COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES
DEPARTMENT OF MATHEMATICS

GRADUATE PROJECT REPORT

ON

DIAGONALLY SINGULARIZABLE MATRICES

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILLMENT OF REQUIREMENTSFOR THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS

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The undersigned hereby certify that they have read and recommend to the department of mathematics for acceptance of this project entitled "Diagonally Singularizable Matrices" by Teumezghi Giovanni in partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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Addis Ababa, Ethiopia

Dedication

This project is dedicated to in loving memory of my beloved Father Ato Giovanni Mehari and Brother AtoTsegabirhan Giovanni!

Acknowledgements

First and foremost, I want to thank the Almighty God for bestowing me health, strength, patience and protection throughout my life.

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List of symbols (notations)

Q(A)	Spectral radius of A
R^{nxn}	the set of all nxn matrices whose entries are real numbers
$\lambda_{min}(A)$	Minimum eigenvalue of a symmetric matrix A
$\|\|C\|\|_F$	Frobenius norm of a matrix A = (a_{ij})
[A-D, A+D]	An interval matrix
$\{-1, 1\}^n$	The set of all ±1 vectors in R^n
T_t	Diagonal matrix with diagonal vector $t \in R^n$
SPD	Symmetric positive definite
min{A,B}	Component wise minimum of matrices (vector)
max{A,B}	Component wise maximum of matrices (vectors)

Abstract

A square matrix A is called diagonally singularizable if $|A - S| \leq I$ *holds for some singular matrix S (I is the identity matrix). The project brings several necessary and/or sufficient conditions for diagonal singularizability and demonstrates another specific feature, namely existence of diagonal singularizability-preserving operations and a theorem of symmetric alternative.*

INTRODUCTION

The main objective of this project is to define diagonal singularizability of a square matrix A and to extend different theorems on this topic.

The project is divided into two Chapters. In chapter one, we collect some important definitions and results which are already known and which will be used in the project. We also stated illustrative examples.

Chaptertwo contains four sections. Section one deals with six equivalent statements (conditions) for diagonal singularizability of a square matrix A andtheir proofs. In section two, sufficient conditions for diagonal singulazibility and its negation is stated and proven.In section three, we show that three matrix operations preserve diagonal singularizability if the square matrix is diagonally singularizable. In the last sectionsymmetric alternative is discussed.

CHAPTER ONE

Preliminaries

In this chapter**,** we collect some important definitions and results which are already known and which will be used throughout the project.

1.1 Preliminary Definitions

Definition 1.1: If A= [a_{ij}] is an *mxn* matrix, then the *nxm* matrix A^T = [b_{ij}],

where $b_{ij} = a_{ji} (1 \leq i \leq m, 1 \leq j \leq n)$is called the transpose of A.

Theorem 1.1: Let A and B be matrices of the same size and α be a scalar. Then

1. $(A + B)^T = A^T + B^T$
2. $(AB)^T = B^T A^T$
3. $(\alpha A)^T = \alpha A^T$
4. $(A^T)^T = A$

Definition 1.2: A square matrix A with real entries is symmetric if $A^T = A$

Definition 1.3: Let A be an nxn matrix. If $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$are all the eigenvalues of A, then the spectral radius of A is max $\{|\lambda_i|$;i = 1, 2, … ,n$\}$ (Real, complex)

Theorem 1.2: Let M be any real symmetric n x n matrix. Then:

1. M has n real eigenvalues λ_1, λ_2 , …, λ_n (not necessarily distinct) .
2. M has an orthonormal basis of eigenvectors for $\Re^n$.

Definition 1.4: An n x n matrix A is called nonsingular or invertible if there exists an n x n matrix B such that AB =I_n= BA.

Any matrix B with the above property is called an inverse of A. If A does not have an inverse, A is called singular.

If a square matrix has an inverse, then its inverse is unique. The unique inverse of A is denoted by A^{-1}.

If A is an invertible n x n matrix, then for every b $\in \Re^n$the equation Ax = b has exactly one solution x = A^{-1}b

Remark: A matrix A is invertible if and only if det(A) $\neq$ 0.

Definition 1.5: Let A= (a_{ij}) and B= (b_{ij}) be mxn matrices. Then

i) The maximum of A and B denoted by max{A,B} is an mxn matrix defined by max{A,B} is = (max {a_{ij}, b_{ij}} for each i, j)

ii) The minimum of A and B denoted by Min{A,B} is an mxn matrix defined by min{A,B}= (min {a_{ij}, b_{ij}}) for each i,j.

Example: Let A=$\begin{pmatrix} -3 & 2 & 7 & 5 \\ 9 & -4 & 6 & 1 \\ -8 & 10 & 13 & 0 \end{pmatrix}$ and B=$\begin{pmatrix} 5 & 1 & 6 & 11 \\ 8 & -2 & -14 & 16 \\ 1 & 3 & -7 & -9 \end{pmatrix}$, then

Max{A, B} =$\begin{pmatrix} 5 & 2 & 7 & 11 \\ 9 & -2 & 6 & 16 \\ 1 & 10 & 13 & 0 \end{pmatrix}$ and min{A, B} =$\begin{pmatrix} -3 & 1 & 6 & 5 \\ 8 & -4 & -14 & 1 \\ -8 & 3 & -7 & -9 \end{pmatrix}$

Definition 1.6: (Absolute value of a vector)

Let x =$\begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix}$ $\in \mathfrak{R}^n$.We define the absolute value of x, denoted by |x|=$\begin{pmatrix} |x_1| \\ |x_2| \\ \vdots \\ |x_n| \end{pmatrix}$

Example: Let x =$\begin{pmatrix} -6 \\ 10 \\ 0 \\ 8 \\ -20 \\ 11 \end{pmatrix}$. Then |x|=$\begin{pmatrix} |-6| \\ |10| \\ |0| \\ |8| \\ |-20| \\ |11| \end{pmatrix}$=$\begin{pmatrix} 6 \\ 10 \\ 0 \\ 8 \\ 20 \\ 11 \end{pmatrix}$

Definition 1.7: Let A= (a_{ij}) be an $m \times n$ matrix. The absolute value of A written as |A|, is defined as |A| = ($|a_{ij}|$)

Example: Let A =$\begin{pmatrix} -2 & 1 & 5 & -6 \\ 9 & 10 & 4 & -11 \\ 12 & -14 & 7 & -8 \\ -20 & 15 & 0 & 3 \end{pmatrix}$.

Then |A|=$\begin{pmatrix} |-2| & |1| & |5| & |-6| \\ |9| & |10| & |4| & |-11| \\ |12| & |-14| & |7| & |-8| \\ |-20| & |15| & |0| & |3| \end{pmatrix} = \begin{pmatrix} 2 & 1 & 5 & 6 \\ 9 & 10 & 4 & 11 \\ 12 & 14 & 7 & 8 \\ 20 & 15 & 0 & 3 \end{pmatrix}$

Definition 1.8: Let x=$\begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix} \in \mathfrak{R}^n$.We define its sign vector sgnx by

sgnx = $\begin{cases} 1\ if x_i \geq 0 \\ -1\ if x_i < 0 \end{cases}$

Example: Let x =$\begin{pmatrix} 2 \\ -3 \\ 5 \\ 0 \\ -4 \end{pmatrix}$. Then Sgnx=$\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$

Theorem 1.3: Let A, B be square matrices, then |AB|≤|A||B|.

Definition 1.10: For n∈ ℕ, we define$\{-1,1\}^n = \{(x_1, x_2 \dots x_n): x_i \in \{-1,1\}\ for\ each\ i\}$

Example:$\{-1,1\}^3$={(1,1,1),(1,1,-1),(1,-1,-1),(1,-1,1),(-1,-1,-1),(-1,-1,1),(-1,1,-1),(-1,1,1)}

Remark: $\{-1,1\}^n$ has 2^n elements.

Definition 1.11: Let A= (a_{ij}) be an nxn matrix. Then the Frobenius norm of A is defined as $||A||_F = \sqrt{\sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2}$

Example: Let A=$\begin{pmatrix} -1 & 2 & 3 \\ 4 & 0 & 1 \\ -2 & -4 & 5 \end{pmatrix}$. Then

$$||A||_F = \sqrt{\sum_{i=1}^{3} \sum_{j=1}^{3} |a_{ij}|^2}$$

$$=\sqrt{\sum_{i=1}^{3}(|a_{i1}|^2 + |a_{i2}|^2 + |a_{i3}|^2)}$$

$$=\sqrt{|-1|^2 + |2|^2 + |3|^2 + |4|^2 + |0|^2 + |1|^2 + |-2|^2 + |-4|^2 + |5|^2}$$

$$=\sqrt{76}= 8.7$$

Definition 1.12: Let y $\in \mathfrak{R}^n$**.** T_y (the diagonal matrix with diagonal vector y$\in \mathfrak{R}^n$ is defined as

$$T_y = \text{diag}\ (y_1, y_2, y_3, \dots, y_n) = \begin{pmatrix} y_1 & 0 & 0 & 0 \dots 0 \\ 0 & y_2 & 0 & 0 \dots 0 \\ 0 & 0 & y_3 & 0 \dots 0 \\ & & \vdots & \\ 0 & 0 & 0 & \dots y_n \end{pmatrix}$$

Example: Let y=$\begin{pmatrix} 1 \\ -3 \\ 9 \\ 2 \end{pmatrix} \in \mathbb{R}^4$. Then

T_y = diag(1,-3,9,2) = $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

Lemma 1.1: If S is a singular matrix, then S^T is also singular.

Proof: The statement can be proved by using contrapositive.

Suppose S^T is not singular. Then there exists a matrix B such that $S^TB = BS^T = I$. Transposing this gives $B^TS= SB^T = I$. This shows that S is nonsingular.

Hence S^T is singular

Theorem 1.3: For a square matrix A of order n, the following statements are equivalent:

1. A is invertible.
2. $Ax = 0$ has only the trivial solution.
3. $Ax = b$ has a unique solution x for every b.

Theorem 1.4: If A is an nxnmatrix, the homogeneous system Ax = 0 has a nontrivial solution if and only if A is not invertible.

Definition 1.13: Let A be a square nxn matrix and k is a non-negative integer, then the kth power of A is defined as:

$$A^k = \begin{cases} I_n & if\ k = 0 \\ A.A.A \ldots\ldots A\ (k\ times) & if\ k > 0 \end{cases}$$

As a consequence of this definition, for non-negative integer i, j and a square matrix A we have:

i. $A^{i+j}=A^i.A^j$
ii. $A^{ij}=(A^i)^j$

Theorem 1.5: A matrix A is singular iff 0 is an eigenvalue of A.

Proof: Suppose A is singular then there exists x≠0 such that Ax = 0 = 0x. This shows that 0 is an eigenvalue of A

Conversely, suppose that 0 is an eigenvalue of A. Then there exists $x \neq 0$ such that

Ax= 0x = 0. It follows that A is singular

Definition 1.14: A symmetric matrix A is

i) Positive definite if and only if all of its eigenvalues are greater than 0.

ii) Positive semi-definite if and only if all its eigenvalues are greater than or equal to 0.

Theorem: The eigenvalues of a real symmetric matrix are real.

Theorem A symmetric matrix has an orthonormal basis of eigenvectors for$\Re^n$.

Theorem 1.6 (Diagonalization of a matrix) If an nxn matrix a has basis of eigenvectors, then $D = X^{-1}AX$ is diagonal, with the eigenvalues of A as entries on the main diagonal. Here X is the matrix with these eigenvectors as column vectors.

Theorem 1.6: (Diagonalization) Let M be a real symmetric $n \times n$ matrix with eigen values $\lambda_1, \lambda_2, \lambda_3, \dots \lambda_n$andcorresponding orthonormal eigenvectors u_1 , u_2, …., u_n. Then

i. $D=X^{-1}MX^T \Rightarrow M= XDX^T$

is diagonal, with the eigenvalues of M as the entries on the main diagonal. Here X is the matrix with these eigenvectors as column vectors.

ii. $M=\sum_{i=1}^{d} \lambda_i u_i u_i^T$

Definition1.15.Given a list of vectors $u_1,u_2,\dots,u_k \in \mathbb{R}^n$.Then the matrix $\begin{pmatrix} u_1.u_1 & \cdots & u_1.u_k \\ \vdots & \dots & \vdots \\ u_k.u_1 & \cdots & u_k u_k \end{pmatrix}$ is called the Gram matrix.

Theorem 1.7: For a symmetric nxn matrix M, the following are equivalent:

i.$v^TMv \geq 0$ for all v∈V.

ii. All the eigenvalues are non negative.

iii. There exists a matrix B such that $B^TB = M$.

iv. Gram matrix of vectors $u_1,u_2,\dots,u_n \in U$ where U is some vector space,

Hence $\forall_{i,j}$, $M_{i,j}=u_i^T u_j$

Proof: (i$\Rightarrow ii$) Let λ be an eigen value of M. Then there exists eigenvector v$\in V$ such that Mv=λv.

Hence $0 \leq v^TMv = v^T(\lambda v) = \lambda(v^Tv)$ Since v^Tv is non negativefor all v, λ is non negative.

(ii) ⇒ (iii) Since the matrix M is symmetric, it has a spectral decomposition

$$M = \sum_i \lambda_i x_i x_i^T$$

Define if $y_i = \sqrt{\lambda_i} x_i$. This definition is possible because$\lambda_i{}' s$ are non-negative. Then,

$$M = \sum_i y_i y_i^T$$

Define B to be the matrix whose columns are y_i. Then it is clear that $B^TB=M$ from this construction B's columns are orthogonal.
In general, any matrix of the form B^TB is positive semi definite. The matrix B need not have orthogonal columns (it can even be rectangular). But this representation is not unique and there always exists a matrix B with orthogonal columns for M, such that $B^TB= M$.
This decomposition is unique if B is positive semi definite. The positive semi definite B, such that $B^TB=M$, is called the square root of M.
(iii) ⇒ (iv) Suppose there exists a matrix B such that $B^TB = M$.

Let the rows of B be $u_1,u_2,\dots,u_n$. Then from the definition of matrix multiplication,

$\forall_{ij}$: $M_{i,j} = u_i^T u_j$

(iv) ⇒ (i) Suppose M is the Gram matrix of vectors $u_1, u_2,\dots,u_n$. Let $x \in V$.Then,

$x^T M x = \sum_{i,j} M_{i,j} x_i x_j = \sum x_i x_j \, (u_i^T u_j)$,

Where x_i is the i^th^ element of vector x.

Define y =$\sum x_i u_i$, then,$0 \le y^T y = \sum_{i,j} x_i x_j \left(u_i^T u_j\right) = x^T M x$

Hence $x^T M x \geq 0$ for all x.

CHAPTER TWO

Diagonally Singularizable Matrices

Definition 2.1: Let A = (a_{ij}) and B= (b_{ij}) be matrices of the same size. We say A is less than or equal to B written as A ≤B if, $\boldsymbol{a_{ij}} \leq \boldsymbol{b_{ij}}$ for each i, j

Example: Let A=$\begin{pmatrix} 1 & 2 & 7 & 4 \\ 5 & 9 & -3 & 5 \\ 6 & -8 & 0 & 10 \end{pmatrix}$and B=$\begin{pmatrix} 3 & 5 & 7 & 6 \\ 5 & 10 & -2 & 8 \\ 7 & -6 & 0 & 12 \end{pmatrix}$,then

A≤B as $a_{ij} \leq b_{ij}$ for each i, j.

Definition 2.2: An interval matrix is a set of matrices of the form

$$[A - D, A + D] = \{C: |A-C| \leq D\} \text{ with } D \geq 0.$$

Example: Let $A = \begin{pmatrix} 2 & 4 & 1 \\ 3 & 0 & 2 \\ 4 & 1 & 3 \end{pmatrix}$, and $D = \begin{pmatrix} 10 & 12 & 16 \\ 11 & 9 & 14 \\ 6 & 15 & 10 \end{pmatrix}$. Then

$$[A - D, A + D] = \left[\begin{pmatrix} 2 & 4 & 1 \\ 3 & 0 & 2 \\ 4 & 1 & 3 \end{pmatrix} - \begin{pmatrix} 10 & 12 & 16 \\ 11 & 9 & 14 \\ 6 & 15 & 10 \end{pmatrix}, \begin{pmatrix} 2 & 4 & 1 \\ 3 & 0 & 2 \\ 4 & 1 & 3 \end{pmatrix} + \begin{pmatrix} 10 & 12 & 16 \\ 11 & 9 & 14 \\ 6 & 15 & 10 \end{pmatrix}\right]$$

$$= \left[\begin{pmatrix} -8 & -8 & -15 \\ -8 & -9 & -12 \\ -2 & -14 & -7 \end{pmatrix}, \begin{pmatrix} 12 & 16 & 17 \\ 14 & 9 & 16 \\ 10 & 16 & 13 \end{pmatrix}\right]$$

$$C = \begin{pmatrix} 9 & 4 & 14 \\ 5 & 7 & 10 \\ 6 & 11 & 8 \end{pmatrix} \in \left[\begin{pmatrix} -8 & -8 & -15 \\ -8 & -9 & -12 \\ -2 & -14 & -7 \end{pmatrix}, \begin{pmatrix} 12 & 16 & 17 \\ 14 & 9 & 16 \\ 10 & 16 & 13 \end{pmatrix}\right]$$

Definition 2.3: A square interval matrix $\mathbf{A}$= [A-D, A+D] = {C | |A-C|≤D} where D≥ 0 is called

i) singular if it contains a singular matrix, and

ii) regular if every element of the interval matrix is non singular.

Theorem 2.1: Let A,D∈ $\mathbb{R}^{n \times n}, D \geq 0.$ The interval matrix $\mathbf{A}$**=[**A+D,A-D**]**is regular if $\varrho(G_R) < 1$ hold, for some matrixR, where G_R = |I – RA| + |R|D (I is the identity matrix)

Definition 2.4: A square matrix A is called diagonally singularizable if $|A - S| \leq I$ holds for some singular matrix S, where I is the identify matrix…………………………(1)

Example: Let A be any singular matrix. Take S = A. Then |A - S| = |A - A| = |0| = 0$\leq I$.

Hence any singular matrix is diagonally singularizable.

This means that A can be brought to singularity by shifting (only!) its diagonal entries by an amount of at most 1 each.

Example: Let A = I. Take S = 0. Hence S is singular. Moreover, |A-S| = |I - 0|=|I|$\leq I$.

Therefore, I isdiagonallysingularizable.

2.1. Theorems on Diagonally Singularizable Matrices

Lemma (Beek's Condition) 2.1: If A is nonsingular and $\varrho(|A^{-1}||B|) < 1$ holds, then

$\mathbf{A}$= [A - |B|, A + |B|] is regular

Theorem 2.2: Let A and B be nxn matrices. If the interval matrix [A-|B|, A+|B|] is regular then for each right hand side b the equation Ax + B|x| = b has a unique solution

Proof: Since due to Beek's condition implies regularity of [A- |B|, A + |B|], and **according to theorem 2.2 we are done**(Rex Georg and Rhon Jiri, n.d).

Theorem 2.3: An interval matrix $\mathbf{A}$ = [A- D, A+D]for D≥ 0 is singular if and only if the inequality

|Ax|≤D|x| has a nontrivial solution ………………… (2.1)

Proof: Suppose that the interval matrix $\mathbf{A}$ = [A-D, A+D] is singular. If $\mathbf{A}$ contains a singular matrix S, then Sx = 0 for some $x \neq 0$, which implies

|Ax| = |(A-S)x|≤ |A-S||x| ≤D|x| (Rex Georg and Rhon Jiri, n.d)

Conversely, suppose $|Ax| \leq D|x|$holds for some x$\neq 0$.

Define $y\epsilon\mathbb{R}^n$ and $z\epsilon y_n$ by

$y_i = \begin{cases} (Ax)_i/(D/x)_i if\ (D|x|)_i > 0 \\ 1 \qquad\qquad\quad if\ (D|x|)_i = 0 \end{cases}$ (i = 1, … ,n) ……………………
(2.2)

And z = sign$x$

Then $T_z x = |x|$, hence

$((A-T_yDT_z)x)_i = (Ax)_i - y_i\,(D|x|)_i=0$

For each i so that $A-T_yDT_z$ is singular, and since $|y_i| \leq 1$ for each i due to (2.1) it follows that

$|(A-T_yDT_z)-A| = |T_yDT_z| \leq D$, hence $A - T_yDT_z \epsilon \mathbf{A}$ and $\mathbf{A}$ is singular.

Theorem 2.4: If $\mathfrak{A}_1=[A-D,\ A+D]$ is regular, then the interval matrix

$$\mathfrak{A}_2 =[DA^{-1}- I,\ DA^{-1}+I] \text{ is singular.}$$

Proof: Suppose that $\mathfrak{A}_1= [A-D,\ A+D]$ is regular. To prove the singularity of $\mathfrak{A}_2$, we use the fact that by theorem 2.2 regularity of

$[A-D,\ A+D]=[A-|-D,\ [A+|-D|]$ implies existence of a unique solution x^* of the absolute value equation

$Ax-D|x|=e$, Where **e** denotes the vector of whose all entries are all ones.

Then $Ax^*=D|x^*|+e >D|x^*|$, so that $x=Ax^*$ satisfies

$|DA^{-1}x| \leq D|A^{-1}x|<x =I|x|$ and theorem 2.3 proves singularity of $\mathfrak{A}_2$.

Theorem 2.5: For each non singular square matrix A, either A or A^{-1} is diagonallysingularizable.

Proof: Let A be a non singular square matrix. Consider the auxiliary interval matrix

$\mathfrak{B} = [A^{-1} - I, A^{-1} + I]$. If $\mathfrak{B}$ is singular, then there exists a singular matrix S such that

$|A^{-1} - S| \leq I$which means that A^{-1} is diagonally singularizable.

If $\mathfrak{B}$is regular, then by theorem 2.4 the interval matrix

$$[I(A^{-1})^{-1} - I, I(A^{-1})^{-1} + I] = [A - I, A + I] \text{ is singular.}$$

This implies that $[A - I, A + I]$ contains a singular matrix S, Hence A is diagonally Singularizable

2.2 Diagonally singularizing a square matrix A

Example: Let nxn $-$matrix A be given, and consider the diagonal matricesof the same size as A.

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \ldots & 0 \\ 0 & 1 & 0 & 0 & 0 \ldots & 0 \\ & & & \cdot & & \\ & & & \cdot & & \\ & & & \cdot & & \\ 0 & 0 & 0 & 0 & 0 & 0 \ldots 1 \end{pmatrix} \text{and} J = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \ldots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \ldots & 0 \\ & & & \cdot & & & \\ & & & \cdot & & & \\ & & & \cdot & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots 1 \end{pmatrix}$$

Then $\det(I) = 1$ and$\det(J) = -1$.

If the number $k > 1$ is sufficiently large, then the matrix $A + kI$ is almost equal to kI, and$det\ (A + \theta I)$ should be close to $det(kI) = k^n$.

Similarly, if the number $\theta > 1$ is sufficiently large, then the matrix $A + J$ is almost equal to θJ, and $det(A + \theta J)$ should be close to

$det(\theta J) = -\theta^n < 0$.

Therefore, the continuous real function

$g(t) = \det(A + tkI + (1 - t)\theta J)$

Changes its sign at the interval $t \in [0,1]$ in so far as

$g(0) = \det(A + \theta J)\ and g(1) = \det(A + kJ)$

By the intermediate value theorem, there is a number

t*∈ [0, 1] such that the matrix $A + t^*kI + (1 - t^*)\ \theta J$ has zero determinant, i. e., it is singular.

Moreover, it is made of the matrix A by the perturbation $t^*kI + (1 - t^*)\theta J$ affecting only diagonal elements of A.

So, any matrix can be made singular by a suitable perturbation of its diagonal, but, of course, the magnitude of this perturbation plays an important role in practice. In our approach, it makes sense to confine ourselves to a fixed finite level of the perturbation magnitude, and the unit is the most natural choice. Then, being interested in investigating the specific magnitude of the diagonal perturbation, we can scale it up to a perturbation of a unit value and do the same with the matrix under study.

The formula in equation (1) is used for formulating a nontrivial assertion (see Theorem 2.5 above): for each nonsingular matrix A, either A or A^{-1} is diagonally singularizable. It was just this remarkable property that prompted this author to investigate the concept of diagonal singularizability in more detail, thus giving rise to the present project which brings several necessary and/or sufficient conditions for diagonal singularizability and demonstrates another specific feature, namely existence of diagonal-singularizability-preserving operations and a theorem of symmetric alternative (Rohn and Shary, 2018).

Let A be an nxn matrix, $\varrho(A)$stands for the spectral radius of A and $\lambda_{\min}(A)$ denotes the minimum eigenvalue of a symmetric matrix A. Let us recall that by the Courant-Fischer theorem (Golub and van Loan, 1996).

$$\lambda_{\min}(A) = \min_{x \neq 0} \frac{x^T A x}{x^T x}$$

Continuity of the minimum eigenvalue follows from the Wielandt-Hoffman theorem (see (Golub and van Loan, 1996).

$$|\lambda_{\min}(A) - \lambda_{\min}(B)| \leq \|A - B\|_F$$

This holds for any two symmetric matrices $A, B \in R^{nxn}$ where we use the Frobenius matrix norm

$$\|C\|_F = \left(\sum_{ij} C_{ij}^2\right)^{1/2} .$$

The definition (1) corresponds to the diagonal perturbation of the unit magnitude. The point is that, with the help of a suitable perturbation of only diagonal elements, any matrix can be made singular, but the magnitude of the perturbation required may be arbitrarily large.

2.3. **Conditions for Diagonal Singularizability**

2.3.1.Necessary and Sufficient Conditions

First, we have several necessary and sufficient conditions for diagonal singularizability.

Theorem 2.6: For a matrix A$\in R^{nxn}$, the following assertions are equivalent.

i. A is diagonally singularizable,
ii. [A-I, A+ I] is singular,
iii. $|Ax| \leq |x|$ for some $x \neq 0$
iv. $Det(A)\, det(A - T_y) \leq 0$ for some y$\in \{-1,1\}^n$,
v. $A - \tau T_y$ is singular for some$\tau \in [0,1] and y \in \{-1,1\}^n$,
vi. $|Ax| = \tau |x|$ for some $\tau \in [0,1] and x \neq 0$.

Proof:$Let\ A \in R^{nxn}$. we shall prove that i ⇒ ii ⇒ iii ⇒ iv ⇒ v ⇒ vi ⇒ i

(i)⇒ (ii) Suppose that A is diagonally singularizable. Then | A – S|≤ I holds for some singular matrix S. Then S ∈ [A-I, A+ I]. Hence [A-I, A+ I] is singular.

(ii)⇒ (iii) Suppose [A-I, A+ I] is singular. Then there exists a singular matrix [A-I, A+ I] is singular.S such that $|A - S| \le I$ with $Sx = 0$ for some $x \neq 0$which implies that $|Ax| = |(A - S)x| \le |A - S||x| \le |x|$

$\boldsymbol{iii} \Rightarrow \boldsymbol{iv}$: $Let |Ax| \le |x|\ for\ some\ x \neq 0. Put t_i = \frac{(Ax)_i}{(x)_i} if x_i \neq 0\ and$

$t_i = 1\ otherwise (i = 1,2,3,4, \ldots, n), then t_i \in [-1,1] and (Ax)_i$

$= t_i x_i\ for\ each\ i\ which\ can\ be\ witten\ as\ (A - T_t)x = 0\ where$

$t = (t_i), implying,$ det(A-T_t) $= 0$ (1)

Now define a function f :$\mathbb{R}^n \to \mathbb{R}byf(s) = \det(A)\det(A - T_s)$, $s \in \mathbb{R}^n$ (2)

We shall construct by induction numbers $y_i \in \{-1,1\}$, i =1,2,3,…,n

Such *that* $f(y_1, \ldots y_i, t_i +_1 \ldots, t_n) \le 0$ (3)

will hold for *i=0…n. for i=0* this follows from (1).

Thus assume that numbers$y_1, \ldots, y_{i-1}$ satisfying

$f(y_1, \ldots, y_{i-1}, t_i, t_{i+1}, \ldots, t_n) \le 0$ *(4)*

have been already constructed for some *i, 1*$\le i \le n$. Define a function of one variable.

$\emptyset_i(\sigma) = f(y_1, \ldots, y_{i-1}, \sigma, t_{i+1}, \ldots, t_n)$

and construct y_i *as follows: Let* $y_i = -1$ *if* $\emptyset_i(-1) \le \emptyset_i(1)$ *(5)*

and set$y_i = 1$ otherwise. It follows from the Laplace expansion of the second determinant in (2) along the i^th^ row that $\emptyset_i(\sigma)$ is linear in σ. Thus,if (5) holds, then $\emptyset_i(\sigma)$is nondecreasing and because $t_i \in [-1,1]$ (see the definition of t_i above), we have

$\emptyset_i(y_i) = \emptyset_i(-1) \le \emptyset_i(t_i) \le 0.$

by induction assumption (4); if

$\emptyset_i(-1) > \emptyset_i(1)$

holds, then$\emptyset_i(\sigma)$ is decreasing and we have

$\emptyset_i(y_i) = \emptyset_i(1) \le \emptyset_i(t_i) \le 0.$

So that in both cases (3) holds which concludes the proof by induction. Now from (3) for i = n we obtain that f(y) ≤0 which was to be proved.

iv ⇒ v: Let $det(A) det(A - T_y) \le 0$ for some y$\in \{-1,1\}^n$.

Define a real function $g$by$g(t) = det(A - tTy), t \in [0,1]$.

Then g is continuous in$[0,1] and\ g(0)g(1) = det(A) det(A - Ty) \le 0$. Hence by the intermediate value theorem means that $g(\tau) = 0$ for some$\tau \epsilon$ $[0, 1]$. Thus $A - \tau T_y$ is singular.

v ⇒ vi:If $A - \tau$ is singular for some $\tau\epsilon$ [0, 1] and y$\in \{-1,1\}^n$,

then $(A - \tau T_y)x = 0$ for some x$\neq$ 0 which implies that $|Ax| = \tau|x|$.

vi ⇒ i: Let $|Ax| = \tau|x|$ for some$\tau\epsilon$ [0, 1] and x$\neq$ 0. Define y, $z \in \{-1,1\}^n$, by $y_i = 1$ if$(Ax)_i \geq 0$and$y_i = -1$ otherwise, and $z_i = 1$ if$x_i \geq 0$ and $z_i = -1$ *otherwise* $(i = 1,2, \dots, n)$, *then* $T_y Ax = \tau T_z x$ implying $(A - \tau T_y T_z)\, x = 0$

which shows that the matrix $S = A - \tau T_y T_z$ is singular and satisfies$|A - S| = \left|\tau T_y T_z\right| \leq I$, hence, A is diagonally singularizable.

2.3.2.Sufficient Conditions

Next we have two sufficient conditions for diagonal singularizabilityand its negation.

Theorem 2.7: If A is nonsingular and

$\max_j |A^{-1}|_{jj} \geq 1$holds, then A is diagonally singularizable.

<u>Proof.</u>Choose k for which$|A^{-1}|_{kk}$ ≥1. Then

$$|A^{-1}e_k| = |A^{-1}|e_k \geq e_k = |e_k| \qquad (6)$$

Wheree_k is the kthcolumn of the identity matrix I. Put x =$A^{-1}e_k$. Then

$x \neq 0$ and from (6) we obtain$|Ax| \leq |x|$

which by Theorem 1, (iii) means that A is diagonally singularizable.

Theorem 2.8: If A is nonsingular and

$$\varrho(|A^{-1}|) \leq 1 \qquad (7)$$

holds, then A is not diagonally singularizable

Proof

Let us recall that (7) implies $(I - |A^{-1}|)^{-1} \geq 0$ (Horn and Johnson). Assume to the contrary that A is diagonally singularizable, so that

$|Ax| \leq |x|$for some $x \neq 0$ (Theorem 1, (iii)). Put x'= Ax, then $x' \neq 0$ and

it satisfies$|x'| \leq |A^{-1}x'|$ which implies

$|x'| \leq |A^{-1}||x'|$,

hence$(I - |A^{-1}|)|x'| \leq 0$

Moreover, premultiplying this inequality by the nonnegative matrix $(I - |A^{-1}|)^{-1}$

yields$|x'| \leq 0$ hence$|x'| = 0$ which contradicts the previously mentioned fact

that$x' \neq 0$.

2.4.Operations Preserving Diagonal Singularizability

In this section we show that three well-known matrix operations preserve diagonal singularizability.

Theorem 2.9: If square matrix A is diagonally singularizable, then so are

A^T, $(A^TA)^{2^k}$and $(AA^T)^{2^k}$for k=0, 1, 2,.….

Proof.

(a) If A is diagonally singularizable, then from $|A-S| \leq I$ for some singular S it follows that $|A^T - S^T| \leq I$ where S^T is again singular, hence A^T is diagonally singularizable.

(b) We shall first prove diagonal singularizability of A^TA. By Theorem 1,

(iii) a diagonally singularizable A satisfies $|Ax| \leq |x|$for some $x \neq 0$. Then we have

$x^TA^TAx = (Ax)^T(Ax) \leq |Ax|^T|Ax| \leq |x|^T|x| = x^Tx$

hence$x^T(A^TA - I)x \leq 0$

Since AA^T-I is a symmetric matrix, then we have

$$\lambda_{min}(A^TA - I) = \min_{X' \neq 0} \frac{x'^{T((A^TA-I)X'}}{x'^Tx'} \leq \frac{x^T(A^TA-I)x}{x^Tx} \leq 0$$

Now define a real function h by

$h(t) = \lambda_{min}(A^TA - tI)$, t ϵ [0,1].

It is well defined because $A^TA - tI$ is symmetric for each $t\epsilon[0,1]$, and the above reasoning implies that $h(1) \leq 0$. Next, we have that $h(0) = \lambda_{min}(A^TA) \geq 0$

because A^TA is symmetric positive semi definite, and h is continuous in [0; 1] since for each $t_1, t_2\epsilon[0,1]$, we have by the Wielandt-Hoffman theorem that

$|h(t_1) - h(t_2)| \leq \left||(t_1 - t_2|\right|_F = n^{1/2}|t_1 - t_2|$.

Hence the intermediate value theorem implies existence of a $\tau\epsilon$[0,1]suchthat $h(\tau)$ = 0 which gives that $\lambda_{min}(A^TA - \tau I) = 0$. Thus $A^TA - \tau I$ is singular and Theorem 1, (v) (with y = e, the vector of all ones) proves A^TA to be diagonally singularizable.

(c) Next we prove by induction on k that $(A^TA)^{2^k}$ is diagonally singularizable for

$k = 0,1,2,...$ The case of k = 0 has been proved in part (b).

Thus assume that $(A^TA)^{2^k}$is diagonally singularizable for some $k \geq 0$.

Then, again by part (b),$((A^TA)^{2^k})^T(A^TA)^{2^k} = ((A^TA)^{2^k})^2 = (A^TA)^{2^{k+1}}$is diagonally singularizable which concludes the proof by induction.

(d) If we apply the previous result to A^T, we obtain that $(AA^T)^{2^k}$ is diagonally singularizable for $k = 0,1,2,...$

We shall first prove diagonal singularizability of AA^T. By Theorem 1,

(iii) a diagonally singularizable A^T satisfies $|A^Tx| \leq |x|$for some $x \neq 0$. Then we have

$x^T(A^T)^TA^Tx = (A^Tx)^T(A^Tx) \leq |A^Tx|^T|A^Tx| \leq |x|^T|x| = x^Tx$

$hence, \quad x^T(AA^T - I)x \leq 0$

Which in view of symmetry of AA^T-I implies that

$$\lambda_{min}(AA^T - I) = \min_{X' \neq 0} \frac{x'^{T((AA^T-I)X'}}{x'^Tx'} \leq \frac{x^T(AA^T - I)x}{x^Tx} \leq 0$$

Now define a real function h by

$h(t) = \lambda_{min}(AA^T - tI)$, t ϵ [0,1].

It is well defined because $AA^T - tI$ is symmetric for each $t\epsilon[0,1]$, and

the above reasoning implies that $h(1) \leq 0$. Next, we have that $h(0) = \lambda_{min}(AA^T) \geq 0$ because AA^T is symmetric positive semi definite, and h is continuous in [0; 1] since for each $t_1, t_2\epsilon[0,1]$, we have by the Wielandt-Hoffman theorem that

$|h(t_1) - h(t_2)| \leq \left||(t_1 - t_2|\right|_F = n^{1/2}|t_1 - t_2|$.

Hence the intermediate value theorem implies existence of a $\tau\epsilon[0,1]$suchthat $h(\tau)$ = 0 which gives that $\lambda_{min}(AA^T - \tau I) = 0$. Thus $AA^T - \tau I$ is

singular and Theorem 1, (v) (with y = e, the vector of all ones) proves AA^T

to be diagonally singularizable.

(e) Next we prove by induction on k that $(AA^T)^{2^k}$ is diagonally singularizable

for k=1, 2,....

The case of k = 0 has been proved previously in (d).

Thus assume that $(AA^T)^{2^k}$is diagonally singularizable for some$k \geq 0$.

Then, again by part (b), $((AA^T)^{2^k})^T(AA^T)^{2^k} = ((AA^T)^{2^k})^2 = (AA^T)^{2^{k+1}}$

is diagonally singularizable which concludes the proof by induction.

2.5. Symmetric Alternative

The situation with matrix inverse is different. The following theorem was proved in **[1].**

Theorem 2.10: For each nonsingular matrix A at least one of the matrices A,
A^{-1} is diagonally singularizable.
We derive two consequences of this result. The minimum/maximum of
Two matrices is understood entry wise.

Theorem 2.11: For each nonsingular matrix A the interval matrix
$[min\,\{A, A^{-1}\} - I, max\{A, A^{-1}\} + I\,]$issingular. (8)
Proof
By Theorem 2.4 at least one of the interval matrices $[A - I, A\, +\, I], [A^{-1} - I;\; A^{-1} +\, I]$
is singular, therefore (8), the minimal (w.r.t. inclusion)
interval matrix enclosing both these interval matrices, contains a singular
matrix. Finally, we have this “symmetric alternative”.
Theorem 2.12: For each square matrix A at least one of the inequalities

$i. |Ax| \leq |x|$

$ii. |x| \leq |Ax|$has a nontrivial solution.

Proof

If A is singular, then a nontrivial solution to Ax = 0 solves (i). If

A is nonsingular and (i) does not possess a nontrivial solution, then A is not diagonallysingularizable, hence the inverse A^{-1} is diagonally singularizableby Theorem 2-4.

So that $|A^{-1}x'| \leq |x'|$has a nontrivial solution x'then

$x\; =\; A^{-1}x'$is a nontrivial solution to (ii)

Summary

A square matrix A is called diagonally singularizable if $|A - S| \leq I$ holds for some singular matrix S ($I\ is\ the\ identity\ matrix$).This means that A can be brought to a singularity by shifting (only!) its diagonal entries by an amount of at most 1 each.

The above definition corresponds to the diagonal perturbation of the unit magnitude.

The point is that, with the help of a suitable perturbation of only diagonal elements, any matrix can be made singular, but the magnitude of the perturbation required maybe arbitrarily large.

The project brings several necessary and/or sufficient conditions for diagonal singularizability and demonstrates another specific feature, namely existence of diagonal singularizability preserving operations and a theorem of symmetric alternative.

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