscr{D}^2 u = (\mathscr{P}_b A u, T_a^+ u)^T$, i.e., the operator $\mathscr{D}^2$ is continuous. On the other hand, if $(\mathscr{F}_1^{D2}, \mathscr{F}_2^{D2}) \in H^{1,0}(\Omega, A) \times H^{-\frac{1}{2}}(\partial\Omega)$, then $\mathscr{F}_1^{D2} = \mathscr{P}_b f_* - W_b \Phi_*$ due to Lemma 3.4, $f_* = \hat{C}_{\Phi 1}\mathscr{F}_1^{D2}$ where $\hat{C}_{\Phi 1} : H^{1,0}(\Omega; A) \to L_2(\Omega)$ is a linear and bounded operator. Then the equivalence theorem, Theorem 3.12 and invertibility of the BVP operator given by Theorem 3.11 imply that $\mathscr{D}^2 u = \mathscr{F}^{D2}$ has a unique solution

$$u = \left(A^D\right)^{-1}(f_*, \mathscr{F}^{D2})^T = \left(A^D\right)^{-1} diag(\hat{C}_{\Phi 1}, I)\mathscr{F}^{D2},$$

Here, $\left(A^D\right)^{-1} : L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^{1,0}(\Omega;A)$ is a bounded inverse of the operator A^D of the Dirichlet BVP. Thus, $\left(A^D\right)^{-1} diag(\hat{C}_{\Phi 1}, I)$ is a bounded inverse of $\mathscr{D}^2$.

3.3 Neumann BVP

Let Ω be a domain in $\mathbb{R}^2$ bounded by simple closed infinitely differentiable curve $\partial\Omega$. Consider the following second order elliptic PDE with scalar variable coefficient in two-dimensional bounded domain Ω defined as:

$$Au(x) := \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left[a(x)\frac{\partial u(x)}{\partial x_i}\right] = f(x), \quad x \in \Omega,$$

where u is an unknown function and f is a given function in Ω and assume $a \in C^\infty(\mathbb{R}^2)$, where $0 < a_{min} \le a(x) \le a_{max} < \infty, \quad \forall x \in \mathbb{R}^2$ and $n(x)$ be the exterior unit normal vector, which is defined for all x in $\partial\Omega$.

In this subsection, we shall derive and investigate the two-operator boundary-domain integral equation systems for the following Neumann boundary value problem.
Find a function $u \in H^1(\Omega)$ subject to the Neumann boundary condition:

$$Au = f \quad \text{in} \quad \Omega, \tag{3.98}$$

$$T_a^+ u = \psi_0 \quad \text{on} \quad \partial\Omega \tag{3.99}$$

where $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ and $f \in L_2(\Omega)$.

Remark 3.3.1 *The differential equation (3.98) is understood in the distributional sense and condition (3.99) is understood in the functional sense.*

Here the BVP defined (3.98) - (3.99) has important applications in engineering. As an example, it may describe a steady-state temperature distribution in plane body Ω, which is thermally anisotropic and inhomogeneous. Where $u(x)$ is an unknown temperature, $a(x)$ is a known variable thermo-conductivity coefficient, $f(x)$ is a known distributed heat source, T_a^+ is a surface flux operator where $T_a^+ u(x) = a(x)\frac{\partial u(x)}{\partial n(x)}$, $n(x)$ is the external normal vector to $\partial\Omega, \quad \psi_0(x)$ is known heat flux on the boundary.

A function $u_0(x) = 1$ for $x \in \Omega$ solves the corresponding homogeneous problem for the BVP (3.98) - (3.99) which gives as follows

$$Au = 0 \quad \text{in} \quad \Omega, \tag{3.100}$$

$$T_a^+ u = 0 \quad \text{on} \quad \partial\Omega. \tag{3.101}$$

Applying the Green's second formula, we obtain solvability condition

$$\langle 1, f \rangle_\Omega = \langle \psi_0, 1 \rangle_{\partial\Omega} \tag{3.102}$$

for the non-homogeneous Neumann BVP.

Here we follow the same procedure as it is done in ([ADM17]). Since there exists a non trivial solution of the homogeneous Neumann BVP if u is a solution of the Neumann BVP, then $u + c, \quad \forall c \in \mathbb{R}$ is also solution. Hence the application of LaxMilgram theorem needs a great attention; it depends on the quotient space, $H^1(\Omega)/\mathbb{R}$, where

$$H^1(\Omega)/\mathbb{R} = \{u + c : u \in H^1(\Omega), c \in \mathbb{R}\}.$$

$H^1(\Omega)/\mathbb{R}$ is a Banach space with norm

$$\|[u]\|_{H^1(\Omega)/\mathbb{R}} = \inf_{c \in \mathbb{R}} \|u + c\|_{H^1(\Omega)}.$$

The infimum on the right-hand side is in fact attained by direct differentiation with respect to c of the function $\|u + c\|$ (of one variable c), we find out that

$$c = -\int_\Omega u dx.$$

Thus, the norm of the equivalence class of u coincides with the norm in $H^1(\Omega)$ of the representative with a zero mean value,

$$\|[u]\|_{H^1(\Omega)/\mathbb{R}} = \|u\|_{H^1(\Omega)} \quad \text{where} \quad u \in [u], \quad \int_\Omega u dx = 0.$$

Note that the quotient space is taken from the Hilbert space $H^1(\Omega)$, one can verify that, $H^1(\Omega)/\mathbb{R}$ is also a Hilbert space. Define the bilinear form on the quotient space as

$$\overline{\mathscr{E}}_a([u],[v]) = \int_\Omega a(x) \nabla u \cdot \nabla v dx, \quad u \in [u], v \in [v].$$

Remark 3.3.2 *$\overline{\mathscr{E}}_a$ is bounded and elliptic on the given qoutient space.*

Since the right-hand side is independent of the representative u & v, the bilinear form is well-defined. It is also continuous. Indeed, if $a(x) \in L_\infty(\Omega)$, then $\overline{\mathscr{E}}_a$ is well-defined.

$$\begin{aligned}
|\overline{\mathscr{E}}_a([u],[v])| = |\int_\Omega a(x)\nabla u \cdot \nabla v dx| &\leq \int_\Omega |a(x)||\nabla u||\nabla v| dx \\
&\leq C\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} \\
&\leq \inf_{u\in[u]} \|u\|_{H^1(\Omega)/\mathbb{R}} \inf_{v\in[v]} \|v\|_{H^1(\Omega)/\mathbb{R}}
\end{aligned}$$

which implies $\overline{\mathscr{E}}_a$ is bounded.

From the Cauchy-Schwarz inequality

$$|\int_\Omega u dx| \leq \left(\int_\Omega u^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega dx\right)^{\frac{1}{2}}$$

$$\text{which implies} \quad |\int_\Omega u dx|^2 \leq \int_\Omega u^2 dx |\Omega| \qquad |\Omega| = \text{meas}(\Omega)$$

consequently,

$$\int_\Omega |\nabla u|^2 dx + |\int_\Omega u dx|^2 \leq \int_\Omega |\nabla u|^2 dx + |\Omega| \int_\Omega u^2 dx \leq max\{1, |\Omega|\}\|u\|^2_{H^1(\Omega)}$$

for every $u \in H^1(\Omega)$. Since Ω is smooth and bounded, it follows form the Sobolev Embedding Theorems that the inverse inequality (sometimes called the second Poincar*é* inequality) holds, (see e.g., [HCW08], inequality 4.1.40), i.e., there is a positive number $C > 0$ such that

$$\|u\|^2_{H^1(\Omega)} \leq C\left[\int_\Omega |\nabla u|^2 dx + |\int_\Omega u dx|^2\right]$$

for every $u \in H^1(\Omega)$. Thus, from this inequality, the bi-linear form, $\overline{\mathscr{E}}_a$ is elliptic on the representative quotient space , i.e.,

$$\begin{aligned}
\overline{\mathscr{E}}_a([u],[u]) &= \int_\Omega a(x)\nabla u \cdot \nabla u dx = \int_\Omega |a(x)||\nabla u|^2 dx \\
&= \int_\Omega |\nabla u|^2 dx + |\int_\Omega u dx|^2, \quad \text{provided that} \quad \int_\Omega \text{udx} = 0 \\
&\geq \frac{1}{C}\|u\|^2_{H^1(\Omega)} = \frac{1}{C}\|u\|^2_{H^1(\Omega)/\mathbb{R}}
\end{aligned}$$

which shows $\overline{\mathscr{E}}_a$ is elliptic on the representative quotient space.

Define a linear functional $\mathscr{J}$ on the quotient space $H^1(\Omega)/\mathbb{R}$ as

$$\langle \mathscr{J}, [v] \rangle = \langle f, v \rangle_\Omega - \langle \psi_0, \gamma^+ v \rangle_{\partial\Omega}$$

where $v \in [v]$ and γ^+ is the trace operator. One can see that, it is well-defined and bounded. i.e., it is independent of the particular representative, $v \in [v]$. The right hand side vanishes for $v \equiv$ constant. This is equivalent to the solvability condition (3.102). With this condition satisfied, functional $\mathscr{J}$ is well-defined. Then, we have the estimates.

$$|\langle f, v \rangle_\Omega| \leq \|f\|_{L_2(\Omega)} \|v\|_{H^1(\Omega)} \quad \text{and}$$
$$|\langle \psi_0, \gamma^+ v \rangle_{\partial\Omega}| \leq \|\psi_0\|_{H^{-\frac{1}{2}}(\partial\Omega)} \|\gamma^+ v\|_{H^{\frac{1}{2}}(\partial\Omega)}$$

and from the continuity of γ^+, we have

$$\|\gamma^+ v\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C \|v\|_{H^1(\Omega)} \qquad \text{for some constant} \quad C$$

Therefore,

$$\langle \mathscr{J}, [v] \rangle \leq \left(\|f\|_{L_2(\Omega)} + \|\psi_0\|_{H^{-\frac{1}{2}}(\partial\Omega)} \right) \|v\|_{H^1(\Omega)}$$

which upon taking the infimum with respect to $v \in [v]$ on the right-hand side, implies that

$$\langle \mathscr{J}, [v] \rangle \leq \left(\|f\|_{L_2(\Omega)} + \|\psi_0\|_{H^{-\frac{1}{2}}(\partial\Omega)} \right) \|v\|_{H^1(\Omega)/\mathbb{R}}.$$

Hence, all assumptions of the LaxMilgram Theorem are satisfied, and therefore, there exists a unique solution in the quotient space $H^1(\Omega)/\mathbb{R}$ to the Neumann BVP.

Theorem 3.14. *Given $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ and $f \in L_2(\Omega)$ satisfying the solvability condition (3.102), the variational form of the BVP (3.98) - (3.99)*

$$\overline{\mathscr{E}_a}([u],[v]) = -\langle f, v \rangle_\Omega + \langle \psi_0, \gamma^+ v \rangle_{\partial\Omega}, \forall v \in H^1(\Omega)$$

where $u \in [u]$, has a unique solution in $H^1(\Omega)/\mathbb{R}$.

Remark 3.3.3 *The Neumann homogeneous problem, (3.100) - (3.101), admits only one linearly independent solution $u^0 = 1$ in $H^1(\Omega)$. Then non-homogeneous Neumann problem (3.98) - (3.99) is solvable if and only if the following solvability condition is satisfied*

$$\langle f, u^0 \rangle_\Omega - \langle \psi_0, \gamma^+ u^0 \rangle_{\partial\Omega} = 0.$$

3.3.1 Invertibility of the Hypersingular operator in 2D

The conormal derivative of the double layer potential $W_b v$ for $v \in H^{\frac{1}{2}}(\partial\Omega)$ defines a bounded operator

$$\mathscr{L}^+_{ab} : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega).$$

Inserting $u^0(y) \equiv 1$ in Ω in to the conormal derivative of the Green's third identity for the case $b = 1$ we obtain $\mathscr{L}^+_\Delta \gamma^+ u^0 = 0$ on $\partial\Omega$. Which shows the kernel of $\mathscr{L}^+_\Delta$ consists a non-zero function. Hence, we can not ensure the elipticity of the Hypersingular Boundary Integral Operator $\mathscr{L}^+_\Delta$ in $H^{\frac{1}{2}}(\partial\Omega)$. Instead we have to consider a suitable subspace to obtain a subspace where $\mathscr{L}^+_\Delta$ is elliptic implying that its invertiblity(see e.g., [Ste07]).

$$H^{\frac{1}{2}}_*(\partial\Omega) := \{g \in H^{\frac{1}{2}}(\partial\Omega) : \langle g, w_{eq}\rangle_{\partial\Omega} = 0\}$$

where $w_{eq} \in H^{-\frac{1}{2}}(\partial\Omega)$ is the natural density as defined in ([Ste07], Equation 6.36).

Theorem 3.15. *The hypersingular Boundary Integral Operator $\mathscr{L}^+_\Delta$ in $H^{\frac{1}{2}}(\partial\Omega)$ is $H^{\frac{1}{2}}_*(\partial\Omega)$- elliptic. i.e.,*

$$\langle \mathscr{L}^+_\Delta v, v\rangle_{\partial\Omega} \geq C\|v\|^2_{H^{\frac{1}{2}}(\partial\Omega)}, \qquad v \in H^{\frac{1}{2}}_*(\partial\Omega).$$

Proof. Let $v \in H^{\frac{1}{2}}_*(\partial\Omega), a(x) = 1$, consider the double layer potential $u(x) = -W_\Delta v(x)$, for $x \in \Omega \cup \Omega^c$ which is a solution of the interior Dirichlet BVP

$$\begin{aligned} \Delta u &= 0 && \text{in} \quad \Omega, \\ \gamma^+ u &= \frac{1}{2} v - \mathscr{W}_\Delta v && \text{on} \quad \partial\Omega \end{aligned}$$

and applying the first Green identity, we have

$$\int_\Omega \nabla u \cdot \nabla w dx = \langle T^+_a u, \gamma^+ w\rangle_{\partial\Omega}, \quad \forall w \in H^1(\Omega).$$

For $y_0 \in \Omega$, let $B_R(y_0)$ be a disc of radius $R > 2diam(\Omega)$ which circumscribes Ω, i.e., $\Omega \subset B_R(y_0)$. Then $u(x) = -W_\Delta v(x)$ is also a unique solution to the Dirichlet BVP

$$
\begin{aligned}
\Delta u &= 0 && \text{in} \quad G = B_R(y_0) \setminus \overline{\Omega} \\
\gamma^- u &= -\frac{1}{2} v - \mathscr{W}_\Delta v && \text{on} \quad \partial\Omega \\
\gamma^- u &= -W_\Delta v(x) && \text{on} \quad \partial B_R(y_0)
\end{aligned}
$$

and the corresponding Green's first formula

$$\int_G \nabla u \cdot \nabla w dx = -\langle T_a^- u, \gamma^- w \rangle_{\partial\Omega} + \langle T_a^+ u, \gamma^+ w \rangle_{\partial B_R(y_0)}, \quad \forall w \in H^1(\Omega).$$

For $x \notin \partial\Omega$, we have by definition

$$u(x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{\langle y - x, n_y \rangle}{|x - y|^2} v(y) dS_y.$$

In particular, for $x \in \partial B_R(y_0)$, we then obtain the estimates

$$|u(x)| \leq C_1(v) R^{-1} \quad \text{and} \quad |\nabla u(x)| \leq C_2(v) R^{-2}.$$

By choosing $w = u = -W_\Delta v$ and taking the limit $R \to \infty$ we finally obtain Green's first formula with respect to the exterior domain,

$$\int_{\Omega^c} |\nabla u|^2 dx = -\langle T_a^- u, \gamma^- u \rangle_{\partial\Omega}.$$

By taking the sum of both Green's formulae with respect to the interior and to the exterior domain, and considering the jump relations of the boundary integral operators involved, we obtain for the bilinear form of the hypersingular boundary integral operator

$$
\begin{aligned}
\langle \mathscr{L}_\Delta^+ v, v \rangle_{\partial\Omega} &= \langle T_a^+ u, \gamma^+ u - \gamma^- u \rangle_{\partial\Omega} = \langle T_a^+ u, \gamma^+ u \rangle_{\partial\Omega} - \langle T_a^- u, \gamma^- u \rangle_{\partial\Omega}, \\
&= \int_\Omega |\nabla u|^2 dx + \int_{\Omega^c} |\nabla u|^2 dx = |u|^2_{H^1(\Omega)} + |u|^2_{H^1(\Omega^c)}.
\end{aligned}
$$

For the exterior domain Ω^c we find from the far field behavior of the double layer potential $u = -W_\Delta v$ as $|x| \to \infty$ the norm equivalent

$$C_1 \|u\|^2_{H^1(\Omega^c)} \leq |u|^2_{H^1(\Omega^c)} \leq C_2 \|u\|^2_{H^1(\Omega^c)}.$$

For $v \in H_*^{\frac{1}{2}}(\partial\Omega)$, for $w_{eq} \in H^{-\frac{1}{2}}(\partial\Omega)$ by using the symmetry relation (see e.g., [Ste07, Corollary 6.19, Equation 6.26]), we obtain

$$\begin{aligned}
\langle\gamma^+u,w_{eq}\rangle_{\partial\Omega} &= \langle\left(\frac{1}{2}I-\mathscr{W}_\Delta\right)v,w_{eq}\rangle_{\partial\Omega} = \langle v,w_{eq}\rangle_{\partial\Omega} - \langle\left(\frac{1}{2}I+\mathscr{W}_\Delta\right)v,w_{eq}\rangle_{\partial\Omega} \\
&= -\langle\left(\frac{1}{2}I+\mathscr{W}_\Delta\right)v,w_{eq}\rangle_{\partial\Omega} = -\langle\left(\frac{1}{2}I+\mathscr{W}_\Delta\right)v,V^{-1}1\rangle_{\partial\Omega} \\
&= -\langle V^{-1}\left(\frac{1}{2}I+\mathscr{W}_\Delta\right)v,1\rangle_{\partial\Omega} = -\langle\left(\frac{1}{2}I+\mathscr{W'}_\Delta\right)V^{-1}v,1\rangle_{\partial\Omega} \\
&= -\langle V^{-1}v,\left(\frac{1}{2}I+\mathscr{W}_\Delta\right)1\rangle_{\partial\Omega} = 0
\end{aligned}$$

and hence $\gamma^+u \in H_*^{\frac{1}{2}}(\partial\Omega)$.
By using the norm equivalent theorem (see [Ste07, Theorem 2.6])

$$\|u\|^2_{H_*^{\frac{1}{2}}(\partial\Omega)} := \left\{\langle\gamma^+u,w_{eq}\rangle^2_{\partial\Omega} + \|u\|^2_{L_2(\Omega)}\right\}$$

to be an equivalent norm in $H^1(\Omega)$. For $v \in H_*^{\frac{1}{2}}(\partial\Omega)$, we have $\gamma^+u \in H_*^{\frac{1}{2}}(\partial\Omega)$ and therefore,

$$|u|^2_{H^1(\Omega)} = \left[\langle\gamma^+u,w_{eq}\rangle_{\partial\Omega}\right]^2 + \|u\|^2_{L_2(\Omega)} = \|u\|^2_{H_*^1(\Omega)} \geq C\|u\|^2_{H^1(\Omega)}.$$

By using the trace theorem and the jump relation of the double layer potential we obtain

$$\begin{aligned}
\langle\mathscr{L}_\Delta^+v,v\rangle_{\partial\Omega} &\geq C\left\{\|u\|^2_{H^1(\Omega)} + \|u\|^2_{H^1(\Omega^c)}\right\} \geq \tilde{C}\left\{\|\gamma^+u\|^2_{H^{\frac{1}{2}}(\partial\Omega)} + \|\gamma^-u\|^2_{H^{\frac{1}{2}}(\partial\Omega)}\right\} \\
&\geq \frac{1}{2}\tilde{C}\|\gamma^+u-\gamma^-u\|^2_{H^{\frac{1}{2}}(\partial\Omega)} = \frac{1}{2}\|v\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \qquad \forall v \in H_*^{\frac{1}{2}}(\partial\Omega)
\end{aligned}$$

and therefore $\mathscr{L}_\Delta^+$ is $H_*^{\frac{1}{2}}(\partial\Omega)$- elliptic.

To prove the ellipticity of the hypersingular boundary integral operator $\mathscr{L}_\Delta^+$ on a more wider space, we have to restrict the functions to a suitable subspace, i.e. orthogonal to the constants. When considering the orthogonality with respect to different inner products this gives the ellipticity of the hypersingular boundary integral operator $\mathscr{L}_\Delta^+$ with respect to different subspaces. As in the norm equivalence theorem of Sobolev (see [Ste07, Theorem 2.6]) we define

$$\|v\|^2_{H_*^{\frac{1}{2}}(\partial\Omega)} := \left[\langle v,w_{eq}\rangle_{\partial\Omega}\right]^2 + |v|^2_{H^{\frac{1}{2}}(\partial\Omega)}$$

to be an equivalent norm in $H^{\frac{1}{2}}(\partial\Omega)$.

Corollary 3.3.1 *The hypersingular boundary integral operator* $\mathscr{L}^+_\Delta$ *is* $H_*^{\frac{1}{2}}(\partial\Omega)$*- semi-elliptic. i.e.,*

$$\langle \mathscr{L}^+_\Delta v, v\rangle_{\partial\Omega} \geq |v|^2_{H^{\frac{1}{2}}(\partial\Omega)}, \qquad \forall v \in H^{\frac{1}{2}}(\partial\Omega).$$

We have seen, the $H_*^{\frac{1}{2}}(\partial\Omega)$- ellipticity and $H^{\frac{1}{2}}(\partial\Omega)$ semi-ellipticity of $\mathscr{L}^+_\Delta$, which is from a practical point of view, this seems not to be very convenient for a computational realization. Hence we may use a subspace which is induced by a much simpler inner product.

In order to have invertibility for the hypersingular operator, we define the following subspace of the space $H^{\frac{1}{2}}(\partial\Omega)$

$$H_{**}^{\frac{1}{2}}(\partial\Omega) := \{g \in H^{\frac{1}{2}}(\partial\Omega) : \langle g, 1\rangle = 0\}. \tag{3.103}$$

From the Corollary 3.3.1, we then have for $v \in H_{**}^{\frac{1}{2}}(\partial\Omega)$

$$\langle \mathscr{L}^+_\Delta v, v\rangle_{\partial\Omega} \geq C|v|^2_{H^{\frac{1}{2}}(\partial\Omega)} = C\left\{|v|^2_{H^{\frac{1}{2}}(\partial\Omega)} + [\langle v, 1\rangle_{\partial\Omega}]^2\right\} \geq C\|v\|^2_{H^{\frac{1}{2}}(\partial\Omega)},$$

i.e.,

$$\langle \mathscr{L}^+_\Delta v, v\rangle_{\partial\Omega} \geq C\|v\|^2_{H^{\frac{1}{2}}(\partial\Omega)}, \quad \forall v \in H_{**}^{\frac{1}{2}}(\partial\Omega). \tag{3.104}$$

Hence, $\mathscr{L}^+_\Delta$ is $H_{**}^{\frac{1}{2}}(\partial\Omega)$-elliptic.

Using equations (3.15) and (3.21), let us introduce the operator

$$\widehat{\mathscr{L}_{ab}} := \left[\frac{b}{a}\mathscr{L}^+_{ab} + \frac{\partial b}{\partial n}\left(-\frac{1}{2}I + \mathscr{W}_b\right)\right] g = \mathscr{L}^+_\Delta(bg) \quad \text{on} \quad \partial\Omega. \tag{3.105}$$

Theorem 3.16. *Let* $\partial\Omega$ *be an infinitely smooth boundary curve.*

(i.) *The pseudo-differential operator*

$$\widehat{\mathscr{L}_{ab}} : H_{**}^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega) \quad \text{on} \quad \partial\Omega \tag{3.106}$$

is invertible.

(ii.) *The operator*

$$\frac{b}{a}\mathscr{L}^+_{ab} - \widehat{\mathscr{L}_{ab}} : H^{\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega) \tag{3.107}$$

is bounded and the operator

$$\frac{b}{a}\mathscr{L}_{ab}^{+} - \widehat{\mathscr{L}_{ab}} : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega) \tag{3.108}$$

is compact.

Proof. i. Let $g \in H^{\frac{1}{2}}(\partial\Omega)$, using the jump relation, we can write from (3.105)that

$$\mathscr{L}_{\Delta}^{+}(bg) = \widehat{\mathscr{L}_{ab}} := \frac{b}{a}\mathscr{L}_{ab}^{+} - \frac{\partial b}{\partial n}\gamma^{+}W_{b}g$$

From Theorem 3.2, the operator (3.106) is bounded and from the $H^{\frac{1}{2}}_{**}(\partial\Omega)$-ellipticity of $\widehat{\mathscr{L}_{ab}}$, applying the LaxMilgram Lemma, we can conclude $\widehat{\mathscr{L}_{ab}}$ is $H^{\frac{1}{2}}_{**}(\partial\Omega)$-invertible.

ii. From the continuity of the operator $\mathscr{W}_b$ follows that

$$\|\mathscr{W}_b g\|_{H^{\frac{3}{2}}(\partial\Omega)} \le C_1 \|g\|_{H^{\frac{1}{2}}(\partial\Omega)}$$

holds true for some constant C_1. And since $H^{\frac{3}{2}}(\partial\Omega)$ is continuously embedded in $H^{\frac{1}{2}}(\partial\Omega)$

$$\|g\|_{H^{\frac{1}{2}}(\partial\Omega)} \le C_2 \|g\|_{H^{\frac{3}{2}}(\partial\Omega)}$$

holds for any $g \in H^{\frac{3}{2}}(\partial\Omega)$ and some constant C_2.
Using the relation

$$\left(\frac{b}{a}\mathscr{L}_{ab}^{+} - \widehat{\mathscr{L}_{ab}}\right) g = -\frac{\partial b}{\partial n}\left(-\frac{1}{2}I + \mathscr{W}_b\right) g \quad \text{on} \quad \partial\Omega$$

the following inequalities hold

$$\begin{aligned}
\| \left(\frac{b}{a}\mathscr{L}_{ab}^{+} - \widehat{\mathscr{L}_{ab}}\right) g\|_{H^{\frac{1}{2}}(\partial\Omega)} &\le \max_{\partial\Omega} |\frac{\partial b}{\partial n}| \| -\frac{1}{2}g + \mathscr{W}_b g\|_{H^{\frac{1}{2}}(\partial\Omega)} \\
&\le C\left\{\frac{1}{2}\|g\|_{H^{\frac{1}{2}}(\partial\Omega)} + \|\mathscr{W}_b g\|_{H^{\frac{1}{2}}(\partial\Omega)}\right\} \\
&\le C\left\{\frac{1}{2}\|g\|_{H^{\frac{1}{2}}(\partial\Omega)} + \|\mathscr{W}_b g\|_{H^{\frac{3}{2}}(\partial\Omega)}\right\} \\
&\le C\left\{\frac{1}{2}\|g\|_{H^{\frac{1}{2}}(\partial\Omega)} + C_3\|g\|_{H^{\frac{1}{2}}(\partial\Omega)}\right\} \\
&\le \tilde{C}\|g\|_{H^{\frac{1}{2}}(\partial\Omega)}
\end{aligned}$$

Hence, the operator (3.107) is bounded.

The embeding $H^{\frac{1}{2}}(\partial\Omega) \subset H^{-\frac{1}{2}}(\partial\Omega)$ is compact. Using the Rellich compact embedding theorem, the operator (3.108) is compact.

Corollary 3.3.2 *The operator*

$$\mathscr{L}^{+}_{ab} : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$$

is Fredholm operator of index zero.

Proof. The operator (3.106) is invertible which is a Fredholm operator of index zero, observe that

$$\frac{b}{a}\mathscr{L}^{+}_{ab} = \left(\frac{b}{a}\mathscr{L}^{+}_{ab} - \widehat{\mathscr{L}_{ab}}\right) + \widehat{\mathscr{L}_{ab}}$$

thus the sum of compact operator $\frac{b}{a}\mathscr{L}^{+}_{ab} - \widehat{\mathscr{L}_{ab}}$ and a Fredholm operator $\widehat{\mathscr{L}_{ab}}$. Hence, $\frac{b}{a}\mathscr{L}^{+}_{ab}$ is Fredholm operator of index zero. Which implies $\mathscr{L}^{+}_{ab}$ is Fredholm operator of index zero.

3.3.2 Two-operator BDIEs for Neumann BVP

We will explore different possibilities of reducing the variable-coefficient Neumann BVP (3.98) -(3.99) to a segregated boundary-domain integral equation system. Let us denote the unknown trace as $\varphi = \gamma^{+}u \in H^{\frac{1}{2}}(\partial\Omega)$ and will further consider φ as formally independent on u. Assuming that the function u satisfies PDE $Au = f$, by substituting the Neumann condition into the third Green identity (3.53) and either into its trace (3.54) or into its co-normal derivative (3.55) on $\partial\Omega$, we can reduce the BVP (3.98)-(3.99) to two different systems of Boundary Domain-Integral Equations for the unknown function $u \in H^{1,0}(\Omega;A)$ and $\varphi = \gamma^{+}u \in H^{\frac{1}{2}}(\partial\Omega)$ as formally segregated of u.

Boundary-Domain Integral Equation system N1

To obtain a system, we use equation (3.53) in Ω, and equation (3.55) on the whole boundary $\partial\Omega$

$$u + \mathscr{Z}_b u + \mathscr{R}_b u - V_b T^{+}_{a} u + W_b \gamma^{+} u = \mathscr{P}_b f \;\text{ in }\; \Omega.$$

Putting $T^{+}_{a}u := \psi_0$ and $\gamma^{+}u := \varphi$ taking the above equation, its conormal derivative and then using the jump relation Theorem 3.4, we will arrive at the following two-operator segregated system of BDIE system N1:

$$u + \mathscr{Z}_b u + \mathscr{R}_b u + W_b \varphi = G_0 \quad \text{in} \quad \Omega, \tag{3.109}$$

$$T_a^+ \mathscr{Z}_b u + T_a^+ \mathscr{R}_b u + \mathscr{L}_{ab}^+ \varphi = T_a^+ G_0 - \psi_0 \quad \text{on} \quad \partial\Omega. \tag{3.110}$$

where

$$G_0 = \mathscr{P}_b f + V_b \psi_0. \tag{3.111}$$

System (3.109)-(3.110) can be rewritten in the form

$$\mathscr{N}^1 \mathscr{U} = \mathscr{G}^1,$$

where $\mathscr{U} = (u, \varphi) \in H^{1,0}(\Omega; A) \times H^{\frac{1}{2}}(\partial\Omega)$ and

$$\mathscr{N}^1 := \begin{bmatrix} I + \mathscr{Z}_b + \mathscr{R}_b & W_b \\ T_a^+[\mathscr{Z}_b + \mathscr{R}_b] & \mathscr{L}_{ab}^+ \end{bmatrix}, \qquad \mathscr{G}^1 := \begin{bmatrix} G_0 \\ T_a^+ G_0 - \psi_0 \end{bmatrix}. \tag{3.112}$$

From the mapping properties of V_a and $\mathscr{P}_a$ in (see, e.g., [DM15, Theorem 1 and Theorem 3]) respectively and applying the relation (3.18)- (3.20) we get the inclussion $G_0 \in H^{1,0}(\Omega, A)$, and the definition of co-normal derivative implies $T_a^+ G_0 \in H^{-\frac{1}{2}}(\partial\Omega)$.
Therefore, $\mathscr{G}^1 \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$. Due to the mapping properties of the operators involved in $\mathscr{N}^1$, the operator $\mathscr{N}^1 : H^{1,0}(\Omega, A) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ is bounded.

Remark 3.3.4 : *Let $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$. Then $\mathscr{G}^1 = 0$ if and only if $(f, \psi_0) = 0$. Indeed, the latter equality evidently implies the former. Inversely, if $\mathscr{G}^1 = 0$, then $G_0 = 0$ and $T_a^+ G_0 = 0$ which implies $\psi_0 = 0$. Then, $G_0 = 0$ implies $\mathscr{P}_b f + V_b \psi_0 = 0$ which is $\mathscr{P}_b f = 0$ in Ω. Multiplying by b, and applying Laplace operator, we get $f = 0$.*

Boundary-Domain Integral Equation system N2

To obtain another system, we use equation (3.53) in Ω, and equation (3.54) on the whole boundary $\partial\Omega$,

$$u + \mathscr{Z}_b u + \mathscr{R}_b u - V_b T_a^+ u + W_b \gamma^+ u = \mathscr{P}_b f \ \text{ in } \ \Omega.$$

Putting $T_a^+ u := \psi_0$ and $\gamma^+ u := \varphi$ taking the above equation, its trace and then using the jump relation, i.e., Theorem 3.4, we will arrive at the following two-operator segregated system of BDIE system N2:

$$u + \mathscr{Z}_b u + \mathscr{R}_b u + W_b \varphi = G_0 \quad \text{in} \quad \Omega, \tag{3.113}$$

$$\gamma^+ \mathscr{Z}_b u + \gamma^+ \mathscr{R}_b u + \frac{1}{2}\varphi + \mathscr{W}_b \varphi = \gamma^+ G_0 \ \text{ on } \ \partial\Omega. \tag{3.114}$$

where G_0 is given by the relation (3.111) System (3.113)-(3.114) can be rewritten in the form

$$\mathscr{N}^2 \mathscr{U} = \mathscr{G}^2,$$

$$\text{where} \ \ \mathscr{N}^2 := \begin{bmatrix} I + \mathscr{Z}_b + \mathscr{R}_b & W_b \\ \gamma^+[\mathscr{Z}_b + \mathscr{R}_b] & \frac{1}{2}I + \mathscr{W}_b \end{bmatrix}, \qquad \mathscr{G}^2 := \begin{bmatrix} G_0 \\ \gamma^+ G_0 \end{bmatrix}. \tag{3.115}$$

By the trace theorem $\gamma^+ G_0 \in H^{\frac{1}{2}}(\partial\Omega)$. Therefore, $\mathscr{G}^2 \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$. Due to the mapping properties of the operators involved in $\mathscr{N}^2$, the operator $\mathscr{N}^2 : H^{1,0}(\Omega, A) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ is bounded.

Remark 3.3.5 *: Let $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ and diam$(\Omega) < r_0$, or $\psi_0 \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$. Then $\mathscr{G}^2 = 0$ if and only if $(f, \psi_0) = 0$. Indeed, the latter equality evidently implies the former. Inversely, if $\mathscr{G}^2 = 0$, then $G_0 = 0$ and $\gamma^+ G_0$. Then, $G_0 = 0$ implies $\mathscr{P}_b f + V_b \psi_0 = 0$ in Ω. Multiplying by b, taking into consideration that $bV_b = V_\Delta$ is harmonic and applying Laplace operator, we get $f = 0$. And hence, $V_b \psi_0 = 0$ in Ω. Then by Lemma 3.3 (i) $\psi_0 = 0$ on $\partial\Omega$.*

In the following theorem we shall see the equivalence of the original Neumann BVP (3.98)-(3.99) with the BDIE systems N1 and N2.

3.3.3 Equivalence and invertibility

In the following theorems we shall see the equivalence and invertibility of the original Neumann BVP to the two-operator BDIEs. Let us first prove the Equivalence Theorem.

Theorem 3.17. *Let $f \in L_2(\Omega)$ and $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ satisfy the solvability condition(3.102)*

i. If some $u \in H^1(\Omega)$ solves the Neumann BVP (3.98)-(3.99) in Ω then the pair (u, φ) where

$$\varphi = \gamma^+ u \in H^{\frac{1}{2}}(\partial\Omega) \tag{3.116}$$

solves BDIEs (N1) and (N2).

ii. If a pair $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solves BDIE system (N1), then the function u solves BDIEs (N2), BVP(3.98)-(3.99) and the relation (3.116) holds.

iii. *If a pair $(u,\varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solves BDIE system (N2) and diam$(\Omega) < r_0$, then the function u solves BDIEs (N1), BVP(3.98)-(3.99) and the relation (3.116) holds.*

iv. *The homogeneous BDIE systems (N1) and (N2) have linearly independent solutions spanned by $\mathscr{U}^0 = (u^0,\varphi^0)^T = (1,1)^T$ in $H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$. Condition (3.102) is necessary and sufficient for solvability of the nonhomogeneous BDIE system (N1) and, if diam$(\Omega) < r_0$, also of the system (N2), in $H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.*

Proof. i. Let $u \in H^1(\Omega)$ be a solution of the Neumann BVP (3.98)-(3.99), then since $f \in L_2(\Omega)$, we have that $u \in H^{1,0}(\Omega;A)$. Setting $\varphi = \gamma^+ u$ and recalling how the BDIE systems (N1) and (N2) were constructed, we obtain that (u,φ) solves them.

Let now a couple $(u,\varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solves the BDIE systems (N1) or (N2). Due to the first equations in the BDIE systems, the hypotheses of Lemma 3.2 are satisfied implying that u belongs to $H^{1,0}(\Omega;A)$ and solves PDE (3.98) in Ω, while the following equation also holds,

$$V_b(\psi_0 - T_a^+ u)(y) - W_b(\varphi - \gamma^+ u)(y) = 0 \qquad y \in \Omega. \tag{3.117}$$

ii. Let now $(u,\varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solve system (N1). Taking the conormal derivative of the first equation in (N1) and subtracting the second equation from it, we get $\psi_0 = T_a^+ u$ on $\partial\Omega$. Thus the Neumann condition is satisfied, and using it in (3.117) gives $W_b(\varphi - \gamma^+ u)(y) = 0$ for $y \in \Omega$ and Lemma 3.3 implies $\varphi = \gamma^+ u$ on $\partial\Omega$.

iii. Let $(u,\varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solve system (N2). Taking the trace of the first equation in (N2) and subtracting the second equation from it, we get $\gamma^+ u = \varphi$ on $\partial\Omega$. Then inserting it in (3.117) gives $V_b(\psi_0 - T^+ u)(y) = 0$ for $y \in \Omega$. Lemma 3.3 then implies $\psi_0 = T_a^+ u$. Hence the Neumann condition is satisfied.

iv. Remark 3.3.3 (the solvability condition) along with (i.) - (iii.) implies the claim of item (iv.).

Note that Theorem 3.17 (iv.) implies that the operators

$$\mathscr{N}^1 : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega), \tag{3.118}$$

$$\mathscr{N}^2 : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{3.119}$$

are not injective since their kernels contain non-zero functions.

The kernels (null-spaces) of these operator are spanned by the element $(u^0, \varphi^0)^T = (1,1)^T$ and thus the kernels of the operators is one-dimensional.

The claims that ker $\mathscr{N}^1$ is one dimensional and the couple $(u^0, \varphi^0)^T = (1,1)^T$ belongs to ker $\mathscr{N}^1$ directly follow from Theorem 3.17(iii). The proof for operator (3.119) is similar. To describe in more details the ranges of operators (3.118) and (3.119), i.e., to give more information about the co-kernels of these operators, we will need several auxiliary assertions(see, e.g., [ADM17], [Mik17]).

First of all, let us remark that for any $v \in H^{s-\frac{3}{2}}(\partial\Omega), s < \frac{3}{2}$ the single layer potential can be defined as

$$V_b v(y) = -\langle \gamma P_b(\cdot, y), v\rangle_{\partial\Omega} = -\langle P_b(\cdot, y), \gamma^* v\rangle_{\mathbb{R}^2} = -\mathbf{P}_b \gamma^* v(y) \quad y \in \mathbb{R}^2 \backslash \partial\Omega \quad (3.120)$$

where for some scalar function g,

$$\mathbf{P}_b g(y) = \int_{\mathbb{R}^2} P_b(x, y) g(x) dx,$$

and $\gamma^* : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-2}_{\partial\Omega}, s < \frac{3}{2}$ is the operator adjoined to the trace operator $\gamma : H^{2-s}(\mathbb{R}^2) \to H^{\frac{3}{2}-s}(\partial\Omega)$ and

$$H^s_{\partial\Omega} = \left\{ v \in H^s(\mathbb{R}^2) : supp(v) \subset \partial\Omega \right\}.$$

Lemma 3.6. *Let $f \in \tilde{H}^{s-2}(\Omega)$, for $s > \frac{1}{2}$ where $\frac{1}{2} < s < \frac{3}{2}$ we assume either $diam(\Omega) < r_0$ or $\langle f, 1\rangle_\Omega = 0$. If*

$$r_\Omega \mathbf{P}_b f = 0, \quad (3.121)$$

then $f = 0$ in $\mathbb{R}^2$.

Proof. Multiplying (3.121) by $b(y)$, taking into account the relation $\mathbf{P}_b = \frac{1}{b}\mathbf{P}_\Delta$ and applying the Laplace operator, we obtain $r_\Omega f = 0$, which means $f \in H^{s-2}_{\partial\Omega}$. If $s \geq \frac{3}{2}$, then $f = 0$ by [Mik11, theorem 2.10] which is the characterization of the space $H^t_{\partial\Omega} = \{0\}$ for $t \geq -\frac{1}{2}$. If $\frac{1}{2} < s < \frac{3}{2}$, then by the same theorem, there exists $v \in H^{s-\frac{3}{2}}(\partial\Omega)$ such that $f = \gamma^* v$. This gives $\mathbf{P}_b f = \mathbf{P}_b \gamma^* v = -V_b v$ in $\mathbb{R}^2$. Then (3.121) reduces to $V_b v = 0$ in Ω which implies $v = 0$ on $\partial\Omega$. (see, e.g., for $s = 1$, which can be generalized to $\frac{1}{2} < s < \frac{3}{2}$) for $diam(\Omega) < r_0$ or $v \in H^{-\frac{1}{2}}_{**}(\partial\Omega)$ and thus, $f = 0$ in $\mathbb{R}^2$.

Theorem 3.18. *Let $\frac{1}{2} < s < \frac{3}{2}$*

i. The operator

$$r_\Omega \mathbf{P}_b : \tilde{H}^{s-2}(\Omega) \to H^s(\Omega) \quad (3.122)$$

is continuous.

ii. If diam(Ω) $< r_0$, the inverse of operator (3.122)

$$(r_\Omega \mathbf{P}_b)^{-1} : H^s(\Omega) \to \tilde{H}^{s-2}(\Omega)$$

is continuous and

$$(r_\Omega \mathbf{P}_b)^{-1} g = \left[\Delta \mathring{E}(I - r_\Omega V_\Delta \mathcal{V}_\Delta^{-1}\gamma^+) - \gamma^* \mathcal{V}_\Delta^{-1}\gamma^+\right](bg) \tag{3.123}$$

in $\mathbb{R}^2, \forall g \in H^s(\Omega)$.

Proof. The continuity of (3.122) is well known from the mapping properties of operators, (cf. [CMN09a, Theorem 3.8]). By Lemma 3.6, operator (3.122) is injective. Let us prove its surjectivity. To this end, for arbitrary $g \in H^s(\Omega)$, let us consider the following equation with respect to $\tilde{H}^{s-2}(\Omega)$,

$$r_\Omega \mathbf{P}_\Delta f = g \quad \text{in} \quad \Omega. \tag{3.124}$$

Let $g_1 \in H^s(\Omega)$ be the (unique) solution of the following Dirichlet problem:

$$\Delta g_1 = 0 \quad \text{in} \quad \Omega,$$
$$\gamma^+ g_1 = \gamma^+ g \quad \text{on} \quad \partial\Omega$$

which can be particularly presented as $g_1 = V_\Delta \mathcal{V}_\Delta^{-1}\gamma^+ g$, (see, e.g., [Mik11, Lemma 2.6]). Let $g_0 = g - r_\Omega g_1$. Then $g_0 \in H^s(\Omega)$ and $\gamma^+ g_0 = 0$ and thus g_0 can be uniquely extended to $\mathring{E}g_0 \in \tilde{H}^s(\Omega)$ where $\mathring{E}$ is the operator of extension by 0 outside Ω. Thus by (3.120), equation (3.124) takes form

$$r_\Omega \mathbf{P}_\Delta \left[f + \gamma^* \mathcal{V}_\Delta^{-1}\gamma^+ g\right] = g_0 \quad \text{in} \quad \Omega. \tag{3.125}$$

Any solution $f \in \tilde{H}^{s-2}(\Omega)$ of the corresponding equation on $\mathbb{R}^2$

$$\mathbf{P}_\Delta \left[f + \gamma^* \mathcal{V}_\Delta^{-1}\gamma^+ g\right] = \mathring{E}g_0 \quad \text{in} \quad \mathbb{R}^2 \tag{3.126}$$

will evidently solve (3.125). If f solves (3.126) then acting with the Laplace operator on (3.126), we obtain

$$f = \tilde{Q}g = \Delta \mathring{E}g_0 - \gamma^* \mathcal{V}_\Delta^{-1}\gamma^+ g = \Delta \mathring{E}(g - r_\Omega V_\Delta \mathcal{V}_\Delta^{-1}\gamma^+ g) - \gamma^* \mathcal{V}_\Delta^{-1}\gamma^+ g \tag{3.127}$$

in $\mathbb{R}^2$.

On the other hand, substituting f given by (3.127) to (3.126) and taking into account that $\mathbf{P}_\Delta \Delta \tilde{h} = \tilde{h}$ for any $\tilde{h} \in \tilde{H}^s(\Omega), s \in \mathbb{R}$, we obtain that $\tilde{Q}g$ is indeed a solution of equation (3.126) and thus (3.125). By Lemma 3.6 the solution of (3.125) is unique, which means that the operator $\tilde{Q}$ is inverse to operator (3.122), i.e., $\tilde{Q} = (r_\Omega \mathbf{P}_b)^{-1}$. Since Δ is a continuous operator from $\tilde{H}^s(\Omega)$ to $\tilde{H}^{s-2}(\Omega)$ equation (3.127) imply that

$$(r_\Omega \mathbf{P}_b)^{-1} = \tilde{Q} : H^s(\Omega) \to \tilde{H}^{s-2}(\Omega)$$

is continuous. The relations $\mathbf{P}_b = \frac{1}{b}\mathbf{P}_\Delta$ and $b(x) > C > 0$ then imply invertibility of operator (3.122) and ansatz (3.123).

Lemma 3.7. *Let $diam(\Omega) < r_0$, for any couple $(\mathscr{F}_1, \mathscr{F}_2) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$, there exists a unique couple $(f_*, \psi_*) = \mathscr{C}_\psi(\mathscr{F}_1, \mathscr{F}_2) \in L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ such that*

$$\mathscr{F}_1 = \mathscr{P}_b f_* + V_b \psi_* \quad \text{in} \quad \Omega \tag{3.128}$$

$$\mathscr{F}_2 = T_a^+ \mathscr{F}_1 - \psi_* \quad \text{on} \quad \partial\Omega \tag{3.129}$$

and $\mathscr{C}_\psi : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ is a linear continuous operator. Moreover,

$$\mathring{E} f_* - \gamma^* \psi_* = (r_\Omega \mathbf{P}_b)^{-1} \mathscr{F}_1.$$

Proof. Let $f_{**} = \mathring{E} f_* - \gamma^* \psi_*$, then $\mathscr{P}_b f_{**} = \mathscr{P}_b \mathring{E} f_* - \mathscr{P}_b \gamma^* \psi_*$ by (3.120) we have $\mathscr{P}_b f_{**} = \mathscr{P}_b \mathring{E} f_* + V_b \psi_*$. Hence, equation (3.128) can be rewritten as

$$\mathscr{P}_b f_{**} = \mathscr{F}_1 \quad \text{in} \quad \Omega \tag{3.130}$$

and $T_a^+ \mathscr{P}_b f_{**} = T_a^+ \mathscr{F}_1 = \mathscr{F}_2 + \psi_*$ on $\partial\Omega$, thus (3.129) can be rewritten as

$$T_a^+ \mathscr{P}_b f_{**} - \psi_* = \mathscr{F}_2 \quad \text{on} \quad \partial\Omega. \tag{3.131}$$

By Theorem 3.18, equation (3.130) has a unique solution

$$f_{**} = (\mathscr{P}_b)^{-1} \mathscr{F}_1 \quad \text{in} \quad \Omega \tag{3.132}$$

equation (3.131) gives

$$\psi_* = T_a^+ \mathscr{P}_b f_{**} - \mathscr{F}_2 \quad \text{on} \quad \partial\Omega \tag{3.133}$$

and finally

$$\mathring{E} f_* = f_{**} + \gamma^* \psi_* \tag{3.134}$$

equation (3.132) proves (3.130), while the linearity and continuity of the operators in (3.132) - (3.134) imply the linearity and continuity of the operator $\mathscr{C}_\psi$.

Theorem 3.19. *Let $diam(\Omega) < r_0$ and $u^0 = 1$. The co-kernel of operator (3.118) is spanned over the functional $g^{*1} \in L_2(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ defined as*

$$g^{*1} = \begin{pmatrix} -b\gamma^* \mathscr{V}_\Delta^{-1} \gamma^+ u^0 \\ 0 \end{pmatrix} \tag{3.135}$$

i.e., $g^{*1}(\mathscr{F}_1, \mathscr{F}_2) = \langle -b\gamma^* \mathscr{V}_\Delta^{-1} \gamma^+ u^0, \mathscr{F}_1 \rangle_\Omega$ where $u^0 = 1$.

Proof. Let us consider the equation $\mathcal{N}^1\mathcal{U} = (\mathcal{F}_1, \mathcal{F}_2)^T$ i.e., the BDIE system $\mathcal{N}^1$, with arbitrary $(\mathcal{F}_1, \mathcal{F}_2) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$. By Lemma 3.7, the right hand side of the system has form (3.128)-(3.129), i.e., up to the notations, as in system $(\mathcal{N}^1)$,

$$u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b \varphi = \mathcal{P}_b f_* + V_b \psi_* \quad \text{in} \quad \Omega, \tag{3.136}$$

$$T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u + \mathcal{L}_{ab}^+ \varphi = T_a^+ \mathcal{F}_1 - \psi_* \quad \text{on} \quad \partial\Omega. \tag{3.137}$$

Then, Theorem 3.17 and Theorem 3.18 imply that BDIE system (3.136)-(3.137) is solvable if and only if

$$\begin{aligned}
\langle f_*, u^0\rangle_\Omega - \langle \psi_*, \gamma^+ u^0\rangle_{\partial\Omega} &= \langle f_*, u^0\rangle_\Omega - \langle \gamma^* \psi_*, u^0\rangle_\Omega \\
&= \langle f_* - \gamma^* \psi_*, u^0\rangle_\Omega \\
&= \langle (r_\Omega \mathbf{P}_b)^{-1} \mathcal{F}_1, u^0\rangle_\Omega \\
&= \langle \left[\Delta \mathring{E}(I - r_\Omega V_\Delta \mathcal{V}_\Delta^{-1}\gamma^+) - \gamma^* \mathcal{V}_\Delta^{-1}\gamma^+\right](b\mathcal{F}_1), U^0\rangle_{\mathbb{R}^2} \\
&= \langle \mathring{E}(I - r_\Omega V_\Delta \mathcal{V}_\Delta^{-1}\gamma^+)(b\mathcal{F}_1), U^0\rangle_{\mathbb{R}^2} - \langle \gamma^* \mathcal{V}_\Delta^{-1}\gamma^+(b\mathcal{F}_1), U^0\rangle_{\mathbb{R}^2} \\
&= -\langle \gamma^+(b\mathcal{F}_1), \mathcal{V}_\Delta^{-1}\gamma^+ u^0\rangle_{\partial\Omega} \\
&= -\langle \mathcal{F}_1, -b\gamma^* \mathcal{V}_\Delta^{-1}\gamma^+ u^0\rangle_{\partial\Omega} = 0
\end{aligned}$$

where U^0 is the zero extension of u^0 to $\mathbb{R}^2$ and we took into account that $\Delta U^0 = 0$ in $\mathbb{R}^2$. Thus the functional g^{*1} defined by (3.135) generates the necessary and sufficient solvability condition of equation

$$\mathcal{N}^1\mathcal{U} = (\mathcal{F}_1, \mathcal{F}_2)^T.$$

Hence, g^{*1} is a basis of the co-kernel of $\mathcal{N}^1$.

Theorem 3.20. *Let diam$(\Omega) < r_0$ and $u^0 = 1$. The co-kernel of operator (3.119) is spanned over the functional $g^{*2} \in L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ defined as*

$$g^{*2} = \begin{pmatrix} -b\gamma^*(-\frac{1}{2}I + \mathcal{W}_\Delta')\mathcal{V}_\Delta^{-1}\gamma^+ u^0 \\ -b(-\frac{1}{2}I + \mathcal{W}_\Delta')\mathcal{V}_\Delta^{-1}\gamma^+ u^0 \end{pmatrix} \tag{3.138}$$

i.e., $g^{*2}(\mathcal{F}_1, \mathcal{F}_2) = \langle -b\gamma^*(-\frac{1}{2}I + \mathcal{W}_\Delta')\mathcal{V}_\Delta^{-1}\gamma^+ u^0, \mathcal{F}_1\rangle_\Omega + \langle -b(-\frac{1}{2}I + \mathcal{W}_\Delta')\mathcal{V}_\Delta^{-1}\gamma^+ u^0, \mathcal{F}_2\rangle_{\partial\Omega}$ *where* $u^0 = 1$.

Proof. Let us consider the equation $\mathcal{N}^2\mathcal{U} = (\mathcal{F}_1, \mathcal{F}_2)^T$ i.e., the BDIE system $\mathcal{N}^2$,

$$u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b \varphi = \mathcal{F}_1 \quad \text{in} \quad \Omega, \tag{3.139}$$

$$\gamma^+ \mathcal{Z}_b u + \gamma^+ \mathcal{R}_b u + \frac{1}{2}\varphi + \mathcal{W}_b \varphi = \mathcal{F}_2 \qquad \text{on } \partial\Omega \tag{3.140}$$

with arbitrary $(\mathscr{F}_1, \mathscr{F}_2) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, for $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.

Introducing new variable, $\varphi' = \varphi - (\mathscr{F}_2 - \gamma^+ \mathscr{F}_1)$ the BDIE system (3.139) - (3.140) takes the form

$$u + \mathscr{Z}_b u + \mathscr{R}_b u + W_b \varphi' = \mathscr{F}_1^{'} \quad \text{in} \quad \Omega, \tag{3.141}$$

$$\gamma^+ \mathscr{Z}_b u + \gamma^+ \mathscr{R}_b u + \frac{1}{2}\varphi' + \mathscr{W}_b \varphi' = \gamma^+ \mathscr{F}_1^{'} \qquad \text{on} \quad \partial\Omega. \tag{3.142}$$

On the other hand, by Theorem 3.18 we can always represent $\mathscr{F}_1^{'} = \mathscr{P}_b f_*$ with

$$f_* = \left[\Delta \mathring{E}(I - r_\Omega V_\Delta \mathscr{V}_\Delta^{-1} \gamma^+) - \gamma^* \mathscr{V}_\Delta^{-1} \gamma^+\right] (b \mathscr{F}_1^{'}) \in L_2(\Omega).$$

For $\mathscr{F}_1^{'} = \mathscr{P}_b f_*$, the right hand side of the BDIEs (3.139) - (3.140) is the same as the system $\mathscr{N}^2$ with $f = f_*$ and $\psi_0 = 0$. Then by Theorem 3.17 implies that the BDIE system (3.141) - (3.142) is solvable if and only if

$$\begin{aligned}
\langle f_*, u^0 \rangle &= \langle \left[\Delta \mathring{E}(I - r_\Omega V_\Delta \mathscr{V}_\Delta^{-1} \gamma^+) - \gamma^* \mathscr{V}_\Delta^{-1} \gamma^+\right] (b \mathscr{F}_1^{'}), u^0 \rangle_{\mathbb{R}^2} \\
&= \langle \mathring{E}(I - r_\Omega V_\Delta \mathscr{V}_\Delta^{-1} \gamma^+)(b \mathscr{F}_1^{'}), \Delta u^0 \rangle_{\mathbb{R}^2} - \langle \gamma^* \mathscr{V}_\Delta^{-1} \gamma^+ (b \mathscr{F}_1^{'}), u^0 \rangle_{\mathbb{R}^2} \\
&= -\langle \gamma^+ (b \mathscr{F}_1^{'}), \mathscr{V}_\Delta^{-1} \gamma^+ u^0 \rangle_{\mathbb{R}^2} \\
&= -\langle \frac{1}{2}[\gamma^+ (b \mathscr{F}_1) + (b \mathscr{F}_2)] - \mathscr{W}_\Delta [b(\mathscr{F}_2 - \gamma^+ \mathscr{F}_1)], \mathscr{V}_\Delta^{-1} \gamma^+ u^0 \rangle_{\partial\Omega} \\
&= \langle -b\gamma^* \left(\frac{1}{2} I + \mathscr{W}_\Delta\right) \mathscr{V}_\Delta^{-1} \gamma^+ u^0, \mathscr{F}_1 \rangle_\Omega + \langle -b \left(\frac{1}{2} I - \mathscr{W}'_\Delta\right) \mathscr{V}_\Delta^{-1} \gamma^+ u^0, \mathscr{F}_2 \rangle_{\partial\Omega} \\
&= 0
\end{aligned}$$

Thus the functional g^{*2} defined by (3.138) generates the necessary and sufficient solvability condition of equation

$$\mathscr{N}^2 \mathscr{U} = (\mathscr{F}_1, \mathscr{F}_2)^T.$$

Hence, g^{*2} is a basis of the co-kernel of $\mathscr{N}^2$.

3.3.4 Perturbed segregated BDIE systems for the Neumann problem

For the given original Neumann BVP, even the solvability condition (3.102) is satisfied, the equivalence theorem, Theorem 3.17, implies that the solutions of both BDIE systems, (N1) and (N2), are not unique. Moreover, Theorem 3.17 (iv.) in turn, the BDIE left-hand operators, $\mathscr{N}^1$ and $\mathscr{N}^2$, have non-zero kernels and thus are not invertible. To avoid such difficulty, it is possible to add a finite-dimensional operator to the BDIEs and obtain an unconditionally and uniquely perturbed BDIE. To find a solution (u,φ) from uniquely solvable BDIE systems with continuously invertible left hand side operators, let us consider, following [Mik99], some BDIE systems obtained from (N1) and (N2) by finite-dimensional operator perturbations.

Below we use the notations $\mathscr{U} = (u,\varphi)^T$ and $|\partial\Omega| = \int_{\partial\Omega} dS$.

Perturbation of BDIE system (N1)

Let us introduce the perturbed counterparts of the BDIE system (N1),

$$\hat{\mathscr{N}}^1\mathscr{U} = \mathscr{G}^1 \tag{3.143}$$

where

$$\hat{\mathscr{N}}^1 := \mathscr{N}^1 + \mathring{\mathscr{N}}^1 \quad \text{and} \quad \mathring{\mathscr{N}}^1\mathscr{U}(y) = g^0(\mathscr{U})\mathscr{Y}^1(y) = \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\varphi(x)dS\begin{pmatrix}\frac{1}{b(y)}\\ 0\end{pmatrix}$$

i.e.,

$$g^0(\mathscr{U}) := \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\varphi(x)dS, \quad \mathscr{Y}^1(y) = \begin{pmatrix}\frac{1}{b(y)}u^0(y)\\ 0\end{pmatrix}$$

with $u^0(y) = 1$

For the functional g^{*1} given by (3.135) in Theorem 3.19, since the operator $\mathscr{V}_\Delta^{-1} : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is positive definite (with additional condition $diam(\Omega) < r_0$) and $u^0(x) = 1$, there exists a positive constant C such that

$$\begin{aligned}
g^{*1}(\mathscr{Y}^1) &= \langle -b\gamma^*\mathscr{V}_\Delta^{-1}\gamma^+u^0, \frac{1}{b}u^0\rangle_\Omega \\
&= \langle -\mathscr{V}_\Delta^{-1}\gamma^+u^0, \gamma^+u^0\rangle_{\partial\Omega} \\
&\le -C\|\gamma^+u^0\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \\
&\le -C\|\gamma^+u^0\|^2_{L_2(\partial\Omega)} = -C|\partial\Omega|^2 < 0
\end{aligned}$$

Thus,

$$g^{*1}(\mathscr{Y}^1) < -C|\partial\Omega|^2 < 0 \tag{3.144}$$

Further, for $\mathscr{U}^0 = (u^0,\varphi^0)^T = (1,1)^T$ in $H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$

$$g^0(\mathscr{U}) = \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\gamma^+u^0 dS = 1. \tag{3.145}$$

Due to (3.144) - (3.145) Theorem 2.18 ([Mik99, Theorem 2.11]) Perturbation of Fredholm operators implies the following assertions.

Theorem 3.21. *Let $diam(\Omega) < r_0$, then*

i. *The operator $\hat{\mathscr{N}}^1 : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ is continuous and continuously invertible.*

ii. *If condition $g^{*1}(\mathscr{G}^1) = 0$ (or solvability condition (3.102)for $\mathscr{G}^1$ in the form (3.112)) is satisfied, then the unique solution of perturbed BDIDE system (3.143) gives a solution of orginal BDIE system (N1) such that*

$$g^0(\mathscr{U}) = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} \varphi dS = 0.$$

Perturbation of BDIE system (N2)

Let us introduce the perturbed counterparts of the BDIE system (N2),

$$\hat{\mathscr{N}}^2 \mathscr{U} = \mathscr{G}^2 \tag{3.146}$$

where

$$\hat{\mathscr{N}}^2 := \mathscr{N}^2 + \mathring{\mathscr{N}}^2 \quad \text{and} \quad \mathring{\mathscr{N}}^2 \mathscr{U} = g^0(\mathscr{U})\mathscr{Y}^2(y) = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} \varphi(x)dS \begin{pmatrix} \frac{1}{b(y)} \\ \gamma^+ \frac{1}{b(y)} \end{pmatrix}$$

i.e.,

$$g^0(\mathscr{U}) = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} \varphi(x)dS, \quad \mathscr{Y}^2(y) = \begin{pmatrix} \frac{1}{b(y)} u^0(y) \\ \gamma^+ \frac{1}{b(y)} u^0(y) \end{pmatrix}$$

with $u^0(y) = 1$.

For the functional g^{*2} given by (3.138) in Theorem 3.20, since the operator $\mathscr{V}_\Delta^{-1} : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is positive definite (with additional condition $diam(\Omega) < r_0$ and $u^0(x) = 1$, there exists a positive constant C such that

$$\begin{aligned}
g^{*2}(\mathscr{Y}^2) &= \langle -b\gamma^* \left(\frac{1}{2}I + \mathscr{W}_\Delta\right) \mathscr{V}_\Delta^{-1}\gamma^+ u^0, b^{-1}u^0\rangle_\Omega + \langle -b\left(\frac{1}{2}I - \mathscr{W}'_\Delta\right) \mathscr{V}_\Delta^{-1}\gamma^+ u^0, \gamma^+(b^{-1}u^0)\rangle_{\partial\Omega} \\
&= \langle \left(\frac{1}{2}I + \mathscr{W}_\Delta\right) \mathscr{V}_\Delta^{-1}\gamma^+ u^0 + \left(\frac{1}{2}I - \mathscr{W}'_\Delta\right) \mathscr{V}_\Delta^{-1}\gamma^+ u^0, \gamma^+ u^0\rangle_{\partial\Omega} \\
&= -\langle \mathscr{V}_\Delta^{-1}\gamma^+ u^0, \gamma^+ u^0\rangle_{\partial\Omega} \\
&\le -C\|\gamma^+ u^0\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \\
&\le -C\|\gamma^+ u^0\|^2_{L_2(\partial\Omega)} = -C|\partial\Omega|^2 < 0.
\end{aligned}$$

Thus,

$$g^{*2}(\mathscr{Y}^2) < -C|\partial\Omega|^2 < 0. \tag{3.147}$$

Further, for $\mathscr{U}^0 = (u^0, \varphi^0)^T = (1,1)^T$ in $H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$

$$g^0(\mathscr{U}) = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} \gamma^+ u^0 dS = 1. \tag{3.148}$$

Due to (3.147) - (3.148) and Theorem 2.18 ([Mik99, Theorem 2.11]), Perturbation of Fredholm operators implies the following assertions.

Theorem 3.22. *Let $diam(\Omega) < r_0$, then*

i. The operator $\hat{\mathscr{N}}^2 : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ is continuous and continuously invertible.

ii. If condition $g^{*2}(\mathscr{G}^2) = 0$ (or solvability condition (3.102)for $\mathscr{G}^2$ in the form (3.115)) is satisfied, then the unique solution of perturbed BDIDE system (3.146) gives a solution of orginal BDIE system (N2) such that

$$g^0(\mathscr{U}) = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} \gamma^+ u dS = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} \varphi dS = 0.$$

3.4 Mixed BVP in 2D

Let Ω be a domain in $\mathbb{R}^2$ bounded by simple closed infinitely differentiable curve $\partial\Omega$, and let $\partial\Omega = \overline{\partial_D\Omega} \cup \overline{\partial_N\Omega}$, where $\overline{\partial_D\Omega}$ and $\overline{\partial_N\Omega}$ are non-empty and non-intersecting parts of $\partial\Omega$. Consider the following second order elliptic PDE with scalar variable coefficient in two-dimensional bounded domain Ω defined as:

$$Au(x) := \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left[a(x)\frac{\partial u(x)}{\partial x_i}\right] = f(x), \quad x \in \Omega,$$

where u is an unknown function and f is a given function in Ω and assume $a \in C^\infty(\mathbb{R}^2)$, where $0 < a_{min} \le a(x) \le a_{max} < \infty, \quad \forall x \in \mathbb{R}^2$ and $n(x)$ be the exterior unit normal vector, which is defined for all x in $\partial\Omega$.

In this subsection, we shall derive and investigate the two-operator boundary-domain integral equation systems for the following mixed boundary value problem. Find a function $u \in H^1(\Omega)$ subject to the Mixed boundary condition:

$$Au = f \qquad \text{in} \quad \Omega, \tag{3.149}$$
$$\gamma^+ u = \varphi_0 \qquad \text{on} \quad \partial_D\Omega, \tag{3.150}$$
$$T_a^+ u = \psi_0 \qquad \text{on} \quad \partial_N\Omega, \tag{3.151}$$

where $\varphi_0 \in H^{\frac{1}{2}}(\partial_D\Omega)$, $\psi_0 \in H^{-\frac{1}{2}}(\partial_N\Omega)$ and $f \in L_2(\Omega)$.

Remark 3.4.1 *The differential equation (3.149) is understood in the distributional sense and condition (3.150) is understood in the trace sense while equality (3.151) is understood in the functional sense(cf. 3.4).*

Here $H^{\frac{1}{2}}(\partial_D\Omega) = \{r_{\partial_D\Omega} g : g \in H^{\frac{1}{2}}(\partial\Omega)\}$ where $r_{\partial_D\Omega}$ is the restriction operator on $\partial_D\Omega$ and $H^{-\frac{1}{2}}(\partial_N\Omega)$ is the dual space of the subspace $\tilde{H}^{\frac{1}{2}}(\partial_N\Omega) = \{g : g \in H^{\frac{1}{2}}(\partial\Omega), supp(g) \subset \overline{\partial_N\Omega}\}$.

Here the BVP defined (3.149) - (3.151) has important applications in engineering. As an example, it may describe a steady-state temperature distribution in plane body Ω, which is thermally anisotropic and inhomogeneous. Where $u(x)$ is an unknown temperature, $a(x)$ is a known variable thermo-conductivity coefficient, $f(x)$ is a known distributed heat source, $\varphi_0(x)$ is the known heat on the displacement boundary, T_a^+ is a surface flux operator where $T_a^+ u(x) = a(x)\frac{\partial u(x)}{\partial n(x)}$, $n(x)$ is the external normal vector to $\partial\Omega$, $\quad \psi_0(x)$ is known heat flux on the boundary.

Theorem 3.23. *The homogeneous version of BVP (3.149) - (3.151), i.e., with* $f = 0, \varphi_0 = 0, \psi_0 = 0$, *has only the trivial solution.*

Proof. we follow the idea of the proof forthe 3D case in ([CMN09a, Theorem 2.1]). The proof immediately follows from Green's formula with $u = v$ as a solution of the homogeneous mixed BVP. i.e., the variational setting for the BVP (3.149) - (3.151) is obtained from the Green's first formula , find $u \in H^1(\Omega)$ with $\gamma^+ u = \varphi_0$ on $\partial_D\Omega$ such that

$$\mathscr{E}_a(u,v) = -\langle f, v\rangle_\Omega + \langle \psi_0, \gamma^+ v\rangle_{\partial_N\Omega},$$

is satisfied for $v \in H_0^1(\Omega, \partial_D\Omega) := \{v \in H^1(\Omega) : \gamma^+ v = 0 \text{ on } \partial_\mathrm{D}\Omega\}$. Let $u = v$ be a solution of the corresponding homogeneous mixed BVP.

$$Au = 0 \qquad \text{in} \quad \Omega,$$
$$\gamma^+ u = 0 \qquad \text{on} \quad \partial_D\Omega,$$
$$T_a^+ u = 0 \qquad \text{on} \quad \partial_N\Omega,$$

Then the associated variational form is

$$\mathscr{E}_a(u,u) = -\langle f,u\rangle_\Omega + \langle \psi_0, \gamma^+ u\rangle_{\partial\Omega_N} = 0$$

The bi-linear form $\mathscr{E}_a(u,u) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is bounded and is $H^1(\Omega)$-elliptic. Hence,

$$0 = \mathscr{E}_a(u,u) \geq C\|u\|^2_{H^1(\Omega)}$$

is true if and only if $u = 0$.

Theorem 3.24. *Let $s \in \mathbb{R}$. Let S_1 and S_2 be nonempty, non-intersecting and $\partial\Omega = \bar{S}_1 \cup \bar{S_2}$. Then the following operators*

$$r_{S_2}\mathscr{V}_b : \tilde{H}^s(S_1) \to H^s(S_2)$$
$$r_{S_2}\mathscr{W}_b : \tilde{H}^s(S_1) \to H^s(S_2)$$
$$r_{S_2}\mathscr{W}^{'}_{ab} : \tilde{H}^s(S_1) \to H^s(S_2)$$

are compact.

Proof. Theorem 3.2 implies that the operators $\mathscr{V}_b, \mathscr{W}_b, \mathscr{W}'_b$ have the following properties

$$r_{S_2}\mathscr{V}_b : \tilde{H}^s(S_1) \to H^{s+1}(S_2)$$
$$r_{S_2}\mathscr{W}_b : \tilde{H}^s(S_1) \to H^{s+1}(S_2)$$
$$r_{S_2}\mathscr{W}^{'}_{ab} : \tilde{H}^s(S_1) \to H^{s+1}(S_2)$$

Since the embedding $H^{s+1}(S_2) \subset H^s(S_2)$ is compact by the Rellich compact embedding theorem (see, e.g., [McL00, Theorem 3.7], [DM15, Corollary 1]), the Theorem follows.

Theorem 3.25. *Let S_1 and $\partial\Omega\backslash\overline{S}_1$ be nonempty, open with smooth part of $\partial\Omega$. Then*

$$\mathscr{L}^+_{ab} + \frac{a}{b}\frac{\partial b}{\partial n}\left(-\frac{1}{2}I + \mathscr{W}_b\right) = \mathscr{L}^-_{ab} + \frac{a}{b}\frac{\partial b}{\partial n}\left(\frac{1}{2}I + \mathscr{W}_b\right) \qquad on \quad \partial\Omega$$

Moreover, the pseudo-differential operator $r_{S_1}\widehat{\mathscr{L}_{ab}} : \tilde{H}^{\frac{1}{2}}(S_1) \to H^{-\frac{1}{2}}(S_1)$ where

$$\widehat{\mathscr{L}_{ab}} := \left[\frac{b}{a}\mathscr{L}^+_{ab} + \frac{\partial b}{\partial n}\left(-\frac{1}{2}I + \mathscr{W}_b\right)\right] g = \mathscr{L}_\Delta(bg) \qquad on \quad \partial\Omega$$

is invertible, while the operators

$$r_{S_1}\left(\frac{b}{a}\mathscr{L}^+_{ab} - \widehat{\mathscr{L}_{ab}}\right) : \tilde{H}^{\frac{1}{2}}(S_1) \to H^{\frac{1}{2}}(S_1)$$

are bounded and the operators

$$r_{S_1}\left(\frac{b}{a}\mathscr{L}^{+}_{ab}-\widehat{\mathscr{L}_{ab}}\right):\tilde{H}^{\frac{1}{2}}(S_1)\to H^{-\frac{1}{2}}(S_1)$$

are compact.

Lemma 3.8. *i) Let $\Psi^*\in H^{-\frac{1}{2}}(\partial\Omega)$, and $diam(\Omega)<r_0$ or $\Psi^*\in H^{-\frac{1}{2}}_{**}(\partial\Omega)$. If $V_b\Psi^*=0$ in Ω, then $\Psi^*=0$.*

ii) Let $\Phi^*\in H^{\frac{1}{2}}(\partial\Omega)$. If $W_b\Phi^*=0$ in Ω, then $\Phi^*=0$.

iii) Let $\partial\Omega=\overline{S_1}\cup\overline{S_2}$, where S_1,S_2 are non-intersecting and S_1 is nonempty. Let $diam(\Omega)<r_0$ or $\Psi^*\in H^{-\frac{1}{2}}_{**}(\partial\Omega)$ and $\Psi^*\in\tilde{H}^{-\frac{1}{2}}(S_1)$, $\Phi^*\in\tilde{H}^{\frac{1}{2}}(S_2)$. If

$$V_b\Psi^*-W_b\Phi^*=0,\quad y\in\Omega,$$

then $\Psi^*=0$ and $\Phi^*=0$ on $\partial\Omega$

Proof. (i.) Let us take the trace of equation (i) on $\partial\Omega$, by the jump relation (3.23) we have

$$\gamma^+V_b\Psi^*=\mathscr{V}_b\Psi^*=0\quad\text{on}\quad\partial\Omega.$$

Then the result follows from the invertibility of the single layer potential given in Theorem 3.8.

(ii.) Let us take the trace of equation (ii) on $\partial\Omega$, and use the jump relation (3.24) to obtain,

$$W_b\Phi^*=-\frac{1}{2}\Phi^*+\mathscr{W}_b\Phi^*=0\quad\text{on}\quad\partial\Omega$$

Multiplying this equation by $b(y)$, denoting $\hat{\Phi}^*=b\Phi^*$ and we obtain equation

$$-\frac{1}{2}\hat{\Phi}^*+\mathscr{W}_\Delta\hat{\Phi}^*=0\quad\text{on}\quad S$$

Since this equation for Φ^* is uniquely solvable and $b(y)\neq 0$,this implies point (ii).i.e., it has only the trivial solution, see. e.g.([DL90], Chapter XI, Part B, §2, Remark 8)

(iii.) To prove (iii), multiplying the equation by $b(y)$, we have

$$V_\Delta\Psi^*-W_\Delta(b\Phi^*)=0\text{ in }\Omega$$

Take the traces of this equation and its co-normal derivative on S_1 and S_2, respectively, to obtain

$$\begin{cases}r_{S_1}\mathscr{V}_\Delta\Psi^*-r_{S_1}\left(-\frac{1}{2}\hat{\Phi}^*+\mathscr{W}_\Delta\hat{\Phi}^*\right)=0\quad\text{on}\quad S_1\\ r_{S_2}\left(\frac{1}{2}\Psi^*+\mathscr{W}'_\Delta\Psi^*\right)-r_{S_2}\mathscr{L}^{+}_\Delta\hat{\Phi}^*=0\quad\text{on}\quad S_2\end{cases}$$

since $\Psi^*\in\tilde{H}^{-\frac{1}{2}}(S_1)$ and $\Phi^*\in\tilde{H}^{\frac{1}{2}}(S_2)$ follows

$$\begin{cases} r_{S_1}\mathscr{V}_\Delta \Psi^* - r_{S_1}\mathscr{W}_\Delta \hat{\Phi}^* = 0 \quad \text{on} \quad S_1 \\ r_{S_2}\mathscr{W}'_\Delta \Psi^* - r_{S_2}\mathscr{L}^+_\Delta \hat{\Phi}^* = 0 \quad \text{on} \quad S_2 \end{cases} \tag{3.152}$$

where $\hat{\Phi}^* = b\Phi^*$.
The system of equations (3.152) can be written in a matrix form as

$$\mathscr{A}_\Delta \chi = 0$$

where

$$\mathscr{A}_\Delta := \begin{bmatrix} r_{S_1}\mathscr{V}_\Delta & -r_{S_1}\mathscr{W}_\Delta \\ r_{S_2}\mathscr{W}'_\Delta & -r_{S_2}\mathscr{L}^+_\Delta \end{bmatrix}; \qquad \chi = \begin{bmatrix} \Psi^* \\ \hat{\Phi}^* \end{bmatrix}$$

We shall prove that the operator $\mathscr{A}_\Delta$ is $\tilde{H}_{**}^{-\frac{1}{2}}(S_1) \times \tilde{H}^{\frac{1}{2}}(S_2)$- elliptic and also if $diam(S_1) < r_0$ it is also $\tilde{H}^{-\frac{1}{2}}(S_1) \times \tilde{H}^{\frac{1}{2}}(S_2)$-elliptic.
Infact,

$$\langle \mathscr{A}_\Delta \chi, \chi \rangle_{\partial\Omega} = \left\langle r_{S_1}\mathscr{V}_\Delta \Psi^* - r_{S_1}\mathscr{W}_\Delta \hat{\Phi}^*, \Psi^* \right\rangle_{S_1} + \left\langle r_{S_2}\mathscr{W}'_\Delta - r_{S_2}\mathscr{L}^+_\Delta, \hat{\Phi}^* \right\rangle_{S_2} \tag{3.153}$$

$$= \langle r_{S_1}\mathscr{V}_\Delta \Psi^*, \Psi^* \rangle_{S_1} + \left\langle r_{S_1}\mathscr{W}_\Delta \hat{\Phi}^*, \Psi^* \right\rangle_{S_1} + \left\langle r_{S_2}\mathscr{W}'_\Delta, \hat{\Phi}^* \right\rangle_{S_2} + \left\langle -r_{S_2}\mathscr{L}^+_\Delta, \hat{\Phi}^* \right\rangle_{S_2} \tag{3.154}$$

On section 3.3.1, from Theorem 3.15, Corollary 3.3.1 and equation (3.104) follows the hypersingular operator $\mathscr{L}^+_\Delta$ in the space $H^{\frac{1}{2}}_{**}(\partial\Omega)$, i.e., 3.103 is $H^{\frac{1}{2}}_{**}(\partial\Omega)$- elliptic (see e.g., [Ste07, Section 6.6.2, particularly eqn. 6.38]). i.e.,

$$\forall \Phi^* \in H^{\frac{1}{2}}_{**}(\partial\Omega), \quad \langle \mathscr{L}^+_\Delta \Phi^*, \Phi^* \rangle \geq C \|\Phi^*\|^2_{H^{\frac{1}{2}}(\partial\Omega)}$$

Using the norm equivalence Theorem of Sobolev (see [Ste07, Theorem 2.6]), let $S_2 \subset \partial\Omega$ be an open part. For a given $\hat{\Phi}^* \in \tilde{H}^{\frac{1}{2}}(S_2)$, let $\tilde{\Phi}^* \in H^{\frac{1}{2}}(\partial\Omega)$ denote the extension defined by:

$$\tilde{\Phi}^*(x) = \begin{cases} \hat{\Phi}^*(x) \qquad for \quad x \in S_2 \\ 0 \qquad elsewhere \end{cases}$$

As in the norm equivalence Theorem of Sobolev, (see e.g., [Ste07, Theorem 2.6]) we defined

$$\|w\|^2_{H^{\frac{1}{2}}(\partial\Omega), S_2} = \|w\|^2_{L_2(\partial\Omega)\setminus S_2} + |w|^2_{H^{\frac{1}{2}}(\partial\Omega)}$$

to be equivalent norm in $H^{\frac{1}{2}}(\partial\Omega)$. Hence we have for $\hat{\Phi}^* \in \tilde{H}^{\frac{1}{2}}(S_2)$

$$\langle \mathscr{L}_{\Delta}^{+}\hat{\Phi}^{*},\hat{\Phi}^{*}\rangle_{S_2} = \langle \mathscr{L}_{\Delta}^{+}\tilde{\Phi}^{*},\tilde{\Phi}^{*}\rangle_{S_2} \geq C|\tilde{\Phi}^{*}|^2_{H^{\frac{1}{2}}(\partial\Omega)} = C\left\{\|\tilde{\Phi}^{*}\|^2_{L_2(\partial\Omega)\backslash S_2} + |\tilde{\Phi}^{*}|^2_{H^{\frac{1}{2}}(\partial\Omega)}\right\}$$
$$= C\|\tilde{\Phi}^{*}\|^2_{H^{\frac{1}{2}}(\partial\Omega),S_2} \geq C\|\tilde{\Phi}^{*}\|^2_{H^{\frac{1}{2}}(\partial\Omega)} = C\|\hat{\Phi}^{*}\|^2_{\tilde{H}^{\frac{1}{2}}(S_2)}$$

which implies that

$$\langle \mathscr{L}_{\Delta}^{+}\hat{\Phi}^{*},\hat{\Phi}^{*}\rangle_{S_2} \geq C\|\hat{\Phi}^{*}\|^2_{\tilde{H}^{\frac{1}{2}}(S_2)} \qquad \text{for} \quad \hat{\Phi}^{*} \in H^{\frac{1}{2}}_{**}(S_2)$$

and therefore the $\tilde{H}^{\frac{1}{2}}(S_2)$-ellipticity of the hypersingular boundary integral operator $\mathscr{L}_{\Delta}^{+}$.

In addition, the ellipticity of single layer potential operator $r_{S_1}\mathscr{V}_{\Delta}$ in $\tilde{H}^{-\frac{1}{2}}(S_1)$ can be investigated as follows. The operator $\mathscr{V}_{\Delta} : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is bounded, which implies the operator $\mathscr{V}_{\Delta} : \tilde{H}^{-\frac{1}{2}}(S_1) \to H^{\frac{1}{2}}(S_1)$ is also bounded.

Suppose that $\Psi^{*} \in \tilde{H}^{-\frac{1}{2}}_{**}(S_1)$ i.e., $\langle \Psi^{*},1\rangle_{S_1} = \langle \Psi^{*},1\rangle_{\partial\Omega} = 0$ which implies $\Psi^{*} \in \tilde{H}^{-\frac{1}{2}}_{**}(\partial\Omega)$, then from Theorem 3.7 (i.) we have that the operator

$$r_{S_1}\mathscr{V}_{\Delta} : \tilde{H}^{-\frac{1}{2}}_{**}(S_1) \to \tilde{H}^{\frac{1}{2}}(S_1)$$

is $\tilde{H}^{-\frac{1}{2}}_{**}(S_1)$- elliptic.i.e., for some positive constant C, there holds

$$\langle r_{S_1}\mathscr{V}_{\Delta}\Psi^{*},\Psi^{*}\rangle_{S_1} = \langle \mathscr{V}_{\Delta}\Psi^{*},\Psi^{*}\rangle_{\partial\Omega} \geq c\|\Psi^{*}\|^2_{\tilde{H}^{-\frac{1}{2}}_{**}(\partial\Omega)}, \forall \Psi^{*} \in \tilde{H}^{-\frac{1}{2}}(S_1)$$

if and only if $diam(\Omega) < r_0$.

Moreover, since the operators

$$r_{S_1}\mathscr{W}_{\Delta} : \tilde{H}^{\frac{1}{2}}(S_2) \longrightarrow H^{\frac{1}{2}}(S_1)$$
$$r_{S_2}\mathscr{W}'_{\Delta} : \tilde{H}^{-\frac{1}{2}}(S_1) \longrightarrow H^{-\frac{1}{2}}(S_2)$$

are mutually adjoint, i.e

$$\langle r_{S_1}\mathscr{W}_{\Delta}\hat{\Phi}^{*},\Psi^{*}\rangle_{S_1} = \langle \hat{\Phi}^{*}, r_{S_2}\mathscr{W}'_{\Delta}\Psi^{*}\rangle_{S_2}$$

for arbitrary $\Psi^{*} \in \tilde{H}^{-\frac{1}{2}}(S_1)$ and $\hat{\Phi}^{*} \in \tilde{H}^{\frac{1}{2}}(S_2)$, they are also mutually adjoint for $\Psi^{*} \in \tilde{H}^{-\frac{1}{2}}_{**}(S_1)$ and $\hat{\Phi}^{*} \in \tilde{H}^{\frac{1}{2}}_{**}(S_2)$.

Then the expression in the middle of the right hand side of equation (3.154) vanish. This implies $\tilde{H}^{-\frac{1}{2}}_{**}(S_1) \times \tilde{H}^{\frac{1}{2}}(S_2)$- ellipticity of the operator $\mathscr{A}_{\Delta}$. Consequently we drive the inequality

$$\langle \mathscr{A}_\Delta \chi, \chi \rangle \geq C \left(\|\Psi^*\|^2_{H^{-\frac{1}{2}}(\partial\Omega)} + \|\hat{\Phi}^*\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \right) = \|\chi\|^2_{H_{**}^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)}$$

Due to (3.152) this implies $\Psi^* = 0$ and $\hat{\Phi}^* = 0$, keeping in mind that $b(y) \neq 0$, we have $\Phi^* = 0$ on $\partial\Omega$, which completes the proof.

Representation Lemma

To prove invertibility of the BDIE operators we need the following representation statements following ([CMN09a], Lemma 5.13).

Lemma 3.9. *Let diam*$(\Omega) < r_0$ *or* $\Psi \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$, *and let* $\partial\Omega = \overline{S}_1 \cup \overline{S}_2$, *where* S_1 *and* S_2 *are nonintersecting curves of* $\partial\Omega$. *For any triple* $\mathscr{F} = (F, \Psi, \Phi)^T \in H^{1,0}(\Omega, A) \times H^{-\frac{1}{2}}(S_1) \times H^{\frac{1}{2}}(S_2)$. *There exists a unique triple*

$$(f_*, \Psi_*, \Phi_*) = \tilde{\mathscr{C}}_{S_1,S_2}\mathscr{F} \in L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

such that

$$F = \mathscr{P}_b f_* + V_b \Psi_* - W_b \Phi_* \quad in \quad \Omega, \tag{3.155}$$

$$\Psi = r_{S_1} \Psi_* \qquad on \quad S_1, \tag{3.156}$$

$$\Phi = r_{S_2} \Phi_* \qquad on \quad S_2. \tag{3.157}$$

Moreover, the operator

$$\tilde{\mathscr{C}}_{S_1,S_2} : H^{1,0}(\Omega, A) \times H^{-\frac{1}{2}}(S_1) \times H^{\frac{1}{2}}(S_2) \to L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

is linear and continuous.

Proof. Let Ψ^0 be a fixed extension of the function Ψ from S_1 onto the whole boundary $\partial\Omega$, similarly let Φ^0 be a fixed extension of the function Φ from S_2 onto the whole boundary $\partial\Omega$. We assume that the extensions preserve the space. i.e., $\Psi^0 \in H^{-\frac{1}{2}}(\partial\Omega), \Phi^0 \in H^{\frac{1}{2}}(\partial\Omega)$ and moreover,

$$\|\Psi^0\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq C_0 \|\Psi\|_{H^{-\frac{1}{2}}(S_1)},$$
$$\|\Phi^0\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C_0 \|\Phi\|_{H^{\frac{1}{2}}(S_2)}$$

with some positive constant C_0 independent of Ψ and Φ. (see, e.g., [HT78, Chapter 4, subsection 4.2]). Then arbitrary extensions of the functions Ψ and Φ in the spaces $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$ respectively can be represented as :

$$\Psi_* = \Psi^0 + \tilde{\psi}, \qquad \tilde{\psi} \in \tilde{H}^{-\frac{1}{2}}(S_2)$$
$$\Phi_* = \Phi^0 + \tilde{\varphi}, \qquad \tilde{\varphi} \in \tilde{H}^{\frac{1}{2}}(S_1).$$

If we look for the unknown functions Ψ_* and Φ_* in the form given above, we see that the condition (3.156)-(3.157) are automatically satisfied for arbitrary $\tilde{\psi}$ and $\tilde{\varphi}$. Thus we have shown that functions $f_*, \tilde{\psi}$ and $\tilde{\varphi}$ can be choosen in such a way that (3.155) is satisfied. Due to a relation on logarithmic and layer potential operators, Equation (3.155) can be rewritten in the following equivalent form.

$$bF = \mathscr{P}_\Delta f_* + V_\Delta(\Psi^0 + \tilde{\psi}) - W_\Delta\left[b(\Phi^0 + \tilde{\varphi})\right] \quad \text{in} \quad \Omega. \tag{3.158}$$

Applying the Laplace operator to (3.158) above, we obtain

$$f_* = \Delta(bF) \quad \text{in} \quad \Omega \tag{3.159}$$

which show that the function f_* is uniquely defined and belongs to $L_2(\Omega)$ since $F \in H^{1,0}(\Omega, \Delta)$. Further, substitute (3.159) in (3.158) and rewrite it in the form

$$V_\Delta \tilde{\psi} - W_\Delta(b\tilde{\varphi}) = bF - \mathscr{P}_\Delta(\Delta(bF)) - V_\Delta \Psi^0 + W_\Delta(b\Phi^0) \quad \text{in} \quad \Omega. \tag{3.160}$$

Denote the known right hand side expression in (3.160) by $\tilde{Q}$.

$$\tilde{Q} := bF - \mathscr{P}_\Delta(\Delta(bF)) - V_\Delta \Psi^0 + W_\Delta(b\Phi^0) \quad \text{in} \quad \Omega.$$

It is easy to check that $\tilde{Q}$ is a harmonic function in Ω, as well as the sum of the layer potentials in the left hand side of (3.160). Let us choose the yet unknown functions $\tilde{\psi}$ and $\tilde{\varphi}$ by the conditions:

$$r_{S_2}\gamma^+\left[V_\Delta \tilde{\psi} - W_\Delta(b\tilde{\varphi})\right] = r_{S_2}\gamma^+\tilde{Q} \quad \text{in} \quad \mathrm{S}_2, \tag{3.161}$$
$$r_{S_1}T_a^+\left[V_\Delta \tilde{\psi} - W_\Delta(b\tilde{\varphi})\right] = r_{S_1}T_a^+\tilde{Q} \quad \text{in} \quad \mathrm{S}_1. \tag{3.162}$$

The operator generated by the left hand side of the above system (3.161) - (3.162) is an isomorphism from $\tilde{H}^{-\frac{1}{2}}(S_2) \times \tilde{H}^{\frac{1}{2}}(S_1)$ onto $H^{\frac{1}{2}}(S_2) \times H^{-\frac{1}{2}}(S_1)$. Therefore, the system (3.161) - (3.162) is uniquely solvable with respect to $\tilde{\psi}$ and $\tilde{\varphi}$ for arbitrary right hand side. Denote this solution by $\tilde{\psi}^0$ and $\tilde{\varphi}^0$. From (3.161) - (3.162) due to the uniqueness theorem for the mixed BVP for harmonic functions, it follows that

$$V_\Delta \tilde{\psi}^0 - W_\Delta(b\tilde{\varphi}^0) = bF - \mathscr{P}_\Delta f_* - V_\Delta \Psi^0 + W_\Delta(b\Phi^0) \quad \text{in} \quad \Omega$$

which implies

$$\mathscr{P}_b f_* + V_b(\Psi^0 + \tilde{\psi}^0) - W_b\left[b(\Phi^0 + \tilde{\varphi}^0)\right] = F \quad \text{in} \quad \Omega.$$

This yeilds the existence of the triple $(f_*, \Psi_*, \Phi_*)^T$ satisfying the condition (3.155). The uniqueness is a consequence of the fact that f_* is defined uniquely by (3.159). Indeed, if $F = 0, \Psi = 0$ and $\Phi = 0$, then $f_* = 0$ and $V_b\Psi_* - W_b\Phi_* = 0$ in Ω where $\Psi_* \in \tilde{H}^{-\frac{1}{2}}(S_2), \Phi_* \in \tilde{H}^{\frac{1}{2}}(S_1)$. Hence, we conclude $\Psi_* = 0$ and $\Phi_* = 0$ by Lemma 3.8(ii).
From the above argument it is evident that the $\tilde{\mathscr{C}}_{S_1,S_2}$ is linear and that the norm of the triple

$$\tilde{\mathscr{C}}_{S_1,S_2}\mathscr{F} = (f_*, \Psi_*, \Phi_*)^T \in L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

can be controlled by the norm of the triple

$$\mathscr{F} = (F, \Psi, \Phi)^T \in H^{1,0}(\Omega, A) \times H^{-\frac{1}{2}}(S_1) \times H^{\frac{1}{2}}(S_2)$$

in the corresponding function spaces.

The cases when $S_1 = \varnothing$ or $S_2 = \varnothing$ need to be considered separately.

Lemma 3.10. *Let diam*$(\Omega) < r_0$ *or* $\Psi_* \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$. *For any function* $\mathscr{F}_{\Psi_*} \in H^{1,0}(\Omega, A)$, *there exists a unique couple* $(f_*, \Psi_*) = \mathscr{C}_\Psi \mathscr{F}_{\Psi_*} \in L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ *such that*

$$\mathscr{F}_{\Psi_*} = \mathscr{P}_b f_* + V_b\Psi_* \qquad \text{in } \Omega \tag{3.163}$$

and $\mathscr{C}_\Psi : H^{1,0}(\Omega, A) \to L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ *is a bounded linear operator.*

Proof. We adapt here the proof scheme from [Mik06, Lemma 5.5].
Suppose first there exist some functions $f_*(y), \Psi_*(y)$ satisfying (3.163) and find their expressions in terms of $\mathscr{F}_{\Psi_*}(y)$. Taking into account definitions for the volume and the single layer potentials, ansatz (3.163) can be rewritten as :

$$b(y)\mathscr{F}_{\Psi_*}(y) = \mathscr{P}_\Delta f_* + V_\Delta(\Psi_*)(y) \qquad y \in \Omega. \tag{3.164}$$

Applying the Laplace operator to (3.164) we obtain that

$$\Delta(b\mathscr{F}_{\Psi_*})(y) = f_*(y) \qquad y \in \Omega. \tag{3.165}$$

Then (3.164) can be rewritten as:

$$V_\Delta(\Psi_*)(y) = Q(y) \qquad y \in \Omega \tag{3.166}$$

where

$$Q(y) = b(y)\mathscr{F}_{\Psi_*}(y) - \mathscr{P}_\Delta[\Delta(b\mathscr{F}_{\Psi_*})](y) \qquad y \in \Omega. \tag{3.167}$$

It is easy to check that Q is harmonic functions in Ω as well as (3.165). Then Trace of (3.166) on the boundary gives

$$\gamma^+ V_\Delta(\Psi_*)(y) = \gamma^+ Q(y) \quad \text{implies} \quad \mathscr{V}_\Delta(\Psi_*)(\mathrm{y}) = \gamma^+ \mathrm{Q}(\mathrm{y}) \quad \mathrm{y} \in \partial\Omega. \tag{3.168}$$

Since $\mathscr{V}_\Delta : H^s(\partial\Omega) \to H^{s+1}(\partial\Omega), s \in \mathbb{R}$ is an isomorphism, (see, e.g., [DL90, chap XI, Part B, sec. 2, Remark 1]) and $b(y) \neq 0$ we obtain the following expression of Ψ_*,

$$\Psi_*(y) = \mathscr{V}_\Delta^{-1} \gamma^+ Q(y) \quad y \in \partial\Omega. \tag{3.169}$$

Relation (3.165) and (3.169) implies uniqueness of the couple f_*, Ψ_* Now we have to prove that $f_*(y), \Psi_*(y)$ given by (3.165) and (3.169) do satisfy (3.163). Indeed, the potential $V_\Delta \Psi_*(y)$ with $\Psi_*(y)$ given by (3.169) is harmonic function, and one can check that Q given by (3.167) is also harmonic. Since (3.168) implies that they coincide on the boundary, the two harmonic functions should also coincide in the domain, i.e, (3.166) holds true, which implies (3.163). Thus, (3.165), (3.169) , (3.167) give

$$(f_*, \Psi_*) = \mathscr{C}_\Psi \mathscr{F}_{\Psi_*} = \big(\Delta(b\mathscr{F}_{\Psi_*}), \mathscr{V}_\Delta^{-1}\gamma^+ Q\big) = \big(\Delta(b\mathscr{F}_{\Psi_*}), \mathscr{V}_\Delta^{-1}\gamma^+[b(y)\mathscr{F}_{\Psi_*} - \mathscr{P}_\Delta[\Delta(b\mathscr{F}_{\Psi_*})]\big)$$

and thus we constructed a bounded operator

$$C_\Psi : H^{1,0}(\Omega;A) \to L_2(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

Considering a couple $(F, \Phi)^T = \mathscr{F} \in H^{1,0}(\Omega, A) \times H^{\frac{1}{2}}(\partial\Omega)$ and employing Lemma 3.10 for $\mathscr{F}_{\Psi_*} = F + W_b \Phi \in H^{1,0}(\Omega, A)$ we arrive at the following statement.

Corollary 3.4.1 *Let $diam(\Omega) < r_0$ or $\Psi \in H^{-\frac{1}{2}}_{**}(\partial\Omega)$. For any couple*

$$(F, \Phi)^T = \mathscr{F} \in H^{1,0}(\Omega, A) \times H^{\frac{1}{2}}(\partial\Omega)$$

there exists a unique triple

$$(f_*, \Psi_*, \Phi_*)^T = \tilde{\mathscr{C}}_{\Phi_*} \mathscr{F}_* \in L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

such that

$$F = \mathscr{P}_b f_* + V_b \Psi_* - W_b \Phi_* \qquad in \;\; \Omega, \tag{3.170}$$
$$\Phi = \Phi_* \qquad on \;\; \partial\Omega. \tag{3.171}$$

Moreover, the operator

$$\tilde{\mathscr{C}}_{\Phi_*} : H^{1,0}(\Omega, A) \times H^{\frac{1}{2}}(\partial\Omega) \to L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

is linear and continuous.

Proof. Taking $\Phi_* = \gamma^+ F - \Phi$ and applying Lemma 3.10 for $\mathscr{F}_{\Psi_*} = F + W_b\Phi \in H^{1,0}(\Omega, A)$ we obtain that the existence of equations (3.170)- (3.171). To prove the uniqueness, we consider its homogeneous case. i.e., with $F = 0$ and $\Phi = 0$. then (3.171) implies $\Phi_* = 0$ and thus by (3.170) and Lemma 3.10 we also ontain $\Psi_* = 0, f_* = 0$.

Considering a couple $(F, \Psi)^T \in H^{1,0}(\Omega, A) \times H^{-\frac{1}{2}}(\partial\Omega)$ and employing Lemma 3.4 for $\mathscr{F}_{\Phi_*} = F - V_b\Psi \in H^{1,0}(\Omega, A)$ we arrive at the following statement.

Corollary 3.4.2 *For any couple*

$$(F, \Psi)^T = \mathscr{F}_* \in H^{1,0}(\Omega, A) \times H^{-\frac{1}{2}}(\partial\Omega)$$

there exists a unique triple

$$(f_*, \Psi_*, \Phi_*)^T = \tilde{\mathscr{C}}_{\Psi_*}\mathscr{F}_* \in L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

such that

$$F = \mathscr{P}_b f_* + V_b\Psi_* - W_b\Phi_* \quad in \;\; \Omega \tag{3.172}$$

$$\Psi = \Psi_* \quad on \;\; \partial\Omega \tag{3.173}$$

Moreover, the operator

$$\tilde{\mathscr{C}}_{\Psi_*} : H^{1,0}(\Omega, A) \times H^{-\frac{1}{2}}(\partial\Omega) \to L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

is linear and continuous.

Proof. Taking $\Psi_* = T_a^+ F - \Psi$ and applying Lemma 3.4 for $\mathscr{F}_* = F - V_b\Phi \in H^{1,0}(\Omega, A)$ we prove the existence of equations (3.172)- (3.173). To prove the uniqueness, we consider its homogeneous case. i.e., with $F = 0$ and $\Psi = 0$. then (3.173) implies $\Psi_* = 0$ and thus by (3.172) and Lemma 3.4 we also ontain $\Phi_* = 0, f_* = 0$.

Now consider the original mixed BVP for $u \in H^1(\Omega)$ given on (3.149) -(3.151). We can rewrite the given mixed BVP in a matrix form

$$A^{DN} u = F^{DN}$$

$$\text{where} \quad A^{DN} := \begin{bmatrix} A \\ \gamma^+ \\ T_a^+ \end{bmatrix}. \qquad F^{DN} := \begin{bmatrix} f \\ \varphi_0 \\ \psi_0 \end{bmatrix}.$$

The operator $A^{DN} : H^{1,0}(\Omega, A) \to L_2(\Omega) \times H^{\frac{1}{2}}(\partial_D\Omega) \times H^{-\frac{1}{2}}(\partial_N\Omega)$ is evidentely continuous and due to the uniqueness theorem for BVP, it is also injective.

The following assertions are based on [CMN09a]. The following assertion is well-known and can be proved using variational setting and the Lax-Milgram Lemma

Theorem 3.26. *The operator*

$$A^{DN} : H^{1,0}(\Omega,A) \to L_2(\Omega) \times H^{\frac{1}{2}}(\partial_D\Omega) \times H^{-\frac{1}{2}}(\partial_N\Omega) \tag{3.174}$$

is continuous and continuously invertible.

3.4.1 Two-operator BDIE systems

Let $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$, be some extensions of the given data $\varphi_0 \in H^{\frac{1}{2}}(\partial_D\Omega)$ from $\partial_D\Omega$ to $\partial\Omega$ and $\psi_0 \in H^{-\frac{1}{2}}(\partial_N\Omega)$ from $\partial_N\Omega$ to $\partial\Omega$ respectively.
Let us denote

$$F_0 := \mathscr{P}_b f + V_b\Psi_0 - W_b\Phi_0 \text{ in } \Omega \tag{3.175}$$

Note that for $f \in L_2(\Omega), \Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ we have the inclusion $F_0 \in H^{1,0}(\Omega,A)$ due to the mapping properties of the Newtonian and layer potentials. To reduce BVP (3.149)-(3.151) to one or another two-operator BDIE system, we shall use equation (3.53) in Ω, and restrictions of equation (3.54) or (3.55) to appropriate parts of the boundary. We shall always substitute $\Phi_0 + \varphi$ for $\gamma^+ u$ and $\Psi_0 + \psi$ for $T_a^+ u$, where $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$, are considered as known, while ψ belongs to $\tilde{H}^{-\frac{1}{2}}(\partial_D\Omega)$ and φ belongs to $\tilde{H}^{\frac{1}{2}}(\partial_N\Omega)$ due to the boundary conditions (3.150)-(3.151) and are to be found along with $u \in H^{1,0}(\Omega,\Delta)$. This will lead us to segregated BDIE systems with respect to the unknown triple

$$\mathscr{U} := [u,\psi,\varphi]^T \in H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial_D\Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N\Omega).$$

Boundary-Domain Integral Equation system M11

To formulate the systems,first let us use equation (3.53) in Ω, the restriction of equation (3.54) on $\partial_D\Omega$ and the restriction of equation (3.55) on $\partial_N\Omega$, by putting $\gamma^+ u := \Phi_0 + \varphi, \quad T_a^+ u := \Psi_0 + \psi$ in the equation (3.53), which can be simplified as :

$$u + \mathscr{Z}_b u + \mathscr{R}_b u - V_b T_a^+ u + W_b \gamma^+ u = \mathscr{P}_b f \quad \text{in} \quad \Omega$$
$$u + \mathscr{Z}_b u + \mathscr{R}_b u - V_b(\Psi_0 + \psi) + W_b(\Phi_0 + \varphi) = \mathscr{P}_b f \quad \text{in} \quad \Omega$$
$$u + \mathscr{Z}_b u + \mathscr{R}_b u - V_b \psi + W_b \varphi = \mathscr{P}_b f + V_b \Psi_0 - W_b \Phi_0 \quad \text{in} \quad \Omega$$
$$u + \mathscr{Z}_b u + \mathscr{R}_b u - V_b \psi + W_b \varphi = F_0 \quad \text{in} \quad \Omega$$

where $F_0 = \mathscr{P}_b f + V_b \Psi_0 - W_b \Phi_0$.

Next, taking the trace and the co-normal derivative,of the above equation. Then we arrive at the following two-operator segregated system of BDIEs:

$$u + \mathscr{Z}_b u + \mathscr{R}_b u - V_b \psi + W_b \varphi = F_0 \quad \text{in} \quad \Omega \tag{3.176}$$
$$\gamma^+ \mathscr{Z}_b u + \gamma^+ \mathscr{R}_b u - \mathscr{V}_b \psi + \mathscr{W}_b \varphi = \gamma^+ F_0 - \varphi_0 \ \text{ in } \partial_D \Omega \tag{3.177}$$
$$T_a^+ \mathscr{Z}_b u + T_a^+ \mathscr{R}_b u - \mathscr{W}'_{ab} \psi + \mathscr{L}_{ab}^+ \varphi = T_a^+ F_0 - \psi_0 \ \text{ in } \partial_N \Omega \tag{3.178}$$

which we call BDIE M11, where M stands for the mixed problem and 11 hints that the integral equations on the Dirichlet and Neumann parts of the boundary are of the first kind. Note that due to Lemma 3.2 all terms of equation (3.176) belongs to $H^{1,0}(\Omega;\Delta)$ and their co-normal derivatives are well defined.
System (3.176)-(3.178) can be rewritten in the form

$$\mathscr{M}^{11} \mathscr{U} = \mathscr{G}^{11}$$

where

$$\mathscr{M}^{11} := \begin{bmatrix} I + \mathscr{Z}_b + \mathscr{R}_b & -V_b & W_b \\ r_{\partial_D \Omega} \gamma^+ [\mathscr{Z}_b + \mathscr{R}_b] & -r_{\partial_D \Omega} \mathscr{V}_b & r_{\partial_D \Omega} \mathscr{W}_b \\ r_{\partial_N \Omega} T_a^+ [\mathscr{Z}_b + \mathscr{R}_b] & -r_{\partial_N \Omega} \mathscr{W}'_{ab} & r_{\partial_N \Omega} \mathscr{L}_{ab}^+ \end{bmatrix} \quad \mathscr{G}^{11} := \begin{bmatrix} F_0 \\ r_{\partial_D \Omega} \gamma^+ F_0 - \varphi_0 \\ r_{\partial_N \Omega} T_a^+ F_0 - \psi_0 \end{bmatrix}.$$

Due to the mapping properties of $V_b, \mathscr{V}_b, W_b, \mathscr{W}_b, \mathscr{P}_b, \mathscr{R}_b, \gamma^+ \mathscr{R}_b, T_a^+ \mathscr{R}_b, \mathscr{Z}_b, \gamma^+ \mathscr{Z}_b$, and $\mathrm{T}_\mathrm{a}^+ \mathscr{Z}_\mathrm{b}$ we have $\mathscr{G}^{11} \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ and the operator

$$\mathscr{M}^{11} : H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial_D \Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N \Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial_D \Omega) \times H^{-\frac{1}{2}}(\partial_N \Omega)$$

is continuous.

Remark 3.1. $\mathscr{G}^{11} = 0$ if and only if $(f, \Phi_0, \Psi_0) = 0$. We can show this in the same way as in 3D case (see e.g.,[CMN09a, Remark 5.1]). The latter equality clearly implies the former. Conversely, let $\mathscr{G}^{11} = 0$. Then by relation (3.175) $F_0 = 0$ in turn $\mathscr{P}_b f + V_b \Psi_0 - W_b \Phi_0 = 0$ in Ω which implies $\mathscr{P}_b f = -V_b \Psi_0 + W_b \Phi_0$ in Ω. Due to Lemma 3.8, $f = 0$ and $V_b \Psi_0 - W_b \Phi_0 = 0$ in Ω.
The equality $r_{\partial_D \Omega} \gamma^+ F_0 - \varphi_0 = 0$ implies $\varphi_0 = 0$ on $\partial_D \Omega$ and $r_{\partial_N \Omega} T_a^+ F_0 - \psi_0 = 0$ implies $\psi_0 = 0$ on $\partial_N \Omega$ i.e., $\Phi_0 \in \tilde{H}^{\frac{1}{2}}(\partial_N \Omega), \Psi_0 \in \tilde{H}^{-\frac{1}{2}}(\partial_D \Omega)$. If the extensions Φ_0

and Ψ_0 are obtained from φ_0 and ψ_0 by some linear operators , i.e., $\Phi_0 = E_1\varphi_0$ and $\Psi_0 = E_2\psi_0$, then $\varphi_0 = 0$ and $\psi_0 = 0$ which implies $\Phi_0 = 0$ and $\Psi_0 = 0$ on $\partial\Omega$.

Theorem 3.27. *Let $f \in L_2(\Omega)$ and $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ be some fixed extensions of $\varphi_0 \in H^{\frac{1}{2}}(\partial_D\Omega)$ and $\psi_0 \in H^{-\frac{1}{2}}(\partial_N\Omega)$ respectively.*

i. If some $u \in H^1(\Omega)$ solves the mixed BVP(3.149)-(3.151) in Ω, then the solution is unique and the triple $(u,\psi,\varphi)^T \in H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial_D\Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N\Omega)$ where

$$\psi = T_a^+ u - \Psi_0, \qquad \varphi = \gamma^+ u - \Phi_0 \qquad \text{on } \partial\Omega \tag{3.179}$$

solves the BDIE system M11.

ii. Assume $diam(\Omega) < r_0$, if a triple $(u,\psi,\varphi) \in H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial_D\Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N\Omega)$ solves BDIE system M11, then the solution is unique, the function u solves BVP(3.150)-(3.151), and relation (3.179) holds.

Proof. i. Let $u \in H^1(\Omega)$ be a solution of BVP (3.149)-(3.151). Then by Theorem 3.23 (see, e.g., [CMN09a, Theorem 2.1]), it is unique. Next, set $\varphi = \gamma^+ u - \Phi_0$, $\psi = T_a^+ u - \Psi_0$. Evidently, $\psi \in \tilde{H}^{-\frac{1}{2}}(\partial_D\Omega)$ and $\varphi \in \tilde{H}^{\frac{1}{2}}(\partial_N\Omega)$. And recalling how The BDIE system M11 was constracted, we obtain that the triple $(u,\psi,\varphi)^T$ satisfies the BDIE system M11.

ii. Let the triple $(u,\psi,\varphi)^T$ solves the BDIE system M11. Let us consider the trace of the equation (3.176) on $\partial_D\Omega$ taking into account the jump properties (3.23) - (3.24), and subtruct equation (3.177) to obtain

$$r_{\partial_D\Omega}\gamma^+ u = \varphi_0 \quad \text{on } \partial_D\Omega \tag{3.180}$$

i.e., u satisfies the Dirichlet condition. (3.150).

Taking the co-normal derivative of (3.176) on $\partial_N\Omega$, again with account of the jump properties (3.25), and subtracting equation (3.178) , we obtain:

$$r_{\partial_N\Omega}T_a^+ u = \psi_0 \quad \text{on } \partial_N\Omega \tag{3.181}$$

i.e., u satisfies the Neumann condition (3.151).

Taking into account that $\varphi = 0, \Phi_0 = \varphi_0$ on $\partial_D\Omega$ and $\psi = 0, \Psi_0 = \psi_0$ on $\partial_N\Omega$ equation (3.180) and (3.181) imply that the first equation of (3.176) is satisfied on $\partial_N\Omega$ and the second equation of (3.179) satisfied on $\partial_D\Omega$.

Equation (3.176) and Lemma 3.2 with $\Psi = \psi + \Psi_0; \Phi = \varphi + \Phi_0$ imply that u is a solution to (3.149) and

$$V_b\Psi^* - W_b\Phi^* = 0 \quad \text{in } \Omega$$

where $\Psi^* = \Psi_0 + \psi - T_a^+ u$ and $\Phi^* = \Phi_0 + \varphi - \gamma^+ u$. Since the first equation of (3.179) on $\partial_N\Omega$ and the second equation of (3.179) on $\partial_D\Omega$ already proved, we have $\Psi^* \in \tilde{H}^{-\frac{1}{2}}(\partial_D\Omega)$; $\Phi^* \in \tilde{H}^{\frac{1}{2}}(\partial_N\Omega)$. Then Lemma 3.8 (iii.) with

$$S_1 = \partial_D\Omega; S_2 = \partial_N\Omega \text{ implies that } \psi = \varphi = 0.$$

Unique solvability of the BDIE systems M11 then follows from the already proved relations (3.179) and the unique solvability of BVP (3.149)-(3.151) stated in item (i).

The mapping properties of operators in (3.27), (3.29), (3.33), (3.35) and (3.37) and Theorem 3.27 imply the following statement.

Corollary 3.4.3 *The operator*

$$\mathscr{M}^{11} : H^{1,0}(\Omega,A) \times \tilde{H}^{-\frac{1}{2}}(\partial_D\Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N\Omega) \to H^{1,0}(\Omega,A) \times H^{\frac{1}{2}}(\partial_D\Omega) \times H^{-\frac{1}{2}}(\partial_N\Omega) \tag{3.182}$$

is continuous and injective.

Now we are in the position to analyse the invertibility of the operators $\mathscr{M}^{11}$.

Theorem 3.28. *The Operator (3.182) is continuously invertible.*

Proof. To prove the invertibility of operator (3.182), let us consider BDIE system M11 with an arbitrary right hand side $\mathscr{F}^{11} = \left(\mathscr{F}_1^{11}, \mathscr{F}_2^{11}, \mathscr{F}_3^{11}\right)^T \in H^{1,0}(\Omega,A) \times H^{\frac{1}{2}}(\partial_D\Omega) \times H^{-\frac{1}{2}}(\partial_N\Omega)$. Taking $S_1 = \partial_N\Omega, S_2 = \partial_D\Omega$ and

$$F = \mathscr{F}_1^{11}, \;\; \Psi = r_{\partial_N\Omega} T_a^+ \mathscr{F}_1^{11} - \mathscr{F}_3^{11}, \quad \Phi = r_{\partial_D\Omega}\gamma^+ \mathscr{F}_1^{11} - \mathscr{F}_2^{11}.$$

From Lemma 3.9,(see, e.g., [CMN09a, Lemma 5.13]), $\mathscr{F}^{11}$ can be represented as

$$\begin{aligned}
\mathscr{F}_1^{11} &= \mathscr{P}_b f_* + V_b \Psi_* - W_b \Phi_* \quad \text{in} \quad \Omega \\
\mathscr{F}_2^{11} &= r_{\partial_D\Omega}[\gamma^+ \mathscr{F}_1^{11} - \Phi_*] \quad \text{on} \quad \partial_D\Omega \\
\mathscr{F}_3^{11} &= r_{\partial_N\Omega}[T_a^+ \mathscr{F}_1^{11} - \Psi_*] \quad \text{on} \quad \partial_N\Omega
\end{aligned}$$

where the triple

$$(f_*, \Psi_*, \Phi_*)^T = \mathscr{C}_{\partial_N\Omega,\partial_D\Omega} \mathscr{F}^{11} \in L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{3.183}$$

is unique and the operator

$$\mathscr{C}_{\partial_N\Omega,\partial_D\Omega} : H^{1,0}(\Omega,A) \times H^{\frac{1}{2}}(\partial_D\Omega) \times H^{-\frac{1}{2}}(\partial_N\Omega) \to L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{3.184}$$

is linear and continuous.
Applying Theorem 3.27 with

$$f = f_*, \ \ \Psi_0 = \Psi_*, \ \ \Phi_0 = \Phi_*, \ \ \psi_0 = r_{\partial_N \Omega} \Psi_0, \ \ \varphi_0 = r_{\partial_D \Omega} \Phi_0 \tag{3.185}$$

we obtain that the system M11 is uniquely solvable and its solution is

$$\mathscr{U}_1 = u = (A^{DN})^{-1} \left(f_*, r_{\partial_D \Omega} \Phi_*, r_{\partial_N \Omega} \Psi_* \right)^T, \ \ \mathscr{U}_2 = \psi = T_a^+ \mathscr{U}_1 - \Psi_*, \ \ \mathscr{U}_3 = \varphi = \gamma^+ \mathscr{U}_1 - \Phi_* \tag{3.186}$$

While $r_{\partial_N \Omega} \mathscr{U}_2 = 0; r_{\partial_D \Omega} \mathscr{U}_3 = 0$. Here, by Theorem 3.26, $(A^{DN})^{-1}$ is the continuous inverse operator to the left-hand-side operator of the mixed BVP (3.149)-(3.151), $A^{DN} : H^{1,0}(\Omega, A) \to H^{1,0}(\Omega, A) \times H^{\frac{1}{2}}(\partial_D \Omega) \times H^{-\frac{1}{2}}(\partial_N \Omega)$ (see, e.g., [CMN09a, Corollary 5.16]), Representation (3.183), and continuity of operator (3.184) complete the proof for $\mathscr{M}^{11}$.

Boundary-Domain Integral Equation system M12

To obtain a second system, we use equation (3.53) in Ω, and equation (3.54) on the whole boundary $\partial\Omega$. Putting $\gamma^+ u := \Phi_0 + \varphi, \ \ T_a^+ u := \Psi_0 + \psi$ and taking the trace of the above equation. Then we arrive at the following two-operator segregated system of BDIE system M12:

$$u + \mathscr{Z}_b u + \mathscr{R}_b u - V_b \psi + W_b \varphi = F_0 \quad \text{in} \quad \Omega, \tag{3.187}$$

$$\frac{1}{2}\varphi + \gamma^+ \mathscr{Z}_b u + \gamma^+ \mathscr{R}_b u - \mathscr{V}_b \psi + \mathscr{W}_b \varphi = \gamma^+ F_0 - \Phi_0 \quad \text{on} \quad \partial\Omega. \tag{3.188}$$

System (3.187)-(3.188) can be rewritten in the form

$$\mathscr{M}^{12} \mathscr{U} = \mathscr{G}^{12}$$

where

$$\mathscr{M}^{12} := \begin{bmatrix} I + \mathscr{Z}_b + \mathscr{R}_b & -V_b & W_b \\ \gamma^+[\mathscr{Z}_b + \mathscr{R}_b] & -\mathscr{V}_b & \frac{1}{2} I + \mathscr{W}_b \end{bmatrix} \qquad \mathscr{G}^{12} := \begin{bmatrix} F_0 \\ \gamma^+ F_0 - \Phi_0]^T \end{bmatrix}.$$

Due to the mapping properties of $V_b, \mathscr{V}_b, W_b, \mathscr{W}_b, \mathscr{P}_b, \mathscr{R}_b, \gamma^+ \mathscr{R}_b, \mathscr{Z}_b$, and $\gamma^+ \mathscr{Z}_{\mathrm{b}}$ we have $\mathscr{G}^{12} \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ and the operator

$$\mathscr{M}^{12} : H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial_D \Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N \Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

is continuous.

Remark 3.2. Let $diam(\Omega) < r_0$ or $\Psi_0 \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$. Then $\mathscr{G}^{12} = 0$ if and only if $(f, \Phi_0, \Psi_0) = 0$.

Indeed, the latter equality clearly implies the former. Conversely, let $\mathscr{G}^{12} = 0$. Then by relation (3.175) $F_0 = 0$ in turn $\mathscr{P}_b f + V_b \Psi_0 - W_b \Phi_0 = 0$ in Ω which implies $\mathscr{P}_b f = -V_b \Psi_0 + W_b \Phi_0$ in Ω. Due to Lemma 3.8, $f = 0$ and $V_b \Psi_0 - W_b \Phi_0 = 0$ in Ω.
The equality $\gamma^+ F_0 - \Phi_0 = 0$ implies $\Phi_0 = 0$ on $\partial\Omega$. Thus, $V_b \Psi_0 = 0$, hence from invertibility of single layer operator, Theorem 3.8 follows $\Psi_0 = 0$ on $\partial\Omega$.

Now let us prove the equivalence of BVP (3.149)-(3.151) with the BDIE systems M12.

Theorem 3.29. *Let $f \in L_2(\Omega)$ and $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ be some fixed extensions of $\varphi_0 \in H^{\frac{1}{2}}(\partial_D\Omega)$ and $\psi_0 \in H^{-\frac{1}{2}}(\partial_N\Omega)$ respectively.*

i. If some $u \in H^1(\Omega)$ solves the mixed BVP(3.149)-(3.151) in Ω, then the solution is unique and the triple $(u, \psi, \varphi) \in H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial_D\Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N\Omega)$ where

$$\psi = T_a^+ u - \Psi_0, \qquad \varphi = \gamma^+ u - \Phi_0 \quad \text{on } \partial\Omega \tag{3.189}$$

solves the BDIE system M12.

ii. Let $diam(\Omega) < r_0$, if a triple $(u, \psi, \varphi) \in H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial_D\Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N\Omega)$ solves BDIE system M12, then the solution is unique, the function u solves BVP(3.149)-(3.151), and relation (3.189) holds.

Proof. i. Let $u \in H^1(\Omega)$ be a solution of BVP (3.149)-(3.151). Then by Theorem 3.23 (see, e.g., [CMN09a, Theorem 2.1]), it is unique. Next, set $\varphi = \gamma^+ u - \Phi_0$, $\psi = T_a^+ u - \Psi_0$. Evidently, $\psi \in \tilde{H}^{-\frac{1}{2}}(\partial_D\Omega)$ and $\varphi \in \tilde{H}^{\frac{1}{2}}(\partial_N\Omega)$. Then immediately follow from the relation (3.187)-(3.188) that the triple (u, ψ, φ) satisfies the BDIE system M12.

ii. Let (u, ψ, φ) solves the BDIE (3.187)-(3.188). Consider the trace of the equation (3.187) on $\partial\Omega$ taking into account the jump properties (3.23) -(3.24), and subtruct equation (3.188) to obtain

$$u + \mathscr{Z}_b u + \mathscr{R}_b u - V_b \psi + W_b \varphi = F_0$$

which implies $$\gamma^+ u + \gamma^+ \mathscr{Z}_b u + \gamma^+ \mathscr{R}_b u - \mathscr{V}_b \psi - \frac{1}{2}\varphi + \mathscr{W}_b \varphi = \gamma^+ F_0$$

subtracting equation (3.188) from the above equation will give us

$$\gamma^+ u = \Phi_0 + \varphi \quad \text{on} \quad \partial\Omega \tag{3.190}$$

which implies the second equation of (3.189) holds. Since $\varphi = 0, \Phi_0 = \varphi_0$ on $\partial_D\Omega$ we see that the Dirichlet condition (3.150) is satisfied.
Equation (3.187) and Lemma 3.2 with $\Psi = \psi + \Psi_0; \Phi = \varphi + \Phi_0$ imply that u is a solution to (3.149) and

$$V_b\left(\Psi_0+\psi-T_a^+u\right)-W_b\left(\Phi_0+\varphi-\gamma^+u\right)=0 \quad \text{in} \quad \Omega. \tag{3.191}$$

Due to (3.190) the second term in (3.191) vanishes, and by Lemma 3.8(i) we obtain:

$$\Psi_0+\psi-T_a^+u=0 \quad \text{on} \quad \partial\Omega \tag{3.192}$$

i.e., the first equation in (3.189) is satisfied as well. Since $\psi=0, \Psi_0=\psi_0$ on $\partial_N\Omega$ equation (3.192) implies that u satisfies the Neumann Boundary Condition (3.151).

The mapping properties of operators in (3.27), (3.29), (3.33), (3.35) and (3.37) and Theorem 3.29 imply the following statement.

Corollary 3.4.4 *The operator*

$$\mathscr{M}^{12}: H^{1,0}(\Omega,A)\times\tilde{H}^{-\frac{1}{2}}(\partial_D\Omega)\times\tilde{H}^{\frac{1}{2}}(\partial_N\Omega)\to H^{1,0}(\Omega,A)\times H^{\frac{1}{2}}(\partial\Omega) \tag{3.193}$$

is continuous and injective.

Now we are in the position to analyse the invertibility of the operators $\mathscr{M}^{12}$.

Theorem 3.30. *The Operators (3.193) is continuously invertible.*

Proof. To prove the invertibility of operator (3.193), let us consider BDIE system M12 with an arbitrary right hand side $\mathscr{F}^{12}=\left(\mathscr{F}_1^{12},\mathscr{F}_2^{12}\right)^T\in H^{1,0}(\Omega,A)\times H^{\frac{1}{2}}(\partial\Omega)$. Taking

$$\begin{aligned} F&=\mathscr{F}_1^{12}, && \text{in} \quad \Omega \\ \Phi&=\gamma^+\mathscr{F}_1^{12}-\mathscr{F}_2^{12} && \text{on} \quad \partial\Omega \end{aligned}$$

in Corollary 3.4.1, we obtain the representation

$$\begin{aligned} \mathscr{F}_1^{12}&=\mathscr{P}_bf_*+V_b\Psi_*-W_b\Phi_* && \text{in} \quad \Omega \\ \mathscr{F}_2^{12}&=\gamma^+\mathscr{F}_1^{12}-\Phi_* && \text{on} \quad \partial\Omega \end{aligned}$$

where the triple

$$(f_*,\Psi_*,\Phi_*)^T=\tilde{\mathscr{C}}_{\Phi_*}\mathscr{F}^{12}\in L_2(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega) \tag{3.194}$$

is unique and the operator

$$\tilde{\mathscr{C}}_{\Phi_*}: H^{1,0}(\Omega,A)\times H^{\frac{1}{2}}(\partial\Omega)\to L_2(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega) \tag{3.195}$$

is linear and continuous.
Applying Theorem 3.29) with (3.185) i.e.,

$$f = f_*, \quad \Psi_0 = \Psi_*, \quad \Phi_0 = \Phi_*, \quad \psi_0 = r_{\partial_N\Omega}\Psi_0, \quad \varphi_0 = r_{\partial_D\Omega}\Phi_0$$

we obtain that the system M12 is uniquely solvable and its solution is given by (3.186) which is:

$$\mathscr{U}_1 = u = (A^{DN})^{-1}\left(f_*, r_{\partial_D\Omega}\Phi_*, r_{\partial_N\Omega}\Psi_*\right)^T, \quad \mathscr{U}_2 = \psi = T_a^+\mathscr{U}_1 - \Psi_*, \quad \mathscr{U}_3 = \phi = \gamma^+\mathscr{U}_1 - \Phi_* \tag{3.196}$$

Representation (3.194), and continuity of operator (3.195) complete the proof for $\mathscr{M}^{12}$.

Boundary-Domain Integral Equation system M21

To obtain a third system, we use equation (3.53) in Ω, and equation (3.55) on the whole boundary $\partial\Omega$. Putting $\gamma^+u := \Phi_0 + \varphi, \quad T_a^+u := \Psi_0 + \psi$ and taking the co-normal derivative of the above equation. Then we arrive at the following two-operator segregated system of BDIE system M21:

$$u + \mathscr{Z}_b u + \mathscr{R}_b u - V_b\psi + W_b\varphi = F_0 \qquad \text{in} \quad \Omega, \tag{3.197}$$

$$\left(1 - \frac{a}{2b}\right)\psi + T_a^+\mathscr{Z}_b u + T_a^+\mathscr{R}_b u - \mathscr{W}'_{ab}\psi + \mathscr{L}_{ab}^+\varphi = T_a^+F_0 - \Psi_0 \quad \text{on} \quad \partial\Omega. \tag{3.198}$$

System (3.197)-(3.198) can be rewritten in the form

$$\mathscr{M}^{21}\mathscr{U} = \mathscr{F}^{21}$$

where

$$\mathscr{M}^{21} := \begin{bmatrix} I + \mathscr{Z}_b + \mathscr{R}_b & -V_b & W_b \\ T_a^+[\mathscr{Z}_b + \mathscr{R}_b] & \left(1 - \frac{a}{2b}\right)I - \mathscr{W}'_{ab} & \mathscr{L}_{ab}^+ \end{bmatrix} \qquad \mathscr{G}^{21} := \begin{bmatrix} F_0 \\ T_a^+F_0 - \Phi_0 \end{bmatrix}.$$

Due to the mapping properties of $V_b, W_b, \mathscr{W}_b, \mathscr{P}_b, \mathscr{R}_b, T_a^+\mathscr{R}_b, \mathscr{Z}_b$, and $\mathrm{T_a^+}\mathscr{Z}_\mathrm{b}$ we have $\mathscr{G}^{21} \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ and the operator

$$\mathscr{M}^{21} : H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial_D\Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N\Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$$

is continuous.

Remark 3.3. $\mathscr{G}^{21} = 0$ if and only if $(f, \Phi_0, \Psi_0) = 0$. Indeed, the latter equality clearly implies the former. Conversely, let $\mathscr{G}^{21} = 0$. Then by relation (3.175) $F_0 = 0$ in turn $\mathscr{P}_b f + V_b\Psi_0 - W_b\Phi_0 = 0$ in Ω which implies $\mathscr{P}_b f = -V_b\Psi_0 + W_b\Phi_0$ in Ω. Due to Lemma 3.8, $f = 0$ and $V_b\Psi_0 - W_b\Phi_0 = 0$ in Ω.

The equality $T_a^+ F_0 - \Psi_0 = 0$ implies $\Psi_0 = 0$ on $\partial\Omega$, Thus, $W_b \Phi_0 = 0$ hence by the same Lemma 3.8(ii.) follows $\Phi_0 = 0$ on $\partial\Omega$.

Now let us prove the equivalence of BVP (3.149)-(3.151) with the BDIE systems M21.

Theorem 3.31. *Let $f \in L_2(\Omega)$ and $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ be some fixed extensions of $\varphi_0 \in H^{\frac{1}{2}}(\partial_D\Omega)$ and $\psi_0 \in H^{-\frac{1}{2}}(\partial_N\Omega)$ respectively.*

i. If some $u \in H^1(\Omega)$ solves the mixed BVP(3.149)-(3.151) in Ω, then the solution is unique and the triple $(u,\psi,\varphi) \in H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial_D\Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N\Omega)$ where

$$\psi = T_a^+ u - \Psi_0, \quad \varphi = \gamma^+ u - \Phi_0 \quad on \;\; \partial\Omega \tag{3.199}$$

solves the BDIE system M21.

ii. Assume a triple $(u,\psi,\varphi) \in H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial_D\Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N\Omega)$ solves BDIE system M21, then the solution is unique, the function u solves BVP(3.149)-(3.151), and relation (3.199) holds.

Proof. i. Let $u \in H^1(\Omega)$ be a solution of BVP (3.149)-(3.151). Then by Theorem 3.23 (see, e.g., [CMN09a, Theorem 2.1]), it is unique. Next, set $\varphi = \gamma^+ u - \Phi_0$, $\psi = T_a^+ u - \Psi_0$. Evidently, $\psi \in \tilde{H}^{-\frac{1}{2}}(\partial_D\Omega)$ and $\varphi \in \tilde{H}^{\frac{1}{2}}(\partial_N\Omega)$. Then immediately follow from the relation (3.197)-(3.198) that the triple (u,ψ,φ) satisfies the BDIE system M21.

ii. Let (u,ψ,φ) solves the BDIE (3.197)-(3.198) . Taking the co-normal derivative of the equation (3.197) on $\partial\Omega$ taking into account the jump properties (3.25), and subtruct equation (3.198) we obtain

$$u + \mathscr{Z}_b u + \mathscr{R}_b u - V_b \psi + W_b \varphi = F_0$$

which implies $$T_a^+ u + T_a^+ \mathscr{Z}_b u + T_a^+ \mathscr{R}_b u - \frac{a}{2b}\psi - \mathscr{W}'_{ab}\psi + \mathscr{L}_{ab}^+ \varphi = T_a^+ F_0$$

subtracting equation (3.198) from the above equation will give us

$$T_a^+ u - \psi = \Psi_0 \;\; \text{on} \;\; \partial\Omega \tag{3.200}$$

which proves the first equation of (3.199), Since $\psi = 0, \Psi_0 = \psi_0$ on $\partial_N\Omega$ we see that the Neumann condition (3.151) is satisfied.
Equation (3.197) and Lemma 3.2 with $\Psi = \psi + \Psi_0; \Phi = \varphi + \Phi_0$ imply that u is a solution to (3.149) and

$$V_b\left(\Psi_0 + \psi - T_a^+ u\right) - W_b\left(\Phi_0 + \varphi - \gamma^+ u\right) = 0 \;\; \text{in} \;\; \Omega \tag{3.201}$$

Due to (3.200) the first term in (3.201) vanishes, and by Lemma 3.8(ii), we obtain:

$$\Phi_0 + \varphi - \gamma^+ u = 0 \quad \text{on} \quad \partial\Omega$$

Which means that the second term in (3.199) holds as well. Taking into account $\varphi = 0, \Phi_0 = \varphi_0$ on $\partial_D\Omega$ equation (3.192) implies that u satisfies the Dirichlet Boundary Condition (3.150).

Unique solvability of the BDIE system M21 then follows from the already proved relations (3.199) and the unique solvability of BVP (3.149)-(3.151) stated in item (i).

The mapping properties of operators in (3.27), (3.29), (3.33), (3.35) and (3.37) and Theorem 3.31 imply the following statement.

Corollary 3.4.5 *The operator*

$$\mathscr{M}^{21} : H^{1,0}(\Omega,A) \times \tilde{H}^{-\frac{1}{2}}(\partial_D\Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N\Omega) \to H^{1,0}(\Omega,A) \times H^{-\frac{1}{2}}(\partial\Omega) \tag{3.202}$$

is continuous and injective.

Now we are in the position to analyse the invertibility of the operators $\mathscr{M}^{21}$.

Theorem 3.32. *The Operator (3.202) is continuously invertible.*

Proof. To prove the invertibility of operator (3.202), let us consider BDIE system M21 with an arbitrary right hand side $\mathscr{F}^{21} = \left(\mathscr{F}_1^{21}, \mathscr{F}_2^{21}\right)^T \in H^{1,0}(\Omega,A) \times H^{-\frac{1}{2}}(\partial\Omega)$.
Taking

$$F = \mathscr{F}_1^{21} \quad \text{in} \quad \Omega, \quad \Psi = T_a^+ \mathscr{F}_1^{21} - \mathscr{F}_2^{21} \quad \text{on} \quad \partial\Omega$$

following Corollary 3.4.1, we obtain that

$$\mathscr{F}_1^{21} = \mathscr{P}_b f_* + V_b \Psi_* - W_b \Phi_* \quad \text{in} \ \Omega$$
$$\mathscr{F}_2^{21} = T_a^+ \mathscr{F}_1^{21} - \Psi_* \quad \text{on} \ \partial\Omega$$

where the triple

$$(f_*, \Psi_*, \Phi_*) = \tilde{\mathscr{C}}_{\Psi_*} \mathscr{F}^{12} \in L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{3.203}$$

is unique and the operator

$$\tilde{\mathscr{C}}_{\Psi_*} : H^{1,0}(\Omega,A) \times H^{-\frac{1}{2}}(\partial\Omega) \to L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{3.204}$$

is linear and continuous.
Applying Theorem 3.31 with substitutions (3.185)i.e.,

$$f = f_*, \ \Psi_0 = \Psi_*, \ \Phi_0 = \Phi_*, \ \psi_0 = r_{\partial_N\Omega}\Psi_0, \ \varphi_0 = r_{\partial_D\Omega}\Phi_0$$

we obtain that the system M21 is uniquely solvable and its solution is given by (3.186) i.e.,

$$\mathscr{U}_1 = u = (A^{DN})^{-1}\left(f_*, r_{\partial_D\Omega}\Phi_*, r_{\partial_N\Omega}\Psi_*\right)^T, \quad \mathscr{U}_2 = \psi = T_a^+\mathscr{U}_1 - \Psi_*, \quad \mathscr{U}_3 = \varphi = \gamma^+\mathscr{U}_1 - \Phi_* \tag{3.205}$$

Representation (3.203), and continuity of operator (3.204) complete the proof for $\mathscr{M}^{21}$.

Boundary-Domain Integral Equation system M22

To reduce (3.53)-(3.55) to a BDIE system of almost the second kind (up to the spaces), we use equation (3.53) in Ω, the restriction of equation (3.55) to $\partial_D\Omega$, and the restriction of equation (3.54) to $\partial_N\Omega$. Putting $\gamma^+u := \Phi_0 + \varphi,\quad T_a^+u := \Psi_0 + \psi$ and first taking the co-normal derivative and then the trace of the above equation. Then we arrive at the following two-operator segregated system of BDIE system M22:

$$u + \mathscr{Z}_b u + \mathscr{R}_b u - V_b\psi + W_b\varphi = F_0 \ \text{ in } \ \Omega, \tag{3.206}$$

$$\left(1 - \frac{a}{2b}\right)\psi + T_a^+\mathscr{Z}_b u + T_a^+\mathscr{R}_b u - \mathscr{W}'_{ab}\psi + \mathscr{L}_{ab}^+\varphi = T_a^+F_0 - \Psi_0 \ \text{ on } \ \partial_D\Omega, \tag{3.207}$$

$$\frac{1}{2}\varphi + \gamma^+\mathscr{Z}_b u + \gamma^+\mathscr{R}_b u - \mathscr{V}_b\psi + \mathscr{W}_b\varphi = \gamma^+F_0 - \Phi_0 \ \text{ in } \ \partial_N\Omega. \tag{3.208}$$

System (3.206)-(3.208) can be rewritten in the form

$$\mathscr{M}^{22} := \begin{bmatrix} I + \mathscr{Z}_b + \mathscr{R}_b & -V_b & W_b \\ r_{\partial_D\Omega}T^+[\mathscr{Z}_b + \mathscr{R}_b] & \left(1 - \frac{a}{2b}\right)I - r_{\partial_D\Omega}\mathscr{W}'_{ab} & r_{\partial_D\Omega}\mathscr{L}_{ab}^+ \\ r_{\partial_N\Omega}\gamma^+[\mathscr{Z}_b + \mathscr{R}_b] & -r_{\partial_N\Omega}\mathscr{V}_b & \frac{1}{2}I + r_{\partial_N\Omega}\mathscr{W}_b \end{bmatrix}$$

$$\mathscr{G}^{22} := \begin{bmatrix} F_0 \\ r_{\partial_D\Omega}T_a^+F_0 - \Psi_0 \\ r_{\partial_N\Omega}\gamma^+F_0 - \Phi_0 \end{bmatrix}.$$

Due to the mapping properties of $V_b, \mathscr{V}_b, W_b, \mathscr{W}_b, \mathscr{P}_b, \mathscr{R}_b, \gamma^+\mathscr{R}_b, T_a^+\mathscr{R}_b, \mathscr{Z}_b, \gamma^+\mathscr{Z}_b$, and $\mathrm{T}_a^+\mathscr{Z}_b$ we have $\mathscr{G}^{22} \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial_D\Omega) \times H^{\frac{1}{2}}(\partial_N\Omega)$ and the operator

$$\mathscr{M}^{22} : H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial_D\Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N\Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial_D\Omega) \times H^{\frac{1}{2}}(\partial_N\Omega)$$

is continuous.

Remark 3.4. $\mathscr{G}^{22} = 0$ if and only if $(f, \Phi_0, \Psi_0) = 0$.
The latter equality clearly implies the former. Conversely, let $\mathscr{G}^{22} = 0$. Then by relation (3.175) $F_0 = 0$ in turn $\mathscr{P}_b f + V_b\Psi_0 - W_b\Phi_0 = 0$ in Ω which implies $\mathscr{P}_b f =$

$-V_b\Psi_0 + W_b\Phi_0$ in Ω. Due to Lemma 3.8, $f = 0$ and $V_b\Psi_0 - W_b\Phi_0 = 0$ in Ω.
The equality $r_{\partial_D\Omega}\{T_a^+ F_0 - \Psi_0\} = 0$ implies $\Psi_0 = 0$ on $\partial_D\Omega$ and $r_{\partial_N\Omega}\{\gamma^+ F_0 - \Phi_0\} = 0$ implies $\Phi_0 = 0$ on $\partial_N\Omega$. i.e., $\Phi_0 \in \tilde{H}^{\frac{1}{2}}(\partial_N\Omega), \Psi_0 \in \tilde{H}^{-\frac{1}{2}}(\partial_D\Omega)$.

Now let us prove the equivalence of BVP (3.149)-(3.151) with the BDIE systems M22.

Theorem 3.33. *Let $f \in L_2(\Omega)$ and $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ be some fixed extensions of $\varphi_0 \in H^{\frac{1}{2}}(\partial_D\Omega)$ and $\psi_0 \in H^{-\frac{1}{2}}(\partial_N\Omega)$ respectively.*

i. If some function $u \in H^1(\Omega)$ solves the mixed BVP(3.149)-(3.151) in Ω then the solution is unique and the triple $(u,\psi,\varphi)^T \in H^1(\Omega)\times\tilde{H}^{-\frac{1}{2}}(\partial_D\Omega)\times\tilde{H}^{\frac{1}{2}}(\partial_N\Omega)$ where

$$\psi = T_a^+ u - \Psi_0, \quad \varphi = \gamma^+ u - \Phi_0 \quad on\ \partial\Omega \tag{3.209}$$

Solves the BDIE system M22.

ii. Let diam$(\Omega) < r_0$, if a triple $(u,\psi,\varphi)^T \in H^1(\Omega)\times\tilde{H}^{-\frac{1}{2}}(\partial_D\Omega)\times\tilde{H}^{\frac{1}{2}}(\partial_N\Omega)$ solves BDIE system M22, then the solution is unique, the function u solves BVP(3.149)-(3.151), and relation (3.209) holds.

Proof. i. Let $u \in H^1(\Omega)$ be a solution of BVP (3.149)-(3.151). Then by Theorem 3.23 (see, e.g., [CMN09a, Theorem 2.1]), it is unique. Next, set $\varphi = \gamma^+ u - \Phi_0$, $\psi = T_a^+ u - \Psi_0$. Evidently, $\psi \in \tilde{H}^{-\frac{1}{2}}(\partial_D\Omega)$ and $\varphi \in \tilde{H}^{\frac{1}{2}}(\partial_N\Omega)$. Then immediately follow from the relation (3.206)-(3.208) that the triple (u,ψ,φ) satisfies the BDIE system M22.

ii. Let (u,ψ,φ) solves the BDIE (3.206)-(3.208). Taking the co-normal derivative of the equation (3.208) on $\partial_D\Omega$ taking into account the jump properties (3.25), and subtruct equation (3.207) we obtain

$$u + \mathscr{Z}_b u + \mathscr{R}_b u - V_b\psi + W_b\varphi = F_0$$

which implies $$T_a^+ u + T_a^+\mathscr{Z}_b u + T_a^+\mathscr{R}_b u - \frac{a}{2b}\psi - \mathscr{W}'_{ab}\psi + \mathscr{L}_{ab}^+\varphi = T_a^+ F_0$$

subtracting equation (3.207) from the above equation will give us

$$T_a^+ u - \psi = \Psi_0 \quad \text{on } \partial_D\Omega \Rightarrow \psi = T_a^+ u - \Psi_0 \quad \text{on } \partial_D\Omega.$$

Further take the trace of equation (3.206) on $\partial_N\Omega$ and subtract it from (3.208) we get

$$\varphi = \gamma^+ u - \Phi_0 \quad \text{on } \partial_N\Omega.$$

These equations imply that the first equation of (3.209) is satisfied on $\partial_D\Omega$ and the second equation of (3.209) is satisfied on $\partial_N\Omega$
Equation (3.206) and Lemma 3.2 with $\Psi = \psi + \Psi_0; \Phi = \varphi + \Phi_0$ imply that u is a solution to (3.149) and

$$V_b\Psi^* - W_b\Phi^* = 0 \quad \text{in} \quad \Omega$$

where $\Psi^* = \Psi_0 + \psi - T_a^+ u$ and $\Phi^* = \Phi_0 + \varphi - \gamma^+ u$.
Due to (3.209) and (3.206) we have $\Psi^* \in \tilde{H}^{-\frac{1}{2}}(\partial_N\Omega), \;\; \Phi^* \in \tilde{H}^{\frac{1}{2}}(\partial_D\Omega)$ and by Lemma 3.8 (iii) with

$$S_1 = \partial_N\Omega; S_2 = \partial_D\Omega \;\; \text{implies that} \;\; \Psi^* = \Phi^* = 0$$

which completes the proof of condition (3.209) on the whole boundary $\partial\Omega$ Taking into account that $\varphi = 0$ on $\partial_D\Omega$ and $\Phi_0 = \varphi_0$ on $\partial_D\Omega$ and $\psi = 0$ on $\partial_N\Omega$ and $\Psi_0 = \psi_0$ on $\partial_N\Omega$, equation (3.209) imply the Boundary Conditions (3.150-)(3.151).

Unique solvability of the BDIE system M22 then follows from the already proved relations (3.209) and the unique solvability of BVP (3.149)-(3.151) stated in item (i).

The mapping properties of operators in (3.27), (3.29), (3.33), (3.35) and (3.37) and Theorem 3.33 imply the following statement.

Corollary 3.4.6 *The operator*

$$\mathscr{M}^{22} : H^{1,0}(\Omega,A) \times \tilde{H}^{-\frac{1}{2}}(\partial_D\Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N\Omega) \to H^{1,0}(\Omega,A) \times H^{-\frac{1}{2}}(\partial_D\Omega) \times H^{\frac{1}{2}}(\partial_N\Omega) \tag{3.210}$$

is continuous and injective.

Now we are in the position to analyse the invertibility of the operators $\mathscr{M}^{22}$.

Theorem 3.34. *The Operator (3.210) is continuously invertible.*

Proof. To prove the invertibility of operator (3.210), we apply similar argument as (3.182). Let us consider BDIE system M22 with an arbitrary right hand side $\mathscr{F}^{22} = \left(\mathscr{F}_1^{22}, \mathscr{F}_2^{22}, \mathscr{F}_3^{22}\right)^T \in H^{1,0}(\Omega,A) \times H^{-\frac{1}{2}}(\partial_D\Omega) \times H^{\frac{1}{2}}(\partial_N\Omega)$ Taking $S_1 = \partial_D\Omega, S_2 = \partial_N\Omega$ and

$$F = \mathscr{F}_1^{22}, \;\; \Psi = r_{\partial_D\Omega} T_a^+ \mathscr{F}_1^{22} - \mathscr{F}_2^{22}, \quad \Phi = r_{\partial_N\Omega}\gamma^+ \mathscr{F}_1^{22} - \mathscr{F}_3^{22}$$

in [CMN09a, Lemma 5.13], , presented as Lemma 3.9, we obtain that $\mathscr{F}^{22}$ can be represented as

$$\begin{aligned} \mathscr{F}_1^{22} &= \mathscr{P}_b f_* + V_b\Psi_* - W_b\Phi_* && \text{in} \quad \Omega \\ \mathscr{F}_2^{22} &= r_{\partial_D\Omega}[T_a^+\mathscr{F}_1^{22} - \Psi_*] && \text{on} \quad \partial_D\Omega \\ \mathscr{F}_3^{22} &= r_{\partial_N\Omega}[\gamma^+\mathscr{F}_1^{22} - \Phi_*] && \text{on} \quad \partial_N\Omega \end{aligned}$$

where the triple

$$(f_*, \Psi_*, \Phi_*) = \mathscr{C}_{\partial_D\Omega,\partial_N\Omega}\mathscr{F}^{22} \in L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

is unique and the operator

$$\mathscr{C}_{\partial_N\Omega,\partial_D\Omega} : H^{1,0}(\Omega,A) \times H^{-\frac{1}{2}}(\partial_D\Omega) \times H^{\frac{1}{2}}(\partial_N\Omega) \to L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

is linear and continuous.
Applying Theorem 3.33 with the same substisution as (3.185) we obtain that the system M22 is uniquely solvable and its solution is given by (3.186) i.e.,

$$\mathscr{U}_1 = u = (A^{DN})^{-1}\left(f_*, r_{\partial_D\Omega}\Phi_*, r_{\partial_N\Omega}\Psi_*\right)^T, \quad \mathscr{U}_2 = \psi = T_a^+\mathscr{U}_1 - \Psi_*, \quad \mathscr{U}_3 = \varphi = \gamma^+\mathscr{U}_1 - \Phi_*$$

Representation (3.4.1), and continuity of operator (3.205) complete the proof for $\mathscr{M}^{22}$.

Conclusion and future work

In this dissertation, we have considered a second-order elliptic partial differential equation with a variable coefficient in a 2D bounded domain, in appropriate Sobolev space. The right-hand side functions were from $L_2(\Omega)$, and equations of the type Lamé system of anisotropic elasticity is considered, where the fundamental solution of the "frozen" coefficient PDE is not known explicitly. We have used a two- operator approach, formulated in [AM10] and employed a parametrix of another (second) auxiliary PDE, not related with the PDE in question, for reducing the BVP to BDIE systems.

The properties of a parametrix-based potential operator that contain logarithmic singularity were investigated. Unlike properties in 3D case in (see e.g., [CMN09a]). The single layer potential needs special consideration to be invertible, which is critical on this study. We separately investigated BVPs with Dirichlet, Neumann and mixed boundary conditions in the bounded domain. In the formulations considered, we obtain on the third Green formula an operator that is not compact, which lead in turn that the BDIE systems are not compact. Hence, we used the representation formula on the right-hand side of the formulated BDIEs to analyze the invertibility of the BDIE systems.

For the Dirichlet BVP, two segregated two-operator BDIE systems were formulated and analyzed, their equivalence to the original Dirichlet BVP was proved in the case of PDEs right-hand side functions from $L_2(\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$ (see e.g., [CMN09a], [DM15]).

On Neumann BVP, two segregated two-operator BDIE systems were formulated and investigated. Their equivalence to the original Dirichlet BVP was proved in the case of PDEs right-hand side functions from $L_2(\Omega)$ and $H^{-\frac{1}{2}}(\partial\Omega)$ (see e.g., [CMN09a], [ADM17]). The two-operator BDIEs solvability, uniqueness/ non-uniqueness of the solution as well as the Fredholm property and invertibility of the boundary domain integral operator are analyzed. Moreover, the two-operator boundary domain integral operators for Neumann BVP are not invertible, and hence appropriate finite-dimensional perturbations are constructed leading to the invertibility of the perturbed operator.

For the mixed BVP, we formulated four segregated BDIE systems and analyzed them. Their equivalence to the original Mixed BVP was proved. It was shown that these four BDIEs are continuously invertible.

Having said this, on our work on the thesis, we recommend the following problems to be investigated as a future research direction:

- In this thesis, we considered only smooth domain in a plane, need to investigate less regular domains such as Lipschitz domains.
- We can consider the two dimensional version of the two-operator BDIEs for the BVP in the exterior domains , United BDIEs as well as Localized BDIEs, which were investigated for 3D in [CMN09a], [CMN09b], [Mik06], [CMN13].

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