Addis Ababa University College of Natural and Computational Sciences Department of Mathematics

Thesis on

Approximate Solution of Linear Differential Equation

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DECLARATION

This thesis has been submitted to the Department of Mathematics, College of Natural and Computational Sciences, Addis Ababa University in partial fulfillment of the degree of Master of Science in Mathematics. I therefore declare that this thesis has not been submitted to any other institution and anywhere for the awarded of any academic diploma or degree certificate.

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ADVISOR APPROVAL SHEET

This is to certify that this thesis is written by Getahun Aragaw in the Department of Mathematics, Addis Ababa University, under my supervision. I hear by also confirm that I have read all the part of the thesis, so it can be submitted for evaluation by examiners and eventual defense.

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Approved by the Board of Examiner

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Abstract

This thesis presents an efficient approach for determining approximate solutions of ordinary linear differential equations at ordinary points, near regular singular points and near irregular singular points. Test examples demonstrate the effectiveness and inefficiency of a given method at different points.

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Introduction

The history of mathematics shows that people devoted much of their lives to solve equations like algebraic equations at first and then eventually differential equations. Thus problems of differential equations that existing in Science and Engineering may not be solved easily, so that one of the differential equations is ordinary differential equations that are used to solve these problems.

When mathematical modeling is used to describe physical, biological or chemical phenomena, one of the most common results of the modeling process is a system of ordinary differential equations. Finding and interpreting the solutions of differential equations is therefore a central part of applied mathematics, and a thorough understanding of differential equations is essential for any applied mathematics.

Although the importance of studying differential equations is not generally in question, exactly how the theory of differential equations should be known, and what aspects should be emphasized, is more controversial.

Ordinary differential equations certainly solve some kind of equation that contains derivatives $y', \quad y''$ . . . which is analogously used to solve algebra and trigonometry, our task we use here to solve which is not solved analytically of the ordinary differential equations. The difficulties that surround higher-order linear differential equations and the few methods that yield analytic solutions are examined. But the rest of these can be estimated by using numerical methods to approximate it.

In this thesis we review three chapters. The first chapter deals with some basic notions and definitions, classification of singular points of homogeneous linear equations of ordinary differential equations together their properties; the second chapter discusses about local series expansion of the solution of at ordinary and near regular singular points, and in the third chapter we look at local series expansions about irregular singular points of homogeneous linear equations and non-homogeneous linear ordinary differential equations by the means of dominance balance.

Chapter One

Preliminaries

1.1 Classifications of Singular Points of Homogenous Linear Equations

In this section we begin the process of local analysis by classifying a point x_o, which may be complex, as an ordinary point, a regular singular point, or an irregular singular point of a homogenous linear equation. This classification gives the first indication of the nature of the solution near x_o, and suggests the appropriate route for further systematic analysis.

Let us define the homogeneous linear differential equations whose order n by

$$y^{(n)}(x)+p_{n-1}(x)y^{(n-1)}+\cdots+p_1(x)y^{(1)}+p_0(x_0)y(x)=0\ldots\ldots\ldots\ldots\ldots \textbf{(1)}$$

where $y^{(n)}(x)=\frac{d^ny}{dx^n}$ .The classification scheme we are about to describe assume that $p_0 \ldots p_{n-1}$ have been defined for complex as well as for real values of their arguments.

Ordinary points

The point x_0 ($x_0 \neq \infty$) is called an ordinary point of (**1**) if the coefficient functions

$p_0(x),,\ldots,p_{n-1}(x)$ are all analytic in a neighborhood of x_0 in the complex plane.

Example 1: Ordinary points

(a) $y''=e^xy$

$\Rightarrow y-e^xy=0$

The function $p_0(x)=e^x$ is analytic everywhere (entire)

$\Rightarrow$ Every point $x_0 \neq \infty$ is an ordinary point

(b) $x^5y''' = y$

$\Rightarrow x^5y''' - y = 0$

$\Rightarrow y''' - \frac{1}{x^5}y = 0$

Where $p_0(x) = -\frac{1}{x^5}$ is the coefficient function and $x_0 \neq 0$ and $x \neq \infty$ are ordinary point because $p(x) = -\frac{1}{x^5}$ is analytic on the complex plane.

(c). $y' = |x|y$.

There are no ordinary points in the complex$-x$ plane because $|x|$ is nowhere analytic. Fuchs proved in 1866 that all n linearly independent solution of (1) are analytic in a neighborhood of an ordinary point. Moreover, he proved that if any solution is expanded in a Taylor series about the ordinary point x_0, y(x)=$\sum_{n=0}^{\infty} a_n(x - x_{0)})^n$, then the radius of convergent of this series at least as large as the distant to the nearest singularity of the coefficient function in the complex plane .The location of a singularity of a solution must coincide with the location of singularity of a coefficient function. The solution of a linear equation cannot have singularities at any other points.

Example 2: Taylor series at any an ordinary point. The equation $(x^2+1)y' + 2xy = 0$ has an ordinary point at 0. The solution $y = c(1+x^2)$ can be expanded in a Taylor series whose radius of convergence is 1; this is the distance to the coefficient singularity at $x = \pm i$ When the differential equation is written in the form of (1).

Regular singular points

The point $x_0(x_0 \neq \infty)$ is called a regular singular point of (1) if not all of the $p_0(x), \ldots p_{n-1}(x)$ are analytic but if all of $(x-x_o)^n p_o(x), (x-x_o)^{n-1}p_1(x), \ldots, (x-x_o)p_{n-1}(x)$ are analytic in a neighborhood of x_0.

Example 3: Regular singular points

(a) $(x-1)\,y''' = y$

$\Rightarrow (x-1)y''' - y = 0$

$\Rightarrow y''' - \frac{1}{x-1}y = 0$ which is 3^{rd} order equation , $n = 3$

The only coefficient functions is $p_0(x) = -\frac{1}{x-1}$ not analytic at $x = 1$.

Now $(x-x_0)^n p_0(x) = (x-1)^3\left(-\frac{1}{x-1}\right) = -(x-1)^2$ which is analytic at $x = 1$. This $x = 1$ is a regular singular point.

(b) $x^2y'' + xy' = y$

$\Rightarrow x^2y" + xy' - y = 0$

$\Rightarrow y" + \frac{1}{x}y' - \frac{1}{x^2}y = 0$ it is 2^{nd} order $\Rightarrow n = 2$

The coefficient functions are $p_1(x) = \frac{1}{x}$ and $p_0(x) = -\frac{1}{x^2}$ both are not analytic at $x = 0$

$(x-0)^1 p_1(x) = x.\frac{1}{x} = 1$ Where is analytic at $x = 0$.

$(x-0)^2 p_0(x) = x^2.\left(-\frac{1}{x^2}\right) = -1$ Where is analytic at $x = 0$.

This the equation has a regular singular point at $x = 0$.

(c) $x^3y' = (x+1)y$

$\Rightarrow x^3y' - (x+1)y = 0$

$\Rightarrow y' - \frac{x+1}{x^3}y = 0$ where is 1^{st} order $n = 1$

The only coefficient function is $p_0(x) = -\frac{x+1}{x^3}$ which is not analytic at $x = 0$.

$$(x-0)^n p_0(x) = (x-0)^1.\left(-\frac{x+1}{x^3}\right)$$

$$= -\frac{x+1}{x^3} \quad \text{not analytic at } x = 0.$$

Thus $x = 0$ is not a regular singular point.

A solution of (1) may be analytic at a regular singular point. If it is not analytic, its singularity must be either a pole or an algebraic or logarithmic branch point. Fuchs showed that there is always at least one solution of the form

$$y = (x - x_0)^a \, A\,(x), \qquad (2)$$

Where a is a number called the indicial exponent and A(x) is a function which is analytic at x_0 and which has a Taylor series whose radius of convergence is at least as large as the distance to the nearest singularity of the coefficient functions in the complex plane.

Example 4: Taylor series at a regular singular point. The equation $y' = y/sinh\, x$ has a regular singular point at 0. The solution $y(x) = c \tan h\,(x/2)$ is analytic at $x = 0$ but has poles at $x = \pm\, i\pi$. Thus, the radius of convergence of the Taylor expansion of $y(x)$ is $\pi,$ the distance to the nearest singularities in the complex plane of the coefficient function $1/sinh\ x$.

If (1) is of order $n \geq 2,$ then there is a second linearly independent solution having one of two possible forms:

$$y = (x - x_0)^{\beta} B(x) \qquad (3)$$

$$or \qquad y = (x - x_0)^{a} A(x)\; ln\,(x - x_0) + C(x)(x - x_0)^{\beta}\,. \qquad (4)$$

Equations (3) and (4) are generalizations of the solutions of an Euler equation for non-repeated and repeated indicial exponents. $B(x)$ and $C(x)$ are new functions which are also analytic at x_0 and which have radii of convergence at least as large as the distance to the nearest singularity of the coefficient functions. $A(x)$ is the same function that appears in (2).

In general, for each new linearly independent solution there is a new analytic function of x and either a new indicial exponent or another power of $(x - x_0)$. Thus, the form of the nth solution is at worst

$$y(x) = (x - x_0)^{y} \sum_{i=0}^{n-1} [ln(x - x_0)]^{i} \, A_i(x), \qquad (5)$$

Where all the function $A_i(x)\; are\; analytic\; at\; x_0$. Conversely, Fuchs showed that if all n solutions have the forms (2) to (5), then x_0 is at worst a regular singular point of the equation.

Irregular singular points

The point $x_0(\, x_0 \neq \infty\,)$ is called an irregular singular point of (**1**) if it is neither an ordinary point nor a regular singular point.

There is no comprehensive theory of irregular singular points , but we can say that at an irregular singular point at least one solution is not the forms (2) to (5). Typically, at an irregular singular point, all solutions exhibit an essential singularity often in combination with a pole or an algebraic or logarithmic branch point. But this is not always the case. Some of the solutions may not have an essential singularity and may even be analytic at x_0.

Classification of the point $\boldsymbol{x_0} = \infty$

We have completed the classification of points $\boldsymbol{x_0}$ in the finite complex plane, but it is also useful to classify the point $\boldsymbol{x_0} = \infty$. We do this by analytically mapping the point at infinity into the origin using the inversion transformation.

$$x = \frac{1}{t}$$

$$\frac{d}{dx} = -t^2 \frac{d}{dt}, \qquad (6)$$

$$\frac{d^2}{dx^2} = t^4 \frac{d^2}{dt^2} + 2t^3 \frac{d}{dt}$$

and so on, and then classifying the point $t = 0$. The point $x_0 = \infty$ is called ordinary, a regular singular or an irregular singular point if the point at $t = 0$ is correspondingly classified.

1.2 Illustrative Example

In the following examples observe how solutions behave in the neighborhood of ordinary, regular singular and irregular singular points.

Example 5: Comparison of ordinary, regular singular, and irregular singular points. Consider the three equations.

$$\frac{dy}{dx} - \frac{1}{2}y = 0, \tag{7}$$

$$\frac{dy}{dx} - \frac{1}{2x}y = 0, \tag{8}$$

$$\frac{dy}{dx} - \frac{1}{2x^2}y = 0, \tag{9}$$

The transformation x=1/t gives three new equations which are respectively

$\frac{dy}{dt}$+$\frac{y}{2t^2} = 0, \frac{dy}{dt} + \frac{y}{2} = 0.$ Every point of (7) is an ordinary point except for ∞. Which is an irregular singular point? As expected, the solution $y(x) = \mathrm{ce}^{\mathrm{x}/2}$ is analytic expect for an essential singularity at $x = \infty$ every point of (8) is an ordinary point except for 0 and ∞ which are regular singular points. The solution $y(x) = cx^{1/2}$ is analytic expect for branch points at $x = 0$ and $x = \infty$. Every point of (9) is an ordinary point except for 0 which is an irregular singular point. The solution $y(x) = cx^{-1/2x}$is analytic in the extended plane except for an essential singularity at $x = 0$.

Example 6: Taylor series about an ordinary point. The equation $y' + \frac{y}{x} - 1 = 0$ has regular singular points at 1 and ∞. The solution $y(x) = c/(1 - x)$ has a pole $x = 1$ and is analytic at ∞. The Taylor series of the solution about

$$x = 0, \qquad y(x) = c\sum_{n=0}^{\infty} x^n \,,$$

has radius of convergence 1, which is the distance to the regular singular point at 1.

Example7: Taylor series solutions which converge beyond the nearest singular point of the differential equation. The equation $(x - 1)(2\mathrm{x} - 1)\mathrm{y}'' + 2\mathrm{xy}' - 2\mathrm{y} = 0$ has regular singular points at $\frac{1}{2}, 1,$ and ∞. One solution of this equation is $y(x) = 1 \,/\, (x - 1)$. A Taylor series expansion of this solution about the ordinary point at 0 converges beyond the first singular point at $\frac{1}{2}$ but ceases to converge at $|x| = 1$. A linearly independent solution is $y(x) = x$ whose Taylor series about $x = 0$ converges for all x.

Example 8: Essential singular behavior near an irregular singular point. The equation

$$y'' + 3y'/2x + y/4x^3 = 0$$

has an irregular singular point at 0 and a regular singular point at ∞. Two linearly independent solutions are $y(x) = sin\, x^{-1/2}, y(x) = cost\, x^{-1/2}$. Both of these solutions have an essential singularity at the origin. The first of these also has branch points at $x = 0\; and\; x = \infty$. The second solution has no branch cut and is analytic at $x = \infty$..

Example 9: Analytic solutions near singular points. At regular or irregular singular points. One or even several linearly independent solution may be analytic. The equation

$$y'' + (1 - x)\, y'/x - y\, /\, x^2 = 0$$

has a regular singular point at 0 and an irregular singular point at ∞. One solution

$y(x) = (e^x - 1 - x)/x,$ has a pole at $x = 0$ but has an essential singularity at $x = \infty$. A linearly independent solution, $y(x) = (1 + x)/x,$ has a pole at $x = 0$ but is analytic at $x = \infty$. Moreover the equation

$$y'' - \frac{1 + x}{x} y + \frac{1}{x} y = 0. \tag{10}$$

again has a regular singular point at 0 and an irregular singular point at ∞. Both linearly independent solutions

$$y(x) = e^x, \quad y(x) = 1 + x \tag{11}$$

are analytic at $x = 0$. The first has an essential singularly at $x = \infty$. And the second has a pole at $x = \infty$. In general, all linearly independent solution may be analytic at a regular singular point but at least one solution must be singular at an irregular singular point.

Sometimes it is possible to alter the character of a singular point by a transformation of the independent or dependent variable.

Example 10: Removing a singularity by transforming the independent variable. The irregular singular point at 0 of $y' - \frac{1}{2} x^{-1/2} y = 0$ disappears if we introduce a new independent variable $t = x^{1/2}$. The resulting equation, $dy/dt - y = 0,$ has an ordinary point at $t = 0$.

Example 11: Removing a singularity by transforming the independent variable.

$$y'' + \frac{2}{x}y' - y = 0 \qquad (12)$$

has a regular singular point at 0 and an irregular singular point at ∞ . The singular point at 0 may be removed by the transformation $y(x) = w(x)/x$, where $w(x)$ satisfies an equation which still has an irregular singularity at ∞: $w'' - w = 0$.

Example 12: Removing a singularity by converting to a linear system. Ostensibly the reason why the singularity of (12) is removable is that the two solutions have the same kind of singularity at $x = 0$: $y = {}^{e^x}/_{x}$, $y = e^{-x}/x$. However, it is difficult to find a transformation which eliminates the regular singular point at 0 of (10), $y'' - (1 + x)y'/x + y/x = 0$, even though both solutions $y = e^x \; and \; y = 1 + x$ are analytic at $x = 0$! The only way to eliminate this singular point is to covert (10) into a linear system of equation of the form

$$\begin{aligned} y'(x) &= a(x)y(x) + b(x)z(x), \\ z'(x) &= c(x)y(x) + d(x)z(x), \end{aligned} \qquad (13)$$

If the general solution of a second-order differential equation is analytic at x=0, then it is possible to find an equivalent homogenous linear system of equations of the form (13) whose coefficients are analytic at $x = 0$. This fact may be especially helpful to an unhappy numerical analyst who is trying to solve a differential equation which has a singular point where the solutions are all known to be analytic.

Chapter Two

Approximate Solution of Linear Ordinary Differential Equations at Ordinary and Regular Singular Points

2.1 Local Behavior Near Ordinary Points Of Homogeneous Linear Equation

In this chapter, we show how to determine the local behavior that solutions to homogenous linear differential equations interms of infinite series expansions when exact closed-form solutions are not known.

First we classify the point as regular, ordinary or an irregular singular point and then selecting a suitable form of the series based on this classification.

A local expansion a bout x_0 of a solution to a differential equation is an example of perturbation series (a series in powers of a small parameter). For instance the small parameter in the distance between x and x_0. Since $y(x)$ is analytic near x_0 if x_0 is an ordinary point, the perturbation expansion is a Taylor series in powers of $x-x_0$. Here the n^{th} approximant (the sum of the first n terms of the perturbation expansion) to the local behavior near an ordinary point a more accurate approximation to the solution as $|x-x_0|$ becomes smaller, or as n increases, or both.

To obtain a series solution about an ordinary point we substitute the Taylor series $y(x)=\sum_{n=0}^{\infty} a_n(x-x_0)^n$ into the differential equation and determine the coefficients a_n by solving a recursion relation.

Since Taylor series is analytic everywhere, it is convenient to approximate a homogeneous linear differential equation at ordinary point by *Taylor series*.

To obtain a series solution about an ordinary point we substitute the Taylor series of $y(x)=\sum_{n=0}^{\infty} a_n(x-x_0)^n$ in to the differential equation and determine the coefficients a_n by solving a recursion relation.

Example 1: (Taylor series solution of a first order differential equation) consider the initial value problem $y' = 2xy, \quad y(0) = 1.$

$$y' = 2xy \;\Rightarrow\; y' - 2xy = 0$$

$$\Rightarrow p_0(x) = -2x$$

$$\Rightarrow \lim_{x\to 0} p_0(x) = 0.$$

$\Rightarrow x = 0$ is an ordinary point. We may seek a solution in the form of a Taylor series.

Let $y(x) = \sum_{n=0}^{\infty} a_n (x-0)^{\mathrm{n}} = \sum_{n=0}^{\infty} a_n x^{\mathrm{n}}$ be the solution of the equation. Then substituting in to differential equation and differentiating term by term we get

$$\sum_{n=0}^{\infty} n a_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$0.a_0 + a_1 + \sum_{n=2}^{\infty} n a_n x^{n-1} - 2\sum_{n=2}^{\infty} a_{n-2} x^{n+1} = 0$$

$$0.a_0 + a_1 + \sum_{n=2}^{\infty} (n a_n - 2a_{n-2}) x^{n-1} = 0$$

a_0 is an arbitrary constant number

$a_1 = 0$

$n a_n = 2a_{n-2}$ for $n = 2,3,4,\ldots$

i.e $a_n = \frac{2a_{n-2}}{n}, \; n = 2,3,4,\ldots$

In general $a_{2n+1} = 0$ for $n \geq 1$ and $a_{2n} = \frac{a_0}{n!}$ for $n \geq 1$

Thus the general solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^{\mathrm{n}} = \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1} + \sum_{n=0}^{\infty} a_{2n} x^{2n}$$

$$\Rightarrow y(x) = 0 + \sum_{n=0}^{\infty} a_{2n} x^{2n} = \sum_{n=0}^{\infty} \frac{a_0}{n!} x^{2n} = a_0 . \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = a_0 e^{x^2} \text{ where } e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

Since $y(0) = 1$, we get $a_0 = 1$.Therefore, the solution of the equation is $y(x) = e^{x^2}$.

We now state and prove the following theorem, dealing with power series solution near ordinary point.

Theorem 1: If the equation $y'' + p(x)y' + q(x)y = 0$ has an ordinary at point $x = 0$, due to the fact that $p(x)$ and $q(x)$ have power series representations converging for $|x| < R$ (where R is the radius of convergence of the series), then there are independent solutions $y_1(x)$ and $y_2(x)$ satisfying $y_1(0) = 1$, $y'_1(0) = 0$, $y_2(0) = 0$, $y'_2(0) = 1$ and each having a power series representation converging for $|x| < R$.

Proof: Suppose that $y_1(x)$ and $y_2(x)$ be the solutions of the differential equation

$$y'' + p(x)y' + q(x)y = 0.$$

To determine these solutions interms of power series solutions, let us take

$$y(x) = c_0 + c_1x + c_2x^2 + \cdots = \sum_{k=0}^{\infty} c_kx^k$$

be a solution of the equation where the series is known to converge for $|x| < R$. Differentiating twice, we get $y'(x) = \sum_{k=0}^{\infty} kc_kx^{k-1}$ and $y''(x) = \sum_{k=0}^{\infty} k(k-1)c_kx^{k-2}$.
Assuming that $p(x) = p_0 + p_1x + p_2x^2 + \cdots = \sum_{k=0}^{\infty} p_kx^k$ and

$$q(x) = q_0 + q_1x + q_2x^2 + \cdots = \sum_{k=0}^{\infty} q_kx^k$$

By substitution the above in the equation we will get

$$2c_2 + 6c_3x + 12c_4x^2 + \cdots$$

$$+(p_0 + p_1x + p_2x^2 + \cdots)(c_1 + 2c_2x + 3c_3x^2 + \cdots)$$

$$+(q_0 + q_1x + q_2x^2 + \cdots)(c_0 + c_1x + c_2x^2 + \cdots) = 0$$

Equating the coefficients of various powers of x to zero, we have

$$2c_2 + p_0c_1 + q_0c_0 = 0$$

$$6c_3 + 2p_0c_2 + p_1c_1 + q_0c_1 + q_1c_0 = 0$$

$$12c_4 + 3p_0c_3 + 2p_1c_2 + p_2c_1 + q_0c_2 + q_1c_1 + q_2c_0 = 0$$

etc. The first equation defines c_2 in terms of c_0 and c_1 which are arbitrary; the second equation defines c_3; the third equation defines c_4; etc. Since the series for $p(x)$ and $q(x)$ converge absolutely for $|x| \leq r < R$, there are constants of M and N such that $|p_k| \leq Mr^{-k}$ and $|q_k| \leq Nr^{-k}$. Let K be the larger of M and Nr; then $|p_k| \leq Kr^{-k}$ and $|q_k| \leq kr^{-k-1}$. If $|c_0| = a_0$and $|c_1| = a_1$, then

$$2|c_2| \leq a_1|p_0| + a_0|q_0|$$

$$\leq 2Ka_1 + Ka_0r^{-1}$$

So that $|c_2| \leq a_2$ where $2a_2 = K(2a_1 + a_0r^{-1})$. Furthermore

$$6|c_3| \leq 2a_2|p_0| + a_1|p_1| + a_1|q_0| + a_0|q_1|$$

$$\leq 3a_2K + 2a_1Kr^{-1} + a_0Kr^{-2}$$

and

$$12|c_4| \leq 3a_3|p_0| + 2a_2|p_1| + a_1|p_2| + a_2|q_0| + a_1|q_1| + a_0|q_2|$$

$$\leq 4a_3K + 3a_2Kr^{-1} + 2a_1Kr^{-2} + a_0Kr^{-3}$$

Therefore, $|c_3| \leq a_3$ and $|c_4| \leq a_4$, where

$$6a_3 = K(3a_2 + 2a_1r^{-1} + a_0r^{-2})$$

$$12a_4 = K(4a_3 + 3a_2r^{-1} + 2a_1r^{-2} + a_0r^{-3})$$

Continuing in this way, we have $|c_k| \leq a_k$, where $(k-1)ka_k = K[ka_{k-1} + (k-1)a_{k-2}r^{-1} + \cdots + 2a_1r^{-k+2} + a_0r^{-k+1}]$ for $k = 2,3,4, \ldots$ if $k \geq 3$, we can write

$$(k-2)(k-1)a_{k-1}r^{-1} = K[(k-1)a_{k-2}r^{-1} + \cdots + 2a_1r^{-k+2} + a_0r^{-k+1}]$$

Subtracting we have

$$(k-1)ka_k-(k-2)(k-1)a_{k-1}r^{-1}=Kka_{k-1}$$

From which follows that

$$\frac{a_k}{a_{k-1}}=\frac{k-2}{kr}+\frac{K}{k-1}$$

Now consider the series $\sum_{k=0}^{\infty}a_kx^k$. Applying the ratio test, we have

$\lim_{k\to\infty}\frac{a_k|x|}{a_{k-1}}=\frac{|x|}{r}<1$ for $|x|<r$.

Therefore, $\sum_{k=0}^{\infty}a_kx^k$ converges absolutely $|x|<r$, where the series $\sum_{k=0}^{\infty}c_kx^k$ also converges absolutely by comparison. However, r is any positive number less than R, so that $\sum_{k=0}^{\infty}c_kx^k$ converges absolutely for $|x|<R$. It is now simple matter to substitute the series in to the differential equation and using the known properties of power series, to verify that is a solution. Finally, since c_0 and c_1 are arbitrary, we can find two independent solutions $y_1(x)$ and $y_2(x)$ by assuming $c_0=1$, $c_1=0$, for $y_1$and $c_0=0, c_1=1$, for y_2. This completes the proof.

Example 2: (The Airy equation is a classical equation of mathematical physics of the form $y''=xy$.)

To determine the approximate solution of the Airy equation $y''=xy$ near $x=0$.

$y''=xy \Rightarrow y''-xy=0$. Since $p_0(x)=-x$, we get $\lim_{x\to 0}p_0(x)=0$. This implies that $x=0$ is an ordinary point.

Let us assume $y(x)=\sum_{n=0}^{\infty}a_nx^n$ be the solution of the equation. Then we have that

$y'=\sum_{n=0}^{\infty}a_n\,x^{n-1}$ and $y''=\sum_{n=0}^{\infty}\mathrm{n(n-1)}\,a_nx^{n-2}$. Then equation $y''=xy$ becomes

$\sum_{n=0}^{\infty}\mathrm{n(n-1)}\,a_nx^{n-2}-\sum_{n=0}^{\infty}a_n\,x^{n+1}=0$

$\Rightarrow 0.\,a_0+0.\,a_1+2a_2+\sum_{n=3}^{\infty}(\mathrm{n(n-1)}\,a_n-a_{n-3})x^{n-2}=0$

$\Rightarrow a_0$ *and* a_1 are arbitrary constants , $a_2=0$ and $n(n-1)a_n=a_{n-3}$ $(for\ n\geq 3)$.

Then $a_n = \frac{a_{n-3}}{n(n-1)}$. After this we will get the following

$$a_{3n} = \frac{a_0}{3n(3n-1)(3n-3)(3n-4)\ldots 9.8.6.5.3.2}$$

$$= \frac{a_0}{3n[3\left(n-\frac{1}{3}\right)3(n-1)3\left(n-\frac{4}{3}\right)\ldots 3\left(\frac{5}{3}\right)3(1)3(\frac{2}{3})]}$$

$$= \frac{a_0}{3^n n! 3^n \left(n-\frac{1}{3}\right)\left(n-1-\frac{1}{3}\right)\left(n-2-\frac{1}{3}\right)\ldots\frac{5}{3}.\frac{2}{3}}$$

$$= \frac{a_0.\Gamma\ (\frac{2}{3})}{3^n n! 3^n\ \Gamma(n+\frac{2}{3})}$$

$$a_{3n+1} = \frac{a_1}{(3n+1)(3n)(3n-2)(3n-3)\ldots 10.9.7.6.4.3}$$

$$= \frac{a_1}{3^n n! 3^n \left(n+\frac{1}{5}\right)\left(n-1+\frac{1}{3}\right)\left(n-2+\frac{1}{3}\right)\ldots(\frac{7}{3})(\frac{4}{3})}$$

$$= \frac{a_{1.}\ \Gamma\ (\frac{4}{3})}{3^n n! 3^n.\Gamma(n+\frac{4}{3})}$$

and $a_{3n+2} = 0$.

Note that the gamma function $\Gamma(z)$ is a function of complex z which satisfies

$$\Gamma(z+1) = z\Gamma(z).$$

Let c_1=$a_0\Gamma(\frac{2}{3})$ and $c_2 = a_1\Gamma(\frac{4}{3})$. Then $y(x)$=$c_1\sum\frac{x^{3n}}{9^n n!\Gamma(n+\frac{2}{3})}$+$c_2\sum\frac{x^{3n+1}}{9^n n!\Gamma(n+\frac{4}{3})}$

is the general solution of the Airy equation for x→0 the convergence is faster which means fewer terms are required for any desired accuracy.

The constants c_1 and c_2 are determined by initial conditions given at $x = 0$, in this case we have c_1=$\Gamma(\frac{2}{3})y(0)$ and c_2= $\Gamma\ (\frac{4}{3})y'(0)$.

It is conventional to define two special linearly independent solutions of the Airy equation by

$$A_i(x) = 3^{-\frac{2}{3}}\sum\frac{x^{3n}}{9^n n!\Gamma(n+\frac{2}{3})} - 3^{-\frac{4}{3}}\sum\frac{x^{3n+1}}{9^n n!\Gamma(n+\frac{4}{3})} \qquad (14)$$

$$B_i(x) = 3^{-\frac{1}{16}} \sum \frac{x^{3n}}{9^n n! \Gamma(n+\frac{2}{3})} + 3^{-\frac{5}{6}} \sum \frac{x^{3n+1}}{9^n n! \Gamma(n+\frac{4}{3})} \qquad (15)$$

2.2 Local Series Expansions About Regular Singular Points of Homogeneous Linear Equations

In section 2.1 which have seen that Taylor series are a good way to represent the local behavior of a solution to a differential equation near ordinary point.

The Taylor series approximation of a solution of a differential equation is not efficient near a regular singular point.

Example 1: Breakdown of a Taylor series representation at a regular singular point. If we seek solution to $y'' + \frac{y}{4x^2} = 0$ in the form of a Taylor series about $x = 0$, the formula expansion procedure which works for ordinary point is not fruitful here. To wit, if we substitute the Taylor series $y = \sum_{n=0}^{\infty} a_n x^n$ into this equation and differentiate term by term, we obtain the following sequence of equation for

a_n: $\left(n - \frac{1}{2}\right)^2 a_n = 0$ $(n = 0,1,2, \dots)$. The solution to these equations is $a_n = 0$ for all n. Thus, we obtain only the trivial solution $y(x) = 0$. This is not progress .

Taylor series expansion failed in Example 1 because the Taylor series are not general enough to describe the local behavior of near regular singular points. Fortunately, the result of Fuchs stated 1.1 suggests a more general structure than a Taylor series. In particular , we learned that if x_0 is a regular singular point of a linear homogenous differential equation , at least one solution must have the form (2): $y(x) = (x - x_0)^{\alpha} A(x)$, where $A(x)$ is analytic at x_0. This solution has an algebraic branch point at x_0 if α is non-integral and a pole at x_0 if α is a negative integer. $y(x)$ is analytic at x_0 if $\alpha = 0,1,2, ..$ the other $n - 1$ linearly independent solutions of an n^{th} – order equation have the form (3),(4), or (5), so they may also exhibit logarithmic branch points.

Since $A(x)$ is analytic, it can be expanded in a Taylor series:

$$y(x) = (x - x_0)^{\alpha} A(x) = (x - x_0)^{\alpha} \sum_{n=0}^{\infty} a_n \, (x - x_0)^n \qquad (16)$$

We call the right side of 16 a Frobenius series and we call the number α an indicial exponent. It is conventional to assume that $a_0 \neq 0$ in a Frobenius series, which is ensured by a proper choice of α. A Taylor series is a special case of a Frobenius series.

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Definition 2.1 The series$(x - x_0)^{\alpha} \sum_{n=0}^{\infty} a_n (x - x_0)^n$ is called a Frobenius Series and the number α is called an inidicial exponent.

Remark: For $\alpha = 0$, the series is called the Taylor series.

In the solution of $y(x) = (x - x_0)^{\alpha} \sum_{n=0}^{\infty} a_n (x - x_0)^n$ of homogenous linear differential equation, it is conventional to assume a_0 which can be ensured by a proper choice of α.

2.2.1 Frobenius Method for Second- Order Equation

If the differential equation

$$y'' + \frac{p(x)}{x - x_0} y' + \frac{q(x)}{(x - x_0)^2} y = 0 \tag{17}$$

has a regular singular point at x_0, then $p(x)$ and $q(x)$ are analytic at x_0. Thus we may expand $p(x)$ and $q(x)$ in a Taylor series about x_0: $p(x) = \sum_{n=0}^{\infty} p_n (x - x_0)^n$,

$q(x) = \sum_{n=0}^{\infty} q_n (x - x_0)^n$. We now substitute these Taylor expansions into (!7) and obtain a solution $y(x)$ in the form of the Frobenius series (16) by equating the coefficients of $(x - x_0)^{n+\alpha-2}$ for $n = 0,1,2,\ldots$

$$(x - x_0)^{\alpha-2}: \qquad [\alpha^2 + (p_0 - 1)\alpha + q_0]a_0 = 0 \tag{18a}$$

$$(x - x_0)^{n+\alpha-2}: \qquad [(\alpha + n)^2 + (p_0 - 1)(\alpha + n) + q_0]a_n$$

$$= -\sum_{k=0}^{n-1} [(\alpha + k)p_{n-k} + q_{n-k}]\, a_k, \qquad n = 1,2,\ldots. \tag{18b}$$

By assumption a_0 is nonzero, so (18a) requires that α must be a root of the indicial polynomial $p(\alpha)$, where

$$p(\alpha) = \alpha^2 + (p_0 - 1)\alpha + q_0 \tag{19}$$

Given these values of α we must then solve the recursion relation (18b) for a_n in terms of a_0. The constant a_0 is arbitrary and will ultimately appear as an overall multiplicative factor in the solution $y(x)$. However, the recursion relation (18b) can be solved for a_n in terms of a_k for $k < n$ only if $p(\alpha + n) \neq 0$ because the left side of (18b) is precisely $p(\alpha + n)a_n$. If this condition holds for all positive integers n, then it may be shown that the series (16) converges in a circle whose radius is as large as the distance to the nearest complex singularity of $p(x)$ and $q(x)$.

Let α_1 and α_2 denote the two roots of the indicial polynomial $p(\alpha)$ which are observed so that $Re\ \alpha_1 \geq Re\ \alpha_2$. If we let $\alpha = \alpha_1$, then $p(\alpha_1 + n) \neq 0$ for $n = 1,2,..$ because α_2 is the only other root of p. Thus, the recursion relation (18b) can be solved for a_n in terms of a_0 for all n. This explains why there is always at least one solution in the form of a Frobenius series.

Next we take that the differential equation $y'' + p(x)y' + q(x)y = 0$ has a regular singular point at $x = 0$ if $xp(x)$ and $x^2q(x)$ have power series representation converging for $|x| < R$. We assume that

$$xp(x) = a_0 + a_1x + a_2x^2 + \cdots += \sum_{k=0}^{\infty} a_k x^k$$

$$x^2q(x) = b_0 + b_1x + b_2x^2 + \cdots += \sum_{k=0}^{\infty} b_k x^k$$

Where the coefficients a_k and b_k are real. The indicial equation is

$$I(m) = m(m-1) + a_0m + b_0 = 0$$

We assume that roots of indicial equation, m_1 and m_2, are both real and that $m_1 \geq m_2$.

Theorem 2 If the equation $y" + p(x)y' + q(x)y = 0$ has a regular singular point at $x = 0$ due to the fact that $xp(x) = \sum_{k=0}^{\infty} a_k x^k$ and $x^2 q(x) = \sum_{k=0}^{\infty} b_k x^k$ converge for $|x| < R$, if the roots of m_1 and m_2 of indicial equation are both real, $m_1 \geq m_2$, then for $x > 0$ there is a solution of the form $y(x) = x^{m_1} \sum_{k=0}^{\infty} c_k k^x$ where the series converges for $|x| < R$.

Proof: We begin by assigning arbitrary nonzero value to c_0. Then the other coefficients are determined by

$$c_k I(m_1 + k) = -\sum_{j=0}^{k-1} c_j \left[(m_1 + j)a_{k-j} + b_{k-j}\right]$$

Where $I(m) = m(m-1) + a_0 m + b_0 = (m - m_1)(m - m_2)$. Therefore, $I(m_1 + k) = k(k + m_1 - m_2)$. For all $k > m_1 - m_2$ we can write the inequality

$$k(k + m_1 - m_2)|c_k| \leq |I(m_1 + k)||c_k|$$

$$\leq \sum_{j=0}^{k-1} \left||c_j|\right| \left[(|m_1| + j)|a_{k-j}| + |b_{k-j}|\right]$$

Let $|c_j| = c_j$ for $j < n$, where n is some integer greater than $m_1 - m_2$.

Then

$$n(n - m_1 m_2)|c_n| \leq K \sum_{j=o}^{n-1} c_j \left[(|m_1| + j)|a_{k-j}| + |b_{k-j}|\right]$$

There exists a positive constant K such that $|a_k| \leq K r^{-k}$ and $|b_k| \leq K r^{-k}$

for $0 < r < R$. Then

$$n(n - m_1 + m_2)|c_n| \leq K \sum_{j=o}^{n-1} c_j \left[|m_1| + j + 1\right] r^{-n+j}$$

and $|c_n| \leq c_n$, where

$$n(n-m_1+m_2)c_n = K\sum_{j=o}^{n-1} c_j\,[|m_1|+j+1]r^{-n+j}$$

Furthermore, for $k > n$

$$k(k-m_1+m_2)|c_k| \leq K\sum_{j=o}^{k-1} c_j\,[|m_1|+j+1]r^{-k+j}$$

and $|c_k| \leq c_k$, where

$$k(k-m_1+m_2)c_k \leq K\sum_{j=o}^{k-1} c_j\,[|m_1|+j+1]r^{-k+j}$$

Replacing k by $k-1$ and dividing by r, we have

$$(k-1)(k-1-m_1-m_2)c_{k-1}r^{-1} = K\sum_{j=o}^{k-2} c_j\,[|m_1|+j+1]r^{-k+j}$$

Subtracting, we obtain

$$k(k-m_1+m_2)c_k - (k-1)(k-1-m_1-m_2)c_{k-1}r^{-1}$$

$$= Kc_{k-1}(|m_1|+k)r^{-1}$$

Or

$$\frac{c_k}{c_{k-1}} = \frac{(k-1)(k-1-m_1-m_2)}{k(k-m_1+m_2)r} + \frac{K(|m_1|+k)}{k(k-m_1+m_2)}$$

We compare the series $\sum_{k=0}^{\infty} c_k k^x$with $\sum_{k=0}^{\infty} c_k k^x$. The latter converges absolutely for $|x| < r$ by the ratio test, since

$$\lim_{k\to\infty} \frac{c_{k|x|}}{c_{k-1}} = \frac{|x|}{r} < 1$$

Therefore, the series $\sum_{k=0}^{\infty} c_k k^x$ converges absolutely for $|x| < R$, since r is any positive real number less than R. It remain the substitute the proposed solution

$y(x) = x^{m_1}\sum_{k=0}^{\infty} c_k k^x$ and use the properties of power series to that it satisfies the differential equation. This completes the proof.

Example 3: Determine the Frobenius series solution of $y'' + \frac{y}{4x^2} = 0$

Solution :- Let $y(x) = x^{\alpha}\sum_{n=0}^{\infty} a_n x^n$ be a solution of the equation since $x = 0$ is a regular singular point. Then

$$y'(x) = \sum_{n=0}^{\infty}(n+\alpha)a_n x^{n+\alpha-1}$$

$y''(x) = \sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)a_n x^{n+\alpha-2}$.

By substitution of these values in the equation $y'' = -\frac{1}{4x^2}y$ we will getl

$$\sum_{n=0}^{\infty}(n+\alpha)(n+\propto-1)a_n x^{n+\alpha-2} = -\frac{1}{4x^2}\sum_{n=0}^{\infty} a_n x^{n+\alpha}$$

$\Rightarrow\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)a_n x^{n+\alpha-2} = -\frac{1}{4}\sum_{n=0}^{\infty} a_n x^{n+\alpha-2}$

$\Rightarrow(n+\alpha)(n+\alpha-1)a_n = -\frac{1}{4}a_n$

$\Rightarrow[(n+\alpha)(n+\alpha-1) + \frac{1}{4}]a_n = 0.$

Since $a_0 \neq 0$for $= 0$, we have $\alpha(\alpha-1) + \frac{1}{4} = 0$. That is

$\alpha^2 - \alpha + \frac{1}{4} = (\alpha - \frac{1}{2})^2 = 0 \Rightarrow \alpha = \frac{1}{2}$but for $n = 1,2,3,\ldots$we have $a_1 = a_2 = \cdots = 0$. Therefore $y(x) = a_0\sqrt{x}$, where a_0 is arbitrary, is the Frobenius series solution of the differential equation.

Example 4: (The Bessel equation of order $\boldsymbol{\mathcal{V}}$**)**

The Bessel equation $\quad y'' + \frac{1}{x}y' - (1+\frac{\mathcal{V}^2}{x^2})y = 0$……..(5)

has a regular singular point at $x = 0$. Since (5) has singular point at $x = 0$, we use Frobenius series $y(x) = \sum_{n=0}^{\infty} a_n x^{\alpha+n}$ be a solution of (5). Then

$$y' = \sum_{n=0}^{\infty} a_n(\alpha + n)\, x^{n+\alpha-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n(\alpha + n)(\alpha + n - 1)a_n\, x^{n+\alpha-2}$$

Now by substituting in equation (5) we have

$\sum_{n=0}^{\infty} a_n(\alpha + n)(\alpha + n - 1)\, x^{n+\alpha-2} + \frac{1}{x}\sum_{n=0}^{\infty} a_n(\alpha + n)\, x^{n+\alpha-1} - (1+\frac{\mathcal{V}^2}{x^2})$

$\sum_{n=o}^{\infty} a_n\, x^{\alpha+n}$=0.

$\Rightarrow \sum_{n=0}^{\infty} a_n(\alpha + n)(\propto + n - 1).\, x^{n+\alpha-2} + \sum_{n=0}^{\infty} a_n(\propto + n)\, x^{n+\alpha-2}$

$-\sum_{n=o}^{\infty} a_n\, x^{\alpha+n} - \sum_{n=0}^{\infty} \mathcal{V}^2 a_n\, x^{\alpha+n-2}$=0.

$\Rightarrow \sum_{n=0}^{\infty} a_n(\alpha + n)(\alpha + n - 1).\, x^{n+\alpha-2} + \sum_{n=0}^{\infty} a_n(\alpha + n)\, x^{n+\alpha-2}$

$-\sum_{n=o}^{\infty} a_{n-2}\, x^{\alpha+n} - \sum_{n=0}^{\infty} \mathcal{V}^2 a_n\, x^{\alpha+n-2}$=0.

By equating coefficient of like terms of x to zero gives

For $x^{\alpha-2}$ **:** $(\alpha^2 - v^2)a_0 = 0$.

For $x^{\alpha-1}$ **:** $[(\alpha + 1)^2 - v^2]a_1 = 0$.

For $x^{\alpha+n-2}$: $[(\alpha + n)^2 - v^2]\ and\ a_n = a_{n-2}\ \ , n = 2,3, \ldots$

Since $\boldsymbol{\mathcal{V}}$ appears as $\boldsymbol{\mathcal{V}^2}$ in the Bessel equation ,we may assume that $Re(\mathcal{V}) \geq 0$ and denote $\alpha_1 = +\mathcal{V}$ and $\alpha_2 = -\mathcal{V}$.

Thus $p(\alpha_1 + n) \neq 0$ for $n \in \mathbb{N}$

For $n = 1,\ [(\alpha + 1)^2 - v^2]a_1 = 0\ \Rightarrow a_1 = 0$

For $n = 2,\ [(\alpha + 2)^2 - v^2]a_2 = a_0 \Rightarrow a_2 = \frac{a_0}{(\alpha+2)^2 - v^2}$

For $n = 3,\ [(\alpha + 3)^2 - v^2]a_3 = a_1 \Rightarrow a_3 = a_1 = 0$

For $n = 4,\ [(\alpha + 4)^2 - v^2]a_4 = a_2 \Rightarrow a_4 = \frac{a_2}{(\alpha+4)^2 - v^2} = \frac{a_0}{(\alpha+2)^2 - v^2(\alpha+4)^2 - v^2}$

In general odd terms $a_{2n+1} = 0$ and even terms are

$$a_{2n} = \frac{a_{2n-2}}{2^2 n(\mathcal{V}+n)} = \frac{a_{2n-4}}{2^4 n(n-1)(\mathcal{V}+n)(\mathcal{V}+n-1)} = \dots = \frac{a_0\,\Gamma(v+1)}{2^{2n} n!\;\Gamma(\mathcal{V}+n+1)}.$$

Now ,we get $y = \sum_{n=0}^{\infty} a_n x^{n+\alpha_1} = \sum_{n=0}^{\infty} a_{2n} x^{2n+v}$

$y' = \sum_{n=0}^{\infty} a_{2n} x^{2n+v} = \sum_{n=0}^{\infty} \frac{a_0 \Gamma(v+1)}{2^{2n} n! \Gamma(v+n+1)}$

$$= a_0 \Gamma\,(\boldsymbol{\mathcal{V}}+1) \sum_{n=0}^{\infty} \frac{x^{2n}.\,x^{\mathcal{V}}}{2^{2n} n! \Gamma(\mathcal{V}+n+1)}$$

$$= a_0 \Gamma\,(\boldsymbol{\mathcal{V}}+1).\,x^{\mathcal{V}} \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n}}{n! \Gamma(\mathcal{V}+n+1)}$$

By choosing $a_0 = \frac{2^{-\mathcal{V}}}{\Gamma(\mathcal{V}+1)}$ in the above equations, the result is the Frobenius series expansion of the modified Bessel function $I_{\boldsymbol{v}}(x)$:

$$I_{\boldsymbol{v}}(x) = \frac{2^{-\mathcal{V}}}{\Gamma(\mathcal{V}+1)} \,.\, \Gamma(\boldsymbol{\mathcal{V}}+1).x^{\mathcal{V}} \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n}}{n! \Gamma(\mathcal{V}+n+1)}$$

$$= \left(\frac{x}{2}\right)^{\mathcal{V}} . \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n}}{n! \Gamma(\mathcal{V}+n+1)}$$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n+\mathcal{V}}}{n! \Gamma(\mathcal{V}+n+1)}.$$

Using the ratio test, we have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\left(\frac{x}{2}\right)^{2(n+1)+\mathcal{V}}}{(n+1)! \Gamma(\mathcal{V}+n+2)} \,.\, \frac{n! \Gamma(\mathcal{V}+n+1)}{\left(\frac{x}{2}\right)^{2n+\mathcal{V}}}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{x}{2}\right)^{2n+\mathcal{V}} . \left(\frac{x}{2}\right)^{2} .\ n! \Gamma(\mathcal{V}+n+1)}{(n+1) n! \Gamma(\mathcal{V}+n+2)\,.\left(\frac{x}{2}\right)^{2n+\mathcal{V}}}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{x}{2}\right)^{2}}{n+1} \,.\, \frac{\Gamma(\mathcal{V}+n+1)}{n\Gamma(\mathcal{V}+n+1)}$$

$= 0 < 1$ convergent for any x.

Let $\alpha = -\mathcal{V}$ is not an integer $-2\mathcal{V} \notin \mathbb{Z}$. Then $p(-\mathcal{V}+n) \neq 0 \quad \forall_{n} \in \mathbb{N}$.

$\Rightarrow a_{2n} = \frac{a_{2n-2}}{2^2 n(n-1)} = \frac{a_{2n-4}}{2^4 n(n-1)(n-\mathcal{V})(n-\mathcal{V}-1)} = \cdots = \frac{a_0 \Gamma(1-\mathcal{V})}{2^{2n} n!(n-\mathcal{V}+1)}$

$\Rightarrow$y(x)=$a_0 \Gamma(1-\mathcal{V}) x^{-\mathcal{V}} \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n}}{n!\Gamma(n-\mathcal{V}+1)}$

By choosing $a_0 = \frac{2^n}{\Gamma(1-\mathcal{V})}$, the above result gives

$$y(x) = \frac{2^{\mathcal{V}}}{\Gamma(1-\mathcal{V})}\ \Gamma(1-\mathcal{V}) x^{-\mathcal{V}} \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n}}{n!\,\Gamma(n-\mathcal{V}+1)}$$

$=\left(\frac{x}{2}\right)^{-\mathcal{V}} \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n}}{n!\Gamma(n-v+1)}$

$=\sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n-\mathcal{V}}}{n!\Gamma(n-\mathcal{V}+1)}$.

Thus, $I_{-v}(x)$=$\sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n-\mathcal{V}}}{n!\Gamma(n-\mathcal{V}+1)}$ is the other solution of the modified Bessel equation.

Moreover $\mathrm{I}_{v}(x)$ and $\mathrm{I}_{-v}(x)$ are linearly independent because their initial powers of x, x^{2n+v} and $x^{2n-v}$are different.

2.2.2 Frobenius Series for Higher – order Equation

The Frobenius method extends easily to the general n^{th} – order homogenous linear differential equation at a regular singular point x_0:

$\frac{d^n}{dx^n} + \frac{q_{n-1(x_0)}}{x-x_0}\frac{d^{n-1}y}{dx^{n-1}} + \frac{q_{n-1}(x)}{(x-x_0)^2}\frac{d^{n-2}y}{dx^{n-2}} + \ldots + \frac{q_0(x)}{(x-x_0)^n} y = 0,$

Where $q_0(x), \ldots, q_{n-1}(x)$ are analytic at x_0. The indicial equation for α is $\alpha(\alpha-1) \ldots (\alpha-n+1) + q_{n-1}(x_0)\alpha(\alpha-1) \ldots (\alpha-n+2) + q_{n-1}(x_0)\alpha(\alpha-1) \ldots (\alpha - n+3)$+…+$q_0(x_0) = 0$.

If the n roots α do not differ by integers, then there will be n linearly independent solutions of the form (17); otherwise, the form of the solution must be generalized to (4) or (5).

Chapter Three

Local Behavior at Irregular Singular Points of Homogeneous Linear Equations

Our analysis of ordinary and of regular singular points is fundamentally different from the asymptotic analysis of to be used though out this book. The approach of the last two sections has been prosaic and heavy-handed: the series were convergent, the manipulations were mechanical and rigorous, and the treatment was thorough to the point of being devastatingly boring (to us). This mummified style reflects the completeness of the theory; there was no room for imaginative mathematics.

From this point on, our style changes. To analyze the local behavior of solutions near irregular singular points, we will be forced to develop entirely new mathematical tools which as a whole comprise a calculus of approximations. Although it is possible to justify to use of these tools on a rigorous level, such a justification makes for a slow reading and is contrary to sprit of this book. Our intention is to omit time-consuming rigor and to emphasize careful problem solving. In a contrast to the methods of the last two sections which we would describe as exact, rigorous, intuitive, heuristic, powerful, and fascinating.

In this section we will see that formulas which express the local behavior of a near irregular singular points are generalizations of Frobenius series in much the same way that Frobenius series are generalizations of Taylor series. Let us begin our analysis by discovering why Frobenius series are insufficient to describe behavior near irregular singular points. We know that if all solutions to a liner differential equation in the neighborhood of a point x_0 can be expanded in a Frobenius series, then x_0 is a regular singular point. Thus, at an irregular singular point, at least one solution must have a Frobenius series representation. Let us observe explicitly how Frobenius series fail.

Example 1: (Irregular point at which there are no solution of Frobenius form)

The differential equation $x^2y" + (1 + 3x)y' + y = 0$ has an irregular singular point at 0. That is $y''+ \frac{1+3x}{x^2} + \frac{1}{x^2y}=0$ which shows that $p_1(x) = \frac{1+3x}{x^2}$ and $p_0(x) = \frac{1}{x^2}$.

But, $x\ f_1(x) = x\frac{1+3x}{x^2} = \frac{1+3x}{x}$ is not analytic at $x = 0$. Thus $x = 0$ is an irregular singular point.

Assuming the ordinary differential equation has a Frobenius series solution $y = \sum_{n=0}^{\infty} a_n x^{n+\alpha}$. Then

$y' = \sum_{n=0}^{\infty} a_n(n+\alpha)x^{n+\alpha-1}$ and $y'' = \sum_{n=0}^{\infty} a_n(n+\alpha)(n+\alpha-1)x^{n+\alpha-1}$.

By substitution of the value of y'' , y' and y in the series form, the equation becomes

$$x^2\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)a_nx^{n+\alpha-2}+(1+3x)\sum_{n=0}^{\infty}(n+\alpha)a_nx^{n+\alpha-1}+$$

$$\sum_{n=0}^{\infty} a_nx^{n+\alpha}=0.$$

Equating coefficients of power of x to zero, we get **:**

for $x^{\alpha-1}$we have $\alpha a_0 = 0$

for $x^{n+\alpha}$**:** $(n+\alpha+1)a_{n+1} + (\mathrm{n}+\alpha+1)^2a_n = 0$,$n = 0,1, \ldots.$

but, a_0≠0 for Frobenius series, the indicial equation is α=0

$\Rightarrow(n+1)a_{n+1} + (n+1)^2a_n = 0 \text{for} n = 0,1,2 \ldots$

$\Rightarrow a_{n+1} = -(n+1) = 0$

for $n = 0$, $a_1 = -a_0$

for n=1 , $a_2 = -2a_1 = 2a_0$

for $n = 2$, $a_3 = -3a_2 = -6a_0$

for $n = 4$, $a_4 = -4a_3 = 24a_0$

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.

$a_n = (-1)^n n!\, a_0.$ where n = 0, 1, 2, 3, . . .

$\Rightarrow y(x) = a_0 \sum_{n=0}^{\infty} (-1)^n n!\, x^n$ is the solution.

But, the radius of convergence of this series is 0.The solution is the trivial solution $y(x) = 0$. Thus we can conclude that Frobenius series method is not convenient as approximation solution of ordinary differential equation near an irregular singular point. This pushes us to develop a convenient method near an irregular singular point.

3.1 Introduction to Asymptotics

The asymptotic methods we are about to introduce are best understood if we master the mechanical aspects which are explained in this section and which are no more difficult than those used to obtain Taylor or Frobenius series. However, since the formal techniques are more difficult to justify mathematically, we have used asymptotic relation and asymptotic series to solve this problem.

We must introduce two new symbols which express the relative behavior of two functions. The notation

$$f(x) \ll g(x)\,, x \to x_0$$

which is read as " $f(x)$ is much smaller than $g(x)$ as x tends to x_0 ," means

$$\lim_{x \to x_0} \frac{f(x)}{\boldsymbol{g(x)}} = \mathbf{0}$$

Second, the notation

$$f(x) \sim g(x)\,, \quad x \to x_0,$$

Which is read " $f(x)$ is asymptotic to $g(x)$ as x tends to x_0 ," means that the relative error between f and g goes to zero as $x \to x_0$:

$$f(x) - g(x) \ll g(x), \qquad x \to x_0,$$

Or, equivalently,

$$\lim_{x\to x_0} \frac{f(x)}{\boldsymbol{g(x)}} = \mathbf{1}$$

Note that if $f(x) \sim g(x)$, $x \to x_0$ then $g(x) \sim f(x)$, $x \to x_0$

Example 1:

a. $x \ll \frac{1}{x}(x \to 0^+)$ because $\lim_{x\to 0} x^2=0$

b. $\sqrt{x}\sim 2(x \to 4)$ because $\lim_{x\to 4} \frac{\sqrt{x}}{2} = 1$

3.2 Behavior Near Irregular Singular Points

Let us consider the behaviors that we were unable to find using a Frobenius series. Then the differential equation $x^3y" = y$ and $x^2y" + (1 + 3x)y' + y = 0$ have two linear independent solutions. Yet, only one solution out of those four could be expressed as a series in a powers of x, and this one was the divergent series $y(x) = a_0 \sum_{n=0}^{\infty}(-1)^n n!\, x^n$. we will shortly formulate a procedure for discovering the local behavior of the other solutions near $x = 0$, but since it is always easier to drive a result that is already known, let us peek at the answers. One solution to $x^3y" = y$ exhibits the behavior

$$y(x)\sim c_1 x^{\frac{3}{4}} e^{2x^{-\frac{1}{2}}}, \qquad x \to 0+ \qquad (20a)$$

The other solution has the behavior

$$y(x)\sim c_2 x^{\frac{3}{4}} e^{-2x^{-\frac{1}{2}}}, \qquad x \to 0+ \qquad (20b)$$

Also, the missing solution to $x^2y" + (1 + 3x)y' + y = 0$ exhibits the behavior

$$y(x)\sim c_2 x^{-1} e^{x^{-1}}, \qquad x \to 0+ \qquad (21)$$

Observe that these behaviors all involve exponentials of functions which become singular at the irregular singularity of the differential equation. Thus, these three functions have essential singularities at $x = 0$.

The asymptotic behaviors (20) and (21) are actually the first terms of infinite series representations of the local behavior of the solutions. We will refer to the first term in a such a series as the leading behavior of the series. We also refer to the most rapidly changing component of the leading behavior in the limit $x \to x_0$ as the controlling factor.

For example, the controlling factor of the leading behavior given in a (20a) is $e^{2x-\frac{1}{2}}$, in (20b) it is $e^{-2x-\frac{1}{2}}$, and in (21) it is e^{x-1}. Also the leading behavior of the series $y(x) = a_0 \sum_{n=0}^{\infty}(-1)^n n!\, x^n$ is a_0. For the Frobenius series $\sum a_n x^{n+\alpha}$ the leading behavior is $a_0 x^{\alpha}$ and the controlling factor is x^{α}.

Definition 3.1 The power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is said to be asymptotic to function $y(x)$ as $x \to x_0$ and we write $y(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n \ (x \to x_0)$

If $y(x) - \sum_{n=0}^{N} a_n (x - x_0)^n \ll (x - x_0)^N \quad (x \to x_0)$ for every N.

Example 1: In the Frobenius series $\sum_{n=0}^{\infty} a_n x^{n+\alpha}$ the first term is $a_0 x^{\alpha}$.

This $a_0 x^{\alpha}$ is the leading behavior and x^{α} is the controlling factor.

Let $e^{s(x)}$ be a controlling factor in the solution of $y'' + p(x)y' + q(x)y = 0$

Substituting $y(x) = e^{s(x)}$ gives

$y'(x) = s'(x).e^{s(x)}$ and $y''(x) = s''(x).e^{s(x)} + s'(x)^2 e^{s(x)}$

$$\Rightarrow s''(x).e^{s(x)} + s'^{(x)^2 e^{s(x)}} + p(x).s'^{(x)}.e^{s(x)} + q(x)e^{s(x)} = 0$$

$$\Rightarrow e^{s(x)}[s''(x) + \left(s'(x)\right)^2 + p(x).s'(x) + q(x)] = 0$$

$$\Rightarrow s''(x) + \left(s'(x)\right)^2 + p(x).s'^{(x)} + q(x) = 0 \text{ since } e^{s(x)} \neq 0.$$

For an irregular singular point x_o it is usually true that $s'' \ll (s')^2, \quad x \to x_0$.

This gives us an asymptotic differential equation

$\left(s'(x)\right)^2 + p(x).s'(x) + q(x) = 0$

$\Rightarrow \left(s'(x)\right)^2 \sim -p(x)s' - q(x), x \to x_0$

Example 2: Determine the controlling factors in the solution of $x^3y'' = y$.

Solution: $x^3y'' = y$

$\Rightarrow x^3y'' - y = 0$

$\Rightarrow y'' + 0.y' - \frac{1}{x^3}y = 0$

$\Rightarrow p(x) = 0 \; and \; q(x) = -\frac{1}{x^3}$.

But $x^2q(x) = -\frac{1}{x}$ is not analytic at $x = 0$.

$\Rightarrow x = 0$ is an irregular singular point

Now $\left(s'(x)\right)^2 \sim -p(x).s'(x) - q(x)(x \to 0^+)$

$\Rightarrow \left(s'(x)\right)^2 \sim -\left(-\frac{1}{x^3}\right) = x^{-3}$

$\Rightarrow s'(x) \sim \pm x^{-\frac{2}{3}}$

$\Rightarrow s(x) \sim \pm 2x^{-\frac{1}{2}} = \pm \int_0^x x^{-\frac{3}{2}} dx$

$\Rightarrow e^{2x^{-\frac{1}{2}}} \; and \; e^{-2x^{-\frac{1}{2}}}$ are controlling factors in each independent solution of $x^3y'' = y$.

Example 3: Determine the leading behavior of $x^3y'' = y$ near the irregular singular point $x = 0$.

Solution: From example3 we have the controlling factor is

$s(x) = \int x^{-\frac{3}{2}} dx = 2x^{-\frac{1}{2}} + c(x)$ where $(x) \ll 2x^{-\frac{1}{2}}$, $x \to 0^+$

$\Rightarrow s'(x) = -x^{-\frac{3}{2}} + c'(x)$ and $s"(x) = \frac{3}{2}x^{-\frac{5}{2}} + c"(x)$

Substituting these values in $s"(x)+\left(s'(x)\right)^2 + p(x).\, s'^{(x)} + q(x) = 0$

We get $\frac{3}{2}x^{-\frac{5}{2}} + c"(x) + (-x^{-\frac{3}{2}} + c'(x))^2 + 0.\, s'(x) + (-x)^{-3} = 0$

$\Rightarrow \frac{3}{2}x^{-\frac{5}{2}} + c"(x) + x^{-3} - 2x^{\frac{3}{2}}c'(x) + (c'(x))^2 - x^{-3} = 0$

$\Rightarrow \frac{3}{2}x^{-\frac{5}{2}} + c"(x) - 2x^{\frac{3}{2}}c'(x) + (c'(x))^2 = 0$….(*****)

but, $c(x) \ll 2x^{-\frac{1}{2}} \Rightarrow c'(x) \ll x^{-\frac{3}{2}}$ and $c"(x) \ll x^{-\frac{5}{2}} \quad (x \to 0^+)$

$\Rightarrow \left(c'(x)\right)^2 \ll x^{-3} \qquad (x \to 0^+)$

Considering these facts in (*****) we have $\frac{3}{2}x^{-\frac{5}{2}} - 2x^{\frac{3}{2}}c'^{(x)} = 0$

$\Rightarrow \frac{3}{2}x^{-\frac{5}{2}} \sim 2x^{\frac{3}{2}}c'(x)\ , \quad x \to 0^+$

$\Rightarrow c'(x) \sim \frac{3}{4x}\ , \ x \to 0^+$

$\Rightarrow c(x) \sim \frac{3}{4}\ln x + D(x)$

$\Rightarrow s(x) = 2x^{-\frac{1}{2}} + \frac{3}{4}\ln x + D(x)$

Again assuming $D(x) \ll \ln x\ , \ x \to 0^+$

We have $s'(x) = -x^{-\frac{3}{2}} + \frac{3}{4x} + D'(x)$ and

$s"(x) = \frac{3}{2}x^{-\frac{5}{2}} - \frac{3}{4x^2} + D"(x)\ , \ x \to 0^+$

but, $s"(x) + \left(s'(x)\right)^2 + p(x).\, s'^{(x)} + q(x) = 0$ where $p(x)$=0 and $q(x) = x^{-3}$

$\Rightarrow -\frac{3x^{-2}}{16} + D"(x) + \left(D'(x)\right)^2 - 2x^{-\frac{3}{2}}\, D'^{(x)} + 3x^{-1}\, \frac{D'(x)}{2} = 0$

But, $x^{-1} \ll x^{-\frac{3}{2}}$, $(x \to 0^+)$

$$3x^{-1}\frac{D'}{2} \ll 2x^{-\frac{3}{2}}D', \quad x \to 0^+$$

$\mathrm{D}(x) \ll \ln x \Rightarrow \mathrm{D}'(x) \ll \ln x \Rightarrow \mathrm{D}'(x) \ll \frac{1}{x}, x \to 0^+$

$\Rightarrow \left(D'(x)\right)^2 \ll \frac{D'}{x}$, $x \to 0^+$

$\Rightarrow \mathrm{D}''(x) \ll \frac{1}{x^2}$, $x \to 0^+$

Now $-\frac{3x^{-2}}{10} - 2x^{-\frac{3}{2}}D'^{(x)} = 0$

$\Rightarrow -2x^{-\frac{3}{2}}D'^{(x)} \sim \frac{3x^{-2}}{10} \qquad (x \to 0^+)$

$\Rightarrow \mathrm{D}'(x) \sim -\frac{3x^{-\frac{1}{2}}}{32} \qquad (x \to 0^+)$

Integrating this result gives

$\Rightarrow \mathrm{D}(x) \sim -\frac{3x^{\frac{1}{2}}}{16} + \mathrm{d} \qquad (x \to 0^+)$ where ***d*** is a constant

$\Rightarrow \mathrm{D}(x) - \mathrm{d} \sim -\frac{3x^{\frac{1}{2}}}{16}$, $\quad x \to 0^+$

$\Rightarrow \mathrm{D}(x) = \mathrm{d} + \delta(x)$ where, $\delta(x) \sim -\frac{3x^{\frac{1}{2}}}{16}$ as $x \to 0^+$

$\Rightarrow y(x) \sim e^{s(x)} = e^{\left(2x^{-\frac{1}{2}} + \frac{3}{4}\ln x + d\right)} \quad (x \to 0^+)$

$\Rightarrow y(x) \sim e^{2x^{-\frac{1}{2}}}.\, x^{\frac{3}{4}}.\, e^{d}$

$\Rightarrow y(x) \sim c_1 x^{\frac{3}{4}} e^{2x^{\frac{1}{2}}} \qquad (x \to 0^+)$ is the leading behavior

$\Rightarrow y(x) = c_1 x^{\frac{3}{4}} e^{2x^{-\frac{1}{2}}}\left(1 + \varepsilon(x)\right)$ where $\varepsilon(x)$ is a correction function.

3.3 Method of Dominance Balance

The method of dominant balance is used to identify those terms in an equation that may be neglected in an asymptotic limit. The technique of dominant balance consists of three steps.

1. We drop all terms that appear small and replace the exact equation by an asymptotic relation.
2. We replace the asymptotic relation with an equation by exchanging the $\sim$ sign for $a_n \equiv$ sign solve the resulting equation exactly.
3. We check that the solution we have obtained is consistent with the approximation made in **step 1.** If it is consistent we must still show that the equation for the function obtained by factoring off the dominant balance solution from the exact solution itself has a solution that varies less rapidly than the dominant balance solution. When this happens, we conclude that the controlling factor obtained from the dominant balance relation is the same as that of the exact solution.

   The dominant balance argument that we have just outlined may appear circular , and indeed it is! Nevertheless, it is the most general and powerful procedure available for finding approximate solution to equation.

Example 1: Determine the local behavior of $x^2y'' + (1+3x)y' + y = 0$ near the irregular singular point $x = 0$.

Solution: Substitute $y = e^s$ in the equation gives

$$x^2s'' + x^2(s')^2 + (1+3x)s' + 1 = 0 \text{ since } y' = s'e^s$$

$$y'' = s''e^5 + (s')^2e^5$$

Now $x^2s'' \ll x^2(s')^2 \; and \; 3xs' \ll s'$ as $x \to 0^+$

The asymptotic differential equation is

$$x^2s'' + s' + 1 \sim 0 \qquad (x \to 0^+)$$

$$\Rightarrow\ s' \sim \frac{-1 \pm \sqrt{1-4x^2}}{2x^2} \qquad (x \to 0^+)$$

but, x is small we have

$$s' \sim \frac{-1 + \sqrt{1-4x^2}}{2x^2} \sim 0$$

$$\text{and}\quad s' \sim \frac{-1-\sqrt{1-4x^2}}{2x^2} \sim -\frac{2}{2x^2} = -x^{-2}$$

Integrating $s' \sim 0$ gives $s \sim d \quad (x \to 0^+)$ is a constant. This shows the leading behavior $c_1 = e^d$

Integrating $s' \sim -x^{-2}$ gives $s \sim x^{-1}\ (x \to 0^+)$. This gives the controlling factor is $e^{x^{-1}}$ in the leading behavior. To find the full leading behavior substitute

$s(x) = x^{-1} + c(x)\ \ where\ \ c(x) \ll x^{-1}\ \ as\ \ \ x \to 0^+$

$$x^2c'' + x^2(c')^2 - (1-3x)c' - x^{-x} + 1 = 0$$

The asymptotic differential equation become

$$c'^{(x)} \sim -x^{-x} \qquad (x \to 0^+)$$

$$\Rightarrow c(x) \sim -\ln x \qquad (x \to 0^+)$$

Now the full behavior of this solution is obtained by substituting

$y(x) = c_2x^{-1}e^{x^{-1}}\,w(x)$ were $w(x)$in a series of power of $x^{\alpha}(\alpha > 0)$.

3.4 Local Analysis of Non-homogeneous Linear Ordinary Differential Equation

To determine the general solution of a non-homogeneous local ordinary differential equation we can use the direct approach of applying the method of dominant balance to the differential equation. Here we should consider several cases in the homogenous part and in the non-homogeneous.

Example 1: Determine the local behavior of the general solution to

$$y' + xy = x^3 \quad at \quad x = 0.$$

Solution: Since $x = 0$ is an ordinary point of $y' + xy = 0$ and x^3 is analytic at $x = 0$, now we may assume the a Taylor series represents for $y(x)$:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow y'(x) = \textstyle\sum_{n=0}^{\infty} a_{n.n} x^{n-1}$$

By substituting $y(x)\ and\ y'(x)$ in the equation $y' + xy = x^3$, it gives

$\sum_{n=0}^{\infty} a_{n.n} x^{n-1} + x.\sum_{n=0}^{\infty} a_n x^n = x^3$.

Equating coefficients of the same powers of x gives a recursion relation for a_n

$$a_1 = 0$$

$$na_n + a_{n-2} = \begin{cases} 0, & n \geq 2, n \neq 4 \\ 1, & n = 4 \end{cases}$$

The recursion relation determines $a_n (n \geq 1)$ in terms of $a_0 \neq 0$ the first few terms are

$$a_0 \neq 0$$

$$a_1 = 0$$

$$a_2 = \frac{1}{2}a_0$$

$$a_3 = 0$$

$$a_4 = \frac{2 - a_0}{8}$$

$$a_5 = 0$$

$\Rightarrow y(x) = a_0 + \frac{1}{2}a_0 x^2 + \frac{2-a_0}{8}.x^4 + \cdots$

Example 2: Determine the behavior of the solutions to the inhomogenous Airy equation $y" = xy - 1$ at $x = \infty$ which satisfy $y(+\infty) = 0$

Solution: $x = \infty$ is an irregular singular point.

We have $y"\sim xy$and $1 \ll x.y \quad (x \to \infty)$. This balance gives the solution

$y(x) = c_1 \sum_{x=0}^{\infty} \frac{x^{3n}}{9^n n!\Gamma\left(n+\frac{2}{3}\right)} + c_2 \sum_{n=0}^{\infty} \frac{x^{3n+1}}{9^n n!\Gamma\left(n+\frac{4}{3}\right)}$. This is shown in example2 of chapter 2. From this we get the two linearly independent special solutions of the homogenous Airy equation.

$$Ai(x) = 3^{-\frac{2}{3}} \sum_{x=0}^{\infty} \frac{x^{3n}}{9^n n!\Gamma\left(n+\frac{2}{3}\right)} - 3^{-\frac{4}{3}} \sum_{n=0}^{\infty} \frac{x^{3n+1}}{9^n n!\Gamma\left(n+\frac{4}{3}\right)} \text{ and}$$

$$Bi(x) = 3^{-\frac{1}{6}} \sum_{x=0}^{\infty} \frac{x^{3n}}{9^n n!\Gamma\left(n+\frac{2}{3}\right)} + 3^{-\frac{5}{6}} \sum_{n=0}^{\infty} \frac{x^{3n+1}}{9^n n!\Gamma\left(n+\frac{4}{3}\right)}$$

But using $Ai(x)$ we get $xy = xAi(x)$ which violates $1 \ll x.y \quad (x \to \infty)$ and using $Bi(x)$ violates $y(+\infty) = 0$. Thus this balance is in consistent.

The 2^{nd} balance is $y"\sim -1$ and $xy \ll 1 \quad (x \to \infty)$

The solution of this balance is $y = -\frac{x^2}{2} + c_1 x + c_2$ is also in consistent because it violates $y(+\infty) = 0$.

The 3^{rd} balance is $xy \sim 1$ and $y'' \ll 1 \quad (x \to \infty)$.Then

$y \sim \frac{1}{x}, \quad x \to +\infty$, which is consistent since $y(+\infty) = 0$

and$y'' \sim \frac{2}{x^3} \ll 1 \quad (x \to \infty)$.

Let $y(x) = \frac{1}{x} + c(x), \qquad c(x) \ll \frac{1}{x} \qquad (x \to \infty)$

$$y = \frac{2}{x^3} + c'', \quad x.c \ll 1 \quad (x \to \infty)$$

$$\Rightarrow xc = \frac{2}{x^3} + c''.$$

From this we get balance $c(x) \sim \frac{2}{x^4} (x \to \infty)$ continuing in this manner we find the asymptomatic series expansion of $y(x)$. Thus, we have

$$y(x) \sim \frac{1}{x} + \frac{2}{x^4} + \cdots \frac{(3n)!}{3^n n!\, x^{3n+1}} + \cdots, \qquad (x \to \infty).$$

Example 3: (Three term dominant balance)

Determine the local analysis of $y' - \frac{y}{x} = \frac{\cos x}{x^2}$ near $x = 0$.

Solution: The homogenous part is $y' - \frac{y}{x} = 0$ and $p_0(x) = -\frac{1}{x}$ and $n = 1$

$\lim_{x \to 0} p_0(x) = -\frac{1}{x}$ which implies that it doesn't exist.

$\lim_{x \to 0} p_0(x) = -1$. This shows that $x = 0$ is a regular singular point.

Balance 1: If $x^{-2} \cos x \ll x^{-1} y \qquad (x \to \infty)$

We have $y' - \frac{y}{x} - \frac{\cos x}{x^2} = 0$

$$\Rightarrow y' - \frac{y}{x} \sim 0$$

$$\Rightarrow y' \sim \frac{y}{x}$$

$$\Rightarrow \frac{dy}{y} = \frac{dx}{x}$$

$$\Rightarrow \ln y \sim \ln x + c$$

$$\Rightarrow y \sim c.x$$

But this result violates $x^{-2} \cos x \ll x^{-1} y \quad (x \to 0)$

Balance 2: If x^{-1} y $\ll x^{-2} cosx \quad (x \to 0)$

But, $y' - \frac{y}{x} - \frac{cos\, x}{x^2} = 0$

$\Rightarrow y' - \frac{cos\, x}{x^2} \sim 0butcos\, x \sim 1 \quad (x \to 0)$

$\Rightarrow y' \sim \frac{1}{x^2}$

$\Rightarrow dy \sim \frac{1}{x^2} dx$

$\Rightarrow y \sim -\frac{1}{x} (x \to 0)$

This is in consistent with the assumption $x^{-1} y \ll x^{-2} \cos x \quad (x \to 0)$

Balance 3: If $y' \ll x^{-2} cos\, x \quad (x \to 0)$

But , $y' - \frac{y}{x} - \frac{cos\, x}{x^2} = 0$

$\Rightarrow -\frac{y}{x} - \frac{cos\, x}{x^2} \sim 0$

$\Rightarrow \frac{y}{x} \sim \frac{cos\, x}{x^2}$

$\Rightarrow y \sim \frac{cos\, x}{x^2} \quad (x \to 0)$ which doesn't agree with $y' \ll x^{-2} cos\, x \quad (x \to 0)$

Thus all two terms balances are in consistent, we should seek a three term balance in the asymptotic differential equation $y' - \frac{y}{x} \sim \frac{1}{x^2}, \quad (x \to 0)$

This has a solution of the form $y \sim \frac{c}{x} (x \to 0)$

Bu t $y' - \frac{y}{x} \sim \frac{1}{x^2}$

$\Rightarrow x^2 y' - xy \sim 1$

$\Rightarrow x^2\left(-\frac{c}{x^2}\right) - x\frac{c}{x} \sim 1$

$\Rightarrow -c - c \sim 1$

$\Rightarrow -2c \sim 1$

$\Rightarrow c \sim -\frac{1}{2}$

Thus the leading behavior of $y(x)$ is $y(x) \sim \frac{1}{2x}\,(x \to 0)$

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