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Office of Graduate Program

Faculty of Computer and Mathematical Science

Department of Mathematics

Seminar Report

On

Fibonacci numbers, Schur polynomials and The Rogers-Ramanujan Identities

A project Submitted to the Department of Mathematics in Partial fulfillment of the requirements for Master's degree of Science in Mathematics

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June, 2011

Addis Ababa, Ethiopia

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I.Project summary

The Rogers-Ramanujan identities are the most celebrated results in partition theory. These identities are infact q-analogues of the well-known Fibonacci numbers.

The two Rogers-Ramanujan identities state that:

First Rogers-Ramanujan identity;The number of partitions of n in which any two summands differ by at least 2 is equal to the number of partitions of n into parts congruent to 1 or 4 modulo 5. Equivalently,

$$1+\sum_{j=1}^{\infty}\frac{q^{j^2}}{(1-q)(1-q^2)\dots(1-q^j)}=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$$

Second Rogers-Ramanujanidentity;The number of partitions of n in which any two summands differing by at least 2 and all summands are greater than 1 is equal to the number of partitions of n into parts congruent to 2 or 3 modulo 5.Equivalently,

$$1+\sum_{j=1}^{\infty}\frac{q^{j^2+j}}{(1-q)(1-q^2)\dots(1-q^j)}=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+2})(1-q^{5n+3})}$$

In this project we show that shur polynomials are q-analogsof Fibonacci numbers which lead to the Rogers –Ramanujan identities. The project also contains proofs of the Rogers –Ramanujan identities using some concepts of partitions of positive integers.

Finally, the generating functions and operator methods are used to prove Fibonacci number identities and to study Schur Polynomials.

II. Acknowledgements

I would like to express deepest thanks to my advisor ***Professor Melkamu Zeleke*** for his continuous, constructive advices and follow up from start to the end, all of his actions makes me strong in my work.

Next, thanks to ***Arega Habte*** for his best material support and encouragement.

Finally, I would like to express thanks to my ***father*** and ***mother*** for their help through my life next to ***Almighty GOD.***

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June, 2011

Section One: Preliminaries

1.1 Generating functions

Definition: Let $a_{0,}a_{1,}a_{2,\dots,}a_{n,\dots}$ be a sequence of real numbers. The function

$$G(x) = a_0 + a_1x + a_2x^2 + \cdots = \textstyle\sum_{i=0}^{\infty} a_ix^i \tag{1.1}$$

is the generating function for the given sequence and we write

$$a_n = [x^n]G(x). \tag{1.2}$$

This function is known as ordinary generating function (OGF) and plays an important part in this work.

The generating function $G(x)$ differs from a polynomial in x in that it can have infinitely many terms. We regard x as a formal symbol, and do not think of it as standing for some unknown quantity. Thus the generating function $G(x)$ is just a way to represent the sequence $a_{0,}a_1, a_2 \dots$

Example 1: Here are some sequences and their generating functions.

$$\begin{aligned}
&a)\ (1,1,1,1,\dots) &&\leftrightarrow 1 + x + x^2 + x^3 + \cdots = \textstyle\sum_{n=0}^{\infty} x^n\\
&b)\ (1,2,3,\dots\ ) &&\leftrightarrow 1 + 2x + 3x^2 + \cdots\\
&c)\ (0,1,2,3,\dots) &&\leftrightarrow x + 2x^2 + 3x^3 + \cdots\\
&d)\ (1,1,2,5,14,\dots) &&\leftrightarrow 1 + x + 2x^2 + 5x^3 + 14\,x^4 + \cdots
\end{aligned}$$

Consider for instance the product of

$$F(x) = 1 + x + x^2 + x^3 + \cdots \textit{ With } G(x) = 1 - x.$$

The constant term is just $a_0b_0 = 1.1 = 1$. If $n > 1$ then the coefficient of x^n is $a_nb_0 + a_{n-1}b_1 = 1 - 1 = 0$ $(since\ b_i = 0\ for\ i > 1$, so we have only two nonzero terms).

Hence

$$(1 + x + x^2 + x^3 + \cdots)(1 - x) = 1.$$

For this reason the sum (an infinite geometric series) in example 1 (a) is given by

$$1 + x + x^2 + x^3 + \cdots = \textstyle\sum_{n=0}^{\infty} x^{\text{n}} = \frac{1}{1-x} \tag{1.3}$$

Example 2: a) $(1, -1, 1, -1, \dots) \leftrightarrow 1 - x + x^2 - x^3 + x^4 - \cdots = \frac{1}{1+x}$

b) $(1, b, b^2, b^3, \dots) \leftrightarrow 1 + bx + b^2x^2 + b^3x^3 + \cdots = \frac{1}{1-bx}$

c) $(1,2,3, \dots) \leftrightarrow 1 + 2x + 3x^2 + \cdots \qquad = \frac{1}{(1-x)^2}$

d) $(0,1,1,2,3,5, \dots) \leftrightarrow x + x^2 + 2x^3 + 3x^4 \dots = \frac{x}{1-x-x^2}$

Example 3: for any $n \in z^+$,

$(1+x)^n = \binom{n}{o} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$, so $(1+x)^n$ is the generating function for the sequence $\left(\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n}\right)$

1.2 Recurrence Relations

Suppose $a_{0,}a_1, a_{2,\dots}a_n, \dots$is an infinite sequence. A recurrence relation is a set of equations

$$a_n = c_n\left(a_{n-1,}a_{n-2,\dots}a_{n-k}\right) \tag{1.4}$$

The whole sequence is determined by (1.4) and the values of $a_{0,}a_{1,\dots}a_{k-1}$.

Example:

$$a_n - 6a_{n-1} + 9a_{n-2} = 0, n \geq 2,$$
$$a_0 = 1 \ a_1 = 9$$

Applying The Snake-Oil method, we have

$$\sum_{n=2}^{\infty}(a_n - 6a_{n-1} + 9a_{n-2})x^n = 0$$

$$\sum_{n=2}^{\infty} a_n x^n = a(x) - a_0 - a_1 x$$
$$= a(x) - 1 - 9x$$

$$\sum_{n=2}^{\infty} 6a_{n-1}\, x^n = 6x \sum_{n=2}^{\infty} a_{n-1} x^{n-1}$$
$$= 6x(a(x) - a_0)$$
$$= 6x(a(x) - 1)$$

$$\sum_{n=2}^{\infty} 9a_{n-2} x^n = 9x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2}$$

$$= 9x^2a(x)$$

$$a(x) - 1 - 9x - 6x(a(x) - 1) + 9x^2a(x) = 0$$

$$a(x) = \frac{1+3x}{1-6x+9x^2} = \frac{1+3x}{(1-3x)^2}$$

$$= \frac{1}{(1-3x)^2} + \frac{3x}{(1-3x)^2}$$

$$= \sum_{n=0}^{\infty}(n+1)3^nx^n + 3x\sum_{n=0}^{\infty}(n+1)3^nx^n$$

$$= \sum_{n=0}^{\infty}(n+1)\,3^nx^n + \sum_{n=0}^{\infty} n3^n$$

$$= \sum_{n=0}^{\infty}(2n+1)3^nx^n$$

Therefore

$$a_n = (2n+1)3^n$$

1.3. Partitions of Positive integers

A fundamental concept in Combinatorics is that of a partition. In general, a partition of an object is a way of breaking it up into smaller objects. We will be concerned here with partitions of positive integers (positive whole numbers). A partition of a positive integer n is a way of writing n as a sum of positive integers, ignoring the order of the summands.

For instance, $3+4+2+1+1+4$ represents a partition of 15, and $4+3+2+1+4+1$ represents the same partition. A partition is allowed to have only one part (summand), so that 5 is a partition of 5. There are in fact seven partitions of 5, given below

$5,\ 4+1,\ \ 3+2,\ \ 3+1+1,\ \ 2+2+1$
$2+1+1+1,\ \ 1+1+1+1+1.$

We denote the number of partitions of n by $\mathrm{P}(n)$, so for instance $\mathrm{P}(5) = 7$. By convention we set $\mathrm{P}(0) = 1$.

The problem of evaluating $\mathrm{P}(n)$ has a long history. There is no simple formula in general for $\mathrm{P}(n)$, however generating function for $\mathrm{P}(n)$ avoid this problem.

Let $\mathrm{P}(x)$ be the generating function of $\mathrm{P}(n)$. since $\mathrm{P}(0) = 1, \mathrm{P}(1) = 1, \mathrm{P}(2) = 2$, $\mathrm{P}(3) = 3, \mathrm{P}(4) = 5$, we have

$$\mathrm{P}(x) = x + 2x^2 + 3x^3 + 5x^4 + \cdots$$

Using generating function modeling closed form of $\mathrm{P}(x)$ is

$$\mathrm{P}(x) = \prod_{n=1}^{\infty} \frac{1}{1-x^n}$$

A specific partition of an integer is represented by Ferrer's diagram.
Example:

is the Ferrer's diagram representation of $6 = 3 + 2 + 1$.

- ✓ A partition of an integer n is **self-conjugate** if the Ferrer's diagram of the partition is n equal to its own transpose. Example 3+2+1.
- ✓ **Durfee square** of size k ($k \times k$ array of dots) is the largest square of dots in the upper left-hand corner of Ferrers diagram.
  In the above example the size of Durfee square is 2×2.

Facts about partitions of integers

- ➢ The number of partitions of n into odd parts is equal to the number of partition of n into distinct parts.
- ➢ The number of partition of n into at most k parts is equal to the number of partition of n in which the largest part is at most k.
- ➢ The number of partition of n into distinct odd parts is equal to the number of self-conjugate partitions.
- ➢ Jacobi's Triple Product identity obtained using decomposition of the number of all partitions of an integer is given by

$$\sum_{n=-\infty}^{\infty} q^{n^2} z^n = \prod_{n=0}^{\infty} (1 - q^{2n})(1 + zq^{2n+1})(1 + z^{-1}q^{2n+1}) \qquad (1.5)$$

1.4 q-binomial coefficients

q-binomial Coefficient is q-analog of the well- known binomial coefficients and it is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(1-q^n)(1-q^{n-1})(1-q^{n-2})\ldots(1-q^{n-k+1})}{(1-q)(1-q^2)(1-q^3)\ldots(1-q^k)} & \text{if } 0 \le k \le n \\ 0 & \text{otherwise} \end{cases} \qquad (1.6)$$

q-Binomial Coefficient is also known as **Gaussian Polynomial**.

Note that even though the Gaussian polynomial $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is defined as a rational function, it does, in fact reduce to a polynomial. To show this let us look additional concept, q-analog of *n*.

q*-analog of *n, also known as the ***q*-bracket** or ***q*-number** of n, to be

$$[n]_q = \frac{1-q^n}{1-q}$$

From this one can define the q-analog of the factorial, the ***q*-factorial**, as

$$\begin{aligned}[n]_q! &= [1]_q\,[2]_q \dots [n-1]_q \\ &= \frac{1-q}{1-q}\frac{1-q^2}{1-q}\dots\frac{1-q^{n-1}}{1-q}\frac{1-q^n}{1-q} \\ &= 1(1+q)\dots(1+q+\dots+q^{n-2})(1+q+\dots+q^{n-1})\end{aligned}$$

Again, one recovers the usual factorial by taking the limit as $q \to 1$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!} \tag{1.7}$$

So that (1.6) is a polynomial just as $\frac{n!}{(n-k)!k!}$ simplifies to an integer with $k \leq n$.

Example:

a) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_q = 1$

b) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}_q = \frac{1-q}{1-q} = 1$

c) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q = \frac{1-q^2}{1-q} = 1+q$

d) $\begin{bmatrix} 3 \\ 1 \end{bmatrix}_q = \frac{1-q^3}{1-q} = 1+q+q^2$

q- binomial coefficients have also the following property

- $\begin{bmatrix} n \\ 0 \end{bmatrix}_q = \begin{bmatrix} n \\ n \end{bmatrix}_q = 1$
- $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q$

- $\begin{bmatrix} n \\ 1 \end{bmatrix}_q = \begin{bmatrix} n \\ n-1 \end{bmatrix}_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{m-1}$
- $lim_{q \to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}$

Analog of Pascal's identities for q-binomial coefficient.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^r \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \qquad (1.8)$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \qquad (1.9)$$

For $q = 1$ the above two identities gives well known binomial Pascal identities.

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

and

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

q-binomial Series

The q-binomial series is the generalization of binomial series and is given by

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} x^k = (x;q)_n = \prod_{k=0}^{n-1} (1 + xq^k) \qquad (1.10)$$

and

$$\sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q x^k = \frac{1}{(x;q)_n} = \prod_{k=0}^{n-1} \frac{1}{(1-xq^k)} \qquad (1.11)$$

Where

$$(A;q)_n = (1-A)(1-Aq) \ldots (1-Aq^{n-1})$$

1.5 Congruence and basic properties

Definition: Given integers a, b, m with $m > 0$.we say that a is congruent to b modulo m, and we write

$$a \equiv b (mod\ m) \qquad (1.12)$$

if m divides the difference $a - b$. The number m is called the modulus of the congruence. In other words, the congruence (1.12) is equivalent to the divisibility relation $m|(a-b)$.
In particular, $a \equiv 0 (mod\ m)$ if and only if $m|a$ .Hence $a \equiv b(mod\ m)$ if and only if $a - b \equiv 0(mod\ m)$ if m does not divide $(a - b)$ we write a is not congruent to b mod m.

Example 1:

$$19 \equiv 7(\text{mod } 12),\ 1 \equiv -1(\text{mod}2),\ 9 \equiv -1(\text{mod}5)$$

Example 2:

a) n is even if and only if $n \equiv 0(mod\ 2)$
b) n is odd if and only if $n \equiv 1(mod\ 2)$
c) $a \equiv b(mod\ 1)$ for every a and b.
d) if $a \equiv b(mod\ m)$ then $a \equiv b(mod\ d)$ when $d|m,\ d > 0$

Congruence is an equivalence relation, which are we have:

a) $a \equiv a(mod\ m)$… (Reflexivity)
b) $a \equiv b(mod\ m)$ implies $b \equiv a(mod\ m)$ … (symmetry)
c) $a \equiv b(mod\ m)$ and $b \equiv c(mod\ m)$ imply $a \equiv c(mod\ m)$ … (transitivity)

1.6 Algebra of operators

Definition: Operators on function x is denoted by η defined as

$$\eta f(x) = f(xq) \tag{1.11}$$

Example: Consider

$$\frac{1}{1-x\eta} f(x) = \sum_{n=0}^{\infty} (x\eta)^n f(x)$$

$$= \sum_{n=0}^{\infty} (x\eta)(x\eta)\ \dots\ (x\eta) f(x)$$

$$= \sum_{n=0}^{\infty} x^n\, q^{0+1+\cdots(n-1)} f(xq^n)$$

$$= \sum_{n=0}^{\infty} x^n q^{n(n-1)/2} f(xq^n)$$

The inverse operator is $1-x\eta$, and we can easily convince ourselves of this as

$$(1 - x\eta) \sum_{n=0}^{\infty} x^n q^{n(n-1)/2} f(xq^n)$$

$$= \sum_{n=0}^{\infty} x^n\, q^{n(n-1)/2} f(xq^n) - \sum_{n=0}^{\infty} x(xq)^n\, q^{n(n-1)/2} f(xq^{n+1})$$

$$= \sum_{n=0}^{\infty} x^n\, q^{n(n-1)/2} f(xq^n) - \sum_{n=0}^{\infty} x^m\, q^{m(m-1)/2} f(xq^m)$$

(Where we have replaced $n - 1$ by m in the second sum)

$$= f(x).$$

Section Two

Fibonacci numbers and Schur Polynomials

2.1 INTRODUCTION

The Fibonacci sequence first appears in the book **Liber Abaci** (1202) by **Leonardo of Pisa**, known a **Fibonacci**. Fibonacci considers the growth of an idealized (biologically unrealistic) rabbit population, assuming that: a newly-born pair of rabbits, one male, one female, are put in a field; rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits; rabbits never die and a mating pair always produces one new pair (one male, one female) every month from the second month on. The puzzle that Fibonacci posed was: how many pairs will there be in one year?
The original problem that Fibonacci investigated was about how fast rabbits could breed in ideal circumstances.
To solve this problem, let us denote the number of pairs of rabbits at the end of the n^{th} month by F_n. Then,

1. At the end of the first month, they mate, but there is still only 1 pair. So that

   $F_0 = 1$ and $F_1 = 1$.

2. At the end of the second month the female produces a new pair, so now there are

   Pairs of rabbits in the field. So $F_2 = 2$

3. At the end of the third month, the original female produces a second pair, making 3

   Pairs in all in the field. That is,$F_3 = 3$.

4. At the end of the fourth month, the original female has produced yet another new

   Pair, the female born two months ago produces her first pair also, making 5

   pairs. $F_4 = 5$

Continuing this process, in general we can obtain information about the number F_n since at the end of the n^{th} month; the F_{n-1} pairs of rabbits that were alive at the end of the previous month will still be alive, since we assume that no rabbits die. This contributes F_{n-1} pairs of rabbits to the total number of pairs for the n^{th} month. But; there will also be some new born pairs. In fact, each of the F_{n-2} pairs of rabbits that were alive two months prior to the n^{th} month, being at least 2 months old themselves, will bear a new pair of rabbits. This contributes F_{n-2} additional pairs of rabbits to the total for the n^{th} month.

Hence we get:

$$\mathrm{F}_n = \mathrm{F}_{n-1} + \mathrm{F}_{n-2}, \mathrm{F}_0 = \mathrm{F}_1 = 1 \quad \text{for all values of } \mathrm{F}_n\ n \geq 2. \qquad (2.1)$$

Where F_0 and F_1 are the initial conditions.

Therefore the number of pairs of rabbits in the field at the start of each month is 1, 1, 2, 3, 5, 8, 13, 21, 34...

Fibonacci numbers are also defined recursively as

$$\mathrm{F}_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ \mathrm{F}_{n-1} + \mathrm{F}_{n-2} & \text{if } n > 1 \end{cases} \qquad (2.2)$$

Hence, from the recursive part of this definition, it follows that but the role played by F_n's are changed.

$$\begin{aligned} \mathrm{F}_2 &= \mathrm{F}_1 + \mathrm{F}_0 = 1 + 0 = 1 \\ \mathrm{F}_3 &= \mathrm{F}_2 + \mathrm{F}_1 = 1 + 1 = 2 \\ \mathrm{F}_4 &= \mathrm{F}_3 + \mathrm{F}_2 = 2 + 1 = 3 \\ \mathrm{F}_5 &= \mathrm{F}_4 + \mathrm{F}_3 = 3 + 2 = 5 \end{aligned}$$

We also find that $\mathrm{F}_6 = 8, \mathrm{F}_7 = 13, \mathrm{F}_8 = 21, \mathrm{F}_9 = 34, \mathrm{F}_{10} = 55, \mathrm{F}_{11} = 89, \mathrm{F}_{12} = 144.$

2.2 Generating function of Fibonacci numbers

Let $\mathrm{F}(x)$ be the generating functions of $(0,1,1,2,3,5,8,13,21, \dots)$, then

$$\mathrm{F}(x) = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7 + \cdots \qquad (2.3)$$

To find closed form of $\mathrm{F}(x)$ by applying Snake-Oil on recursive part of (2.2)

$$\sum_{\mathrm{n}=2}^{\infty} (\mathrm{F}_{\mathrm{n}} - \mathrm{F}_{\mathrm{n}-1} - \mathrm{F}_{\mathrm{n}-2}) x^n = 0$$

$$\Leftrightarrow \quad \sum_{\mathrm{n}=2}^{\infty} \mathrm{F}_{\mathrm{n}} x^n - \sum_{\mathrm{n}=2}^{\infty} \mathrm{F}_{\mathrm{n}-1} x^n - \sum_{\mathrm{n}=2}^{\infty} \mathrm{F}_{\mathrm{n}-2} x^n = 0$$

$$\Leftrightarrow \quad (\mathrm{F}(x) - \mathrm{F}_0 - \mathrm{F}_1 x) - \big(x(\mathrm{F}(x) - \mathrm{F}_0)\big) - x^2 \mathrm{F}(x) = 0$$

$$\Leftrightarrow \quad \mathrm{F}(x) - x - x\mathrm{F}(x) - x^2 \mathrm{F}(x) = 0$$

$$\Leftrightarrow \ \mathrm{F}(x) = \frac{x}{1-x-x^2} \tag{2.4}$$

The next subsection includes some Fibonacci number identities and their generating function proofs.

2.3. Fibonacci number Identities

The following are some of Fibonacci number identities.

i) $\mathrm{F}_0 + \mathrm{F}_1 + \ldots + \mathrm{F}_n = \mathrm{F}_{n+2} - 1$

ii) $\mathrm{F}_{n+m} = \mathrm{F}_{m-1}\mathrm{F}_n + \mathrm{F}_m\mathrm{F}_{n+1}$

iii) $\mathrm{F}_n = 2^{1-n}\sum_{0\le 2j+1\le n}\binom{n}{2j+1}5^j$

- **Generating function proof of Fibonacci number identities.**

To prove each identity we consider the generating functions for the sequences defined by each side. If these generating functions are identical the identities are proved.

Proof: (i) Start from its left-hand side taking the sum over n and multiplying by x^n.

$$\begin{aligned}\sum_{n=0}^{\infty}(\mathrm{F}_0 + \mathrm{F}_1 + \ldots + \mathrm{F}_n)x^n &= \mathrm{F}_0x^0 + (\mathrm{F}_0 + \mathrm{F}_1)x^1 + (\mathrm{F}_0 + \mathrm{F}_1 + \mathrm{F}_2)x^2 + \cdots \\ &\quad +(\mathrm{F}_0 + \mathrm{F}_1 + \ldots + \mathrm{F}_n)x^n + \cdots \\ &= (\mathrm{F}_1x + \mathrm{F}_1x^2 + \mathrm{F}_1x^3 + \cdots) + (\mathrm{F}_2x^2 + \mathrm{F}_2x^3 + \cdots) + \\ &\quad \ldots(\mathrm{F}_nx^n + \mathrm{F}_nx^{n+1} +)\ldots \quad \text{Since } \mathrm{F}_0 = 0 \\ &= \mathrm{F}_1(x + x^2 + x^3 + \cdots) + \mathrm{F}_2(x^2 + x^3 + \cdots) + \cdots + \\ &\quad \mathrm{F}_n(x^n + x^{n+1} + \cdots) + \cdots \\ &= \mathrm{F}_1\left(\frac{x}{1-x}\right) + \mathrm{F}_2\left(\frac{x^2}{1-x}\right) + \cdots + \mathrm{F}_n\left(\frac{x^n}{1-x}\right) + \cdots \\ &= \frac{1}{1-x}(\mathrm{F}_1x + \mathrm{F}_2x^2 + \cdots \mathrm{F}_nx^n + \cdots) \\ &= \frac{1}{1-x}\sum_{n=0}^{\infty}\mathrm{F}_nx^n\end{aligned}$$

$$= \frac{x}{(1-x)(1-x-x^2)}$$

$$= \frac{1+x}{1-x-x^2} - \frac{1}{1-x}$$

$$= x^{-2}\left(\frac{x}{1-x-x^2} - x\right) - \frac{1}{1-x}$$

$$= x^{-2}\sum_{n=2}^{\infty} \mathrm{F}_n x^n - \sum_{n=0}^{\infty} x^n$$

$$= \sum_{n=2}^{\infty} \mathrm{F}_n x^{n-2} - \sum_{n=0}^{\infty} x^n$$

$$= \sum_{n=0}^{\infty} (\mathrm{F}_{n+2} - 1)x^n$$

Hence by comparing the coefficients of x^n (i) is proved.

Proof :(ii) we apply the following treatment.

By fixing m, we have

$$(1-x-x^2)\sum_{n=0}^{\infty} F_{n+m}\,x^n = \sum_{n=0}^{\infty} F_{n+m}x^n - x\sum_{n=0}^{\infty} F_{n+m}x^n - x^2\sum_{n=0}^{\infty} F_{n+m}x^n$$

$$= F_m + F_{m+1}x + \sum_{n=2}^{\infty} F_{n+m}x^n - F_m x - \sum_{n=1}^{\infty} F_{n+m}\,x^{n+1} - \sum_{n=0}^{\infty} F_{n+m}x^{n+2}$$

$$= F_m + (F_{m+1} - F_m)x + \sum_{n=0}^{\infty} F_{n+m+2}x^{n+2} - \sum_{n=0}^{\infty} F_{n+m+1}\,x^{n+2} -$$

$$\sum_{n=0}^{\infty} F_{n+m}x^{n+2}$$

$$= F_m + xF_{m-1} + \sum_{n=0}^{\infty}(F_{n+m+2} - F_{n+m+1} - F_{n+m})x^{n+2}$$

(where n has been replaced by n + 2
in the first sum and n + 1 in the second)

$$= F_m + xF_{m-1}$$

Since the sum is equal to zero by definition of Fibonacci number.

Hence

$$\sum_{n=0}^{\infty} F_{n+m}x^n = \frac{F_m + xF_{m-1}}{(1-x-x^2)}$$

$$= F_m \sum_{n=0}^{\infty} F_{n+1}\, x^n + F_{m-1} \sum_{n=0}^{\infty} F_n x^n$$

$$= \sum_{n=0}^{\infty} F_m F_{n+1} x^n + \sum_{n=0}^{\infty} F_{m-1} F_n x^n$$

$$= \sum_{n=0}^{\infty} (F_m F_{n+1} + F_{m-1} F_n)\, x^n$$

Consequently, (ii) follows by comparing coefficients of x^n.

Proof: (iii)
Proceed by taking the sum over all n and multiplying by x^n of right-hand side of (iii)

$$\sum_{n=0}^{\infty} 2^{1-n} \sum_{0\le 2j+1\le n} \binom{n}{2j+1} 5^j x^n = \sum_{n,j\ge 0} 2^{1-n-2j-1} \binom{n+2j+1}{2j+1} 5^j x^{n+2j+1}$$

(Where n has been shifted to n+2j+1)

$$= \sum_{n,j\ge 0} \frac{1}{2^n}\frac{1}{2^{2j}} 5^j x^{2j+1} \binom{n+2j+1}{2j+1} x^n$$

$$= \sum_{j\ge 0} \frac{5^j x^{2j+1}}{4^j} \sum_{n\ge 0} \binom{n+2j+1}{n} \left(\frac{x}{2}\right)^n$$

$$= \sum_{j\ge 0} \frac{5^j x^{2j+1}}{4^j} \left(\frac{1}{\left(1-\frac{x}{2}\right)^{2j+2}}\right) \quad \text{by Book keeper's identity.}$$

(Where $2j$ is replaced by $2j+2$)

$$= \sum_{j\ge 0} \frac{5^j x^{2j+1}}{4^j} \frac{1}{\left(1-\frac{x}{2}\right)^{2j}} \frac{x}{\left(1-\frac{x}{2}\right)^{2}}$$

$$= \frac{x}{\left(1-\frac{x}{2}\right)^{2}} \sum_{j\ge 0} \left(\frac{5x^2}{4\left(1-\frac{x}{2}\right)^{2}}\right)^j$$

$$= \left(\frac{x}{\left(1-\frac{x}{2}\right)^{2}}\right)\left(\frac{1}{1-\frac{5x^2}{4\left(1-\frac{x}{2}\right)^{2}}}\right) \quad \text{by Geometric sum.}$$

$$= \frac{1}{\left(1-\frac{x}{2}\right)^{2}} \frac{x}{1-\frac{5x^2}{4\left(1-\frac{x}{2}\right)^{2}}}$$

$$= \frac{x}{\left(1-\frac{x}{2}\right)^{2} - \frac{5x^2}{4}}$$

$$= \frac{x}{1-x-x^2}$$

$$= \sum_{n=0}^{\infty} F_n x^n$$

Hence (iii) is proved

2.4 Schur Polynomials

A sequence of polynomials first considered by I. Schur is given by

$$S_n(q) = S_n = \begin{cases} 0 & if\ n = 0 \\ 1 & if\ n = 1 \\ S_{n-1} + q^{n-2} S_{n-2} & if\ n > 1 \end{cases} \tag{2.5}$$

Example:

$$s_2(q) = S_1 + q^0 S_0 = 1$$

$$S_3 = S_2 + qS_1 = 1 + q$$

$$S_4 = S_3 + q^2 S_2 = 1 + q + q^2$$

$$S_5 = 1 + q + q^2 + q^3 + q^4$$

$$\vdots \qquad \vdots$$

Of course, it is immediately obvious that for q = 1.

$$S_n(1) = F_n. \tag{2.6}$$

Observe from the above example that,

$S_5(q) = 1 + q + q^2 + q^3 + q^4$, for q = 1 we obtain

$S_5(1) = 5 = F_5$

Also the $(n+1)^{th}$ Schur's polynomials given by

$$S_{n+1}(q) = \sum_{0\le 2j\le n} q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix} = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2} \begin{bmatrix} n \\ \lfloor \frac{n-5j}{2} \rfloor \end{bmatrix} \tag{2.7}$$

Note that for real number x, the symbol $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Lemma 1:

$$\sum_{0\le 2j\le n} q^{j^2}\begin{bmatrix} n-j \\ j \end{bmatrix} = \sum_{j=-\infty}^{\infty}(-1)^j\, q^{j(5j+1)/2}\begin{bmatrix} n \\ \left\lfloor\frac{n-5j}{2}\right\rfloor \end{bmatrix} \qquad (2.8)$$

Remark: Both sides of (2.8) are polynomials because all but a finite number of terms in each sum are zero and each term is a polynomial.

Proof: First denote

$$E_n(o;q) = \sum_{0\le 2j\le n} q^{j^2}\begin{bmatrix} n-j \\ j \end{bmatrix}$$

and

$$C_n(0;q) = \sum_{j=-\infty}^{\infty}(-1)^j\, q^{j(5j+1)/2}\begin{bmatrix} n \\ \left\lfloor\frac{n-5j}{2}\right\rfloor \end{bmatrix}$$

To check equality of $E_n(o;q)$ and $C_n(0;q)$, we shall prove that

1) $E_0(o;q) = C_0(0;q) = 1$

2) $E_1(o;q) = C_1(0;q) = 1+q$

3) $E_n(o;q) = E_{n-1}(o;q) + q^n E_{n-2}(o;q)$

4) $C_n(0;q) = C_{n-1}(0;q) + q^n C_{n-2}(0;q) \quad for\ n \ge 2$

Proof:

1) $E_0(o;q) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1$

$$C_0(0;q) = \sum_{j=0}^{\infty}(-1)^j\, q^{j(5j+1)/2}\begin{bmatrix} 0 \\ \left\lfloor\frac{0-5j}{2}\right\rfloor \end{bmatrix} + \sum_{j=-1}^{-\infty}(-1)^j\, q^{j(5j+1)/2}\begin{bmatrix} 0 \\ \left\lfloor\frac{0-5j}{2}\right\rfloor \end{bmatrix} = 1$$

2) $E_1(o;q) = \sum_{0\le 2j\le 1} q^{j^2}\begin{bmatrix} 1-j \\ j \end{bmatrix} = 1+q$

$$C_1(0;q) = \sum_{j=-\infty}^{\infty}(-1)^j\, q^{j(5j+1)/2}\begin{bmatrix} 1 \\ \left\lfloor\frac{1-5j}{2}\right\rfloor \end{bmatrix} = 1+q$$

3) $E_n(o;q) = \sum_{j=0}^{\infty} q^{j^2}\left(\begin{bmatrix} (n-1)-j \\ j \end{bmatrix} + q^{n-2j}\begin{bmatrix} (n-1)-j \\ j-1 \end{bmatrix}\right)$

$$= \sum_{0\le 2j\le n-1} q^{j^2}\begin{bmatrix} (n-1)-j \\ j \end{bmatrix} + q^n \sum_{0\le 2j\le n-2} q^{j^2}\begin{bmatrix} (n-2)-j \\ j \end{bmatrix}$$

$$= E_{n-1}(o;q) + q^n E_{n-2}(o;q)$$

4) First note that

$$C_{2n}(o;q)=\sum_{j=-\infty}^{\infty} q^{j(10j+1)} \begin{bmatrix} 2n+1 \\ n-5j \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{(2j+1)(5j+3)} \begin{bmatrix} 2n+1 \\ n-2-5j \end{bmatrix} \qquad \text{(I)}$$

$$C_{2n+1}(o;q)=\sum_{j=-\infty}^{\infty} q^{j(10j+1)} \begin{bmatrix} 2n+2 \\ n+1-5j \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{(2j+1)(5j+3)} \begin{bmatrix} 2n+2 \\ n-2-5j \end{bmatrix} \qquad \text{(II)}$$

Now from (I)

$$\mathrm{C_{2n}(o;q)} = \sum_{j=-\infty}^{\infty} q^{j(10j+1)} \left(\begin{bmatrix} 2n \\ n-5j \end{bmatrix} + q^{n+1+5j} \begin{bmatrix} 2n \\ n-1-5j \end{bmatrix} \right) -$$

$$\sum_{j=-\infty}^{\infty} q^{(2j+1)(5j+3)} \left(\begin{bmatrix} 2n \\ n-3-5j \end{bmatrix} + q^{n-2-5j} \begin{bmatrix} 2n \\ n-2-5j \end{bmatrix} \right)$$

$$= C_{2n-1}(o;q) + q^{n+1} \left(\sum_{j=-\infty}^{\infty} q^{10j^2+6j} \begin{bmatrix} 2n \\ n-1-5j \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{10j^2+6j} \begin{bmatrix} 2n \\ n-2-5j \end{bmatrix} \right)$$

$$= C_{2n-1}(o;q) + q^{n+1} \begin{pmatrix} \sum_{j=-\infty}^{\infty} q^{10j^2+6j} \left(\begin{bmatrix} 2n-1 \\ n-2-5j \end{bmatrix} + q^{n-1-5j} \begin{bmatrix} 2n-1 \\ n-1-5j \end{bmatrix} \right) - \\ \sum_{j=-\infty}^{\infty} q^{10j^2+6j} \left(\begin{bmatrix} 2n-1 \\ n-2-5j \end{bmatrix} + q^{n+2+5j} \begin{bmatrix} 2n-1 \\ n-3-5j \end{bmatrix} \right) \end{pmatrix}$$

$$= C_{2n-1}(o;q) + q^{2n} C_{2n-2}(o;q),$$

since the first and third sums cancel each other.

On the other hand from (II)

$$C_{2n+1}(o;q)=\sum_{j=-\infty}^{\infty} q^{j(10j+1)} \left(\begin{bmatrix} 2n+1 \\ n-5j \end{bmatrix} + q^{n+1-5j} \begin{bmatrix} 2n+1 \\ n+1-5j \end{bmatrix} \right) -$$

$$\sum_{j=-\infty}^{\infty} q^{(2j+1)(5j+3)} \left(\begin{bmatrix} 2n+1 \\ n-2-5j \end{bmatrix} + q^{n+4+5j} \begin{bmatrix} 2n+1 \\ n-3-5j \end{bmatrix} \right)$$

$$= C_{2n}(o;q) + q^{n+1} \left(\sum_{j=-\infty}^{\infty} q^{10j^2-4j} \begin{bmatrix} 2n+1 \\ n+1-5j \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{10j^2+16j+6} \begin{bmatrix} 2n+1 \\ n-3-5j \end{bmatrix} \right)$$

$$= C_{2n}(o;q) + q^{n+1} \begin{pmatrix} \sum_{j=-\infty}^{\infty} q^{10j^2-4j} \left(\begin{bmatrix} 2n \\ n+1-5j \end{bmatrix} + q^{n+5j} \begin{bmatrix} 2n \\ n-5j \end{bmatrix} \right) - \\ \sum_{j=-\infty}^{\infty} q^{10j^2+16j+6} \left(\begin{bmatrix} 2n \\ n-4-5j \end{bmatrix} + q^{n-3-5j} \begin{bmatrix} 2n \\ n-3-5j \end{bmatrix} \right) \end{pmatrix}$$

$$= C_{2n}(o;q) + q^{2n+1}\, C_{2n-1}(o;q)$$

$$+ q^{n+1}\left(\sum_{j=-\infty}^{\infty} q^{10j^2-4j} \begin{bmatrix} 2n \\ n+1-5j \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{10j^2+16j+6} \begin{bmatrix} 2n \\ n-4-5j \end{bmatrix}\right)$$

Replacing j by j − 1 in the second sum, we see that it is identical with the first.

$$= C_{2n}(o;q) + q^{2n+1}\, C_{2n-1}(o;q)$$

Hence in general

$$C_n(o;q) = C_{n-1}(o;q) + q^n\, C_{n-2}(o;q)$$

Thus, if we allow $n \to \infty$ in Lemma 1 and from the fact that,

$$\begin{bmatrix} n-j \\ j \end{bmatrix} = \begin{bmatrix} n-j \\ n-2j \end{bmatrix}$$

We have

$$\sum_{j\geq 0} q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix} = \sum_{j\geq 0} q^{j^2} \left(\frac{(1-q^{n-j})(1-q^{n-j-1})\ldots(1-q^{j+1})}{(1-q)(1-q^2)\ldots(1-q^j)(1-q^{j+1})\ldots}\right)$$

$$= 1 + \sum_{j=1}^{\infty} \frac{q^{j^2}}{(1-q)(1-q^2)\ldots(1-q^j)}$$

And from right hand side of lemma 1it follows that

$$\begin{bmatrix} n \\ \left\lfloor \frac{n-5j}{2} \right\rfloor \end{bmatrix} = \frac{1}{(1-q)(1-q^2)\ldots}$$

Therefore

$$C_n(0;q) = \sum_{j=-\infty}^{\infty} (-1)^j\, q^{j(5j+1)/2} \begin{bmatrix} n \\ \left\lfloor \frac{n-5j}{2} \right\rfloor \end{bmatrix} = \frac{\sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2}}{\prod_{n=1}^{\infty}(1-q^n)}$$

Then

$$1 + \sum_{j=1}^{\infty} \frac{q^{j^2}}{(1-q)(1-q^2)\ldots(1-q^j)} = \frac{\sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2}}{\prod_{n=1}^{\infty}(1-q^n)}$$

$$= \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})} \qquad (2.9)$$

is the first of the celebrated Rogers-Ramanujan identities.

Schur also considered a sequence slightly different from (2.5):

$$T_n(q) = T_n = \begin{cases} 0 & if\, n = 0 \\ 1 & if\, n = 1 \\ T_{n-1} + q^{n-1}T_{n-2} & if\, n > 1 \end{cases} \tag{2.10}$$

Example:

$$T_2(q) = T_1 + q^0T_0 = 1$$

$$T_3 = T_2 + q^2T_1 = 1 + q^2$$

$$T_4 = T_3 + q^3T_2 = 1 + q^2 + q^3$$

$$T_5 = 1 + q^2 + q^3 + q^4 + q^6$$

$$\vdots \qquad \vdots$$

Of course, it is immediately obvious that for q=1

$$T_n(1) = \mathrm{F}_n \tag{2.11}$$

In the same way as (2.7) the $(n + 1)^{th}$ Shur polynomial is also given by

$$T_{n+1}(q) = \sum_{0\le 2j\le n} q^{j^2+j} \begin{bmatrix} n-j \\ j \end{bmatrix} = \sum_{j=-\infty}^{\infty}(-1)^j\, q^{j(5j-3)/2} \begin{bmatrix} n+j \\ \lfloor\frac{n+1-5j}{2}\rfloor + 1 \end{bmatrix} \tag{2.12}$$

As in (2.7), allowing $n \to \infty$ 1 in (1.9), we obtain the second Rogers-Ramanujan identity

$$1+\sum_{j=1}^{\infty} \frac{q^{j^2+j}}{(1-q)(1-q^2)\ldots(1-q^j)} = \frac{\sum_{j=-\infty}^{\infty}(-1)^j q^{j(5j-3)/2}}{\prod_{n=1}^{\infty}(1-q^n)}$$

$$= \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})} \tag{2.13}$$

The two identities (2.9) and (2.13) are studied and proved in the next section by using concept of partitions of positive integers.

From (2.6) and (2.11) we conclude that Schur polynomial sequences are Polynomial or q- analogs of the Fibonacci numbers. Indeed, for q→1 we see immediately from (2.7) and (2.12) that

$$S_{n+1}(1) = T_{n+1}(1) = \mathrm{F}_{n+1} = \sum_{0\le 2j\le n} \binom{n-j}{j} \tag{2.14}$$

and

$$
\begin{aligned}
\mathrm{F_n} &= \sum\nolimits_{0\le 2j\le n-1}\binom{n-j-1}{j} \\
&= \sum\nolimits_{j=-\infty}^{\infty}(-1)^j\begin{bmatrix} n \\ \left\lfloor\frac{n+5j}{2}\right\rfloor \end{bmatrix} \qquad (2.14) \\
&= \sum\nolimits_{j=-\infty}^{\infty}(-1)^j\begin{bmatrix} n+j \\ \left\lfloor\frac{n+1-5j}{2}\right\rfloor+1 \end{bmatrix}
\end{aligned}
$$

The first line of (2.14) is well-known Fibonacci number Identity but the other two are studied in generalized Fibonacci numbers.

Section Three
The Rogers-Ramanujan identities

3.1 Introduction

The Rogers–Ramanujan identities play an important role in the theory of partitions. They have an unusual history. They were first found by L.J. Rogers in 1894 but remained unnoticed for some time. About 1913 Srinivasa Ramanujan rediscovered the formulas but he had no proof. In 1917 Ramanujan came accidentally across Rogers' paper. He and Rogers then found simpler proofs. At about the same time, Issai Schur independently rediscovered the identities and gave two proofs which differed radically from these other proofs.

3.2 First Rogers-Ramanujan Identities

The following theorem show first Rogers-Ramanujan identity

Theorem 1: The number of partitions of n in which any two summands differ by at least 2 denoted by (D_n) is equal to the number of partitions of n into parts congruent to 1 or 4 modulo 5 (R_n).

$$\Leftrightarrow 1+\sum_{j=1}^{\infty}\frac{q^{j^2}}{(1-q)(1-q^2)\dots(1-q^j)}=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})} \qquad (3.1)$$

Where left-hand side of (3.1) is generating function of D_n and right-hand side is generating function of R_n .Before proving the theorem let us see example.

Example: Partitions of $n = 9$ with the given first restriction are

$$9,\ 8+1,\ 7+2,\ 6+3,\ 5+3+1 \text{ and}$$

Partitions of $n = 9$ with the second restriction are $9,\ 6+1+1+1,\ 4+4+1,\ 4+4+1+1+1+1+1,\ 1+1+1+1+1+1+1+1+1$

Since 1 and 6 are congruent to 1 modulo 5, 4 and 9 are congruent to 4 modulo 5.

So that $D_9 = R_9 = 4$.

Before going to proof of the theorem first drive both sides of (2.14)

Let $D(q)$ and $R(q)$ be the generating functions of D_n and of R_n respectively.

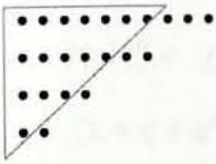
Claim: a) $D(q) = 1+\sum_{j=1}^{\infty}\frac{q^{j^2}}{(1-q)(1-q^2)\dots(1-q^j)}$ and

b) $R(q) = \prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$

Proof: (a)

Take a specific partition α =10 + 7 + 4 + 2 of *n*=23, partition of 23 into parts differing by at least 2.

Ferrer's diagram representation of α is



Since parts must differ by at least 2, each line must have at least two more dots than the one below.

If the partition has exactly j parts, the graph must have at least $1 + 2 + 5 + \cdots 2j - 1 = j^2$ dots (in our diagram above, the dots inside the triangle, which is$1 + 2 + 5 + 7 = 4^2 = 16$ dots). As a result, a partition of n into k parts differing by at least 2 can be diagrammatically represented by a triangle with j^2 dots and a partition of $n - j^2$ into at most j parts.

Then the generating function for the number of partitions of $n - j^2$ into at most parts is the same to the generating function for the number of partitions of $n - j^2$ into parts not exceeding is given by

$$\frac{1}{(1-q)(1-q^2)\dots(1-q^j)} \tag{3.2}$$

And the generating function for the dots inside the triangle is q^{j^2} since the triangle contributes j^2.

Consequently, by multiplication principle the generating function for the number of partition of n into j parts differing by at least 2 is

$$\frac{q^{j^2}}{(1-q)(1-q^2)\dots(1-q^j)} \tag{3.3}$$

To find the generating function for the number of all partitions of n into parts differing by at least 2, take the sum of (3.3) for j = 1,2,3, ... , so that

$$D(q) = 1 + \sum_{j=1}^{\infty} \frac{q^{j^2}}{(1-q)(1-q^2)\dots(1-q^j)} \tag{3.4}$$

Where 1 in (3.4) indicates the empty case for $n = 0$.

Proof: (b)
To find generating function $R(q)$ for R_n
Consider 1, 4, 6, and 9... *m*,... are parts congruent to 1 or 4 modulo 5 and using generating function modeling. We have

$$R(q) = (q^0 + q^1 + q^2 + \cdots) * ((q^4)^0 + (q^4)^1 + (q^4)^2 + \cdots) *$$

$$((q^6)^0 + (q^6)^1 + (q^6)^2 \ \ldots) * \ldots * ((q^m)^0 + (q^m)^1 + (q^m)^2 + \cdots) * \ldots$$

$$= (1 + q + q^2 + \cdots)(1 + q^4 + q^8 + \cdots)(1 + q^6 + q^{12} + \cdots) \ldots$$

$$(1 + q^m + q^{2m} + \ \ldots) \ldots$$

$$= \left(\frac{1}{1-q}\right)\left(\frac{1}{1-q^4}\right)\left(\frac{1}{1-q^6}\right)\left(\frac{1}{1-q^9}\right)\ldots\left(\frac{1}{1-q^m}\right)\ldots$$

$$R(q) = \prod_{n=0}^{\infty} \frac{1}{(1-x^{5n+1})(1-x^{5n+4})} \qquad (3.5)$$

Since *1,4,6,9*, ... , m, ... are of the form $5n + 1$ or $5n + 4$. since $5n + 1 \equiv 1(\text{modulo}5)$ and $5n + 4 \equiv 4(\text{modulo}5)$ etc...

To proof the Rogers-Ramanujan identities we follow the following procedures.

1. Produce recurrence formulae for the partition function $\delta_i(m,n)$ (the number of partition of *n* into *m* distinct parts where 1 appears at most *i* times ,and in which any two summands differ by at least 2) under consideration
2. Considering the related generating functions for the partition function $\delta_1(m,n)$and $\delta_0(m,n)$ in step1
3. Relating "New" function formulated by Rogers and Ramanujan with generating functions of partition function considered in step 2
4. Relating $\delta_1(m,n)$ and $\delta_0(m,n)$ with D_n , using "New function" and Jacobi's triple product identity theorem we obtain first and second Rogers-Ramanujan identities.

i) Recurrence Formulae for the δ − partitions

Consider $\delta_i(m,n)$ (enumerated by D_n) is the number of partitions of n into m distinct parts where 1 appears at most *i* times in which any two summands differ by at least 2 then the following two recurrence equations holds.

$$\delta_1(m,n) = \delta_0(m,n) + \delta_0(m-1,n-m) \qquad (3.6)$$

$$\delta_0(m,n) = \delta_1(m,n-m) \qquad (3.7)$$

First verify (3.6) and (3.7) in the following way
To verify (3.6) divide the partitions enumerated by $\delta_1(m,n)$ into two classes;

I) The class containing 1 as a summand and
II) The class does not contain 1 as a summand.

The elements of (II) are exactly the partitions enumerated by $\delta_0(m,n)$.

Let us transform all partitions in (I) by deleting the summand 1 from each, and subtracting 1 from each of the remaining summands. This transformation leaves each partition in (I) with one less part, and it reduces the number being partitioned to $n-m$ .Furthermore, since each partition originally contained 1 as a summand, all of the other parts must have been larger than 2 (by virtue of the proscription of consecutive integers in the original partition). Thus, after our transformation, all parts are larger than 1. Therefore we obtained partitions of the type enumerated by $\delta_0(m-1,n-m)$.

Reversing the preceding transformation, given any partition of $\mathrm{n-m}$ into $\mathrm{m-1}$ distinct parts, each larger than 1, and with no consecutive integers appearing as summands, we may add 1 to every part and insert 1 as a summand to produce the elements of (I).

As a result we have established that there are exactly $\delta_0(m-1,n-m)$ elements of (I). Since the total number of elements in both classes equals $\delta_1(\mathrm{m,n})$, therefore (3.6) is established.

To verify (3.7), we apply the transformation to all partitions enumerated by $\delta_0(m,n)$.Then, as above, we establish that there are $\delta_1(m,n-m)$ partitions being counted.

Example: (a)

$$\delta_1(3,15)=\delta_0(3,15)+\delta_0(2,12)\quad \textit{Since } \delta_1(3,15)=7, \textit{and}$$

$$\delta_0(3,15)=3, \delta_0(2,12)=4$$

(b) $\delta_0(3,15)=\delta_1(3,12)=3$

If either of classes is empty we may easily validate (3.6) and (3.7) as follow

$$\delta_1(m,n)=\delta_0(m,n)=\begin{cases}1 & \textit{if } m=n=0\\ 0 & \textit{if eithe m or n is negative and}\\ & \text{not both are 0}\end{cases} \qquad (3.8)$$

The equation $\delta_1(0,0)=\delta_0(0,0)=1$ accounts for the empty partition of zero.

ii) Generating functions for the δ-partition functions

Define

$$G_1(x;q) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\delta_1(m,n)x^m q^n \qquad (3.9)$$

$$G_0(x;q) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\delta_0(m,n)x^m q^n \qquad (3.10)$$

Substituting (3.6), (3.7) and (3.8) in (3.9) and (3.10) leads to three important identities:

$$G_1(x;q) = G_0(x;q) + xqG_0(xq;q) \qquad (3.11)$$

$$G_0(x;q) = G_1(xq;q) \qquad (3.12)$$

$$G_1(0;q) = G_0(0;q) \qquad (3.13)$$

Proof:

$$\begin{aligned}
G_1(x;q) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\delta_1(m,n)x^m q^n \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\big(\delta_0(m,n)+\delta_0(m-1,n-m)\big)x^m q^n \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\delta_0(m,n)x^m q^n + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\delta_0(m-1,n-m)x^m q^n \\
&= G_0(x;q) + xq\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\delta_0(m-1,n-m)(xq)^{m-1}q^{n-m} \\
&= G_0(x;q) + xq\sum_{m=-1}^{\infty}\sum_{n=-m-1}^{\infty}\delta_0(m,n)(xq)^m q^n
\end{aligned}$$

$$G_1(x;q) = G_0(x;q) + xqG_0(xq;q)$$

Similarly

$$\begin{aligned}
G_0(x;q) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\delta_0(m,n)x^m q^n \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\delta_1(m,n-m)x^m q^n \\
&= \sum_{m=0}^{\infty}\sum_{n=-m}^{\infty}\delta_1(m,n)x^m q^{n+m} \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\delta_1(m,n)(xq)^m q^n
\end{aligned}$$

$$G_0(x;q) = G_1(xq,q)$$

Furthermore, (3.8) implies that

$$G_1(0;q) = G_0(0;q) = 1$$

New functions: We need some new functions of x and q that satisfy (3.11), (3.12) and (3.13). L.J.Rogers and S.Ramanujan produced the following functions, for any integer i

$$f_i(x;q) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} q^{\frac{1}{2}n(5n+3)-in}\left(1-x^{i+1}q^{(2n+1)(i+1)}\right)}{(1-q)(1-q^2)\dots(1-q^n)\prod_{j=n+1}^{\infty}(1-xq^j)} \tag{3.14}$$

When $n = 0$, the summand is by convention just,

$$\left(1 - x^{i+1}q^{i+1}\right)/\prod_{j=1}^{\infty}\left(1 - xq^j\right)$$

Since for $n = 0$, the empty product is defined as 1.

When $i = -1$,

$$1 - x^{i+1}q^{(2n+1)(i+1)} = 1 - x^0 q^0 = 0$$

Thus, each term in the series for $i = -1$ is zero.

Therefore

$$f_{-1}(x;q) = 0 \tag{3.15}$$

Then we have the following results,

Corollary 1:

a) $f_1(x;q) = f_0(x;q) + xqf_0(xq;q)$ *(3.16)*

b) $f_0(x;q) = f_1(xq;q)$ *(3.17)*

c) $f_0(0;q) = f_1(0;q) = 1$ (3.18)

Proof *:(a)*

$$f_i(x;q) - f_{i-1}(x;q)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} q^{1/2n(5n+3)}\left(q^{-in} - x^{i+1}q^{(2n+1)(i+1)-in} - q^{-(i-1)n} + x^i q^{(2n+1)i-(i-1)n}\right)}{(1-q)(1-q^2)\dots(1-q^n)\prod_{j=n+1}^{\infty}(1-xq^j)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} q^{1/2n(5n+3)}\left(q^{-in}(1-q^n) + x^i q^{(2n+1)i-(i-1)n}\left(1-xq^{n+1}\right)\right)}{(1-q)(1-q^2)\dots(1-q^n)\prod_{j=n+1}^{\infty}(1-xq^j)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} q^{{}^1\!/_2 n(5n+3)} q^{-in}(1-q^n)}{(1-q)(1-q^2)\ldots(1-q^n)\prod_{j=n+1}^{\infty}(1-xq^j)}$$

$$+ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} q^{{}^1\!/_2 n(5n+3)} x^i q^{(2n+1)i-(i-1)n}(1-xq^{n+1})}{(1-q)(1-q^2)\ldots(1-q^n)\prod_{j=n+1}^{\infty}(1-xq^j)}$$

We note that the first term in the first sum of the last expression is simply

$$(1-q^0)/\prod_{j=1}^{\infty}(1-xq^j) = 0$$

Thus, we may really treat the first sum as running from $n = 1$ to ∞. Replacing n by $n+1$ in the first sum, we find that

$$f_i(x;q) - f_{i-1}(x;q) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+2} q^{{}^1\!/_2 n(n+1)(5n+8)} q^{-in-i}}{(1-q)(1-q^2)\ldots(1-q^n)\prod_{j=n+2}^{\infty}(1-xq^j)}$$

$$+ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} q^{{}^1\!/_2 n(5n+3)} x^i q^{(2n+1)i-(i-1)n}}{(1-q)(1-q^2)\ldots(1-q^n)\prod_{j=n+2}^{\infty}(1-xq^j)}$$

$$= x^i q^i \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} q^{{}^1\!/_2 n(5n+3)+in+n}\left(1-x^{2-i}q^{(2n+2)(2-i)}\right)}{(1-q)(1-q^2)\ldots(1-q^n)\prod_{j=n+1}^{\infty}(1-xqq^j)}$$

$$= x^i q^i \sum_{n=0}^{\infty} \frac{(-1)^n (xq)^{2n} q^{{}^1\!/_2 n(5n+3)-(1-i)n}\left(1-(xq)^{2-i}q^{(2n+1)(2-i)}\right)}{(1-q)(1-q^2)\ldots(1-q^n)\prod_{j=n+1}^{\infty}(1-xqq^j)}$$

$$f_i(x;q) - f_{i-1}(x;q) = x^i q^i f_{1-i}(xq;q) \tag{3.19}$$

By setting $i = 1$ in (3.19) above we get

$$f_1(x;q) = f_0(x;q) + xqf_0(xq;q)$$

and setting $i = 0$ in (3.19) together with (3.15),

$$f_0(x;q) = f_1(xq;q)$$

Finally, if we set $x = 0$ in (3.14), we see that

$$f_0(0,q) = f_1(0,q) = 1$$

We show that (3.16), (3.17) and (3.18) are simply (3.11), (3.12) and (3.13) respectively with f replacing G.
Thus we may conclude that

$$f_1(x;q) = G_1(x;q)$$

and

$$f_0(x;q) = G_0(x;q).$$

Now this is the time to proof the Rogers-Ramanujan Identities using the results we obtained so far.

iii) Proof of first Rogers-Ramanujan Identity

Clearly we have

$$\sum_{m=0}^{\infty} \delta_1(m,n) = D_n \tag{3.20}$$

Taking the sum over all n and multiplying both sides by x^n of (3.20)

$$\begin{aligned}\sum_{n=0}^{\infty} D_n q^n &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \delta_1(m,n) q^n \\ &= G_1(1;q) \\ &= f_1(1;q) \\ &= \frac{\sum_{n=0}^{\infty}(-1)^n q^{1/2 n(5n+1)}(1-q^{4n+2})}{\prod_{j=1}^{\infty}(1-q^j)} \\ &= \frac{\sum_{n=0}^{\infty}(-1)^n q^{1/2 n(5n+1)} - \sum_{n=0}^{\infty}(-1)^n q^{1/2 n(5n+9)+2}}{\prod_{j=1}^{\infty}(1-q^j)}\end{aligned}$$

If we replace n by $-n-1$ in the second sum of the last expression, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} D_n q^n &= \frac{\sum_{n=0}^{\infty}(-1)^n q^{1/2 n(5n+1)} - \sum_{n=-1}^{-\infty}(-1)^{-n-1} q^{1/2(-n-1)(-5n+4)+2}}{\prod_{j=1}^{\infty}(1-q^j)} \\ &= \frac{\sum_{n=0}^{\infty}(-1)^n q^{1/2 n(5n+1)} + \sum_{n=-1}^{-\infty}(-1)^n q^{1/2 n(5n+1)}}{\prod_{j=1}^{\infty}(1-q^j)} \\ &= \frac{\sum_{n=-\infty}^{\infty}(-1)^n q^{1/2 n(5n+1)}}{\prod_{j=1}^{\infty}(1-q^j)} \qquad (*)\end{aligned}$$

Replacing q by $q^{5/2}$ and z by - $q^{1/2}$ in Jacobi's Triple Product identities (1.5), we get

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{1/2 n(5n+1)} = \prod_{n=0}^{\infty}(1-q^{5n+5})(1-q^{5n+3})(1-q^{5n+2}) \qquad (**)$$

and

$\prod_{j=1}^{\infty}(1-q^{j}) =$

$$\prod_{n=0}^{\infty}\left(1-q^{5n+5}\right)\left(1-q^{5n+1}\right)\left(1-q^{5n+2}\right)\left(1-q^{5n+3}\right)\left(1-q^{5n+4}\right) \quad (***)$$

Since the sequences $\{5n+5\}, \{5n+1\}, \{5n+2\}, \{5n+3\}$ and $\{5n+4\}$ with n ranging from 0 to ∞ contain all the positive integers without duplications.

By substituting $(**)$ and $(***)$ in $(*)$ we have

$$\sum_{n=0}^{\infty} D_n q^n = \frac{\prod_{n=0}^{\infty}(1-q^{5n+5})(1-q^{5n+2})(1-q^{5n+3})}{\prod_{n=0}^{\infty}(1-q^{5n+5})(1-q^{5n+1})(1-q^{5n+2})(1-q^{5n+3})(1-q^{5n+4})}$$

$$D_n(q) \quad = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$$

Therefore by (3.4) we have

$$1 + \sum_{j=1}^{\infty} \frac{q^{j^2}}{(1-q)(1-q^2)\dots(1-q^j)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$$

3.3 Second Rogers- Ramanujan Identity

Theorem 2 (Second Rogers-Ramanujan identities) The number of partitions of n in which any two summands differing by at least 2 and all summands are greater than1 denoted by (H_n) is equal to the number of partitions of n into parts congruent to 2 or 3 modulo 5 (S_n)

$$\Leftrightarrow \; 1 + \sum_{j=1}^{\infty} \frac{q^{j^2+j}}{(1-q)(1-q^2)\dots(1-q^j)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})} \qquad (3.21)$$

Where $H(q) = 1 + \sum_{j=1}^{\infty} \frac{q^{j^2+j}}{(1-q)(1-q^2)\dots(1-q^j)}$ and $S(q) = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}$ are the generating functions of H_n and S_n respectively.

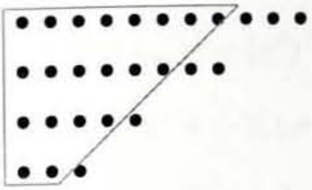
Example: Partitions of $n = 9$ with the first restriction are,

$9,\ 7+2,\ 6+3$

Partitions of 9 with the second restriction are $3+2+2+2,\ 3+3+3,\ 7+2$, since 2 and 7 are congruent to 2 modulo 5, 3 is congruent to 3 modulo 5. So that $H_9 = S_9 = 3$

Take a specific partition of n=27, $\theta = 11+8+5+3$, partition of 27 into parts differing by at least 2, and all summands are greater than 1.

Let us represent θ in Ferrer's diagram



Since parts must differ by at least 2, and each parts are greater than 1, each line must have at least two more dots than the one below and each parts are greater than 1 .If the partition has exactly $j$parts, the graph must have at least $2+4+6+\cdots+2j = j^2 + j$ dots (in our graph above, the dots inside trapezoid), which is $2+4+6+8=4^2+4=20$ dots. As a result, a partition of n into j parts differing by at least 2 and all summands are greater than 1 can be graphically represented by j^2+j dots and a partition of $n-(j^2+j)$ into at most j parts.

Then the generating function for the number of partitions of $n-(j^2+j)$ into at most j parts is the same as the generating function for the number of partitions of $n-(j^2+j)$into parts not exceeding j is given by

$$\frac{1}{(1-q)(1-q^2)\ldots(1-q^j)} \tag{3.22}$$

and the generating function for trapezoid is q^{j^2+j}
Consequently, by multiplication principle the generating function for the number of partition of n into j parts differing by at least 2 and all summands are greater than 1 is

$$\frac{q^{j^2+j}}{(1-q)(1-q^2)\ldots(1-q^j)} \tag{3.23}$$

To find the generating function for the number of all partitions of n into parts differing by at least 2 and all summands are greater than1, take the sum of (3.23) for $j = 1, 2, 3\ldots$
We get

$$H(q) = 1+\sum_{j=1}^{\infty}\frac{q^{j^2+j}}{(1-q)(1-q^2)\ldots(1-q^j)} \tag{3.24}$$

Where 1 in (3.24) indicates the empty case for $n=0$.

To find generating function $S(q)$ for S_n first find the generating function for

$2,3,7,8,\ldots,r,\ldots$each congruent to 2 or 3 modulo 5.

Using generating function modeling

$$S(q) = [(q^2)^0 + (q^2)^1 + (q^2)^2 + \cdots] * [(q^3)^0 + (q^3)^1 + (q^3)^2 + \cdots] *$$

$$[(q^7)^0 + (q^7)^1 + (q^7)^2 + \ \ldots] * \ldots * [(q^r)^0 + (q^r)^1 + (q^r)^2 + \cdots] \ldots$$

$$= (1 + q^2 + q^4 + \cdots)(1 + q^3 + q^6 + \cdots)(1 + q^7 + q^{14} + \cdots) \ldots$$

$$(1 + q^r + q^{2r} + \cdots) \ldots$$

$$= \left(\frac{1}{1-q^2}\right)\left(\frac{1}{1-q^3}\right)\left(\frac{1}{1-q^7}\right)\left(\frac{1}{1-q^8}\right) \ldots \left(\frac{1}{1-q^r}\right) \ldots$$

$$S(q) = \prod_{n=0}^{\infty} \frac{1}{(1-x^{5n+2})(1-x^{5n+3})}$$

Since $2,3,7,8, \ldots, r, \ldots$ are of the form $5n + 2$ or $5n + 3$. since $5n + 2 \equiv 2(modulo\ 5)$ and $5n + 3 \ \equiv 3(modulo\ 5)$.
We have already accomplished enough to establish easily, second Rogers-Ramanujan identity. Since it is similarly derived from our formula for $f_i(x; q)$, we shall only outline the procedure.

Clearly as in (3.20)

$$\sum_{\mathrm{m}=0}^{\infty} \delta_0(\mathrm{m}, \mathrm{n}) = \mathrm{H_n} \qquad (3.25)$$

Applying Snake-Oil method on (3.25)

$$\sum_{n=0}^{\infty} H_n q^n = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \delta_0\,(m, n) q^n$$

$$\sum_{n=0}^{\infty} H_n q^n = G_0(1; q)$$

$$= f_0(x; q)$$

$$= \frac{\sum_{n=-\infty}^{\infty} (-1)^n q^{1/2 n(5n+3)}}{\prod_{j=1}^{\infty} (1-q^j)}$$

$$= \frac{\prod_{n=0}^{\infty} (1-q^{5n+5})(1-q^{5n+1})(1-q^{5n+4})}{\prod_{n=0}^{\infty} (1-q^{5n+5})(1-q^{5n+1})(1-q^{5n+2})(1-q^{5n+3})(1-q^{5n+4})}$$

$$H(q) = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}$$

By (3.24), we conclude that

$$1 + \sum_{j=1}^{\infty} \frac{q^{j^2+j}}{(1-q)(1-q^2) \ldots (1-q^j)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}$$

Section Four

Generating function and Identities of Schur's polynomial sequences

The main task in this section is to find generating function and identities with their proof of Schur's polynomial sequences as q- analog of Fibonacci number identities studied in section two.

In order to find analogs of (2.1) for S_n and T_n, we need the q-analog of the binomial series defined in section 1.

$$\sum_{j=0}^{\infty}\begin{bmatrix} n+j \\ j \end{bmatrix} x^j = \frac{1}{(x\,;q)_{n+1}}, \tag{4.1}$$

Where

$$(x\,;q)_{n+1} = (1-x)(1-xq)\ldots(1-xq^n)$$

4.1 Generating function of Schur Polynomial sequence And q-analog of Fibonacci numbers

Theorem 3:

$\sum_{n=0}^{\infty} S_n(q)x^n = \frac{1}{1-x-x^2\eta}x$, where η is an operator on functions of x.

Proof:

Let us denote by $\delta(x)$ the expression on the left side of the equation in Theorem 1 Hence equivalently, we are to prove,

$$(1-x-x^2\eta)\delta(x) = x$$

Now,

$$(1-x-x^2\eta)\delta(x) = x + \sum_{n=2}^{\infty} S_n(n)x^n - \sum_{n=0}^{\infty} S_n(q)x^{n+1} - \sum_{n=0}^{\infty} S_n(q)\,x^{n+2}q^n$$

$$= x + \sum_{n=2}^{\infty}(S_n(q) - S_{n-1}(q) - q^{n-2}S_{n-2}(q))\,x^n$$

$$= x\,.$$

since the sum is equals to zero, by (2.5)

Hence the theorem is proved.

Lemma 2:

$$(x+x^2\eta)^n x = x^{n+1}\sum_{j=0}^{n} x^j q^{j^2}\begin{bmatrix} n \\ j \end{bmatrix}.$$

Proof: For $n = 0$, this asserts $x = x$. Now assume the result for a particular n n;

Then we need to show

$$(x+x^2\eta)^{n+1}x = x^{n+2}\sum_{j\geq 0} x^j q^{j^2} \begin{bmatrix} n+1 \\ j \end{bmatrix}$$

$$(x+x^2\eta)^{n+1}x = (x+x^2\eta)\,(x+x^2\eta)^n x$$

$$(x+x^2\eta)^{n+1}x = (x+x^2\eta)x^{n+1}\sum_{j=0}^{n} x^j q^{j^2} \begin{bmatrix} n \\ j \end{bmatrix} \qquad \textit{by assumption.}$$

$$= x^{n+2}\sum_{j=0}^{n} x^j q^{j^2} \begin{bmatrix} n \\ j \end{bmatrix} + x^{n+3}\sum_{j=0}^{n} x^j q^{j^2+n+1+j} \begin{bmatrix} n \\ j \end{bmatrix} \quad \textit{by (1.11)}$$

$$= x^{n+2}\sum_{j=0}^{n} x^j q^{j^2} \begin{bmatrix} n \\ j \end{bmatrix} + x^{n+2}\sum_{j=0}^{n} x^{j+1} q^{j^2+n+1+j} \begin{bmatrix} n \\ j \end{bmatrix}$$

$$= x^{n+2}\left(\sum_{j=0}^{n} x^j q^{j^2} \begin{bmatrix} n \\ j \end{bmatrix} + \sum_{j=0}^{n} x^{j+1} q^{j^2+n+1+j} \begin{bmatrix} n \\ j \end{bmatrix}\right)$$

$$= x^{n+2}\left(\sum_{j=0}^{n} x^j q^{j^2} \begin{bmatrix} n \\ j \end{bmatrix} + \sum_{j=0}^{n} x^{(j-1)+1} q^{(j-1)^2+n+1+(j-1)} \begin{bmatrix} n \\ j-1 \end{bmatrix}\right)$$

$$(j \textit{ has been replaced by } j-1)$$

$$= x^{n+2}\left(\sum_{j=0}^{n} x^j q^{j^2} \begin{bmatrix} n \\ j \end{bmatrix} + \sum_{j=0}^{n} x^j q^{j^2-2j+1+n+1+j-1} \begin{bmatrix} n \\ j-1 \end{bmatrix}\right)$$

$$= x^{n+2}\left(\sum_{j\geq 0} x^j q^{j^2}\left(\begin{bmatrix} n \\ j \end{bmatrix} + q^{n+1-j}\begin{bmatrix} n \\ j-1 \end{bmatrix}\right)\right)$$

$$= x^{n+2}\sum_{j\geq 0} x^j q^{j^2} \begin{bmatrix} n+1 \\ j \end{bmatrix}$$

The result follows by Mathematical induction.

Corollary 2:

$$\sum_{n\geq 0} S_n(q)\,x^n = \sum_{j\geq 0} \frac{x^{2j+1}q^{j^2}}{(x;q)_{j+1}}$$

Proof:

$$\sum_{n\geq 0} S_n(q)x^n = \frac{1}{1-x-x^2\eta}x \qquad \textit{by theorem 3}$$

$$= \frac{1}{1-(x+x^2\eta)}x$$

$$= \sum_{n=0}^{\infty}(x+x^2\eta)^n x \quad \textit{by Geometric sum.}$$

$$= \sum_{n=0}^{\infty} x^{n+1} \sum_{j=0}^{n} x^j q^{j^2} \begin{bmatrix} n \\ j \end{bmatrix} \qquad \textit{by the above lemma.}$$

$$= \sum_{n,j\geq 0} x^{n+2j+1} q^{j^2} \begin{bmatrix} n+j \\ j \end{bmatrix} \quad (\nabla)$$

(Where n has been replaced by n + j)

$$= \sum_{j\geq 0} x^{2j+1} q^{j^2} \sum_{n\geq 0} \begin{bmatrix} n+j \\ n \end{bmatrix} x^n$$

$$\sum_{n\geq 0} S_n(q) x^n = \sum_{j\geq 0} \frac{x^{2j+1} q^{j^2}}{(x\,;q)_{j+1}} \; \textit{(By (4.1))}$$

The following corollary is the generalization of Fibonacci number.

Corollary 3:

$S_n(q) = \sum_{j\geq 0} q^{j^2} \begin{bmatrix} n-j-1 \\ j \end{bmatrix}.$ (4.2)

Proof: Fromstep (∇)in the proof of Corollary 2 we have

$$\sum_{n\geq 0} S_n(q) x^n = \sum_{n,j\geq 0} x^{n+2j+1} q^{j^2} \begin{bmatrix} n+j \\ j \end{bmatrix}$$

$$= \sum_{n=0}^{\infty} \left[\sum_{j=0}^{\infty} x^{2j+1} q^{j^2} \begin{bmatrix} n+j \\ j \end{bmatrix} \right] x^n$$

$$= \sum_{n=0}^{\infty} \left[\sum_{j=0}^{\infty} q^{j^2} \begin{bmatrix} n+j-2j-1 \\ j \end{bmatrix} \right] x^n$$

The result follows by extracting the coefficient of x^n.

4.2 Schur Polynomial identities

Theorem 2:

$$S_{n+2}(q) - 1 = \sum_{j=0}^{n} q^j S_j(q). \textit{(4.2)}$$

Proof: Taking the sum over all *n* and multiplying by $x^n$of left hand side of (4.2), we have

$$\sum_{n\geq 0} (S_{n+2}(q) - 1)\, x^n = \sum_{n=0}^{\infty} S_{n+2}(q) x^n - \sum_{n\geq 0} x^n$$

$$= x^{-2} \left(\sum_{n=0}^{\infty} S_n(q) x^n - x\right) - \frac{1}{1-x}$$

$$= x^{-2} \frac{1}{1-x-x^2\eta} x - \frac{x^{-1}}{1-x}$$

$$= \frac{1}{1-x}(-x^{-2}(1-x-x^2\eta)+x^{-2}-x^{-1})\frac{1}{1-x-x^2n}x$$

$$= \frac{1}{1-x}\eta\frac{1}{1-x-x^2n}x$$

$$= \frac{1}{1-x}\sum_{j\geq 0} S_j\,(q)(xq)^j \quad \textit{by (1.11)}$$

$$=\sum_{n\geq 0} x^n \sum_{j\geq 0} S_j\,(q)x^j q^j$$

$$= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} q^j S_j(q)\right) x^n \quad \textit{by Cauchy product rule.}$$

$S_{n+2}(q) - 1 \;=\; \sum_{j=0}^{n} q^j\, S_j(q)$
by extracting the coefficients of x^n

Note that If we let $q \to 1$ in Corollary 3 it reduces directly to (2.14), and if we let $q \to 1$ in Theorem 4.2 it reduces to Fibonacci number identity (i)

Theorem 5: For $n \geq 0, \mathrm{m} > 0,$

$$s_{n+m}(t,q) = S_m(t,q)S_{n+1}(tq^{m-1},q) + tq^{m-1}S_{m-1}(t,q)S_n(tq^m,q)$$

Making direct generalization of (2.3) as in proof of (2.3) is impossible because it vanishes when $q = 1$. In order to overcome this difficulty we must, in fact generalize $s_n(q)$ as follows:

$$S_n(t\,,q) \;=\begin{cases} 0 & \textit{if } n = 0 \\ 1 & \textit{if } n = 1 \\ S_{n-1}(t\,,q) + tq^{n-2}S_{n-2}(t\,,q) & \textit{for } n > 1 \end{cases} \qquad (4.3)$$

Proof:

Now for $m \geq 0$

$(1 - x - tx^2q^m\eta)\sum_{n=0}^{\infty} S_{n+m}(t,q)x^n$

$$= S_m(t\,,q) + xS_{m+1}(t\,,q) + \sum_{n=2}^{\infty} S_{n+m}(t\,,q)\,x^n - xS_m(t\,,q) -$$

$$\sum_{n=1}^{\infty} S_{n+m}(t\,,q)x^{n+1} - \sum_{n=0}^{\infty} S_{n+m}(t\,,q)tq^{m+n}\,x^{n+2}$$

$$= S_m(t\,,q) + x\big(S_{m+1}(t\,,q) - S_m(t\,,q)\big) + \sum_{n=2}^{\infty}\big(S_{n+m}(t,q) - S_{n+m-1}(t,q) - tq^{m+n-2}S_{n+m-2}(t,q)\big)x^n \qquad (\omega)$$

(Where n has been replaced by $n-1$ *n-1 in the second sum and* $n-2$ *in the third)*

Divide (ω) into two cases,

Case 1 if $m = 0$,

$$(1 - x - tx^2\eta)\sum_{n=0}^{\infty} S_n(t,q)x^n = S_0(t\,,q) + x\big(S_1(t\,,q) - S_0(t\,,q)\big) + \sum_{n=2}^{\infty}\big(S_n(t,q) - S_{n-1}(t,q) - tq^{n-2}S_{n-2}(t,q)\big)x^n$$

$$= x \quad \text{by (4.3)}$$

Case 2: if $m > 0$

$$(1 - x - tx^2q^m\eta)\sum_{n=0}^{\infty} S_{n+m}(t,q)x^n$$

$$= S_m(t\,,q) + x\big(S_{m+1}(t\,,q) - S_m(t\,,q)\big) + \sum_{n=2}^{\infty}\big(S_{n+m}(t,q) - S_{n+m-1}(t,q) - tq^{m+n-2}S_{n+m-2}(t,q)\big)x^n$$

$$= S_m(t\,,q) + x\big(S_m(t\,,q) + tq^{m-1}S_{m-1}(t\,,q) - S_m(t\,,q)\big) + 0 \text{ by (4.3)}$$

$$= S_m(t\,,q) + x\,tq^{m-1}S_{m-1}(t\,,q)$$

Therefore

$$(1 - x - tx^2q^m\eta)\sum_{n=0}^{\infty} S_{n+m}(t,q)x^n = \begin{cases} x & \text{if } m = 0 \\ S_m(t\,,q) + xtq^{m-1}S_{m-1}(t\,,q) & \text{if } m > 0. \end{cases} \qquad (4.4)$$

From case 1 we have,

$$\sum_{n=0}^{\infty} S_n(t\,,q)\,x^n = \frac{1}{1-x-tx^2\eta}\,x \qquad (4.5)$$

and from case 2 for $m = 1$

$$(1 - x - tx^2q\eta)\sum_{n=0}^{\infty} S_{n+1}(t,q)x^n = S_1(t\,,q) + x\,tS_0(t\,,q)$$

This implies

$$\sum_{n=0}^{\infty} S_{n+1}(t,q)x^n = \frac{1}{1-x-tx^2q\eta} \tag{4.6}$$

Replacing t by ${}^{t}/_{q}$ in (4.6) we obtain

$$\sum_{n=0}^{\infty} S_{n+1}(t/q,q)x^n = \frac{1}{1-x-tx^2\eta} \tag{4.7}$$

Finally from case 2 we have

$$\sum_{n=0}^{\infty} S_{n+m}(t,q)x^n = \frac{1}{1-x-tx^2q^m\eta}\left(S_m(t,q) + x\,tq^{m-1}S_{m-1}(t,q)\right) \qquad (4.8)$$

$$\sum_{n=0}^{\infty} S_{n+m}(tq^{-m};q)x^n = \frac{1}{1-x-tx^2\eta}\left(S_m(tq^{-m},q) + x\,tq^{-1}S_{m-1}(tq^{-m},q)\right)$$

(Replacing t by tq^{-m} *in (4.8))*

$$= S_m(tq^{-m},q)\frac{1}{1-x-tx^2\eta} + tq^{-1}S_{m-1}(tq^{-m},q)\frac{x}{1-x-tx^2\eta}$$

$$= S_m(tq^{-m},q)\sum_{n=0}^{\infty} S_{n+1}(t/q,q)x^n +.$$

$$tq^{-1}S_{m-1}(tq^{-m},q)\sum_{n=0}^{\infty} S_n(t,q)x^n \quad \text{by (4.5) and (4.7)}$$

Thus

$$\sum_{n=0}^{\infty} S_{n+m}(tq^{-m};q)x^n$$

$$= \sum_{n=0}^{\infty}\left(S_m(tq^{-m},q)S_{n+1}(t/q,q) + tq^{-1}S_{m-1}(tq^{-m},q)S_n(t,q)\right)x^n$$

Now replace t by tq^m and extracting the coefficient of x^n, we have

$$s_{n+m}(t,q) = S_m(t,q)S_{n+1}(tq^{m-1},q) + tq^{m-1}S_{m-1}(t,q)S_n(tq^m,q)$$

This is the required generalization of Fibonacci number identity (ii).

Remark: The new variable t was an essential addition. If it had not been included, it would not have been possible to transform the factor

$(1 - x - tx^2q^m\eta)$ into

$(1 - x - tx^2\eta)$, which formed the essential step in obtaining Theorem 5

Section Five
Conclusion

- Fibonacci numbers and the Fibonacci sequence are prime examples of "how mathematics is connected to seemingly unrelated things, like; the number of pairs of rabbits in the field, the growth of buds on trees, the starfish etc...
- Schur's polynomials are polynomials in q that are extensions of Fibonacci numbers and leads to the Rogers -Ramanujan identities. The following table shows their relationship.

	Fibonacci Number	Schur Polynomial
Generating function	$\mathrm{F}(x) = \frac{x}{1-x-x^2}$	$\mathrm{S}(x) = \frac{1}{1-x-x^2 n} x$
Identities	$I)\ \mathrm{F}_n = \sum_{0\le 2j \le n-1} \binom{n-j-1}{j}$	$i)\ S_n(q) = \sum_{j\ge 0} q^{j^2} \begin{bmatrix} n-j-1 \\ j \end{bmatrix}$
	$\mathrm{II})\ \mathrm{F}_0 + \mathrm{F}_1 + \ldots + \mathrm{F}_n = \mathrm{F}_{n+2} - 1$	$ii)\ S_{\ n+2}(q) - 1 = \sum_{j=0}^{n} q^j S_j(q)$
	$\mathrm{III})\ \mathrm{F}_{n+m} = \mathrm{F}_{m-1}\mathrm{F}_\mathrm{n} + \mathrm{F}_m \mathrm{F}_{n+1}$	$iii)\ s_{n+m}(t,q) = S_m(t,q) S_{n+1}(tq^{m-1},q) + $ $+tq^{m-1} S_{m-1}(t,q) S_n(tq^m,q)$

- The Roger-Ramanujan identities are identities derived from Schur's polynomial as shown in subsection (2.4) and by equating number of certain restricted partitions of positive integers (theorem 1 and 2).
- The Rogers-Ramanujan identities are q-analogues of identities for the Fibonacci numbers.
- Using operator generating function method has power to prove Fibonacci numbers identities and to study Schur polynomials as extension (q-analog) Fibonacci numbers.

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