



**CALCULATION OF ELECTRON SELF ENERGY
AND VACCUM POLARIZATION BY USING THE
METHOD OF DIMENSIONAL
REGULARIZATION**

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Abstract

In this paper interacting quantum fields, electron self-energy, vacuum polarization, methods of calculating electron and photon self-energy and Wick rotation are reviewed. These self-energy calculating methods include Pauli-Villars regularization and dimensional regularization. Electron self-energy is calculated by using Pauli-Villars regularization method while the photon self-energy is by dimensional regularization method.

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Chapter 1

Introduction

In quantum field theories, we encounter many apparent divergences. Of course all physical quantities are finite, and therefore divergences only at intermediate stages of calculations that get cancelled one or the other way. However, such apparent divergences pose technical problems in dealing with them. Obviously we need some methods to add, subtract, multiply and divide apparently divergent quantities and extract finite answers in the end. To do so, we need to “regulate” the divergences, namely to make apparent divergences manifestly finite so that we can manipulate them.

What regularization does is to introduce a new parameter, let’s say ϵ , to the apparently divergent quantity O . The quantity is now a function of ϵ , $O(\epsilon)$. It is supposed to reduce the original quantity in the limit $\epsilon \rightarrow 0$,

$$\lim_{\epsilon \rightarrow 0} O(\epsilon) = O$$

we recover the apparent divergence. Yet for finite but very small ϵ , the quantity is finite, $|O(\epsilon)| < \infty$. Then we say the divergent quantity O is regularized by the regulator ϵ .

One of the main issues of the regularization is that a regulator tends to break certain symmetries of the quantity. The usefulness of a regulator depends on what symmetries it retains, how easy it is to deal with, how widely it can be used etc. In this thesis we will see how Pauli-Villars Regularization and Dimensional Regularization methods are applied in manipulating the divergences in electron self-energy and vacuum polarization.

In a divergent loop integrals Pauli-Villars regularization subtracts off the same loop integral with a much larger mass. This method has the benefit of maintaining the translational invariance in the momentum space. It also maintains the gauge invariance. The subtracted piece is regarded as a contribution of another field (Pauli-Villar field) with the same quantum numbers as the original field, but has the opposite statistics. For instance, in Φ^4 theory, the Pauli-Villar field Φ is fermion and is Grassmann-odd number in path integral even though it has the same quantum number and hence is a scalar field. Of course a fermionic scalar field would break spin-statistics theorem and leads to violation of causality and/or positivity of energy. However as long as we take its mass M very large and deal with physics at energy scales much lower than M such diseases do not appear.

The other method of regulation is Dimensional regularization. This regularization has many advantages: translational invariance in momentum space, gauge invariance. Its drawback is; it is highly artificial to discuss space-time in non-integer dimensions, it is tricky to extend the Dirac gamma matrices to non-integer dimensions. [www.wikipedia encyclopedia]

Chapter 2

Theory of Interacting Quantum Fields

Quantum fluctuations will induce complicated admixtures of particles of different kinds on the level of interacting quantum fields. A free field operator has a very simple dynamics; the expansion into eigenmodes simply leads to constant creation and annihilation operators. The field operator of an interacting field theory, however, can no longer be described in this simple way; in general it will develop in time into a superposition of creation and annihilation operators for infinitely many virtual particles and antiparticles. As a consequence there is no way to find exact solutions for interacting field theories (except for overly simplified mathematical models). Nevertheless, under favourable conditions, i.e., if the strength of the interaction is small, it is possible to obtain reliable approximate solutions. [www.wikipedia encyclopedia]

2.1 Interacting fields

The Schrodinger equation

$$i\hbar \frac{\partial \Psi(t)}{\partial t} = H \Psi(t) \quad (2.1.1)$$

is usually the starting point of the analysis of the interaction problem in quantum mechanics. The complete hamiltonian H of the system is written as the sum of the free motion hamiltonian H_0 and the interaction hamiltonian H_1 :

$$H = H_0 + H_1 \quad (2.1.2)$$

Perturbation theory is usually employed because an exact solution of the Schrodinger Equation (2.1.1) cannot, as a rule, be obtained. The initial step then is to neglect the interaction term, and this yields the problem

$$i\hbar \frac{\partial \Psi_0(t)}{\partial t} = H_0 \Psi_0(t) \quad (2.1.3)$$

which can be solved exactly. In theory of quantized fields, as in nonrelativistic quantum mechanics, an exact solution can be obtained only for relatively simple models. This is why it is again necessary to resort to perturbation theory based on the initial approximation of noninteracting particles. One then has to introduce in to the analysis corresponding idealized free fields, and consider the interaction as an additional factor that has little effect on the properties of the dynamic system and can be “turned on” or “turned off”. At first sight, the problem presented in this form appears to be an objectionable. In fact, elementary particles interact stringly only in sufficiently close encounters (in collision processes). It would therefore seem natural to suppose that the interaction between fields at large distances is unimportant and, to a particular approximation, can be neglected, and real particles can be looked upon as free. However, even free particles interact with the fields produced by them (this is well known, for example, from classical electrodynamics). In quantum case, this situation corresponds to interaction between the particles and vacuum which acts as a kind of physical medium in which they move (for example the phenomenon of vacuum polarization). Although the interaction may be weak, the fact that the particles have small dimensions ensures that the effects of this interaction turn out to be large (infinite, in the limit of point particles). It follows that the use of the idea of “bare” (i.e., non-interacting, even with vacuum) particles turn out to be unsatisfactory, and it is very desirable to deal, right from the start, with real interacting particles without introducing the artificial concept of fictitious free fields.

The emergence of interesting physical processes can only be expected if there are interactions among fields. From a mathematical point of view this happens if the lagrangian

contains more complicated parts. Such terms may consist of higher powers of the field function (describing self-interactions) or of products involving fields of different kinds. Within the framework of classical theory this leads to a non-linear field equation or coupled systems of field equations which are difficult to solve. [4]

2.2 The interaction picture

As we know from ordinary quantum mechanics the interaction picture theory can be formulated in several different, but completely equivalent, “representations” or “pictures” that differ in the way the time dependence is treated. This ambiguity arises since all the physical observables are expectation values or matrix elements of operators \hat{O} evaluated with state vectors $|\alpha\rangle$. These matrix elements remain unchanged if the operators and the state vectors are jointly subjected to a unitary transformation.

The two extreme choices that can be constructed in this way are the Schrodinger picture (operators are constant, state vectors are time dependent) and the Heisenberg picture (state vectors are constant, operators are time dependent). In the context of quantum field theory one often prefers the Heisenberg picture. In the Heisenberg picture the operators satisfy the Heisenberg equation of motion

$$i\partial_t\hat{O}^H(t) = [\hat{O}^H, \hat{H}] \quad (2.2.1)$$

if \hat{H} is the (time independent) Hamiltonian of the system. We assume that the system is closed, otherwise it might experience an additional time dependence driven by external influences.

A formal solution for the time development of an operator \hat{O} can be written down at once;

$$\hat{O}(t) = e^{i\hat{H}(t)}\hat{H}(0)e^{-i\hat{H}t} \quad (2.2.2)$$

which is easily verified by forming the derivative. Instead of $t = 0$ of course we can choose any other time $t = t_0$ as a point of reference. This only amounts to the replacement of t

by $t - t_0$ in the exponent. The state vectors in the Heisenberg picture are constant, i.e.,

$$|\alpha, t\rangle^H = |\alpha, 0\rangle^H \equiv |\alpha\rangle^H \quad (2.2.3)$$

The time dependence now can be transferred to the state vectors, which leads to the Schrodinger picture. This is achieved through the transformation

$$\hat{O}^s = e^{-i\hat{H}t} \hat{O}^H e^{i\hat{H}t} \quad (2.2.4)$$

$$|\alpha, t\rangle^s = e^{-i\hat{H}t} |\alpha, t\rangle^H$$

Because of equations (2.2.2) and (2.2.3) it is obvious that \hat{O}^s is time independent and the pictures agree at the (arbitrarily chosen) time $t = 0$;

$$\hat{O}^s = \hat{O}^H(0) \quad (2.2.5)$$

$$|\alpha, 0\rangle^s = |\alpha\rangle^H$$

The state vectors are constant and satisfy the Schrodinger equation

$$i\partial_t |\alpha, t\rangle^s = \hat{H} |\alpha, t\rangle^s$$

The unitarity transformation, equation (2.2.5) of course leaves the matrix elements invariant, since

$$\begin{aligned} (\langle \beta, t | \hat{O}^s | \alpha, t \rangle)_s &= (\langle \beta | e^{i\hat{H}t} e^{-i\hat{H}t} \hat{O}^H(t) e^{i\hat{H}t} e^{-i\hat{H}t} | \alpha \rangle)^H \\ &= (\langle \beta | O^H(t) | \alpha \rangle)^H \end{aligned}$$

where the Hamiltonian \hat{H} is a hermitian operator, and thus $(e^{-i\hat{H}t})^+ = (e^{i\hat{H}t})$. It is important that the transition between the two pictures is mediated by a canonical transformation which leaves the commutation relations between operators invariant. If two operators in the Heisenberg picture satisfy the commutation relation

$$[\hat{A}^H, \hat{B}^H] = \hat{C}^H$$

then the corresponding Scrodinger operators are immediately found to satisfy

$$\begin{aligned} [\hat{A}^s, \hat{B}^s]_{\pm} &= e^{-i\hat{H}t}(\hat{A}^H e^{i\hat{H}t} e^{-i\hat{H}t} \hat{B}^H \pm \hat{B}^H e^{i\hat{H}t} e^{-i\hat{H}t} \hat{A}^H) e^{i\hat{H}t} \\ &= e^{-i\hat{H}t}[\hat{A}^H, \hat{B}^H] e^{i\hat{H}t} = e^{-i\hat{H}t} \hat{C}^H e^{i\hat{H}t} = \hat{C}^S \end{aligned}$$

In principle it is possible to formulate the perturbation theory for interacting systems within the Schrodinger picture. In such a formulation the field operators $\hat{\Psi}^s(x)$ will depend only on the spatial coordinate x and not time t . This is a draw back if one is intersted in studying relativistic theories since in this way the (manifest) Lorentz covariance is lost. As a starting point for the development of perturbation theory one assumes that the Hamiltonian of the system under study can be split into two parts; as stated in equation (2.1.2). Here H_0 is typically the Hamiltonian of the system of free fields. For this hamiltonian alone an exact solution can be given. The strategy now is to put this "easy" part of the problem into the definition of the state vectors. Then only the "perturbation" operator H_1 will be visible in the equations of motion. This is achieved by transforming to the interaction picture;

$$\begin{aligned} \hat{O}^I(t) &= e^{i\hat{H}_0^s t} O^s e^{-i\hat{H}_0^s t} \\ |\alpha, t\rangle^I &= e^{i\hat{H}_0^s t} |\alpha, t\rangle^s \end{aligned}$$

with the free Hamiltonian $\hat{H}_0^s = \hat{H}_0(t=0)$.

The connection with the Heisenberg picture is given by the following relations, using equation (2.2.4)

$$\begin{aligned} \hat{O}^I(t) &= e^{i\hat{H}_0^s(t)} e^{-iH(t)} \hat{O}^H(t) e^{iH(t)} e^{-iH_0^s(t)} \\ |\alpha, t\rangle^I &= e^{i\hat{H}_0^s(t)} e^{-i\hat{H}(t)} |\alpha\rangle^H \end{aligned}$$

In the case $\hat{H}_1 = 0$ the interaction picture agrees with the Heisenberg picture. Furthermore, at $t = 0$ all three pictures agree.

$$|\alpha, 0\rangle^I = |\alpha, 0\rangle^s = |\alpha, 0\rangle^H$$

$$\hat{O}^I(0) = \hat{O}^H(0) = \hat{O}^s$$

Since the connection between the three pictures is mediated by unitary transformations, they all lead to the same matrix elements.[6]

2.3 The time evolution operator

Within the Dirac picture the effect of the interaction is of course hidden in the state vectors $|\alpha, t\rangle^I$. Thus the Schrodinger equation is the starting point for a perturbative treatment of the problem of interacting fields. For this we define the time-evolution operator U , which describes the connection between the state vectors at the times t_o and t_1 , i.e.,

$$|\alpha, t_1\rangle^I = \hat{U}(t_1, t_0)|\alpha, t_0\rangle^I$$

With the help of equations $|\alpha, t\rangle^s = e^{-i\hat{H}t}|\alpha\rangle^H$ and $|\alpha, t\rangle^I = e^{i\hat{H}_0^s}|\alpha, t\rangle^s$ we can immediately write down a formal solution of this equation (with $\hat{H}_0 \equiv \hat{H}_0^s$)

$$|\alpha, t_1\rangle^I = e^{i\hat{H}_0 t_1}|\alpha, t_1\rangle^s = e^{i\hat{H}_0 t_1} e^{-i\hat{H}_0(t_1-t_0)}|\alpha, t_0\rangle^s = e^{i\hat{H}_0 t_1} e^{-i\hat{H}_0(t_1-t_0)} e^{-i\hat{H}_0 t_0}|\alpha, t_0\rangle^I,$$

and thus

$$\hat{U}(t_1, t_0) = e^{i\hat{H}_0 t_0} e^{-i\hat{H}(t_1-t_0)} e^{-i\hat{H}_0 t_0} \quad (2.3.1)$$

Since the operators \hat{H}_0 and \hat{H} in general do not commute, the ordering of the factors in equation (2.3.1) is very important. The time evolution operator satisfies a number of fundamental relations. A trivial property is

$$\hat{U}(t_0, t_0) = 1 \quad (2.3.2)$$

Furthermore when two time translations are applied consecutively the following group property holds;

$$\hat{U}(t_2, t_1)\hat{U}(t_1, t_0) = \hat{U}(t_2, t_0)$$

For the special case $t_2 = t_0$, this together with equation (2.3.2) gives an expression for the inverse operator

$$\hat{U}^{-1}(t_0, t_1) = \hat{U}^{-1}(t_1, t_0)$$

Finally \hat{U} turns out to be a unitary operator

$$\hat{U}^+(t_1, t_0) = \hat{U}^{-1}(t_1, t_0) \quad (2.3.3)$$

which guarantees that the normalization of state vectors does not depend on time. Equation (2.3.3) can be derived from the hermitian property of the Hamiltonians \hat{H}_0 and \hat{H}_1 , as we can prove with the help of the formal solution equation (1.3.1)

$$\hat{U}^+(t_1, t_0) = e^{i\hat{H}_0^+ t_0} e^{i\hat{H}^+(t_1-t_0)} e^{-i\hat{H}_0^+ t_1} = (e^{i\hat{H}_0^+ t_1} e^{i\hat{H}(t_0-t_1)} e^{-i\hat{H}_0 t_0})^{-1} = \hat{U}^{-1}(t_1, t_0)$$

We note that $\hat{U}(t, 0) = e^{i\hat{H}_0^s t}$ is just the transformation operator linking the Heisenberg and the interaction pictures;

$$\hat{O}^I(t) = \hat{U}(t, 0) \hat{O}^H(t) \hat{U}^{-1}(t, 0)$$

$$|\alpha, t\rangle^I = \hat{U}(t, 0) |\alpha\rangle^H$$

Since the operator \hat{U} determines the time evolution of the state vector $|\alpha, t\rangle^I$ satisfies

$$i\partial_t \hat{U}(t, t_0) = \hat{H}^I(t) \hat{U}(t, t_0)$$

with the boundary condition (2.3.2) In quantum field theory as in ordinary quantum mechanical scattering theory it proves advantageous to transform the differential equation

$$i\partial_t |\alpha, t\rangle^I = H^I |\alpha, t\rangle^I$$

into an equivalent integral equation. Within the boundary condition, equation (2.3.2), this integral equation reads

$$\hat{U}(t, t_0) = 1 + (-i) \int_{t_0}^t dt' \hat{H}^I(t') \hat{U}(t', t_0) \quad (2.3.4)$$

which is immediately verified by differentiating with respect to time. Equation (2.3.4) is an integral equation of Volterra type (i.e., the independent variable enters as an integral

boundary). Equations of this type can be solved by iteration under quite general conditions. The process of successive re-insertion of the left hand side of equation (2.3.4) leads to the Neumann series

$$\begin{aligned} \hat{U}(t, t_0) = & 1 + (-i) \int_{t_0}^t dt_1 \hat{H}_I(t_1) + (-i)^2 \int_{t_0}^t dt_1 \int_{t_0}^t dt_2 \hat{H}_I(t_1) \hat{H}_I(t_2) \\ & + \dots \int_{t_0}^{t_{n-1}} dt_n \hat{H}_I(t_2) \dots \hat{H}_I(t_n) + \dots \end{aligned} \quad (2.3.5)$$

The Neumann series consists of multiple integrals involving products of the interaction Hamiltonian $\hat{H}^I(t_i)$ taken at different times. We notice that the time arguments are sorted in descending order. The appearance of these mutually dependent upper boundaries makes these multiple integrals (equation(2.3.5)) quite difficult to handle. Fortunately, however, following an idea by Dyson the integrations can be rewritten such that they all cover the full time interval $[t_0, t]$. To achieve this we need the time-ordered product (or Dyson product) as a tool. The operators in the product put in the order of descending time argument;

$$T(\hat{H}(t_1)\hat{H}(t_2)\hat{H}(t_3)\dots\dots\dots\hat{H}(t_n)) = \hat{H}(t_{i_1})\hat{H}(t_{i_2})\hat{H}(t_{i_3})\dots\dots\dots\hat{H}(t_{i_n}) \quad (2.3.6)$$

where

$$t_{i_1} \geq t_{i_2} \geq t_{i_3} \dots \dots \dots \geq t_{i_n}$$

The concept of time ordering can be applied to all types of products of operators, in particular to field operators ($\hat{\Phi}(x), \hat{\Psi}(x), \hat{A}_\mu(x)$) and their canonically conjugate fields. There may arise problems if non-commuting operators taken at equal time arguments are involved. For a general definition one adds the rule that a minus sign arises for each interchange of fermion operators upon time ordering. Such sign factors do not arise in equation (2.3.6) since the Hamiltonian is of a scalar nature and therefore always consists of an even number of fermion operators (which involve half integer spin). The second term in equation (2.3.5) is

$$\int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \hat{H}(t_1) \hat{H}(t_2) = \int_{t_0}^t dt_2 \int_{t_2}^t dt_1 \hat{H}(t_1) \hat{H}(t_2)$$

$$= \int_{t_0}^t dt_1 \int_{t_1}^t dt_2 \hat{H}(t_2) \hat{H}(t_1)$$

Adding up the the two alternative but equivalent forms of integration,we arrive at

$$\begin{aligned} 2 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} \hat{H}(t_1) \hat{H}(t_2) &= \int_{t_0}^t dt_2 \int_{t_2}^t dt_1 \hat{H}(t_1) \hat{H}(t_2) + \int_{t_0}^t dt_1 \int_{t_1}^t dt_2 \hat{H}(t_2) \hat{H}(t_1) \\ &= \int_{t_0}^t dt_1 \int_{t_1}^t T(dt_2 \hat{H}(t_2) \hat{H}(t_1)) \end{aligned} \quad (2.3.7)$$

Thus the use of the time ordering operator T has made it possible to extend both integrations to the full integral $[t_0, t]$.In fact the same procedure can be extended to all the higher-order multiple integrals encountered in equation (2.3.5).In the n^{th} order there are $n!$ possible permutations of the time arguments.Therefore equation (2.3.7) gets generalized into

$$n! \int_{t_0}^t dt_1 \dots \int_{t_0}^{t_{n-1}} dt_n \hat{H}(t_1) \dots \hat{H}(t_n) = \int_{t_0}^t dt_1 \dots \int_{t_0}^t dt_n T(\hat{H}(t_1) \dots \hat{H}(t_n))$$

It is not difficult to prove this plausible result formally by employing mathematical induction.Using the above equation we find the perturbation series for the time -evolution operator

$$\begin{aligned} \hat{U}(t, t_0) &= \sum_{n=0}^{\infty} \frac{1}{n!} (-i)^n \int_{t_0}^t dt_1 \dots \int_{t_0}^t dt_n T(\hat{H}(t_1) \dots \hat{H}(t_n)) \quad (2.3.8) \\ i\partial_t \hat{u}(t, t_0) &= i \sum_{n=1}^{\infty} \frac{1}{n!} (-i)^n n \int_{t_0}^t dt_1 \dots \int_{t_0}^t dt_{n-1} T(\hat{H}(t_1) \dots \hat{H}(t_{n-1}) \hat{H}(t)) \\ &= \hat{H}(t) \sum_{n=1}^{\infty} \frac{1}{(n-1)!} (-i)^{n-1} \int_{t_0}^t dt_1 \dots \int_{t_0}^t dt_{n-1} T(\hat{H}(t_1) \dots \hat{H}(t_{n-1})) \\ &= \hat{H}(t) U(t, t_0) \end{aligned}$$

Here in the first step the symmetry of the integrand was used to obtain the factor n .Subsequently $\hat{H}(t)$ could be extracted from the T product since t is larger than all the other time arguements t_i .It is possible to formally sum up the series (2.3.8),arriving at the time ordered exponential function,

$$\hat{U}(t, t_0) = T \exp(-i \int_{t_0}^t dt' \hat{H}(t')) \quad (2.3.9)$$

In essence, however, this is just a compact way of writing the series (2.3.8).

In any local field theory the Hamiltonian can be expressed as an integral over the Hamiltonian density $H(x)$ which consists of products of field operators and possibly their derivatives. Then (2.3.9) can be written as

$$\hat{U}(t, t_0) = T \exp\left(-i \int_{t_0}^t d^4x' \hat{H}(x')\right)$$

The main use of the time evolution operator lies in its application to scattering processes. Furthermore $\hat{U}(t, t_0)$ can be employed to calculate the energy shift of a bound level under the influence of an interaction. [6]

2.4 The scattering matrix (s-matrix)

The scattering matrix (s-matrix) is a central concept in quantum field theory as well as in ordinary quantum mechanics. It describes the probability amplitude for a process in which the system makes a transition from an initial to a final state under the influence of an interaction. If one works in the interaction picture the time evolution operator is the right tool to use to evaluate the scattering matrix.

The typical situation in a scattering experiment is the following; At the initial time (idealized as $t \rightarrow -\infty$), we have widely separated and thus non-interacting particles. These particles approach one another and interact for a short time corresponding to the range of the forces. The particles and possibly newly created ones, that remain after this interaction then travel away from one another and cease to interact. At a much later time (idealized as $t \rightarrow \infty$), these are observed. The scattering process is represented schematically in fig 2.1. The time for which the particles interact is very much shorter than the time taken for the particles to travel from the source to the point of observation (detector), hence, it is reasonable to take the final and initial times as $t \rightarrow \pm\infty$, respectively.

At the initial time $t_i = -\infty$ of the scattering process we have a state $|i\rangle$ corresponding

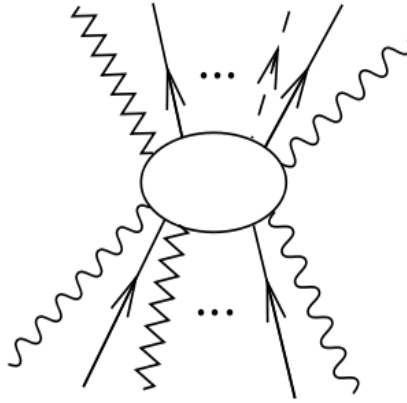


Figure 2.1: Schematic representation of a general scattering process. A number of particles are incident upon one another, interact, and scattered particles leave the interaction region. The number of scattered particles can be greater or smaller than the number of incoming particles

to free, non-interacting particles

$$|\Psi(-\infty)\rangle = |i\rangle$$

After the scattering, the particles that remain are again well separated from one another and are described by

$$|\Psi(\infty)\rangle = U'(\infty, -\infty)|i\rangle$$

The transition amplitude into a particular final state $|f\rangle$ is given by

$$\langle f|\Psi(\infty)\rangle = \langle f|U'(\infty, -\infty)\rangle = \langle f|S|i\rangle = S_{fi}$$

The states $|i\rangle$ and $|f\rangle$ are eigen states of H_0 . One imagines that the interaction is switched off at the beginning and the end. Here we have introduced the S-matrix by way of

$$S = U(-\infty, \infty),$$

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{\infty} dt_3 \dots \int_{-\infty}^{\infty} dt_n T(\hat{H}(t_1)\hat{H}(t_2)\hat{H}(t_3)\dots\hat{H}(t_n))$$

If the Hamiltonian is expressed in terms of the Hamiltonian density, one obtains

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int \dots \int d^4x_1 \int d^4x_2 \int d^4x_3 \dots \int d^4x_n T(\hat{H}(x_1)\hat{H}(x_2)\hat{H}(x_3)\dots\hat{H}(x_n))$$

$$= T(\exp(-i \int d^4x(H(x))))$$

Since the interaction operator is Lorentz invariant and the time ordering does not change under orthochronous Lorentz transformations the scattering matrix is itself invariant with respect to Lorentz transformations, i.e., it is a relativistically invariant quantity. The unitarity of $U(t, t_0)$ implies that the S-matrix is also unitary, $SS^+ = S^+S = 1$.

The unitarity of the S-matrix expresses conservation of probability. If the initial state is $|i\rangle$, then the probability of finding the final state $|f\rangle$ in an experiment is given by $|S_{fi}|^2$. The unitarity of the S-matrix guarantees that the sum of these probabilities over all possible final states is equal to one. Since particles may be created or annihilated, the possible final states may contain particles different from those in the initial states.

The states $|i\rangle$ and $|f\rangle$ have been assumed to be eigenstates of the unperturbed Hamiltonian H_0 , i.e., the interaction was assumed to be switched off. In reality, the physical states of real particles differ from these free states. The interaction turns the "bare" states into "dressed" states. An electron in such a state is surrounded by a cloud of virtual photons that are continually being emitted and reabsorbed, as in fig. 3.3

The calculation of the transition elements between bare states $|i\rangle$ and $|f\rangle$ can be justified by appealing to the adiabatic hypothesis. The interaction Hamiltonian $H_I(t)$ is replaced by $H_I(t)\zeta(t)$, where $\zeta(t) \rightarrow 0$ as $t \rightarrow \pm\infty$ and $\zeta(t) = 1$ for $-T < t < T$, i.e., at time $t = -\infty$ one has free particles. During the time interval $-\infty < t < T$, the interaction turns the free particles into physical particles. Thus in the time interval $[-T, T]$, we have real particles that experience the total interaction $H_I(t)$. Since the particles involved in a scattering process are initially widely separated, they only interact during the time interval $[-\tau, \tau]$, which is determined by the range of the interaction and the speed of the particles. The time T must, of course, be much larger than τ . The assumption of the adiabatic hypothesis is that the scattering cannot depend on the description of the states long before or long after the

interaction. At the end of the calculation one takes the limit $T \rightarrow \infty$. If one is calculating a process only in the lowest order perturbation theory at which it occurs, one uses the entire interaction for the transition and not to convert the bare state into physical state. In this case, one can take the limit $T \rightarrow \infty$ from the outset, and use the full interaction Hamiltonian in the whole time interval.[2]

2.5 Wick's theorem

In order to calculate the transition amplitude from the state $|i\rangle$ to the state $|f\rangle$, one needs to determine the matrix element $\langle f|S|i\rangle$. If one considers a particular order of perturbation theory, one has to evaluate the matrix element of a time-ordered product of interaction Hamiltonians. Of the many terms in the perturbation expansion, contributions come only from those whose application to $|i\rangle$ yields the state $|f\rangle$. Hence, (apart from the possibility of individual particles moving without interaction) the corresponding perturbation-theoretical contribution to the S-matrix must contain those annihilation operators that annihilate the particles in $|i\rangle$ and those creation operators that create the particles in $|f\rangle$. In addition, a general term in S-matrix will also contain further creation and annihilation operators responsible for the creation and subsequent annihilation of virtual particles. These particles are termed virtual because they are not present in the initial or final states; they are emitted and reabsorbed in intermediate processes, e.g., photons in electron-positron pair annihilation. The virtual particles do not obey the energy-momentum relation, $p^2 = m^2$, valid for real particles, i.e., they do not lie on the mass shell. It is possible to calculate the value of S-matrix elements by using the commutation relations to move the annihilation operators to the right. Instead of carrying out this calculation for every single case individually, it is helpful to rewrite the time-ordered products so that they are normal ordered from the start, i.e., so that all annihilation operators are to the left of all creation operators. Wick's theorem tells us how an arbitrary time-ordered product can be represented as a sum of normal ordered products. Wick's theorem is the basis for the

systematic calculation of perturbation theoretical contributions and their representation by means of Feynman diagrams. The problem of calculating correlation functions to that of evaluating expressions of the form $\langle 0|T\Phi_I(x_1)\Phi_I(x_2), \dots\Phi_I(x_n)|0\rangle$ i.e, vacuum expectation values of time ordered products of finite (but arbitrary) numbers of free field operators. For $n=2$ this expression is just the Feynman propagator. For higher n you could evaluate this object by brute force, plugging in the expansion of Φ_I in terms of ladder operators.

Consider the case of two fields, $\langle 0|T\Phi_I(x)\Phi_I(y)|0\rangle$. To calculate this quantity let us rewrite it in a form that is easy to evaluate and also generalizes to the case of more than two fields. To do this we first decompose Φ_I into positive and negative frequency parts;

$$\Phi_I(x) = \Phi_I^+(x) + \Phi_I^-(x), \quad (2.5.1)$$

where

$$\begin{aligned} \Phi_I^+(x) &= \int \frac{d^3x}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p e^{-ip \cdot x} \\ \Phi_I^-(x) &= \int \frac{d^3x}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p^+ e^{+ip \cdot x} \end{aligned}$$

This decomposition can be done for any free field. It is useful because

$$\Phi_I^+(x)|0\rangle = 0$$

and

$$\langle 0|\Phi_I^-(x) = 0$$

For example, consider the case $x^0 \geq y^0$. Time ordered product of two fields is then

$$\begin{aligned} T\Phi_I(x)\Phi_I(y) &= \Phi_I^+(x)\Phi_I^+(y) + \Phi_I^+(x)\Phi_I^-(y) + \Phi_I^-(x)\Phi_I^+(y) + \Phi_I^-(x)\Phi_I^-(y) \\ &= \Phi_I^+(x)\Phi_I^+(y) + \Phi_I^-(x)\Phi_I^+(y) + \Phi_I^-(x)\Phi_I^-(y) + [\Phi_I^+(x), \Phi_I^-(y)] \end{aligned} \quad (2.5.2)$$

In every term except the commutator, all a'_p s are to the right of all the a_p^+ s. Such a term (e.g, $a_p^+ a_q^+ a_k a_l$) is said to be in normal order, and has vanishing vacuum expectation value. Let us also define the normal ordering symbol $N()$ to place whatever operators

it contains in normal order,for example

$$N(a_p a_k^+ a_q) \equiv a_k^+ a_p a_q$$

The order of a_p and a_q on the right hand side makes no difference since they commute.

If we had instead considered the case $y^0 > x^0$,we would get the same four normal-ordered terms,but this time the final commutator would be $[\Phi_I^+(y), \Phi_I^-(x)]$.Let's therefore define one more quantity,the contraction of two fields,as follows

$$: \Phi(x) \Phi(y) : \equiv [\Phi^+(x), \Phi^-(y)]$$

for $x^0 > y^0$ and

$$: \Phi(x) \Phi(y) : \equiv [\Phi^+(y), \Phi^-(x)]$$

for $y^0 > x^0$. This quantity is exactly the Feynman propagator

$$: \Phi(x) \Phi(y) := D_F(x - y)$$

The relation between time ordering and normal ordering is now extremely simple to express,at least for two fields;

$$T\Phi(x)\Phi(y) = N\Phi(x)\Phi(y) + : \Phi(x)\Phi(y) : \quad (2.5.3)$$

But now that we have all this new notation,the generalization to arbitrarily many fields is also easy to write down;

$$T\Phi(x_1)\Phi(x_2)\dots\Phi(x_m) = N\Phi(x_1)\Phi(x_2)\dots\Phi(x_m) + \text{all possible contractions} \quad (2.5.4)$$

This quantity is known as Wick's theorem,and we will prove it in a moment.The phrase all possible contractions means there will be one term for each possible way of contracting the m fields in pairs.Thus for m=4 we have

$$\begin{aligned}
& T(\Phi_1\Phi_2\Phi_3\Phi_4) \\
&= N(\Phi_1\Phi_2\Phi_3\Phi_4 + : \Phi_1\Phi_2 : \Phi_3\Phi_4 + : \Phi_1\Phi_2\Phi_3 : \Phi_4 + : \Phi_1\Phi_2\Phi_3\Phi_4 : + \Phi_1 : \Phi_2\Phi_3 : \Phi_4 \\
&+ \Phi_1 : \Phi_2\Phi_3\Phi_4 : + \Phi_1\Phi_2 : \Phi_3\Phi_4 : + : \Phi_1\Phi_2 : \Phi_3\Phi_4 : + : \Phi_1 : \Phi_2\Phi_3 : \Phi_4 : + : \Phi_1 : \Phi_2\Phi_3 : \Phi_4 :) \\
& \hspace{15em} (2.5.5)
\end{aligned}$$

When the contraction symbol connects two operators that are not adjacent, we still define it to give a factor of D_F . For example,

$$N(:\Phi_1\Phi_2\Phi_3 : \Phi_4) \text{ means } D_F(x_1 - x_3)N(\Phi_2\Phi_4)$$

In the vacuum expectation value of any term in which there remain uncontracted operators gives zero (since $\langle 0|N(\text{any operator})|0\rangle = 0$). Only the three fully contracted terms in the last line survive. So we have

$$\begin{aligned}
& \langle 0|T(\Phi_1\Phi_2\Phi_3\Phi_4)|0\rangle \\
&= D_F(x_1 - x_2)D_F(x_3 - x_4) + D_F(x_1 - x_3)D_F(x_2 - x_4) + D_F(x_1 - x_4)D_F(x_2 - x_3)
\end{aligned}$$

Now let us prove Wick's theorem; Naturally the proof is by induction on m , the number of fields. We have already proved the case $m=2$. So assume the theorem is true for $m-1$ fields, and let us try to prove it for m fields. Without loss of generality, we can restrict ourselves to the case $x_1^0 \geq x_2^0 \geq \dots \geq x_m^0$; if this is not the case we can just relabel the points. Then applying Wick's theorem to Φ_2, \dots, Φ_m , we have

$$T(\Phi_2, \dots, \Phi_m) = \Phi_1 \dots \Phi_m = \Phi_1 N(\Phi_2, \dots, \Phi_m + (\text{all contractions not involving } \Phi_1))$$

$$=(\Phi_1^+ + \Phi_1^- N(\Phi_2 \dots \Phi_m + (\text{all contractions not involving } \Phi_1)))(2.5.6)$$

We want to move the Φ_1^+ and Φ_1^- inside the $N()$. For the Φ_1^- term this is easy; Just move it in, since (being on the left) it is already in normal order. The term with Φ_1^+ must be put in normal orders by commuting it to the right past all the other Φ 's. Consider, for example, the term with no term with no contractions.

$$\begin{aligned} \Phi_1^+ N(\Phi_2 \dots \Phi_m) &= N(\Phi_2 \dots \Phi_m) + \Phi_1^+ + [\Phi_1^+, N(\Phi_2 \dots \Phi_m)] \\ &= N(\Phi_1^+ \Phi_2 \dots \Phi_m) + N([\Phi_1^+, \Phi_2^-] \Phi_3 \dots \Phi_m + \Phi_2 [\Phi_1^+, \Phi_3^+] \Phi_4 \dots \Phi_m + \dots) \\ &= N(\Phi_1^+ \Phi_2 \dots \Phi_m + : \Phi_1 \Phi_2 : \Phi_3 \dots \Phi_m + : \Phi_1 \Phi_2 \Phi_3 : \dots + \dots) \end{aligned}$$

The first term in the last line combines with part of the Φ_1^- term to give $N(\Phi_1 \Phi_2 \dots \Phi_m)$ so we now have the first term on the right hand side of Wick's theorem, as well as all possible terms involving a single contraction of Φ_1 with one of the other fields. Doing this with all the terms we eventually get all possible contractions of all the fields, including Φ_1 . This Wick's theorem would lead to derivation of Feynman diagrams for various processes in quantum field theory. For instance the emission of a photon by an electron and the scattering of an electron by an external potential (Mott scattering) and as examples of second order processes, the scattering of two electrons (Moller scattering) and the scattering of a photon from an electron (Compton scattering). [1,2]

Chapter 3

The Electron Self-Energy

3.1 The electron self-energy

In quantum field theory a particle's self-energy $\Sigma(p)$ represents the contribution to the particle's energy, or effective mass, due to interactions between the particle and the system part of. For example, in electrostatics the self energy of a given charge distribution is the energy required to assemble the distribution by bringing the constituent charges from infinity where the electric force goes to zero. In a condensed matter context relevant to electrons moving in a material, the self energy represents the potential felt by the electron due to the surrounding medium's interactions with it; for example the fact that electrons repel each other means that a moving electron polarizes (causes to displace) the electrons in its vicinity and this in turn changes the potential the moving electron feels; these and other effects are included in the self energy. [www.wikipedia encyclopedia]

An electron interacts with its own radiation field. It can emit and reabsorb photons. These photons which are described by perturbation theoretical contributions of higher order, modify the propagation properties of the electron. If, for example, a fermion line within some diagram is replaced by the diagram in fig 3.1. this means that the propagator as a whole becomes

$$S_f(p) \rightarrow S'_f(p) = S_f(p) + S_f(p)\Sigma(p)S_f(p) \quad (3.1.1)$$

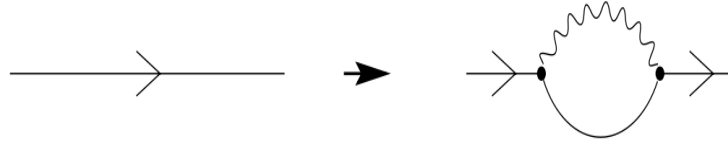


Figure 3.1: (a) Replacement of a fermion propagator by two propagators with inclusion of the self-energy

The bubble consisting of a photon and a fermion line is the self energy $\Sigma(p)$. Summing over processes of this type

$$\begin{aligned} S'_f(p) &= S_f(p) + S_f(p)\Sigma(p)S_f(p) + S_f(p)\Sigma(p)S_f(p)\Sigma(p)S_f(p) + \dots \\ &= S_f(p) + S_f(p)\Sigma(p)[S_f(p) + S_f(p)\Sigma(p)S_f(p)] \\ &= S_f(p) + S_f(p)\Sigma(p)S'_f(p) \end{aligned} \quad (3.1.2)$$

$$\Rightarrow S'_f(p) = \frac{1}{(S_f(p))^{-1} - \Sigma(p)} \quad (3.1.3)$$

Hence, the self-energy $\Sigma(p)$ and its associated self energy diagram give, among other things, a correction to the mass and a modification of the particle's energy, the latter being the source of the name "self-energy". The diagrammatic representation of equation (3.1.2) and (3.1.3) is given in the figure below. To distinguish it from the free (or bare) propagator $S_f(p)$, the propagator $S'_f(p)$ is called the interacting (dressed) propagator.

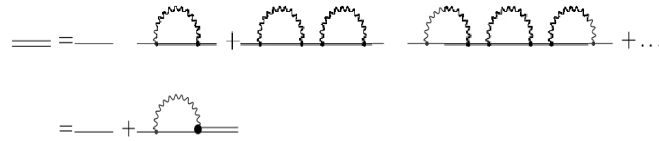


Figure 3.2: Diagrammatic representation of the Dyson equations (3.1.2) and (3.1.3). The propagator $S_F(p)$ is represented by the double line

A few self energy diagrams of higher order have already been given in fig 3.3. In general a part of a diagram is called a self-energy contribution when it is linked to the rest of the diagram only by two $S_f(p)$ propagators.

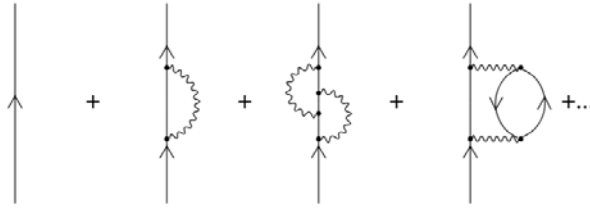


Figure 3.3: The propagation of a real (physical) electron involves the free propagation and the propagation that includes the additional emission and reabsorption of virtual photons.

A proper self energy diagram (one particle-irreducible) is one that cannot be separated into two parts by cutting a single $S_f(p)$ line. Otherwise, one has an improper self-energy diagram. The self energy diagrams in fig.3.3 are all proper ones.

The second summand in the first line of fig.3.3 contains a proper self-energy part; all others are improper. The Dyson equation, equation (3.1.2), can be extended to arbitrary higher orders; then $\Sigma(p)$ in equations (3.1.2) and (3.1.3) consists of the sum of all proper self energy diagrams.[2]

Using the Feynman rules we can write the second order matrix element for an electron in state (p,s) to make a transition to state (p',s') in the absence of any external field as follows:[7]

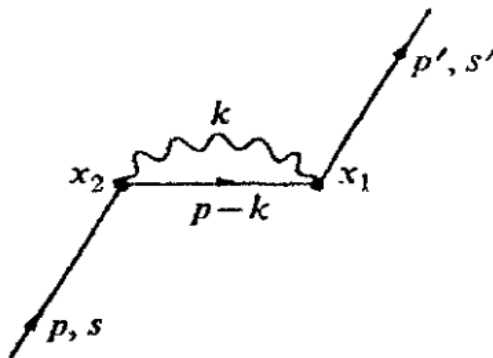


Figure 3.4: Electron Self-Energy

In quantum electrodynamics both $\overline{\Psi}\gamma_\mu\Psi_1\overline{\Psi}_2\gamma_\nu\Psi_2\overline{A}_\mu A_\nu$ and $\overline{\Psi}_1\gamma_\mu\Psi_1\overline{\Psi}_2\gamma_\nu\Psi_2\overline{A}_\mu A_\nu$ represent electron self-energy.

Hence using the scattering matrix the transition amplitude S_{fi} is,

$$S_{fi} = 2 \frac{(-ie)^2}{2} \int d^4x_1 \int d^4x_2 \langle f | \overline{\Psi}_1 \gamma_\mu \Psi_1 \overline{\Psi}_2 \gamma_\nu \Psi_2 \overline{A}_\mu A_\nu | i \rangle$$

Let $|i\rangle = b_1^+ |0\rangle, |f\rangle = b_1^+ |0\rangle$

$$\Psi_1^- = \frac{1}{\sqrt{V}} \Sigma_{p_2} \sqrt{\frac{m}{E_2}} \Sigma_{s_2} b_2^+ \overline{U}_3 e^{(-ip_3)x_2}$$

$$\Psi_2^+ = \frac{1}{\sqrt{V}} \Sigma_{p_3} \sqrt{\frac{m}{E_3}} \Sigma_{s_3} b_3^+ \overline{U}_3 e^{(-ip_3)x_1}$$

$$S_{fi} = -e^2 \frac{1}{V} \Sigma_{p_2, p_3} \Sigma_{s_2, s_3} \sqrt{\frac{m^2}{E_2 E_3}} \int d^4x_1 \int d^4x_2 e^{ip_3 x_2} e^{ip_2 x_1} \overline{U}_3 \gamma_\mu S_F(x_1 - x_2) \gamma_\nu D_F(x_1 - x_2) \langle 0 | b_1' b_3^+ b_2 b_1^+ | 0 \rangle$$

where

$$S_F = \frac{1}{i(2\pi)^4} \int d^4q \frac{-i\gamma \cdot q + m}{q^2 + m^2} e^{iq \cdot (x_1 - x_2)}$$

and

$$D_F = \frac{\delta_{\mu\nu}}{i(2\pi)^4} \int \frac{d^4k}{k^2} e^{ik \cdot (x_1 - x_2)}$$

$$\begin{aligned} S_{fi} &= -e^2 \sqrt{\frac{m^2}{V^2 E_1' E_2}} \frac{\delta_{\mu\nu}}{i(2\pi)^4} \int \frac{d^4k}{k^2} e^{ik \cdot (x_1 - x_2)} \int d^4x_1 e^{ip_1 \cdot x_1} \int d^4x_2 e^{-ip_1' \cdot x_2} \\ &\quad \times \overline{U}_1' \gamma_\mu \frac{1}{i(2\pi)^4} \int d^4q \frac{-i\gamma \cdot q + m}{q^2 + m^2} e^{iq \cdot (x_1 - x_2)} \gamma_\nu U_1 \\ &= -e^2 \sqrt{\frac{m^2}{V^2 E_1' E_2}} \frac{\delta_{\mu\nu}}{i(2\pi)^4} \int \frac{d^4k}{k^2} \int d^4x_1 e^{i(k+p_1+q) \cdot x_1} \int d^4x_2 e^{-i(k+p_1'+q) \cdot x_2} \\ &\quad \times \overline{U}_1' \gamma_\mu \frac{1}{i(2\pi)^4} \int d^4q \frac{-i\gamma \cdot (p_1 - k) + m}{(p_1 - k)^2 + m^2} e^{iq \cdot (x_1 - x_2)} \gamma_\nu U_1 \\ &= -e^2 \sqrt{\frac{m^2}{V^2 E_1' E_1}} \frac{(-i)}{(2\pi)^4} \int d^4q (2\pi)^4 \delta^{(4)}(k + p_1 + q) (2\pi)^4 \delta^{(4)}(-k - p_1' - q) \\ &\quad \times \overline{U}_1' \gamma_\mu \frac{\delta_{\mu\nu}}{i(2\pi)^4} \int \frac{d^4k}{k^2} \frac{-i\gamma \cdot (p_1 - k) + m}{(p_1 - k)^2 + m^2} \gamma_\nu U_1 \\ &= (-i)(2\pi)^4 \delta^{(4)}(p - p') \sqrt{\frac{m^2}{V^2 E_1' E_1}} \overline{U}_1' \frac{(ie)^2}{(2\pi)^4} \int d^4k \frac{\delta_{\mu\nu}}{ik^2} \gamma_\mu \frac{[-i\gamma(p - k) + m]}{i[(p - k)^2 + m^2]} \gamma_\nu U_1 \\ &= (-i)(2\pi)^4 \delta^{(4)}(p - p') \sqrt{\frac{m^2}{V^2 E' E}} \overline{U}_{(p')}^{(s')} \frac{(ie)^2}{(2\pi)^4} \int d^4k \frac{\delta_{\mu\nu}}{ik^2} \gamma_\mu \frac{[-i\gamma(p - k) + m]}{i[(p - k)^2 + m^2]} \gamma_\nu U_{(p)}^{(s)} \end{aligned} \quad (3.1.4)$$

The transition amplitude, S_{fi} , into the final state, (p', s') , is given by

$$S_{fi} = -i(2\pi)^{(4)}\delta^{(4)}(p - p')\sqrt{\frac{m^2}{V^2EE'}}\bar{U}^{(s')}(p')\sum(p)U^{(s)}(p) \quad (3.1.5)$$

where $\sum(p)$ is a 4x4 matrix given by equation (3.1.4)

$$-\sum(p) = (-e^2)\int\frac{d^4k\delta_{\mu\nu}[-i\gamma_\mu(p-k)+m]\gamma_\nu}{(2\pi)^4ik^2i[(p-k)^2+m^2]} \quad (3.1.6)$$

Note that in equation(3.1.6),there is one non-trivial equation to be performed in four-dimensional momentum space.This integration is expected since the virtual-photon can have an arbitrary four momentum even if it is assumed energy-momentum conservation at the two vertices.The self energy is

$$\sum(p) = (ie^2)\int\frac{d^4k[-i\gamma_\mu(p-k)+m]\gamma_\mu}{(2\pi)^4ik^2[(p-k)^2+m^2]} \quad (3.1.7)$$

For use in the calculation of the above self energy,it will be convenient to see wick rotation and to use a method of regularization introduced by Pauli and Villars.

3.2 Wick rotation

Consider the integral of the form

$$I_{nm} = \int\frac{d^d p}{(2\pi)^d}\frac{p^{2n}}{(p^2 - C + i\epsilon)^m}$$

To evaluate the integral I_{nm} we will use integration in the complex plane of the variable p^0 as described in the figure below

We can then write

$$I_{nm} = \int\frac{d^{d-1}p}{(2\pi)^d}\int dp^0\frac{p^{2n}}{((p^0)^2 - |p|^2 - C - i\epsilon)^m}$$

The function under the integral has poles for $p^0 = \pm\sqrt{|p|^2 + C} - i\epsilon$ as shown in fig.3.5.Using the properties of complex variables (Cauchy theorem)we can deform the contour,changing the integration from the real to the imaginary axis plus the two arcs at infinity.This can be done because in deforming the contour we do not cross any pole.Notice the importance of

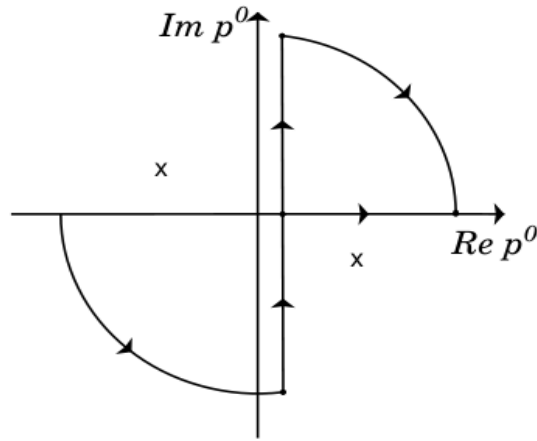


Figure 3.5: Wick Rotation

the $i\epsilon$ prescription to be able to do this. The contribution from the arcs at infinity vanishes in dimension sufficiently low for the integral to converge, as we assume in dimensional regularization. We have then changed the integration along the imaginary axis in the plane of the complex variable p^0 . If we write

$$p^0 = ip_E^0, \int_{-\infty}^{\infty} dp^0 \rightarrow i \int_{-\infty}^{\infty} dp_E^0 \text{ and } p^2 = (p^0)^2 - |p|^2 = -(p_E^0)^2 - |p|^2 \equiv -p_E^2$$

where $p_E = (p_E^0, p)$ is a Euclidean vector.

Then we can write

$$I_{nm} = i(-1)^{n-m} \int \frac{d^d p_E}{(2\pi)^d} \frac{(p_E^2)^n}{(p_E^2 + C)^m}$$

Where we do not need the $i\epsilon$ because the denominator is strictly positive ($C > 0$). This procedure is Wick Rotation. We note that the Feynman prescription for the propagators that originated the $i\epsilon$ rule for the denominators is crucial for the Wick Rotation to be possible. If r is the length of the vector p_E in the Euclidean space in d -dimensions and $d\Omega_{d-1}$ is the solid angle that generalizes spherical coordinates in that Euclidean space then,

$$p_E = r(\cos\theta_1, \sin\theta_1 \cos\theta_2, \sin\theta_1 \sin\theta_2 \cos\theta_3, \dots, \sin\theta_1 \dots \sin\theta_{d-1})$$

$$d^d p_E = r^{d-1} \sin\theta_1^{d-2} \sin\theta_2^{d-3} \sin\theta_3^{d-4} \dots dr d\theta_1 d\theta_2 d\theta_3 \dots d\theta_{d-1}$$

$$d^d p_E = r^{d-1} dr d\Omega_{d-1}$$

where $d\Omega_{d-1} = \sin\theta_1^{d-2} d\theta_1 \sin\theta_2^{d-3} d\theta_2 \sin\theta_3^{d-4} d\theta_3 \dots d\theta_{d-1}$. Hence,

$$\begin{aligned} \int d^d p_E &= \int r^{d-1} dr d\Omega_{d-1} = \int_0^\infty r^{d-1} dr \int d\Omega_{d-1} \\ &= \int_0^\infty r^{d-1} dr \int_0^\pi \sin\theta_1^{d-2} d\theta_1 \int_0^\pi \sin\theta_2^{d-3} d\theta_2 \int_0^\pi \sin\theta_3^{d-4} d\theta_3 \dots \int_0^{2\pi} d\theta_{d-1} \end{aligned} \quad (3.2.1)$$

We can then write

$$\int d\Omega_{d-1} = \int_0^\pi \sin\theta_1^{d-2} d\theta_1 \int_0^\pi \sin\theta_2^{d-3} d\theta_2 \int_0^\pi \sin\theta_3^{d-4} d\theta_3 \dots \int_0^{2\pi} d\theta_{d-1}$$

To compute the above integration we use the following trick:

$$\sqrt{\pi} = \int e^{-x^2} dx \Rightarrow (\sqrt{\pi})^d = \int \exp(-\sum_{i=1}^d x_i^2) d^d x$$

Using equation (3.2.1)

$$\begin{aligned} (\sqrt{\pi})^d &= \int d\Omega_{d-1} \int_0^\infty x^{d-1} e^{-x^2} dx = \left(\int d\Omega_{d-1} \right) \cdot \frac{1}{2} \int_0^\infty (x^2)^{\frac{d}{2}-1} e^{-(x^2)} d(x^2) = \left(\int d\Omega_{d-1} \right) \cdot \frac{1}{2} \Gamma\left(\frac{d}{2}\right) \\ &\Rightarrow \int d\Omega_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \end{aligned} \quad (3.2.2)$$

Hence equation (3.2.1) will be (for d=4)

$$\int d^4 p_E = 2\pi^2 \int_0^\infty r^3 dr = \pi^2 \int_0^\infty r^2 dr^2 \quad (3.2.3)$$

[Jorge Romao, Modern Techniques for One-Loop Calculations (2006)]

3.3 Pauli-Villars regularization

A regularization technique is any mathematical prescription which renders the divergent Feynman amplitudes finite by means of a specific cut-off procedure. The technique is then said to regularize the divergent integrals. One of the earliest procedures is due to Pauli and Villars, who introduced massive auxiliary fields called regulators in order to eliminate the singularities from propagators and other ill-defined field functions. The Pauli-Villars

technique gives a precise prescription for the use of these regulators in quantum electrodynamics in such a way that the theory remains gauge-invariant at each order of perturbation theory. Since the auxiliary fields do not admit a physical interpretation, the Pauli-Villars method must be regarded as a purely formal cut-off procedure. An example of such a procedure is afforded by the so called Feynman regularization of the causal photon Green's function which consists of replacing

$$D_{\mu\nu} = \frac{1}{k^2} \text{ with } \frac{1}{k^2} - \frac{1}{k^2 - \Lambda^2}$$

where Λ is a very large mass. The integrand is unaffected for small k (since Λ is very large), but cuts off smoothly when $k \geq \Lambda$. We can think of the second term as the propagator of a fictitious heavy photon, whose contribution is subtracted from that of the ordinary photon. Note that the fictitious photon has no physical significance, and that this method is only one of many for defining the divergent integrals. [www.wikipedia encyclopedia]

Hence replacing the photon propagator $\frac{1}{k^2}$ with

$$\frac{1}{k^2} \longrightarrow \frac{1}{k^2} - \frac{1}{k^2 - \Lambda^2} \quad (3.3.1)$$

where $\Lambda \gg k$ (later it is possible to drop the regulator by letting the regulator mass Λ go to infinity).

Hence using this regularization method the electron self-energy in equation (3.1.7) can be written as

$$\Sigma(p) = \frac{ie^2}{(2\pi)^4} \int d^4k \left[\frac{1}{k^2} - \frac{1}{k^2 - \Lambda^2} \right] \frac{\gamma_\mu [-i\gamma \cdot (p - k) + m] \gamma_\mu}{[(p - k)^2 + m^2]} \quad (3.3.2)$$

Using the Feynman trick,

$$\frac{1}{AB^n} = \int_0^1 dx \frac{nx^{n-1}}{[Ax + B(1-x)]^{n+1}} \quad (3.3.3)$$

to combine the denominators of equation (3.3.2),

$$\begin{aligned} \frac{1}{k^2[(p-k)^2 + m^2]} &= \int_0^1 \frac{dx}{[(p-k)^2x + mx^2 + k^2(1-x)]^2} \\ &= \int_0^1 \frac{dx}{[(k-px)^2 + p^2x(1-x) + m^2x]^2} \end{aligned} \quad (3.3.4)$$

$$\begin{aligned} \frac{1}{[k^2 - \Lambda^2][(p-k)^2 + m^2]} &= \int_0^1 \frac{dx}{[(p-k)^2 + m^2x + (k^2 - \Lambda^2)(1-x)]^2} \\ &= \int_0^1 \frac{dx}{[(k-px)^2 + p^2x(1-x) + m^2x + \Lambda^2(1-x)]^2} \end{aligned} \quad (3.3.5)$$

Substituting equations (3.3.4) and (3.3.5) in equation (3.3.2),

$$\begin{aligned} \Sigma(p) &= \frac{ie^2}{(2\pi)^4} \int d^4k [2i\gamma \cdot (p-k) + 4m] \\ &\times \int_0^1 dx \left[\frac{1}{[(k-px)^2 + p^2x(1-x) + m^2x]^2} - \frac{1}{[(k-px)^2 + p^2x(1-x) + m^2x + \Lambda^2(1-x)]^2} \right] \end{aligned} \quad (3.3.6)$$

Notice that an integral of the form

$$I = \int \frac{d^4k}{[(k-px)^2 + a^2]^2} \quad (3.3.7)$$

is a divergent integral (according to equation (3.3.12)) and it will be calculated by shifting $k \rightarrow k + px$. Hence,

$$\int \frac{d^4k}{[(k-px)^2 + a^2]^2} = \int \frac{d^4k}{[k^2 + a^2]^2} \quad (3.3.8)$$

Using this property of divergent integrals, equation (3.3.6) can be rewritten as

$$\begin{aligned} \Sigma(p) &= \frac{ie^2}{(2\pi)^4} \int d^4k [2i\gamma \cdot (p - px - k) + 4m] \\ &\times \int_0^1 dx \left[\frac{1}{[k^2 + p^2x(1-x) + m^2x]^2} - \frac{1}{[k^2 + p^2x(1-x) + m^2x + \Lambda^2(1-x)]^2} \right] \end{aligned} \quad (3.3.9)$$

The third term, $-\gamma \cdot k$, in the numerator of the integral will vanish upon symmetrical integration. Hence

$$\begin{aligned} \Sigma(p) &= \frac{ie^2}{(2\pi)^4} \int d^4k [2i\gamma \cdot (p - px) + 4m] \\ &\times \int_0^1 dx \left[\frac{1}{[k^2 + p^2x(1-x) + m^2x]^2} - \frac{1}{[k^2 + p^2x(1-x) + m^2x + \Lambda^2(1-x)]^2} \right] \\ &= \frac{2ie^2}{(2\pi)^4} \int_0^1 dx [i\gamma \cdot (p - px) + 2m] \\ &\times \int d^4k \left[\frac{1}{[k^2 + p^2x(1-x) + m^2x]^2} - \frac{1}{[k^2 + p^2x(1-x) + m^2x + \Lambda^2(1-x)]^2} \right] \end{aligned} \quad (3.3.10)$$

Using equation (3.2.3)

$$\begin{aligned} \Sigma(p) &= \frac{2\pi^2 e^2}{(2\pi)^4} \int_0^1 dx [i\gamma \cdot p(1-x) + 2m] \\ &\times \int_0^\infty k^3 dk \left[\frac{1}{[k^2 + p^2 x(1-x) + m^2 x]^2} - \frac{1}{[k^2 + p^2 x(1-x) + m^2 x + \Lambda^2(1-x)]^2} \right] \end{aligned}$$

Using the properties

$$\frac{1}{\alpha^n} - \frac{1}{\beta^n} = -n \int_0^1 \frac{(\alpha - \beta)}{[\beta + (\alpha - \beta)z]^{n+1}} dz \quad (3.3.11)$$

and

$$\int \frac{(k^2)^{m-2} d^4 k}{(k^2 + a^2)^n} = \frac{i\pi^2}{(a^2)^{n-m}} \frac{\Gamma(m)\Gamma(n-m)}{\Gamma(n)} \quad (3.3.12)$$

for $n > m > 0$ [5].

$$\begin{aligned} \Sigma(p) &= \frac{e^2}{8\pi^2} \int_0^1 dx [i\gamma \cdot p(1-x) + 2m] \ln \frac{p^2 x(1-x) + m^2 x + \Lambda^2(1-x)}{p^2 x(1-x) + m^2 x} \\ &= \frac{\alpha}{2\pi} \int_0^1 dx [i\gamma \cdot p(1-x) + 2m] \ln \frac{p^2 x(1-x) + m^2 x + \Lambda^2(1-x)}{p^2 x(1-x) + m^2 x} \end{aligned} \quad (3.3.13)$$

It can be seen, on invariance grounds, that $\Sigma(p)$ can be expressed as a linear combination of the identity matrix and the 4×4 matrix $\gamma \cdot p$. It is more convenient, however, to consider an expansion of $\Sigma(p)$ about the point $i\gamma \cdot p = -m$ of the form

$$\Sigma(p) = A + (i\gamma \cdot p + m)B + (i\gamma \cdot p + m)^2 \Sigma^f(p) \quad (3.3.14)$$

where A, B and $\Sigma^f(p)$ do not contain gamma matrices [7]. Hence at $i\gamma \cdot p = -m$ the self energy $\Sigma(p)$ is

$$\begin{aligned} \Sigma(p) &= \frac{\alpha}{2\pi} \int_0^1 dx [-m(1-x) + 2m] \ln \frac{-m^2 x(1-x) + m^2 x + \Lambda^2(1-x)}{-m^2 x(1-x) + m^2 x} \\ &= \frac{\alpha m}{2\pi} \int_0^1 dx (1+x) \ln \frac{m^2 x^2 + \Lambda^2(1-x)}{m^2 x^2} \end{aligned} \quad (3.3.15)$$

Since $\Lambda \gg m$

$$\Sigma(p) = \frac{\alpha m}{2\pi} \int_0^1 dx (1+x) \ln \frac{\Lambda^2(1-x)}{m^2 x^2}$$

$$\begin{aligned}
&= \frac{\alpha m}{2\pi} \int_0^1 dx(1+x) \left[2 \ln \frac{\Lambda}{m} + \ln \left(\frac{1-x}{x^2} \right) \right] \\
&= \frac{3\alpha m}{2\pi} \left(\ln \frac{\Lambda}{m} + \frac{1}{4} \right) \approx \frac{3\alpha m}{2\pi} \left(\ln \frac{\Lambda}{m} \right)
\end{aligned} \tag{3.3.16}$$

In classical electrodynamics, we actually have a linearly divergent self-energy. An electron creates a Coulomb field around it, and it feels its own Coulomb field. If, for instance, one imagines the electron to be a sphere of radius r_e with a uniform charge density, the total potential energy is $V = \frac{3e^2}{5r_e}$. The total energy of the electron is the sum of the rest energy $m_o c^2$ and the potential energy V and hence the total mass of the electron we observe is given by

$$mc^2 = m_o c^2 + \frac{3}{5} \frac{e^2}{4\pi r_e}$$

In the limit $r_e \rightarrow 0$, the bare mass m_o needs to be sent negative to cancel the linearly divergent Coulomb self-energy to obtain the observed mass of the electron. In quantum mechanical language, clearly this is an ‘‘ultraviolet’’ divergence as it corresponds to short distance physics $r_e \rightarrow 0$. If we imagine the electron to be as small as the Planck size $r_e = \sqrt{\hbar G_N c^3} = 1.6 \times 10^{-33} \text{cm}$ where presumably the quantum gravity takes over physics, we need to make the bare mass as negative as $-5.34 \times 10^{19} \text{MeV}$ which is cancelled by the self-energy for 20 digits to get 0.511 MeV. This is absurd. The classical electrodynamics therefore breaks down at distance scale where the rest energy and the self energy become comparable $r_e \simeq \frac{e^2}{mc^2} \simeq 10^{-13} \text{cm}$, and a better (deeper) theory needs to take over the classical electrodynamics below this distance scale.

What we have learnt here is that the situation in quantum electrodynamics is much better. The total mass of the electron is

$$mc^2 = m_o c^2 \left(1 + \frac{\alpha}{4\pi} \left[3 \ln \left(\frac{\Lambda^2}{m^2} \right) + \frac{3}{2} \right] \right)$$

The ultraviolet cut off Λ corresponds to the inverse ‘‘size’’ of the electron size in the classical language. The crucial difference from the classical theory is that; the dependence on the ‘size’ of the electron is only logarithmic instead of power and the correction is

proportional to the electron mass itself and hence can never be much larger than the bare mass.

3.4 Dimensional regularization

The idea of dimensional regularization is very simple to state. Compute the feynman diagram as an analytic function of the dimensionality of space-time, d . For sufficiently small d , any loop-momentum integral will converge. The final expression for any observable quantity should have a well defined limit $d \rightarrow 4$. A typical example to do a practice calculation to see how this technique works is shown below. Using the trick in Wick rotation, equation (3.2.2),

$$\int \frac{d^d \ell_E}{[2\pi]^d} \frac{1}{[\ell_E^2 + \Delta]^2} = \int \frac{d\Omega_d}{[2\pi]^d} \int_0^\infty d\ell \frac{\ell^{d-1}}{[\ell^2 + \Delta]^2} \quad (3.4.1)$$

The first factor contains the area of a unit sphere in d dimensions as it has been proved in equation (3.2.2). The second factor is

$$\int_0^\infty d\ell \frac{\ell^{d-1}}{[\ell^2 + \Delta]^2} = \frac{1}{2} \int_0^\infty d(\ell^2) \frac{(\ell^2)^{\frac{d}{2}-1}}{(\ell^2 + \Delta)^2} = \frac{1}{2} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \int_0^\infty dx x^{1-\frac{d}{2}} (1-x)^{\frac{d}{2}-1} \quad (3.4.2)$$

where we have substituted $x = \frac{\Delta}{\ell^2 + \Delta}$. Using the definition of the beta function,

$$\int_0^1 dx x^{\alpha-1} (1-x)^{\beta-1} = B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} \quad (3.4.3)$$

we can easily evaluate the integral over x . The final result for the d -dimensional integral is

$$\int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 + \Delta)^2} = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(2 - \frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \quad (3.4.4)$$

Since $\Gamma(z)$ has isolated poles at $z = 0, -1, -2, -3, \dots$, this integral has isolated poles at $d = 4, 6, 8, \dots$. To find the behavior near $d = 4$, define $\epsilon = 4 - d$, and use the approximation,

$$\Gamma\left(2 - \frac{d}{2}\right) = \Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} - \gamma + O(\epsilon), \quad (3.4.5)$$

where $\gamma \approx 0.5772$ is the Euler-Mascherioni constant. (This constant will always cancel in observable quantities). The integral is then

$$\int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 + \Delta)^2} \xrightarrow{d \rightarrow 4} \frac{1}{(4\pi)^2} \left(\frac{2}{\epsilon} - \ln \Delta - \gamma + \ln 4\pi + O(\epsilon) \right) \quad (3.4.6)$$

When this integral is defined with a Pauli-Villars regulator it is found as

$$\int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 + \Delta)^2} \xrightarrow{\Lambda \rightarrow \infty} \frac{1}{(4\pi)^2} \left(\ln \frac{x\Lambda^2}{\Delta} + O(\Lambda^{-1}) \right) \quad (3.4.7)$$

Thus the $\frac{1}{\epsilon}$ pole in dimensional regularization correspondes to a logarithmic divergence in the momentum integral. Note the curious fact that equation (3.4.6) involves the logarithm of Δ , a dimensionful quantity. The scale of the logarithm is hidden in the $\frac{1}{\epsilon}$ term, and appears explicitly when the divergence is canceled. In general,

$$\int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 + \Delta)^n} = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta} \right)^{(n - \frac{d}{2})} \quad (3.4.8)$$

$$\int \frac{d^d \ell}{(2\pi)^d} \frac{\ell}{(\ell^2 + \Delta)^n} = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{d}{2} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta} \right)^{(n - \frac{d}{2})} \quad (3.4.9)$$

In d dimensions, $g^{\mu\nu}$ obeys $g_{\mu\nu} g^{\mu\nu} = d$. Thus, if the numerator of a symmetric integrand contains $\ell^\mu \ell^\nu$, we should replace

$$\ell^\mu \ell^\nu \longrightarrow \frac{1}{d} \ell^2 g^{\mu\nu} \quad (3.4.10)$$

[1,8]

Chapter 4

Vacuum Polarization

4.1 The photon self-energy

In quantum electrodynamics vacuum polarization describes a process in which a background electromagnetic field produces virtual electron-positron pairs that change the distribution of charges and currents that generated the original electromagnetic field. It is also sometimes referred to as the self energy of the gauge boson (photon) [www.wikipedia encyclopedia]. In addition to the self energy of the electron, which has a classical counterpart, there also appears in quantum electrodynamics a self-energy of photon which does not have a simple classical analogue. Such a self-energy could be caused only by a self-interaction of the electromagnetic field, which has no place in the classical Maxwell equations. The terms in the S-matrix which describe this effect are represented by diagrams which have only two external photon lines and no external electron lines. We calculate the one-loop correction to the photon self-energy amplitude using the dimensional regularization scheme described in this section. This process is also known as the one-loop vacuum polarization diagram since virtual electron-positron pairs are created, polarizing the overall distribution of charge. The figure below displays the only contributing diagram. This diagram is mostly incorporated as a correction to the photon propagator, and therefore we allow the photon to be off-shell with an invariant momentum k^2 and the mass of the

fermion m . We will calculate the tensor amplitude, stripping off the external polarization vectors which can later be contracted with the tensorial quantity.

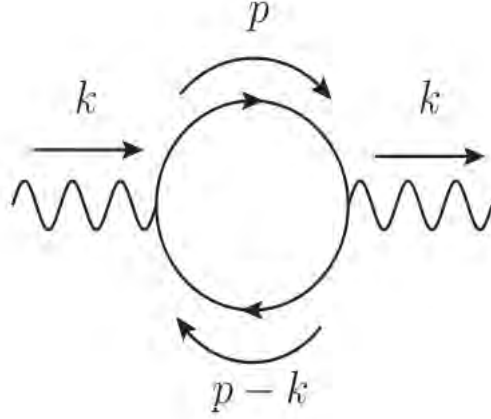


Figure 4.1: Vacuum polarization

On the figure above a photon decays into an electron-positron pair which recombines to a photon. The insertion of a photon self-energy part into an internal photon line has the effect of replacing the photon propagation function

$$\delta_{\mu\nu}D(k) = \delta_{\mu\nu}\left(\frac{1}{k^2}\right) \quad (4.1.1)$$

corresponding to this line by another function $D'_{\mu\nu}(k)$ [3,5,9]. For the second order self-energy parts, we find

$$D'_{\mu\nu}(k) = \delta_{\mu\nu}D(k) + D(k)\Pi_{\mu\nu}(k)D(k) \quad (4.1.2)$$

$$\Pi_{\mu\nu}(k) = ie^2 \int \frac{d^d p}{(2\pi)^d} \text{Tr} \left[\frac{\gamma_\mu(\gamma \cdot p + m)\gamma_\nu(\gamma \cdot p - \gamma \cdot k + m)}{(p^2 - m^2)((p - k)^2 - m^2)} \right] \quad (4.1.3)$$

The trace appearing on the above expression is over the spinor indices. This trace contains a maximum of four Dirac matrices can be evaluated using trace identities. Thus the numerator of equation (4.1.3) is

$$\begin{aligned} \text{Tr}[\gamma_\mu(\gamma \cdot p + m)\gamma_\nu(\gamma \cdot p - \gamma \cdot k + m)] &= \text{Tr}(\gamma_\mu\gamma \cdot p\gamma_\nu\gamma \cdot p - \gamma_\mu\gamma \cdot p\gamma_\nu\gamma \cdot k + m\gamma_\mu\gamma \cdot p\gamma_\nu + m\gamma_\mu\gamma_\nu\gamma \cdot p \\ &\quad - m\gamma_\mu\gamma_\nu\gamma \cdot k + m^2\gamma_\mu\gamma_\nu) \end{aligned}$$

$$\begin{aligned}
&= Tr(\gamma_\mu \gamma \cdot p \gamma_\nu \gamma \cdot p) - Tr(\gamma_\mu \gamma \cdot p \gamma_\nu \gamma \cdot k) + m Tr(\gamma_\mu \gamma \cdot p \gamma_\nu) + m Tr(\gamma_\mu \gamma_\nu \gamma \cdot p) - m Tr(\gamma_\mu \gamma_\nu \gamma \cdot k) + m^2 Tr(\gamma_\mu \gamma_\nu) \\
&= Tr(\gamma_\mu \gamma \cdot p \gamma_\nu \gamma \cdot p) - Tr(\gamma_\mu \gamma \cdot p \gamma_\nu \gamma \cdot k) + m^2 Tr \gamma_\mu \gamma_\nu \\
&\gamma_\mu \gamma \cdot p \gamma_\nu \gamma \cdot p = \gamma_\mu \gamma_\sigma \gamma_\nu \gamma_\rho p_\sigma p_\rho = (2\delta_{\mu\sigma} - \gamma_\sigma \gamma_\mu) \gamma_\nu \gamma_\rho p_\sigma p_\rho \\
&= [2\delta_{\mu\sigma} \gamma_\nu \gamma_\rho - \gamma_\sigma (2\delta_{\mu\nu} - \gamma_\nu \gamma_\mu) \gamma_\rho] p_\sigma p_\rho \\
&= [2\delta_{\mu\sigma} \gamma_\nu \gamma_\rho - 2\delta_{\mu\nu} \gamma_\sigma \gamma_\rho + \gamma_\sigma \gamma_\nu \gamma_\mu \gamma_\rho] p_\sigma p_\rho \\
&= [2\delta_{\mu\sigma} \gamma_\nu \gamma_\rho - 2\delta_{\mu\nu} \gamma_\sigma \gamma_\rho + 2\delta_{\mu\rho} \gamma_\sigma \gamma_\nu - \gamma_\sigma \gamma_\nu \gamma_\rho \gamma_\mu] p_\sigma p_\rho \\
&= 2[\delta_{\mu\sigma} \gamma_\nu \gamma_\rho - \delta_{\mu\nu} \gamma_\sigma \gamma_\rho + \delta_{\mu\rho} \gamma_\sigma \gamma_\nu] p_\sigma p_\rho - \gamma_\sigma \gamma_\nu \gamma_\rho \gamma_\mu p_\sigma p_\rho \\
&= 2[\delta_{\mu\sigma} \gamma_\nu \gamma_\rho - \delta_{\mu\nu} \gamma_\sigma \gamma_\rho + \delta_{\mu\rho} \gamma_\sigma \gamma_\nu] p_\sigma p_\rho - \gamma \cdot p \gamma_\nu \gamma \cdot p \gamma_\mu \\
&\Rightarrow Tr[\gamma_\mu (\gamma \cdot p) \gamma_\nu (\gamma \cdot p)] = Tr[\delta_{\mu\sigma} \gamma_\nu \gamma_\rho - \delta_{\mu\nu} \gamma_\sigma \gamma_\rho + \delta_{\mu\rho} \gamma_\sigma \gamma_\nu] p_\sigma p_\rho \\
&= 4(\delta_{\mu\sigma} \delta_{\nu\rho} - \delta_{\mu\nu} \delta_{\sigma\rho} + \delta_{\mu\rho} \delta_{\sigma\nu}) p_\sigma p_\rho \\
&= 4(p_\mu p_\nu - \delta_{\mu\nu} p^2 + p_\nu p_\mu) = 4(2p_\mu p_\nu - \delta_{\mu\nu} p^2) \tag{4.1.4}
\end{aligned}$$

In the same way

$$\begin{aligned}
&\gamma_\mu \gamma \cdot p \gamma_\nu \gamma \cdot k = 2[\delta_{\mu\sigma} \gamma_\nu \gamma_\rho - \delta_{\mu\nu} \gamma_\sigma \gamma_\rho + \delta_{\mu\rho} \gamma_\sigma \gamma_\nu] p_\sigma k_\rho - \gamma \cdot p \gamma_\nu \gamma \cdot k \gamma_\mu \\
&\Rightarrow Tr[\gamma_\mu \gamma \cdot p \gamma_\nu \gamma \cdot k] = Tr[\delta_{\mu\sigma} \gamma_\nu \gamma_\rho - \delta_{\mu\nu} \gamma_\sigma \gamma_\rho + \delta_{\mu\rho} \gamma_\sigma \gamma_\nu] p_\sigma k_\rho \\
&= 4(\delta_{\mu\sigma} \delta_{\nu\rho} - \delta_{\mu\nu} \delta_{\sigma\rho} + \delta_{\mu\rho} \delta_{\sigma\nu}) p_\sigma k_\rho \\
&= 4(p_\mu k_\nu - \delta_{\mu\nu} p \cdot k + k_\nu p_\mu) \tag{4.1.5}
\end{aligned}$$

And

$$m^2 Tr[\gamma_\mu \gamma_\nu] = 4m^2 \delta_{\mu\nu} \tag{4.1.6}$$

Using equations (44),(45) and (46)

$$\begin{aligned}
Tr[\gamma_\mu (\gamma \cdot p + m) \gamma_\nu (\gamma \cdot p - \gamma \cdot k + m)] &= 4(p_\mu p_\nu - \delta_{\mu\nu} p^2 + p_\nu p_\mu) + 4(p_\mu k_\nu - \delta_{\mu\nu} p \cdot k + k_\nu p_\mu) + 4m^2 \delta_{\mu\nu} \\
&= 4[2p_\mu p_\nu - p_\mu k_\nu - p_\nu k_\mu + \delta_{\mu\nu} (m^2 - k^2) + \delta_{\mu\nu} k \cdot p] \tag{4.1.7}
\end{aligned}$$

Hence equation (4.1.3) will be

$$\Pi_{\mu\nu}(k) = -4ie^2 \int \frac{d^d p}{(2\pi)^d} \frac{[2p_\mu p_\nu - p_\mu k_\nu - p_\nu k_\mu + \delta_{\mu\nu}(m^2 - k^2) + \delta_{\mu\nu} k \cdot p]}{(p^2 - m^2)((p - k)^2 - m^2)} \quad (4.1.8)$$

$$= -4ie^2(2B_{\mu\nu} - B_\mu p_\nu - B_\nu p_\mu - \delta_{\mu\nu} A_1 + \delta_{\mu\nu} k \cdot B) \quad (4.1.9)$$

where

$$B_{\mu\nu} = \int \frac{d^d p}{(2\pi)^d} \frac{p_\mu p_\nu}{(p^2 - m^2)((p - k)^2 - m^2)}, \quad B_\mu = \int \frac{d^d p}{(2\pi)^d} \frac{p_\mu}{(p^2 - m^2)((p - k)^2 - m^2)},$$

$$B_\nu = \int \frac{d^d p}{(2\pi)^d} \frac{p_\nu}{(p^2 - m^2)((p - k)^2 - m^2)}, \quad A_1 = \int \frac{d^d p}{(2\pi)^d} \frac{1}{((p - k)^2 - m^2)}$$

and

$$B = \int \frac{d^d p}{(2\pi)^d} \frac{p}{(p^2 - m^2)((p - k)^2 - m^2)}$$

The next step will be to explicitly evaluate the above integrals.

$B_{\mu\nu}$ tensor integral

$$B_{\mu\nu} = \int \frac{d^d p}{(2\pi)^d} \frac{p_\mu p_\nu}{(p^2 - m^2)((p - k)^2 - m^2)}$$

Applying the Feynman parameters formula to combine the factors in the denominator to one single factor

$$\begin{aligned} x((p-k)^2 - m^2) + (1-x)(p^2 - m^2) &= xp^2 + xk^2 - 2xk \cdot p - xm^2 + p^2 - xp^2 - m^2 + xm^2 = p^2 - 2xk \cdot p + xk^2 - m^2 \\ &= (p - xk)^2 + x(1-x)k^2 - m^2 \end{aligned}$$

Performing the shift $q = p - xk$ and $M = m^2 - x(1-x)k^2$,

$$\begin{aligned} B_{\mu\nu} &= \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{(q_\mu + xk_\mu)(q_\nu + xk_\nu)}{(q^2 - M)^2} \\ &= \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{q_\mu q_\nu}{(q^2 - M)^2} + k_\mu k_\nu \int_0^1 x^2 dx \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 - M)^2} \\ &\quad + \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{(q_\mu p_\nu + p_\mu q_\nu)}{(q^2 - M)^2} \end{aligned}$$

The q part of the second term can be evaluated using dimensional regularization (use equation (3.4.8))

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 - M)^2} = \frac{i}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(2 - \frac{d}{2})}{\Gamma(2)} \left(\frac{1}{M}\right)^{2 - \frac{d}{2}} = \frac{i}{(4\pi)^{2-\omega}} \Gamma(\omega) M^{-\omega}$$

where ω is defined by $\omega = 2 - \frac{d}{2}$

Similarly for the first term

$$\int \frac{d^d q}{(2\pi)^d} \frac{q_\mu q_\nu}{(q^2 - M)^2} = \frac{-i}{(4\pi)^{\frac{d}{2}}} \frac{\delta_{\mu\nu}}{2} \frac{\Gamma(1 - \frac{d}{2})}{\Gamma(2)} \left(\frac{1}{M}\right)^{1 - \frac{d}{2}} = \frac{-1}{2} \frac{i}{(4\pi)^{2-\omega}} \delta_{\mu\nu} \Gamma(-1 + \omega) M^{1-\omega}$$

The third term will vanish because it is odd function of q_μ .

The result of the momentum tensor integral is

$$B_{\mu\nu} = \frac{i}{(4\pi)^{2-\omega}} \left[\frac{-\delta_{\mu\nu} \Gamma(-1 + \omega)}{2} \int_0^1 dx M^{1-\omega} + k_\mu k_\nu \int_0^1 M^{-\omega} x^2 dx \right] \quad (4.1.10)$$

To perform the remaining x integral, M can be up to order ω ,

$$M^{-\omega} = e^{\ln M^{-\omega}} = e^{-\omega \ln M} \simeq 1 - \omega \ln M = 1 - \omega \ln(m^2 - x(1-x)k^2)$$

B_μ tensor integral

Using the same Feynman parameter shift as for $B_{\mu\nu}$ and the dimensional regularization trick

$$\begin{aligned} B_\mu &= \int \frac{d^d p}{(2\pi)^d} \frac{p_\mu}{(p^2 - m^2)((p-k)^2 - m^2)} = \int_0^1 dx \int \frac{d^d p}{(2\pi)^d} \frac{p_\mu}{[(1-x)(p^2 - m^2) + x((p-k)^2 - m^2)]^2} \\ &= \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{q_\mu}{[q^2 - M]^2} + \int_0^1 k_\mu x dx \int \frac{d^d q}{(2\pi)^d} \frac{1}{[q^2 - M]^2} \\ &= \frac{i}{(4\pi)^{2-\omega}} k_\mu \Gamma(\omega) \int_0^1 dx x M^{-\omega} \end{aligned} \quad (4.1.11)$$

A_1 scalar integral

Using dimensional regularization after performing the shift $q = p - k$

$$\begin{aligned} A_1 &= \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p-k)^2 - m^2} = \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 - m^2} \\ &= \frac{-i}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(1 - \frac{d}{2})}{\Gamma(1)} (m^2)^{1-\omega} = \frac{i}{(4\pi)^{2-\omega}} \Gamma(-1 + \omega) (m^2)^{1-\omega} \end{aligned} \quad (4.1.12)$$

And expanding $(m^2)^{-\omega}$

$$(m^2)^{-\omega} \simeq 1 - \omega \ln m^2$$

Now let us calculate the integral terms in equations (4.1.10) and (4.1.11)

$$\begin{aligned} \int_0^1 dx M^{1-\omega} &= \int_0^1 dx M(1 - \omega \ln M) \\ &= \int_0^1 dx (m^2 - x(1-x)k^2) - \omega \int_0^1 dx (m^2 - x(1-x)k^2) \ln(m^2 - x(1-x)k^2) \\ &= m^2 - \frac{k^2}{6} - \omega [m^2 \int_0^1 dx \ln(m^2 - x(1-x)k^2) - \int_0^1 dx x \ln(m^2 - x(1-x)k^2) + \int_0^1 dx x^2 \ln(m^2 - x(1-x)k^2)] \end{aligned} \quad (4.1.13)$$

$$\begin{aligned} \int_0^1 dx \ln(m^2 - x(1-x)k^2) &= \int_0^1 dx \ln m^2 + \int_0^1 dx \ln(1 - x(1-x)\eta) \\ &= \int_0^1 dx \ln m^2 + \int_0^1 dx \ln \eta (x - \frac{1}{2}(1+\beta))(x - \frac{1}{2}(1-\beta)) \end{aligned}$$

Where $\eta = \frac{k^2}{m^2}$, $\beta = \sqrt{1 - \frac{4}{\eta}}$ and

$$(1 - x(1-x)\eta) = \eta (x - \frac{1}{2}(1+\beta))(x - \frac{1}{2}(1-\beta))$$

$$\begin{aligned} \int dx \ln m^2 (1 - x(1-x)\eta) &= x \ln m^2 + x \ln \eta (x - \frac{1}{2}(1+\beta))(x - \frac{1}{2}(1-\beta)) \\ &\quad - 2x + 1 - \frac{1}{2} \ln(\eta^{-1} - x(1-x)) + \frac{\beta}{2} \ln\left(\frac{2x-1+\beta}{2x-1-\beta}\right) \\ &= x \ln m^2 + x \ln(1 - x(1-x)\eta) - 2x + 1 - \frac{1}{2} \ln(\eta^{-1} - x(1-x)) + \frac{\beta}{2} \ln\left(\frac{2x-1+\beta}{2x-1-\beta}\right) \\ &\Rightarrow \int_0^1 dx \ln(m^2 - x(1-x)k^2) = \ln m^2 - 2 + \beta \ln\left(\frac{1+\beta}{1-\beta}\right) \\ &= \ln m^2 - 2 + \sqrt{1 - \frac{4m^2}{k^2}} \ln\left(\frac{1 + \sqrt{1 - \frac{4m^2}{k^2}}}{1 - \sqrt{1 - \frac{4m^2}{k^2}}}\right) \end{aligned} \quad (4.1.14)$$

The fourth term in equation (4.1.13) is

$$\int dx x \ln(m^2 - x(1-x)k^2) = \int \ln m^2 x dx + \int dx x \ln(1 - x(1-x)\eta)$$

Using integration by parts

$$\begin{aligned}
& \int dx x \ln(1 - x(1 - x)\eta) = x \int \ln(1 - x(1 - x)\eta) dx \\
& - \int [x \ln(1 - x(1 - x)\eta) - 2x + 1 - \frac{1}{2} \ln(\eta^{-1} - x(1 - x)) + \frac{\beta}{2} \ln(\frac{2x - 1 + \beta}{2x - 1 - \beta})] dx \\
& = \frac{1}{2}(x + \frac{1}{2}) \int \ln(1 - x(1 - x)\eta) dx + \int x dx - \frac{1}{2} \int dx - \frac{1}{4} \ln \eta \int dx - \frac{\beta}{4} \int \ln(\frac{2x - 1 + \beta}{2x - 1 - \beta}) dx \\
& = \frac{1}{2}(x + \frac{1}{2}) [x \ln(1 - x(1 - x)\eta) - 2x + 1 - \frac{1}{2} \ln(\eta^{-1} - x(1 - x)) + \frac{\beta}{2} \ln(\frac{2x - 1 + \beta}{2x - 1 - \beta})] \\
& \quad + \int x dx - \frac{1}{2} \int dx - \frac{1}{4} \ln \eta \int dx - \frac{\beta}{4} \int \ln(\frac{2x - 1 + \beta}{2x - 1 - \beta}) dx \\
& = \frac{1}{2}(x + \frac{1}{2}) [x \ln(1 - x(1 - x)\eta) - 2x + 1 - \frac{1}{2} \ln(\eta^{-1} - x(1 - x)) + \frac{\beta}{2} \ln(\frac{2x - 1 + \beta}{2x - 1 - \beta})] \\
& \quad + \frac{x^2}{2} - \frac{x}{2} - \frac{x}{4} \ln \eta - \frac{\beta^2}{8} [\ln(2x - 1 + \beta)(2x - 1 - \beta) - 2]
\end{aligned}$$

Substituting the range of x (from 0 to 1)

$$\begin{aligned}
\int_0^1 dx x \ln(1 - x(1 - x)\eta) &= -1 + \frac{1}{2} \beta \ln\left(\frac{1 + \beta}{1 - \beta}\right) = -1 + \frac{1}{2} \sqrt{1 - \frac{4m^2}{k^2}} \ln\left(\frac{1 + \sqrt{1 - \frac{4m^2}{k^2}}}{1 - \sqrt{1 - \frac{4m^2}{k^2}}}\right) \\
\Rightarrow \int_0^1 dx x \ln(m^2 - x(1 - x)k^2) &= \int_0^1 \ln m^2 x dx + \int_0^1 dx x \ln(1 - x(1 - x)\eta) \\
&= \frac{1}{2} \ln m^2 - 1 + \frac{1}{2} \sqrt{1 - \frac{4m^2}{k^2}} \ln\left(\frac{1 + \sqrt{1 - \frac{4m^2}{k^2}}}{1 - \sqrt{1 - \frac{4m^2}{k^2}}}\right) \tag{4.1.15}
\end{aligned}$$

The last term of equation (4.1.13) is

$$\begin{aligned}
\int dx x^2 \ln(m^2 - x(1 - x)k^2) &= \int dx x^2 \ln m^2 + \int dx x^2 \ln(1 - x(1 - x)\eta) \\
&= \frac{1}{3} x^3 \ln m^2 + \int dx x^2 \ln(1 - x(1 - x)\eta)
\end{aligned}$$

Using integration by parts

$$\int dx x^2 \ln(1 - x(1 - x)\eta) = x \int dx x \ln(1 - x(1 - x)\eta) - \int [\int dx x \ln(1 - x(1 - x)\eta)] dx$$

$$\begin{aligned}
&= x \int dx x \ln(1 - x(1 - x)\eta) \\
&-\frac{1}{2} \int (x + \frac{1}{2}) [x \ln(1 - x(1 - x)\eta) - 2x + 1 - \frac{1}{2} \ln(\eta^{-1} - x + x^2) + \frac{\beta}{2} \ln(\frac{2x - 1 + \beta}{2x - 1 - \beta})] dx \\
&\quad + \frac{1}{2} \int [x^2 - x - \frac{x}{2} + \frac{\beta^2}{2}] dx - \frac{\beta^2}{8} \int \ln(2x - 1 + \beta)(2x - 1 - \beta) dx \\
&= x \int dx x \ln(1 - x(1 - x)\eta) - \frac{1}{2} \int x^2 \ln(1 - x(1 - x)\eta) dx - \frac{1}{8} \int \ln(1 - x(1 - x)\eta) dx \\
&\quad + \frac{1}{4} \int (x + \frac{1}{2}) \ln \eta dx - \int (1 - 2x)(x + \frac{1}{2}) dx \\
&-\frac{\beta}{4} \int (x + \frac{1}{2}) \ln(\frac{2x - 1 + \beta}{2x - 1 - \beta}) dx - \frac{1}{2} \int [x^2 - x - \frac{x}{2} + \frac{\beta^2}{2}] dx + \frac{\beta^2}{8} \int \ln(2x - 1 + \beta)(2x - 1 - \beta) dx \\
&= \frac{2}{3} [x \int dx x \ln(1 - x(1 - x)\eta) - \frac{1}{8} \int \ln(1 - x(1 - x)\eta) dx + \frac{1}{4} \int (x + \frac{1}{2}) \ln \eta dx - \int (1 - 2x)(x + \frac{1}{2}) dx \\
&-\frac{\beta}{4} \int (x + \frac{1}{2}) \ln(\frac{2x - 1 + \beta}{2x - 1 - \beta}) dx - \frac{1}{2} \int [x^2 - x - \frac{x}{2} + \frac{\beta^2}{2}] dx + \frac{\beta^2}{8} \int \ln(2x - 1 + \beta)(2x - 1 - \beta) dx]
\end{aligned}$$

Hence, evaluating the range of x (0 to 1) on the above equation

$$\begin{aligned}
\int_0^1 dx x^2 \ln(1 - x(1 - x)\eta) &= -\frac{13}{18} + \frac{2m^2}{3k^2} + \frac{2}{3} (1 - \frac{4m^2}{k^2}) \sqrt{1 - \frac{4m^2}{k^2}} \ln\left(\frac{1 + \sqrt{1 - \frac{4m^2}{k^2}}}{1 - \sqrt{1 - \frac{4m^2}{k^2}}}\right) \\
\Rightarrow \int_0^1 dx x^2 \ln(m^2 - x(1 - x)k^2) &= \frac{1}{3} \ln m^2 - \frac{13}{18} + \frac{2m^2}{3k^2} + \frac{2}{3} (1 - \frac{4m^2}{k^2}) \sqrt{1 - \frac{4m^2}{k^2}} \ln\left(\frac{1 + \sqrt{1 - \frac{4m^2}{k^2}}}{1 - \sqrt{1 - \frac{4m^2}{k^2}}}\right)
\end{aligned} \tag{4.1.16}$$

Substituting equations (4.1.14), (4.1.15) and (4.1.16) in equation (4.1.13)

$$\begin{aligned}
\int_0^1 dx M^{1-\omega} &= m^2 - \frac{k^2}{6} - \omega \left[(\ln m^2 - 2 + \sqrt{1 - \frac{4m^2}{k^2}} \ln\left(\frac{1 + \sqrt{1 - \frac{4m^2}{k^2}}}{1 - \sqrt{1 - \frac{4m^2}{k^2}}}\right)) (m^2 - \frac{1}{2}) \right. \\
&\quad \left. - \frac{13}{18} + \frac{2m^2}{3k^2} + \frac{2}{3} (1 - \frac{4m^2}{k^2}) \sqrt{1 - \frac{4m^2}{k^2}} \ln\left(\frac{1 + \sqrt{1 - \frac{4m^2}{k^2}}}{1 - \sqrt{1 - \frac{4m^2}{k^2}}}\right) \right]
\end{aligned} \tag{4.1.17}$$

And also

$$\int_0^1 dx x M^{-\omega} = \frac{1}{2} - \omega \left[\frac{1}{2} \ln m^2 - 1 + \frac{1}{2} \sqrt{1 - \frac{4m^2}{k^2}} \ln\left(\frac{1 + \sqrt{1 - \frac{4m^2}{k^2}}}{1 - \sqrt{1 - \frac{4m^2}{k^2}}}\right) \right] \tag{4.1.18}$$

$$\int_0^1 dx x^2 M^{-\omega} = \frac{1}{3} - \omega \left[-\frac{13}{18} + \frac{2m^2}{3k^2} + \frac{2}{3} \left(1 - \frac{4m^2}{k^2} \right) \sqrt{1 - \frac{4m^2}{k^2}} \ln \left(\frac{1 + \sqrt{1 - \frac{4m^2}{k^2}}}{1 - \sqrt{1 - \frac{4m^2}{k^2}}} \right) \right] \quad (4.1.19)$$

The next step will be to substitute equations (4.1.17-19) in $B_{\mu\nu}, B_\mu, B_\nu$ and A_1 to find $\Pi_{\mu\nu}(k)$. Finally, combining all the results, the tensor amplitude is,

$$\Pi_{\mu\nu}(k) = (k_\mu k_\nu - \delta_{\mu\nu} k^2) \Pi(k^2) \quad (4.1.20)$$

where up to order $O(1)$

$$\Pi(k^2) = \frac{\alpha}{3\pi} \left(\frac{1}{\omega} + \frac{5}{3} + \frac{4m^2}{k^2} - \left(2 + \frac{4m^2}{k^2} \right) \sqrt{1 - \frac{4m^2}{k^2}} \ln \left(\frac{1 + \sqrt{1 - \frac{4m^2}{k^2}}}{1 - \sqrt{1 - \frac{4m^2}{k^2}}} \right) - \ln m^2 \right) \quad (4.1.21)$$

Dimensional regularization makes the divergences evident. Only the term proportional to $\frac{1}{\omega}$ is ultraviolet divergent. To absorb this divergence some non-physical parameter in the Lagrangian can be renormalized. In this case we are evaluating a correction to the photon propagator, the bare photon field $A_{o,\mu}$ should be renormalized by a proper definition of the Z_A factor (where $A_{o,\mu} = Z_A A_{R,\mu}$, where $A_{R,\mu}$ is the renormalized field). This will add a Feynman rule that will generate a divergent tree diagram (a photon propagator) proportional to $\frac{1}{\omega}$. Of course since we can always add finite terms, there is an ambiguity in the choice of Z_A . In this case we choose Z_A such that the amplitude will receive a contribution $-\Pi(0) = -\frac{\alpha}{3\pi} \left(\frac{1}{\omega} - \ln m^2 \right)$, successfully cancelling the divergence [3]. The renormalized, denoted $\hat{\Pi}(k^2)$ by a hat is,

$$\hat{\Pi}(k^2) = \frac{\alpha}{3\pi} \left(\frac{5}{3} + \frac{4m^2}{k^2} - \left(2 + \frac{4m^2}{k^2} \right) \sqrt{1 - \frac{4m^2}{k^2}} \ln \left(\frac{1 + \sqrt{1 - \frac{4m^2}{k^2}}}{1 - \sqrt{1 - \frac{4m^2}{k^2}}} \right) \right) \quad (4.1.22)$$

Chapter 5

Conclusion

In quantum field theory it is possible to make divergences manifestly finite, so as to manipulate them, using regulating methods such as Pauli-Villars regularization and dimensional regularization method.

In applying regularization technique a regulator introduces a parameter, say ϵ , to the apparently divergent quantity which is a function of ϵ to reduce the original quantity in the limit $\epsilon \rightarrow 0$, $\lim_{\epsilon \rightarrow 0} O(\epsilon) = 0$, the apparent divergence is recovered. Yet for finite but very small ϵ the quantity is finite, $|O(\epsilon)| < \infty$.

The self energy of the free Dirac electron is positive and logarithmically dependent on the cut-off, Λ , with coefficient $\frac{3\alpha m}{2\pi}$.

The photon self-energy is a one loop vacuum polarization since virtual electron-positron pairs are created, polarizing the overall distribution of charge.

It is possible to calculate the one-loop correction to the photon self-energy amplitude using the dimensional regularization method.

In the tensor amplitude of photon self-energy it appears some divergent term. To absorb this divergence some non-physical parameter in the Lagrangian can be renormalized.

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Declaration

This thesis is my original work. It has not been presented for a degree in any other University and that all the sources of material used for the thesis have been fully acknowledged.

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