Constraint Qualification In Vector Optimization

ADDIS ABABA UNIVERSITY
COLLEGE OF COMPUTATIONAL AND NATURAL SCIENCE
DEPARTMENT OF MATHEMATICS

A project Submitted in partial fulfillment of the requirement of the degree of master of science in mathematics

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Stream: Optimization
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Contents

abstract

In this Project some necessary and sufficient optimality conditions for the weakly efficient solutions of vector optimization problems (VOP) with finite inequality constraints are shown by using constraints qualifications in differentiable case. Suitable generalized convexity properties are utilized in the study of constraint qualification in the case of efficient optimality conditions. Constraint qualification establishes the sign of multiplier vectors associated with Fritz-John conditions.The weak constraint qualification,the Generalized Guignared Constraint Qualification (GGCQ), implies necessary and sufficient condition by deriving Kuhn-Tucker and Fritz-John Conditions.

Keywords: Multiobjective optimization, Vector Optimization, Efficient solution, Constraint Qualification, Optimality Condition and convexity for vector function.

Nomenclature

. $Cl(S)$ is closure of S for any set.
. $Clco(S)$ denotes closure of convex hall (S).
. $int(S)$ denotes the set of interior point of (S).
. $\Re^p$ represents Euclidean space of dimension p.
. $\Re^p_-$ denotes the non positive orthant of $\Re^p$.
. $\Re^p_+$ denotes the nonnegative orthant of $\Re^p$
. $f_i = 1, \ldots, p$ p component of function f.
. X^T the superscript T denotes the transpose of the vector $X \in \Re^n$.
. If $f : \Re^n \to \Re^p$, we denote $\nabla f = (\nabla f_1, \ldots, \nabla f_p)$
. $I = \{1, \ldots, p\}$, $J = \{1, \ldots, m\}$ and if $I > 1$, and $i \in I$, we denote $I^i = I \backslash \{i\}$
. Inequalities $x \geq 0$ for arbitrary vectors $x \in R^n$ are defined componentwise.
. $t_n \downarrow 0$ means that t_n approaches zero from the righthand side $(t_n \to 0^+)$.
. $f'(x)$ is derivative of a function f at x.

Chapter 1

Introduction

1.1 Optimization Theory:

Optimization is essentially the art, science and mathematics of choosing the best among a given set of finite or infinite alternatives. It arises in a vast variety of problems, such as Engineering, economics, operation researchers, natural sciences, and many other fields with problems that need optimal decision making.
Mathematical models for these optimization problems can be constructed by specifying a constraint set X, which consists of the available decisions x, and a cost or objective function $f(x)$ that maps each $x \in X$ into a scalar and represents a measure of undesirability of choosing decision x. This problem can then be written as

$$\begin{aligned} &\min f(x) \\ &\text{s.t} \quad x \in X, \end{aligned} \tag{1.1}$$

In this project, we focus on the case where each decision x is an n-dimensional vector, i.e., x is an n-tuple of real numbers $(x_1, \dots, x_n)$. Hence, the constraint set X is a subset of $\Re^n$.
A new inspiration toward the development of optimization began with radical change to the development of linear programming in the late 1940's. A linear programming problem consists of linear functions as objective and constraints. However in the early 1950's it was observed that there are important application problems which involved nonlinear functions as well as constraints represented by inequalities. Thus in order to tackle the inequality constraints some new mathematics was developed and in this direction was given by Fritz John(1948) and later refined by Kuhn and Tucker (1951) [10]. However the use of inequality constraints was also studied by W. Karush

way back in 1939.Thus W. Karush and Kuhn-Tucker conditions together are called the Karush-Kuhn-Tucker (KKT)conditions which extends the Lagrangian theory to include single-objective nonlinear programming problem. In mathematical optimization, the method of Lagrangian Multiplier(named after Joseph Louis Lagrange) is a method for finding the maximum/minimum of a function subject to constraints.
We consider the following mathematical programming problem for inequality constrained optimization problem:

$$\begin{aligned} &\min f(x) \\ &\text{s.t} \quad g_j(x) \leq 0, j \in J = 1, \ldots, m. \end{aligned} \tag{1.2}$$

where $f : S \subseteq \Re^n \to \Re$ and $g : \Re^n \to \Re^m$ are continuously differentiable and the feasible set S define as $S = \{x \in \Re^n : g(x) \leq 0\}$. Then construct the Lagrangian as: $L(x, \mu_0, \mu) = \mu_0 f(x) + \mu^T g(x)$. Hence the multiplier $\mu_0 \geq 0, \mu \geq 0$ and $(\mu_0, \mu) \neq (0, 0)$ is said to be Lagrangian multiplier corresponding to the optimization problem is called Fritz-John Optimality condition and if $\mu_0 > 0$ it is said to be KKT optimality condition. This approach amounts to solving the system of equation for x and μ

$$\begin{aligned} &\nabla f(x) + \mu^T \nabla g(x) = 0 \\ &\mu g_j(x) = 0, \quad for \quad all \quad j = 1, \ldots, m. \\ &\mu \geq 0 \end{aligned} \tag{1.3}$$

where $\mu = (\mu_1, \ldots,_m)$. Later these conditions are extended to multiobjective optimization problem.

1.1.1 Vector Optimization

In vector optimization one investigates optimal elements of a set in a pre-ordered space. The problem of looking for these optimal elements, if they exist at all, is called a vector optimization problem [16]. These problems are also called multiobjective (or multi criteria or Pareto) optimization problems or one speaks of multi criteria decision making. Vector optimization problems arise frequently not only in mathematics but also in engineering and economics. In the last decades vector optimization has been extended to problems with set-valued maps. This field, called set optimization, has important applications to variational inequalities and optimization problems with multivalued data.
Vector optimization problem with inequality constraints in mathematical

terms ,can be described as

$$\begin{aligned} &\min f(x) \\ &\text{s.t} \quad \bar{x} \in S, \end{aligned} \tag{1.4}$$

where $f : \Re^n \to \Re^p, g : \Re^n \to \Re^m, S = \{x \in \Re^n : g_j(x) \leq 0, j \in J = 1, \ldots, m\}$ and is said feasible set.
The concept of an ideal point that a single feasible point that minimizes all the objectives is impossible. In general multiobjective optimization problems could not be combined into a single objective to determine the optimal point as it is in scalar case. Moreover, the objectives of multiobjective optimization problems often conflict with each other in real fact. As a result, to determine the optimal point in vector optimization problems the concept of Pareto optimality, characterizing an efficient solution, has been introduced. The solutions of a VOP are referred as noninferior, efficient, Pareto-optimal,or non dominated solutions. Other variants include weakly efficient solutions, local efficient solutions, etc.
The concept of solution for a VOP was introduced at the turn of the century (1896) by Pareto, a prominent economist, but it is only since 1951, when Kuhn and Tucker published necessary and sufficient conditions for (proper) noninferiority, that considerable effort has been devoted to developing procedures for generating non inferior solutions to a VOP [16] One of the most important application of the vectorial optimization techniques is found in the study of vectorial mathematical programming problems, and as a particular case in multiobjective programming problems.
Consider the vector optimization problem (VOP)(1.2). If $C = \Re^p_+$(C is an ordering cone on $\Re^p$) and $K = \Re^m_+$ (K is an ordering cone on $\Re^m$) we have a multiobjective Pareto program. If $p = 1$ and $C = \Re_+$ we have a scalar program. For the solution concepts in vector optimization problem, we consider only efficient and weak-efficient solutions, but in vector optimization there are other solution concepts as ideal, strong, strict or proper efficient points.
A nonempty subset C of $\Re^n$ is a cone when $td \in C$, for all $t \geq 0$ and $d \in C$,a cone C is pointed if $C \cap (-C) = \{0\}$ and C is a convex cone if C is a cone which is a convex set.
Solution Concepts: Let (X, C) be a linearly ordered space, where C is a pointed convex cone, and let $E \subset Y$ be a nonempty set. Then a point $\bar{x} \in E$ is said to be an efficient or minimal element of E if there is no $x \in E, x \neq \bar{x}$, such that $f(x) \leq f(\bar{x})$. The set of efficient elements of E is denoted by $Min(E, C)$, i.e there is no $x \in E$, such that $\bar{x} \in y + C \setminus \{0\}$ and $(E - \bar{x}) \cap (-C) = \{0\}$.
In VOP, as in scalar case, Lagrangian Multipliers for Fritz-John and KKT

optimality conditions in solution concepts plays similar role and are given as follows.
Consider the following inequality constrained vector optimization problem:

$$\begin{aligned} &\min f(x) \\ &\text{s.t} \quad g(x) \leq 0, \end{aligned} \tag{1.5}$$

where $f : S \subseteq \Re^n \to \Re^p$ and $g : \Re^n \to \Re^m$ are continuously differentiable and the feasible set S is defined as $S = \{x \in \Re^n : g(x) \leq 0\}$. Then the Lagrangian Multipliers for Fritz-John and KKT optimality conditions in VOP solution concepts are given as follows:

$$L(x, \lambda, \mu) = \lambda^T f(x) + \mu^T g(x) \tag{1.6}$$

If $\bar{x} \in X$ is an efficient solution to Problem (VOP), then the Fritz-John optimality condition states that there exist vectors $\lambda, \mu : (\lambda, \mu) \in \Re^p_+ \times \Re^m_+$ such that

$$\begin{aligned} \sum_{i=1}^{p} \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j \nabla g_j(\bar{x}) = 0 \\ \sum_{j=1}^{m} \mu_j g_j(\bar{x}) = 0, \\ \lambda \geq 0 \quad \text{and} \quad \mu \geq 0. \end{aligned} \tag{1.7}$$

And KKT optimality condition would be set by restricting multiplier λ to positive vectors. Lagrangian Multipliers associated with all the components of objective functions to be positive for the KKT conditions. This calls attention, since if the Lagrangian Multiplier corresponding to some components of the objective functions is equal to zero, then the component has no role in the corresponding necessary conditions. Hence in order to ensure that the KKT conditions are necessary for optimality a constraint qualification (CQ) is needed.

1.1.2 Constraint Qualification in Vector Optimization Problem

Constraint qualifications are properties of the algebraic description of the feasible set that allow its local geometry at a feasible point $\bar{x}$ to be recovered from the gradient of the active constraints at $\bar{x}$. It play a fundamental role for both a scalar and a vector optimization problem, since they ensure the validity of the Karush-Kuhn- Tucker necessary optimality conditions, i.e., the positivity of the multiplier associated with the objective functions in Fritz John conditions. Constraint qualifications impose some regularity on the behavior of the feasible region. Guignard, Abadie, Cottle, Slater, Mangasarian-Fromovitz constraint qualifications and linear constraint qualification are among the most utilized constraint qualifications. Among these constraint qualification Guignard constraint qualification is considered as the weakest one in the sense that all the other constraint qualifications imply it. [1] As in the case of sufficient optimality conditions, the study of constraint qualifications has benefited from the developments on generalized convexity, so that several conditions have been established by imposing some suitable generalized convexity properties on the constraint functions.

A vector optimization problem where the objective and the constraint functions are differentiable and the feasible region is defined by inequality constraints are considered. Unlike the scalar case,in vector optimization constraint qualifications must concern the behavior of both the objective functions and the feasible region. Like in the scalar case, generalized convexity plays a key role in deriving new constraint qualifications for vector optimization problems.

This project is organized as follows , section 2 gives some preliminaries and definition, third section discusses optimality conditions of vector optimization and the last section discusses various Constraint Qualification conditions for vector optimization and their relations.

Chapter 2

Important Definition, Notions and Properties

In this section we present some preliminary definitions, results and important cones which are quite relevant to discusses various Constraint Qualification conditions for vector optimization and their relations.

2.1 Cones

We present some relatively important cones, which are crucially important in the discussion of constraint qualification.These are cone, pointed cone,convex cone,etc.

Definition 2.1. *[17]A nonempty subset K of $\Re^n$ is called a cone if it is closed under scalar multiplication, i.e., $\lambda x \in K$ when $x \in K$ and $\lambda > 0$ such a set is a union of half lines emanating from origin. The origin itself may or may not be included. A cone K is pointed if $K \cap (-K) = \{0\}$.*

Definition 2.2. *[17]A convex cone is a cone which is a convex set. (Note:- Many authors do not call K a convex cone unless in addition K contains the origin. Thus for these authors a convex cone is a non-empty convex set which is closed under non-negative scalar multiplication.)*

The concept of contingent cone is very helpful for the investigation of optimality conditions. A contingent cone to a set S at some $\bar{x} \in cl(S)$ describes a local approximation of the set $S - \{\bar{x}\}$. The cones of directions are used to derive optimality conditions and weakest constraint qualifications for the vector optimization problem. Contingent cones, cones of feasible directions and cones of attainable directions are defined as follows.

Definition 2.3. *[9]Let S be a non-empty subset of a real normed space $(X, \| . \|)$.*

- *Let some $\bar{x} \in cl(S)$ be given. An element $h \in X$ is called tangent to S at $\bar{x}$, if there are a sequence $(x_n)_{n\in\mathbb{N}}$ of elements $x_n \in S$ and a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of positive real numbers λ_n so that*

$$\bar{x} = \lim_{n\to\infty} x_n$$
and
$$h = \lim_{n\to\infty} \lambda_n(x_n - \bar{x}).$$

- *The set $T(S, \bar{x})$ of all tangents to S at $\bar{x}$ is called the contingent cone(or the Bouligand tangent cone) to S at $\bar{x}$.*

- *The set $A(S, \bar{x}) = \{v \in \Re^n \mid \exists\delta > 0, \exists\gamma : [0, \delta] \to \Re^n : \gamma(0) = \bar{x}, \gamma(t) \in S, \forall t \in (0, \delta], \gamma'(0) = v\}$ is the cone of the attainable directions to S at $\bar{x}$.*

- *The set $Z(S, \bar{x}) = \{v \in \Re^n\ \exists\delta > 0$ such that $\bar{x} + tv \in S, \forall t \in (0, \delta]\}$ is the cone of the feasible directions to S at $\bar{x}$.*

Definition 2.4. *[17]Let C be a non-empty convex set in $\Re^n$. We shall say that C recedes in the direction D if C include all the half-lines in the direction D which start at points of C. In other words, C recedes in the direction of Y, where $Y \neq 0$, if and only if $x + \lambda Y \in C$ for every $\lambda \geq 0$ and $x \in C$. The set of all vectors $Y \in \Re^n$ satisfying the latter condition including $Y = 0$, will be called the recession cone of C. The recession cone of C will be denoted by O^+C. Direction in which C recedes will also be referred to as directions of recession of C.*

The most usual ordering cone in a finite dimensional space in $\Re^n$ is the non-negative orthant $\Re^n_+$. This set is a pointed, closed and convex cone that defines the componentwise partial ordering on R^n_+, also called a Pareto Order. Let's consider the definition of a set of nonnegative elements in $\Re^n$ can be used to derive a geometric interpretation of properties of orders.

Definition 2.5. *[9]A binary relation $\leq$ on Y is called a partial ordering on Y if the following properties are satisfied (for arbitrary $x; y; z; u \in Y$ and $\lambda \in \Re_+$):*
(i) $x \leq x$;
(ii) $x \leq y; y \leq z \Rightarrow x \leq z$;
(iii) $x \leq y; u \leq z \Rightarrow x + u \leq y + z$;
(iv) $x \leq y \Rightarrow \lambda x \leq \lambda y$:

A partial ordering is called antisymmetric if the following condition holds: $x \leq y; y \leq x \Rightarrow x = y$.

Extending the above definition to $\Re^n$ and for two vectors x and y in $\Re^n$, we shall use the following conventions. If $x = (x_1, \ldots, x_n) \in \Re^n$ and $y = (y_1, \ldots, y_n) \in \Re^n$, then
$x > y$ if and only if $x_i > y_i, i = 1, \ldots, n$
$x \geqq y$ if and only if $x_i \geq y_i, i = 1, \ldots, n$
$x \geq y$ if and only if $x_i \geq y_i, i = 1, \ldots, n$,but $x \neq y$
$x = y$ if and only if $x_i = y_i, i = 1, \ldots, n$
These equivalent views on orders will be extremely useful in multicriteria optimization.
Similarly we consider the equivalent convention for inequalities $<, \leq, \leqq$.

2.2 Convex Functions

Convexity play a crucial (central) role in optimization. Convexity of the objective and constraint functions guarantee the sufficiency of Karush Kahn Tucker (KKT) condition for optimality are assumed to be convex. This is precisely where convexity enters the larger picture of optimization and goes on to play a central role. Modern optimization by large is based on convexity and notions relying on it. A function $f : S \subseteq \Re^n \rightarrow \Re$ is said to be convex over the convex set if for any $x_1, x_2 \in S$ and $\lambda \in [0, 1]$ we have $f(x_1 + \lambda(x_2 - x_1)) \leq \lambda f(x_2) + (1 - \lambda)f(x_1)$
It is clear from the above expression that every critical point (point x with $\nabla f(x) = 0$) of a convex function is a global minima. In fact even without differentiability one can easily show that every local minima of a convex function is global [10]. This simple feature makes convexity such an attractive thing in optimization.
The optimization problem:

$$\text{minimize } f(x) \text{ for } x \in X \subset \Re^n, \\ \text{subject to } g(x) \leqq 0,$$

is called a convex program if the functions involved are convex on some convex subset X of $\Re^n$. Convex programs have many useful properties. Among them,
1. The set of all feasible solutions is convex.
2. Any local minimum is a global minimum.
3.The Karush-Kuhn-Tucker optimality conditions are sufficient for a minima.

Definition 2.6. *[20]A subset X of $\Re^n$ is convex if for every $x_1, x_2 \in X$ and $0 < \lambda < 1$, we have*

$$\lambda x_1 + (1 - \lambda)x_2 \in X. \tag{2.1}$$

and a vector function $f : X \to \Re^p$ defined on a convex subset X of $\Re^n$ is convex if for any $x_1, x_2 \in X$ and $0 < \lambda < 1$, we have

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2). \tag{2.2}$$

An important property of a differentiable convex function is that any stationary point is also a global minimum point.

Definition 2.7. *A function $f : X \to \Re^p$ is quasiconvex on X if*

$$f(x_1) \leq f(x_2) \Rightarrow f(\lambda x_1 + (1 - \lambda)x_2) \leq f(x_2), \forall x_1, x_2 \in X, \forall \lambda \in [0, 1] \tag{2.3}$$

and for differentiable function $f : X \to \Re^p$ is said to be quasiconvex on X if $f(x_1) \leq f(x_2) \to (x_1 - x_2)\nabla f(x_2) \leq 0, \forall x_1, x_2 \in X$

In the definition of some constraint qualifications pseudoconvexity of components of objective functions and constraint functions is required.

Definition 2.8. *[20]Let $f : X \to \Re^p$ be differentiable on the open set $X \subset \Re^n$; then f is pseudoconvex on X if;*

$$f(x_1) < f(x_2) \Rightarrow (x_1 - x_2)\nabla f(x_2) < 0, \forall x_1, x_2 \in X$$

or equivalently if

$$(x_1 - x_2)\nabla f(x_2) \geq f(x_1) \geq f(x_2), \forall x_1, x_2 \in X.$$

The function $f : X \to \Re$ is called pseudoconcave if $-f$ is pseudoconvex. Convexity can be weakened to various kinds of generalized convex. Many classes of generalized convex functions have been defined and studied. Now we recall some definitions.

Definition 2.9. *[20] A set X is said to be invex if there exists a vector-valued function $\eta : X \times X \to \Re^n$ such that for each $x_1, x_2 \in X$ such that $\eta(x_1, x_2) \neq 0$ if $x_1 \neq x_2$, and $x_2 + \lambda\eta(x_1, x_2) \in X, \forall \lambda \in [0, 1]$*
A function $f : X \to \Re$, X open subset of $\Re^n$, is said to be invex on X with respect to η if $f(x_1) - f(x_2) \geq \eta^T(x_1, x_2)\nabla f(x_2), \forall x_1, x_2 \in X$.

The name invex stands for invariant convex. [20]

Chapter 3

Optimality conditions

3.1 Optimality conditions in Vector Optimization

In this section, we derive necessary and sufficient optimality conditions for the weakly efficient solutions of (VOP) for finite differentiable inequality constraints.
We consider the following vector optimization problem (VOP):

$$\begin{aligned} &\min f(x) = (f_1(x), \ldots, f_p(x)) \\ \text{s.t} \quad & g_j(x) \leq 0, j = 1, 2, \ldots, m \\ & x \in X \subseteq \Re^n, \end{aligned} \tag{3.1}$$

where $f : X \to \Re^p$ and $g : X \to \Re^m$ are differentiable functions and a non-empty open set $X \subseteq \Re^n$ and $f_i, g_j : \Re^n \to \Re, \forall i \in I = \{1, \ldots, p\}, \forall j \in J = \{1, \ldots, m\}$ $J(\bar{x}) = \{j \in J : g_j(\bar{x}) = 0\}$ is the index set of the active constraints at $\bar{x}$ and $S = \{x \in X : g_j(x) \leq 0, j \in J\}$ is the feasible set of (VOP). Here the minimization means finding the collection of efficient points. We denote by S the feasible set, that is, the intersection of X with the set of points x in which $g_j(x) \leq 0, j \in J = \{1, \ldots, m\}$. If $x \in S$, we say that x is a feasible point. $f_i(x)$ is the result of the i^{th} objective function if the decision maker chooses the action $x \in S$.We say that $x \in S$ dominates $y \in S$ in (VOP) if $f(x) \leq f(y)$ and $x \in S$ strictly dominates $y \in S$ in (VOP) if $f(x) < f(y)$.
There could not be valid single objective optimal solution for vector optimization problem (VOP). In multiobjective optimization problem objectives could be conflicting and may not be combined into a single objective. Hence, Pareto optimality concept for efficient solutions to multiobjective optimization has been introduced, two of which are defined as follows:

Let X be a non empty subset of $\Re^n$ and $f : \Re^n \to \Re^p$, $g : \Re^n \to \Re^m$. Then the vector optimization problem is to find the efficient point for (VOP)

$$\begin{aligned}&\min f(x)\\&\text{Subject to}\quad g_j(x) \le 0\end{aligned}$$

where condition (3.1) holds

Definition 3.1. *[2]A point $\bar{x}$ in X is said to be efficient (Pareto), if there exists no $x \in X$ such that $f(x) \le f(\bar{x})$.*

Definition 3.2. *A point $\bar{x}$ in X is said to be weak efficient (weak Pareto), if there exists no $x \in X$ such that $f(x) < f(\bar{x})$.*

Obviously every Pareto efficient point is also a weak Pareto efficient point,i.e., if S is convex and if the objective functions are quasiconvex with at least one strictly quasiconvex component, the set of local Pareto efficient points is a subset of the set of weak Pareto efficient points.
Now, we characterize weakly efficient solutions for the convex VOP using concepts similar to Fritz-John and Kuhn-Tucker optimality condition concepts, and assuming that f and g are differentiable functions on the open set X.
A strategy to minimize the problem(VOP (3.1)) consists of finding its Lagrangian and applying the KKT theorem which gives the necessary and sufficient conditions for the optimal solution of a constrained optimization. Lagrangian is defined as follows:

Definition 3.3. *Given the vector optimization problem*

$$\begin{aligned}&min f(x)\\&\textit{Subject to}\quad g_i(x) \le 0, i = 1, \ldots, m\end{aligned}$$

the Lagrangian is defined as

$$L(x, \lambda, \mu) = \sum_{i=1}^{p} \lambda_i f_i(x) + \sum_{j=1}^{m} \mu_j g_j(x)$$

where λ and μ are Lagrangian Multipliers. Lagrange multipliers have long been used in optimality conditions involving constraints, and their role has come to be understood from many different angles. Firtz-John and KKT conditions are expressed in terms of the Lagrangian for the optimal solution.

Definition 3.4. *A feasible point, $\bar{x} \in X$, is said to be a Vector Fritz-John Point (VFJP) to the problem VOP, if there exists a vector $(\bar{\lambda}, \bar{\mu}) \in \Re^p \times \Re^m$, with $(\bar{\lambda}, \bar{\mu}) \geq 0$ such that*

$$\bar{\lambda}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}) = 0,$$
$$\bar{\mu}^T g(\bar{x}) = 0.$$

Definition 3.5. *[3]A feasible point, $\bar{x} \in X$, is said to be a Vector Kuhn-Tucker Point (VKTP) to the problem VOP, if there exists a vector $(\bar{\lambda}, \bar{\mu}) \in \Re^p \times \Re^m$, with $(\bar{\lambda}, \bar{\mu}) \geq 0$ and $\bar{\lambda} \neq 0$ such that*

$$\bar{\lambda}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}) = 0$$
$$\bar{\mu}^T g(\bar{x}) = 0.$$

Note that, when we assume the condition that the multiplier $\lambda > 0$, it means that $\lambda_i \neq 0, \forall i \in I$. If it may happen that some multipliers related to the objectives are zero, then it is not possible to deduce the behavior at x of the objective functions associated with such zero multipliers. Therefore it is important to find necessary and/or sufficient conditions which guarantee $\lambda > 0$.
The following theorem states the Fritz - John optimality conditions. Here, without loss of generality, we will assume that the constraints are binding at a local efficient point of VOP. The first result to be presented is a necessary condition(first- order condition) of the Fritz-John type for pareto optimality.

Theorem 3.1. *(Fritz-John necessary condition for pareto optimality)*
Let the objective and constraint functions of problem(VOP) be differentiable at a point $\bar{x} \in S$. A necessary condition for the point $\bar{x}$ to be pareto optimal is that there exist multipliers $0 \leq \lambda \in \Re^p$ and $0 \leq \mu \in \Re^m$ for which $(\lambda, \mu) \neq (0, 0)$ such that
1. $\sum_{i=1}^{p} \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j \nabla g_j(\bar{x}) = 0$
2. $\mu_j g_j(\bar{x}) = 0$ *for all* $j = 1, \ldots, m$

Proof: See [6]
Let $f : \Re^n \to \Re^p$ $g : \Re^n \to \Re^m$ and consider the following vector optimization problem (VOP)

$$\text{(VOP)} : min f(x)$$
$$x \in S$$

Where $S = \{x \in \Re^n : g_j(x) \leq 0, j = 1, \ldots, m\}$. Let $\bar{x} \in S$ be any feasible solution to (VOP), and let $J(\bar{x})$ be the subset of indices defined by $J(\bar{x}) = \{j \in 1, \ldots, m : g_j(\bar{x}) = 0\}$.

For each $i = 1, \ldots, p$, we shall define the nonempty sets Q^i and Q by
$Q^i(\bar{x}) = \{x \in \Re^n : g(x) \leq 0, f_k(x) \leq f_k(\bar{x}), k = 1, \ldots, p \text{ and } k \neq i\}$
$Q(\bar{x}) = \{x \in \Re^n : g(x) \leq 0, f(x) \leq f(\bar{x})\}$
$C(Q, \bar{x}) = \{d \in \Re^n : \nabla f_i(\bar{x})^T d \leq 0, i = 1, \ldots, p; \nabla g_j(\bar{x})^T d \leq 0, j \in J(\bar{x})\}$
Note that, throughout this project we use the notations Q,Q^i, $C(Q, \bar{x})$ and S as defined above.
The following theorem expresses a necessary optimality conditions which involves the bouligand tangent cone.

Theorem 3.2. *[1]If $\bar{x}$ is a local efficient point for VOP and f and g are differentiable at $\bar{x}$ then we have $\nabla f(\bar{x})^T d \in \Re^p_+, \forall d \in \cap_{i=1}^p ClcoT(Q^i, \bar{x})$*

Proof. *Let $i \in \{1, \ldots, p\}$, and let $d \in T(Q^i, \bar{x})$. Then ,there exist$\{x_n\} \subset Q^i, x_n \to \bar{x}, \{\alpha_n\} \subseteq \Re_+, \alpha_n \to +\infty$, such that $\alpha_n(x_n - \bar{x}) \to d$. By Taylor's expansion,we have $f_i(x_n) - f_i(\bar{x}) = \nabla f_i(\bar{x})^T(x_n - \bar{x}) + o(\|x_n - \bar{x}\|)$,with $\frac{o(\|x_n - \bar{x}\|)}{\|x_n - \bar{x}\|} \to 0$. Since $f_i(x_n) - f_i(\bar{x}) \geq 0, \forall n$, and $lim_{n \to +\infty} a_n(f_i(x_n) - f_i(\bar{x}))$ $= lim_{n \to +\infty}[\nabla f_i(\bar{x})^T \alpha_n(x_n - \bar{x}) + \alpha_n\|x_n - \bar{x}\|[\frac{o(\|x_n - \bar{x}\|)}{\|x_n - \bar{x}|}] = \nabla f_i(\bar{x})^T d$, it results $\nabla f_i(\bar{x})^T d \geq 0, \forall d \in T(Q^i, \bar{x})$. Let $u \in CoT(Q^i, \bar{x})$. Then , there exist $d^h \in T(Q^i, \bar{x}), h = 1, \ldots, k, \lambda_h \geq 0, h = 1, \ldots, k, \sum_{h=1}^k \lambda_h = 1$, such that $u = \sum_{h=1}^k \lambda_h d^h$. Consequently, $\nabla f_i(\bar{x})^T u = \sum_{h=1}^k \lambda_h \nabla f_i(\bar{x})^T d^h \geq 0$, and thus $\nabla f_i(\bar{x})^T u \geq 0, \forall u \in CoT(Q^i, \bar{x})$. At last ,let $z \in ClcoT(Q^i, \bar{x})$. Then, there exist a sequence $\{u_n\} \subset CoT(Q^i, \bar{x})$, such that $u_n \to z$,with $\nabla f_i(\bar{x})^T u_n \geq 0$. By the continuity of scalar product,we have $\nabla f_i(\bar{x})^T z \geq 0$ for every $i \in \{1, \ldots, p\}$,so that $d \in \cap_{i=1}^p ClcoT(Q^i, \bar{x})$.*

3.2 Scalarization of Vector Optimization

In this subtopic we consider the problem of finding weakly efficient points (or weak Pareto minimal elements) of an inequality constrained vector optimization problem.

As described earlier in multiobjective optimization problems objectives could be generally conflicting. All the objectives cannot be solved for their minimum values simultaneously, so a compromise has to be reached. This is the nature of the multiobjective optimization problems. Since such optimization problems involve more than one objective, the objective function is expressed as a vector and the problem becomes a vector optimization problem . Such problems can be expressed as:

$$\begin{aligned} (VOP)\min f(x) &= (f_1(x), \ldots, f_p(x)) \\ x &\in S; \end{aligned} \tag{3.2}$$

where $S = \{x \in \Re^p : g(x) \leq 0\}$ is the feasible set(region) in the decision space and condition VOP(3.1)) holds. This problem (VOP) can be solved by solving single objective problems of scalar problems. The vector optimization problem can be solved by reducing it to a scalar optimization problem. This involves an aggregation of the components of the vector objective function into a single objective function. This process maps the objective space onto a real line. This scalar optimization problem is expected to be equivalent to the vector optimization problem. Different scalarization methods have been known. Among these many possible ways one of the most known scalar problems associated with multiobjective programming problems is the weighting problem whose formulation has the following form.
Consider the following weighted sum scalar problem:

$$\begin{aligned} &\sum_{i=1}^{p} w_i f_i(x) \\ &x \in S \end{aligned} \tag{3.3}$$

where p is the number of the objectives, $\sum_1^p w_i = 1$ and $w_i > 0, i = 1, \ldots, p$. This completely orders the objective space. This allows the resulting scalar objective function to be solved for a single Pareto optimal solution that represents the preference of the decision maker. Weighted sum method involves a linear or convex combination of the objectives $f_i(x), i = 1, \ldots, p$. Each of the objective $f_i(x)$ is multiplied by a normalized weight factor w_i and the product added to give the scalar objective (3.3) of vector optimization (3.2); The scalar objective optimization problem (3.3) has been shown to be equivalent to the vector optimization problem (3.2) if the problem is convex. The weighted sum method is the commonly used scalarization method because of its simplicity, ease of use, and direct translation of weight into the relative importance of the objectives. Let the attainable set of objectives in the objective space be denoted by
$F = \{f_1(x), \ldots, f_p(x) : g(x) \leq 0, \}$,
Consider the weight vector $w = (w_1, \ldots, w_p)^T \in \Re^p$, the vector objective function $f(x) = (f_1(x), \ldots, f_p(x))^T \in \Re^p$, and the map $G(f, w) = G_w$: $\Re^p \times \Re^m \rightarrow \Re$. The weighted sum method derives the scalar objective $G_w(x) = G_w f(x)$, through a convex combination of the objectives $f_i(x), i = 1, \ldots, p$. Thus, with p number of the objectives, the equivalent scalar objec-

tive G(f,w) is given as

$$\begin{aligned} G_w(x) &= \sum_{i=1}^{p} w_i f_i(x) = w^T f(x) \\ \sum_{i}^{p} w_i &= 1, w_i > 0, i = 1, ..., p \end{aligned} \tag{3.4}$$

This transforms the vector optimization to a scalar optimization problem(SOP) form:

$$\begin{aligned} &\min G(x) \\ &\text{s.t.} \quad x \in S \end{aligned} \tag{3.5}$$

This process maps the p-dimensional objective space onto the positive real line R, and all the non-dominated points are mapped to the same point on the real line.

Theorem 3.3. *[11] The solution of weighted optimization problem* (3.4) *is pareto optimal, if the weighting coefficients are positive, that is, $w_i > 0$, for all $i = 1, \ldots, p$*

Proof. *Let $\bar{x} \in S$ be a solution of weighting problem*(3.4)*. Let us suppose that it is not pareto optimal. This means that there exists a solution $x_0 \in S$ such that $f_i(x_0) \leq f_i(\bar{x})$ for all $i = 1, \ldots, p$ and $f_j(x_0) \leq f_j(\bar{x})$ for at least one j, since $w_i > 0$ for all $i = 1, \ldots, p$ we have $\sum_{i=1}^{p} w_i f_i(x_0) < \sum_{i=1}^{p} w_i f_i(\bar{x})$. This contradicts the assumption that $\bar{x}$ is a solution of the weighting problem and thus, $\bar{x}$ must be Pareto Optimal.*

Theorem 3.4. *If f is invex on an open set S, then all weakly efficient solutions of the VOP solve the weighted scalar problem with $\lambda \geq 0$.*

Proof. *Let $\bar{x}$ be a weak efficient point, then there exists $\lambda \in \Re^p$ with $\lambda \geq 0$ such that $\lambda^T \nabla f(\bar{x}) = 0$. As f is invex at $\bar{x}$ with respect to η so is $\lambda^T f$ that is, $\lambda^T f(x) - \lambda^T f(\bar{x}) \geq \lambda^T \nabla f(\bar{x}) \eta(x, \bar{x}) = 0$ Then $\lambda^T f(x) \geq \lambda^T f(\bar{x})$. Therefore, $\bar{x}$ is optimal solution for* (3.4) *with $\lambda \geq 0$*

It is interesting to note that under invexity hypothesis, the vector critical point, the weakly efficient solution and the optimal solution for weighted scalar problems coincide . Now we illustrate the relation of scalar optimization and vector optimization by extending KKT sufficient condition scalar optimality to vector optimality for convex optimization. If the multiobjective optimization problem is convex, then we can state a sufficient condition of optimality in the single objective case,

Theorem 3.5. *(KKT sufficient condition for optimality)*
[11]A sufficient condition for a point $\bar{x} \in \Re^n$ to be (global) minimum for the problem

$$\begin{aligned} &min f_i(x) \\ &g(x) = (g_1(x), g_2(x), \ldots, g_m(x)) \leq 0, i = 1, \ldots, m. \end{aligned} \tag{3.6}$$

where the objective function $f_i : \Re^n \to \Re, j = 1, \ldots, p$ and the constraint function $g_j : \Re^n \to \Re, j = 1, \ldots, m$ are convex and continuously differentiable at $\bar{x}$, is that there exist multipliers $0 \leq \mu \in \Re^m$ such that

$$\begin{aligned} &1)\nabla f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j \nabla g_j(\bar{x}) = 0 \\ &2)\mu_j g_j(\bar{x}) = 0, \quad for \quad all \quad j = 1, \ldots, m. \end{aligned} \tag{3.7}$$

Proof: See [21]
Now we can extend the theorem (3.5) for vector optimization case as in the following theorem.

Theorem 3.6. *(KKT sufficient condition for pareto optimality)*
[11]Let the objective and the constraint functions of VOP(3.1) be convex and continuously differentiable at a decision vector $\bar{x} \in S$. A sufficient condition for $\bar{x}$ to be pareto optimal is that there exist multipliers $0 < \lambda \in \Re^p$ and $0 \leq \mu \in \Re^m$ such that

$$\begin{aligned} &1)\sum_{i=1}^{p} \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j \nabla g_j(\bar{x}) = 0 \\ &2)\mu_j g_j(\bar{x}) = 0, for \quad all \quad j = 1, \ldots, m. \end{aligned}$$

Proof. *Let the vector λ and μ be such that the condition stated are satisfied.We defined a function $F : \Re^n \to \Re$ as $F(x) = \sum_{i=1}^{p} \lambda_i f_i(x)$, where $x \in S$. Trivially F is convex because all the function f_i are and we have $\lambda > 0$. Now from statement (1) and (2), we obtain $\nabla F(\bar{x}) + \sum_{j=1}^{m} \mu_j \nabla g_j(\bar{x}) = 0$ and $\mu_j g_j(\bar{x}) = 0$, for all $j = 1, \ldots, m$. thus according to theorem (3.5) the sufficient condition for F to attain its minimum at $\bar{x}$ is satisfied. So $F(\bar{x}) \leq F(x)$ for all $x \in S$. In other words*

$$\sum_{i=1}^{p} \lambda_i f_i(\bar{x}) \leq \sum_{i=1}^{p} \lambda_i f_i(x)$$

for all $x \in S$.
Let as assume that $\bar{x}$ is not pareto optimal. Then there exists some point

$\hat{x} \in S$ such that $f_i(\hat{x}) \leq f_i(\bar{x})$ for all $i = 1, \ldots, p$ and at least for one index j is $f_j(\hat{x}) < f_j(\bar{x})$. Because all λ_i were assumed to be positive , we have $\sum_{i=1}^{p} \lambda_i f_i(\hat{x}) < \sum_{i=1}^{p} \lambda_i f_i(\bar{x})$. This is contradiction with inequality $\sum_{i=1}^{p} \lambda_i f_i(\bar{x}) \leq \sum_{i=1}^{p} \lambda_i f_i(x)$ and $\bar{x}$ is thus pareto optimal .

Chapter 4

Constraint Qualification in Vector Optimization

4.1 Constraint Qualification

It may happen that some multipliers related to the objectives in Fritz-John conditions are zero and consequently it is not possible to deduce the behavior at $\bar{x}$ of the objective functions associated with such zero multipliers. [1] Therefore, it is important to find necessary and/or sufficient conditions which guarantee that $\lambda \in int\Re^n_+$.
When $\lambda \in int\Re^n_+$ Fritz - John conditions are referred to as Karush Kahn Tucker (KKT) conditions. Now we are able to state necessary and sufficient conditions which ensures $\lambda \in int\Re^n_+$. These condition are called constraint qualification conditions. A necessary and sufficient condition for the validity of Fritz John conditions with positivity of all the multipliers associated with the objective functions. Such a condition can be interpreted as general constraint qualification.
Now, consider a generalized constraint qualification, for which we obtain also Kuhn-Tucker necessary conditions which guarantee that a feasible solution of Problem (VOP(3.1)) is an efficient solution.
Let $f : \Re^n \to \Re^p$, $g : \Re^n \to \Re^m$ and consider the following vector optimization problem (VOP)

$$\begin{aligned} (VOP) \quad & \min f(x) \\ & x \in S \end{aligned} \tag{4.1}$$

where $S = \{x \in \Re^n : g_j(x) \leq 0, j = 1, \ldots, m\}$ and $\bar{x}$ is a feasible solution(vector) of Problem (VOP) and for each $i = 1, \ldots p$, consider the nonempty sets $(Q^i, \bar{x})$ and $(Q, \bar{x})$ defined in preceding chapter and

$C(Q,\bar{x}) = \{d \in \Re^n : \nabla f_i(\bar{x})d \leq 0, i = 1, \ldots, p, \nabla g_j(\bar{x})d \leq 0, j \in J(\bar{x})\}$
$(J(\bar{x}) = \{j \in J : g_j(\bar{x}) = 0\})$.
We suppose that all functions are differentiable at the point taken into consideration, i.e.at $\bar{x} \in S$.

Definition 4.1. *Let $S = \{x \in \Re^n : g_j(x) \leq 0, j = 1, \ldots, m\}$, g_j differentiable at $\bar{x} \in S$. The linearizing cone to S at $\bar{x}$ is defined as:*
$C(S,\bar{x}) = \{d \in \Re^n : \nabla g_j(\bar{x})^T d \leq 0, j = 1, \ldots, m\}$.

We consider also the following linearizing cones to S at $\bar{x}$:
$\bar{C}(S) = \{d \in \Re^n : \nabla g_j(\bar{x})d < 0, j \in J(x)\}$
$C(S) = \{d \in \Re^n : \nabla g_j(\bar{x})d \leq 0, j \in J(x)\}$

Note that, we have the following inclusions

$$Z(Q,\bar{x}) \subset A(Q,\bar{x}) \subset T(Q,\bar{x}) \subset \cap_{i\in I} ClcoT(Q^i,\bar{x}) \subset C(Q).$$

Where,
$Z(Q,\bar{x})$ is a feasible direction cone.
$A(Q,\bar{x})$ is an attainable direction cone.
$T(Q,\bar{x})$ is Bouligand cone.
$ClcoT(Q^i,\bar{x})$ closed convex cone of tangent cone.
C(Q) is linearized cone.
Where $\bar{x}$ is a feasible solution(vector) of Problem (VOP(3.1)) and for each $i \in 1, \ldots, p$, let the nonempty sets $(Q^i,\bar{x})$ and $(Q,\bar{x})$ be as defined above:
$C(Q,\bar{x}) = \{d \in \Re^n : \nabla f_i(\bar{x})d \leq 0, i = 1, \ldots p, \nabla g_i(\bar{x})d \leq 0, j \in J(\bar{x})\}$
In other condition, $C(Q,\bar{x}) = \{d \in \Re^n : f_i'(\bar{x})d \leq 0, i = 1, \ldots p, g_i'(\bar{x})d \leq 0, j \in J(\bar{x})\}$
In other condition, $(J(\bar{x}) = \{j \in J : g_j(\bar{x}) = 0\})$.

Proposition 4.1. *If $f_i'(\bar{x}), i \in 1, \ldots, p$, and $g_j'(\bar{x}), j \in J(x)$, are convex functions on $\Re^n$, then $C(Q,\bar{x})$ is a closed convex cone.*

Proof. *Let $\lambda \geq O$ and $d \in C(Q,\bar{x})$. Then, $\lambda d \in C(Q,\bar{x})$, because $\lambda f_i'(\bar{x})d \leqq 0, i \in 1, \ldots, p$ and similarly, $\lambda g_j'(\bar{x})d \leqq 0$, for $j \in J(\bar{x})$ Now, let $d_1, d_2 \in C(Q,\bar{x})$, and let $\lambda \in [0,1]$. Since the functions $f'(\bar{x})$ and $g'(\bar{x})$ are convex, we have, for $i \in I$ $f_i'(\bar{x}, \lambda d_1 + (1-\lambda)d_2) \leqq \lambda f_i'(\bar{x})d_1 + (1-\lambda)f_i'(\bar{x})d_2 \leqq 0$ and similarly, for $j \in J(\bar{x})$, $g_j'(\bar{x}, \lambda d_1 + (1-\lambda)d_2) \leqq 0$. Finally, $C(Q,\bar{x})$ is closed, due to the fact that, if we take a sequence $(d^k)_k \in C(Q,\bar{x})$ such that $d^k \to d^0$, it follows that $f'(\bar{x})d^k \leqq 0, \forall k$ and therefore, $lim_k f_i'(\bar{x})d^k = f_i'(\bar{x})d^0 \leqq 0, i \in I$ and similarly,we get $g_j'(\bar{x})d^0 \leqq 0, j \in J(\bar{x})$ We used the continuity of the convex functions $f_i'(\bar{x})$ and $g'(\bar{x})$ on $\Re^n$.*

The following result shows the relationship between the tangent cones $T(Q^i, \bar{x})$ and the linearizing cone $C(Q, \bar{x})$.

Lemma 4.1. *[15]We assume that $\bar{x}$ is a feasible solution to Problem (VOP) and that :*
(A1) $f_i'(\bar{x}), i \in I$, and $g_j'(\bar{x}), j \in J(x)$, are convex functions on $\Re^n$;
(A2) $f_i, i \in I$, and $g_j, j \in J(\bar{x})$, are quasiconvex functions at $\bar{x}$.
Then, we have

$$\cap_{i=1}^{p} ClcoT(Q^i, \bar{x}) \subseteq C(Q, (\bar{x})) \tag{4.2}$$

Proof. *We shall give a proof for $p > 1$, the proof for $p = 1$ being similar. For $i \in I$, let us define $C(Q^i, \bar{x})) = \{d \in \Re^n : f_k'(\bar{x})d \leqq 0, k \in I^i$ and $g_j'(\bar{x})d \leqq 0\}$, $j \in J(\bar{x})$ Using a proof similar to that of Proposition (4.1), one can see that $C(Q^i, \bar{x})$ is convex and closed for all $i \in I$.*

$$\cap_{i=1}^{p} C(Q^i, \bar{x})) = C(Q, \bar{x}) \tag{4.3}$$

Now, we shall show that, for every $i \in I$,

$$T(Q^i, \bar{x}) \subseteq C(Q^i, \bar{x})) \tag{4.4}$$

Let $i \in I$ and $h \in T(Q^i, \bar{x})$. We get a sequence $(x_k)_k \subset (Q^i, \bar{x})$ and a sequence $(t_k)_k \in \Re$, with $t_k > 0$, such that $lim_{k\to\infty} x_k = \bar{x}, lim_{k\to\infty} t_k(x_k - \bar{x}) = h$ For $k \geq 1$, we put $h_k = t_k(x_k - \bar{x})$ Then, for all $j \in J(\bar{x})$ and for all k,

$$g_j'(x_k) = g_j(\bar{x} + \frac{1}{t_k} h_k) \leqq 0 = g_j(\bar{x}) \tag{4.5}$$

,

$$f(\bar{x} + \frac{1}{t_k} h_k) \leqq f(\bar{x}) \tag{4.6}$$

Using (4.5) and (4.6), and the quasiconvexity of the functions $f_i, i \in I$, and $g_j, j \in J(\bar{x})$, according to Assumption (A2) we obtain, for all k,

$$g_j'(\bar{x}, \frac{1}{t_k} h_k) \leqq 0, j \in J(\bar{x}) \tag{4.7}$$

$$f_s'(\bar{x}, \frac{1}{t_k} h_k) \leqq 0, s \in I^i \tag{4.8}$$

Using (4.7) and (4.8), and the homogeneity property of f_s' and g_j', we get, for all k

$$g_j'(\bar{x}) h_k \leqq 0, j \in J(\bar{x}) \tag{4.9}$$

$$f_s'(\bar{x})h_k \leqq 0, s \in I^i \tag{4.10}$$

Due to the continuity of the convex functions $g_j'(\bar{x}), f_s'(\bar{x})$ *[see Assumption (A1)] and to the fact that* $lim_{k\to\infty}h_k = h$ *One gets that,*

$$g_j'(\bar{x})h \leqq 0, j \in J(\bar{x}) \tag{4.11}$$

$$f_s'(\bar{x})h \leqq 0, s \in I^i \tag{4.12}$$

so that (4.5) *is true. Hence, due to the fact that every* $C(Q^i, \bar{x})$ *is convex and closed, one obtains* $CoT(Q^i, \bar{x}) \subseteq C(Q^i, \bar{x})$, $ClcoT(Q^i, \bar{x}) \subseteq \cap_{i\in I}C(Q^i, \bar{x})$ *hence,* $\cap_{i\in I}ClcoT(Q^i, \bar{x}) \subseteq \cap_{i\in I}C(Q^i, \bar{x})$ *According to (2), the proof is complete for* $p > 1$*. For* $p = 1$*, we take* $C(Q, \bar{x})$ *instead of* $C(Q^i, \bar{x})$*, and with similar arguments we obtain* $ClcoT(Q, \bar{x}) \subseteq \cap_{i\in I}C(Q, \bar{x})$ *The proof is now complete for* $p \geq 1$*.*

In general, the converse inclusion in Lemma 4.2 does not hold, as can be seen in the next example.

Example 4.1. *Consider problem VOP*(3.1)*,where*
$f(x_1, x_2) = (x_1^3 - x_2, x_1x_2 - 3x_2)$ *and*
$S = \{(x_1, x_2) \in \Re^2 : g_1(x_1, x_2) = x_1^2 - x_2 \leq 0, g_2(x_1, x_2) = x_2 \leq 0\}$.
It is to verify that $S = \{(0, 0)\}$*, so that* $\bar{x} = (0, 0)$ *is efficient for VOP*(3.1)*.*
By simple calculation , we have $C(Q, \bar{x}) = \{(d_1, 0) : d_1 \in \Re\}$ *and*
$\cap_{i=1}^2 ClcoT(Q^i, \bar{x}) = \{(0, 0)\}$*. So that the,* $C(Q, \bar{x}) = \cap_1^2 ClcoT(Q^i, \bar{x})$ *does not hold. On the other hand KKT conditions are verified (for instance, we can choose* $\lambda_1 = \lambda_2 = \mu_1 = 2$ *and* $\mu_2 = 10$*)*

In order to obtain the necessary conditions that a feasible solution to Problem (VOP(3.1)) be an efficient solution, it is reasonable to assume that $C(Q, \bar{x}) \subseteq \cap_{i\in I}ClcoT(Q^i, \bar{x})$. This condition is considered as a generalization of the Guignard constraint qualification in mathematical programming.
As we have previously pointed out in vector optimization, constraint qualification must involve both the objective function and the constraints, so, we need to consider the sets Q^i and Q with this respect, it is important to observe that, when $\bar{x}$ is a local efficient point for problem (VOP) we have $int(Q, \bar{x}) = \emptyset$ [14].
Assume that f and g are differentiable and $\nabla f_i(\bar{x}), i = 1, \ldots, p$ and $\nabla g_i(\bar{x}), j \in J(\bar{x})$ are convex function on $\Re^n$, then the following conditions are among the common constraint qualifications that can be proven by utilizing convexity property for none empty interior cone following the same line of the scalar case.

Definition 4.2. *[18]For vector optimization problem VOP*(3.1)*), if the inclusion*

$$C(Q, \bar{x}) \subseteq \cap_{i=1}^{p} Cl(co(T(Q^i, \bar{x}))) \tag{4.13}$$

*holds, we call that VOP*3.1 *satisfies the Generalized Guignard constraint qualification (GGCQ) at* $\bar{x}$*, where* $C(Q, \bar{x})$ *is the linearizing cone to* Q *at* $\bar{x} \in Q$*,* $T(Q^i, \bar{x}), i = 1, \ldots, p$ *are the tangent cones to* Q^i *at* $\bar{x} \in ClQ^i$*.*
i.e the Generalized Guignard constraint qualification, if $C(Q, \bar{x}) \subseteq \cap_{i=1}^{p} Cl(co(T(Q^i, \bar{x})))$*.*

Definition 4.3. *[11]Let the objective and constraint function of VOP*(3.1)*) be differentiable at point* $\bar{x} \in S$*. VOP satisfies the Cottle Constraint qualification at* $\bar{x}$ *if either* $g_j(\bar{x}) < 0$ *for all* $j = 1, \ldots, m$ *or* $O \notin Conv\{\nabla g_j(\bar{x}) : g_j(\bar{x}) = 0\}$
i.e Cottle - type constraint qualification if

$$intC(Q^i, \bar{x}) \neq \emptyset, \forall i. \tag{4.14}$$

Definition 4.4. *[15] Let the objective and constraint function of VOP*(3.1)*) be differentiable at point* $\bar{x} \in S$*.If the inclusion*

$$C(Q, \bar{x}) \subseteq T(Q, \bar{x}) \tag{4.15}$$

holds, we call that VOP(3.1) *satisfies the Abadie constraint qualification(ACQ)*
General Abadie Constraint qualification if

$$C(Q, \bar{x}) \subseteq \cap_{i=1}^{n} T(Q^i, \bar{x}) \tag{4.16}$$

Definition 4.5. *[15] Slater-Type Constraint Qualification(SCQ): For* $p > 1$*, the functions* $f_i, i \in \{1, \ldots, p\}$*, and* $g_j(x), j \in \{1, \ldots, m\}$ *are all pseudoconvex on* $\Re^n$*, and for each* $i \in 1, \ldots, p$*, the system*

$$f_k(x) < f_k(\bar{x}), k \in \{1, \ldots, p\} \quad and \quad k \neq i \quad , g_j(x) < 0, j \in \{1, \ldots, m\} \tag{4.17}$$

has a solution at $x \in \Re^n$*.*

Definition 4.6. *[15] we assume that* $\bar{x}$ *is a feasible solution to VOP*(3.1)*)*
Mangasarian-Fromovitz Constraint Qualification(MFCQ): The system

$$\begin{aligned} \nabla f_i(\bar{x})d \leq 0, i = 1, \ldots, p. \\ \nabla g_j(\bar{x})d < 0, j \in J(\bar{x}) \end{aligned} \tag{4.18}$$

has a solution $d \in \Re^n$*, and the following property holds: there does not exist* $k \in \{1, \ldots, p\}$ *such that* $\nabla f_k(\bar{x})d = 0$ *for any solution* d *to the system* (4.18)*.*

Definition 4.7. *Linear Constraint Qualification(LCQ): The functions $f_i, i \in I$, and $g_j, j \in J(\bar{x})$, are all linear.*

It is obvious that in order to avoid that some of the Lagrange multipliers vanish, we need to consider a constraint qualification.

Theorem 4.1. *[14]Let $\bar{x} \in S$ be any feasible solution for problem (VOP(3.1)) where f, g are differentiable at $\bar{x}$ and assume that the General Guignard Constraint Qualification (GGCQ) (4.13) holds at $\bar{x}$. If $\bar{x} \in X$ is an efficient solution to Problem (VOP(3.1)), then there exist vectors $\lambda, \mu : (\lambda, \mu) \in \Re^p_+ \times \Re^m_+$ such that*

$$\begin{aligned} \sum_{i=1}^{p} \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j \nabla g_j(\bar{x}) &= 0 \\ \sum_{j=1}^{m} \mu_j g_j(\bar{x}) &= 0, \\ \lambda > 0 \quad and \quad \mu &\geq 0 \end{aligned} \tag{4.19}$$

Proof. *Let $\bar{x} \in X$ be an efficient solution to VOP(3.1). Then, from theorem (3.2) we have , the inequality system $\nabla f_i(\bar{x})^T d \leq 0, i = 1, \ldots, p$ $\nabla g_j(\bar{x})^T d \leq 0, j \in J(\bar{x})$ has no solution. From Kahn Tucker's Theorem, there exist $\lambda > 0, \mu_j \geq 0, j \in J(\bar{x})$, such that $\sum_{i=1}^{p} \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j \nabla g_j(\bar{x}) = 0$. By setting $\mu_j = 0, j \notin J(\bar{x})$, we have $\sum_{i=1}^{p} \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j \nabla g_j(\bar{x}) = 0$, $\lambda > 0, \mu \geq 0$. Since $g_j(\bar{x}) = 0$, for $j \in J(\bar{x})$, we have $\mu_j g_j(\bar{x}) = 0$, for $j = 1, \ldots, m$ which completes the proof.*

Sufficient condition for Guignard constraint qualification
We note that, in the case when $f_i, i \in I$ and $g_j, j \in J$ are continuously differentiable , ACQ, CCQ,SCQ and MFCQ are satisfied according to the related condition(assumptions) guarantees that GGCQ holds. Consequently, they validate also the KKT conditions represent first order optimality conditions.

With reference to VOP(3.1), and all the functions(objective functions and constraint functions(inequality)) are differentiable on a common open set, "weaker" or a "stronger" constraint qualification condition has been considered. Consider the following Abadie constraint qualification(ACQ):

$$C(Q, \bar{x}) = T(Q, \bar{x}) \tag{4.20}$$

i.e the linearized cone (cone of locally constrained direction) in a nonempty closed convex polyhedral cone determined by the active constraints of VOP(3.1). The following lemma was proved by Abadie (1967).

Lemma 4.2. *[7]In problem (VOP(3.1)) let* $\bar{x} \in S$*, then it results* $A(S, \bar{x}) \subset (C, \bar{x})$ *i.e* $(C^*, \bar{x}) \subset T^*(S, \bar{x})$

Proof. *It is sufficient to prove that* $T(S, \bar{x}) \subset (C, \bar{x})$*, as being that* $(C, \bar{x})$ *is a closed convex cone, then* $A(S, \bar{x}) = Cl(Conv(T(S, \bar{x}))) \subset (C, \bar{x})$*.*

Let $d \in T(S, \bar{x})$. Then there exist a sequence $\{x_n\} \in S$ that converges to $\bar{x}$ and a nonnegative sequence $\{\lambda_n\} \in \Re$, such that $d = lim_{n \to \infty^+} \lambda_n(x_n - \bar{x})$. We have, for $j \in J(\bar{x})$, $\lambda_n g_j(x_n) = \lambda_n(x_n - \bar{x})\nabla g_j(\bar{x}) + \lambda_n 0 \parallel x_n - \bar{x} \parallel$. If n is large enough and if $d\nabla g_j(\bar{x}) > 0$ for $\forall j \in J(\bar{x})$, then the right hand side of the last expression is positive, so that $g_i(x_n) > 0$, which is a contradiction to $x_n \in S$.
As a counter example to the converse of the above lemma, consider the following system $n = 2 \in \Re^2$:
$g_1(x) = x_2 - x_1^3 \leq 0$
$g_2(x) = -x_2 \leq 0$ and take $\bar{x} = (0, 0)$
The cone of tangents is the half-line $x_2 = 0, x_1 \geq 0$, while the linearizing cone is the whole line $x_2 = 0$. It is worthwhile to emphasize that the cone of tangents is a geometrical concept, while the linearizing cone depends only upon the analytical description of the feasible set S. For example if we add to the above two constraints the third equality $g_3(x) = -x_1 - x_2 \leq 0$
The feasible set S remain the same as well as the cone of tangents at $\bar{x} = (0, 0)$, but the linearizing cone coincides with the tangent cone.
Contrary to the scalar case, where the Abadie constraint qualification assures the positivity of the multiplier associated to $\nabla f(\bar{x})$, in the Fritz John necessary optimality conditions, for vop(3.1) this does not happen. Obviously, this holds true also for all other which imply the Abadie constraint qualification condition Consider the following example [1].

Example 4.2. *Consider the problem*

$$\begin{aligned} &minf(x_1, x_2) = ((x_1 - x_2), -(x_1 - x_2)^3) \\ &\quad s.t: \\ &\quad S = \{(x_1, x_2) \in \Re^2 \mid x_1 \leq 0\} \end{aligned}$$

Every feasible point is an eficient solution. In particular $\bar{x} = (0, 0) \in S$ *is efficient, it holds* $C(S, \bar{x}) = T(S, \bar{x})$*, but we have*

$$\lambda_1(1, -1)^T + \lambda_2(0, 0)^T + \mu_1(1, 0)^T = (0, 0)^T \text{ for } \lambda_1 = 0, \lambda_2 > 0, \mu_1 = 0, \text{ so}$$
$$\lambda = (0, \lambda_2) \notin int\Re^2_+$$

Hence, we obtain the following result.

If $\bar{x}$ is a local efficient point for vop(3.1), or also a local weak efficient point, and the Abadie constraint qualification (4.15) is satisfied, then there exist $\lambda \in \Re^p, \lambda \geq 0, \lambda \neq 0, \mu \in \Re^m$ and $\mu \geq 0$ such that

$$\sum_{i \in I} \lambda_i \nabla f_i(\bar{x}) + \sum_{j \in J} \mu_j \nabla g_i(\bar{x}) = 0$$
$$\sum_{j \in J} \mu_j g_j(\bar{x}) = 0$$

We have to note that the above Weak Kuhn-Tucker conditions are useful, together with appropriate (generalized) convexity assumptions, to obtain sufficient conditions for the weak efficiency . Assuming the Cottle Constraint qualification, we obtain the KKT necessary condition for pareto optimality.

Theorem 4.2. *[11](KKT necessary condition for pareto optimality) let the assumption of theorem* (3.1) *be satisfied by Cottle Constraint qualification. theorem* (3.1) *is then valid with the addition that* $\lambda \neq 0$.

Let $\bar{x} \in S$ be pareto optimal. At first we define an additional function $F : \Re^n \rightarrow \Re$ by $F(x) = max\{[f_i(x) - f_i(\bar{x})], g_j(x) : i = 1, \ldots, p; j = 1, \ldots, m\}$ and for all $x \in \Re; F(x) \geq 0$ Theorem (3.1) is here valid up to the observation $0 \in \partial f(\bar{x})$. From the definition of F we know that $F(\bar{x}) = 0$

Proof. *We continue by first assuming the* $g_j(\bar{x}) < 0$ *for all* $j = 1, \ldots, m$. *In this case,* $f(\bar{x}) > g_j(\bar{x})$ *for all* j, *obtain* $O \in Conv\{\partial f_i(\bar{x}) : i = 1, \ldots, p\}$ *From the definition of convex hull we know that there exists a vector* $O \leq \lambda \in \Re^p$ *of multipliers for which* $\sum_{i=1}^p \lambda_i = 1$ *(thus* $\lambda \neq 0$*) such that* $0 \in \sum_{i=1}^p \lambda_i \partial f_i(\bar{x})$ *we obtain the statement to be proved(denoted by (1) in theorem* (3.1) *by setting* $\mu_j = 0, \forall j = 1, \ldots, m$. *on the otherhand, if there exists some index* j *such that* $g_j(\bar{x}) = 0$, *we denote the set of such indices by* $J(\bar{x})$. *by the Cottle constraint qualification we know that* $0 \notin Conv\{\partial g_i(\bar{x}) : j \in J(\bar{x})\}$ *In this case, we deduce that* $0 \in Conv\{\partial f_i(\bar{x}), \partial g_j(\bar{x}) : i = 1, \ldots, p; j \in J(\bar{x})\}$ *Appling the definition of a convex hull, we know that there exist multipliers* $\lambda_i \geq 0, i = 1, \ldots, p$ *and* $\mu_j \geq 0, j \in J(\bar{x})$ *for which* $\sum_{i=1}^p \lambda_i + \sum_{j \in J(\bar{x})} \mu_j = 1$ *and by assumption of theorem*(3.1) *,especially* $\lambda \neq 0$, *such that*
$0 \in \sum_{i=1}^p \lambda_i \partial f_i(\bar{x}) + \sum_{j \in J(\bar{x})} \mu_j \partial g_j(\bar{x})$. *Again, we obtain the statement to be proved by setting* $\mu_j = 0$, *for all* $j \in J \mid J(x)$. *The proof of part (ii) is similar to proof of Theorem* (3.1).

4.2 Relations between Constraint Qualification conditions

Lemma 4.3. *Let* $\bar{x}$ *be a feasible solution for Problem (VOP). Then, (SCQ) implies (CCQ).*

Proof. *Since (SCQ) holds, for each $i \in I$, there exists $x_i \in \Re^n$ such that*

$$\begin{aligned} f_k(x_i) &< f_k(\bar{x}), k \in I^i \\ g_j(x_i) &< 0, j \in J \end{aligned} \tag{4.21}$$

Also, the functions $f_k, k \in I$, and $g_j, j \in J$, are semidifferentiable and pseudoconvex at $\bar{x}$. Then, we have

$$\begin{aligned} f'_k(\bar{x}, x_i - \bar{x}) &< 0, k \in I^i \\ g'_j(\bar{x}, x_i - \bar{x}) &< 0, j \in J(\bar{x}) \end{aligned} \tag{4.22}$$

i.e $d_i = x_i - \bar{x}, i \in I$, is a solution for the system given by (CCQ). Therefore, (SCQ) implies (CCQ) is proved.

Lemma 4.4. *Let $\bar{x}$ be a feasible solution for Problem (VOP(3.1)). Then, (CCQ) implies (GGCQ) [15].*

Proof. *To proof that (GGCQ) holds , let $h \in C(Q, \bar{x})$. Then we get*

$$\nabla f_i(\bar{x})h \leqq 0, i \in \{1, \dots, p\} \tag{4.23}$$

$$\nabla g_j(\bar{x})h \leqq 0, j \in J(\bar{x}) \tag{4.24}$$

Let $i \in I$ be a fixed element. Since (CCQ) holds, there exists $h_i \in \Re^n$, such that

$$\nabla f_k(\bar{x})h_i < 0, k \in I^i \tag{4.25}$$

$$\nabla g_j(\bar{x})h_i < 0, j \in J(\bar{x}) \tag{4.26}$$

Now, we shall define a sequence $\{h^n\}_n$ given by $d^n = h + t_n d_i$, where $\{t_n\}_n$ is a positive sequence converging to zero. We note that $lim_{n\to\infty} d^n = h$. Using the linearity property of the functions $\nabla f_k(\bar{x})$ and $\nabla g_j(\bar{x})$, and using relations (4.23) − (4.26), we get $\nabla f_k(\bar{x})h^n \leqq \nabla f_k(\bar{x})h + t_n \nabla f_k(\bar{x})h_i < 0$, $\nabla g_j(\bar{x})d^n \leqq \nabla g_j(\bar{x}h + t_n \nabla g_j(\bar{x})h_i < 0$ for any n and $k \in I^i, j \in B(\bar{x})$. Now, if $\{\mu_{nk}\}_k$ is a positive sequence of real numbers, converging to zero, then the sequence $\{x^{nk}\}_k$, given by $x^{nk} = \bar{x} + \mu_{nk} h^n$, is converging to $\bar{x}$,i.e $lim_{k\to\infty} x^{nk} = \bar{x}$ and $f_i(x^{nk}) \leq f_i(\bar{x})$, we get that, for any k sufficiently large,

$$f_s(x^{nk}) < f_s(\bar{x}), s \in I^i \tag{4.27}$$

$$g_j(x^{nk}) < 0, j \in J(\bar{x}) \tag{4.28}$$

For $j \in J \mid J(\bar{x})$, $g_j(x^{nk}) \leq 0$, for any k sufficiently large. Now, by (4.27) and (4.28), we have that $x^{nk} \in (Q^i, \bar{x})$, for any k sufficiently large.

The relations between some of these constraint qualifications can be summarized as follows for differentiable objective and inequality constraint function:

- Slater constraint qualification implies Cottle constraint qualification implies Guignard constraint qualification
- Linear constraint qualification implies Abadie constraint qualification implies General Abadie constraint qualification implies Guignard constraint qualification
- Mangasarian-Formovitz constraint qualification implies Cottle constraint qualification implies Guignard constraint qualification

Now, we can summarize as follows:
For multiobjective optimization programming with inequality constraint, if $\bar{x} \in X$ is an efficient(or weakly efficient) solution to VOP(3.1). and an appropriate constraint qualification is satisfied at $\bar{x}$,then there exist vector $\lambda \in \Re^p_+, \lambda > 0$, and $\mu \in \Re^m_+$ such that

$$\begin{aligned} \sum_{i=1}^{p} \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j \nabla g_j(\bar{x}) = 0 \\ \mu_j g_j(\bar{x}) = 0, j = 1, \ldots, m \\ \lambda > 0, \mu \geq 0 \end{aligned} \tag{4.29}$$

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