Dynamics of Three-Level Laser Pumped by Electron Bombardment

A project submitted to graduate program AAU, in partial fulfillment of the
requirement for the degree of Master of Science in Physics.

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This is to certify that the project prepared by Zeleke Behailu, entitled **Dynamics of Three-Level Laser Pumped by Electron Bombardment** and submitted to graduate program AAU, in partial fulfillment of the requirements for the degree of Master of Science in Physics complies with the regulations of the University and meets the accepted standards with respect to its originality.

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Abstract

In this project, we discuss the quantum properties of the light generated by a three-level laser with the closed cavity and coupled to a vacuum reservoir. The three-level atoms available in the closed cavity are pumped from the bottom to the top level by means of electron bombardment. Our analysis is carried out by putting the noise operators associated with the vacuum reservoir in normal order. We have found that the light emitted from the top level is in a chaotic state in any regime of laser operation. On the other hand, the light emitted from the intermediate level is in a coherent state when the laser is operating well above threshold and is chaotic when the laser is operating at threshold. The maximum quadrature squeezing of the superposed light modes is found to be 50 % below the vacuum-state level. We have shown that the maximum local quadrature squeezing is 64.5 %. Moreover the local quadrature squeezing increases with the frequency interval and approaches to the global quadrature squeezing.

Keywords: Stimulated emission, Photon statistics, Global and Local Quadrature squeezing.

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1 Introduction

A three-level laser is a quantum optical system in which light is generated by three-level atoms inside a cavity usually coupled to a vacuum reservoir. In one model of such a laser, three-level atoms initially prepared in a coherent superposition of the top and bottom levels are injected into a cavity and then removed after they have decayed due to spontaneous emission [1,2]. In another model, the top and bottom levels of the three-level atoms injected into a cavity are coupled by coherent light [3,4]. The statistical and squeezing properties of the light generated by three-level lasers have been investigated by several authors [5-11]. It is found that a three-level laser in either model generates squeezed light under certain conditions. It appears to be quite difficult to prepare the atoms in a coherent superposition of the top and bottom levels before they are injected into the laser cavity. In addition, it should certainly be hard to find out that the atoms have decayed spontaneously before they are removed from the cavity. On the other hand, the degree of squeezing of the light generated by the three-level laser, with the top and bottom levels coupled by coherent light, is relatively large when the mean photon number is relatively small [4].

Moreover, the quantum analysis of a three-level laser is usually carried out by including the interaction of the atoms inside the cavity with the vacuum reservoir outside the cavity. It may be possible to justify the feasibility of such interaction for a laser with an open cavity into which and from which atoms are injected and removed. However, there cannot be any valid justification for leaving open the laser cavity in which the available atoms are pumped to the top level by electron bombardment or by coherent light. Therefore, the aforementioned interaction is not feasible for a laser in which the atoms available in a closed cavity are pumped to the top level by means of electron bombardment.

We seek here to analyze the quantum properties of the light emitted by the three-level atoms available in a closed cavity and pumped to the top level at a constant rate. Thus taking into account the interaction of the three-level atoms with a resonant cavity light mode and the damping of the cavity light by a vacuum reservoir, we study the photon statistics of the cavity light and the quadrature squeezing of the cavity (output) light. We also determine the quadrature squeezing of the cavity (output) light in any frequency interval. We carry out our calculation by putting the noise operators associated with the vacuum reservoir in normal order and without considering the interaction of the three-level atoms with the vacuum reservoir outside the cavity.

2 Operator Dynamics

In this section, we consider the case in which N degenerate three-level atoms in cascade configuration are available in a closed cavity. We denote the top, intermediate, and bottom levels of these atoms by $|a\rangle_k$, $|b\rangle_k$, and $|c\rangle_k$, respectively. We prefer to call the light emitted from the top level light mode a and the one emitted from the intermediate level light mode b. In addition, we assume the cavity modes to be at resonance with the two transitions $|a\rangle_k \rightarrow |b\rangle_k$ and $|b\rangle_k \rightarrow |c\rangle_k$, with direct transition between levels $|a\rangle_k$ and $|c\rangle_k$ to be dipole forbidden. The interaction of one of the three-level atoms with the cavity modes can be described at resonance by the Hamiltonian

$$\hat{H} = ig(\hat{\sigma}_a^{\dagger k}\hat{a} - \hat{a}^\dagger\hat{\sigma}_a^k + \hat{\sigma}_b^{\dagger k}\hat{b} - \hat{b}^\dagger\hat{\sigma}_b^k), \tag{2.1}$$

where

$$\hat{\sigma}_a^k = |b\rangle_k\,{}_k\langle a| \tag{2.2}$$

and

$$\hat{\sigma}_b^k = |c\rangle_k\,{}_k\langle b| \tag{2.3}$$

are lowering atomic operators, and $\hat{a}$ and $\hat{b}$ are the annihilation operators for light modes a and b, with g being the coupling constant between the atom and the light mode a or b. We assume that the laser cavity is coupled to a vacuum reservoir via a single-port mirror.
In addition, we carry out our calculation by putting the noise operators associated with the vacuum reservoir in normal order. Thus the noise operators will not have any effect on the dynamics of the cavity mode operators.

We next seek to obtain the equation of evolution for $\langle\hat{a}\rangle$ using the master equation. The master equation for a cavity mode coupled to a vacuum reservoir has the form

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H},\hat{\rho}] + \frac{\kappa}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) + \frac{\kappa}{2}(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}), \tag{2.4}$$

where κ is the cavity damping constant. On account of Eqs. (2.1), Eq.(2.4) can be written as

$$\begin{aligned}\frac{d\hat{\rho}}{dt} = g(&\hat{\sigma}_a^{\dagger k}\hat{a}\hat{\rho} - \hat{a}^\dagger\hat{\sigma}_a^k\hat{\rho} + \hat{\sigma}_b^{\dagger k}\hat{b}\hat{\rho} - \hat{b}^\dagger\hat{\sigma}_b^k\hat{\rho}\\ &-\hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{a} + \hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^k - \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{b} + \hat{\rho}\hat{b}^\dagger\hat{\sigma}_b^k)\\ &+\frac{\kappa}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}).\end{aligned} \tag{2.5}$$

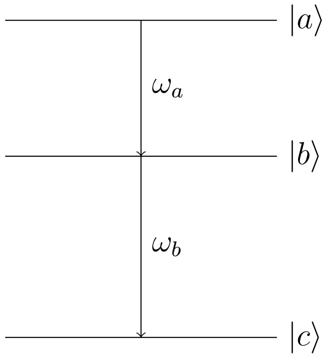

Figure 1: A three-level atom in a cascade configuration.

Employing the relation

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr\left(\frac{d\hat{\rho}}{dt}\hat{A}\right), \tag{2.6}$$

we can write the time evolution of $\langle\hat{a}\rangle$ as

$$\frac{d}{dt}\langle\hat{a}\rangle = Tr\left(\frac{d\hat{\rho}}{dt}\hat{a}\right). \tag{2.7}$$

On account of Eq.(2.5), Eq.(2.7) can be written as

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}\rangle = {}& gTr(\hat{\sigma}_a^{\dagger k}\hat{a}\hat{\rho}\hat{a} - \hat{a}^\dagger\hat{\sigma}_a^k\hat{\rho}\hat{a} + \hat{\sigma}_b^{\dagger k}\hat{b}\hat{\rho}\hat{a} - \hat{b}^\dagger\hat{\sigma}_b^k\hat{\rho}\hat{a} \\ & -\hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{a}^2 + \hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^k\hat{a} - \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{b}\hat{a} + \hat{\rho}\hat{b}^\dagger\hat{\sigma}_b^k\hat{a}), \\ & +\frac{\kappa}{2}Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2) \\ = {}& g(Tr(\hat{\rho}\hat{a}\hat{\sigma}_a^{\dagger k}\hat{a} - \hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{a}^2) + Tr(-\hat{\rho}\hat{a}\hat{a}^\dagger\hat{\sigma}_a^k + \hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^k\hat{a}) \\ & +Tr(\hat{\rho}\hat{a}\hat{\sigma}_b^{\dagger k}\hat{b} - \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{b}\hat{a}) + Tr(-\hat{\rho}\hat{a}\hat{b}^\dagger\hat{\sigma}_b^k + \hat{\rho}\hat{b}^\dagger\hat{\sigma}_b^k\hat{a})) \\ & +\frac{\kappa}{2}Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2). \end{aligned} \tag{2.8}$$

Evaluating the terms in the above equation, one easily gets

$$Tr(\hat{\rho}\hat{a}\hat{\sigma}_a^{\dagger k}\hat{a} - \hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{a}^2) = Tr(\hat{\rho}\hat{a}|a\rangle_{k\;k}\langle b|\hat{a} - \hat{\rho}|a\rangle_{k\;k}\langle b|\hat{a}^2). \tag{2.9}$$

On using cyclic property of the trace operation, there follows

$$Tr(\hat{\rho}\hat{a}\hat{\sigma}_a^{\dagger k}\hat{a} - \hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{a}^2) = {}_k\langle b|\hat{a}^2|a\rangle_k - {}_k\langle b|\hat{a}^2|a\rangle_k = 0. \tag{2.10}$$

$$Tr(-\hat{\rho}\hat{a}\hat{a}^\dagger\hat{\sigma}_a^k + \hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^k\hat{a}) = Tr(-\hat{\rho}\hat{a}\hat{a}^\dagger\hat{\sigma}_a^k) + Tr(\hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^k\hat{a}). \tag{2.11}$$

Upon substituting

$$\hat{a}\hat{a}^\dagger = 1 + \hat{a}^\dagger\hat{a}, \tag{2.12}$$

we have

$$\begin{aligned}
Tr(-\hat{\rho}\hat{a}\hat{a}^\dagger\hat{\sigma}_a^k + \hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^k\hat{a}) &= -Tr(\hat{\rho}(1 + \hat{a}^\dagger\hat{a})\hat{\sigma}_a^k) + Tr(\hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^k\hat{a}), \\
&= -Tr(\hat{\rho}\hat{\sigma}_a^k) - Tr(\hat{\rho}\hat{a}^\dagger\hat{a}|b\rangle_{kk}\langle a|) \\
&\quad +Tr(\hat{\rho}\hat{a}^\dagger|b\rangle_{kk}\langle a|\hat{a}), \\
&= -\langle\hat{\sigma}_a^\kappa\rangle - {}_k\langle a|\hat{a}^\dagger\hat{a}|b\rangle_k + {}_k\langle a|\hat{a}\hat{a}^\dagger|b\rangle_k, \\
&= -\langle\hat{\sigma}_a^\kappa\rangle - n\,{}_k\langle a|b\rangle_k + (1+n)\,{}_k\langle a|b\rangle_k, \\
&= -\langle\hat{\sigma}_a^\kappa\rangle.
\end{aligned} \tag{2.13}$$

$$Tr(\hat{\rho}\hat{a}\hat{\sigma}_b^{\dagger k}\hat{b} - \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{b}\hat{a}) = Tr(\hat{\rho}\hat{a}|b\rangle_k\,{}_k\langle c|\hat{b} - \hat{\rho}|b\rangle_k\,{}_k\langle c|\hat{b}\hat{a}). \tag{2.14}$$

Applying cyclic property of the trace operation, we get

$$Tr(\hat{\rho}\hat{a}\hat{\sigma}_b^{\dagger k}\hat{b} - \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{b}\hat{a}) = {}_k\langle c|\hat{b}\hat{a}|b\rangle_k - {}_k\langle c|\hat{b}\hat{a}|b\rangle_k = 0. \tag{2.15}$$

$$Tr(-\hat{\rho}\hat{a}\hat{b}^\dagger\hat{\sigma}_b^k + \hat{\rho}\hat{b}^\dagger\hat{\sigma}_b^k\hat{a}) = Tr(-\hat{\rho}\hat{a}\hat{b}^\dagger|c\rangle_k\,{}_k\langle b| + \hat{\rho}\hat{b}^\dagger|c\rangle_k\,{}_k\langle b|\hat{a}). \tag{2.16}$$

Using the cyclic property of the trace operation, one easily finds

$$Tr(-\hat{\rho}\hat{a}\hat{b}^\dagger\hat{\sigma}_b^k + \hat{\rho}\hat{b}^\dagger\hat{\sigma}_b^k\hat{a}) = -\,{}_k\langle b|\hat{a}\hat{b}^\dagger|c\rangle_k + {}_k\langle b|\hat{a}\hat{b}^\dagger|c\rangle_k = 0. \tag{2.17}$$

$$\frac{\kappa}{2}Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2) = \frac{\kappa}{2}Tr(2\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2), \tag{2.18}$$

$$= \frac{\kappa}{2}Tr(\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}). \tag{2.19}$$

Introducing Eq (2.12) into the above equation, we have

$$\begin{aligned}
\frac{\kappa}{2}Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2) &= \frac{\kappa}{2}Tr(\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}(\hat{a}^\dagger\hat{a}+1)\hat{a}), \\
&= \frac{\kappa}{2}Tr(\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}\hat{a}), \\
&= -\frac{\kappa}{2}Tr(\hat{\rho}\hat{a}), \\
&= -\frac{\kappa}{2}\langle\hat{a}\rangle.
\end{aligned} \tag{2.20}$$

Now upon substituting Eqs. (2.13) and (2.20) into Eq (2.8), we arrive at

$$\frac{d}{dt}\langle \hat{a}\rangle = -\frac{\kappa}{2}\langle \hat{a}\rangle - g\langle \hat{\sigma}_a^k\rangle. \tag{2.21}$$

Based on this result, we can write that

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - g\hat{\sigma}_a^k + \hat{F}(t),$$

where $\hat{F}(t)$ is a noise operator. In this project, we carry out our analysis for a cavity mode coupled to a vacuum reservoir and by putting the noise operator in normal order, then the noise operator associated with the vacuum reservoir has no effect on the dynamics of the cavity mode operators. In view of this, we drop the the noise operator $\hat{F}(t)$ and rewrite Eq. (2.21) as

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - g\hat{\sigma}_a^k. \tag{2.22}$$

Following a similar procedure, one can establish that

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} - g\hat{\sigma}_b^k. \tag{2.23}$$

Furthermore, applying the relation

$$\frac{d}{dt}\langle \hat{A}\rangle = -i\langle[\hat{A}, \hat{H}]\rangle, \tag{2.24}$$

we can write the time evolution of $\langle\hat{\sigma}_a\rangle$ as

$$\frac{d}{dt}\langle \hat{\sigma}_a^k\rangle = -i\langle[\hat{\sigma}_a^k, \hat{H}]\rangle, \tag{2.25}$$

$$= -i\langle(\hat{\sigma}_a^k\hat{H} - \hat{H}\hat{\sigma}_a^k)\rangle. \tag{2.26}$$

In view of Eq. (2.1), one can write the first term on the right side of the above equation as

$$\hat{\sigma}_a^k\hat{H} = ig(\hat{\sigma}_a^k\hat{\sigma}_a^{\dagger k}\hat{a} - \hat{\sigma}_a^k\hat{a}^\dagger\hat{\sigma}_a^k + \hat{\sigma}_a^k\hat{\sigma}_b^{\dagger k}\hat{b} - \hat{\sigma}_a^k\hat{b}^\dagger\hat{\sigma}_b^k). \tag{2.27}$$

Evaluating the terms in the above equation, one easily gets

$$\begin{aligned}\hat{\sigma}_a^k\hat{\sigma}_a^{\dagger k}\hat{a} &= |b\rangle_{k\,k}\langle a|a\rangle_{k\,k}\langle b|\hat{a},\\ &= |b\rangle_{k\,k}\langle b|\hat{a},\\ &= \hat{\eta}_b^k\hat{a},\end{aligned} \tag{2.28}$$

where

$$\hat{\eta}_b^k = |b\rangle_k\,{}_k\langle b|. \tag{2.29}$$

In addition,

$$\hat{\sigma}_a^k \hat{a}^\dagger \hat{\sigma}_a^k = |b\rangle_k\,{}_k\langle a|\hat{a}^\dagger|b\rangle_k\,{}_k\langle a| = a|b\rangle_k\,{}_k\langle a|b\rangle_k\,{}_k\langle a| = 0, \tag{2.30}$$

$$\hat{\sigma}_a^k \hat{\sigma}_b^{\dagger k} \hat{b} = |b\rangle_k\,{}_k\langle a|b\rangle_k\;{}_k\langle c|\hat{b} = 0, \tag{2.31}$$

$$\hat{\sigma}_a^k \hat{b}^\dagger \hat{\sigma}_b^k = |b\rangle_k\,{}_k\langle a|\hat{b}^\dagger|c\rangle_k\,{}_k\langle b| = 0. \tag{2.32}$$

Upon substituting Eq. (2.28) into Eq. (2.27), we get

$$\hat{\sigma}_a^k \hat{H} = ig\hat{\eta}_b^k \hat{a}. \tag{2.33}$$

Moreover, the second term in Eq. (2.26) can be written as

$$\hat{H}\hat{\sigma}_a^k = ig(\hat{\sigma}_a^{\dagger k}\hat{a}\hat{\sigma}_a^k - \hat{a}^\dagger\hat{\sigma}_a^k\hat{\sigma}_a^k + \hat{\sigma}_b^{\dagger k}\hat{b}\hat{\sigma}_a^k - \hat{b}^\dagger\hat{\sigma}_b^k\hat{\sigma}_a^k). \tag{2.34}$$

Evaluating the terms in the above equation, we have

$$\begin{aligned}
\hat{\sigma}_a^{\dagger k}\hat{a}\hat{\sigma}_a^k &= |a\rangle_k\,{}_k\langle b|\hat{a}|b\rangle_k\,{}_k\langle a|,\\
&= |a\rangle_k\,{}_k\langle a|\hat{a},\\
&= \hat{\eta}_a^k\hat{a},
\end{aligned} \tag{2.35}$$

where

$$\hat{\eta}_a^k = |a\rangle_k\,{}_k\langle a|. \tag{2.36}$$

In addition, we see that

$$\hat{a}^\dagger\hat{\sigma}_a^k\hat{\sigma}_a^k = \hat{a}^\dagger|b\rangle_k\,{}_k\langle a|b\rangle_k\,{}_k\langle a| = 0, \tag{2.37}$$

$$\hat{\sigma}_b^{\dagger k}\hat{b}\hat{\sigma}_a^k = |b\rangle_k\,{}_k\langle c|\hat{b}|b\rangle_k\,{}_k\langle a| = 0, \tag{2.38}$$

$$\begin{aligned}
\hat{b}^\dagger\hat{\sigma}_b^k\hat{\sigma}_a^k &= \hat{b}^\dagger|c\rangle_k\,{}_k\langle b|b\rangle_k\;{}_k\langle a|,\\
&= \hat{b}^\dagger|c\rangle_k\,{}_k\langle a|,\\
&= \hat{b}^\dagger\hat{\sigma}_c^k,
\end{aligned} \tag{2.39}$$

where

$$\hat{\sigma}_c^k = |c\rangle_k\,{}_k\langle a|. \tag{2.40}$$

Substitution of Eqs. (2.35) and (2.39) into (2.34) yields

$$\hat{H}\hat{\sigma}_a^k = ig(\hat{\eta}_a^k\hat{a} - \hat{b}^\dagger\hat{\sigma}_c^k). \tag{2.41}$$

Finally introducing Eqs.(2.33) and (2.41) into (2.26), we arrive at

$$\frac{d}{dt}\langle\hat{\sigma}_a^k\rangle = -i\langle ig\hat{\eta}_b^k\hat{a} - ig(\hat{\eta}_a^k\hat{a} - \hat{b}^\dagger\hat{\sigma}_c^k)\rangle,$$

$$= g\langle(\hat{\eta}_b^k - \hat{\eta}_a^k)\hat{a}\rangle + g\langle\hat{b}^\dagger\hat{\sigma}_c^k\rangle. \tag{2.42}$$

In similar manner, we can also establish that

$$\frac{d}{dt}\langle\hat{\sigma}_b^k\rangle = g\langle(\hat{\eta}_c^k - \hat{\eta}_b^k)\hat{b}\rangle - g\langle\hat{a}^\dagger\hat{\sigma}_c^k\rangle, \tag{2.43}$$

where $\hat{\eta}_c^k = |c\rangle_{k\,k}\langle c|$.

We next proceed to drive the equation of evolution for $\langle\hat{\eta}_a^k\rangle$. Once more, application of Eq. (2.24) for $\langle\hat{\eta}_a^k\rangle$ gives

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = -i\langle[\hat{\eta}_a^k, \hat{H}]\rangle, \tag{2.44}$$

$$= -i\langle(\hat{\eta}_a^k\hat{H} - \hat{H}\hat{\eta}_a^k)\rangle. \tag{2.45}$$

The first term in the above equation can be written as

$$\hat{\eta}_a^k\hat{H} = ig(\hat{\eta}_a^k\hat{\sigma}_a^{\dagger k}\hat{a} - \hat{\eta}_a^k\hat{a}^\dagger\hat{\sigma}_a + \hat{\eta}_a^k\hat{\sigma}_b^{\dagger k}\hat{b} - \hat{\eta}_a^k\hat{b}^\dagger\hat{\sigma}_b). \tag{2.46}$$

Evaluating the terms on the right side of the above equation, we have

$$\hat{\eta}_a^k\hat{\sigma}_a^{\dagger k}\hat{a} = |a\rangle_{k\,k}\langle a|a\rangle_{k\,k}\langle b|\hat{a},$$

$$= |a\rangle_{k\,k}\langle b|\hat{a},$$

$$= \hat{\sigma}_a^{\dagger k}\hat{a}, \tag{2.47}$$

$$\hat{\eta}_a^k\hat{a}^\dagger\hat{\sigma}_a^k = |a\rangle_{k\,k}\langle a|\hat{a}^\dagger|b\rangle_{k\,k}\langle a| = 0,$$

$$\hat{\eta}_a^k\hat{\sigma}_b^{\dagger k}\hat{b} = |a\rangle_{k\,k}\langle a|b\rangle_{k\,k}\langle c|\hat{b} = 0,$$

and

$$\hat{\eta}_a^k\hat{b}^\dagger\hat{\sigma}_b^k = |a\rangle_{k\,k}\langle a|\hat{b}^\dagger|c\rangle_{k\,k}\langle b| = 0.$$

Substituting Eq. (2.47) into (2.46), we get

$$\hat{\eta}_a^k\hat{H} = ig\hat{\sigma}_a^{\dagger k}\hat{a}. \tag{2.48}$$

Moreover, the second term in Eq. (2.45) can be written as

$$\hat{H}\hat{\eta}_a^k = ig(\hat{\sigma}_a^{\dagger k}\hat{a}\hat{\eta}_a^k - \hat{a}^\dagger\hat{\sigma}_a^k\hat{\eta}_a^k + \hat{\sigma}_b^{\dagger k}\hat{b}\hat{\eta}_a^k - \hat{b}^\dagger\hat{\sigma}_b^k\hat{\eta}_a^k). \tag{2.49}$$

Evaluating the terms in the above equation, one easily gets

$$\hat{\sigma}_a^{\dagger k}\hat{a}\hat{\eta}_a^k = |a\rangle_{k\,k}\langle b|\hat{a}|a\rangle_{k\,k}\langle a| = |a\rangle_{k\,k}\langle b|a\rangle_{k\,k}\langle a|\hat{a} = 0,$$

$$\begin{aligned}\hat{a}^\dagger\hat{\sigma}_a^k\hat{\eta}_a^k &= \hat{a}^\dagger|b\rangle_{k\,k}\langle a|a\rangle_{k\,k}\langle a|,\\ &= \hat{a}^\dagger|b\rangle_{k\,k}\langle a|,\\ &= \hat{a}^\dagger\hat{\sigma}_a^k,\end{aligned} \tag{2.50}$$

$$\hat{\sigma}_b^{\dagger k}\hat{b}\hat{\eta}_a^k = |b\rangle_{k\,k}\langle c|\hat{b}|a\rangle_{k\,k}\langle a| = |b\rangle_{k\,k}\langle c|a\rangle_{k\,k}\langle a|\hat{b} = 0,$$

and

$$\hat{b}^\dagger\hat{\sigma}_b^k\hat{\eta}_a^k = \hat{b}^\dagger|c\rangle_{k\,k}\langle b|a\rangle_{k\,k}\langle a| = 0.$$

With the aid of Eq. (2.50), one can write Eq. (2.49) as

$$\hat{H}\hat{\eta}_a^k = ig\hat{a}^\dagger\hat{\sigma}_a^k. \tag{2.51}$$

On account of Eqs. (2.47) and (2.50), Eq. (2.44) can be written as

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = -i\langle(ig\hat{\sigma}_a^{\dagger k}\hat{a} + ig\hat{a}^\dagger\hat{\sigma}_a^k)\rangle, \tag{2.52}$$

$$= g\langle\hat{\sigma}_a^{\dagger k}\hat{a}\rangle + g\langle\hat{a}^\dagger\hat{\sigma}_a^k\rangle. \tag{2.53}$$

Following a similar procedure, one can establish that

$$\frac{d}{dt}\langle\hat{\eta}_b^k\rangle = g\langle\hat{\sigma}_b^{\dagger k}\hat{b}\rangle + g\langle\hat{b}^\dagger\hat{\sigma}_b^k\rangle - g\langle\hat{\sigma}_a^{\dagger k}\hat{a}\rangle - g\langle\hat{a}^\dagger\hat{\sigma}_a^k\rangle \tag{2.54}$$

and

$$\frac{d}{dt}\langle\hat{\eta}_c^k\rangle = -g\langle\hat{\sigma}_b^{\dagger k}\hat{b}\rangle - g\langle\hat{b}^\dagger\hat{\sigma}_b^k\rangle. \tag{2.55}$$

We see Eqs. (2.42), (2.43), (2.53), (2.54) and (2.55) are nonlinear differential equations and hence it is not possible to find the exact time-dependent solutions of these equations. We intend to overcome this problem by applying the large-time approximation [12]. Then using this approximation scheme, we get from Eqs, (2.22) and (2.23) the approximately valid relations

$$\hat{a} = -\frac{2g}{\kappa}\hat{\sigma}_a^k \tag{2.56}$$

and

$$\hat{b} = -\frac{2g}{\kappa}\hat{\sigma}_b^k. \tag{2.57}$$

Evidently, these turns out to be exact relations at steady state. We next combine Eqs (2.56) and (2.57) with Eq (2.42).

$$\frac{d}{dt}\langle\hat{\sigma}_a^k\rangle = g\langle(\hat{\eta}_b^k - \hat{\eta}_a^k)(-\frac{2g}{\kappa}\hat{\sigma}_a^k)\rangle + g\langle\frac{-2g}{\kappa}\hat{\sigma}_b^{\dagger k}\hat{\sigma}_c^k\rangle,$$

$$= -\frac{2g^2}{\kappa}\langle(\hat{\eta}_b^k\hat{\sigma}_a^k - \hat{\eta}_a^k\hat{\sigma}_a^k + \hat{\sigma}_b^{\dagger k}\hat{\sigma}_c^k)\rangle. \tag{2.58}$$

Evaluating the terms on the right side of Eq. (2.58), we readily get

$$\hat{\eta}_b^k\hat{\sigma}_a^k = |b\rangle_{k\,k}\langle b|b\rangle_{k\,k}\langle a|,$$

$$= |b\rangle_{k\,k}\langle a|,$$

$$= \hat{\sigma}_a^k, \tag{2.59}$$

$$\hat{\eta}_a^k\hat{\sigma}_a^k = |a\rangle_{k\,k}\langle a|b\rangle_{k\,k}\langle a| = 0,$$

and

$$\hat{\sigma}_b^{\dagger k}\hat{\sigma}_c^k = |b\rangle_{k\,k}\langle c|c\rangle_{k\,k}\langle a|,$$

$$= |b\rangle_{k\,k}\langle a|,$$

$$= \hat{\sigma}_a^k. \tag{2.60}$$

Now taking into account Eqs (2.59) and (2.60), we arrive at

$$\frac{d}{dt}\langle\hat{\sigma}_a^k\rangle = -\frac{2g^2}{\kappa}\langle(\hat{\sigma}_a^k + \hat{\sigma}_a^k)\rangle,$$

$$= -\frac{4g^2}{\kappa}\langle\hat{\sigma}_a^k\rangle,$$

$$= -\gamma_c\langle\hat{\sigma}_a^k\rangle, \tag{2.61}$$

where

$$\gamma_c = \frac{4g^2}{\kappa} \tag{2.62}$$

is the stimulated emission decay constant.
One can also show that

$$\frac{d}{dt}\langle\hat{\sigma}_b^k\rangle = -\frac{\gamma_c}{2}\langle\hat{\sigma}_b^k\rangle. \tag{2.63}$$

Furthermore, introducing Eqs (2.56) and (2.57) into Eq (2.53), we find

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = g\langle\hat{\sigma}_a^{\dagger k}(-\frac{2g}{\kappa}\hat{\sigma}_a^k)\rangle + g\langle\frac{-2g}{\kappa}\hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k\rangle, \tag{2.64}$$

$$= -\frac{2g^2}{\kappa}\langle \hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k + \hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k\rangle,$$

$$= -\frac{4g^2}{\kappa}\langle \hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k\rangle. \tag{2.65}$$

Applying

$$\hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k = |a\rangle_{k\,k}\langle b|b\rangle_{k\,k}\langle a| = \hat{\eta}_a^k \tag{2.66}$$

along with Eq (2.62) in (2.65), we get

$$\frac{d}{dt}\langle \hat{\eta}_a^k\rangle = -\gamma_c\langle \hat{\eta}_a^k\rangle. \tag{2.67}$$

In similar manner, one can establish that

$$\frac{d}{dt}\langle \hat{\eta}_b^k\rangle = -\gamma_c\langle \hat{\eta}_b^k\rangle + \gamma_c\langle \hat{\eta}_a^k\rangle \tag{2.68}$$

and

$$\frac{d}{dt}\langle \hat{\eta}_c^k\rangle = \gamma_c\langle \hat{\eta}_b^k\rangle. \tag{2.69}$$

The three-level atoms available in the cavity are pumped from the bottom to the top level by means of electron bombardment. The pumping process must surely affect the dynamics of $\langle \hat{\eta}_a^k\rangle$ and $\langle \hat{\eta}_c^k\rangle$. If r_a represents the rate at which a single atom is pumped from the bottom to the top level, then $\langle \hat{\eta}_a^k\rangle$ increases at the rate of $r_a\langle \hat{\eta}_c^k\rangle$ and $\langle \hat{\eta}_c^k\rangle$ decreases at the same rate. In view of this, we rewrite Eqs. (2.67) and (2.69) as [13]

$$\frac{d}{dt}\langle \hat{\eta}_a^k\rangle = -\gamma_c\langle \hat{\eta}_a^k\rangle + r_a\langle \hat{\eta}_c^k\rangle \tag{2.70}$$

and

$$\frac{d}{dt}\langle \hat{\eta}_c^k\rangle = \gamma_c\langle \hat{\eta}_b^k\rangle - r_a\langle \hat{\eta}_c^k\rangle. \tag{2.71}$$

We next sum Eqs (2.61), (2.63), (2.68), (2.70), and (2.71) over the N three-level atoms, so that

$$\frac{d}{dt}\langle \hat{m}_a\rangle = -\gamma_c\langle \hat{m}_a\rangle, \tag{2.72}$$

$$\frac{d}{dt}\langle \hat{m}_b\rangle = -\frac{\gamma_c}{2}\langle \hat{m}_b\rangle, \tag{2.73}$$

$$\frac{d}{dt}\langle \hat{N}_a\rangle = -\gamma_c\langle \hat{N}_a\rangle + r_a\langle \hat{N}_c\rangle, \tag{2.74}$$

$$\frac{d}{dt}\langle\hat{N}_b\rangle = -\gamma_c\langle\hat{N}_b\rangle + \gamma_c\langle\hat{N}_a\rangle, \tag{2.75}$$

$$\frac{d}{dt}\langle\hat{N}_c\rangle = \gamma_c\langle\hat{N}_b\rangle - r_a\langle\hat{N}_c\rangle, \tag{2.76}$$

in which

$$\hat{m}_a = \sum_{k=1}^{N}\hat{\sigma}_a^k, \tag{2.77}$$

$$\hat{m}_b = \sum_{k=1}^{N}\hat{\sigma}_b^k, \tag{2.78}$$

$$\hat{N}_a = \sum_{k=1}^{N}\hat{\eta}_a^k, \tag{2.79}$$

$$\hat{N}_b = \sum_{k=1}^{N}\hat{\eta}_b^k, \tag{2.80}$$

$$\hat{N}_c = \sum_{k=1}^{N}\hat{\eta}_c^k, \tag{2.81}$$

with the operators $\hat{N}_a, \hat{N}_b$ and $\hat{N}_c$ representing the number of atoms in the top, intermediate, and bottom levels. In view of Eqs. (2.79), (2.80), (2.81), one can write

$$\langle\hat{N}_a\rangle = \sum_{k=1}^{N}\langle\hat{\eta}_a^k\rangle, \tag{2.82}$$

$$\langle\hat{N}_b\rangle = \sum_{k=1}^{N}\langle\hat{\eta}_b^k\rangle, \tag{2.83}$$

$$\langle\hat{N}_c\rangle = \sum_{k=1}^{N}\langle\hat{\eta}_c^k\rangle. \tag{2.84}$$

Summation of the above three equations gives

$$\langle\hat{N}_a\rangle + \langle\hat{N}_b\rangle + \langle\hat{N}_c\rangle = \sum_{k=1}^{N}\langle\hat{\eta}_a^k\rangle + \sum_{k=1}^{N}\langle\hat{\eta}_b^k\rangle + \sum_{k=1}^{N}\langle\hat{\eta}_c^k\rangle, \tag{2.85}$$

$$= \sum_{k=1}^{N}(\langle\hat{\eta}_a^k\rangle + \langle\hat{\eta}_b^k\rangle + \langle\hat{\eta}_c^k\rangle). \tag{2.86}$$

Employing the completeness relation

$$\hat{\eta}_a^k + \hat{\eta}_b^k + \hat{\eta}_c^k = \hat{I} \tag{2.87}$$

in Eq (2.86), we get

$$\langle \hat{N}_a \rangle + \langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle = \sum_{k=1}^{N} \langle \hat{I} \rangle = N. \tag{2.88}$$

Furthermore, setting for any k in the definition given by Eq (2.2), we have

$$\hat{\sigma}_a^k = |b\rangle\langle a|. \tag{2.89}$$

Applying Eq (2.89) in (2.77), one can write

$$\hat{m}_a = \sum_{k=1}^{N} |b\rangle\langle a|. \tag{2.90}$$

Upon replacing $\sum\limits_{k=1}^{N}$ by N, we see that

$$\hat{m}_a = N|b\rangle\langle a|. \tag{2.91}$$

Following the same procedure, one can easily establish that

$$\hat{m}_b = N|c\rangle\langle b|, \tag{2.92}$$

$$\hat{m}_c = N|c\rangle\langle a|, \tag{2.93}$$

where

$$\hat{m}_c = \sum_{k=1}^{N} \hat{\sigma}_c^k, \tag{2.94}$$

and $\hat{m}_b$ is defined by Eq (2.78).

Moreover, according to Eq (2.82), we have

$$\hat{N}_a = \sum_{k=1}^{N} \hat{\eta}_a^k. \tag{2.95}$$

Upon setting $\hat{\eta}_a^k = |a\rangle\langle a|$, we see that

$$\hat{N}_a = \sum_{k=1}^{N} |a\rangle\langle a|. \tag{2.96}$$

It then follows that

$$\hat{N}_a = N|a\rangle\langle a|. \tag{2.97}$$

Following the same procedure, we can easily establish that

$$\hat{N}_b = N|b\rangle\langle b|, \tag{2.98}$$

$$\hat{N}_c = N|c\rangle\langle c|. \tag{2.99}$$

Moreover, using the definition

$$\hat{m} = \hat{m}_a + \hat{m}_b, \tag{2.100}$$

we see that

$$\hat{m}^\dagger = \hat{m}_a^\dagger + \hat{m}_b^\dagger, \tag{2.101}$$

where

$$\hat{m}_a^\dagger = N|a\rangle\langle b|, \tag{2.102}$$

and

$$\hat{m}_b^\dagger = N|b\rangle\langle c|. \tag{2.103}$$

Combination of Eqs. (2.100) and (2.101) yields

$$\hat{m}^\dagger\hat{m} = (\hat{m}_a^\dagger + \hat{m}_b^\dagger)(\hat{m}_a + \hat{m}_b), \tag{2.104}$$

$$= \hat{m}_a^\dagger\hat{m}_a + \hat{m}_a^\dagger\hat{m}_b + \hat{m}_b^\dagger\hat{m}_a + \hat{m}_b^\dagger\hat{m}_b. \tag{2.105}$$

Upon evaluating the terms on the right side of the above equation, we get

$$\hat{m}_a^\dagger\hat{m}_a = N|a\rangle\langle b|N|b\rangle\langle a|,$$

$$= N^2|a\rangle\langle b|b\rangle\langle a|,$$

$$= N^2|a\rangle\langle a|,$$

$$= N\hat{N}_a, \tag{2.106}$$

$$\hat{m}_a^\dagger\hat{m}_b = N|a\rangle\langle b|N|c\rangle\langle b| = N^2|a\rangle\langle b|c\rangle\langle b| = 0,$$

$$\hat{m}_b^\dagger\hat{m}_a = N|b\rangle\langle c|N|b\rangle\langle a| = N^2|b\rangle\langle c|b\rangle\langle a| = 0,$$

$$\hat{m}_b^\dagger\hat{m}_b = N|b\rangle\langle c|N|c\rangle\langle b|,$$

$$= N^2|b\rangle\langle c|c\rangle\langle b|,$$

$$= N^2|b\rangle\langle b|,$$

$$= N\hat{N}_b. \tag{2.107}$$

Substitution of Eq. (2.106) and (2.107) into (2.105), gives

$$\hat{m}^\dagger\hat{m} = N(\hat{N}_a + \hat{N}_b). \tag{2.108}$$

We next seek to establish the expression for $\hat{m}\hat{m}^\dagger$. Applying Eqs. (2.100) and (2.101), one can write

$$\hat{m}\hat{m}^\dagger = (\hat{m}_a + \hat{m}_b)(\hat{m}_a^\dagger + \hat{m}_b^\dagger), \tag{2.109}$$

$$= \hat{m}_a\hat{m}_a^\dagger + \hat{m}_a\hat{m}_b^\dagger + \hat{m}_b\hat{m}_a^\dagger + \hat{m}_b\hat{m}_b^\dagger. \tag{2.110}$$

Evaluating the terms on the right side of the above equation, we have

$$\hat{m}_a\hat{m}_a^\dagger = N|b\rangle\langle a|N|a\rangle\langle b|,$$

$$= N^2|b\rangle\langle a|a\rangle\langle b|,$$

$$= N^2|b\rangle\langle b| = N\hat{N}_b,$$

$$= N\hat{N}_b, \tag{2.111}$$

$$\hat{m}_a\hat{m}_b^\dagger = N|b\rangle\langle a|N|b\rangle\langle c| = N^2|b\rangle\langle a|b\rangle\langle c| = 0,$$

$$\hat{m}_b\hat{m}_a^\dagger = N|c\rangle\langle b|N|a\rangle\langle b| = N^2|c\rangle\langle b|a\rangle\langle b| = 0,$$

$$\hat{m}_b\hat{m}_b^\dagger = N|c\rangle\langle b|N|b\rangle\langle c|,$$

$$= N^2|c\rangle\langle b|b\rangle\langle c|,$$

$$= N^2|c\rangle\langle c|,$$

$$= N\hat{N}_c. \tag{2.112}$$

Introducing Eqs. (2.111) and (2.112) in (2.110), we have

$$\hat{m}\hat{m}^\dagger = N(\hat{N}_b + \hat{N}_c). \tag{2.113}$$

We next proceed to determine $\hat{m}^2$. Using the definition given by Eq (2.100), one can write

$$\hat{m}^2 = (\hat{m}_a + \hat{m}_b)(\hat{m}_a + \hat{m}_b), \tag{2.114}$$

$$= \hat{m}_a\hat{m}_a + \hat{m}_a\hat{m}_b + \hat{m}_b\hat{m}_a + \hat{m}_b\hat{m}_b. \tag{2.115}$$

Evaluating the terms on the right side of the above equation, we get

$$\hat{m}_a\hat{m}_a = N|b\rangle\langle a|N|b\rangle\langle a| = N^2|b\rangle\langle a|b\rangle\langle a| = 0,$$

$$\hat{m}_a\hat{m}_b = N|b\rangle\langle a|N|c\rangle\langle b| = N^2|b\rangle\langle a|c\rangle\langle b| = 0,$$

$$\begin{aligned} \hat{m}_b\hat{m}_a &= N|c\rangle\langle b|N|b\rangle\langle a|, \\ &= N^2|c\rangle\langle b|b\rangle\langle a|, \\ &= N^2|c\rangle\langle a|, \\ &= N\hat{m}_c, \end{aligned} \tag{2.116}$$

$$\hat{m}_b\hat{m}_b = N|c\rangle\langle b|N|c\rangle\langle b| = NN|c\rangle\langle b|c\rangle\langle b| = 0.$$

Upon Substituting Eq (2.116) into Eq (2.115), there follows

$$\hat{m}^2 = N\hat{m}_c. \tag{2.117}$$

In the presence of N three-level atoms, we rewrite Eq. (2.22) as [13]

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} + \lambda\hat{m}_a, \tag{2.118}$$

in which λ is a constant whose value remains to be fixed. Applying the steady-state solution of Eq. (2.22), we get

$$[\hat{a}, \hat{a}^\dagger]_k = \frac{\gamma_c}{\kappa}(\hat{\eta}_b^k - \hat{\eta}_a^k) \tag{2.119}$$

and on summing over all atoms, we have

$$[\hat{a}, \hat{a}^\dagger] = \frac{\gamma_c}{\kappa}(\hat{N}_b - \hat{N}_a), \tag{2.120}$$

where

$$[\hat{a}, \hat{a}^\dagger] = \sum_{k=1}^{N}[\hat{a}, \hat{a}^\dagger]_k \tag{2.121}$$

stands for the commutator of $\hat{a}$ and $\hat{a}^\dagger$ when light mode a is interacting with all the N three-level atoms. The steady state solution of Eq (2.118) is found to be

$$\hat{a} = \frac{2\lambda\hat{m}_a}{\kappa}. \tag{2.122}$$

Using Eq (2.122) along with its adjoint, we can write the commutator of $\hat{a}$ and $\hat{a}^\dagger$ as

$$\begin{aligned}[\hat{a},\hat{a}^\dagger] &= \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a},\\ &= \frac{2\lambda\hat{m}_a}{\kappa}\frac{2\lambda\hat{m}_a^\dagger}{\kappa} - \frac{2\lambda\hat{m}_a^\dagger}{\kappa}\frac{2\lambda\hat{m}_a}{\kappa},\\ &= \left(\frac{2\lambda}{\kappa}\right)^2(\hat{m}_a\hat{m}_a^\dagger - \hat{m}_a^\dagger\hat{m}_a). \qquad (2.123)\end{aligned}$$

Application of Eq (2.91) along with its adjoint in the above equation yields

$$\begin{aligned}[\hat{a},\hat{a}^\dagger] &= \left(\frac{2\lambda}{\kappa}\right)^2(N|b\rangle\langle a|N|a\rangle\langle b| - N|a\rangle\langle b|N|b\rangle\langle a|),\\ &= \left(\frac{2\lambda}{\kappa}\right)^2 N^2(|b\rangle\langle a|a\rangle\langle b| - a\rangle\langle b|b\rangle\langle a|),\\ &= \left(\frac{2\lambda}{\kappa}\right)^2 N^2(|b\rangle\langle b| - a\rangle\langle a|),\\ &= \left(\frac{2\lambda}{\kappa}\right)^2 N(N\hat{\eta}_b^K - N\hat{\eta}_a^K),\\ &= N\left(\frac{2\lambda}{\kappa}\right)^2(\hat{N}_b - \hat{N}_a). \qquad (2.124)\end{aligned}$$

Thus on account of Eqs (2.120) and (2.124), we see that

$$\lambda = \pm\frac{g}{\sqrt{N}} \qquad (2.125)$$

and in veiw of this result, Eq (2.118) can be written as

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} + \frac{g}{\sqrt{N}}\hat{m}_a. \qquad (2.126)$$

Following a similar procedure, one can also readily establish that

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} + \frac{g}{\sqrt{N}}\hat{m}_b, \qquad (2.127)$$

and

$$[\hat{b},\hat{b}^\dagger] = \frac{\gamma_c}{\kappa}(\hat{N}_c - \hat{N}_b). \qquad (2.128)$$

Now adding Eqs. (2.120) and (2.128), we have

$$[\hat{c},\hat{c}^\dagger] = \frac{\gamma_c}{\kappa}(\hat{N}_c - \hat{N}_a) \qquad (2.129)$$

and adding Eqs. (2.126) and (2.127), we have

$$\frac{d\hat{c}}{dt} = -\frac{\kappa}{2}\hat{c} + \frac{g}{\sqrt{N}}\hat{m}, \tag{2.130}$$

in which

$$\hat{c} = \hat{a} + \hat{b} \tag{2.131}$$

and $\hat{m}$ is defined by Eq (2.100). It is not hard to realize that the operators $\hat{c}$ and $\hat{c}^\dagger$ represent the superposition of light mode a and b . We notice that the commutation relation for the operators $\hat{a}$ and $\hat{a}^\dagger$ (or $\hat{b}$ and $\hat{b}^\dagger$) involves energy levels between which transition is dipole allowed. This may be taken as a signature that the operators $\hat{a}$ and $\hat{a}^\dagger$ (or $\hat{b}$ and $\hat{b}^\dagger$) represent a single mode light. On the other hand, we observe that the commutation relation for the operators $\hat{c}$ and $\hat{c}^\dagger$ involves energy levels between which transition is dipole forbidden. This my be taken as a signature that the operators $\hat{c}$ and $\hat{c}^\dagger$ represent a two-mode light [13].

3 Photon Statistics

We next seek to calculate the steady-state mean and variance of the photon number for light modes a and b and for the superposition of these light modes. However, we need first to establish several important relations. On account of Eq (2.88), we have

$$\langle \hat{N}_c \rangle = N - \langle \hat{N}_a \rangle - \langle \hat{N}_b \rangle. \tag{3.1}$$

Substitution of Eq (3.1) into Eq (2.74) yields

$$\frac{d}{dt}\langle \hat{N}_a \rangle = -\gamma_c \langle \hat{N}_a \rangle + r_a (N - \langle \hat{N}_a \rangle - \langle \hat{N}_b \rangle), \tag{3.2}$$

$$= -\gamma_c \langle \hat{N}_a \rangle + r_a N - r_a \langle \hat{N}_a \rangle - r_a \langle \hat{N}_b \rangle), \tag{3.3}$$

$$= -(\gamma_c + r_a)\langle \hat{N}_a \rangle + r_a (N - \langle \hat{N}_b \rangle). \tag{3.4}$$

Now application of the large-time approximation scheme to Eq (2.75) gives

$$\langle \hat{N}_b \rangle = \langle \hat{N}_a \rangle. \tag{3.5}$$

Thus on taking into account this result, we find the steady-state solution of Eq (3.4) to be

$$\langle \hat{N}_a \rangle = \frac{r_a N}{\gamma_c + 2 r_a}. \tag{3.6}$$

Using the steady-state solution of Eq (2.76), we have

$$\langle \hat{N}_c \rangle = \frac{\gamma_c}{r_a} \langle \hat{N}_a \rangle. \tag{3.7}$$

Applying the identity given by Eq (2.87), the state vector of a three-level atom can be put in the form

$$|\psi\rangle_k = C_a |a\rangle_k + C_b |b\rangle_k + C_c |c\rangle_k, \tag{3.8}$$

in which

$$C_a = {}_k\langle a|\psi\rangle_k, \tag{3.9}$$

$$C_b = {}_k\langle b|\psi\rangle_k, \tag{3.10}$$

and

$$C_c = {}_k\langle c|\psi\rangle_k. \tag{3.11}$$

The state vector described by Eq (3.8) can be used to determine the expectation value of an atomic operator formed by a pair of identical energy levels or by two distinct energy levels between which transition with the emission of a photon is dipole forbidden. Employing the relation

$$\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle, \tag{3.12}$$

we can write the expectation value of $\hat{\eta}_a^k$ as

$$\langle \hat{\eta}_a^k \rangle = {}_k\langle \psi | \hat{\eta}_a^k | \psi \rangle_k. \tag{3.13}$$

On account of Eq (2.36), the above equation takes the form

$$\langle \hat{\eta}_a^k \rangle = {}_k\langle \psi | a \rangle_k \, {}_k\langle a | \psi \rangle_k. \tag{3.14}$$

In view of Eq (3.9), the above equation can be written as

$$\langle \hat{\eta}_a^k \rangle = C_a^* C_a. \tag{3.15}$$

In a similar manner, we can also establish that

$$\langle \hat{\eta}_c^k \rangle = C_c^* C_c. \tag{3.16}$$

Employing once more Eq (3.12), the expectation value of $\hat{\sigma}_c^k$ can be written as

$$\langle \hat{\sigma}_c^k \rangle = {}_k\langle \psi | \hat{\sigma}_c^k | \psi \rangle_k. \tag{3.17}$$

On account of Eq (2.40) the above equation takes the form

$$\langle \hat{\sigma}_c^k \rangle = {}_k\langle \psi | c \rangle_k \, {}_k\langle a | \psi \rangle_k. \tag{3.18}$$

Applying Eq (3.9) and the adjoint of (3.11) in the above equation, we find

$$\langle \hat{\sigma}_c^k \rangle = C_c^* C_a. \tag{3.19}$$

Now on account of the above equation, we have

$$\langle \hat{\sigma}_c^k \rangle \langle \hat{\sigma}_c^k \rangle^* = C_c^* C_a C_c C_a^*, \tag{3.20}$$

$$|\langle \hat{\sigma}_c^k \rangle|^2 = C_a^* C_a C_c^* C_c, \tag{3.21}$$

$$= \langle \hat{\eta}_a^k \rangle \langle \hat{\eta}_c^k \rangle, \tag{3.22}$$

$$\langle \hat{\sigma}_c^k \rangle = \sqrt{\langle \hat{\eta}_a^k \rangle \langle \hat{\eta}_c^k \rangle}. \tag{3.23}$$

Finally on summing over k from 1 up to N, we get

$$\langle \hat{m}_c \rangle = \sqrt{\langle \hat{N}_a \rangle \langle \hat{N}_c \rangle}. \tag{3.24}$$

Furthermore, adding Eqs. (2.72) and (2.73), we have

$$\frac{d}{dt}\langle\hat{m}\rangle = -\frac{1}{2}\gamma_c\langle\hat{m}\rangle - \frac{1}{2}\gamma_c\langle\hat{m}_a\rangle. \tag{3.25}$$

In order to include the effect of pumping process, we rewrite this equation as

$$\frac{d\hat{m}}{dt} = -\frac{1}{2}\mu\hat{m} + \frac{1}{2}\mu\hat{m}_a + \hat{F}_m(t), \tag{3.26}$$

in which $\hat{F}_m(t)$ is a noise operator with vanishing mean and μ is a parameter whose value remains to be determined.

We next seek to obtain equation of evolution for $\left\langle\hat{m}^\dagger\hat{m}\right\rangle$. From Eq (3.26), we notice that

$$\frac{d\hat{m}^\dagger}{dt} = -\frac{1}{2}\mu\hat{m}^\dagger + \frac{1}{2}\mu\hat{m}_a^\dagger + \hat{F}_m^\dagger(t). \tag{3.27}$$

Upon introducing Eq (3.26) and (3.27) into the relation

$$\frac{d}{dt}\left\langle\hat{m}^\dagger\hat{m}\right\rangle = \left\langle\frac{d\hat{m}^\dagger}{dt}\hat{m}\right\rangle + \left\langle\hat{m}^\dagger\frac{d\hat{m}}{dt}\right\rangle, \tag{3.28}$$

there follows

$$\frac{d}{dt}\left\langle\hat{m}^\dagger\hat{m}\right\rangle = \left\langle(-\frac{1}{2}\mu\hat{m}^\dagger + \frac{1}{2}\mu\hat{m}_a^\dagger + \hat{F}_m^\dagger(t))\hat{m}\right\rangle$$

$$+\left\langle\hat{m}^\dagger(-\frac{1}{2}\mu\hat{m} + \frac{1}{2}\mu\hat{m}_a + \hat{F}_m(t))\right\rangle, \tag{3.29}$$

$$= \left\langle(-\frac{1}{2}\mu\hat{m}^\dagger\hat{m} + \frac{1}{2}\mu\hat{m}_a^\dagger\hat{m} + \hat{F}_m^\dagger(t)\hat{m}(t) - \frac{1}{2}\mu\hat{m}^\dagger\hat{m} + \frac{1}{2}\mu\hat{m}^\dagger\hat{m}_a + \hat{m}^\dagger(t)\hat{F}_m(t))\right\rangle, \tag{3.30}$$

$$= -\mu\langle\hat{m}^\dagger\hat{m}\rangle + \frac{1}{2}\mu(\langle\hat{m}_a^\dagger\hat{m}\rangle + \langle\hat{m}^\dagger\hat{m}_a\rangle) + \langle\hat{F}_m^\dagger(t)\hat{m}(t)\rangle + \langle\hat{m}^\dagger(t)\hat{F}_m(t)\rangle. \tag{3.31}$$

On account of Eqs.(2.100), (2.91), and (2.92) along with their adjoint, we see that

$$\frac{1}{2}\mu(\langle\hat{m}_a^\dagger\hat{m}\rangle + \langle\hat{m}^\dagger\hat{m}_a\rangle) = \frac{1}{2}\mu(\langle\hat{m}_a^\dagger\hat{m}_a + \hat{m}_a^\dagger\hat{m}_b\rangle + \langle\hat{m}_a^\dagger\hat{m}_a + \hat{m}_b^\dagger\hat{m}_a\rangle), \tag{3.32}$$

$$= \frac{1}{2}\mu(\langle N^2|a\rangle\langle b|b\rangle\langle a| + N^2|a\rangle\langle b|c\rangle\langle b|\rangle$$

$$+\langle N^2|a\rangle\langle b|b\rangle\langle a| + N^2|b\rangle\langle c|b\rangle\langle a|\rangle), \tag{3.33}$$

$$= \frac{1}{2}\mu(N^2\langle|a\rangle\langle a|\rangle + N^2\langle|a\rangle\langle a|\rangle), \tag{3.34}$$

$$= \mu N^2\langle|a\rangle\langle a|\rangle. \tag{3.35}$$

Furthermore, one can easily establish that

$$\langle\hat{m}_a^\dagger\hat{m}_a\rangle = \langle N|a\rangle\langle b|N|b\rangle\langle a|\rangle, \tag{3.36}$$

$$= N^2\langle|a\rangle\langle b|b\rangle\langle a|\rangle, \tag{3.37}$$

$$= N^2\langle|a\rangle\langle a|\rangle. \tag{3.38}$$

Hence, in view of the above equation, Eq (3.35) takes the form

$$\frac{1}{2}\mu(\langle\hat{m}_a^\dagger\hat{m}\rangle + \langle\hat{m}^\dagger\hat{m}_a\rangle) = \mu\langle\hat{m}_a^\dagger\hat{m}_a\rangle. \tag{3.39}$$

Substitution of Eq (3.39) into (3.31) yields

$$\frac{d}{dt}\left\langle\hat{m}^\dagger\hat{m}\right\rangle = -\mu\langle\hat{m}^\dagger\hat{m}\rangle + \mu\langle\hat{m}_a^\dagger\hat{m}_a\rangle + \langle\hat{F}_m^\dagger(t)\hat{m}(t)\rangle + \langle\hat{m}^\dagger(t)\hat{F}_m(t)\rangle. \tag{3.40}$$

In view of Eq (2.106) and (2.108), the above equation takes the form

$$\frac{d}{dt}\langle\hat{N}_a + \hat{N}_b\rangle = -\mu\langle\hat{N}_a + \hat{N}_b\rangle + \mu\langle\hat{N}_a\rangle + \frac{1}{N}\langle\hat{F}_m^\dagger(t)\hat{m}(t)\rangle + \frac{1}{N}\langle\hat{m}^\dagger(t)\hat{F}_m(t)\rangle. \tag{3.41}$$

On the other hand, on account of Eqs. (2.74) and (2.75) we see that

$$\frac{d}{dt}\langle\hat{N}_a + \hat{N}_b\rangle = -\gamma_c\langle\hat{N}_b\rangle + r_a\langle\hat{N}_c\rangle. \tag{3.42}$$

Using Eqs. (2.88) and (3.5), the above equation can be rewritten as

$$\frac{d}{dt}\langle\hat{N}_a + \hat{N}_b\rangle = -(\gamma_c + 2r_a)\langle\hat{N}_a + \hat{N}_b\rangle + (\gamma_c + 2r_a)\langle\hat{N}_a\rangle + r_aN. \tag{3.43}$$

Hence comparison of Eq (3.41) and (3.43) shows that

$$\mu = \gamma_c + 2r_a \tag{3.44}$$

and

$$\langle\hat{F}_m^\dagger(t)\hat{m}(t)\rangle + \langle\hat{m}^\dagger(t)\hat{F}_m(t)\rangle = r_aN^2. \tag{3.45}$$

We next seek to establish correlation function $\langle\hat{F}_m^\dagger(t)\hat{F}_m(t')\rangle$. The solution of Eq (3.26) is given by

$$\hat{m}(t) = \hat{m}(0)e^{-\frac{1}{2}\mu t} + \int_0^t e^{-\frac{1}{2}\mu(t-t')}\hat{F}_m(t')dt'. \tag{3.46}$$

Multiplication of both sides of the above equation from the left side by $\hat{F}^{\dagger}_{m}(t)$ and taking the expectation value of the resulting expression, we have

$$\langle\hat{F}^{\dagger}_{m}(t)\hat{m}(t)\rangle = \langle\hat{F}^{\dagger}_{m}(t)\hat{m}(0)\rangle e^{-\frac{1}{2}\mu t} + \int_{0}^{t} e^{-\frac{1}{2}\mu(t-t')}\langle\hat{F}^{\dagger}_{m}(t)\hat{F}_{m}(t')\rangle dt'. \quad (3.47)$$

In view of the fact that a noise operator at certain time should not affect a light mode operator at an earlier time, we have

$$\langle\hat{F}^{\dagger}_{m}(t)\hat{m}(0)\rangle = 0. \quad (3.48)$$

Hence on account of (3.48), we get

$$\langle\hat{F}^{\dagger}_{m}(t)\hat{m}(t)\rangle = \int_{0}^{t} e^{-\frac{1}{2}\mu(t-t')}\langle\hat{F}^{\dagger}_{m}(t)\hat{F}_{m}(t')\rangle dt'. \quad (3.49)$$

Moreover, the solution of Eq. (3.27) is given by

$$\hat{m}^{\dagger}(t) = \hat{m}^{\dagger}(0)e^{-\frac{1}{2}\mu t} + \int_{0}^{t} e^{-\frac{1}{2}\mu(t-t')}\hat{F}^{\dagger}_{m}(t')dt'. \quad (3.50)$$

Multiplication of both sides of the above equation from the right side by $\hat{F}_{m}(t)$ and taking the expectation value of the resulting expression, we have

$$\langle\hat{m}^{\dagger}(t)\hat{F}_{m}(t)\rangle = \langle\hat{m}^{\dagger}(0)\hat{F}_{m}(t)\rangle e^{-\frac{1}{2}\mu t} + \int_{0}^{t} e^{-\frac{1}{2}\mu(t-t')}\langle\hat{F}^{\dagger}_{m}(t')\hat{F}_{m}(t)\rangle dt'. \quad (3.51)$$

For the same reason given previously, we have

$$\langle\hat{m}^{\dagger}(0)\hat{F}_{m}(t)\rangle = 0, \quad (3.52)$$

which leads to

$$\langle\hat{m}^{\dagger}(t)\hat{F}_{m}(t)\rangle = \int_{0}^{t} e^{-\frac{1}{2}\mu(t-t')}\langle\hat{F}^{\dagger}_{m}(t')\hat{F}_{m}(t)\rangle dt'. \quad (3.53)$$

Adding Eqs. (3.49) and (3.53) and comparing the result with Eq (3.45), one gets

$$\langle\hat{F}^{\dagger}_{m}(t)\hat{m}(t)\rangle + \langle\hat{m}^{\dagger}(t)\hat{F}_{m}(t)\rangle = \int_{0}^{t} e^{-\frac{1}{2}\mu(t-t')}\langle\hat{F}^{\dagger}_{m}(t)\hat{F}_{m}(t')\rangle dt' + \int_{0}^{t} e^{-\frac{1}{2}\mu(t-t')}\langle\hat{F}^{\dagger}_{m}(t')\hat{F}_{m}(t)\rangle dt'. \quad (3.54)$$

$$r_aN^2=\int_0^t e^{-\frac{1}{2}\mu(t-t')}\langle\hat{F}^{\dagger}_{m}(t)\hat{F}_m(t')\rangle dt'+\int_0^t e^{-\frac{1}{2}\mu(t-t')}\langle\hat{F}^{\dagger}_{m}(t')\hat{F}_m(t)\rangle dt'. \tag{3.55}$$

Assuming that

$$\langle\hat{F}^{\dagger}_{m}(t)\hat{F}_m(t')\rangle=\langle\hat{F}^{\dagger}_{m}(t')\hat{F}_m(t)\rangle, \tag{3.56}$$

we have

$$2\int_0^t e^{-\frac{1}{2}\mu(t-t')}\langle\hat{F}^{\dagger}_{m}(t)\hat{F}_m(t')\rangle dt=r_aN^2, \tag{3.57}$$

from which we conclude that

$$\langle\hat{F}^{\dagger}_{m}(t)\hat{F}_m(t')\rangle=r_aN^2\delta(t-t'). \tag{3.58}$$

We next seek to obtain the equation of evolution for $\langle\hat{m}\hat{m}^{\dagger}\rangle$. Upon introducing Eq (3.26) and (3.27) into the relation

$$\frac{d}{dt}\langle\hat{m}\hat{m}^{\dagger}\rangle=\left\langle\frac{d\hat{m}}{dt}\hat{m}^{\dagger}\right\rangle+\left\langle\hat{m}\frac{d\hat{m}^{\dagger}}{dt}\right\rangle, \tag{3.59}$$

there follows

$$\frac{d}{dt}\langle\hat{m}\hat{m}^{\dagger}\rangle=\left\langle(-\frac{1}{2}\mu\hat{m}+\frac{1}{2}\mu\hat{m}_a+\hat{F}_m(t))\hat{m}^{\dagger}\right\rangle+\left\langle\hat{m}(-\frac{1}{2}\mu\hat{m}^{\dagger}+\frac{1}{2}\mu\hat{m}^{\dagger}_{a}+\hat{F}^{\dagger}_{m}(t))\right\rangle, \tag{3.60}$$

$$=\left\langle(-\frac{1}{2}\mu\hat{m}\hat{m}^{\dagger}+\frac{1}{2}\mu\hat{m}_a\hat{m}^{\dagger}+\hat{F}_m(t)\hat{m}^{\dagger}(t)-\frac{1}{2}\mu\hat{m}\hat{m}^{\dagger}+\frac{1}{2}\mu\hat{m}\hat{m}^{\dagger}_{a}+\hat{m}(t)\hat{F}^{\dagger}_{m}(t))\right\rangle, \tag{3.61}$$

$$=-\mu\langle\hat{m}\hat{m}^{\dagger}\rangle+\frac{1}{2}\mu(\langle\hat{m}_a\hat{m}^{\dagger}\rangle+\langle\hat{m}\hat{m}^{\dagger}_{a}\rangle)+\langle\hat{F}_m(t)\hat{m}^{\dagger}(t)\rangle+\langle\widehat{m(t)}F^{\dagger}_{m}(t)\rangle. \tag{3.62}$$

On using Eqs.(2.100), (2.91), and (2.92) along with their adjoint, we see that

$$\frac{1}{2}\mu(\langle\hat{m}_a\hat{m}^{\dagger}\rangle+\langle\hat{m}\hat{m}^{\dagger}_{a}\rangle)=\frac{1}{2}\mu(\langle\hat{m}_a\hat{m}^{\dagger}_{a}\rangle+\langle\hat{m}_a\hat{m}^{\dagger}_{b}\rangle+\langle\hat{m}_a\hat{m}^{\dagger}_{a}\rangle+\langle\hat{m}_b\hat{m}^{\dagger}_{a}\rangle),$$

$$=\mu N\langle\hat{N}_b\rangle. \tag{3.63}$$

Substituting the above equation into Eq (3.62), one gets

$$\frac{d}{dt}\langle\hat{m}\hat{m}^{\dagger}\rangle=-\mu\langle\hat{m}\hat{m}^{\dagger}\rangle+\mu N\langle\hat{N}_b\rangle+\langle\hat{F}_m(t)\hat{m}^{\dagger}(t)\rangle+\langle\hat{m}(t)\hat{F}^{\dagger}_{m}(t)\rangle. \tag{3.64}$$

On account of the fact that $\hat{m}\hat{m}^{\dagger}=N(\hat{N}_b+\hat{N}_c)$, the above equation takes the form

$$\frac{d}{dt}\langle\hat{N}_b+\hat{N}_c\rangle=-\mu\langle\hat{N}_b+\hat{N}_c\rangle+\mu\langle\hat{N}_b\rangle+\frac{1}{N}\langle\hat{F}_m(t)\hat{m}^{\dagger}(t)\rangle+\frac{1}{N}\langle\hat{m}(t)\hat{F}^{\dagger}_{m}(t)\rangle. \tag{3.65}$$

On the other hand, in view of Eqs. (2.75) and (2.76) we see that

$$\frac{d}{dt}\langle \hat{N}_b + \hat{N}_c\rangle = \gamma_c\langle \hat{N}_a\rangle - r_a\langle \hat{N}_c\rangle. \tag{3.66}$$

Using Eq (2.88) the above equation can be rewritten as

$$\frac{d}{dt}\langle \hat{N}_b + \hat{N}_c\rangle = \gamma_c N - \gamma_c\langle \hat{N}_b\rangle - \gamma_c\langle \hat{N}_c\rangle - r_a\langle \hat{N}_c\rangle. \tag{3.67}$$

On account of Eq. (3.5), one can write Eq. (3.67) as

$$\frac{d}{dt}\langle \hat{N}_b + \hat{N}_c\rangle = \gamma_c N - \gamma_c\langle \hat{N}_a\rangle - \gamma_c\langle \hat{N}_c\rangle - r_a\langle \hat{N}_c\rangle.$$

With the aid of Eq. (3.7), the above equation takes the form

$$\frac{d}{dt}\langle \hat{N}_b + \hat{N}_c\rangle = -(\gamma_c + 2r_a)\langle \hat{N}_c\rangle + \gamma_c N.$$

We can easily rewrite the above equation in the form

$$\frac{d}{dt}\langle \hat{N}_b + \hat{N}_c\rangle = -(\gamma_c + 2r_a)\langle \hat{N}_b + \hat{N}_c\rangle + (\gamma_c + 2r_a)\langle \hat{N}_b\rangle + \gamma_c N. \tag{3.68}$$

Now comparison of Eq (3.65) and (3.68) shows that μ has the value given by (3.44) and

$$\langle \hat{F}_m(t)\hat{m}^{\dagger}(t)\rangle + \langle \hat{m}(t)\hat{F}_m^{\dagger}(t)\rangle = \gamma_c N^2. \tag{3.69}$$

Furthermore, multiplying both sides of Eq (3.46) from the right side by $\hat{F}_m^{\dagger}(t)$ and taking the expectation value of the resulting expression, we have

$$\langle \hat{m}\hat{(t)}F_m^{\dagger}(t)\rangle = \langle \hat{m}(0)\hat{F}_m^{\dagger}(t)\rangle e^{-\frac{1}{2}\mu t} + \int_0^t e^{-\frac{1}{2}\mu(t-t')}\langle \hat{F}_m(t')\hat{F}_m^{\dagger}(t)\rangle dt'. \tag{3.70}$$

In view of the fact that a noise operator at certain time should not affect a light mode operator at an earlier time, we have

$$\langle \hat{m}(t)\hat{F}_m^{\dagger}(t)\rangle = \int_0^t \langle \hat{F}_m(t')\hat{F}_m^{\dagger}(t)\rangle dt'. \tag{3.71}$$

In addition, multiplying both sides of the Eq (3.50) from the left side by $\hat{F}_m(t)$ and taking the expectation value of the resulting expression, we have

$$\langle \hat{F}_m(t)\hat{m}^{\dagger}(t)\rangle = \langle \hat{F}_m(t)\hat{m}^{\dagger}(0)\rangle e^{-\frac{1}{2}\mu t} + \int_0^t e^{-\frac{1}{2}\mu(t-t')}\langle \hat{F}_m(t)\hat{F}_m^{\dagger}(t')\rangle dt'. \tag{3.72}$$

Upon setting

$$\langle \hat{F}_m(t)\hat{m}^\dagger(0)\rangle = 0, \tag{3.73}$$

we easily get

$$\langle \hat{F}_m(t)\hat{m}^\dagger(t)\rangle = \int_0^t e^{-\frac{1}{2}\mu(t-t')}\langle \hat{F}_m(t)\hat{F}^\dagger_m(t')\rangle dt'. \tag{3.74}$$

Adding Eqs. (3.71) and (3.74) and comparing the result with Eq (3.69), one gets

$$\langle \hat{F}_m(t)\hat{m}^\dagger(t)\rangle + \langle \hat{m}(t)\hat{F}^\dagger_m(t)\rangle = \int_0^t e^{-\frac{1}{2}\mu(t-t')}\langle \hat{F}_m(t')\hat{F}^\dagger_m(t)\rangle dt' + \int_0^t e^{-\frac{1}{2}\mu(t-t')}\langle \hat{F}_m(t)\hat{F}^\dagger_m(t')\rangle dt'. \tag{3.75}$$

$$\gamma_c N^2 = \int_0^t e^{-\frac{1}{2}\mu(t-t')}\langle \hat{F}_m(t')\hat{F}^\dagger_m(t)\rangle dt' + \int_0^t e^{-\frac{1}{2}\mu(t-t')}\langle \hat{F}_m(t)\hat{F}^\dagger_m(t')\rangle dt'. \tag{3.76}$$

Then, upon assuming that

$$\langle \hat{F}_m(t)\hat{F}^\dagger_m(t')\rangle = \langle \hat{F}_m(t')\hat{F}^\dagger_m(t)\rangle, \tag{3.77}$$

there follows

$$2\int_0^t e^{-\frac{1}{2}\mu(t-t')}\langle \hat{F}_m(t)\hat{F}^\dagger_m(t')\rangle dt = \gamma_c N^2. \tag{3.78}$$

Now in view of Eq (3.78), one can conclude that

$$\langle \hat{F}_m(t)\hat{F}^\dagger_m(t')\rangle = \gamma_c N^2 \delta(t-t'). \tag{3.79}$$

On the other hand, up on casting Eqs (2.72) and (2.73) into the form

$$\frac{d\hat{m}_a}{dt} = -\frac{1}{2}\mu\hat{m}_a + \hat{F}_a(t) \tag{3.80}$$

and

$$\frac{d\hat{m}_b}{dt} = -\frac{1}{2}\mu\hat{m}_b + \hat{F}_b(t), \tag{3.81}$$

and following a similar procedure, we can easily show that μ has the value given by (3.44),

$$\langle \hat{F}^\dagger_a(t)\hat{F}_a(t')\rangle = r_a N^2 \delta(t-t') \tag{3.82}$$

and

$$\langle \hat{F}_b(t)\hat{F}^\dagger_b(t')\rangle = \gamma_c N^2 \delta(t-t'). \tag{3.83}$$

With the atoms considered to be initially in the bottom level, the expectation value of the solution of Eq (3.80) happens to be

$$\langle \hat{m}_a(t) \rangle = 0. \tag{3.84}$$

Hence the expectation value of the solution of Eq (2.126) turns out to be

$$\langle \hat{a}(t) \rangle = 0. \tag{3.85}$$

In view of the linear equation given by (2.126) as well as Eq.(3.85), we claim that $\hat{a}(t)$ is a Gaussian variable with zero mean. We can also verify that

$$\langle \hat{b}(t) \rangle = 0. \tag{3.86}$$

Then on account of the linear equation described by (2.127) and Eq (3.86), we realize $\hat{b}(t)$ to be Gaussian variable with zero mean. We now observe that

$$\langle \hat{c}(t) \rangle = 0. \tag{3.87}$$

Thus in view of Eqs. (2.130) and (3.87), we see that $\hat{c}(t)$ to be Gaussian variable with zero mean.

The steady state solution of Eq. (2.126) is given by

$$\hat{a} = \frac{2g}{\kappa\sqrt{N}}\hat{m}_a. \tag{3.88}$$

Hence the mean photon number of light mode a is expressible as

$$\bar{n}_a = \langle \hat{a}^\dagger \hat{a} \rangle = \left(\frac{2g}{\kappa\sqrt{N}}\right)^2 \langle \hat{m}_a^\dagger \hat{m}_a \rangle. \tag{3.89}$$

Using Eq (2.62) along with (2.106) we have

$$\bar{n}_a = \frac{\gamma_c}{\kappa}\langle \hat{N}_a \rangle. \tag{3.90}$$

We also find the mean photon number of light mode b to be

$$\bar{n}_b = \frac{\gamma_c}{\kappa}\langle \hat{N}_b \rangle. \tag{3.91}$$

On account of (3.5), we notice that

$$\bar{n}_a = \bar{n}_b. \tag{3.92}$$

The mean photon number of light mode b reduces to

$$\bar{n}_b = \frac{\gamma_c}{\kappa}\left(\frac{N}{2}\right) \tag{3.93}$$

for $\gamma_c \ll r_a$, and to

$$\bar{n}_b = \frac{\gamma_c}{\kappa}\left(\frac{N}{3}\right) \tag{3.94}$$

for $\gamma_c = r_a$.

The steady-state solution of Eq (2.130) has the form

$$\hat{c} = \frac{2g}{\kappa\sqrt{N}}\hat{m}. \tag{3.95}$$

Thus using Eq (3.95) along with (2.108), the mean photon number of the two-mode cavity light can be written as

$$\bar{n} = \langle \hat{c}^\dagger \hat{c}\rangle = \frac{\gamma_c}{\kappa}\left(\langle \hat{N}_a\rangle + \langle \hat{N}_b\rangle\right). \tag{3.96}$$

It proves to be convenient to refer to a regime of laser operation with more atoms in the top level than in the bottom level as above threshold, the regime of laser operation with equal number of atoms in the top and bottom levels as threshold, and the regime of laser operation with less atom in the top level than in the bottom level as below threshold. Thus according to Eq (3.7) for the laser operating above threshold $\gamma_c < r_a$, for the laser operating at threshold $\gamma_c = r_a$ and for the laser operating below threshold $\gamma_c > r_a$.

The variance of photon number for light mode a can be written as

$$(\Delta n_a)^2 = \langle \hat{a}^\dagger \hat{a}\hat{a}^\dagger \hat{a}\rangle - \langle \hat{a}^\dagger \hat{a}\rangle^2 \tag{3.97}$$

and using the fact that a is a Gaussian variable with zero mean, we readily get

$$(\Delta n_a)^2 = \langle \hat{a}^\dagger \hat{a}\rangle\langle \hat{a}\hat{a}^\dagger\rangle + \langle \hat{a}^{+2}\rangle\ \langle \hat{a}^2\rangle. \tag{3.98}$$

Applying Eq (3.88) into the last term of the above equation, we get

$$\langle \hat{a}^2\rangle = \left(\frac{2g}{\kappa\sqrt{N}}\right)^2 \langle \hat{m}_a^2\rangle. \tag{3.99}$$

Using

$$\langle \hat{m}_a^2\rangle = \langle N|b\rangle\langle a|N|b\rangle\langle a|\rangle = N^2\langle |b\rangle\langle a|b\rangle\langle a|\rangle = 0 \tag{3.100}$$

in (3.99), one gets

$$\langle \hat{a}^2 \rangle = 0. \tag{3.101}$$

Moreover, from (3.90) we have

$$\langle \hat{a}^\dagger \hat{a} \rangle = \frac{\gamma_c}{\kappa} \langle \hat{N}_a \rangle. \tag{3.102}$$

And using Eq (3.88) in $\langle \hat{a}\hat{a}^\dagger \rangle$, we have

$$\langle \hat{a}\hat{a}^\dagger \rangle = \left(\frac{2g}{\kappa\sqrt{N}} \right)^2 \langle \hat{m}_a \hat{m}_a^\dagger \rangle. \tag{3.103}$$

Taking into account Eqs (2.111) along with (2.62), we find

$$\langle \hat{a}\hat{a}^\dagger \rangle = \frac{\gamma_c}{\kappa} \langle \hat{N}_b \rangle. \tag{3.104}$$

Therefor on account of Eqs (3.101), (3.102) and (3.104), we arrive at

$$(\Delta n_a)^2 = \bar{n}_a^2. \tag{3.105}$$

This represents the normally-ordered variance of the photon number for a chaotic state. One can also establish in a similar manner that the photon-number variance for light mode b has the form

$$(\Delta n_b)^2 = \left(\frac{\gamma_c}{\kappa} \right)^2 \langle \hat{N}_b \rangle \langle \hat{N}_c \rangle. \tag{3.106}$$

This takes the form

$$(\Delta n_b)^2 = 0 \tag{3.107}$$

for $\gamma_c \ll r_a$ and

$$(\Delta n_b)^2 = \bar{n}_b^2 \tag{3.108}$$

for $\gamma_c = r_a$, with $\bar{n}_b$ is given by (3.94). These results show that light mode b is in coherent state for $\gamma_c \ll r_a$ and in chaotic state for $\gamma_c = r_a$.

Furthermore, the variance of the photon number for the two-mode cavity light is expressible as

$$(\Delta n)^2 = \langle \hat{c}^\dagger \hat{c} \hat{c}^\dagger \hat{c} \rangle - \langle \hat{c}^\dagger \hat{c} \rangle^2, \tag{3.109}$$

and using the fact that $\hat{c}$ is a Gaussian variable with zero mean, we readily get

$$(\Delta n)^2 = \langle \hat{c}^\dagger \hat{c} \rangle \langle \hat{c} \hat{c}^\dagger \rangle + \langle \hat{c}^{+2} \rangle \ \langle \hat{c}^2 \rangle. \tag{3.110}$$

Employing once more Eq (3.95) and taking into account (2.113), we find

$$\langle \hat{c}\hat{c}^{\dagger} \rangle = \frac{\gamma_c}{\kappa} \left(\langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle \right). \tag{3.111}$$

In addition, applying (3.7), one can express Eq (3.24) in the form

$$\langle \hat{m}_c \rangle = \sqrt{\frac{\gamma_c}{r_a}} \langle \hat{N}_a \rangle. \tag{3.112}$$

Hence with the aid of Eqs. (3.95), (2.117), and (3.112), we easily get

$$\langle \hat{c}^2 \rangle = \frac{\gamma_c}{\kappa} \sqrt{\frac{\gamma_c}{r_a}} \langle \hat{N}_a \rangle. \tag{3.113}$$

Now on account of Eqs. (3.96), (3.111), and (3.113) along with (3.5), (3.6), and (3.7), we arrive at

$$(\Delta n)^2 = \frac{1}{4} \bar{n}^2 (3\eta + 2). \tag{3.114}$$

with $\eta = \dfrac{\gamma_c}{r_a}$.

4 Global Quadrature Squeezing

We seek here to drive the global quadrature variance (in the entire frequency interval) of light mode a and b and that of the two-mode cavity (output) light. The squeezing properties of light mode a are described by two quadrature operators defined by

$$\hat{a}_+ = \hat{a}^\dagger + \hat{a} \tag{4.1}$$

and

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}). \tag{4.2}$$

We find the commutation relation for the above two operators to be

$$\begin{aligned}
[\hat{a}_-, \hat{a}_+] &= [i(\hat{a}^\dagger - \hat{a}), (\hat{a}^\dagger + \hat{a})], \\
&= i[\hat{a}^\dagger, \hat{a}^\dagger] + i[\hat{a}^\dagger, \hat{a}] - i[\hat{a}, \hat{a}^\dagger] - i[\hat{a}, \hat{a}], \\
&= i[\hat{a}^\dagger, \hat{a}] - i[\hat{a}, \hat{a}^\dagger], \\
&= -2i[\hat{a}, \hat{a}^\dagger].
\end{aligned} \tag{4.3}$$

Using Eq. (2.120) in Eq (4.3), one gets

$$[\hat{a}_-, \hat{a}_+] = 2i\frac{\gamma_c}{\kappa}(\hat{N}_a - \hat{N}_b). \tag{4.4}$$

It then follows that

$$\Delta a_+ \Delta a_- \geq \frac{\gamma_c}{\kappa}\Big|\langle \hat{N}_a \rangle - \langle \hat{N}_b \rangle\Big|. \tag{4.5}$$

The variance of the quadrature operators is expressible as

$$(\Delta a_\pm)^2 = \pm\langle [\hat{a}^\dagger \pm \hat{a}]^2 \rangle \mp [\langle \hat{a}^\dagger \rangle \pm \langle \hat{a} \rangle]^2, \tag{4.6}$$

so that on account of Eq. (3.85), we have

$$(\Delta a_\pm)^2 = \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a} \hat{a}^\dagger \rangle \pm \langle \hat{a}^{\dagger 2} \rangle \pm \langle \hat{a}^2 \rangle. \tag{4.7}$$

Now employing (3.101), (3.102) and (3.104) along with (3.5), we arrive at

$$(\Delta a_+)^2 = (\Delta a_-)^2 = 2\bar{n}_a, \tag{4.8}$$

with $\bar{n}_a$ given by Eq (3.90). We thus realize that light mode a is in a chaotic state. Moreover, following the same procedure, one can readily verify that the quadrature variance of light mode b has the form

$$(\Delta b_+)^2 = (\Delta b_-)^2 = \frac{\gamma_c}{\kappa}(\langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle). \tag{4.9}$$

We then see that for the laser operating well above threshold

$$(\Delta b_+)^2 = (\Delta b_-)^2 = \frac{\gamma_c}{\kappa}\left(\frac{N}{2}\right) \tag{4.10}$$

and for a laser operating at threshold

$$(\Delta b_+)^2 = (\Delta b_-)^2 = 2\bar{n}_b, \tag{4.11}$$

in which $\bar{n}_b$ is given by Eq (3.94).

Following similar steps of derivation of (4.4), one can easily show that

$$[\hat{b}_-, \hat{b}_+] = 2i\frac{\gamma_c}{\kappa}(\hat{N}_b - \hat{N}_c). \tag{4.12}$$

On account of this commutation relation, we have

$$\Delta b_+ \Delta b_- \geq \frac{\gamma_c}{\kappa}\Big|\langle\hat{N}_b\rangle - \langle\hat{N}_c\rangle\Big|, \tag{4.13}$$

so that for $\gamma_c \ll r_a$, this uncertainity relation takes the form

$$\Delta b_+ \Delta b_- \geq \frac{\gamma_c}{\kappa}\left(\frac{N}{2}\right). \tag{4.14}$$

We realize from Eqs. (4.10) and (4.14) that light mode b is in a coherent state for $\gamma_c \ll r_a$. In addition, Eqs. (3.108) and (4.11) indicate that light mode b is in a chaotic state for $\gamma_c = r_a$.

We now proceed to calculate the global quadrature squeezing of the two-mode cavity and output light. The squeezing properties of the two-mode cavity light are described by two quadrature operators defined by

$$\hat{c}_+ = \hat{c}^\dagger + \hat{c} \tag{4.15}$$

and

$$\hat{c}_- = i(\hat{c}^\dagger - \hat{c}). \tag{4.16}$$

It can be readily established that

$$[\hat{c}_-, \hat{c}_+] = 2i\frac{\gamma_c}{\kappa}(\hat{N}_a - \hat{N}_c). \tag{4.17}$$

It then follows that

$$\Delta c_+ \Delta c_- \geq \frac{\gamma_c}{\kappa}\Big|\langle\hat{N}_a\rangle - \langle\hat{N}_c\rangle\Big|. \tag{4.18}$$

The variance of the quadrature operators is expressible as

$$(\Delta c_{\pm})^2 = \pm\langle[\hat{c}^{\dagger} \pm \hat{c}]^2\rangle \mp [\langle\hat{c}^{\dagger}\rangle \pm \langle\hat{c}\rangle]^2, \tag{4.19}$$

so that on account of (3.87), we have

$$(\Delta c_{\pm})^2 = \langle\hat{c}^{\dagger}\hat{c}\rangle + \langle\hat{c}\hat{c}^{\dagger}\rangle \pm \langle\hat{c}^{+2}\rangle \pm \langle\hat{c}^2\rangle. \tag{4.20}$$

And on account of Eq (3.96), (3.111) and (3.113), we have

$$(\Delta c_{\pm})^2 = \frac{\gamma_c}{\kappa}\left(N + \langle\hat{N}_a\rangle \pm 2\sqrt{\frac{\gamma_c}{r_a}}\langle\hat{N}_a\rangle\right).$$

Hence employing (3.6), we arrive at

$$(\Delta c_{\pm})^2 = \frac{\gamma_c}{\kappa}\left(N + \frac{r_a N}{\gamma_c + 2r_a} \pm 2\sqrt{\frac{\gamma_c}{r_a}}\left(\frac{r_a N}{\gamma_c + 2r_a}\right)\right), \tag{4.21}$$

$$= \frac{\gamma_c}{\kappa}\left(N + \frac{r_a N}{\gamma_c + 2r_a} \pm 2\sqrt{\gamma_c r_a}\left(\frac{N}{\gamma_c + 2r_a}\right)\right). \tag{4.22}$$

Moreover, on setting $r_a = 0$ in Eqs. (4.22), we get

$$(\Delta c_{+})^2_{\nu} = (\Delta c_{-})^2_{\nu} = \frac{\gamma_c}{\kappa}N. \tag{4.23}$$

This indeed represents the quadrature variance of a two-mode vacuum state. We seek to calculate the quadrature squeezing of the two-mode cavity (output) light relative to the quadrature variance of the two-mode cavity (output) vacuum state. We then define the quadrature squeezing of the two-mode cavity light by

$$S = \frac{(\Delta c_{-})^2_{\nu} - (\Delta c_{-})^2}{(\Delta c_{-})^2_{\nu}}. \tag{4.24}$$

Now applying (4.22) and (4.23) into Eq (4.24), and taking into account Eq. (3.6), we have

$$\begin{aligned} S &= 1 - \frac{(\Delta c_{-})^2}{(\Delta c_{-})^2_{\nu}}, \\ &= 1 - \frac{\frac{\gamma_c}{\kappa}\left(N + \frac{r_a N}{\gamma_c + 2r_a} - 2\sqrt{\gamma_c r_a}\left(\frac{N}{\gamma_c + 2r_a}\right)\right)}{\frac{\gamma_c}{\kappa}N}, \\ &= 1 - 1 - \frac{r_a}{\gamma_c + 2r_a} + 2\frac{\sqrt{\gamma_c r_a}}{\gamma_c + 2r_a}, \end{aligned}$$

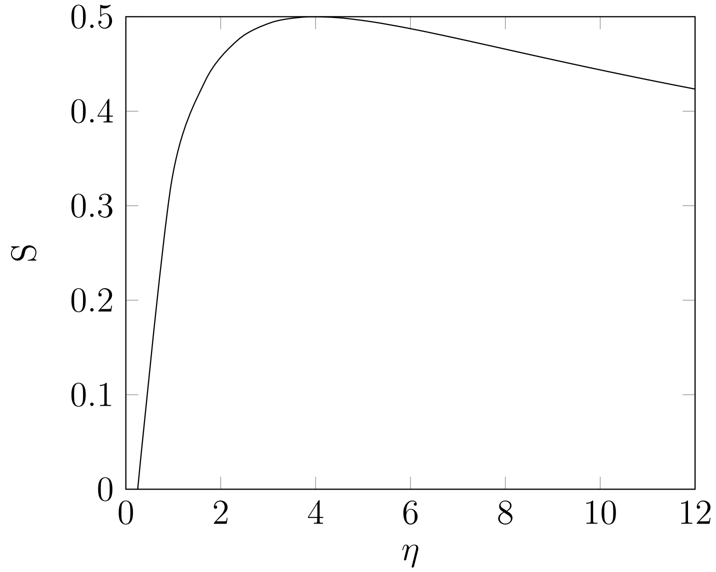

Figure 2: A plot of global quadrature squeezing versus η.

$$= \frac{2\sqrt{\gamma_c r_a} - r_a}{\gamma_c + 2r_a},$$

$$= \frac{2\sqrt{\frac{\gamma_c}{r_a}} - 1}{\frac{\gamma_c}{r_a} + 2},$$

$$= \frac{2\sqrt{\eta} - 1}{\eta + 2}, \tag{4.25}$$

where $\eta = \frac{\gamma_c}{r_a}$. We note that, unlike the mean photon number, the quadrature squeezing does not depend on the number of atoms. The plot in Figure 2 shows that the light generated by the three-level laser operating under the condition $\gamma_c > \frac{1}{4} r_a$ is in squeezed state, with the maximum quadrature squeezing being 50 % below the vacuum state level. This occurs when the three-level laser is operating below threshold at $\gamma_c = 4r_a$. Since the squeezed light is generated when the laser is operating far below threshold, the mean photon number of the squeezed light is relatively small.

On the other hand, we define the quadrature squeezing of the two-mode output light by

$$S^{out} = \frac{(\Delta c_-^{out})^2_\nu - (\Delta c_-^{out})^2}{(\Delta c_-^{out})^2_\nu}, \tag{4.26}$$

where $(\Delta c_{-}^{out})^2_{\nu}$ is the quadrature variance of the two-mode output vacuum state. And applying $\hat{c}_{-}^{out} = \sqrt{\kappa}\hat{c}_{-}$ we can easily see that

$$(\Delta c_{-}^{out})^2_{\nu} = \kappa(\Delta c_{-})^2_{\nu} \tag{4.27}$$

and

$$(\Delta c_{-}^{out})^2 = \kappa(\Delta c_{-})^2. \tag{4.28}$$

Now in view of Eqs. (4.27),(4.28) and (4.24), we arrive at

$$S^{out} = S. \tag{4.29}$$

We observe that the quadrature squeezing of output light is equal to that of the cavity light.

5 Local Quadrature Squeezing

We finally seek to obtain the local quadrature squeezing (in a given frequency interval) of the two-mode cavity (output) light when both light mode a and light mode b have the same frequency. To this end, we first determine the spectrum of quadrature fluctuations for the two-mode cavity light. We define this spectrum for a two-mode light with central frequency ω_o by

$$S_{\pm}(\omega) = \frac{1}{\pi} R_e \int_0^{\infty} \langle \hat{c}_{\pm}(t), \hat{c}_{\pm}(t+\tau) \rangle_{ss} e^{i(\omega-\omega_o)\tau} d\tau. \tag{5.1}$$

Upon integrating both sides of Eq (5.1) over ω, we get

$$\int_{-\infty}^{\infty} S_{\pm}(\omega) d\omega = (\Delta c_{\pm})^2, \tag{5.2}$$

in which

$$(\Delta c_{\pm})^2 = \langle \hat{c}_{\pm}(t), \hat{c}_{\pm}(t) \rangle_{ss} \tag{5.3}$$

is the quadrature variance of the two-mode light at steady state. On the basis of the result given by (5.2), we assert that $S_{\pm}(\omega) d\omega$ is the steady-state quadrature variance of the two-mode light in the interval between ω and $\omega + d\omega$. In view Eq. (3.87), we note that

$$\langle \hat{c}_{\pm}(t), \hat{c}_{\pm}(t+\tau) \rangle = \langle \hat{c}_{\pm}(t) \hat{c}_{\pm}(t+\tau) \rangle. \tag{5.4}$$

We now proceed to determine the two-time correlation function that appears in Eq (5.4) for the cavity light. To this end, we realize that the solution of Eq. (2.130) can also be written as

$$\hat{c}(t+\tau) = \hat{c}(t) e^{-\kappa\tau/2} + \frac{g}{\sqrt{N}} e^{-\kappa\tau/2} \int_0^{\tau} e^{\kappa\tau'/2} \hat{m}(t+\tau') d\tau'. \tag{5.5}$$

On the other hand, the solution of Eq (3.26) is expressible as

$$\begin{aligned} \hat{m}(t+\tau') = \hat{m}(t) e^{-\mu\tau'/2} + e^{-\mu\tau'/2} \int_0^{\tau'} & e^{\mu\tau''/2} \\ & \times \left(\frac{1}{2} \mu \hat{m}_a(t+\tau'') + \hat{F}_m(t+\tau'') \right) d\tau''. \end{aligned} \tag{5.6}$$

Applying the large-time approximation scheme to Eq (3.80), we get

$$\hat{m}_a(t+\tau) = \frac{2}{\mu} \hat{F}_a(t+\tau), \tag{5.7}$$

so that on introducing this into Eq (5.6), there follows

$$\hat{m}(t+\tau') = \hat{m}(t)e^{-\mu\tau'/2} + e^{-\mu\tau'/2}\int_0^{\tau'} e^{\mu\tau''/2} \times \left(\hat{F}_a(t+\tau'') + \hat{F}_m(t+\tau'')\right) d\tau''. \tag{5.8}$$

Now combination of Eqs (5.5) and (5.8) yields

$$\hat{c}(t+\tau) = \hat{c}(t)e^{-\kappa\tau/2} + \frac{2g\hat{m}(t)}{\sqrt{N}(\kappa-\mu)}\left[e^{-\mu\tau/2} - e^{-\kappa\tau/2}\right] + \frac{g}{\sqrt{N}}e^{-\kappa\tau/2}\int_0^{\tau} d\tau' e^{(\kappa-\mu)\tau'/2} \times \int_0^{\tau'} d\tau'' e^{\mu\tau''/2}\left(\hat{F}_a(t+\tau'') + \hat{F}_m(t+\tau'')\right). \tag{5.9}$$

Multiplying both sides of the above equation by $\hat{c}^\dagger(t)$ and taking the expectation value of the resulting expression, one gets

$$\langle\hat{c}^\dagger(t)\hat{c}(t+\tau)\rangle = \langle\hat{c}^\dagger(t)\hat{c}(t)\rangle e^{-\kappa\tau/2} + \frac{2g\langle\hat{c}^\dagger(t)\hat{m}(t)\rangle}{\sqrt{N}(\kappa-\mu)}\left[e^{-\mu\tau/2} - e^{-\kappa\tau/2}\right] + \frac{g}{\sqrt{N}}e^{-\kappa\tau/2}\int_0^{\tau} d\tau' e^{(\kappa-\mu)\tau'/2} \times \int_0^{\tau'} d\tau'' e^{\mu\tau''/2}\left(\langle\hat{c}^\dagger(t)\hat{F}_a(t+\tau'')\rangle + \langle\hat{c}^\dagger(t)\hat{F}_m(t+\tau'')\rangle\right). \tag{5.10}$$

On account of the assertion that a noise operator at certain time should not affect light mode operator at earlier time, we have

$$\langle\hat{c}^\dagger(t)\hat{c}(t+\tau)\rangle = \langle\hat{c}^\dagger(t)\hat{c}(t)\rangle e^{-\kappa\tau/2} + \frac{2g}{\sqrt{N}(\kappa-\mu)} \times\langle\hat{c}^\dagger(t)\hat{m}(t)\rangle[e^{-\mu\tau/2} - e^{-\kappa\tau/2}]. \tag{5.11}$$

Applying once more the large-time approximation, one gets from Eq (2.130)

$$\hat{m}(t) = \frac{\kappa\sqrt{N}}{2g}\hat{c}(t). \tag{5.12}$$

With this substituted into (5.11), there emerges

$$\langle \hat{c}^\dagger(t)\hat{c}(t+\tau)\rangle = \langle \hat{c}^\dagger(t)\hat{c}(t)\rangle \left(\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\right). \tag{5.13}$$

We next seek to establish the two-time correlation function $\langle \hat{c}(t)\hat{c}^+(t+\tau)\rangle$. On account of Eq. (2.130), we see that

$$\frac{d\hat{c}^\dagger}{dt} = -\frac{\kappa}{2}\hat{c}^\dagger + \frac{g}{\sqrt{N}}\hat{m}^\dagger. \tag{5.14}$$

The solution of Eq (5.14) can be written as

$$\hat{c}^\dagger(t+\tau) = \hat{c}^\dagger(t)e^{-\kappa\tau/2} + \frac{g}{\sqrt{N}}e^{-\kappa\tau/2}\int_0^\tau e^{-\kappa\tau'/2}\hat{m}^\dagger(t+\tau')d\tau'. \tag{5.15}$$

On the other hand, the solution of Eq (3.27) is expressible as

$$\begin{aligned}\hat{m}^\dagger(t+\tau') = \hat{m}^\dagger(t)e^{-\mu\tau'/2} + e^{-\mu\tau'/2}\int_0^{\tau'} e^{\mu\tau''/2}\\ \times\left(\frac{1}{2}\mu\hat{m}_a^\dagger(t+\tau'') + \hat{F}_m^\dagger(t+\tau'')\right)d\tau''.\end{aligned} \tag{5.16}$$

On account of Eq (5.7), we have

$$\hat{m}_a^\dagger(t+\tau) = \frac{2}{\mu}\hat{F}_a^\dagger(t+\tau). \tag{5.17}$$

On introducing Eq (5.17) into (5.16), there follows

$$\begin{aligned}\hat{m}^\dagger(t+\tau') = \hat{m}^\dagger(t)e^{-\mu\tau'/2} + e^{-\mu\tau'/2}\int_0^{\tau'} e^{\mu\tau''/2}\\ \times\left(\hat{F}_a^\dagger(t+\tau'') + \hat{F}_m^\dagger(t+\tau'')\right)d\tau''.\end{aligned} \tag{5.18}$$

Now combination of Eqs (5.15) and (5.18) yields

$$\begin{aligned}\hat{c}^\dagger(t+\tau) = \hat{c}^\dagger(t)e^{-\kappa\tau/2} + \frac{2g\hat{m}^\dagger(t)}{\sqrt{N}(\kappa-\mu)}\left[e^{-\mu\tau/2} - e^{-\kappa\tau/2}\right]\\ +\frac{g}{\sqrt{N}}e^{-\kappa\tau/2}\int_0^\tau d\tau' e^{(\kappa-\mu)\tau'/2}\\ \times\int_0^{\tau'} d\tau'' e^{\mu\tau''/2}\left(\hat{F}_a^\dagger(t+\tau'') + \hat{F}_m^\dagger(t+\tau'')\right).\end{aligned} \tag{5.19}$$

Multiplying both sides of Eq (5.19) by $\hat{c}(t)$ from left and taking the expectation value of the resulting expression, one gets

$$\begin{aligned}\langle\hat{c}(t)\hat{c}^\dagger(t+\tau)\rangle &= \langle\hat{c}(t)\hat{c}^\dagger(t)\rangle e^{-\kappa\tau/2} + \frac{2g\langle\hat{c}(t)\hat{m}^\dagger(t)\rangle}{\sqrt{N}(\kappa-\mu)}\left[e^{-\mu\tau/2} - e^{-\kappa\tau/2}\right] \\ &+\frac{g}{\sqrt{N}}e^{-\kappa\tau/2}\int_0^\tau d\tau' e^{(\kappa-\mu)\tau'/2} \\ &\times\int_0^{\tau'} d\tau'' e^{\mu\tau''/2}\left(\langle\hat{c}(t)\hat{F}_a^\dagger(t+\tau'')\rangle + \langle\hat{c}(t)\hat{F}_m^\dagger(t+\tau'')\rangle\right).\end{aligned} \tag{5.20}$$

On account of the assertion that a noise operator at certain time should not affect a light mode operator at an earlier time, we have

$$\begin{aligned}\langle\hat{c}(t)\hat{c}^\dagger(t+\tau)\rangle &= \langle\hat{c}(t)\hat{c}^\dagger(t)\rangle e^{-\kappa\tau/2} + \frac{2g}{\sqrt{N}(\kappa-\mu)} \\ &\times\langle\hat{c}(t)\hat{m}^\dagger(t)\rangle\left[e^{-\mu\tau/2} - e^{-\kappa\tau/2}\right].\end{aligned} \tag{5.21}$$

Application of the large-time approximation to Eq (5.14) yields

$$\hat{m}^\dagger(t) = \frac{\kappa\sqrt{N}}{2g}\hat{c}^\dagger(t). \tag{5.22}$$

Substituting Eq (5.22) into (5.21), one gets

$$\begin{aligned}\langle\hat{c}(t)\hat{c}^\dagger(t+\tau)\rangle &= \langle\hat{c}(t)\hat{c}^\dagger(t)\rangle e^{-\kappa\tau/2} + \frac{2g}{\sqrt{N}(\kappa-\mu)}\frac{\kappa\sqrt{N}}{2g} \\ &\times\langle\hat{c}(t)\hat{c}^\dagger(t)\rangle\left[e^{-\mu\tau/2} - e^{-\kappa\tau/2}\right], \\ &= \langle\hat{c}(t)\hat{c}^\dagger(t)\rangle e^{-\kappa\tau/2} + \langle\hat{c}(t)\hat{c}^\dagger(t)\rangle\frac{\kappa}{\kappa-\mu}\left[e^{-\mu\tau/2} - e^{-\kappa\tau/2}\right], \\ &= \langle\hat{c}(t)\hat{c}^\dagger(t)\rangle\left(e^{-\kappa\tau/2} + \frac{\kappa}{\kappa-\mu}\left[e^{-\mu\tau/2} - e^{-\kappa\tau/2}\right]\right), \\ &= \langle\hat{c}(t)\hat{c}^\dagger(t)\rangle\frac{1}{\kappa-\mu}\left(\kappa e^{-\kappa\tau/2} - \mu e^{-\kappa\tau/2} + \kappa e^{-\mu\tau/2} - \kappa e^{-\kappa\tau/2}\right), \\ &= \langle\hat{c}(t)\hat{c}^\dagger(t)\rangle\left(\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\right).\end{aligned} \tag{5.23}$$

Following a similar procedure, one can also establish that

$$\langle\hat{c}(t)\hat{c}(t+\tau)\rangle = \langle\hat{c}^2(t)\rangle\left(\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\right). \tag{5.24}$$

Therefore, on a account of Eqs. (5.13), (5.23), and (5.24), there follows

$$\langle \hat{c}_\pm(t), \hat{c}_\pm(t+\tau)\rangle_{ss} = (\Delta c_\pm)^2 \left(\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\right). \tag{5.25}$$

Now on introducing Eq. (5.25) into Eq. (5.1) and carrying out the integration, we find the spectrum of the minus quadrature fluctuations for the two-mode cavity light to be

$$\begin{aligned} S_-(\omega) = (\Delta c_-)^2 \Bigg[&\frac{\kappa}{\kappa-\mu}\left(\frac{\mu/2\pi}{(\omega-\omega_o)^2+[\mu/2]^2}\right) \\ &- \frac{\mu}{\kappa-\mu}\left(\frac{\kappa/2\pi}{(\omega-\omega_o)^2+[\kappa/2]^2}\right)\Bigg], \end{aligned} \tag{5.26}$$

in which $(\Delta c_-)^2$ is given by (4.22).

We realize that the variance of the minus quadrature in the interval between $\omega' = -\lambda$ and $\omega' = \lambda$ is expressible as

$$(\Delta c_-)^2_{\pm\lambda} = \int_{-\lambda}^{\lambda} S_-(\omega')d\omega', \tag{5.27}$$

in which $\omega' = \omega - \omega_o$. Hence applying Eq (5.26) and taking into account (4.22) along with (3.44), we easily get

$$\begin{aligned} (\Delta c_-)^2_{\pm\lambda} = &\left[\frac{2\kappa/\pi}{\kappa-(\gamma_c+2r_a)}tan^{-1}\left(\frac{2\lambda}{\gamma_c+2r_a}\right) - \frac{2(\gamma_c+2r_a)/\pi}{\kappa-(\gamma_c+2r_a)}tan^{-1}\left(\frac{2\lambda}{\kappa}\right)\right] \\ &\times \frac{\gamma_c}{\kappa}\left(N + \langle\hat{N}_a\rangle - 2\sqrt{\frac{r_a}{\gamma_c}}\langle\hat{N}_c\rangle\right). \end{aligned} \tag{5.28}$$

On account of (3.6) and (3.7), Eq (5.28) is also expressible as

$$\begin{aligned} (\Delta c_-)^2_{\pm\lambda} = &\left[\frac{2\kappa/\pi}{\kappa-(\gamma_c+2r_a)}tan^{-1}\left(\frac{2\lambda}{\gamma_c+2r_a}\right) - \frac{2(\gamma_c+2r_a)/\pi}{\kappa-(\gamma_c+2r_a)}tan^{-1}\left(\frac{2\lambda}{\kappa}\right)\right] \\ &\times \frac{\gamma_c}{\kappa}N\left(1 + \frac{r_a - 2\sqrt{\gamma_c r_a}}{\gamma_c+2r_a}\right). \end{aligned} \tag{5.29}$$

Furthermore, upon setting $r_a = 0$ in Eq (5.29), we find the local quadrature variance of a two-mode vacuum state to be

$$(\Delta c_-)^2_{\nu\pm\lambda} = \frac{\gamma_c}{\kappa}N\left[\frac{2\kappa/\pi}{\kappa-\gamma_c}tan^{-1}\left(\frac{2\lambda}{\gamma_c}\right) - \frac{2\gamma_c/\pi}{\kappa-\gamma_c}tan^{-1}\left(\frac{2\lambda}{\kappa}\right)\right]. \tag{5.30}$$

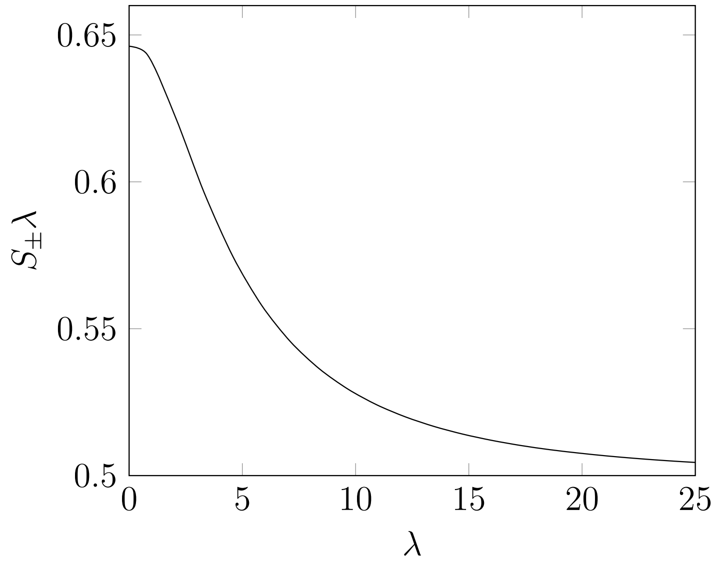

Figure 3: A plot of local quadrature squeezing versus λ for $\gamma_c = 0.1,\ r_a = 0.025, \kappa = 0.8$ *and* $N = 50$.

We define the quadrature squeezing of the cavity light in the $\lambda_\pm$ frequency interval by

$$S_{\pm\lambda} = \frac{(\Delta c_-)^2_{\nu\pm\lambda} - (\Delta c_-)^2_{\pm\lambda}}{(\Delta c_-)^2_{\nu\pm\lambda}}, \tag{5.31}$$

so that on account of (5.29) and (5.30), there follows

$$S_{\pm\lambda} = 1 - \frac{\dfrac{2\kappa/\pi}{\kappa-(\gamma_c+2r_a)} tan^{-1}\left(\dfrac{2\lambda}{\gamma_c+2r_a}\right) - \dfrac{2(\gamma_c+2r_a)/\pi}{\kappa-(\gamma_c+2r_a)} tan^{-1}\left(\dfrac{2\lambda}{\kappa}\right)}{\dfrac{2\kappa/\pi}{\kappa-\gamma_c} tan^{-1}\left(\dfrac{2\lambda}{\gamma_c}\right) - \dfrac{2\gamma_c/\pi}{\kappa-\gamma_c} tan^{-1}\left(\dfrac{2\lambda}{\kappa}\right)}$$

$$\times\left(1 + \frac{r_a - 2\sqrt{\gamma_c r_a}}{\gamma_c + 2r_a}\right). \tag{5.32}$$

The plot in Figure 3 indicates that the maximum squeezing is 64.5 % below the vacuum-state level and happens to be in the $\lambda = \pm 0.1$ frequency interval. We also notice that as λ increases the local quadrature squeezing approaches the global quadrature squeezing. It is not hard to realize that the mean photon number is very small when the quadrature squeezing is relatively large. Evidently, one can increase the mean photon number by superposing several squeezed light beam.

Finally, defining the quadrature squeezing of the output light in the aforementioned frequency interval by

$$S^{out}_{\pm\lambda} = \frac{(\Delta c^{out}_{-})^2_{\nu\pm\lambda} - (\Delta c^{out}_{-})^2_{\pm\lambda}}{(\Delta c^{out}_{-})^2_{\nu\pm\lambda}}, \tag{5.33}$$

and taking in to account the fact that

$$(\Delta c^{out}_{-})^2_{\nu\pm\lambda} = \kappa(\Delta c_{-})^2_{\nu\pm\lambda}, \tag{5.34}$$

together with

$$(\Delta c^{out}_{-})^2_{\pm\lambda} = \kappa(\Delta c_{-})^2_{\pm\lambda}, \tag{5.35}$$

we arrive at

$$S^{out}_{\pm\lambda} = S_{\pm\lambda}. \tag{5.36}$$

We see that the quadrature squeezing of the output light in a given frequency interval is the same as that of the cavity light in the same frequency interval.

6 Conclusion

We considered a three-level laser in which the three-level atoms available in a closed cavity are pumped from the bottom to the top level by means of electron bombardment. We have carried out our analysis by putting the vacuum noise operators in normal order and applying the large-time approximation scheme. The procedure of normal ordering the noise operators renders the vacuum reservoir to be a noiseless physical entity. We maintain the standpoint that the notion of a noiseless vacuum reservoir would turn out to be compatible with observation.

We have seen that the light generated by the three-level laser operating under the condition $\gamma_c > \frac{1}{4}r_a$ is in a squeezed state, with the maximum quadrature squeezing being 50 % below the vacuum-state level. This occurs when the three-level laser is operating below threshold at $\gamma_c = 4r_a$. We have also established that the quadrature squeezing of the output light in a given frequency interval is the same as that of the cavity light in the same frequency interval. On the basis of this result, we come to the conclusion that the quadrature squeezing of the laser light is an intrinsic property of the individual photons.

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Declaration

I, the undersigned, declare that this project is my original work and that all sources of materials used for the project have been correctly acknowledged.

Name: Zeleke Behailu.

Signature ____________________

Date: March 2017

The project has been submitted with my approval as University advisor.

Name: Fesseha Kassahun (PhD).

Signature ____________________

Date: March 2017