NON-DIPOLAR COMPONENTS OF PULSAR MAGNETIC FIELDS

By

Feyisso Sado

A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE IN PHYSICS
ADDIS ABABA UNIVERSITY
ADDIS ABABA, ETHIOPIA
JULY 2010

ADDIS ABABA UNIVERSITY
FACULTY OF SCIENCE
DEPARTMENT OF PHYSICS

The undersigned hereby certify that they have read and recommended to the Faculty of Science School of Graduate Studies for acceptance a thesis entitled "**Non-dipolar components of Pulsar magnetic fields** " by **Feyisso Sado** in partial fulfillment of the requirements for the degree of **Master of Science in Physics**.

Advisor: ______________________
Dr. LEGESSE WOTRO

Examiners: ______________________
PROF. MAL′NEV

____
PROF. GHOLAP

ADDIS ABABA UNIVERSITY

Date: **July 2010**

Author: **Feyisso Sado**

Title: **NON-DIPOLAR COMPONENTS OF PULSAR MAGNETIC FIELDS**

Department: **Physics**

Degree: **M.Sc.** Convocation: **July** Year: **2010**

____

Signature of Author

For those relatives and friends of mine who have unlimited potential but can not attend the school and university.

Table of Contents

List of Figures

Acknowledgements

Glory to the almighty **Allah**, the lord of '**Alamin**' and the ultimate source of success and wisdom.

I am deeply indebted to my advisor, **Dr. Legesse Wetro**, for his many suggestions, countless support and friendly approach during this thesis research.

My secret heartfelt gratitude goes to my sister, **Jemila Beriso**, for her endless wishes to my success and material support through out my study.

Abstract

The recent observations indicate that the presence of non-dipolar pulsars magnetic fields. Post-Newtonian approximation(PN) used in solving Einstein's field equation in general relativity was made to derive the much more complicated non-dipolar pulsar magnetic field. In this work, we analyze the presence of non-dipolar components of the pulsar magnetic field, as derived from a recent model by Kebede[5]. We will derive both the dipolar and non dipolar components of the field. We consider the quadrupole and octopole fields, which result in the non dipolar components of the pulsar's magnetic field.

Introduction

Since their discovery by Jocelyn Bell Burnel and Antony Hewish at Cambridge in 1967(Hewish et al.1968), pulsars have assumed a central role in astronomy and astrophysices. They offer an apportunity to explore theoretical physics under extreme condition. Pulsars provide a wealth of information about NS physics, general relativity, the Galactic gravitational potential and magnetic field, the interstellar medium, celestial mechanics, planetary physics and even cosmology.

Magnetic fields allow pulsars to be distinguished from each other and classified into phenomenologically very different groups. They classified as Radio pulsars, X-ray binaries, Magnetars and Thermal X-ray emitters.

Radio pulsars are regularly pulsating sources of radio waves, interpreted as highly magnetized, rotating neutron stars (Pacini 1967; Gold 1968). Beams of radiation emerging from the poles of a roughly dipolar magnetic field misaligned with respect to the rotation axis appear as pulses every time they sweep the location of the Earth. These pulses reveal rotation periods (P) from 1.55 milliseconds (ms) to several seconds. In terms of rotation rate(spin) parameters "spin-down age" for the pulsar, radio pulsars fall into, young ($\tau 10^{3-7}$ yrs), relatively slow ($P \sim$ 16ms to several seconds), and strongly magnetized ($B \sim 10^{11-13}$ G) classical pulsars, and old (10^{8-10} yrs), fast (1.55 to several ms), and weakly magnetized (10^{8-9} G) millisecond pulsars. The first millisecond pulsar **B1937+21** by Shrinivas Kulkarni, Donald Bcker at Arecibo(Backer et al.1982). This remarkable NS spins at (64Hz), very close to the theoretical rotation limit; **PSR B1937+21** remains the most rapidly spinning NS known.

NS with young, high-mass companions (high-mass X-ray binaries or HMXBs) tend to appear as X-ray pulsars, in which the accreted material is presumably channelled by the magnetic field onto the polar caps. In some cases, cyclotron features have been found in the X-ray spectrum, corresponding to magnetic fields $B \sim (1-4)\times\ 10^{12}$ G (e.g., Makishima et al. 1999; Coburn et al. 2002). Note that these spectral features (found also in magnetars and thermal emitters, see below) are the only direct measurements of neutron star magnetic fields, akin to the many measurements of magnetic fields on white dwarfs and other stars. The first binary pulsar B1913+16 by Russell Hulse and Joseph Taylor at Arecibo in 1974 (Hules and Taylor 1975).

Two intriguing kinds of astronomical objects have in recent years found a likely interpretation as very highly magnetized Magnetars NS (Kouveliotou et al. 2003): Soft gamma-ray repeaters (SGRs) are objects which repeatedly emit bursts of gamma-rays, in addition to persistent X-rays. For three of these sources, regular pulses have been observed in the persistent X-ray emission, allowing the measurement of a rotation period and period derivative (Kouveliotou et al. 1998; Hurley et al. 1999). Anomalous X-ray pulsars (AXPs) show persistent X-ray emission, modulated at a stable, slowly lengthening period. Contrary to the standard, binary X-ray pulsars, they show no evidence for a companion star (Mereghetti 2000).

The dipole fields inferred from the spin-down rate are 10^{14-15} G (Kouveliotou et al. 1998; Hurley et al. 1999), much larger than in previously known classical pulsars, though radio pulsars with similar inferred dipole fields have recently been found (Camilo et al. 2000; McLaughlin et al. 2003). In addition, features in the X-ray spectra of both SGR 1806-20 (Ibrahim et al. 2002) and AXP 1RXS J170849-400910 (Rea et al. 2003), interpreted as proton cyclotron resonance lines (Ibrahim et al. 2003), indicate $B \sim 10^{15}$ G, in reasonable agreement with the inferred dipole fields with additional non-dipolar field.

It is widely accepted that from different observational hints pulsars have non-dipolar magnetic field. Thus we are aimed at the analytical study of non-dipolar magnetic field based on post-Newtonian approximation potential fields.

The first chapter of the paper dicsuss basic pulsar properties and some relevant theoretical background about pulsar's formation and magnetic field. The Post-Newtonian Dynamics briefly discussed in chapter two. The derivation of simple dipolar field is given in Chapter 3. We present the derivation of non-dipolar field in chapter 4. Finally, we discuss our results and conclude in the last chapter.

Chapter 1

Pulsars and Neutron Stars

1.1 Pulsar

Neutron stars are widely believed to be formed in the gravitational angular-momentum conserving supernova collapse of the core of massive stars ($M \geq 8M_{\odot}$),where $M_{\odot}$ solar masses. The result is a repidly rotating neutron of roughly nuclear density whose existence was predicted by Bade and Zwicky(1934) and whose magnetic properties were predicted by Pacini(1968). Pulsar is rapidly rotating highly magnetized NS. The most visible manifestation of NS is during their generally rather short phases as radio pulsers, the first of which was discored by Hewish etal.(1968). While over thusand of these are now know, many NSs are now observed at X-ray wavelengths and higher energies. It is beleived that the radio emission originates in electrodynamic processes in the open magnetic field lines of the magnetosphere above the magnetic poles. The result is a large flux of highly relativistic charged particles which radiate coherently to produce a beam of radiation along the magnetic field lines, which is observed as pulses as the rotation of the star passes the beam over the Earth. Neutron stars are very dense and spin very fast and are typically only 10-15 km in radius. Because neutron stars form from burnt-out stars, they do not glow. The collapse of the star causes the matter to be converted into mostly neutrons, hence the name neutron star. Some neutron stars emit radio waves that pulse on and off. These stars are called **pulsars**. Pulsars don't really turn radio waves on and off it just appears that way to observers on Earth because the star is spinning. What happens is that the radio waves only escape from the North and South magnetic poles of the neutron star. If the spin axis is tilted with respect to the magnetic poles, the escaping radio waves sweep around like the light beam from a **lighthouse**. Far away on Earth, radio astronomers pick up the radio waves only when the beam sweeps across the Earth. A pulsar emits two very high-energy beams into space, concentrated along its magnetic axis (the magnetic field is around one trillion times that of the Earth's). The beams are made of material usually stolen from a companion star, and the particles are accelerated to speeds as great as 20 perecent that of light.

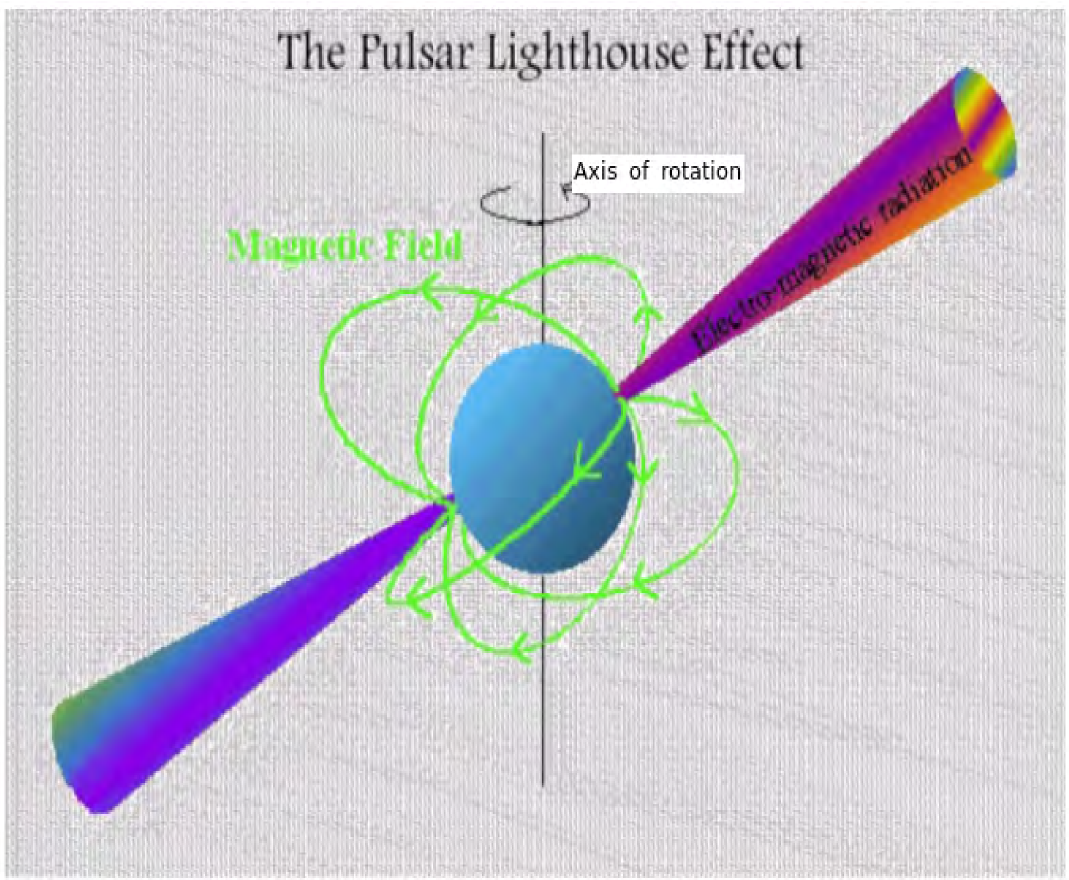

Figure 1.1: The Pulsar Lighthouse Effect

These pulses allow us to detect the objects and to observe their rotation with very high precision. Simple electrodynamic arguments allow the estimation of the surface magnetic-field strength from the rate of slow-down, which is a million times stronger than the strongest magnetic fields produced on Earth.

Pulsars are rapidly spinning and highly magnetized, beam electromagnetic radiation out along its magnetic axis. The beam is framed by the magnetic field lines emanating from the pulsar. Because the beam is tilted slightly from the rotation axis, it revolves as the pulsar spins. Like a light-house beacon, if the magnetic axis happens to point in the general direction of the Earth, we may see a "**pulse**" of radiation across the spectrum, from radio waves to visible light to gamma rays.

The "pulses" of high-energy radiation we see from a pulsar are due to a misalignment of the Pulsars's rotation axis and its magnetic axis. Pulsars pulse because the rotation of the neutron star causes the radiation generated within the magnetic field to sweep in and out of our line of sight with a regular period. Neutron stars for which we see such pulses are called **pulsars**, or sometimes **spin-powered pulsars**, indicating that the source of energy is the rotation of the magnetized neutron star. Pulsars are rotating magnetized neutron star that can produce radiation by spinning its powerful magnetic field through space and their magnetic moments inclined to spin axes. There are also '**accreting pulsars**' which funnel matter from a companion star onto their magnetic polar caps as they rotate.

A pulsar uses up a lot of its rotational energy moving its magnetic field around this way, and so it gradually slows down. When it slows down enough, it no longer radiates very much energy, and so it is no longer considered a pulsar. This usually happens within a few million years. If a neutron star had only a weak magnetic field, it would also not be a pulsar. Pulsars will eventually slow down and stop spinning, due to energy lost as it sends off ripples in space (also called gravitational waves, that emanate from all moving massive objects; the waves travel at the speed of light). It will then be seen as an ordinary neutron star because if the beams no longer sweep past Earth we can't see it "pulsing".

The pulsars have incredibly high magnetic fields of strength up to a thousand million Tesla. These produce strong radio signals from the star in two opposite directions. As the star rotates these radio signals are swept around the sky in a circle. This was the 'lighthouse' explanation of the pulsars which Gold had proposed. However, the axis of the magnetic field is not aligned with the neutron star's rotation axis. The combination of this strong magnetic field and the rapid rotation of the neutron star produces extremely powerful electric fields, with electric potential in excess of a thousand million volts. Electrons are accelerated to high velocities by these strong electric fields.

Interior Structure Of Neutron Stars

The interior structure of a neutron star consists of iron, neutron rich nuclei and electrons in the outer crystalline solid crust. The inner crust contains neutron rich nuclei, free superfluid neutrons and electrons and the interior, superfluid neutrons, superfluid protons and electrons. The makeup of the core is unknown. It is mystery how matter behaves at these amazingly high densities.

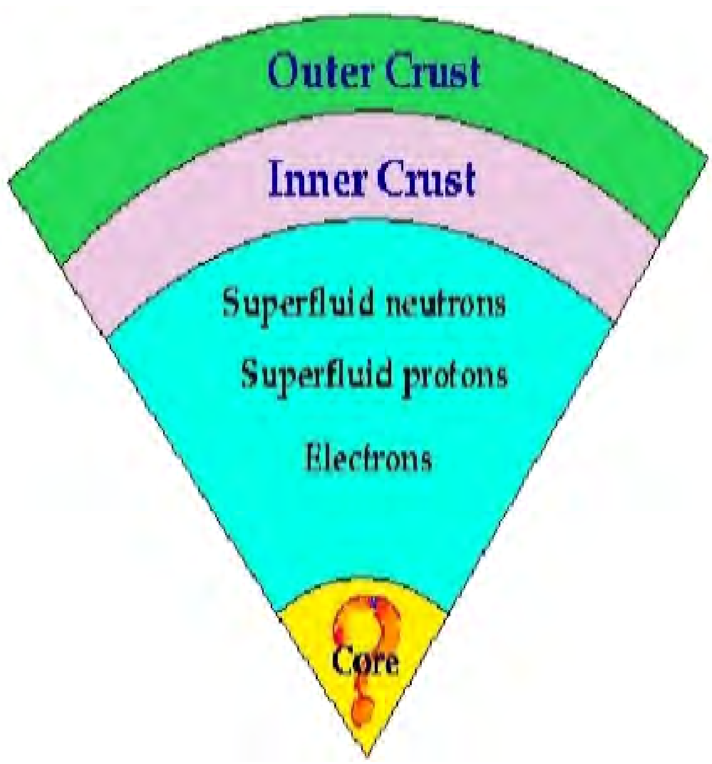

Figure 1.2: structure of neutron star

The outer layer of a neutron star has a complex structure which depends strongly on the nuclear density. In the inner crust of the star, due to the high density and pressure, a large fraction of neutrons occupy unbound states.

1.2 Pulsar Formation

A neutron star is a star made entirely out of neutrons, as the name suggests. After a star goes nova, the remaining core collapses while the outer layers are blasted off into space to create a nebula(cloud). Gravity shrinks and condenses the core into a sphere about the size of Manhattan (25 km diameter) within a few seconds. This shows that neutron stars have very high density($10^{18} kg/m^3$). A neutron star is so dense that a pinhead's worth of material from one would weight as much as a supertanker. Because of its small size and high density, a neutron star possesses a surface gravitational field about 2×10^{11} times that of Earth[4]. During core collapse, the angular momentum is conserved, there will be an increase in the number of rotation.

Electro-Magnetic(EM) Force keeps the electrons out of the atom's nucleus. However, the most important principle that comes into play is the Pauli Exclusion Principle and electron degeneracy. This says that no two electrons can occupy the same quantum state. This boils down to no two electrons being able to occupy the same place. Therefore, Pauli Exclusion Principle is broken. As more mass is piled on (up to 1.44 solar masses, also called the Chandraskhar Limit), the densities get higher and the electrons are boosted to higher energy levels. Eventually, the required speed for the electrons is greater than that of light in order to support the electron degeneracy. Once this happens, the star collapses, for the electron degeneracy pressure can no longer support it. In a few brief seconds, the star collapses down to a neutron star, supported by neutron degeneracy, where no two neutrons can occupy the same quantum state. This keeps the neutron star from collapsing further, until its critical mass is reached at about 3 solar masses. Then, the collapse is to a black hole.

1.3 The Magnetic Field

Neutron stars contain persistent, ordered magnetic fields that are the strongest known in the Universe. Pulsars are very strongly magnetized neutron stars. Their dipole field strengths are consistent with the collapse of a normal star with a polar field of order 100gauss, the flux being conserved, in the collpsing stellar material. Since the interior of the star is super fluid, the decay time is long compared with the life time of a pulsar. Polar field strengths reaching 10^{12}G occur in young pulsars; fileds of order 10^{10} gauss are found in the main body of pulsars, while in the order evolved millisecond pulsars it is relativily as low as 10^9. The active pulsars in the galaxy have surface magnetic fields in the range of $\sim 2-6\times 10^{12}$G: This is still thousands of times larger than fields attainable in the laboratory. Despite the intensity of the magnetic field, it has very little effect on the structure of the star. Outside the star, however the magnetic field **B** completely dominates all physical processes, even outweighing gravitation by a very large factor.

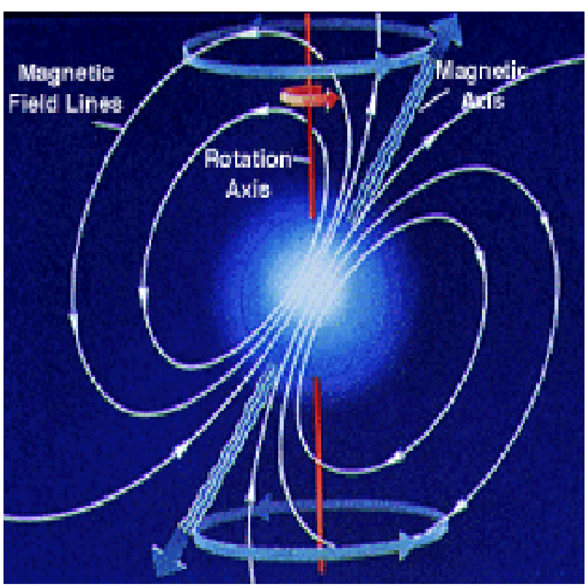

Figure 1.3: Pulsar's rotation axis, magnetic axis, and magnetic fieled

The external magnetic field plays acrucial part in several observable characteristics of pulsars, which is our area of interest. The field is substantially misalgned with the rotation axis and therefore generates a radiating electromagnetic wave at the rotation frequency; this accounts for the main loss of rotational energy and the observed slow down.

Lines of magnetic force will be carried along with the collapse of the plasma core of the star. As the core collapses the magnetic field lines are pulled more closely together, intensifying the magnetic field to valuess of the order of 10^{12} gauss, which leads to magnetic flux Conservation. Pulsars produce a Lighthouse Effect due to an intense magnetic field whose axis is misaligned with its rotational axis is generally accepted. One popular view is that the rapid rotation and intense magnetic field of the neutron star generate strong electric fields, which accelerate charged particles (principally electrons because they are less massive) near the magnetic poles where the magnetic field is most intense.

Chapter 2

Post-Newtonian Dynamics

2.1 Introduction

In the PN approximation we attempt to study the equations of motion and the field equations in Einsteins gravity in the Newtonian way as closely as possible. The Newtonian order potential and 1PN order metric variables are determined by Einsteins equations interms of the Newtonian matter and potential variables. Thus, the metric (or relativistic) contributions are interpreted as small correction terms to the well-known Newtonian equations. In the PN approximation, by assuming that the relativistic effects are small, we expand relativistic corrections in powers of small parameters $(v/c)^2$. Here, and in what follows , we use the notation of Ref.[6] with the system of units c =1.

$$\bar{v}^2 \sim \frac{G\bar{M}}{\bar{r}} \tag{2.1.1}$$

Where, $\bar{v}$, $\bar{M}$ and $\bar{r}$ denote the typical velocity, mass and separation of the particles in the system under the consideration, respectively.

In this chapter, we aim at a derivation of equation of motion in gravitation and the post-Newtonian fields.

2.2 The Equation of Motion in Gravitation

Now we can start to look at the dynamics. The slow reaction of matter to the inertio-gravitational field depends on the gravity term affine connection[6]: Consider a particle which moves freely through the neighborhood of an accelerated observer. As seen in an inertial reference frame, the particle moves through spacetime on a straight line, also called a geodesic of curved spacetime. Let $\mathbf{u}$ be the tangent 4-velocity field defined along spacetime-like curve parametrized by the prefered affine parameter τ, the components of $\mathbf{u}$ in this frame are $\mathbf{u}^\alpha = \dfrac{dx^\alpha}{d\tau}$, where τ is the particles proper time. Correspondingly, a geometric, frame-independent version of its geodesic law of motion takes the form

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{2.2.1}$$

Suppose for simplicity that the particle is moving slowly relative to the observer, so its ordinary velocity $v^j = \dfrac{dx^j}{dt}$ is very nearly equal to $u^j = \dfrac{dx^j}{d\tau}$ and is very small compared to unity (the speed of light), and $u^0 = \dfrac{dx^0}{d\tau}$ is very nearly unity. Then to first order in the ordinary velocity v^j , the spatial part of the geodesic equation (2.2.1) becomes

$$\frac{d^2x^i}{dt^2} = -\Gamma^i_{00} - 2\Gamma^i_{0j}v^j \tag{2.2.2}$$

Here, from the formula of affine connection[6] the Newtonian approximation gives Γ^i_{00} to order $\dfrac{v^2}{r}$ and Γ^i_{0j} to order $\dfrac{v^3}{r}$. Let us the general relativistic gravitational field generated by some material source,i.e. the solution $g_{\mu\nu}$ of the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.2.3}$$

where, $T_{\mu\nu}$ denotes the stress-energy tensor of the matter, satisfying some suitable no-incoming-radiation condition. The post-Newtonian expansion of $g_{\mu\nu}$ is a combined weak-field expansion in powers(2.1.1) of Φ, where $\Phi \sim G\bar{M}/\bar{r}$ is a gravitational potential. The so-called 1PN approximation consists of keeping the following terms in the various components of the space-time metric $g_{\mu\nu}$:

$$g_{00} = -1 + g^{(2)}_{00} + ... \tag{2.2.4}$$

$$g_{ij} = \delta_{ij} + g_{ij}^{(2)} + ... \quad (2.2.5)$$

$$g_{i0} = g_{i0}^{(3)} + g_{i0}^{(5)} + ... \quad (2.2.6)$$

and the inverse of the metric tensor

$$g^{00} = -1 + g^{00(2)} + ... \quad (2.2.7)$$

$$g^{ij} = \delta_{ij} + g^{ij(2)} + ... \quad (2.2.8)$$

$$g^{i0} = g^{i0(3)} + g^{i0(5)} + ... \quad (2.2.9)$$

where the symbols $g_{\mu\nu}^{(N)}$ denoting the terms in $g_{\mu\nu}$ of order $\bar{v}^{(N)}$.

The harmonic coordinate condition $g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0$, which reduces to

$$4\frac{\partial \phi}{\partial t} + \nabla.A = 0 \quad (2.2.10)$$

has been imposed and the resulting simplfyed field equation(2.2.3)[6] can be written as

$$\nabla^2 g_{00}^{(2)} = -8\pi G T^{00(0)} \quad (2.2.11)$$

$$\nabla^2 g_{i0}^{(3)} = 16\pi G T^{i0(1)} \quad (2.2.12)$$

$$\nabla^2 g_{ij} = -8\pi G \delta_{ij} T^{00(0)} \quad (2.2.13)$$

From (2.2.11) we find as expected

$$g_{00}^{(2)} = -2\phi \quad (2.2.14)$$

where ϕ is the Newtonian scalar potential, defined by poission's equation

$$\nabla^2 \phi = 4\pi G T^{00(0)} \quad (2.2.15)$$

for $g_{00}^{(2)}$ must vanish at infinity (2.2.15) has the solution

$$\phi(\mathbf{x}, t) = -G \int \frac{T^{00(0)}}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}' \quad (2.2.16)$$

From (2.2.13) we obtain the solution for $g_{ij}^{(2)}$ that vanish at infinity is

$$g_{ij}^{(2)} = -2\delta_{ij}\phi \tag{2.2.17}$$

Finally, $g_{i0}^{(3)}$ is a new post-Newtonian vector potential ζ:

$$g_{i0}^{(3)} \equiv \zeta_i \tag{2.2.18}$$

has the solution (2.2.12) that vanish at infinity is

$$\zeta(\mathbf{x}, t) = -4G \int \frac{T^{i0(1)}(x', t)}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}' \tag{2.2.19}$$

As the result of the PN approximation to Einstein's field equations the reduced equations can be summerized as

$$\nabla . g = -4\pi G\rho \tag{2.2.20}$$

$$\nabla \times \mathbf{B} = -4\pi G\mathbf{K} \tag{2.2.21}$$

$$\nabla \times g = 0 \tag{2.2.22}$$

Where, $g \equiv -\nabla\phi$, the "gravitoelectric field"

$\mathbf{B} \equiv \nabla \times \zeta$, the " gravitomagnetic field"

$\rho \equiv T^{00(0)}$, mass density

$k_i \equiv T^{i0(1)}$, the momentum density

we see that the correspodence with Maxwell's equation is satisfyed.

Using the coordinate condition(2.2.10) together with (2.2.14),(2.2.17) and (2.2.18) in $\Gamma_{0i}^{0(2)}$ and $\Gamma_{0j}^{i(3)}$ gives the desired components of the affine connection from the metric[6]

$$\Gamma_{0i}^{0(2)} = \frac{\partial\phi}{\partial x^i} \tag{2.2.23}$$

$$\Gamma_{0j}^{i(3)} = \frac{1}{2}\left(\frac{\partial\zeta_i}{\partial x^j} - \frac{\partial\zeta_j}{x^i}\right) \tag{2.2.24}$$

Therefore, we bring the low-velocity geodesic law of motion(1.2.2) into the form

$$\frac{d^2x^i}{dt^2} = -\frac{\partial\phi}{\partial x^i} + v^j\left(\frac{\partial\zeta_i}{\partial x^j} - \frac{\partial\zeta_j}{x^i}\right)$$

we can write in vector form as

$$\frac{d\mathbf{v}}{dt} = \mathbf{g} + \mathbf{V} \times \mathbf{B} \tag{2.2.25}$$

This is the standard form of the **law of motion**. In(2.2.25), the first term on the right-hand side corresponds physically to the gravitational electric field due to the failure of the frame to fall freely, and the second term is magnetic field acceleration due to the frames rotation, and expect to be generated by moving mass density.

2.3 Particle Dynamics

We shall now determine the acceleration of a particle (it should be understood that the sun, the stars and planets are regared as point particles). Eq.(2.2.1) gives the acceleration of a particle as

$$\frac{d^2U^i}{dt} = -\Gamma^i_{00} - 2\Gamma^i_{0j}v^j - \Gamma^i_{jk}v^jv^k + [\Gamma^0_{00} + 2\Gamma^0_{0j}v^j]v^i \tag{2.3.1}$$

where dU^i/dt is the i^{th} component of the acceleration of a particle. In the post-Newtonian approximation to Einstein's theory, the above equation is reduced to

$$\frac{du^i}{dt} = -\frac{\partial\phi}{\partial x^i} + 4u^i\frac{\partial\phi}{\partial x^k}u^k - u^2\frac{\partial\phi}{\partial x^i} - \left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x^i}\right)u^j - 3\frac{\partial\phi}{\partial t}u^i \tag{2.3.2}$$

If the gravito-electromagnetic feild of a system is static, then the last term in(2.3.2) vanishes. In this case, Eq.(2.3.2) can be written for static system in vector notation as

$$\frac{d\mathbf{u}}{dt} = -(1+u^2)\nabla\phi + 4\mathbf{u}(\mathbf{u}.\nabla\phi) + \mathbf{u} \times (\nabla \times \zeta) \tag{2.3.3}$$

where the first two terms are the part of the acceleration of a particle given by the total mass of the system and last term is the contribution of new Newtonian gravitatioal field, produced by moving masses. Eq.(2.3.3) is the **Lorentz force density** for a particle in post-Newtonian gravitational field, known as a "gravito-electromagnetic feild."

Using post-Newtonian fields the acceleration of a photon in gravitational field, can be determined by replacing the velocity by unity in (2.3.3)

$$\frac{d\mathbf{u}}{dt} = -2\nabla\phi + 4\mathbf{u}(\mathbf{u}\ .\nabla\phi) + \mathbf{u} \times (\nabla \times A) \tag{2.3.4}$$

and clearly noted that the speed of photon from the metric, is given by

$$|\mathbf{u}| = 1 + 2\phi + 0(v^3) \tag{2.3.5}$$

One popular view is that the rapid rotation and intense magnetic field of the neutron star generate strong electric fields, which accelerate charged particles (principally electrons because they are less massive) near the magnetic poles where the magnetic field is most intense. The charged particles, accelerated along the curved magnetic field lines produce a type of light called curvature radiation. "Whenever charged particles are accelerated electromagnetic radiation (light) is produced."

Chapter 3

Dipolar Field

Introduction

The post-Newtonian approximation is applicable in the slow motion and weak gravitational field regime. This formalism is valid all over the weak field region outside the source including the wave zone (up to future null infinity). This is the reason, why different authors ignore this field. In calculating the magnitude of the separated charges and the resulting dipole fields, Kebede (2002) ignored gravity. For this case, we assume a uniform spherical surface charge distribution as a simple case in magnetostatics. In the absence of gravity we expect to have dipolar surface fields for the assumed symmetry. In the following sections we will derive the vector potential and magnetic field by ignoring gravity.

3.1 The Vector Potential

The vector potentianal of rotating charged pulsar in an observetion point say p characterized by the radius vector $\mathbf{r}$ can be written according to [5] the solution for Poisson equation and taking into account time-dependent surface current density $J(x,t)$, defined as, is given by

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{J(x')}{|x - x'|} d^3x' \tag{3.1.1}$$

using the spherical harmonics expansion[5]

$$\frac{1}{|x-x'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\phi') Y_{lm}(\theta,\phi) \tag{3.1.2}$$

which gives the potential in a completely factorized form in the coordinates x' and x, which are the variable of integration and the coordinate of the observation point, respectively. The vector potential will be

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \sum_{lm} \frac{4\pi}{2l+1} \int \mathbf{J}(x') \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\phi') Y_{lm}(\theta,\phi) d^3x' \tag{3.1.3}$$

In a uniform spherical charge distribution with surface charge density σ, the current density in the system is given by

$$\mathbf{J} = \sigma\delta(r'-R)\mathbf{v}$$

$$\mathbf{J} = \sigma\delta(r'-R)\vec{\omega}\times\mathbf{r}$$

and

$$\vec{\omega}\times\mathbf{r} = \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & \omega \\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\theta' \end{bmatrix}$$

we obtain

$$\begin{aligned} \vec{\omega}\times\mathbf{r} &= -R\omega\sin\theta'\sin\phi'\hat{\mathbf{x}} + R\omega\sin\theta'\cos\phi'\hat{\mathbf{y}} \\ &= R\omega\sin\theta'\hat{e}'_\phi \end{aligned} \tag{3.1.4}$$

where, $\hat{e}'_\phi = -\sin\phi'\hat{\mathbf{x}} + \cos\phi'\hat{\mathbf{y}}$ Therefore, the vectorial current density $\mathbf{J}$ can be written

$$\mathbf{J} = -\mathbf{J}_\phi\sin\phi'\hat{\mathbf{x}} + \mathbf{J}_\phi\cos\phi'\hat{\mathbf{y}} \tag{3.1.5}$$

since the geometry is spherical symmetric, we may choose the observation point in the xz-plane($\phi = 0$) for explicity. The azimuthal integration in (3.1.1) is symmetric about $\phi' = 0$, terms

$$\int_0^{2\pi} \cos\phi' d\phi' = \int_0^{2\pi} \sin\phi' d\phi' = 0 \tag{3.1.6}$$

have no contribution. This leaves only the y component, which is $\mathbf{A}_\phi$. Thus

$$\mathbf{A}_\phi = \frac{\mu_0}{4\pi}\sigma\omega R\sum_{lm}\frac{4\pi}{2l+1}\int\frac{Y^*_{lm}(\theta',\phi')}{r_>^{l+1}}r_<^l Y_{lm}(\theta,\phi)\delta(r'-R)\cos\phi'\sin\theta' r'^2 dr' d\Omega'$$

using the solid angle $d\Omega' = \sin\theta' d\theta' d\phi'$

$$\mathbf{A}_\phi = \frac{\mu_0}{4\pi}\sigma\omega R\sum_{lm}\frac{4\pi}{2l+1}\int\frac{r_<^l}{r_>^{l+1}}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)\delta(r'-R)\cos\phi'\sin^2\theta' r'^2 dr' d\theta' d\phi' \tag{3.1.7}$$

Eq.(3.1.7) is called multipoles vector potential for a system that is spherically symmetric and rotates with angular frequency ω.

Now let us calculate the field inside and outside the sphere.

Vector potential inside the sphere

For the vector potential inside the sphere, taking the expansion $(\frac{r}{R})$ with $r_< = r$ and $r_> = R$. Then

$$\mathbf{A}_{\phi,in} = \frac{\mu_0}{4\pi}\sigma\omega R\sum_{lm}\frac{4\pi}{2l+1}\int\frac{r^l}{R^{l+1}}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)\delta(r'-R)\cos\phi'\sin^2\theta' r'^2 dr' d\theta' d\phi' \tag{3.1.8}$$

Integration over the delta function becomes

$$\int\delta(r'-R)r'^2 dr' = R^2$$

and the angular factor can be written[5]

$$\sin\theta'\cos\phi' = -\sqrt{\frac{8\pi}{3}}Re[Y_{11}(\theta,\phi)] \tag{3.1.9}$$

$$\mathbf{A}_{\phi,in} = \frac{\mu_0}{4\pi}\sigma\omega R^3\sum_{lm}\frac{4\pi}{2l+1}\frac{r^l}{R^{l+1}}\left(-\sqrt{\frac{8\pi}{3}}\right)Re[Y_{11}(\theta,\phi)]\int Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)\sin\theta' d\theta' d\phi'$$

The normalization and orthogonality conditions are

$$\int_0^{2\pi}d\phi'\int_0^{\pi}\sin\theta' d\theta' Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi) = \delta_{ll'}\delta_{mm'} \tag{3.1.10}$$

we see that, only the $l = 1$ and $m = 1$ terms will survive in the integration. Consequently

$$\mathbf{A}_{\phi,in} = \frac{\mu_0}{4\pi}\sigma\omega R^3\frac{4\pi}{3}\frac{r}{R^2}\left(-\sqrt{\frac{8\pi}{3}}\right)Re[Y_{11}(\theta,\phi)]$$

but the angular factor

$$Re[Y_{11}(\theta,\phi)] = \sqrt{\frac{3}{8\pi}}\sin\theta\cos\phi$$

Thus, the vector potential inside the sphere can be

$$\mathbf{A}_{\phi,in} = \frac{\mu_0\sigma R\omega}{3} r\sin\theta \tag{3.1.11}$$

evidently we can write the surface charge distribution as $Q = 4\pi R^2\delta$

The Vector potential outside the sphere

The Vector potential everywhere outside the sphere will be obtianed by using the multipole Vector potential (3.1.7) substituting $r_< = R$ and $r_> = r$ we get

$$\mathbf{A}_{\phi,out} = \frac{\mu_0\sigma\omega R^3}{3}\sum_{lm}\frac{4\pi}{2l+1}\frac{R^l}{r^{l+1}}\Big(-\sqrt{\frac{8\pi}{3}}\Big)Y_{lm}(\theta,\phi)\int Y^*_{lm}(\theta',\phi')Y_{11}(\theta',\phi')\sin\theta' d\theta' d\phi'$$

again employing the normalization and orthogonality conditions, we obtain

$$\mathbf{A}_{\phi,out} = \frac{\mu_0\sigma\omega R^3}{3}\frac{R}{r^2}\Big(-\sqrt{\frac{8\pi}{3}}\Big)\Big(-\sqrt{\frac{3}{8\pi}}\Big)\sin\theta\cos\phi$$

Finally the Vector potential everywhere outside the sphere's crust, takes the form

$$\mathbf{A}_{\phi,out} = \frac{\mu_0\sigma R^4}{3r^3}\omega r\sin\theta \tag{3.1.12}$$

3.2 The Magnetic Field

In basic differential laws of magnetostatics, we see that due to the absence of free magnetic poles(charges) the divergence of magnetic field **B** vanishes. Therefore, **B** must be the curl of some vertor field **A(x)**, called the vector potential[5].

Hence, we can write as

$$\mathbf{B}(\mathbf{x}) = \vec{\nabla}\times\mathbf{A}(\mathbf{x}) \tag{3.2.1}$$

But

$$\vec{\nabla}\times\mathbf{A}(\mathbf{x}) = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi)-\frac{\partial A_\theta}{\partial\phi}\right]\hat{e}_r+\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right]\hat{e}_\theta+\left[\frac{1}{r}\frac{\partial(rA_\theta)}{\partial r}-\frac{1}{r}\frac{\partial A_r}{\partial\theta}\right]\hat{e}_\phi$$

In our case, we have only the ϕ component of the vector potential.

Therefore,

$$\mathbf{B}(\mathbf{x}) = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi)\right]\hat{e}_r - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\hat{e}_\theta \tag{3.2.2}$$

we immediately note that the components of magnetic field,

$$\begin{aligned}\mathbf{B}_r &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi)\right]\\ \mathbf{B}_\theta &= -\frac{1}{r}\frac{\partial(rA_\phi)}{\partial r}\\ \mathbf{B}_\phi &= 0\end{aligned} \tag{3.2.3}$$

The magnetic field inside the sphere

We now seek to obtain the field inside the sphere by solving (3.2.1) for $A_{\phi,in}$

$$\begin{aligned}\mathbf{B}_{in} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_{\phi,in})\right]\hat{e}_r - \frac{1}{r}\frac{\partial}{\partial r}(rA_{\phi,in})\hat{e}_\theta\\ &= \left[\frac{\cos\theta A_{\phi,in}}{r\sin\theta} + \frac{1}{r}\frac{\partial A_{\phi,in}}{\partial\theta}\right]\hat{e}_r - \left[\frac{A_{\phi,in}}{r} + \frac{\partial A_{\phi,in}}{\partial r}\right]\hat{e}_\theta\end{aligned}$$

substituting for $A_{\phi,in}$, it then follows

$$\begin{aligned}\mathbf{B}_{in} &= \frac{\mu_0\sigma R\omega}{3}\left[\frac{\cos\theta r\sin\theta}{r\sin\theta} + \frac{1}{r}\frac{\partial r\sin\theta}{\partial\theta}\right]\hat{e}_r - \frac{\mu_0\sigma R\omega}{3}\left[\frac{r\sin\theta}{r} + \frac{\partial r\sin\theta}{\partial r}\right]\hat{e}_\theta\\ &= \frac{2\mu_0\sigma R\omega}{3}\left(\cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta\right)\end{aligned} \tag{3.2.4}$$

Employing the unit vectors $\hat{e}_r$ and $\hat{e}_\theta$, the radial and the angular components respectively

$$\hat{e}_r = \sin\theta\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}}$$

$$\hat{e}_\theta = \cos\theta\hat{\mathbf{j}} - \sin\theta\hat{\mathbf{k}} \tag{3.2.5}$$

Eq.(3.2.4) can be simplified using (3.2.5)

$$\begin{aligned}\cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta &= \cos\theta(\sin\theta\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}}) - \sin\theta(\cos\theta\hat{\mathbf{j}} - \sin\theta\hat{\mathbf{k}}) \\ &= \hat{\mathbf{k}}\end{aligned}$$

As stated above, a sphere of radius R having a uniform surface charge density

$$\sigma = \frac{-Q}{4\pi R^2} \tag{3.2.6}$$

which is spinning round at uniform angular velocity $\omega = \omega\hat{\mathbf{k}}$ obeys $R\omega << C$, and C is the speed of light in vacuum. While the interior of the sphere contains equal and opposite charge densities $\pm\rho$. These charge densities are at rest with respect to the rotating sphere (as an external energy source would be required to maintain any motion of charges with respect to the conducting surface). There is no (macroscopic) electric field inside the sphere, but the rotating surface charge, with surface current density (3.1.5)

$$\mathbf{J} = \sigma R\omega\sin\theta\hat{\phi} = \frac{-\mathbf{Q}\omega\sin\theta}{4\pi R}\hat{\phi}$$

in a spherical coordinate system (r, θ, ϕ) centered on the sphere. This current distribution is the same as that of a sphere with uniform magnetization density $\mathbf{M} = \mathbf{M}\hat{\mathbf{k}}$, where

$$\mathbf{M} = \frac{\mathbf{J}}{C\sin\theta} \tag{3.2.7}$$

in Gaussian units, and so the interior of the sphere has a uniform magnetic field ($C = 1$)

$$\mathbf{B}_{in} = \frac{-2\mathbf{Q}\omega}{3R}\hat{\mathbf{k}} \tag{3.2.8}$$

The Magnetic field outsde the Sphere

The field outside the crust can be obtained from (3.2.1), on account of the vector potential outside the sphere as

$$\begin{aligned}\mathbf{B}_{out} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_{\phi,out})\right]\hat{e}_r - \frac{1}{r}\frac{\partial}{\partial r}(rA_{\phi,out})\hat{e}_\theta \\ &= \left[\frac{\cos\theta A_{\phi,out}}{r\sin\theta} + \frac{1}{r}\frac{\partial A_{\phi,out}}{\partial\theta}\right]\hat{e}_r - \left[\frac{A_{\phi,out}}{r} + \frac{\partial A_{\phi,out}}{\partial r}\right]\hat{e}_\theta\end{aligned}$$

Recalling that(3.1.11) for $A_{\phi,out}$

$$\begin{aligned}\mathbf{B}_{out} &= \frac{\mu_0\sigma R^4\omega}{3}\left[\frac{\cos\theta}{r^3}+\frac{1}{r^3}\frac{\partial\sin\theta}{\partial\theta}\right]\hat{e}_r-\frac{\mu_0\sigma R^4\omega}{3}\left[\frac{\sin\theta}{r^3}+\frac{\partial}{\partial r}(\frac{\sin\theta}{r^2})\right]\hat{e}_\theta \\ &= \frac{\mu_0\sigma R^4\omega}{3r^3}\left(2\cos\theta\hat{e}_r+\sin\theta\hat{e}_\theta\right)\end{aligned} \tag{3.2.9}$$

we can Simplify this using the unit vectors (3.1.14)

$$2\cos\theta\hat{e}_r+\sin\theta\hat{e}_\theta = 2\cos\theta(\sin\theta\hat{\mathbf{j}}+\cos\theta\hat{\mathbf{k}})+\sin\theta(\cos\theta\hat{\mathbf{j}}-\sin\theta\hat{\mathbf{k}})$$

$$2\cos\theta\hat{e}_r+\sin\theta\hat{e}_\theta = 2\cos\theta\sin\theta\hat{\mathbf{j}}+2\cos^2\theta\hat{\mathbf{k}}+\sin\theta\cos\theta\hat{\mathbf{j}}-\sin^2\theta\hat{\mathbf{k}}$$

$$2\cos\theta\hat{e}_r+\sin\theta\hat{e}_\theta = 3\cos\theta\sin\theta\hat{\mathbf{j}}+2\cos^2\theta\hat{\mathbf{k}}-\sin^2\theta\hat{\mathbf{k}}$$

$$2\cos\theta\hat{e}_r+\sin\theta\hat{e}_\theta = 3\cos\theta\sin\theta\hat{\mathbf{j}}+3\cos^2\theta\hat{\mathbf{k}}-(\cos^2\theta+\sin^2\theta)\hat{\mathbf{k}}$$

$$2\cos\theta\hat{e}_r+\sin\theta\hat{e}_\theta = 3\cos\theta(\sin\theta\hat{\mathbf{j}}+\cos\theta\hat{\mathbf{k}})-\hat{\mathbf{k}}$$

$$2\cos\theta\hat{e}_r+\sin\theta\hat{e}_\theta = 3\cos\theta\hat{e}_r-\hat{\mathbf{k}}$$

we thus, find the magnetic field outside the crust as

$$\mathbf{B}_{out} = \frac{\mu_0\sigma R^4\omega}{3r^3}\left(3\cos\theta\hat{e}_r-\hat{\mathbf{k}}\right) \tag{3.2.10}$$

Equation (3.2.10) clearly shows that NS magnetic fields generated by the spinning separated charges are dipolar. It can be concluded that the main characteristic of these fields is that it is constant inside (3.2.8) the sphere while outside it is dipolar-like (3.2.10) nearby to radaition zone.

Chapter 4

Non dipolar Field

In this chapter we will derive the non-dipolar components of pulsar magnetic fields resulting from the rapidly spinning of pulsar with angular frequency ω, using Post-Newtonian potential field.

4.1 The Vector Potential

First, we review the post-Newtonian vector potential as derived in[6], which is given by

$$\zeta(\mathbf{x},t) = -4G\sum_{lm}\frac{4\pi}{2l+1}\int\frac{r_<^l}{r_>^{l+1}}Y^*_{lm}(\theta^{'},\varphi^{'})Y_{lm}(\theta,\varphi)T^{io(1)}(x^{\prime},t)d^3x^{\prime} \tag{4.1.1}$$

is called a multipole gravitational vector potentail in PN approximation: When $l=0$ we have a monopole. When $l=1$ we have a dipole, with the dipole moment $p_i=\int d^3r^{\prime}\rho(r^{\prime})r^{\prime}_i$. When $l=2$ we have a quadrupole, $Q_{ij}=\frac{1}{2}\int d^3r^{\prime}\rho(r^{\prime})(3r^{\prime}_ir^{\prime}_j-\delta_{ij}r^{\prime 2})$ is a symmetric trace-free tensor. When $l=3$ we have the octopole moment and when $l=4$ the hexadecapole moment,etc. The dipole term comes as the second term next to the monopole(absent in magnetostatics) field which is basically the source of Newtonian gravity in the multipole expansion from the linearized weak field solution to general relativity. Here in the expansion vector potential, terms $l=1$ and $l=2$ are true physical effects of great interest.

Considering a cosmological object consisting of a gravity and mass for example, a star that is at rest spherically symmetric, but rotates with angular frequency $\omega(r)$, the gravitational fields source, energy momemtum tensor $T^{\mu\nu}$, leads to the momentum density $T^{i0(1)}(\mathbf{x},t)$ which is given by

$$T^{i0(1)}(x',t) = T^{00(0)}(r')[\omega(r') \times \mathbf{x}'] \tag{4.1.2}$$

evidently $T^{i0(1)}(x,t)$ is the momentum flux density playing the role of current density $\mathbf{J}(x,t)$.

(4.1.2) then gives

$$\zeta(x,t) = -4G \sum_{lm} \frac{4\pi}{2l+1} \int \frac{[\omega(r') \times \mathbf{x}']}{r_>^{l+1}} Y^*_{lm}(\theta', \varphi') Y_{lm}(\theta,\varphi) r_<^l T^{00(0)}(r') d^3x' \tag{4.1.3}$$

and

$$\omega(r') \times \vec{x}' = \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & \omega \\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\theta' \end{bmatrix}$$

$$\begin{aligned} \omega \times \vec{x}' &= -R\omega\sin\theta'\sin\phi'\hat{\mathbf{x}} + R\omega\sin\theta'\cos\phi'\hat{\mathbf{y}} \\ &= R\omega\sin\theta'(-\sin\phi'\hat{\mathbf{x}} + \cos\phi'\hat{\mathbf{y}}) \\ &= R\omega\sin\theta' e_{\hat{\phi}'} \end{aligned}$$

Since the geometry is spherical symmetric, we may choose the observation point in the x-z plane ($\phi = 0$) for explicity. The azimuthal integration in (4.1.1) is symmetric about $\phi' = 0$, terms

$$\int_0^{2\pi} \cos\phi' d\phi' = \int_0^{2\pi} \sin\phi' d\phi' = 0$$

have no contribution. This leaves only the y component, which is ζ_ϕ. Thus

$$\zeta_\phi = -G \sum_{lm} \frac{16\pi R\omega T^{00(0)}}{2l+1} \int \frac{Y^*_{lm}(\theta',\phi')}{r_>^{l+1}} r_<^l Y_{lm}(\theta,\phi) \cos\phi' \sin\phi' d^3x' \tag{4.1.4}$$

This multipole field clearly shows, the post-Newtonian law of gravity correspond quite closely to the Maxwellian laws of electromagnetism, with matter density replacing charge density.

Using the spherical harmonics(3.1.9), we see that only $m = 1$ will contribute to the sum. From (3.1.10), it follows

$$\zeta_\phi = -16\pi G R\rho\omega \sum_{l=1} \frac{Y_{l1}(\theta,0)}{2l+1} \frac{r_<^l}{r_>^{l+1}} \Big(Y_{l1}(\theta,0) \Big)$$

we observe that $Y_{l1}(\theta, 0)$ is a number depending on l; Therefore, on account of[5]

$$Y_{l1}(\theta,0) = \sqrt{\frac{2l+1}{4\pi l(l+1)}} P_l^1(\cos\theta)$$

using gamma funcion[16], we can write

$$\zeta_\phi = -16\pi G R\rho\omega \sum_{l=0} \frac{Y_{l1}(\theta,0)}{2l+1} \frac{r_<^l}{r_>^{l+1}} \left[\sqrt{\frac{2l+1}{4\pi l(l+1)}} \frac{(-1)^{n+1}\Gamma(n+\frac{3}{2})}{\Gamma(n+1)\Gamma(\frac{3}{2})} \right]$$

but

$$Y_{l1}(\theta,0) = \sqrt{\frac{2l+1}{4\pi l(l+1)}} P_l^1(\cos\theta)$$

for $l = 2n + 1$, it can be written as

$$\zeta_\phi = -16\pi G R\rho\omega \sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} \frac{r_<^l}{r_>^{l+1}} \frac{(-1)^{n+1}\Gamma(n+\frac{3}{2})}{\Gamma(n+1)\Gamma(\frac{3}{2})} P_l^1(\cos\theta)$$

so that applying the form of gamma function to the equivalent double factorials, gives the multipole vector potential of a spinning sphere as

$$\zeta_\phi = -16\pi G R\rho\omega \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n-1)!!}{2^n(n+1)!} \frac{r_<^{2n+1}}{r_>^{2n+2}} P_{2n+1}^1(\cos\theta) \qquad (4.1.5)$$

Furthermore, the vector potential inside and outside the sphere can be calculated in similar way as (3.1.6)

Vector Potential inside the Sphere

The new post-Newtonian vector field inside a spinning sphere can be obtained by replacing $r_< = r$ and $r_> = R$, and using angulr spherical harmonics $Y_{lm}(\theta, \phi)$[5], then

$$\zeta_\phi = -G \sum_{lm} \frac{16\pi R}{2l+1} \left[-\sqrt{\frac{8\pi}{3}} Re Y_{11}(\theta,\phi) \right] \int \frac{Y_{lm}^*(\theta',\phi')}{R^{l+1}} r^l Y_{lm}(\theta,\phi) T^{00(0)}(r')\omega(r') r'^2 dr' \sin\theta' d\theta' d\phi' \qquad (4.1.6)$$

The normalization and orthogonality condition gives

$$\int Y^*_{lm}(\theta', \phi')Y_{lm}(\theta, \phi)\sin\theta' d\theta' d\phi' = \delta_{ll'}\delta_{mm'}$$

only $l = 1$ and $m = 1$ terms are survive

$$\zeta_\phi = -G\frac{16\pi}{3R}r\sin\theta \int T^{00(0)}(r')\omega(r')r'^2 dr'$$

$$\zeta_\phi = -G\frac{16\pi}{3R}\left[\mathbf{r} \times \int T^{00(0)}(r')\omega(r')r'^2 dr'\right] \tag{4.1.7}$$

we can rewrite the field inside the sphere as

$$\zeta_\phi = \mathbf{r} \times \Omega \tag{4.1.8}$$

where

$$\Omega \equiv \frac{16\pi G}{3R}\int T^{00(0)}(r')\omega(r')r'^2 dr' \tag{4.1.9}$$

Vector Potential Outside the Sphere

The vector potential outside the sphere can be calculated from (4.1.4) by replacing $r_< = R = r'$ and $r_> = r$. In similar argument for normalization and orthogonality condition, the integral in (4.1.5) becomes

$$\zeta_\phi(\mathbf{r}) = \frac{16\pi G r'}{3}\frac{r'}{r^2}\sin\theta \int T^{00(0)}(r')\omega(r')r'^2 dr'$$

$$\zeta_\phi(\mathbf{r}) = \frac{16\pi G}{3r^3}\left[\mathbf{r} \times \int \omega(r')T^{00(0)}(r')r'^4 dr'\right] \tag{4.1.10}$$

(4.1.9) is called the post-Newtonian vector potential outside the rotating sphere. We can express the integral in (4.1.6) in terms of the angular momentum, given by

$$\mathbf{J} = \int T^{00(0)}(r')(x' \times [\omega(r') \times x'])d^4x'$$

$$\mathbf{J} = \int [r'^2\omega(r') - x'(x' \cdot \omega(r')]T^{00(0)}(r')d^3x'$$

considering the angular momentum of a homogeneously charged rotating sphere, with axis of rotation coinsides with z-axis

$$\mathbf{J} = \int (r'^2 - z^2)\omega(r')T^{00(0)}(r')d^3x'$$

$$\mathbf{J} = \int (r'^2 - z^2)d\cos\theta \int_0^{2\pi} d\phi \int \omega(r')T^{00(0)}(r')r'^2 dr'$$

but $z = r\cos\theta$

$$\mathbf{J} = 2\pi \int_{-1}^{1} (1 - \cos^2\theta)d\cos\theta \int \omega(r')T^{00(0)}(r')r'^4 dr'$$

$$\mathbf{J} = 2\pi \left[\cos\theta \Big|_{\pi}^{0} - \frac{\cos^3\theta}{3}\Big|_{\pi}^{0} \right] \int \omega(r')T^{00(0)}(r')r'^4 dr'$$

$$\mathbf{J} = \frac{8\pi}{3} \int \omega(r')T^{00(0)}(r')r'^4 dr' \tag{4.1.11}$$

which implies

$$\frac{3\mathbf{J}}{8\pi} = \int \omega(r')T^{00(0)}(r')r'^4 dr'$$

substituting for the integral in (4.1.9) we obtain the field outside the crust as

$$\zeta_\phi(\mathbf{r}) = \frac{2G}{r^3}(\mathbf{r} \times \mathbf{J}) \tag{4.1.12}$$

It indicated that, the term in the expression responsible for the surface field.

4.2 The Magnetic Field

The Non-dipolar components of pulsar magnetic fields outside crust, can be determined by using (3.2.1)

$$\mathbf{B} = \nabla \times \mathbf{A}$$

With only ϕ component of ζ, the components of the non-dipolar magnetic field is given by

$$\begin{aligned}
\mathbf{B}_{out} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_{\phi,out})\right]\hat{e}_r - \frac{1}{r}\frac{\partial}{\partial r}(rA_{\phi,out})\hat{e}_\theta \\
&= \left[\frac{\cos\theta A_{\phi,out}}{r\sin\theta} + \frac{1}{r}\frac{\partial A_{\phi,out}}{\partial\theta}\right]\hat{e}_r - \left[\frac{A_{\phi,out}}{r} + \frac{\partial A_{\phi,out}}{\partial r}\right]\hat{e}_\theta
\end{aligned}$$

4.2.1 The Quadrupole Magnetic Fields

The quadrupole magnetic field obtained from (4.1.4) in the expansion for $l = 2$ and $m = 2$

$$\zeta_\phi = -G\sum_{lm}\frac{16\pi R\omega T^{00(0)}}{2l+1}\int\frac{Y^*_{lm}(\theta',\phi')}{r_>^{l+1}}r_<^l Y_{lm}(\theta,\phi)\sin^2\theta' d^3x' \tag{4.2.1}$$

The angular spherical harmonics will be

$$Y_{22}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta' \tag{4.2.2}$$

for $\phi = 0$

using the() the integration become

$$\begin{aligned}
\int \sin^2\theta' Y^*_{lm}(\theta',\phi')\sin\theta' d\theta' d\phi' &= 4\sqrt{\frac{2\pi}{15}}\int Y_{22}(\theta,\phi)Y^*_{lm}(\theta',\phi')\sin\theta' d\theta' d\phi' \\
&= 4\sqrt{\frac{2\pi}{15}}\delta_{l,2}\delta_{m,2}
\end{aligned}$$

here only $l = 2$ and $m = 2$ terms are non-vanishing.

Therefore, we obtain the potential as:

$$\zeta_{\phi,out} = -G\frac{16\pi\omega}{5}T^{00(0)}\frac{R^3}{r^3}\sin^2\theta' \tag{4.2.3}$$

The quadrupole magnetic field will be

$$\begin{aligned}\mathbf{B}_{qua,out} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\zeta_{\phi,out})\right]\hat{e}_r - \frac{1}{r}\frac{\partial}{\partial r}(r\zeta_{\phi,out})\hat{e}_\theta \\ &= \left[\frac{\cos\theta\zeta_{\phi,out}}{r\sin\theta} + \frac{1}{r}\frac{\partial\zeta_{\phi,out}}{\partial\theta}\right]\hat{e}_r - \left[\frac{\zeta_{\phi,out}}{r} + \frac{\partial\zeta_{\phi,out}}{\partial r}\right]\hat{e}_\theta\end{aligned}$$

substituting for $\zeta_{\phi,out}$ in(4.2.3), we get

$$\begin{aligned}\mathbf{B}_{qua,out} &= -G\frac{16\pi R^3\omega}{5}T^{00(0)}\left[\frac{3\cos\theta\sin\theta}{r^4}\right]\hat{e}_r + G\frac{16\pi R^3\omega}{5}T^{00(0)}\left[\frac{-2\sin^2\theta}{r^4}\right]\hat{e}_\theta \\ &= -G\frac{16\pi\omega}{5}T^{00(0)}\frac{R^3}{r^4}\left[3\cos\theta\sin\theta\hat{e}_r + 2\sin^2\theta\hat{e}_\theta\right] \end{aligned}\tag{4.2.4}$$

4.2.2 The Octapole Magnetic Fields

The Octapole fields can be determined by noting that with only a ϕ component of vector potential given by multipole field (4.1.5). Therefore, we can determine the radial component B_r

$$\mathbf{B}_r = -\frac{1}{r}\frac{d}{dx}\left[(1-x^2)^{\frac{1}{2}}\zeta_\phi\right]$$

for $l = 2n+1$ one can obtain

$$\mathbf{B}_r = \frac{4G\omega}{2r}T^{00(0)}R^3\sum_{n=0}^{\infty}\frac{d}{dx}\left[(1-x^2)^{\frac{1}{2}}\frac{(-1)^{n+2}(2n-1)!!}{2^n(n+1)!}\frac{r_<^{2n+1}}{r_>^{2n+2}}P_{2n+1}^1(\cos\theta)\right] \tag{4.2.5}$$

using Legendre differential equation[5], (4.2.5) gives the radial field as

$$\mathbf{B}_r = \frac{4G\omega}{2r}T^{00(0)}R^3\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)!!}{2^n(n)!}\frac{r_<^{2n+1}}{r_>^{2n+2}}P_{2n+1}(\cos\theta)\Big] \tag{4.2.6}$$

similarly the θ component of B, can be evaluated from

$$\mathbf{B}_\theta = -\frac{1}{r}\frac{d(r\zeta_\phi)}{dr}$$

from (4.1.5), we have

$$\mathbf{B}_\theta = \frac{4G\omega}{3}T^{00(0)}R^4\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+1)!!}{2^n(n+1)!}\left[-\frac{r_<^{2n+1}}{rr_>^{2n+2}} - \frac{\partial}{\partial r}\left(\frac{r_<^{2n+1}}{r_>^{2n+2}}\right)\right]P^1_{2n+1}(\cos\theta)$$

Taking the expansion $(\frac{R}{r})$ for $r >> R$ outside the source

$$\mathbf{B}_\theta = \frac{4G\omega}{3}T^{00(0)}R^4\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+1)!!}{2^n(n+1)!}\left[-\frac{R^{2n+1}}{r^{2n+3}} - \left(-(2n+2)\right)\frac{R^{2n+1}}{r^{2n+3}}\right]P^1_{2n+1}(\cos\theta)$$

one can get the angular field as

$$\mathbf{B}_\theta = \frac{4G\omega}{3}T^{00(0)}R^4\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+1)!!}{2^n(n+1)!}\left(\frac{R^{2n}}{r^{2n+3}}\right)P^1_{2n+1}(\cos\theta) \qquad (4.2.7)$$

We obtain the octapole field when $n = 1$ in the expansion which gives $l = 3$. Thus, from the radial field (4.2.6), we have

$$\mathbf{B}_r = -\frac{4G\omega}{2r}T^{00(0)}R^3\frac{(3)!!}{2}\frac{r_<^3}{r_>^4}P_3(\cos\theta)$$

using Legendre polynomial[5]

$$P_3(\cos\theta) = \frac{1}{8}(20\cos^3\theta - 12\cos\theta) \qquad (4.2.8)$$

The radial field immediately written as

$$\mathbf{B}_r = -\frac{G\omega}{8}T^{00(0)}R^3\frac{r_<^3}{r_>^5}(20\cos^3\theta - 12\cos\theta) \qquad (4.2.9)$$

And for $n = 1$ the angular component (4.2.7) will be

$$\mathbf{B}_\theta = -\frac{4G\omega}{3}T^{00(0)}R^4\frac{(3)!!}{4}\frac{r_<^2}{r_>^5}P^1_3(\cos\theta)$$

The associated Legendre function[5], gives the B_θ as:

$$\mathbf{B}_\theta = G\omega T^{00(0)} R^4 \frac{r_<^2}{r_>^5} \frac{\sin\theta}{2} (5\cos^2\theta - 1) \tag{4.2.10}$$

Therefore, the outside octapole field will be

$$\begin{aligned} \mathbf{B}_{oct} &= \mathbf{B}_r + \mathbf{B}_\theta \\ &= -G\omega T^{00(0)} \frac{R_<^6}{2r^5} \Big[5\cos^3\theta - 3\cos\theta) \Big] \hat{e}_r + G\omega T^{00(0)} \frac{R^6}{2r^5} \Big[\sin\theta(5\cos^2\theta - 1) \Big] \hat{e}_\theta \end{aligned}$$

The Non-dipolar octapole pulsar's magnetic field can be written as

$$\mathbf{B}_{oct} = G\omega T^{00(0)} \frac{R^6}{2r^5} \Big[(3\cos\theta - 5\cos^3\theta)\hat{e}_r + (5\cos^2\theta\sin\theta - \sin\theta)\hat{e}_\theta \Big] \tag{4.2.11}$$

from the unit vectors (3.2.5) one can find the field,

$$\mathbf{B}_{oct} = G\omega T^{00(0)} \frac{R^6}{2r^5} \Big[\sin 2\theta\hat{\mathbf{j}} + (1 - 3\cos^2\theta)\hat{\mathbf{k}} \Big] \tag{4.2.12}$$

which is much more complicated than a dipolar pulsar magnetic field derived in chaper three.

Chapter 5

Discussion And Conclusion

5.1 Discussion

Eq.(3.2.10) indicate the outside magnetic field is generated by the spinning separated surface charges. However, the internal fields are constant and relatively weaker then the surface fields. The dipolar field is constant inside, while dipolar outside crust as shown in (3.2.10).

The induced surface-charge density from spinning separated charges accumulated on the crust, have an angular dependence. The total induced surface charge σ, can be written

$$\sigma = \sigma_0 + \sigma_1 \tag{5.1.1}$$

Where, σ_0 is constant related to the dipolar component of the pulsar magnetic field, while σ_1 is a small, zenith angle dependent, which determine the non-dipolar field. The dipole will rotate to minimize its potential energy, defining the angular dependence in equilibrium. From both outside fields, we see that for the values of $\theta = 0$ and 180 the magnetic field decay as $B \propto \frac{1}{r^5}$.

A pulsar uses up a lot of its rotational energy moving its magnetic field, and so it gradually slows down, may be long periods order of $\sim 10^6$ years, leads to field decay.

5.2 Conclusion

It can be concluded that Equations (4.2.4) and (4.2.12) suggest that the outside magnetic fields of the pulsar is non-dipolar. These fields could be generated by an acquired angular momentum from the basic field as suggested in kebede, 2002. In post-Newtonian approximation, gravitational terms indicate that the gravitoelecric field related to mass density, while magnetic fields are determined by the magnetic moment densities, $(\mathbf{x} \times T^{(i0)}(x,t)/2)$, which is responsible for the magnetic fields.

Finally, depending on current observational hint for the presence of non-dipolar field, it is advisible to include the non-dipolar components of pulsar's magnetic field, fore more accurate explanation to pulsar's magnetic field.

Bibliography

[1]. Hartle, J. B., 2003. Gravity: An Introduction to Einsteins General Relativity, San Francisco: Addison-Wesley.

[2]. Michael. E. Pati and C. M. Will, Phys. Rev. D **62**, 124015 (2002).

[3]. Weber J 2004 General Relativity and Gravitational Waves (New York: Dover)

[4]. L.W.Kebede 2002

[5]. Classical electro dynamics, 3rd edition,John David Jackson.

[6]. Weinberg, S.: 1972, Wiley, New York, New York.

[7]. E. Racine and E. E. Flanagan, Phys. Rev. D **71**, 044010 (2005).

[8]. R. D. Blandford and K. S. Thorne, Applications of Classical Physics http://www.pma.caltech.edu/Courses/ph136/yr2008/.

[9]. B. Mashhoon, Classical Quantum Gravity **25**, 085014 (2008).

[10]. E. Goulart and F. T. Falciano, Int. J. Mod. Phys. A **24**, 4589 (2009).

[11]. L. F. O. Costa and C. A. R. Herdeiro, Phys. Rev. D **78**,024021 (2008).

[12]. Jeffrey D. Kaplan, David A. Nichols PHYSICAL REVIEW D **80**, 124014 (2009)

[13]. T. Damour, M. Soffel, and C. Xu, Phys. Rev. D **45**, 1017 (1992).

[14]. http://imagine.gsfc.nassa.gov/index.html.

[15]. Collins H 2004 Gravitys Shadow: The Search for Gravitational Waves (Chicago, IL: University Of Chicago Press)

Declaration

This thesis is my original work, has not been presented for a degree in any other University and that all the sources of material used for the thesis have been dully acknowledged.

Name: Feyisso Sado

Signature: ....

This thesis has been submitted for examination with my approval as
University advisor.

Name: Dr. Legesse Wotro

Signature:................

Addis Ababa University

Department of Physics

July, 2010