



DYNAMICS OF DEGENERATE THREE-LEVEL LASER

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In Partial Fulfillment for the
Degree of Master of Science in Physics

By
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Abstract

We analyze the quantum properties of the light generated by a three-level laser coupled to a vacuum reservoir. The three-level atoms available in the cavity are pumped from the bottom to the top level at a rate r_{da} .

Employing the pertinent master equation, we obtained the equation of evolution for cavity mode and atomic operators. Moreover, we have calculated the mean and variance of the photon number, the quadrature variance and the quadrature squeezing as well as the power spectrum for the cavity mode.

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Chapter 1

Introduction

A three-level laser is a quantum optical system in which light is generated by three-level atoms inside a cavity usually coupled to a vacuum reservoir [1 - 4]. The statistical and squeezing properties of the light generated by three-level laser have been investigated by several authors [5 - 10]. These studies show that the degenerate three-level laser can generate squeezed light under certain conditions.

In this thesis, we consider N degenerate three-level atoms in a cavity coupled to a vacuum reservoir via a single port-mirror. We study the case in which the atoms as well as the cavity mode interact with the vacuum reservoir. We denote the top, intermediate, bottom and ground levels of a three-level atom in a cascade configuration by $|a_j\rangle$, $|b_j\rangle$, $|c_j\rangle$, $|d_j\rangle$ respectively. We also consider the case in which a three-level atom is pumped from the ground level $|d_j\rangle$ to the top level $|a_j\rangle$ at the rate of r_{da} . A degenerate three-level atom may make a transition from level $|a_j\rangle$ to level $|b_j\rangle$ and then from level $|b_j\rangle$ to level $|c_j\rangle$ by emitting two photons of the same frequency ω . Alternatively, the atom may decay from level $|a_j\rangle$, $|b_j\rangle$, or $|c_j\rangle$ spontaneously to the ground level $|d_j\rangle$ at the rate γ .

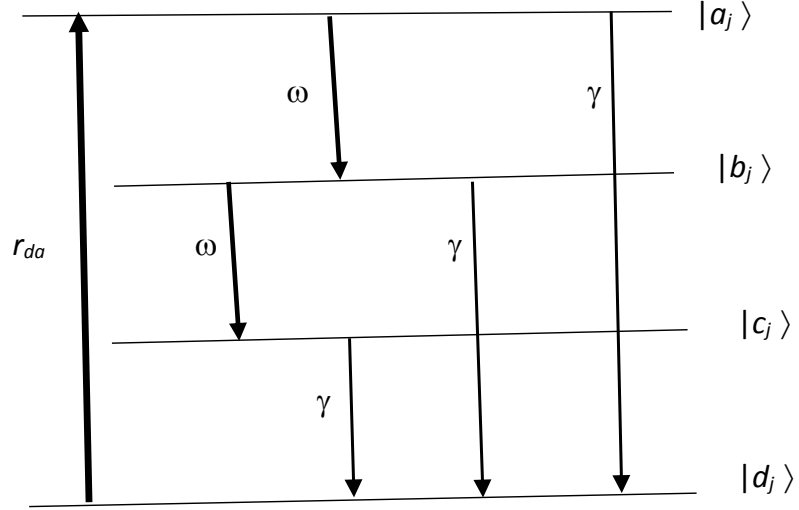


Figure 1.1: Degenerate three-level atom

In this thesis, we study the statistical and squeezing properties of the light produced by degenerate three-level laser in which degenerate three-level atoms inside a cavity coupled to a vacuum reservoir are pumped to the top level at a rate r_{da} . Employing the master equation for the system under consideration, we obtain the quantum Langevin equations for the cavity mode and atomic operators.

Applying the solutions of the quantum Langevin equations along with the correlation properties of noise operators, we calculate the mean and variance of the photon number as well as the power spectrum for the cavity mode. We also obtain the quadrature variance and the quadrature squeezing for the cavity mode.

Chapter 2

Dynamics of Cavity Mode and Atomic Operators

The Hamiltonian describing the interaction of a degenerate three-level atom with a cavity mode is describable at resonance by the

$$\hat{H} = ig [\hat{a}^\dagger(|b_j\rangle\langle a_j| + |c_j\rangle\langle b_j|) - (|a_j\rangle\langle b_j| + |b_j\rangle\langle c_j|)\hat{a}], \quad (2.0.1)$$

where g is the atom-cavity mode coupling constant, taken to be the same for both transitions, \hat{a} is the annihilation operators for the cavity mode.

The master equation describing the interaction of a degenerate three-level atom and cavity mode with the vacuum reservoir is expressible as

$$\begin{aligned} \frac{d\hat{\rho}}{dt} = & -i[\hat{H}, \hat{\rho}] + \frac{\kappa}{2} (2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) \\ & + \frac{\gamma}{2} [2|d_j\rangle\langle a_j|\hat{\rho}|a_j\rangle\langle d_j| - |a_j\rangle\langle a_j|\hat{\rho} - \hat{\rho}|a_j\rangle\langle a_j| + 2|d_j\rangle\langle b_j|\hat{\rho}|b_j\rangle\langle d_j| \\ & - |b_j\rangle\langle b_j|\hat{\rho} - \hat{\rho}|b_j\rangle\langle b_j| + 2|d_j\rangle\langle c_j|\hat{\rho}|c_j\rangle\langle d_j| - |c_j\rangle\langle c_j|\hat{\rho} - \hat{\rho}|c_j\rangle\langle c_j|], \end{aligned} \quad (2.0.2)$$

where κ is cavity mode damping constant. So that employing the Hamiltonian described by Eq. (2.0.1), one can write the above master equation as

$$\begin{aligned}
\frac{d\hat{\rho}}{dt} = & g [\hat{a}^\dagger |b_j\rangle \langle a_j| \hat{\rho} - \hat{\rho} \hat{a}^\dagger |b_j\rangle \langle a_j| + \hat{a}^\dagger |c_j\rangle \langle b_j| \hat{\rho} - \hat{\rho} \hat{a}^\dagger |c_j\rangle \langle b_j| - \hat{a} |a_j\rangle \langle b_j| \hat{\rho} \\
& + \hat{\rho} \hat{a} |a_j\rangle \langle b_j| - \hat{a} |b_j\rangle \langle c_j| \hat{\rho} + \hat{\rho} \hat{a} |b_j\rangle \langle c_j|] + \frac{\kappa}{2} [2\hat{a} \hat{\rho} \hat{a}^\dagger - \hat{a}^\dagger \hat{a} \hat{\rho} - \hat{\rho} \hat{a}^\dagger \hat{a}] \\
& + \frac{\gamma}{2} (2|d_j\rangle \langle a_j| \hat{\rho} |a_j\rangle \langle d_j| - |a_j\rangle \langle a_j| \hat{\rho} - \hat{\rho} |a_j\rangle \langle a_j| + 2|d_j\rangle \langle b_j| \hat{\rho} |b_j\rangle \langle d_j| \\
& - |b_j\rangle \langle b_j| \hat{\rho} - \hat{\rho} |b_j\rangle \langle b_j| + 2|d_j\rangle \langle c_j| \hat{\rho} |c_j\rangle \langle d_j| - |c_j\rangle \langle c_j| \hat{\rho} - \hat{\rho} |c_j\rangle \langle c_j|) \quad (2.0.3)
\end{aligned}$$

or

$$\begin{aligned}
\frac{d\hat{\rho}}{dt} = & g \left(\hat{a}^\dagger \hat{\sigma}_a^j \hat{\rho} - \hat{\rho} \hat{a}^\dagger \hat{\sigma}_a^j + \hat{a}^\dagger \hat{\sigma}_b^j \hat{\rho} - \hat{\rho} \hat{a}^\dagger \hat{\sigma}_b^j - \hat{a} \hat{\sigma}_a^{\dagger j} \hat{\rho} + \hat{\rho} \hat{a} \hat{\sigma}_a^{\dagger j} - \hat{a} \hat{\sigma}_b^{\dagger j} \hat{\rho} + \hat{\rho} \hat{a} \hat{\sigma}_b^{\dagger j} \right) \\
& + \frac{\kappa}{2} (2\hat{a} \hat{\rho} \hat{a}^\dagger - \hat{a}^\dagger \hat{a} \hat{\rho} - \hat{\rho} \hat{a}^\dagger \hat{a}) + \frac{\gamma}{2} (2|d_j\rangle \langle a_j| \hat{\rho} |a_j\rangle \langle d_j| - \hat{\eta}_a^j \hat{\rho} - \hat{\rho} \hat{\eta}_a^j \\
& + 2|d_j\rangle \langle b_j| \hat{\rho} |b_j\rangle \langle d_j| - \hat{\eta}_b^j \hat{\rho} - \hat{\rho} \hat{\eta}_b^j + 2|d_j\rangle \langle c_j| \hat{\rho} |c_j\rangle \langle d_j| - \hat{\eta}_c^j \hat{\rho} - \hat{\rho} \hat{\eta}_c^j), \quad (2.0.4)
\end{aligned}$$

where

$$\begin{aligned}
\hat{\sigma}_a^j = & |b_j\rangle \langle a_j|, \quad \hat{\sigma}_b^j = |c_j\rangle \langle b_j|, \quad \hat{\sigma}_c^j = |c_j\rangle \langle a_j|, \\
\hat{\eta}_a^j = & |a_j\rangle \langle a_j|, \quad \hat{\eta}_b^j = |b_j\rangle \langle b_j|, \quad \hat{\eta}_c^j = |c_j\rangle \langle c_j| \quad (2.0.5)
\end{aligned}$$

are atomic operators.

2.1 Time evolution for the expectation values of the cavity mode and atomic operators

Employing the master Eq. (2.0.4) along with the relation

$$\frac{d}{dt} \langle \hat{A} \rangle = Tr \left(\frac{d\hat{\rho}}{dt} \hat{A} \right), \quad (2.1.1)$$

it is possible to obtain the equation of evolution for the expectation values of the cavity mode and atomic operators.

We thus see that

$$\begin{aligned}
\frac{d}{dt}\langle\hat{a}(t)\rangle &= gTr(\hat{a}^\dagger\hat{\sigma}_a^j\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^j\hat{a} + \hat{a}^\dagger\hat{\sigma}_b^j\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{\sigma}_b^j\hat{a} - \hat{a}\hat{\sigma}_a^{\dagger j}\hat{\rho}\hat{a} \\
&\quad + \hat{\rho}\hat{a}\hat{\sigma}_a^{\dagger j}\hat{a} - \hat{a}\hat{\sigma}_b^{\dagger j}\hat{\rho}\hat{a} + \hat{\rho}\hat{a}\hat{\sigma}_b^{\dagger j}\hat{a}) + \frac{\kappa}{2}Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2) \\
&\quad + \frac{\gamma}{2}Tr(2|d_j\rangle\langle a_j|\hat{\rho}|a_j\rangle\langle d_j|\hat{a} - \hat{\eta}_a^j\hat{\rho}\hat{a} - \hat{\rho}\hat{\eta}_a^j\hat{a} + 2|d_j\rangle\langle b_j|\hat{\rho}|b_j\rangle\langle d_j|\hat{a} \\
&\quad - \hat{\eta}_b^j\hat{\rho}\hat{a} - \hat{\rho}\hat{\eta}_b^j\hat{a} + 2|d_j\rangle\langle c_j|\hat{\rho}|c_j\rangle\langle d_j|\hat{a} - \hat{\eta}_c^j\hat{\rho}\hat{a} - \hat{\rho}\hat{\eta}_c^j\hat{a}). \tag{2.1.2}
\end{aligned}$$

Applying the cyclic property of the trace operation together with the commutation relation

$$[\hat{a}, \hat{a}^\dagger] = 1, \tag{2.1.3}$$

we obtain

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = -\frac{\kappa}{2}\langle\hat{a}(t)\rangle + g(\langle\hat{\sigma}_a^j\rangle + \langle\hat{\sigma}_b^j\rangle). \tag{2.1.4}$$

In a similar manner one can readily establish that

$$\frac{d}{dt}\langle\hat{\sigma}_a^j\rangle = -\gamma\langle\hat{\sigma}_a^j\rangle + g(\langle\hat{\eta}_a^j\hat{a}\rangle - \langle\hat{\eta}_b^j\hat{a}\rangle - \langle\hat{a}^\dagger\hat{\sigma}_c^j\rangle), \tag{2.1.5}$$

$$\frac{d}{dt}\langle\hat{\sigma}_b^j\rangle = -\gamma\langle\hat{\sigma}_b^j\rangle + g(\langle\hat{\eta}_b^j\hat{a}\rangle - \langle\hat{\eta}_c^j\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_c^j\rangle), \tag{2.1.6}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c^j\rangle = -\gamma\langle\hat{\sigma}_c^j\rangle + g(\langle\hat{\sigma}_a^j\hat{a}\rangle - \langle\hat{\sigma}_b^j\hat{a}\rangle), \tag{2.1.7}$$

$$\frac{d}{dt}\langle\hat{\eta}_a^j\rangle = -\gamma\langle\hat{\eta}_a^j\rangle - g(\langle\hat{a}^\dagger\hat{\sigma}_a^j\rangle + \langle\hat{\sigma}_a^{\dagger j}\hat{a}\rangle), \tag{2.1.8}$$

$$\frac{d}{dt}\langle\hat{\eta}_b^j\rangle = -\gamma\langle\hat{\eta}_b^j\rangle + g(\langle\hat{a}^\dagger\hat{\sigma}_a^j\rangle - \langle\hat{a}^\dagger\hat{\sigma}_b^j\rangle + \langle\hat{\sigma}_a^{\dagger j}\hat{a}\rangle - \langle\hat{\sigma}_b^{\dagger j}\hat{a}\rangle), \tag{2.1.9}$$

$$\frac{d}{dt}\langle\hat{\eta}_c^j\rangle = -\gamma\langle\hat{\eta}_c^j\rangle + g(\langle\hat{a}^\dagger\hat{\sigma}_b^j\rangle + \langle\hat{\sigma}_b^{\dagger j}\hat{a}\rangle), \tag{2.1.10}$$

$$\frac{d}{dt}\langle\hat{\eta}_d^j\rangle = \gamma(\langle\hat{\eta}_a^j\rangle + \langle\hat{\eta}_b^j\rangle + \langle\hat{\eta}_c^j\rangle). \tag{2.1.11}$$

We see that Eqs. (2.1.5), (2.1.6), (2.1.7), (2.1.8), (2.1.9) and (2.1.10) are nonlinear differential equations and hence it is not possible to find the exact time dependent

solutions of these equations. We intended to overcome this problem by applying the large-time approximation scheme [6].

On the basis of Eq. (2.1.4) we can write

$$\frac{d}{dt}\hat{a}(t) = -\frac{\kappa}{2}\hat{a}(t) + g(\hat{\sigma}_a^j + \hat{\sigma}_b^j) + \hat{g}_a(t), \quad (2.1.12)$$

where $\hat{g}_a(t)$ is cavity mode noise operator when the cavity mode interacts with a single three-level atom and whose correlation properties remain to be determined.

Applying the large-time approximation scheme to equation (2.1.12), we obtain

$$\hat{a}(t) = \frac{2g}{\kappa}(\hat{\sigma}_a^j + \hat{\sigma}_b^j) + \frac{2}{\kappa}\hat{g}_a(t). \quad (2.1.13)$$

Now introducing Eq. (2.1.13) and its conjugate into Eqs. (2.1.5), (2.1.6), (2.1.7), (2.1.8), (2.1.9) and (2.1.10), one finds

$$\frac{d}{dt}\langle\hat{\sigma}_a^j\rangle = -(\gamma + \gamma_c)\langle\hat{\sigma}_a^j\rangle + \frac{2g}{\kappa}(\langle\hat{\eta}_a^j(t)\hat{g}_a(t)\rangle - \langle\hat{\eta}_b^j(t)\hat{g}_a(t)\rangle - \langle\hat{g}_a^\dagger(t)\hat{\sigma}_c^j(t)\rangle), \quad (2.1.14)$$

$$\begin{aligned} \frac{d}{dt}\langle\hat{\sigma}_b^j\rangle &= -\left(\gamma + \frac{\gamma_c}{2}\right)\langle\hat{\sigma}_b^j\rangle + \gamma_c\langle\hat{\sigma}_c^j\rangle \\ &+ \frac{2g}{\kappa}(\langle\hat{\eta}_b^j(t)\hat{g}_a(t)\rangle - \langle\hat{\eta}_c^j(t)\hat{g}_a(t)\rangle + \langle\hat{g}_a^\dagger(t)\hat{\sigma}_c^j(t)\rangle), \end{aligned} \quad (2.1.15)$$

$$\frac{d}{dt}\langle\hat{\sigma}_c^j\rangle = -\left(\gamma + \frac{\gamma_c}{2}\right)\langle\hat{\sigma}_c^j\rangle + \frac{2g}{\kappa}(\langle\hat{\sigma}_a^j(t)\hat{g}_a(t)\rangle - \langle\hat{\sigma}_b^j(t)\hat{g}_a(t)\rangle), \quad (2.1.16)$$

$$\frac{d}{dt}\langle\hat{\eta}_a^j\rangle = -(\gamma + \gamma_c)\langle\hat{\eta}_a^j\rangle - \frac{2g}{\kappa}(\langle\hat{\sigma}_a^{\dagger j}(t)\hat{g}_a(t)\rangle + \langle\hat{g}_a^\dagger(t)\hat{\sigma}_a^j(t)\rangle), \quad (2.1.17)$$

$$\begin{aligned} \frac{d}{dt}\langle\hat{\eta}_b^j\rangle &= -(\gamma + \gamma_c)\langle\hat{\eta}_b^j\rangle + \gamma_c\langle\hat{\eta}_a^j\rangle + \frac{2g}{\kappa}(\langle\hat{\sigma}_a^{\dagger j}(t)\hat{g}_a(t)\rangle \\ &+ \langle\hat{g}_a^\dagger(t)\hat{\sigma}_a^j(t)\rangle - \langle\hat{\sigma}_b^{\dagger j}(t)\hat{g}_a(t)\rangle - \langle\hat{g}_a^\dagger(t)\hat{\sigma}_b^j(t)\rangle), \end{aligned} \quad (2.1.18)$$

$$\frac{d}{dt}\langle\hat{\eta}_c^j\rangle = -\gamma\langle\hat{\eta}_c^j\rangle + \gamma_c\langle\hat{\eta}_b^j\rangle + \frac{2g}{\kappa}(\langle\hat{\sigma}_b^{\dagger j}(t)\hat{g}_a(t)\rangle + \langle\hat{g}_a^\dagger(t)\hat{\sigma}_b^j(t)\rangle), \quad (2.1.19)$$

where

$$\gamma_c = \frac{4g^2}{\kappa} \quad (2.1.20)$$

is the stimulated emission decay constant.

The pumping process must surely affect the dynamics of $\langle \hat{\eta}_a^j \rangle$ and $\langle \hat{\eta}_d^j \rangle$. If r_{da} represents the rate at which a single atom is pumped from the ground level to the top level, then $\langle \hat{\eta}_a^j \rangle$ increases at the rate of $r_{da}\langle \hat{\eta}_d^j \rangle$ and $\langle \hat{\eta}_d^j \rangle$ decreases at the same rate [6].

In view of this, we rewrite Eqs. (2.1.17) and (2.1.11) as

$$\frac{d}{dt}\langle \hat{\eta}_a^j \rangle = -(\gamma + \gamma_c)\langle \hat{\eta}_a^j \rangle + r_{da}\langle \hat{\eta}_d^j \rangle - \frac{2g}{\kappa}(\langle \hat{\sigma}_a^{\dagger j}(t)\hat{g}_a(t) \rangle + \langle \hat{g}_a^\dagger(t)\hat{\sigma}_a^j(t) \rangle), \quad (2.1.21)$$

$$\frac{d}{dt}\langle \hat{\eta}_d^j \rangle = \gamma(\langle \hat{\eta}_a^j \rangle + \langle \hat{\eta}_b^j \rangle + \langle \hat{\eta}_c^j \rangle) - r_{da}\langle \hat{\eta}_d^j \rangle. \quad (2.1.22)$$

In the presence of N three-level atoms, we rewrite Eq. (2.1.12) as

$$\frac{d}{dt}\hat{a}(t) = -\frac{\kappa}{2}\hat{a}(t) + \lambda\hat{m}(t) + \alpha\hat{g}_a(t), \quad (2.1.23)$$

in which λ and α are constants whose values remains to be fixed and

$$\hat{m}(t) = \sum_{j=1}^N (\hat{\sigma}_a^j + \hat{\sigma}_b^j) = \hat{m}_a + \hat{m}_b. \quad (2.1.24)$$

With the aid of Eq. (2.1.13), we get

$$\begin{aligned} [\hat{a}, \hat{a}^\dagger]_j &= \frac{\gamma_c}{\kappa} (\hat{\eta}_c^j - \hat{\eta}_a^j) + \frac{\gamma_c}{g\kappa} [(\hat{\sigma}_a^j(t) + \hat{\sigma}_b^j(t)), \hat{g}_a^\dagger(t)] \\ &+ \frac{\gamma_c}{g\kappa} [\hat{g}_a(t), (\hat{\sigma}_a^{\dagger j}(t) + \hat{\sigma}_b^{\dagger j}(t))] + \frac{4}{\kappa^2} [\hat{g}_a(t), \hat{g}_a^\dagger(t)], \end{aligned} \quad (2.1.25)$$

and summing over all atoms, we have

$$\begin{aligned} [\hat{a}, \hat{a}^\dagger] &= \frac{\gamma_c}{\kappa} (\hat{N}_c - \hat{N}_a) + \frac{\gamma_c}{g\kappa} ([\hat{m}(t), \hat{G}^\dagger(t)] + [\hat{G}(t), \hat{m}^\dagger(t)]) \\ &+ \frac{4}{\kappa^2} N [\hat{g}_a(t), \hat{g}_a^\dagger(t)], \end{aligned} \quad (2.1.26)$$

where $\hat{G}(t)$ is cavity mode noise operator when the cavity mode is interacting with N three-level atoms and

$$[\hat{a}, \hat{a}^\dagger] = \sum_{j=1}^N [\hat{a}, \hat{a}^\dagger]_j \quad (2.1.27)$$

stands for the commutator of \hat{a} and \hat{a}^\dagger when the cavity mode is interacting with all the three-level atoms.

Assuming that the cavity mode noise operator and atomic operators commute with each other, Eq. (2.1.26) reduces to

$$[\hat{a}, \hat{a}^\dagger] = \frac{\gamma_c}{\kappa} (\hat{N}_c - \hat{N}_a) + \frac{4}{\kappa^2} N [\hat{g}_a(t), \hat{g}_a^\dagger(t)], \quad (2.1.28)$$

On the other hand, applying the large-time approximation scheme to Eq. (2.1.23), we note that

$$\hat{a}(t) = \frac{2\lambda}{\kappa} \hat{m}(t) + \frac{2}{\kappa} \alpha \hat{g}_a(t) \quad (2.1.29)$$

and

$$[\hat{a}, \hat{a}^\dagger] = \left(\frac{2\lambda}{\kappa} \right)^2 N (\hat{N}_c - \hat{N}_a) + \frac{4}{\kappa^2} \alpha^2 [\hat{g}_a(t), \hat{g}_a^\dagger(t)]. \quad (2.1.30)$$

Thus on account of Eqs. (2.1.28) and (2.1.30), we see that

$$\lambda = \pm \frac{g}{\sqrt{N}} \quad \text{and} \quad \alpha = \sqrt{N} \quad (2.1.31)$$

and in view of this result, Eq. (2.1.23) can be rewritten as

$$\frac{d}{dt} \hat{a}(t) = -\frac{\kappa}{2} \hat{a}(t) + \frac{g}{\sqrt{N}} \hat{m}(t) + \sqrt{N} \hat{g}_a(t) \quad (2.1.32)$$

or

$$\frac{d}{dt} \hat{a}(t) = -\frac{\kappa}{2} \hat{a}(t) + \frac{g}{\sqrt{N}} \hat{m}(t) + \hat{G}(t), \quad (2.1.33)$$

in which $\hat{G}(t) = \sqrt{N} \hat{g}_a(t)$ is cavity mode noise operator when the cavity mode is interacting with N three-level atoms.

We next sum Eqs. (2.1.14), (2.1.15), (2.1.16), (2.1.18), (2.1.19), (2.1.21) and (2.1.22) over the N three-level atoms. It then follows that

$$\begin{aligned} \frac{d}{dt} \langle \hat{m}_a(t) \rangle &= -(\gamma + \gamma_c) \langle \hat{m}_a(t) \rangle \\ &+ \frac{2g}{\kappa} \left(\langle \hat{N}_a(t) \hat{G}(t) \rangle - \langle \hat{N}_b(t) \hat{G}(t) \rangle - \langle \hat{G}^\dagger(t) \hat{m}_c(t) \rangle \right), \end{aligned} \quad (2.1.34)$$

$$\begin{aligned} \frac{d}{dt} \langle \hat{m}_b(t) \rangle &= -\left(\gamma + \frac{\gamma_c}{2}\right) \langle \hat{m}_b(t) \rangle + \gamma_c \langle \hat{m}_a(t) \rangle \\ &+ \frac{2g}{\kappa} \left(\langle \hat{N}_b(t) \hat{G}(t) \rangle - \langle \hat{N}_c(t) \hat{G}(t) \rangle + \langle \hat{G}^\dagger(t) \hat{m}_c(t) \rangle \right), \end{aligned} \quad (2.1.35)$$

$$\frac{d}{dt} \langle \hat{m}_c(t) \rangle = -\left(\gamma + \frac{\gamma_c}{2}\right) \langle \hat{m}_c(t) \rangle + \frac{2g}{\kappa} \left(\langle \hat{m}_a(t) \hat{G}(t) \rangle - \langle \hat{m}_b(t) \hat{G}(t) \rangle \right), \quad (2.1.36)$$

$$\begin{aligned} \frac{d}{dt} \langle \hat{N}_a(t) \rangle &= -(\gamma + \gamma_c) \langle \hat{N}_a(t) \rangle + r_{da} \langle \hat{N}_d(t) \rangle \\ &- \frac{2g}{\kappa} \left(\langle \hat{m}_a^\dagger(t) \hat{G}(t) \rangle + \langle \hat{G}^\dagger(t) \hat{m}_a(t) \rangle \right), \end{aligned} \quad (2.1.37)$$

$$\begin{aligned} \frac{d}{dt} \langle \hat{N}_b(t) \rangle &= -(\gamma + \gamma_c) \langle \hat{N}_b(t) \rangle + \gamma_c \langle \hat{N}_a(t) \rangle + \frac{2g}{\kappa} \left[\langle \hat{m}_a^\dagger(t) \hat{G}(t) \rangle \right. \\ &\left. + \langle \hat{G}^\dagger(t) \hat{m}_a(t) \rangle - \langle \hat{m}_b^\dagger(t) \hat{G}(t) \rangle - \langle \hat{G}^\dagger(t) \hat{m}_b(t) \rangle \right], \end{aligned} \quad (2.1.38)$$

$$\frac{d}{dt} \langle \hat{N}_c(t) \rangle = -\gamma \langle \hat{N}_c(t) \rangle + \gamma_c \langle \hat{N}_b(t) \rangle + \frac{2g}{\kappa} \left(\langle \hat{m}_b^\dagger(t) \hat{G}(t) \rangle + \langle \hat{G}^\dagger(t) \hat{m}_b(t) \rangle \right), \quad (2.1.39)$$

$$\frac{d}{dt} \langle \hat{N}_d(t) \rangle = \gamma (\langle \hat{N}_a(t) \rangle + \langle \hat{N}_b(t) \rangle + \langle \hat{N}_c(t) \rangle) - r_{da} \langle \hat{N}_d(t) \rangle, \quad (2.1.40)$$

where

$$\hat{m}_a = \sum_{j=1}^N \hat{\sigma}_a^j, \quad (2.1.41)$$

$$\hat{m}_b = \sum_{j=1}^N \hat{\sigma}_b^j, \quad (2.1.42)$$

$$\hat{m}_c = \sum_{j=1}^N \hat{\sigma}_c^j, \quad (2.1.43)$$

and

$$\hat{N}_a = \sum_{j=1}^N \hat{\eta}_a^j, \quad (2.1.44)$$

$$\hat{N}_b = \sum_{j=1}^N \hat{\eta}_b^j, \quad (2.1.45)$$

$$\hat{N}_c = \sum_{j=1}^N \hat{\eta}_c^j, \quad (2.1.46)$$

$$\hat{N}_d = \sum_{j=1}^N \hat{\eta}_d^j \quad (2.1.47)$$

are atomic operators in which \hat{N}_a , \hat{N}_b , \hat{N}_c and \hat{N}_d represent number of atoms in the upper, intermediate, bottom and ground levels respectively.

We now proceed to determine the expectation values of the product of the atomic operators and cavity mode noise operator that appear in Eqs. (2.1.34) - (2.1.39).

Employing the master equation described by Eq. (2.0.4) along with Eq. (2.1.1) we see that

$$\begin{aligned} \frac{d}{dt} \langle |b_j\rangle \langle a_j | \hat{a} \rangle &= g Tr \left[\hat{a}^\dagger \hat{\sigma}_a^j \hat{\rho} |b_j\rangle \langle a_j | \hat{a} - \hat{\rho} \hat{a}^\dagger \hat{\sigma}_a^j |b_j\rangle \langle a_j | \hat{a} + \hat{a}^\dagger \hat{\sigma}_b^j \hat{\rho} |b_j\rangle \langle a_j | \hat{a} - \hat{\rho} \hat{a}^\dagger \hat{\sigma}_b^j |b_j\rangle \langle a_j | \hat{a} \right. \\ &\quad \left. - \hat{a} \hat{\sigma}_a^{\dagger j} \hat{\rho} |b_j\rangle \langle a_j | \hat{a} + \hat{\rho} \hat{a} \hat{\sigma}_a^{\dagger j} |b_j\rangle \langle a_j | \hat{a} - \hat{a} \hat{\sigma}_b^{\dagger j} \hat{\rho} |b_j\rangle \langle a_j | \hat{a} + \hat{\rho} \hat{a} \hat{\sigma}_b^{\dagger j} |b_j\rangle \langle a_j | \hat{a} \right] \\ &+ \frac{\kappa}{2} Tr \left[2 \hat{a} \hat{\rho} \hat{a}^\dagger |b_j\rangle \langle a_j | \hat{a} - \hat{a}^\dagger \hat{a} \hat{\rho} |b_j\rangle \langle a_j | \hat{a} - \hat{\rho} \hat{a}^\dagger \hat{a} |b_j\rangle \langle a_j | \hat{a} \right] \\ &+ \frac{\gamma}{2} Tr \left[2 |d_j\rangle \langle a_j | \hat{\rho} |a_j\rangle \langle d_j | b_j \rangle \langle a_j | \hat{a} - \hat{\eta}_a^j \hat{\rho} |b_j\rangle \langle a_j | \hat{a} - \hat{\rho} \hat{\eta}_a^j |b_j\rangle \langle a_j | \hat{a} \right. \\ &+ 2 |d_j\rangle \langle b_j | \hat{\rho} |b_j\rangle \langle d_j | b_j \rangle \langle a_j | \hat{a} - \hat{\eta}_b^j \hat{\rho} |b_j\rangle \langle a_j | \hat{a} - \hat{\rho} \hat{\eta}_b^j |b_j\rangle \langle a_j | \hat{a} \\ &\left. + 2 |d_j\rangle \langle c_j | \hat{\rho} |c_j\rangle \langle d_j | b_j \rangle \langle a_j | \hat{a} - \hat{\eta}_c^j \hat{\rho} |b_j\rangle \langle a_j | \hat{a} - \hat{\rho} \hat{\eta}_c^j |b_j\rangle \langle a_j | \hat{a} \right]. \quad (2.1.48) \end{aligned}$$

Applying the cyclic property of the trace operation, Eq. (2.1.48) can be written as

$$\begin{aligned}
\frac{d}{dt}\langle |b_j\rangle\langle a_j|\hat{a}\rangle &= gTr [\hat{\rho}|b_j\rangle\langle a_j|\hat{a}\hat{a}^\dagger\hat{\sigma}_a^j - \hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^j|b_j\rangle\langle a_j|\hat{a} + \hat{\rho}|b_j\rangle\langle a_j|\hat{a}\hat{a}^\dagger\hat{\sigma}_b^j \\
&\quad - \hat{\rho}\hat{a}^\dagger\hat{\sigma}_b^j|b_j\rangle\langle a_j|\hat{a} - \hat{\rho}|b_j\rangle\langle a_j|\hat{a}^2\hat{\sigma}_a^{+j} + \hat{\rho}\hat{a}\hat{\sigma}_a^{+j}|b_j\rangle\langle a_j|\hat{a} \\
&\quad - \hat{\rho}|b_j\rangle\langle a_j|\hat{a}^2\hat{\sigma}_b^{+j} + \hat{\rho}\hat{a}\hat{\sigma}_b^{+j}|b_j\rangle\langle a_j|\hat{a}] \\
&\quad + \frac{\kappa}{2}Tr [2\hat{\rho}\hat{a}^\dagger|b_j\rangle\langle a_j|\hat{a}^2 - \hat{\rho}|b_j\rangle\langle a_j|\hat{a}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}|b_j\rangle\langle a_j|\hat{a}] \\
&\quad + \frac{\gamma}{2}Tr [2\hat{\rho}|a_j\rangle\langle d_j|b_j\rangle\langle a_j|\hat{a}|d_j\rangle\langle a_j| - \hat{\rho}|b_j\rangle\langle a_j|\hat{a}\hat{\eta}_a^j - \hat{\rho}\hat{\eta}_a^j|b_j\rangle\langle a_j|\hat{a} \\
&\quad + 2\hat{\rho}|b_j\rangle\langle d_j|b_j\rangle\langle a_j|\hat{a}|d_j\rangle\langle b_j| - \hat{\rho}|b_j\rangle\langle a_j|\hat{a}\hat{\eta}_b^j - \hat{\rho}\hat{\eta}_b^j|b_j\rangle\langle a_j|\hat{a} \\
&\quad + 2\hat{\rho}|c_j\rangle\langle d_j|b_j\rangle\langle a_j|\hat{a}|d_j\rangle\langle c_j| - \hat{\rho}|b_j\rangle\langle a_j|\hat{a}\hat{\eta}_c^j - \hat{\rho}\hat{\eta}_c^j|b_j\rangle\langle a_j|\hat{a}]. \quad (2.1.49)
\end{aligned}$$

Now taking into account Eq. (2.0.5), Eq. (2.1.49) reduces to

$$\frac{d}{dt}\langle \hat{\sigma}_a^j(t)\hat{a}(t)\rangle = gTr (\hat{\rho}\hat{a}^2(\hat{\eta}_a^j - \hat{\eta}_b^j) - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{\sigma}_c^j) - \frac{\kappa}{2}Tr(\hat{\rho}\hat{\sigma}_a^j\hat{a}) - \gamma Tr(\hat{\rho}\hat{\sigma}_a^j\hat{a}) \quad (2.1.50)$$

or

$$\frac{d}{dt}\langle \hat{\sigma}_a^j(t)\hat{a}(t)\rangle = -\frac{\kappa}{2}\langle \hat{\sigma}_a^j\hat{a}\rangle - \gamma\langle \hat{\sigma}_a^j\hat{a}\rangle + g(\langle \hat{a}^2(\hat{\eta}_a^j - \hat{\eta}_b^j)\rangle - \langle \hat{a}^\dagger\hat{a}\hat{\sigma}_c^j\rangle). \quad (2.1.51)$$

On the basis of Eq. (2.1.5), we can write

$$\frac{d}{dt}\hat{\sigma}_a^j(t) = -\gamma\hat{\sigma}_a^j + g(\hat{\eta}_a^j\hat{a} - \hat{\eta}_b^j\hat{a} - \hat{a}^\dagger\hat{\sigma}_c^j) + \hat{f}_{aj}(t), \quad (2.1.52)$$

where $\hat{f}_{aj}(t)$ is atomic noise operator whose correlation properties remain to be determined.

Employing Eqs. (2.1.52) and (2.1.12) in the relation

$$\frac{d}{dt}\langle \hat{\sigma}_a^j\hat{a}\rangle = \left\langle \frac{d\hat{\sigma}_a^j}{dt}\hat{a}\right\rangle + \left\langle \hat{\sigma}_a^j\frac{d\hat{a}}{dt}\right\rangle \quad (2.1.53)$$

we note that

$$\begin{aligned}
\frac{d}{dt}\langle \hat{\sigma}_a^j(t)\hat{a}(t)\rangle &= -\frac{\kappa}{2}\langle \hat{\sigma}_a^j\hat{a}\rangle - \gamma\langle \hat{\sigma}_a^j\hat{a}\rangle + g(\langle \hat{a}^2(\hat{\eta}_a^j - \hat{\eta}_b^j)\rangle - \langle \hat{a}^\dagger\hat{a}\hat{\sigma}_c^j\rangle) \\
&\quad + \langle \hat{f}_{aj}(t)\hat{a}(t)\rangle + \langle \hat{\sigma}_a^j(t)\hat{g}_a(t)\rangle. \quad (2.1.54)
\end{aligned}$$

Here upon comparing Eqs. (2.1.51) and (2.1.54), we find that

$$\langle \hat{f}_{aj}(t)\hat{a}(t) \rangle + \langle \hat{\sigma}_a^j(t)\hat{g}_a(t) \rangle = 0. \quad (2.1.55)$$

The formal solution of Eqs. (2.1.52) and (2.1.12) are expressible as

$$\begin{aligned} \hat{\sigma}_a^j(t) &= \hat{\sigma}_a^j(0)e^{-\gamma t} + ge^{-\gamma t} \left(\int_0^t e^{\gamma t'} (\hat{\eta}_a^j(t')\hat{a}(t') - \hat{\eta}_b^j(t')\hat{a}(t') - \hat{a}^\dagger(t')\hat{\sigma}_c^j(t')) dt' \right) \\ &+ e^{-\gamma t} \int_0^t e^{\gamma t'} \hat{f}_{aj}(t') dt' \end{aligned} \quad (2.1.56)$$

and

$$\begin{aligned} \hat{a}(t) &= \hat{a}(0)e^{-\frac{\kappa}{2}t} + ge^{-\frac{\kappa}{2}t} \left(\int_0^t e^{\frac{\kappa}{2}t'} (\hat{\sigma}_a^j(t') + \hat{\sigma}_b^j(t')) dt' \right) \\ &+ e^{-\frac{\kappa}{2}t} \int_0^t e^{\frac{\kappa}{2}t'} \hat{g}_a(t') dt' \end{aligned} \quad (2.1.57)$$

respectively.

Now multiplying Eq. (2.1.57) on the left by $\hat{f}_{aj}(t)$ and taking the expectation value of the resulting expression, we obtain

$$\begin{aligned} \langle \hat{f}_{aj}(t)\hat{a}(t) \rangle &= \langle \hat{f}_{aj}(t)\hat{a}(0) \rangle e^{-\frac{\kappa}{2}t} + ge^{-\frac{\kappa}{2}t} \int_0^t e^{\frac{\kappa}{2}t'} \left(\langle \hat{f}_{aj}(t)\hat{\sigma}_a^j(t') \rangle + \langle \hat{f}_{aj}(t)\hat{\sigma}_b^j(t') \rangle \right) dt' \\ &+ e^{-\frac{\kappa}{2}t} \int_0^t e^{\frac{\kappa}{2}t'} \langle \hat{f}_{aj}(t)\hat{g}_a(t') \rangle dt'. \end{aligned} \quad (2.1.58)$$

Since a noise operator at some time t does not affect the cavity mode and atomic operators at earlier times, we see that

$$\langle \hat{f}_{aj}(t)\hat{a}(0) \rangle = \langle \hat{f}_{aj}(t) \rangle \langle \hat{a}(0) \rangle = 0, \quad \langle \hat{f}_{aj}(t)\hat{\sigma}_a^j(t') \rangle = \langle \hat{f}_{aj}(t) \rangle \langle \hat{\sigma}_a^j(t') \rangle = 0. \quad (2.1.59)$$

In view of Eq. (2.1.59), Eq. (2.1.58) is expressible as

$$\langle \hat{f}_{aj}(t)\hat{a}(t) \rangle = e^{-\frac{\kappa}{2}t} \int_0^t e^{\frac{\kappa}{2}t'} \langle \hat{f}_{aj}(t)\hat{g}_a(t') \rangle dt'. \quad (2.1.60)$$

Similarly, multiplying Eq. (2.1.56) on the right by $\hat{g}_a(t)$ and taking the expectation value of the resulting expression and taking into account the assertion that the noise operator at some time does not affect the cavity mode and atomic operators at earlier time, we arrive at

$$\langle \hat{\sigma}_a^j(t) \hat{g}_a(t) \rangle = e^{-\gamma t} \int_0^t e^{\gamma t'} \langle \hat{f}_{aj}(t') \hat{g}_a(t) \rangle dt'. \quad (2.1.61)$$

Introducing Eqs. (2.1.60) and (2.1.61) into Eq. (2.1.55), one obtains

$$e^{-\frac{\kappa}{2}t} \int_0^t e^{\frac{\kappa}{2}t'} \langle \hat{f}_{aj}(t) \hat{g}_a(t') \rangle dt' + e^{-\gamma t} \int_0^t e^{\gamma t'} \langle \hat{f}_{aj}(t') \hat{g}_a(t) \rangle dt' = 0. \quad (2.1.62)$$

Assuming that

$$\langle \hat{f}_{aj}(t) \hat{g}_a(t') \rangle = \langle \hat{f}_{aj}(t') \hat{g}_a(t) \rangle, \quad (2.1.63)$$

we assert that

$$\int_0^t [e^{-\frac{\kappa}{2}(t-t')} + e^{-\gamma(t-t')}] \langle \hat{f}_{aj}(t) \hat{g}_a(t') \rangle dt' = 0. \quad (2.1.64)$$

It then follows that

$$\langle \hat{f}_{aj}(t) \hat{g}_a(t') \rangle = 0. \quad (2.1.65)$$

Taking into account Eq. (2.1.65), we can write Eq. (2.1.61) as

$$\langle \hat{\sigma}_a^j(t) \hat{g}_a(t) \rangle = 0. \quad (2.1.66)$$

Now summing over the N three-level atoms, Eq. (2.1.66) goes over into

$$\langle \hat{m}_a(t) \hat{G}(t) \rangle = 0. \quad (2.1.67)$$

In a similar manner, one can readily establish that

$$\langle \hat{m}_b(t) \hat{G}(t) \rangle = 0, \quad (2.1.68)$$

$$\langle \hat{m}_c(t) \hat{G}(t) \rangle = 0, \quad (2.1.69)$$

$$\langle \hat{N}_a(t) \hat{G}(t) \rangle = 0, \quad (2.1.70)$$

$$\langle \hat{N}_b(t) \hat{G}(t) \rangle = 0, \quad (2.1.71)$$

$$\langle \hat{N}_c(t) \hat{G}(t) \rangle = 0. \quad (2.1.72)$$

Finally, employing Eqs. (2.1.67) - (2.1.72), we can rewrite Eqs. (2.1.34) - (2.1.39) in the form

$$\frac{d}{dt} \langle \hat{m}_a(t) \rangle = -(\gamma + \gamma_c) \langle \hat{m}_a(t) \rangle, \quad (2.1.73)$$

$$\frac{d}{dt} \langle \hat{m}_b(t) \rangle = -(\gamma + \frac{\gamma_c}{2}) \langle \hat{m}_b(t) \rangle + \gamma_c \langle \hat{m}_a(t) \rangle, \quad (2.1.74)$$

$$\frac{d}{dt} \langle \hat{m}_c(t) \rangle = -(\gamma + \frac{\gamma_c}{2}) \langle \hat{m}_c(t) \rangle, \quad (2.1.75)$$

$$\frac{d}{dt} \langle \hat{N}_a(t) \rangle = -(\gamma + \gamma_c) \langle \hat{N}_a(t) \rangle + r_{da} \langle \hat{N}_d(t) \rangle, \quad (2.1.76)$$

$$\frac{d}{dt} \langle \hat{N}_b(t) \rangle = -(\gamma + \gamma_c) \langle \hat{N}_b(t) \rangle + \gamma_c \langle \hat{N}_a(t) \rangle, \quad (2.1.77)$$

$$\frac{d}{dt} \langle \hat{N}_c(t) \rangle = -\gamma \langle \hat{N}_c(t) \rangle + \gamma_c \langle \hat{N}_b(t) \rangle. \quad (2.1.78)$$

On the basis of Eqs. (2.1.73) and (2.1.74), one can write

$$\frac{d}{dt} \hat{m}_a(t) = -(\gamma + \gamma_c) \hat{m}_a(t) + \hat{f}_a(t) \quad (2.1.79)$$

and

$$\frac{d}{dt} \hat{m}_b(t) = -(\gamma + \frac{\gamma_c}{2}) \hat{m}_b(t) + \gamma_c \hat{m}_a(t) + \hat{f}_b(t), \quad (2.1.80)$$

where $\hat{f}_a(t)$ and $\hat{f}_b(t)$ are the atomic noise operators with vanishing mean and whose correlation properties remains to be determined.

In addition, in view of the completeness relation

$$\hat{\eta}_a^j + \hat{\eta}_b^j + \hat{\eta}_c^j + \hat{\eta}_d^j = \hat{I}, \quad (2.1.81)$$

we easily arrive at

$$\langle \hat{N}_a \rangle + \langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle + \langle \hat{N}_d \rangle = N. \quad (2.1.82)$$

Hence with the aid of Eq. (2.1.82) one can put Eqs. (2.1.76) and (2.1.40) in the form

$$\frac{d}{dt} \langle \hat{N}_a \rangle = -(\gamma + \gamma_c + r_{da}) \langle \hat{N}_a \rangle + r_{da} (N - \langle \hat{N}_b \rangle - \langle \hat{N}_c \rangle), \quad (2.1.83)$$

and

$$\frac{d}{dt} \langle \hat{N}_d \rangle = -(\gamma + r_{da}) \langle \hat{N}_d \rangle + \gamma N. \quad (2.1.84)$$

At steady-state, the solutions of Eqs. (2.1.83), (2.1.77) and (2.1.78) are expressible as

$$\langle \hat{N}_a \rangle = \frac{r_{da}}{(\gamma + \gamma_c + r_{da})} (N - \langle \hat{N}_b \rangle - \langle \hat{N}_c \rangle), \quad (2.1.85)$$

$$\langle \hat{N}_b \rangle = \frac{\gamma_c}{(\gamma + \gamma_c)} \langle \hat{N}_a \rangle, \quad (2.1.86)$$

and

$$\langle \hat{N}_c \rangle = \frac{\gamma_c}{\gamma} \langle \hat{N}_b \rangle = \frac{\gamma_c^2}{\gamma(\gamma + \gamma_c)} \langle \hat{N}_a \rangle \quad (2.1.87)$$

respectively.

Thus on taking into account Eqs. (2.1.86) and (2.1.87), we can write Eq. (2.1.85) in the form

$$\langle \hat{N}_a \rangle = \left(\frac{\gamma r_{da}}{(\gamma + \gamma_c)(\gamma + r_{da})} \right) N. \quad (2.1.88)$$

Now substituting Eq. (2.1.88) into (2.1.86), we get

$$\langle \hat{N}_b \rangle = \left(\frac{\gamma \gamma_c r_{da}}{(\gamma + \gamma_c)^2 (\gamma + r_{da})} \right) N, \quad (2.1.89)$$

in view of which Eq. (2.1.87) becomes

$$\langle \hat{N}_c \rangle = \left(\frac{\gamma_c^2 r_{da}}{(\gamma + \gamma_c)^2 (\gamma + r_{da})} \right) N. \quad (2.1.90)$$

We also notice that the steady-state solution of Eq. (2.1.84) is

$$\langle \hat{N}_d \rangle = \left(\frac{\gamma}{\gamma + r_{da}} \right) N. \quad (2.1.91)$$

Thus at steady-state the expressions for the expectation values of the operators representing the number of atoms in the upper, intermediate, bottom and ground levels are given by Eqs. (2.1.88), (2.1.89), (2.1.90) and (2.1.91) respectively.

2.2 The correlation properties of the noise operators

In this section we seek to determine the correlation properties of cavity mode noise operator $\hat{G}(t)$ and atomic noise operators $\hat{f}_a(t)$ and $\hat{f}_b(t)$.

We note that for N three-level atoms Eq. (2.1.4) is expressible as

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = -\frac{\kappa}{2}\langle\hat{a}(t)\rangle + \frac{g}{\sqrt{N}}\langle\hat{m}(t)\rangle. \quad (2.2.1)$$

This equation and expectation value of the Eq. (2.1.33) will have the same form if

$$\langle\hat{G}(t)\rangle = 0. \quad (2.2.2)$$

Employing the relation

$$\frac{d}{dt}\langle\hat{a}\hat{a}^\dagger\rangle = \left\langle\frac{d\hat{a}}{dt}\hat{a}^\dagger\right\rangle + \left\langle\hat{a}\frac{d\hat{a}^\dagger}{dt}\right\rangle \quad (2.2.3)$$

along with Eq. (2.1.33) and its conjugate, we note that

$$\frac{d}{dt}\langle\hat{a}\hat{a}^\dagger\rangle = -\kappa\langle\hat{a}\hat{a}^\dagger\rangle + \frac{g}{\sqrt{N}}\langle\hat{m}\hat{a}^\dagger\rangle + \frac{g}{\sqrt{N}}\langle\hat{a}\hat{m}^\dagger\rangle + \langle\hat{G}(t)\hat{a}^\dagger(t)\rangle + \langle\hat{a}(t)\hat{G}^\dagger(t)\rangle. \quad (2.2.4)$$

On the other hand using the master equation described by Eq. (2.0.4) along with Eq. (2.1.1), we see that

$$\begin{aligned} \frac{d}{dt}\langle\hat{a}\hat{a}^\dagger\rangle &= gTr [\hat{a}^\dagger\hat{\sigma}_a^j\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^j\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{\sigma}_b^j\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{\sigma}_b^j\hat{a}\hat{a}^\dagger \\ &\quad - \hat{a}\hat{\sigma}_a^{+j}\hat{\rho}\hat{a}\hat{a}^\dagger + \hat{\rho}\hat{a}\hat{\sigma}_a^{+j}\hat{a}\hat{a}^\dagger - \hat{a}\hat{\sigma}_b^{+j}\hat{\rho}\hat{a}\hat{a}^\dagger + \hat{\rho}\hat{a}\hat{\sigma}_b^{+j}\hat{a}\hat{a}^\dagger] \\ &\quad + \frac{\kappa}{2}Tr [2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a}^2\hat{a}^\dagger] \\ &\quad + \frac{\gamma}{2}Tr [2|d_j\rangle\langle a_j|\hat{\rho}|a_j\rangle\langle d_j|\hat{a}\hat{a}^\dagger - \hat{\eta}_a^j\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{\eta}_a^j\hat{a}\hat{a}^\dagger \\ &\quad + 2|d_j\rangle\langle b_j|\hat{\rho}|b_j\rangle\langle d_j|\hat{a}\hat{a}^\dagger - \hat{\eta}_b^j\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{\eta}_b^j\hat{a}\hat{a}^\dagger \\ &\quad + 2|d_j\rangle\langle c_j|\hat{\rho}|c_j\rangle\langle d_j|\hat{a}\hat{a}^\dagger - \hat{\eta}_c^j\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{\eta}_c^j\hat{a}\hat{a}^\dagger]. \end{aligned} \quad (2.2.5)$$

Applying the cyclic property of the trace operation to Eq. (2.2.5), we obtain

$$\frac{d}{dt}\langle\hat{a}\hat{a}^\dagger\rangle = gTr\left(\rho(\hat{\sigma}_a^j + \hat{\sigma}_b^j)\hat{a}^\dagger + \rho\hat{a}(\hat{\sigma}_a^{\dagger j} + \hat{\sigma}_b^{\dagger j})\right) - \kappa Tr(\rho\hat{a}^\dagger\hat{a})$$

or

$$\frac{d}{dt}\langle\hat{a}\hat{a}^\dagger\rangle = -\kappa\langle\hat{a}\hat{a}^\dagger\rangle + g\left(\langle(\hat{\sigma}_a^j + \hat{\sigma}_b^j)\hat{a}^\dagger\rangle + \langle\hat{a}(\hat{\sigma}_a^{\dagger j} + \hat{\sigma}_b^{\dagger j})\rangle\right) + \kappa, \quad (2.2.6)$$

which for N three-level atoms becomes

$$\frac{d}{dt}\langle\hat{a}\hat{a}^\dagger\rangle = -\kappa\langle\hat{a}\hat{a}^\dagger\rangle + \frac{g}{\sqrt{N}}\langle\hat{m}\hat{a}^\dagger\rangle + \frac{g}{\sqrt{N}}\langle\hat{a}\hat{m}^\dagger\rangle + \kappa N. \quad (2.2.7)$$

Upon comparing Eqs. (2.2.4) and (2.2.7), we see that

$$\langle\hat{G}(t)\hat{a}^\dagger(t)\rangle + \langle\hat{a}(t)\hat{G}^\dagger(t)\rangle = \kappa N. \quad (2.2.8)$$

The formal solution of Eq. (2.1.33) can be expressed in the form

$$\hat{a}(t) = \hat{a}(0)e^{-\frac{\kappa}{2}t} + \frac{g}{\sqrt{N}}\int_0^t e^{-\frac{\kappa}{2}(t-t')}\hat{m}(t')dt' + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\hat{G}(t')dt'. \quad (2.2.9)$$

Now multiplying Eq. (2.2.9) on the right by $\hat{G}^\dagger(t)$ and taking the expectation value of the resulting expression, we note that

$$\begin{aligned} \langle\hat{a}(t)\hat{G}^\dagger(t)\rangle &= \langle\hat{a}(0)\hat{G}^\dagger(t)\rangle e^{-\frac{\kappa}{2}t} + \frac{g}{\sqrt{N}}\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{m}(t')\hat{G}^\dagger(t)\rangle dt' \\ &+ \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{G}(t')\hat{G}^\dagger(t)\rangle dt'. \end{aligned} \quad (2.2.10)$$

Since a noise operator at time t should not affect cavity mode and atomic operators at earlier times, we can write

$$\langle\hat{a}(0)\hat{G}^\dagger(t)\rangle = \langle\hat{a}(0)\rangle\langle\hat{G}^\dagger(t)\rangle = 0, \quad (2.2.11)$$

$$\langle\hat{m}(t')\hat{G}^\dagger(t)\rangle = \langle\hat{m}(t')\rangle\langle\hat{G}^\dagger(t)\rangle = 0. \quad (2.2.12)$$

With the aid of Eqs. (2.2.11) and (2.2.12), Eq. (2.2.10) reduces to

$$\langle \hat{a}(t)\hat{G}^\dagger(t) \rangle = \int_0^t e^{-\frac{\kappa}{2}(t-t')} \langle \hat{G}(t')\hat{G}^\dagger(t) \rangle dt'. \quad (2.2.13)$$

Upon multiplying the conjugate of Eq. (2.2.9) on the left by $\hat{G}(t)$ and taking the expectation value of the resulting expression and taking into account Eqs. (2.2.11) and (2.2.12), we arrive at

$$\langle \hat{G}(t)\hat{a}^\dagger(t) \rangle = \int_0^t e^{-\frac{\kappa}{2}(t-t')} \langle \hat{G}(t)\hat{G}^\dagger(t') \rangle dt'. \quad (2.2.14)$$

Introduction of Eqs. (2.2.13) and (2.2.14) into Eq. (2.2.8) leads to

$$\int_0^t e^{-\frac{\kappa}{2}(t-t')} \langle \hat{G}(t')\hat{G}^\dagger(t) \rangle dt' + \int_0^t e^{-\frac{\kappa}{2}(t-t')} \langle \hat{G}(t)\hat{G}^\dagger(t') \rangle dt' = \kappa N \quad (2.2.15)$$

and assuming that

$$\langle \hat{G}(t')\hat{G}^\dagger(t) \rangle = \langle \hat{G}(t)\hat{G}^\dagger(t') \rangle, \quad (2.2.16)$$

one obtains

$$2 \int_0^t e^{-\frac{\kappa}{2}(t-t')} \langle \hat{G}(t)\hat{G}^\dagger(t') \rangle dt' = \kappa N. \quad (2.2.17)$$

Now on the basis of the relation [1]

$$\int_0^t e^{-s(t-t')} \langle \hat{f}(t)\hat{g}(t') \rangle dt' = D, \quad (2.2.18)$$

we assert that

$$\langle \hat{f}(t)\hat{g}(t') \rangle = 2D\delta(t-t'), \quad (2.2.19)$$

where s and D are constants, or D is some function of time t .

On account of this relation and Eq. (2.2.17), we note that

$$\langle \hat{G}(t)\hat{G}^\dagger(t') \rangle = \kappa N\delta(t-t'). \quad (2.2.20)$$

In a similar fashion, one can easily verify that

$$\langle \hat{G}^\dagger(t) \hat{G}(t') \rangle = 0, \quad (2.2.21)$$

$$\langle \hat{G}(t) \hat{G}(t') \rangle = 0. \quad (2.2.22)$$

We now proceed to obtain the correlation properties of the atomic noise operators $\hat{f}_a(t)$ and $\hat{f}_b(t)$.

We note that Eq. (2.1.73) and the expectation value of Eq. (2.1.79) as well as Eq. (2.1.74) and the expectation value of Eq. (2.1.80) will have the same form if

$$\langle \hat{f}_a(t) \rangle = \langle \hat{f}_b(t) \rangle = 0. \quad (2.2.23)$$

Employing the relation

$$\frac{d}{dt} \langle \hat{m}_a^\dagger(t) \hat{m}_a(t) \rangle = \left\langle \frac{d\hat{m}_a^\dagger(t)}{dt} \hat{m}_a(t) \right\rangle + \left\langle \hat{m}_a^\dagger(t) \frac{d\hat{m}_a(t)}{dt} \right\rangle \quad (2.2.24)$$

along with Eq. (2.1.79) and its conjugate, we see that

$$\frac{d}{dt} \langle \hat{m}_a^\dagger(t) \hat{m}_a(t) \rangle = -2(\gamma + \gamma_c) \langle \hat{m}_a^\dagger(t) \hat{m}_a(t) \rangle + \langle \hat{f}_a^\dagger(t) \hat{m}_a(t) \rangle + \langle \hat{m}_a^\dagger(t) \hat{f}_a(t) \rangle \quad (2.2.25)$$

and taking Eq. (2.1.41) into account, one obtains

$$\frac{d}{dt} \langle \hat{N}_a(t) \rangle = -2(\gamma + \gamma_c) \langle \hat{N}_a(t) \rangle + \frac{1}{N} \left(\langle \hat{f}_a^\dagger(t) \hat{m}_a(t) \rangle + \langle \hat{m}_a^\dagger(t) \hat{f}_a(t) \rangle \right). \quad (2.2.26)$$

Now comparison of Eqs. (2.1.76) and (2.2.26) shows that

$$\langle \hat{f}_a^\dagger(t) \hat{m}_a(t) \rangle + \langle \hat{m}_a^\dagger(t) \hat{f}_a(t) \rangle = (\gamma + \gamma_c) N \langle \hat{N}_a(t) \rangle + r_{da} N \langle \hat{N}_d(t) \rangle. \quad (2.2.27)$$

The formal solution of Eqs. (2.1.79) and (2.1.80) can be written in the form

$$\hat{m}_a(t) = \hat{m}_a(0) e^{-(\gamma + \gamma_c)t} + \int_0^t e^{-(\gamma + \gamma_c)(t-t')} \hat{f}_a(t') dt' \quad (2.2.28)$$

and

$$\begin{aligned}\hat{m}_b(t) &= \hat{m}_b(0)e^{-(\gamma+\frac{\gamma_c}{2})t} + \int_0^t e^{-(\gamma+\frac{\gamma_c}{2})(t-t')} \hat{m}_a(t') dt' \\ &+ \int_0^t e^{-(\gamma+\frac{\gamma_c}{2})(t-t')} \hat{f}_b(t') dt'.\end{aligned}\quad (2.2.29)$$

Multiplying Eq. (2.2.28) on the left by $\hat{f}_a^\dagger(t)$ and taking the expectation values of the resulting expressions, we obtain

$$\langle \hat{f}_a^\dagger(t) \hat{m}_a(t) \rangle = \langle \hat{f}_a^\dagger(t) \hat{m}_a(0) \rangle e^{-(\gamma+\gamma_c)t} + \int_0^t e^{-(\gamma+\gamma_c)(t-t')} \langle \hat{f}_a^\dagger(t) \hat{f}_a(t') \rangle dt'. \quad (2.2.30)$$

Since noise operators at time t has no effect on atomic operators at an earlier time, we can write

$$\langle \hat{f}_a^\dagger(t) \hat{m}_a(0) \rangle = \langle \hat{f}_a^\dagger(t) \rangle \langle \hat{m}_a(0) \rangle = 0. \quad (2.2.31)$$

One can then rewrite Eq. (2.2.30) as

$$\langle \hat{f}_a^\dagger(t) \hat{m}_a(t) \rangle = \int_0^t e^{-(\gamma+\gamma_c)(t-t')} \langle \hat{f}_a^\dagger(t) \hat{f}_a(t') \rangle dt'. \quad (2.2.32)$$

Upon multiplying the conjugate of Eq. (2.2.28) on the right by $\hat{f}_a(t)$ and taking the expectation values of the resulting expression and applying the fact that a noise operator at some time has no effect on atomic operators at earlier times, we obtain

$$\langle \hat{m}_a^\dagger(t) \hat{f}_a(t) \rangle = \int_0^t e^{-(\gamma+\gamma_c)(t-t')} \langle \hat{f}_a^\dagger(t') \hat{f}_a(t) \rangle dt'. \quad (2.2.33)$$

Now substitution of Eqs. (2.2.32) and (2.2.33) into Eq. (2.2.27) leads to

$$\begin{aligned}\int_0^t e^{-(\gamma+\gamma_c)(t-t')} \langle \hat{f}_a^\dagger(t) \hat{f}_a(t') \rangle dt' + \int_0^t e^{-(\gamma+\gamma_c)(t-t')} \langle \hat{f}_a^\dagger(t') \hat{f}_a(t) \rangle dt' \\ = (\gamma + \gamma_c)N \langle \hat{N}_a \rangle + r_{da}N \langle \hat{N}_d \rangle\end{aligned}\quad (2.2.34)$$

and on assuming

$$\langle \hat{f}_a^\dagger(t) \hat{f}_a(t') \rangle = \langle \hat{f}_a^\dagger(t') \hat{f}_a(t) \rangle, \quad (2.2.35)$$

one obtains

$$2 \int_0^t e^{-(\gamma+\gamma_c)(t-t')} \langle \hat{f}_a^\dagger(t) \hat{f}_a(t') \rangle dt' = (\gamma + \gamma_c)N \langle \hat{N}_a \rangle + r_{da}N \langle \hat{N}_d \rangle. \quad (2.2.36)$$

Taking into account Eqs. (2.1.88) and (2.1.91), we can rewrite Eq. (2.2.36) in the form

$$\int_0^t e^{-(\gamma+\gamma_c)(t-t')} \langle \hat{f}_a^\dagger(t') \hat{f}_a(t) \rangle dt' = \frac{1}{2} \left(\frac{2\gamma r_{da}}{(\gamma + r_{da})} N^2 \right). \quad (2.2.37)$$

On account of the relation described by Eq. (2.2.19), Eq. (2.2.37) reduces to

$$\langle \hat{f}_a^\dagger(t) \hat{f}_a(t') \rangle = \frac{2\gamma r_{da}}{\gamma + r_{da}} N^2 \delta(t - t'). \quad (2.2.38)$$

Once more, using the relation

$$\frac{d}{dt} \langle \hat{m}_a(t) \hat{m}_a^\dagger(t) \rangle = \left\langle \frac{d\hat{m}_a(t)}{dt} \hat{m}_a^\dagger(t) \right\rangle + \left\langle \hat{m}_a(t) \frac{d\hat{m}_a^\dagger(t)}{dt} \right\rangle \quad (2.2.39)$$

along with Eq. (2.1.79) and its conjugate, we see that

$$\frac{d}{dt} \langle \hat{m}_a(t) \hat{m}_a^\dagger(t) \rangle = -2(\gamma + \gamma_c) \langle \hat{m}_a(t) \hat{m}_a^\dagger(t) \rangle + \langle \hat{m}_a(t) \hat{f}_a^\dagger(t) \rangle + \langle \hat{f}_a(t) \hat{m}_a^\dagger(t) \rangle, \quad (2.2.40)$$

and taking into account of Eq. (2.1.41), one obtains

$$\frac{d}{dt} \langle \hat{N}_b(t) \rangle = -2(\gamma + \gamma_c) \langle \hat{N}_b(t) \rangle + \frac{1}{N} \left(\langle \hat{m}_a(t) \hat{f}_a^\dagger(t) \rangle + \langle \hat{f}_a(t) \hat{m}_a^\dagger(t) \rangle \right). \quad (2.2.41)$$

Hence comparison of Eqs. (2.1.77) and (2.2.41) yields

$$\langle \hat{m}_a(t) \hat{f}_a^\dagger(t) \rangle + \langle \hat{f}_a(t) \hat{m}_a^\dagger(t) \rangle = (\gamma + \gamma_c)N \langle \hat{N}_b \rangle + \gamma_c N \langle \hat{N}_a \rangle. \quad (2.2.42)$$

With the aid of Eqs. (2.1.88) and (2.1.89) we can rewrite Eq. (2.2.42) as

$$\langle \hat{m}_a(t) \hat{f}_a^\dagger(t) \rangle + \langle \hat{f}_a(t) \hat{m}_a^\dagger(t) \rangle = \left(\frac{2\gamma\gamma_c r_{da}}{(\gamma + \gamma_c)(\gamma + r_{da})} \right) N^2. \quad (2.2.43)$$

Multiplying Eq. (2.2.28) on the right by $\hat{f}_a^\dagger(t)$ and taking the expectation value of the resulting expression, one finds

$$\langle \hat{m}_a(t) \hat{f}_a^\dagger(t) \rangle = \langle \hat{m}_a(0) \hat{f}_a^\dagger(t) \rangle e^{-(\gamma+\gamma_c)t} + \int_0^t e^{-(\gamma+\gamma_c)(t-t')} \langle \hat{f}_a(t') \hat{f}_a^\dagger(t) \rangle dt'. \quad (2.2.44)$$

Taking into account of Eq. (2.2.31), Eq. (2.2.44) can be rewritten as

$$\langle \hat{m}_a(t) \hat{f}_a^\dagger(t) \rangle = \int_0^t e^{-(\gamma+\gamma_c)(t-t')} \langle \hat{f}_a(t') \hat{f}_a^\dagger(t) \rangle dt'. \quad (2.2.45)$$

Similarly, multiplying the conjugate of Eq. (2.2.28) on the left by $\hat{f}_a(t)$ and taking the expectation value of the resulting expression and with the aid of Eq. (2.2.31), we arrive at

$$\langle \hat{f}_a(t) \hat{m}_a^\dagger(t) \rangle = \int_0^t e^{-(\gamma+\gamma_c)(t-t')} \langle \hat{f}_a(t) \hat{f}_a^\dagger(t') \rangle dt'. \quad (2.2.46)$$

Upon substituting Eqs. (2.2.45) and (2.2.46) into Eq. (2.2.43), we get

$$\begin{aligned} \int_0^t e^{-(\gamma+\gamma_c)(t-t')} \langle \hat{f}_a(t') \hat{f}_a^\dagger(t) \rangle dt' + \int_0^t e^{-(\gamma+\gamma_c)(t-t')} \langle \hat{f}_a(t) \hat{f}_a^\dagger(t') \rangle dt' \\ = \left(\frac{2\gamma\gamma_c r_{da}}{(\gamma+\gamma_c)(\gamma+r_{da})} \right) N^2 \end{aligned} \quad (2.2.47)$$

and on assuming

$$\langle \hat{f}_a(t') \hat{f}_a^\dagger(t) \rangle = \langle \hat{f}_a(t) \hat{f}_a^\dagger(t') \rangle, \quad (2.2.48)$$

one obtains

$$2 \int_0^t e^{-(\gamma+\gamma_c)(t-t')} \langle \hat{f}_a(t) \hat{f}_a^\dagger(t') \rangle dt' = \left(\frac{2\gamma\gamma_c r_{da}}{(\gamma+\gamma_c)(\gamma+r_{da})} \right) N^2. \quad (2.2.49)$$

On account of the relation given by Eq. (2.2.19) and Eq. (2.2.49), we see that

$$\langle \hat{f}_a(t) \hat{f}_a^\dagger(t') \rangle = \left(\frac{2\gamma\gamma_c r_{da}}{(\gamma+\gamma_c)(\gamma+r_{da})} \right) N^2 \delta(t-t'). \quad (2.2.50)$$

Following a similar procedure, we can easily establish that

$$\langle \hat{f}_b(t) \hat{f}_b^\dagger(t') \rangle = \left(\frac{\gamma_c^2 r_{da} (2\gamma + \gamma_c)}{(\gamma + \gamma_c)^2 (\gamma + r_{da})} \right) N^2 \delta(t - t'), \quad (2.2.51)$$

$$\langle \hat{f}_b^\dagger(t) \hat{f}_b(t') \rangle = \left(\frac{\gamma \gamma_c r_{da} (2\gamma + \gamma_c)}{(\gamma + \gamma_c)^2 (\gamma + r_{da})} \right) N^2 \delta(t - t'), \quad (2.2.52)$$

$$\langle \hat{f}_a(t) \hat{f}_a(t') \rangle = 0, \quad (2.2.53)$$

$$\langle \hat{f}_b(t) \hat{f}_b(t') \rangle = 0, \quad (2.2.54)$$

$$\langle \hat{f}_a(t) \hat{f}_b(t') \rangle = 0, \quad (2.2.55)$$

$$\langle \hat{f}_a^\dagger(t) \hat{f}_b(t') \rangle = - \left(\frac{\gamma \gamma_c r_{da}}{(\gamma + \gamma_c) (\gamma + r_{da})} \right) N^2 \delta(t - t'). \quad (2.2.56)$$

It is worth mentioning that expressions (2.2.23), (2.2.38), (2.2.50), (2.2.51), (2.2.52), (2.2.53), (2.2.54), (2.2.55) and (2.2.56) describe the correlation properties of the atomic noise operators $\hat{f}_a(t)$ and $\hat{f}_b(t)$.

Chapter 3

Photon Statistics

In this chapter we study the statistical properties of a light produced by degenerate three-level laser. To this end, we calculate the mean and variance of the photon number.

Employing the relation

$$\hat{m} = \hat{m}_a + \hat{m}_b \quad (3.0.1)$$

and taking into account Eqs. (2.1.41) - (2.1.47), it can be established that

$$\hat{m}^\dagger \hat{m} = N(\hat{N}_a + \hat{N}_b), \quad (3.0.2)$$

$$\hat{m} \hat{m}^\dagger = N(\hat{N}_b + \hat{N}_c), \quad (3.0.3)$$

$$\hat{m}^2 = N\hat{m}_c. \quad (3.0.4)$$

3.1 Mean of the photon number

Here we calculate the mean photon number of light produced by the three-level laser employing the solution of Eq. (2.1.33). With the atoms considered to be initially in the ground level, the expectation value of the solution of Eq. (3.0.1) happens to be

$$\langle \hat{m}_a(t) \rangle = 0 \quad (3.1.1)$$

and

$$\langle \hat{m}_b(t) \rangle = 0. \quad (3.1.2)$$

Then on account of these results, we see that

$$\langle \hat{m}(t) \rangle = 0. \quad (3.1.3)$$

On the other hand, the solution of Eq. (2.1.33) is expressible as

$$\hat{a}(t) = \hat{a}(0)e^{-\frac{\kappa}{2}t} + \frac{g}{\sqrt{N}}e^{-\frac{\kappa}{2}t} \int_0^t e^{-\frac{\kappa}{2}t'} \hat{m}(t') dt' + e^{-\frac{\kappa}{2}t} \int_0^t e^{-\frac{\kappa}{2}t'} \hat{G}(t') dt'. \quad (3.1.4)$$

It then follows that

$$\hat{a}^\dagger(t) = \hat{a}^\dagger(0)e^{-\frac{\kappa}{2}t} + \frac{g}{\sqrt{N}}e^{-\frac{\kappa}{2}t} \int_0^t e^{-\frac{\kappa}{2}t''} \hat{m}^\dagger(t'') dt'' + e^{-\frac{\kappa}{2}t} \int_0^t e^{-\frac{\kappa}{2}t''} \hat{G}^\dagger(t'') dt''. \quad (3.1.5)$$

Now in view of Eqs. (3.1.3) and (2.2.2) and the assumption that the cavity light is initially in a vacuum state, the expectation values of Eqs. (3.1.4) and (3.1.5) becomes

$$\langle \hat{a}(t) \rangle = 0 \quad (3.1.6)$$

and

$$\langle \hat{a}^\dagger(t) \rangle = 0. \quad (3.1.7)$$

We observe on the basis of Eqs. (2.1.33) and (3.1.6) that \hat{a} is a Gaussian variable with zero mean.

In view of Eqs. (3.1.4) and (3.1.5), the mean photon number of the cavity mode is

$$\begin{aligned}
\bar{n}(t) = \langle \hat{a}^\dagger(t)\hat{a}(t) \rangle &= \langle \hat{a}^\dagger(0)\hat{a}(0) \rangle e^{-\kappa t} + \frac{g}{\sqrt{N}} e^{-\kappa t} \int_0^t e^{\kappa t'} \langle \hat{a}^\dagger(0)\hat{m}(t') \rangle dt' \\
&+ e^{-\kappa t} \int_0^t e^{\kappa t'} \langle \hat{a}^\dagger(0)\hat{G}(t') \rangle dt' + \frac{g}{\sqrt{N}} e^{-\kappa t} \int_0^t e^{\kappa t''} \langle \hat{m}^\dagger(t'')\hat{a}(0) \rangle dt'' \\
&+ \frac{g^2}{N} e^{-\kappa t} \int_0^t \int_0^t e^{\frac{\kappa}{2}(t'+t'')} \langle \hat{m}^\dagger(t'')\hat{m}(t') \rangle dt' dt'' \\
&+ \frac{g}{\sqrt{N}} e^{-\kappa t} \int_0^t \int_0^t e^{\frac{\kappa}{2}(t'+t'')} \langle \hat{m}^\dagger(t'')\hat{G}(t') \rangle dt' dt'' \\
&+ e^{-\kappa t} \int_0^t e^{\frac{\kappa}{2}(t')} \langle \hat{G}^\dagger(t'')\hat{a}(0) \rangle dt'' \\
&+ \frac{g}{\sqrt{N}} e^{-\kappa t} \int_0^t \int_0^t e^{\frac{\kappa}{2}(t'+t'')} \langle \hat{G}^\dagger(t'')\hat{m}(t') \rangle dt' dt'' \\
&+ e^{-\kappa t} \int_0^t \int_0^t e^{\frac{\kappa}{2}(t'+t'')} \langle \hat{G}^\dagger(t'')\hat{G}(t') \rangle dt' dt''. \tag{3.1.8}
\end{aligned}$$

Since the expectation value of the cavity mode initially in the vacuum state is zero and the expectation values of the product of cavity mode noise operators and atomic operators are zero, we can write

$$\langle \hat{a}^\dagger(0)\hat{a}(0) \rangle = \langle \hat{a}^\dagger(0) \rangle \langle \hat{a}(0) \rangle = 0, \tag{3.1.9}$$

$$\langle \hat{m}^\dagger(t'')\hat{G}(t') \rangle = \langle \hat{G}^\dagger(t'')\hat{m}(t') \rangle = 0 \tag{3.1.10}$$

and at initial time the atomic and cavity mode operators are not correlated. Thus

$$\langle \hat{m}^\dagger(t'')\hat{a}(0) \rangle = \langle \hat{m}^\dagger(t'') \rangle \langle \hat{a}(0) \rangle = 0, \quad \langle \hat{a}^\dagger(0)\hat{m}(t') \rangle = \langle \hat{a}^\dagger(0) \rangle \langle \hat{m}(t') \rangle = 0. \tag{3.1.11}$$

With the aid of Eqs. (3.1.9), (3.1.10) and (3.1.11), Eq. (3.1.8) is expressible as

$$\begin{aligned}
\bar{n}(t) &= e^{-\kappa t} \int_0^t e^{\kappa t'} \langle \hat{a}^\dagger(0)\hat{G}(t') \rangle dt' + \frac{g^2}{N} e^{-\kappa t} \int_0^t \int_0^t e^{\frac{\kappa}{2}(t'+t'')} \langle \hat{m}^\dagger(t'')\hat{m}(t') \rangle dt' dt'' \\
&+ e^{-\kappa t} \int_0^t e^{\frac{\kappa}{2}(t')} \langle \hat{G}^\dagger(t'')\hat{a}(0) \rangle dt'' + e^{-\kappa t} \int_0^t \int_0^t e^{\frac{\kappa}{2}(t'+t'')} \langle \hat{G}^\dagger(t'')\hat{G}(t') \rangle dt' dt''. \tag{3.1.12}
\end{aligned}$$

Taking into account Eq. (2.2.21) and assuming that the cavity mode noise operator at some time has no effect on cavity mode operators at an earlier time, Eq. (3.1.12) reduces to

$$\bar{n}(t) = \frac{g^2}{N} e^{-\kappa t} \int_0^t \int_0^t e^{\frac{\kappa}{2}(t'+t'')} \langle \hat{m}^\dagger(t'') \hat{m}(t') \rangle dt' dt'' \quad (3.1.13)$$

or

$$\bar{n}(t) = \langle \hat{a}'^\dagger(t) \hat{a}'(t) \rangle, \quad (3.1.14)$$

where

$$\hat{a}'(t) = \frac{g}{\sqrt{N}} e^{-\frac{\kappa}{2}t} \int_0^t e^{\frac{\kappa}{2}t'} \hat{m}(t') dt' \quad (3.1.15)$$

and

$$\hat{a}'^\dagger(t) = \frac{g}{\sqrt{N}} e^{-\frac{\kappa}{2}t} \int_0^t e^{\frac{\kappa}{2}t''} \hat{m}^\dagger(t'') dt''. \quad (3.1.16)$$

Now taking the time derivative of Eqs. (3.1.15) and (3.1.16) and applying the relation

$$\frac{d}{dx} \int_a^x f(x, x') dx' = f(x, x) - f(x, a) + \int_a^x \frac{d}{dx} f(x, x') dx', \quad (3.1.17)$$

we obtain

$$\frac{d}{dt} \hat{a}'(t) = -\frac{\kappa}{2} \hat{a}'(t) + \frac{g}{\sqrt{N}} \hat{m}(t) - \frac{g}{\sqrt{N}} \hat{m}(0) e^{-\frac{\kappa}{2}t} \quad (3.1.18)$$

and

$$\frac{d}{dt} \hat{a}'^\dagger(t) = -\frac{\kappa}{2} \hat{a}'^\dagger(t) + \frac{g}{\sqrt{N}} \hat{m}^\dagger(t) - \frac{g}{\sqrt{N}} \hat{m}^\dagger(0) e^{-\frac{\kappa}{2}t}. \quad (3.1.19)$$

Applying the large-time approximation scheme to Eqs. (3.1.18) and (3.1.19), we see that

$$\hat{a}'(t) = \frac{2g}{\kappa\sqrt{N}} \hat{m}(t) \quad (3.1.20)$$

and

$$\hat{a}'^\dagger(t) = \frac{2g}{\kappa\sqrt{N}} \hat{m}^\dagger(t). \quad (3.1.21)$$

With the aid of Eqs. (3.1.20) and (3.1.21), Eq. (3.1.16) can be rewritten as

$$\bar{n}(t) = \frac{4g^2}{\kappa^2 N} \langle \hat{m}^\dagger \hat{m} \rangle \quad (3.1.22)$$

Taking into account Eq. (3.0.2), Eq. (3.1.22) can be expressible as

$$\bar{n} = \frac{\gamma_c}{\kappa} \left(\langle \hat{N}_a \rangle + \langle \hat{N}_b \rangle \right). \quad (3.1.23)$$

Therefore, with the aid of Eqs. (2.1.88) and (2.1.89) the mean of photon number can be expressed in terms of total number of atoms N as

$$\bar{n} = \frac{\gamma_c}{\kappa} \left(\frac{\gamma + 2\gamma_c}{(\gamma + \gamma_c)^2 (\gamma + r_{da})} \right) \gamma r_{da} N. \quad (3.1.24)$$

3.2 The variance of the photon number

The variance of the photon number for the a cavity light is expressible as

$$(\Delta n)^2 = \langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}^\dagger \hat{a} \rangle^2 \quad (3.2.1)$$

and using the fact that \hat{a} is Gaussian variable with vanishing mean, we readily get

$$(\Delta n)^2 = \langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{a} \hat{a}^\dagger \rangle + \langle \hat{a}^{\dagger 2} \rangle \langle \hat{a}^2 \rangle. \quad (3.2.2)$$

Once multiplying Eqs. (3.1.4) by (3.1.5) and taking the expectation value of the resulting expressions, we obtain

$$\begin{aligned}
\langle \hat{a}(t)\hat{a}^\dagger(t) \rangle &= \langle \hat{a}(0)\hat{a}^\dagger(0) \rangle e^{-\kappa t} + \frac{g}{\sqrt{N}} e^{-\kappa t} \int_0^t e^{\kappa t''} \langle \hat{a}(0)\hat{m}^\dagger(t'') \rangle dt'' \\
&+ e^{-\kappa t} \int_0^t e^{\kappa t'} \langle \hat{a}(0)\hat{G}^\dagger(t') \rangle dt' + \frac{g}{\sqrt{N}} e^{-\kappa t} \int_0^t e^{\kappa t'} \langle \hat{m}(t')\hat{a}^\dagger(0) \rangle dt' \\
&+ \frac{g^2}{N} e^{-\kappa t} \int_0^t \int_0^t e^{\frac{\kappa}{2}(t'+t'')} \langle \hat{m}(t')\hat{m}^\dagger(t'') \rangle dt' dt'' \\
&+ \frac{g}{\sqrt{N}} e^{-\kappa t} \int_0^t \int_0^t e^{\frac{\kappa}{2}(t'+t'')} \langle \hat{m}(t')\hat{G}^\dagger(t'') \rangle dt' dt'' \\
&+ e^{-\kappa t} \int_0^t e^{\frac{\kappa}{2}t'} \langle \hat{G}(t')\hat{a}^\dagger(0) \rangle dt' \\
&+ \frac{g}{\sqrt{N}} e^{-\kappa t} \int_0^t \int_0^t e^{\frac{\kappa}{2}(t'+t'')} \langle \hat{G}(t')\hat{m}^\dagger(t'') \rangle dt' dt'' \\
&+ e^{-\kappa t} \int_0^t \int_0^t e^{\frac{\kappa}{2}(t'+t'')} \langle \hat{G}(t')\hat{G}^\dagger(t'') \rangle dt' dt''. \tag{3.2.3}
\end{aligned}$$

Because of the expectation value of the cavity mode initially in the vacuum state is zero and with the aid of Eqs. (3.1.9), (3.1.10) and (3.1.11), Eq. (3.2.3) will be rewritten as

$$\begin{aligned}
\langle \hat{a}(t)\hat{a}^\dagger(t) \rangle &= \frac{g^2}{N} e^{-\kappa t} \int_0^t \int_0^t e^{\frac{\kappa}{2}(t'+t'')} \langle \hat{m}(t')\hat{m}^\dagger(t'') \rangle dt' dt'' \\
&+ e^{-\kappa t} \int_0^t \int_0^t e^{\frac{\kappa}{2}(t'+t'')} \langle \hat{G}(t')\hat{G}^\dagger(t'') \rangle dt' dt'', \tag{3.2.4}
\end{aligned}$$

so that in view of the relation given by Eq. (2.2.20), one obtains

$$\begin{aligned}
\langle \hat{a}(t)\hat{a}^\dagger(t) \rangle &= \frac{g^2}{N} e^{-\kappa t} \int_0^t \int_0^t e^{\frac{\kappa}{2}(t'+t'')} \langle \hat{m}(t')\hat{m}^\dagger(t'') \rangle dt' dt'' \\
&+ \kappa N e^{-\kappa t} \int_0^t e^{\frac{\kappa}{2}t'} dt' \int_0^t e^{\frac{\kappa}{2}t''} \delta(t'' - t') dt''. \tag{3.2.5}
\end{aligned}$$

or

$$\langle \hat{a}(t)\hat{a}^\dagger(t) \rangle = \langle \hat{a}'(t)\hat{a}'^\dagger(t) \rangle + \kappa N e^{-\kappa t} \int_0^t e^{\frac{\kappa}{2}t'} dt' \int_0^t e^{\frac{\kappa}{2}t''} \delta(t'' - t') dt'', \tag{3.2.6}$$

in which $\hat{a}'(t)$ and $\hat{a}'^\dagger(t)$ are given by Eqs. (3.1.15) and (3.1.16).

With the aid of Eqs. (3.1.20) and (3.1.21), Eq. (3.2.6) reduces to

$$\langle \hat{a}(t)\hat{a}^\dagger(t) \rangle = \frac{4g^2}{\kappa^2 N} \langle \hat{m}(t)\hat{m}^\dagger(t) \rangle + \kappa N e^{-\kappa t} \int_0^t e^{\frac{\kappa}{2}t'} dt' \int_0^t e^{\frac{\kappa}{2}t''} \delta(t'' - t') dt'', \quad (3.2.7)$$

Now taking Eq. (3.0.3) into account and carrying out the integration, we note that

$$\langle \hat{a}(t)\hat{a}^\dagger(t) \rangle = \frac{\gamma_c}{\kappa} \left(\langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle \right) + N (1 - e^{-\kappa t}), \quad (3.2.8)$$

which at steady-state becomes

$$\langle \hat{a}\hat{a}^\dagger \rangle = \frac{\gamma_c}{\kappa} \left(\langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle \right) + N. \quad (3.2.9)$$

Moreover, on the basis of Eq. (3.1.4), we see that

$$\begin{aligned} \langle \hat{a}^2(t) \rangle &= \langle \hat{a}^2(0) \rangle e^{-\kappa t} + \frac{g}{\sqrt{N}} e^{-\kappa t} \int_0^t e^{\kappa t''} \langle \hat{a}(0)\hat{m}(t'') \rangle dt'' \\ &+ e^{-\kappa t} \int_0^t e^{\kappa t''} \langle \hat{a}(0)\hat{G}(t'') \rangle dt'' + \frac{g}{\sqrt{N}} e^{-\kappa t} \int_0^t e^{\kappa t'} \langle \hat{m}(t')\hat{a}(0) \rangle dt' \\ &+ \frac{g^2}{N} e^{-\kappa t} \int_0^t \int_0^t e^{\frac{\kappa}{2}(t'+t'')} \langle \hat{m}(t')\hat{m}(t'') \rangle dt' dt'' \\ &+ \frac{g}{\sqrt{N}} e^{-\kappa t} \int_0^t \int_0^t e^{\frac{\kappa}{2}(t'+t'')} \langle \hat{m}(t')\hat{G}(t'') \rangle dt' dt'' \\ &+ e^{-\kappa t} \int_0^t e^{\frac{\kappa}{2}t'} \langle \hat{G}(t')\hat{a}(0) \rangle dt' \\ &+ \frac{g}{\sqrt{N}} e^{-\kappa t} \int_0^t \int_0^t e^{\frac{\kappa}{2}(t'+t'')} \langle \hat{G}(t')\hat{m}(t'') \rangle dt' dt'' \\ &+ e^{-\kappa t} \int_0^t \int_0^t e^{\frac{\kappa}{2}(t'+t'')} \langle \hat{G}(t')\hat{G}(t'') \rangle dt' dt''. \end{aligned} \quad (3.2.10)$$

With the aid of Eqs. (3.1.9), (3.1.10), (3.1.11) and (2.2.22), Eq. (3.2.10) reduces to

$$\langle \hat{a}^2(t) \rangle = \frac{g^2}{N} e^{-\kappa t} \int_0^t \int_0^t e^{\frac{\kappa}{2}(t'+t'')} \langle \hat{m}(t')\hat{m}(t'') \rangle dt' dt''. \quad (3.2.11)$$

Taking into account Eq. (3.1.15), Eq. (3.2.11) can be reduces to

$$\langle \hat{a}^2(t) \rangle = \langle \hat{a}'(t)\hat{a}'(t) \rangle \quad (3.2.12)$$

and with the aid of Eqs. (3.1.20) and (3.0.4), one finds that

$$\langle \hat{a}^2(t) \rangle = \frac{\gamma_c}{\kappa} \langle \hat{m}_c \rangle \quad (3.2.13)$$

We now proceed to determine the expectation value of the atomic operator \hat{m}_c . We assume that the state vector of three-level atom, put in the form

$$|\psi_j\rangle = C_a|a_j\rangle + C_b|b_j\rangle + C_c|c_j\rangle, \quad (3.2.14)$$

can be used to determine the expectation value of an atomic operator formed by a pair of identical energy levels or by two distinct energy levels between which transition of the emission of photon is dipole forbidden [6].

One multiplying Eq. (3.2.14) on the left by $\langle a_j|$ and $\langle c_j|$, we note that

$$C_a = \langle a_j|\psi_j\rangle \quad (3.2.15)$$

and

$$C_c = \langle c_j|\psi_j\rangle \quad (3.2.16)$$

respectively.

Now multiplying Eqs. (3.2.15) and (3.2.16) on the right by their conjugates, it then follows that

$$C_a C_a^* = \langle \hat{n}_a^j \rangle \quad (3.2.17)$$

and

$$C_c C_c^* = \langle \hat{n}_c^j \rangle. \quad (3.2.18)$$

Upon multiplying Eq. (3.2.15) by the conjugate of Eq. (3.2.16), we see that

$$\begin{aligned} C_a C_c^* &= \langle a_j|\psi_j\rangle \langle \psi_j|c_j\rangle \\ &= \langle \hat{\sigma}_c^j \rangle. \end{aligned} \quad (3.2.19)$$

In order to have a mathematically manageable analysis, we take $\langle \hat{\sigma}_c^j \rangle$ to be real, so that

$$C_a C_c^* = C_a^* C_c \quad (3.2.20)$$

and on subtracting $C_a C_c$ from both sides of this equation, we have

$$C_a(C_c^* - C_c) + C_c(C_a - C_a^*) = 0. \quad (3.2.21)$$

Hence for all possible values of C_a and C_c , we see that

$$C_a = C_a^* \quad (3.2.22)$$

and

$$C_c = C_c^*. \quad (3.2.23)$$

Now on account of Eqs. (3.2.17) and (3.2.18) together with Eqs. (3.2.22) and (3.2.23), Eq. (3.2.19) is expressible as in the form

$$\langle \hat{\sigma}_c^j \rangle = \sqrt{\langle \hat{\eta}_a^j \rangle \langle \hat{\eta}_c^j \rangle}. \quad (3.2.24)$$

Finally on summing over j from 1 upon N and taking into account Eqs. (2.1.44) and (2.1.46), we note that

$$\langle \hat{m}_c \rangle = \sqrt{\langle \hat{N}_a \rangle \langle \hat{N}_c \rangle}. \quad (3.2.25)$$

Now putting Eq. (3.2.25) into Eq. (3.2.13), we arrive at

$$\langle \hat{a}^2(t) \rangle = \frac{\gamma_c}{\kappa} \sqrt{\langle \hat{N}_a \rangle \langle \hat{N}_c \rangle}. \quad (3.2.26)$$

Following a similar procedure, it can be verified that

$$\langle \hat{a}^{\dagger 2}(t) \rangle = \frac{\gamma_c}{\kappa} \sqrt{\langle \hat{N}_a \rangle \langle \hat{N}_c \rangle}. \quad (3.2.27)$$

Now on account of Eqs. (3.1.23), (3.2.9), (3.2.26) and (3.2.27), we can express Eq. (3.2.2) as

$$(\Delta n)^2 = \left(\frac{\gamma_c}{\kappa}\right)^2 \left(\langle \hat{N}_a \rangle + \langle \hat{N}_b \rangle\right) \left(\langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle\right) + \frac{\gamma_c}{\kappa} \left(\langle \hat{N}_a \rangle + \langle \hat{N}_b \rangle\right) N + \left(\frac{\gamma_c}{\kappa}\right)^2 \left(\langle \hat{N}_a \rangle \langle \hat{N}_c \rangle\right), \quad (3.2.28)$$

or

$$(\Delta n)^2 = \left(\frac{\gamma_c}{\kappa}\right)^2 \left(\langle \hat{N}_a \rangle \langle \hat{N}_b \rangle + 2\langle \hat{N}_a \rangle \langle \hat{N}_c \rangle + \langle \hat{N}_b \rangle^2 + \langle \hat{N}_b \rangle \langle \hat{N}_c \rangle\right) + \frac{\gamma_c}{\kappa} \left(\langle \hat{N}_a \rangle + \langle \hat{N}_b \rangle\right) N. \quad (3.2.29)$$

With the aid of Eqs. (2.1.88), (2.1.89) and (2.1.90) one can express Eq. (3.2.29) in

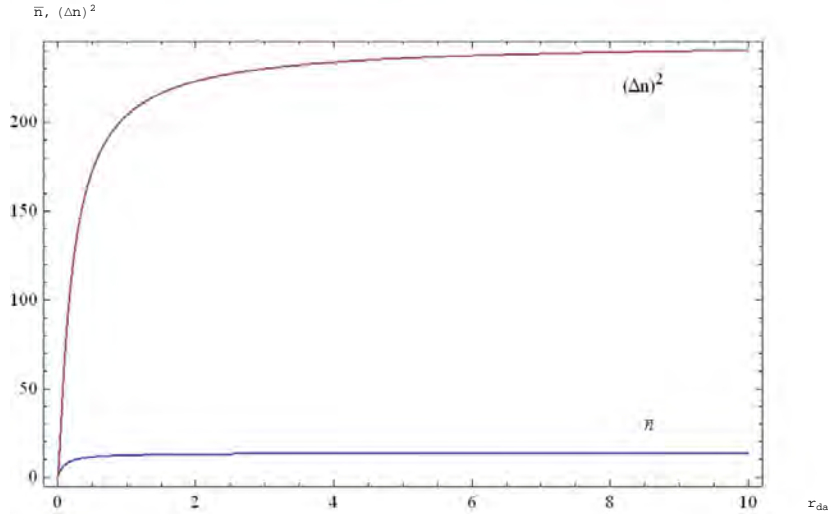


Figure 3.1: A plot of the mean (Eq. (3.1.24)) and variance (Eq. (3.2.30)) of photon number versus pump rate

the form

$$\begin{aligned}
 (\Delta n)^2 &= \left(\frac{\gamma_c}{\kappa}\right)^2 \left(\frac{(\gamma + 3\gamma_c)}{(\gamma + \gamma_c)^3(\gamma + r_{da})^2}\right) \gamma\gamma_c r_{da}^2 N^2 \\
 &+ \left(\frac{\gamma_c}{\kappa}\right) \left(\frac{(\gamma + 2\gamma_c)}{(\gamma + \gamma_c)^2(\gamma + r_{da})}\right) \gamma r_{da} N^2.
 \end{aligned} \tag{3.2.30}$$

Finally, the variance of the photon number can be described in terms of the mean photon number as

$$(\Delta n)^2 = \left(\frac{\gamma_c}{\gamma}\right) \left(\frac{(\gamma + \gamma_c)(\gamma + 3\gamma_c)}{(\gamma + 2\gamma_c)^2}\right) \bar{n}^2 + \bar{n}N, \tag{3.2.31}$$

where \bar{n} is given by Eq. (3.1.24).

From Figure (3.1) we observe that the mean and variance of the photon number increase as the pump rate increases. We also note that the variance of the photon number is greater than the mean of the number and hence the light produced by degenerate three-level laser has super Poissonian photon statistics.

Chapter 4

Quadrature Squeezing

In this chapter we wish to calculate the quadrature variance and quadrature squeezing of a light generated by degenerate three-level laser. Moreover, the power spectrum for the cavity mode is also discussed. The squeezing properties of the cavity light are described by plus and minus quadrature operators defined by [6]

$$\hat{a}_+ = \hat{a}^\dagger + \hat{a} \quad (4.0.1)$$

and

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}). \quad (4.0.2)$$

Taking into account Eqs. (4.0.1) and (4.0.2), it can be readily established that

$$[\hat{a}_+, \hat{a}_-] = 2i[\hat{a}, \hat{a}^\dagger] = 2i[\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}]. \quad (4.0.3)$$

In view of the relations given by Eqs. (3.1.23) and (3.2.9), it can be verified that

$$\langle [\hat{a}_+, \hat{a}_-] \rangle = 2i \left[N + \frac{\gamma_c}{\kappa} \left(\langle \hat{N}_c \rangle - \langle \hat{N}_a \rangle \right) \right]. \quad (4.0.4)$$

Now on the basis of Eq. (4.0.4), the uncertainty relation for \hat{a}_+ and \hat{a}_- can be written as

$$\Delta a_+ \Delta a_- \geq \frac{1}{2} |\langle [\hat{a}_+, \hat{a}_-] \rangle|, \quad (4.0.5)$$

so that using Eq. (4.0.4), we obtain

$$\Delta a_+ \Delta a_- \geq \left| N + \frac{\gamma_c}{\kappa} \left(\langle \hat{N}_c \rangle - \langle \hat{N}_a \rangle \right) \right|. \quad (4.0.6)$$

4.1 The quadrature variance

The variance of the quadrature operators is expressible as

$$(\Delta a_{\pm})^2 = \langle \hat{a}_{\pm}^2 \rangle - \langle \hat{a}_{\pm} \rangle^2, \quad (4.1.1)$$

so that taking into account Eqs. (4.0.1) and (4.0.2), we find that

$$(\Delta a_{\pm})^2 = \pm \langle [\hat{a}^{\dagger} \pm \hat{a}]^2 \rangle \mp [\langle \hat{a}^{\dagger} \rangle \pm \langle \hat{a} \rangle]^2. \quad (4.1.2)$$

Now in view of Eq. (3.1.6), one obtains

$$(\Delta a_{\pm})^2 = \langle \hat{a}^{\dagger} \hat{a} \rangle + \langle \hat{a} \hat{a}^{\dagger} \rangle \pm \langle \hat{a}^{\dagger 2} \rangle \pm \langle \hat{a}^2 \rangle. \quad (4.1.3)$$

Upon introducing Eqs. (3.1.23), (3.2.9), (3.2.26) and (3.2.27), into Eq. (4.1.3) we arrive at

$$(\Delta a_{\pm})^2 = \frac{\gamma_c}{\kappa} \left(\langle \hat{N}_a \rangle + 2\langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle \right) \pm 2\frac{\gamma_c}{\kappa} \sqrt{\langle \hat{N}_a \rangle \langle \hat{N}_c \rangle} + N. \quad (4.1.4)$$

Now with the aid of Eqs. (2.1.88), (2.1.89) and (2.1.90), we can express Eq. (4.1.4) in terms of the total number of three-level atoms in the cavity as

$$\begin{aligned} (\Delta a_{\pm})^2 &= \frac{\gamma_c}{\kappa} \left(\frac{\gamma^2 + \gamma_c^2 + 3\gamma\gamma_c}{(\gamma + \gamma_c)^2(\gamma + r_{da})} \right) r_{da} N + N \\ &\pm 2\frac{\gamma_c}{\kappa} \left(\frac{\gamma_c r_{da}}{(\gamma + \gamma_c)(\gamma + r_{da})} \right) \sqrt{\frac{\gamma}{\gamma + \gamma_c}} N. \end{aligned} \quad (4.1.5)$$

Moreover, we see for $\gamma \ll \gamma_c$, Eq. (4.1.5) reduces to

$$(\Delta a_{\pm})^2 = \frac{\gamma_c}{\kappa} \left(\frac{r_{da}}{\gamma + r_{da}} \right) N + N. \quad (4.1.6)$$

Employing Eq. (4.0.6) along with the relations given by Eqs. (2.1.88) and (2.1.90), the minimum uncertainty relation for \hat{a}_+ and \hat{a}_- can be expressed as

$$\Delta a_+ \Delta a_- = \left| N + \frac{\gamma_c}{\kappa} \left(\frac{(\gamma_c^2 - \gamma^2 - \gamma\gamma_c)}{(\gamma + \gamma_c)^2(\gamma + r_{da})} \right) r_{da} N \right|. \quad (4.1.7)$$

Now for $\gamma \ll \gamma_c$, Eq. (4.1.7) goes over into

$$\Delta a_+ \Delta a_- = \frac{\gamma_c}{\kappa} \left(\frac{r_{da}}{(\gamma + r_{da})} \right) N + N. \quad (4.1.8)$$

On the basis of Eqs. (4.1.6) and (4.1.8), we note that the uncertainties in the plus and minus quadratures are equal and their product satisfies the minimum uncertainty relation for $\gamma \ll \gamma_c$. This shows that for $\gamma \ll \gamma_c$, the light produced by degenerate three-level is coherent [2, 6].

4.2 The quadrature squeezing

In this section we calculate the quadrature squeezing of the cavity light produced by three-level laser. According to the analysis presented in Ref [6], the quadrature squeezing of a cavity light is calculated relative to the quadrature variance of the cavity coherent light. We then define the quadrature squeezing of the cavity light by

$$S = \frac{(\Delta a_{\pm})_c^2 - (\Delta a_-)^2}{(\Delta a_{\pm})_c^2}, \quad (4.2.1)$$

where $(\Delta a_{\pm})_c^2$ given by Eq. (4.1.6), is the quadrature variance of the coherent light.

Hence employing Eqs. (4.1.5) and (4.1.6) along with Eq. (4.2.1), we see that

$$S = \left(\frac{\frac{\gamma_c}{\kappa} \left(\frac{r_{da}}{\gamma + r_{ca}} \right) N - \frac{\gamma_c}{\kappa} \left(\frac{\gamma^2 + \gamma_c^2 + 3\gamma\gamma_c}{(\gamma + \gamma_c)^2(\gamma + r_{da})} \right) r_{da} N + 2\frac{\gamma_c}{\kappa} \left(\frac{\gamma_c r_{da} N}{(\gamma + \gamma_c)(\gamma + r_{da})} \right) \sqrt{\frac{\gamma}{\gamma + \gamma_c}}}{\frac{\gamma_c}{\kappa} \left(\frac{r_{da}}{\gamma + r_{da}} \right) N + N} \right). \quad (4.2.2)$$

This equation can be reduced to

$$S = \frac{\frac{\gamma_c^2 r_{da}}{\gamma + \gamma_c} N \left[2\sqrt{\frac{\gamma}{\gamma + \gamma_c}} - \frac{\gamma}{\gamma + \gamma_c} \right]}{\gamma_c r_{da} N + \kappa(\gamma + r_{da}) N} \quad (4.2.3)$$

or

$$S = \frac{\frac{\gamma_c r_{da}}{1+\eta} \left[2\sqrt{\frac{\eta}{1+\eta}} - \frac{\eta}{1+\eta} \right]}{\gamma_c \left[r_{da} + \kappa \left(\eta + \frac{r_{da}}{\gamma_c} \right) \right]}, \quad (4.2.4)$$

where

$$\eta = \frac{\gamma}{\gamma_c}. \quad (4.2.5)$$

Thus, we observe from Eq. (4.2.4) that unlike the mean photon number, the quadra-

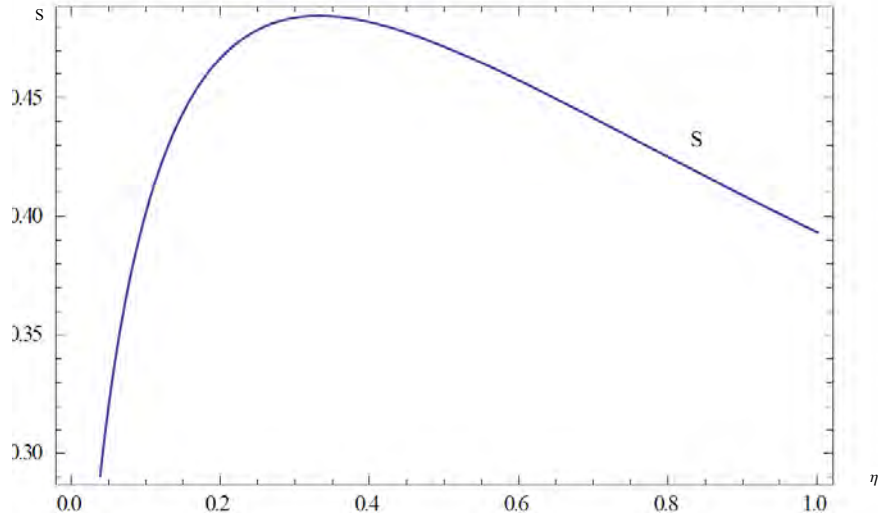


Figure 4.1: A plot of quadrature squeezing (Eq. (4.2.4)) versus η

ture squeezing does not depend on the number of atoms. The plot in Fig. (4.1) shows that the maximum squeezing of the cavity light is 49% below the coherent level when $\eta = 0.33$.

4.3 The power spectrum

We now seek to obtain the power spectrum of the cavity light. The power spectrum of a single-mode light with central frequency ω_0 is defined as [7]

$$P(\omega) = \frac{1}{\pi} \text{Re} \int_0^\infty d\tau e^{i(\omega - \omega_0)\tau} \langle \hat{a}^\dagger(t) \hat{a}(t + \tau) \rangle_{ss}, \quad (4.3.1)$$

where the subscript "ss" stands for the steady-state.

Up on integration both sides of Eq. (4.3.1) over ω , we readily get

$$\int_{-\infty}^\infty P(\omega) d\omega = \bar{n}, \quad (4.3.2)$$

in which \bar{n} is the steady-state mean photon number.

On the basis of this result, we assert that $P(\omega)d\omega$ is the steady-state mean photon number in the frequency interval between ω and $\omega + d\omega$ [6, 7].

We now proceed to determine the two-time correlation function that appears in Eq. (4.3.1) for the cavity light. To this end, we realize that the solution of Eq. (2.1.33) can also be written as

$$\begin{aligned} \hat{a}(t + \tau) = & \hat{a}(t) e^{-\frac{\kappa}{2}\tau} + \frac{g}{\sqrt{N}} e^{-\frac{\kappa}{2}\tau} \int_0^\tau e^{\frac{\kappa}{2}\tau'} \hat{m}(t + \tau') d\tau' \\ & + e^{-\frac{\kappa}{2}\tau} \int_0^\tau e^{-\frac{\kappa}{2}\tau'} \hat{G}(t + \tau') d\tau'. \end{aligned} \quad (4.3.3)$$

On the other hand, the solution of the sum of Eqs. (2.1.79) and (2.1.80) is expressible as

$$\begin{aligned} \hat{m}(t + \tau) = & \hat{m}(t) e^{-(\gamma + \frac{\gamma_c}{2})\tau} + e^{-(\gamma + \frac{\gamma_c}{2})\tau} \int_0^\tau d\tau'' e^{(\gamma + \frac{\gamma_c}{2})\tau''} \left(\frac{\gamma_c}{2} \hat{m}_a(t + \tau'') \right. \\ & \left. + \hat{f}_a(t + \tau'') + \hat{f}_b(t + \tau'') \right) \end{aligned} \quad (4.3.4)$$

or

$$\hat{m}(t + \tau) = \hat{m}(t) e^{-\beta\tau} + e^{-\beta\tau} \int_0^\tau d\tau'' e^{\beta\tau''} \left(\frac{\gamma_c}{2} \hat{m}_a(t + \tau'') + \hat{F}_m(t + \tau'') \right), \quad (4.3.5)$$

where

$$\beta = \gamma + \frac{\gamma_c}{2}, \quad (4.3.6)$$

$$\hat{F}_m(t) = \hat{f}_a(t) + \hat{f}_b(t), \quad (4.3.7)$$

$$\hat{m}(t) = \hat{m}_a(t) + \hat{m}_b(t). \quad (4.3.8)$$

Applying the large-time approximation scheme to Eq. (2.1.79), we get

$$\hat{m}_a(t + \tau) = \frac{1}{(\gamma + \gamma_c)} \hat{f}_a(t + \tau), \quad (4.3.9)$$

and on introducing this into Eq. (4.3.5), there follows

$$\hat{m}(t + \tau') = \hat{m}(t)e^{-\beta\tau'} + e^{-\beta\tau'} \int_0^{\tau'} d\tau'' e^{\beta\tau''} \left(\frac{\gamma_c}{2(\gamma + \gamma_c)} \hat{f}_a(t + \tau'') + \hat{F}_m(t + \tau'') \right). \quad (4.3.10)$$

Now combination of Eqs. (4.3.3) and (4.3.10) yields

$$\begin{aligned} \hat{a}(t + \tau) &= \hat{a}(t)e^{-\frac{\kappa}{2}\tau} + \frac{g}{\sqrt{N}} e^{-\frac{\kappa}{2}\tau} \hat{m}(t) \int_0^\tau d\tau' e^{(\frac{\kappa}{2}-\beta)\tau'} \\ &+ \frac{g}{\sqrt{N}} e^{-\frac{\kappa}{2}\tau} \int_0^\tau e^{(\frac{\kappa}{2}-\beta)\tau'} d\tau' \int_0^{\tau'} e^{\beta\tau''} \left(\frac{\gamma_c}{2(\gamma + \gamma_c)} \hat{f}_a(t + \tau'') + \hat{F}_m(t + \tau'') \right) d\tau'' \\ &+ e^{-\frac{\kappa}{2}\tau} \int_0^\tau e^{\frac{\kappa}{2}\tau'} \hat{G}(t + \tau') d\tau', \end{aligned} \quad (4.3.11)$$

and on carrying out the first integration, we arrive at

$$\begin{aligned} \hat{a}(t + \tau) &= \hat{a}(t)e^{-\frac{\kappa}{2}\tau} + \frac{2g}{\sqrt{N}(\kappa - 2\beta)} \hat{m}(t) [e^{-\beta\tau} - e^{-\frac{\kappa}{2}\tau}] \\ &+ \frac{g}{\sqrt{N}} e^{-\frac{\kappa}{2}\tau} \int_0^\tau d\tau' \int_0^{\tau'} e^{((\frac{\kappa}{2}-\beta)\tau' + \beta\tau'')} \left(\frac{\gamma_c}{2(\gamma + \gamma_c)} \hat{f}_a(t + \tau'') + \hat{F}_m(t + \tau'') \right) d\tau'' \\ &+ e^{-\frac{\kappa}{2}\tau} \int_0^\tau e^{\frac{\kappa}{2}\tau'} \hat{G}(t + \tau') d\tau'. \end{aligned} \quad (4.3.12)$$

Now multiplying both sides of this equation by $\hat{a}^\dagger(t)$ from the left and taking the expectation value of the resulting expression, we have

$$\begin{aligned}
\langle \hat{a}^\dagger(t)\hat{a}(t+\tau) \rangle &= \langle \hat{a}^\dagger(t)\hat{a}(t) \rangle e^{-\frac{\kappa}{2}\tau} + \frac{2g}{\sqrt{N}(\kappa-2\beta)} \langle \hat{a}^\dagger(t)\hat{m}(t) \rangle [e^{-\beta\tau} - e^{-\frac{\kappa}{2}\tau}] \\
&+ \frac{g}{\sqrt{N}} e^{-\frac{\kappa}{2}\tau} \int_0^\tau d\tau' \int_0^{\tau'} e^{((\frac{\kappa}{2}-\beta)\tau'+\beta\tau'')} \left(\frac{\gamma_c}{2(\gamma+\gamma_c)} \langle \hat{a}^\dagger(t)\hat{f}_a(t+\tau'') \rangle \right) \\
&+ \frac{g}{\sqrt{N}} e^{-\frac{\kappa}{2}\tau} \int_0^\tau d\tau' \int_0^{\tau'} e^{((\frac{\kappa}{2}-\beta)\tau'+\beta\tau'')} \left(\langle \hat{a}^\dagger(t)\hat{F}_m(t+\tau'') \rangle \right) d\tau'' \\
&+ e^{-\frac{\kappa}{2}\tau} \int_0^\tau e^{\frac{\kappa}{2}\tau'} \langle \hat{a}^\dagger(t)\hat{G}(t+\tau') \rangle d\tau'. \tag{4.3.13}
\end{aligned}$$

Since a noise operators at some time t has no effect on the cavity mode operator at an earlier time, Eq. (4.3.13) reduces to

$$\langle \hat{a}^\dagger(t)\hat{a}(t+\tau) \rangle = \langle \hat{a}^\dagger(t)\hat{a}(t) \rangle e^{-\frac{\kappa}{2}\tau} + \frac{2g}{\sqrt{N}(\kappa-2\beta)} \langle \hat{a}^\dagger(t)\hat{m}(t) \rangle [e^{-\beta\tau} - e^{-\frac{\kappa}{2}\tau}]. \tag{4.3.14}$$

Applying once more the large-time approximation, one gets from Eq. (2.1.33)

$$\hat{m}(t) = \frac{\kappa\sqrt{N}}{2g}\hat{a}(t) - \sqrt{N}\hat{G}(t). \tag{4.3.15}$$

With this substituted into Eq. (4.3.14), there emerges

$$\langle \hat{a}^\dagger(t)\hat{a}(t+\tau) \rangle = \langle \hat{a}^\dagger(t)\hat{a}(t) \rangle e^{-\frac{\kappa}{2}\tau} + \frac{\kappa}{(\kappa-2\beta)} \langle \hat{a}^\dagger(t)\hat{a}(t) \rangle [e^{-\beta\tau} - e^{-\frac{\kappa}{2}\tau}], \tag{4.3.16}$$

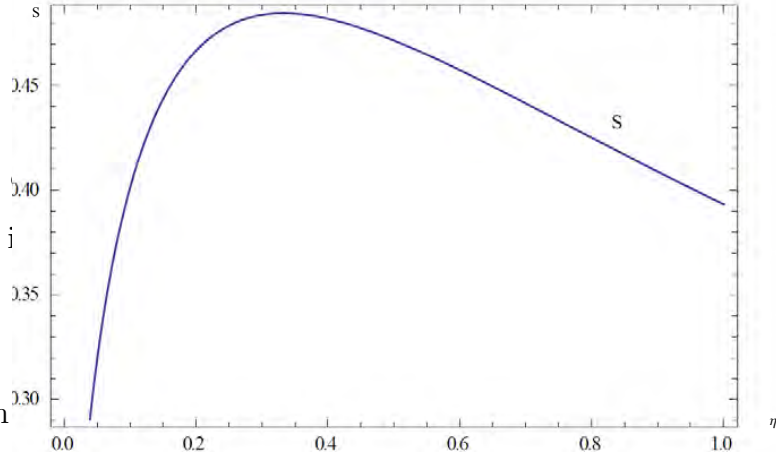
or

$$\langle \hat{a}^\dagger(t)\hat{a}(t+\tau) \rangle = \frac{\kappa\bar{n}}{(\kappa-2\beta)} e^{-\beta\tau} - \frac{2\beta\bar{n}}{(\kappa-2\beta)} e^{-\frac{\kappa}{2}\tau}. \tag{4.3.17}$$

Finally, on substituting Eq. (4.3.17) into Eq. (4.3.1) and carrying out the integration, we readily arrive at

$$P(\omega) = \frac{\kappa\bar{n}}{(\kappa-2\beta)} \left[\frac{\beta/\pi}{(\omega-\omega_0)^2 + \beta^2} \right] - \frac{2\beta\bar{n}}{(\kappa-2\beta)} \left[\frac{\kappa/2\pi}{(\omega-\omega_0)^2 + (\frac{\kappa}{2})^2} \right]. \tag{4.3.18}$$

In view
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(4.3.19)

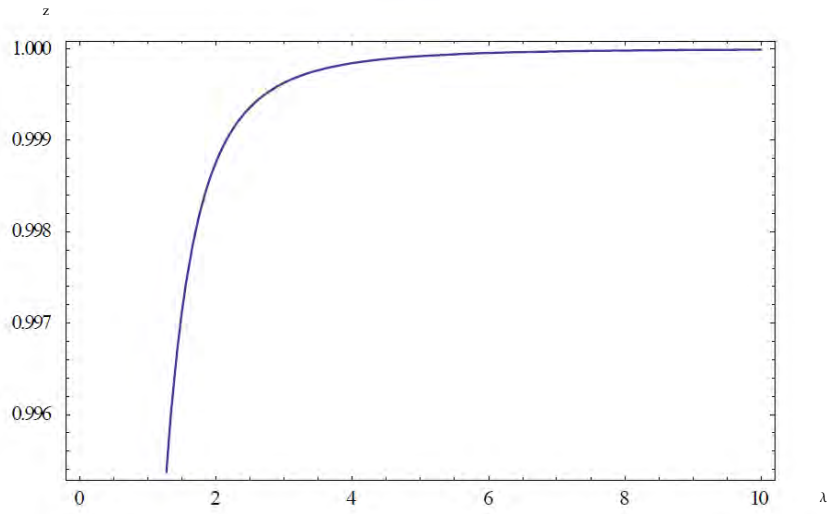


Figure 4.2: A plot of $z(\lambda)$ (Eq. (4.3.22)) versus λ

Therefore, upon substituting Eq. (4.3.18) into Eq. (4.3.19) and carrying out the integration applying the relation

$$\int_{-\lambda}^{\lambda} \frac{dx}{x^2 + a^2} = \frac{2}{a} \tan^{-1} \left(\frac{\lambda}{a} \right), \quad (4.3.20)$$

we arrive at

$$\bar{n}_{\pm\lambda} = \bar{n}z(\lambda), \quad (4.3.21)$$

where

$$z(\lambda) = \frac{2\kappa/\pi}{\kappa - 2\beta} \tan^{-1} \left(\frac{\lambda}{\beta} \right) - \frac{4\beta/\pi}{\kappa - 2\beta} \tan^{-1} \left(\frac{2\lambda}{\kappa} \right). \quad (4.3.22)$$

From Fig. (4.2) the value of $z(\lambda)$ described by Eq. (4.3.22) approaches 1 for a relatively small values of λ . This indicates that the total mean photon number is confined in a relatively small frequency interval near the central frequency.

Chapter 5

Conclusion

In this thesis we have studied the statistical and squeezing properties of the light produced by degenerate three-level laser in a cavity coupled to vacuum reservoir via a single port-mirror. We considered a three-level laser in which the three-level atoms available in a cavity are pumped from the bottom to the top level at a rate r_{da} . Employing the master equation for the system under consideration, we obtained the quantum Langevin equations for the cavity mode and atomic operators. Employing the solutions of the quantum Langevin equations, we have calculated the mean photon number, the variance of the photon number, the quadrature variance and the quadrature squeezing as well as the power spectrum for the cavity mode. From Fig. (3.1) we observe that the mean and variance of the photon number increase as the pump rate increases. We also note that the variance of the photon number is greater than the mean photon number and hence the light produced by degenerate three-level laser has super Poissonian photon statistics. The plot in Fig. (4.1) shows that the maximum squeezing of the cavity light is 49% below the coherent level when $\eta = 0.33$. From Fig. (4.2) the value of $z(\lambda)$ described by Eq. (4.3.22) approaches 1 for a relatively small values of λ . This indicates that the total mean photon number is confined in a relatively small frequency interval near the central frequency.

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DECLARATION

I hereby declare that this Thesis is my original work and has not been presented for a degree in any other universities, and that all sources of material used for the Thesis have been duly acknowledged.

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This Thesis has been submitted to for examination with my approval as University advisor.

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