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The study of soft groups based on soft binary operations

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Declaration

I, Tesfaye Degife Weldetekle, with student number GSR/6601/13 hereby declare that this dissertation is my original work under the supervision of Dr. Berhanu Bekele, Dr. Zelalem Teshome and Dr. Gezahagne Mulat and has not been presented for a degree in any other university and that all sources of information used for the thesis have been fully acknowledged.

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This is to certify that the dissertation prepared by Tesfaye Degife Weldetekle entitled: " The study of soft groups based on soft binary operations" submitted in fulfillment of the requirements for the Degree of Doctor of Philosophy in Mathematics complies with the regulations of the University and meets the accepted standards with respect to originality and quality.

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Abstract

In this thesis, we propose a new definition for soft groups based on soft binary operations. The idea is to bring the archetype of 'softness' into the spectrum of algebraic structures using soft binary operations parametrized by a given set of suitable parameters. One of our achievement is that we obtain an ordinary group model representing our soft group. The existing classical group serves as a model to describe and characterize the overall internal properties of our soft groups. In this vein, we further investigate the soft subgroups (respectively, normal soft subgroups) and proved some structural theorems.

In this thesis, we also study soft homomorphisms on soft groups and investigate their properties. Given a soft mapping $\langle f, A\rangle$ from G to G', we obtain an ordinary map $\breve{f}$ from the set $SE_A(G)$ of soft elements of G to the set $SE_A(G')$ of soft elements of G', and show that $\langle f, A\rangle$ is a soft homomorphism (respectively, soft isomorphism) if and only if $\breve{f}$ is an ordinary group homomorphism (respectively, isomorphism). We apply this concept to study soft isomorphism theorems on soft groups. In addition, we study those soft automorphisms of soft groups and the particular class of soft inner automorphisms.

Moreover, we study a few soft group-related findings based on soft binary operations, including soft orbits, soft stabilizers, and the action of a soft group on a set. Given a soft mapping $\langle \mu, A\rangle$ from $G \times X$ to X, we obtain an ordinary map $\widehat{\mu}$ from the set $SE_A(G) \times SE_A(X)$ to the set $SE_A(X)$ and show that $\langle \mu, A\rangle$ is a soft action if and only if $\widehat{\mu}$ is an ordinary action. Finally, we present the fundamental ideas and characteristics of normal fuzzy soft subgroups.

List of Publications

From this dissertation, the following two papers are published and the last two papers are communicated:

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3. Action of a soft group on a set, (communicated).
4. Fuzzy Soft Subgroups of a Soft Group, (communicated).

List of abbreviations

$SE_A(G)$	The collection of all soft elements of G with a set of parameter A.
$SS_A(G)$	The collection of all soft subgroups of G with a set of parameter A
$Sg_A(X)$	The soft subgroup of G generated by the soft set $\langle X, A\rangle$
$Sg_A(\breve{a})$	The soft subgroup of G generated by the soft element $\langle \breve{a}, A\rangle$
$SN_A(G)$	The collection of all normal soft subgroups of G with the set of parameters A
$SCon_A(G)$	The collection of all soft congruence relations on G with the set of parameters A
$SAut(G)$	The collection of all soft automorphisms of G
$SI_A(G)$	The set of all inner soft isomorphisms of G with the given set of parameters A
$FP(X)$	The set of all fuzzy sets on X

List of Symbols

$\breve{\subseteq}$	The subset of soft sets
$\breve{=}$	The equality of soft sets
$\breve{\cap}$	The intersection of soft sets
$\breve{\cup}$	The union of soft sets
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^+$	The set of positive real numbers
$\mathbb{Q}$	The set of rational numbers
$\mathbb{Z}$	The set of integers
$\mathbb{N}$	The set of natural numbers

Contents

Chapter I

Introduction

Molodtsov developed soft set theory in 1999 as a generic mathematical method to address uncertainty [29]. In recent years, the concept of "softness" has gained significant attention in various fields, including mathematics and computer science. Soft computing techniques have been successfully applied to solve complex problems that involve uncertainty and imprecision. Several researchers have been extensively working on the development of the theory of soft sets. As an example, Maji et al. [24, 25] defined some basic operations like equality of two soft sets, subset and super set of a soft set, etc. and applied them to decision making problems. Many hybrid structures incorporating soft sets were proposed as an extension of these concepts. The following are a few of them: fuzzy N-soft sets by Akram and Adeel [3], intuitionistic fuzzy soft set by Xu et al. [41], hesitant fuzzy soft sets by Wang et al. [37], rough soft sets by Roy and Bera [34] and so on.

Traditional algebraic structures, such as groups, rings, and fields, have been extensively studied and utilized to model and analyze various mathematical phenomena. However, these structures often assume precise and deterministic values, which may not adequately capture the inherent uncertainty and imprecision present in real-world scenarios. Soft computing, on the other hand, provides a framework to handle such uncertainties by incorporating fuzzy logic, probability theory, and other mathematical tools. With the idea of integrating soft computing techniques to the algebraic structures have introduced the concept of fuzzy subgroups of a group by A. Rosenfeld in 1971 [33]. R. Biswas et al. [12] applied the idea of roughness in group theory and Aktas et al. [4] initiated the study of soft groups. According to Aktas et al. [4] soft groups over a given ordinary group G are defined as a soft set $\langle F, A \rangle$ over

G for which the values $F(\alpha)$ are subgroups of G for all $\alpha \in A$. After a decade, in 2016, J. Ghosh et al. [18] came up with a new idea of soft groups using the concept of soft elements as defined by D. Wardowski [38]. They define a soft group as the collection of nonempty soft elements of a soft set $\langle F, A\rangle$ over G along with the binary operation induced by the binary operations $*$ and $\circ$ of G and E respectively, and satisfying all the classical group axioms. This is given a group $\langle G, *\rangle$ as an initial set of universe and a group $\langle E, \circ\rangle$ as a set of parameters. This idea was somewhat expanded upon a few years later by G. Yaylali [42] et al., who defined soft groups as the set of nonempty soft elements of a soft set $\langle F, A\rangle$ over G along with a binary operation that satisfies all the defining characteristics of a classical group, where G and E are not assumed to be a group or to have binary operations.

One can easily observe that all the above discussed soft groups are defined based on classical binary operation which is the central unit determining all the algebraic properties of the structure. Taking this into consideration, in this thesis, we propose a new approach to define soft groups based on soft binary operations. A soft group is defined as a triple $\langle G, *, A\rangle$ consisting of a nonempty set G equipped with a soft binary operation $\langle *, A\rangle$ respecting all the group axioms in a soft setting. These soft binary operations are parameterized by a given set of suitable parameters, allowing us to model and analyze the inherent imprecision and uncertainty in group operations. By doing so, we bridge the gap between traditional algebraic structures and the realm of soft computing.

The significance of our research lies in its potential applications in various domains. Soft groups can serve as a powerful tool for modeling and analyzing complex systems that involve uncertainty and imprecision. By incorporating soft binary operations, we can capture and manipulate imprecise data, allowing us to make informed decisions in real-world scenarios. Moreover, the study of soft subgroups and normal soft subgroups opens up new avenues for exploring the interplay between classical group theory and soft computing techniques.

This thesis is organized in five chapters. The first chapter contains basic concepts of soft set and fuzzy set theory collected from literature. One of the key achievements of our research is we develop an ordinary group model that represents our soft group. This result is proved in chapter two. This model serves as a foundation for describing and characterizing the internal properties of our soft groups. By leveraging the existing classical group theory, we can establish a solid theoretical framework for our soft groups, enabling us to study their properties and behavior in a rigorous manner. Furthermore, we delve into the study of soft subgroups and normal soft subgroups within our proposed framework. Soft subgroups are soft subsets of a soft group that retain the essential properties of a group. Soft normal subgroups, on the other hand, possess additional properties that make them particularly interesting for applications in various areas, such as cryptography and data analysis. We present several structural theorems that shed light on the properties and relationships between soft subgroups and normal soft subgroups, providing valuable insights into the structure and behavior of our soft groups.

In chapter three, we study the basic notion of soft homomorphisms on soft groups based on soft binary operation. Moreover, we state and prove several soft isomorphism theorems on soft groups. We further study the soft automorphisms of soft groups and particularly those soft inner automorphisms. From chapter two since $SE_A(G)$ is the collection of all soft elements of G with a set A of parameters, we build an embedding of the group of soft automorphisms of a soft group G into the group of automorphisms of $SE_A(G)$. Finally, it is shown that for every soft group G, the soft group of its inner automorphisms is soft isomorphic with the quotient of G by its center $Z_A(G)$.

The fourth chapter examines several findings on soft groups based on soft binary operations, including the effect of a soft group on a set, soft orbits and soft stabilizers as well as some of their characteristics. Furthermore, given a soft mapping $\langle \mu, A \rangle$ from $G \times X$ to X, we obtain an ordinary map $\widehat{\mu}$ from the set $SE_A(G)$ of soft elements of G to the set $SE_A(X)$ of soft elements of X, and demonstrate that $\langle \mu, A \rangle$ is a soft action (soft orbits and soft stabilizers, respectively) if and only if $\widehat{\mu}$ is an ordinary action (orbits and stabilizers, respectively). We also look at the idea of primitive and transitive actions on soft set.

In the final chapter, we present fuzzy soft subgroups of a soft group, which are a generalization of the soft groups , and examine their different characteristics. Given a fuzzy soft set $\langle \mu, A \rangle$ over a soft group $\langle G, *, A \rangle$. Define a mapping $\widehat{\mu} : SE_A(G) \to I$ as for each

$\breve{a} \in SE_A(G)$ and $\alpha \in A$:

$$\widehat{\mu}(\breve{a}) = \wedge\{t_\alpha : \langle \alpha, a_\alpha, t_\alpha \rangle \in \mu \ \ where\ \breve{a}(\alpha) = \{a_\alpha\}\}.$$

We proved the equivalent conditions of fuzzy soft subgroup, fuzzy subgroup, soft subgroup, and subgroups in this chapter using this model, which is one of the primary findings in this thesis. Lastly, We present the fundamental ideas and characteristics of normal fuzzy soft subgroups.

Chapter 1

Preliminaries

In this chapter, we present basic notions and results that will be used throughout the thesis.

1.1 Soft sets

In this section, we give some basic definitions which will be used in this thesis. Throughout this section, X and Y are assumed to be nonempty sets considered as an initial universe set and A is the set of all convenient parameters. Most of the results in this section are standard and are collected from [2, 5, 16, 17, 20, 24, 25, 27, 29, 35, 36, 44].

Definition 1.1. *[29] A pair $\langle F, A\rangle$ is called a soft set over X if F is a mapping from A into power set of X, which is given by*

$$F : A \to P(X),$$

where $P(X)$ is a power set of X.

Definition 1.2. *[25] Given two soft sets $\langle F, A\rangle$ and $\langle E, A\rangle$ over X : we say that*

(1) $\langle F, A\rangle$ is included in $\langle E, A\rangle$, written as $\langle F, A\rangle \breve{\subseteq} \langle E, A\rangle$, provided that $(\forall x)[x \in F(\alpha) \Rightarrow x \in E(\alpha)]$ for all $\alpha \in A$;

(2) $\langle F, A\rangle$ and $\langle E, A\rangle$ are equal provided that $(\forall x)[x \in F(\alpha) \Leftrightarrow x \in E(\alpha)]$ for all $\alpha \in A$. We write $\langle F, A\rangle \breve{=} \langle E, A\rangle$ to say that $\langle F, A\rangle$ and $\langle E, A\rangle$ are equal.

Definition 1.3. *[25] Given that $\langle F, A\rangle$ and $\langle E, A\rangle$ are soft sets over X. Then*

(1) their union denoted by $\langle F, A\rangle \breve{\cup} \langle E, A\rangle$, is a soft set $\langle H, A\rangle$ over X defined as follows: for all $\alpha \in A$ and $x \in X$, $x \in H(\alpha)$ if and only if either $x \in F(\alpha)$ or $x \in E(\alpha)$;

(2) their intersection denoted by $\langle F, A\rangle \breve{\cap} \langle E, A\rangle$*, is a soft set* $\langle H, A\rangle$ *over* X *defined as follows: for all* $\alpha \in A$ *and* $x \in X$*,* $x \in H(\alpha)$ *if and only if* $x \in F(\alpha)$ *and* $x \in E(\alpha)$*.*

Definition 1.4. *[25] Given that* $\langle F, A\rangle$ *and* $\langle E, A\rangle$ *are soft sets over* X*. Then*

(a) $\langle F, A\rangle$ *and* $\langle E, A\rangle$ *are said to be disjoint soft sets is* $F(\alpha) \cap E(\alpha) = \emptyset$ *for all* $\alpha \in A$*.*

(b) $\langle F, A\rangle$ *and* $\langle E, A\rangle$ *are said to be weakly disjoint soft sets is* $F(\alpha) \cap E(\alpha) = \emptyset$ *for some* $\alpha \in A$*.*

Definition 1.5. *[25] A null soft set over* X *is a soft set denoted by* $\langle 0_X, A\rangle$ *such that for each* $\alpha \in A$*,* $0_X(\alpha) = \emptyset$*. Moreover, the absolute soft set over* X *is a soft set denoted by* $\langle 1_X, A\rangle$ *such that* $1_X(\alpha) = X$ *for all* $\alpha \in A$*.*

Definition 1.6. *[44] A soft relation from* X *and* Y *is defined as a soft set* $\langle R, A\rangle$ *over* $X \times Y$*. That is* $R : A \to P(X \times Y)$ *such that* $R(\alpha) \subseteq X \times Y$ *for all* $\alpha \in A$*. The notation* $\langle \alpha, x, y\rangle \in R$ *indicates that* $\langle x, y\rangle \in R(\alpha)$ *for* $\alpha \in A$ *and* $(x, y) \in X \times Y$*.*

Definition 1.7. *[2] A soft mapping from* X *to* Y *is a soft relation* $\langle f, A\rangle$ *from* X *to* Y *such that:*

(1) for each $\alpha \in A$ *and* $x \in X$ *there exists some* $y \in Y$ *such that* $\langle \alpha, x, y\rangle \in f$*;*

(2) for each $\alpha \in A$*,* $x \in X$ *and* $y_1, y_2 \in Y$*,* $\langle \alpha, x, y_1\rangle \in f$ *and* $\langle \alpha, x, y_2\rangle \in f$ *implies* $y_1 = y_2$*.*

Definition 1.8. *[2] A soft mapping* $\langle f, A\rangle$ *from* X *to* Y *is said to be :*

(1) injective if for each $\alpha \in A$*,* $x_1, x_2 \in X$ *and* $y \in Y$*;* $\langle \alpha, x_1, y\rangle \in f$ *and* $\langle \alpha, x_2, y\rangle \in f$ *together imply* $x_1 = x_2$*.*

(2) surjective if for each $\alpha \in A$ *and* $y \in Y$ *there is* $x \in X$ *such that* $\langle \alpha, x, y\rangle \in f$*.*

(3) bijective if it is both injective and surjective.

Definition 1.9. *[2] Let* $\langle f, A\rangle$ *represent a soft mapping from* X *to* Y *and* $\langle g, A\rangle$ *represent a soft mapping from* Y *to* Z*. Then, their composition is a soft mapping from* X *to* Z *given by for all* $\alpha \in A$*, and* $x \in X$*,* $z \in Z$*, :* $\langle \alpha, x, z\rangle \in g \circ f$ *if and only if* $y \in Y$ *such that* $\alpha, x, y\rangle \in f$ $\alpha, y, z\rangle \in g$*.*

Definition 1.10. *[2] A soft equivalence relation on* X *is a soft set* $\langle \theta, A\rangle$ *over* $X \times X$ *such that*

(1) $\langle \alpha, x, x\rangle \in \theta$ *for all* $\alpha \in A$ *and* $x \in X$;

(2) $\langle \alpha, x, y\rangle \in \theta \Rightarrow \langle \alpha, y, x\rangle \in \theta$ *for any* $\alpha \in A$ *and* $x, y \in X$;

(3) $\langle \alpha, x, y\rangle \in \theta$ *and* $\langle \alpha, y, z\rangle \in \theta \Rightarrow \langle \alpha, x, z\rangle \in \theta$ *for any* $\alpha \in A$ *and* $x, y, z \in X$.

1.2 Soft groups

A parameterized family of subgroups is what Aktas and Cagman [4] characterized as a soft group in 2007. Extending this notion of soft group, many authors also defined soft (ring, field, ideal) (see [1, 4, 7, 14, 15, 18, 19, 23, 30, 31, 42]) etc.

Throughout this section unless otherwise stated, S(U) denotes the collection of all soft sets on U, E is the set of parameters with regard to U, and U is the universal set.

Definition 1.11. *[4] Given a group* G*, let* $\langle F, A\rangle$ *be a soft set over* G*. Assuming that* $F(\alpha) < G$ *for every* $\alpha \in A$*, then* $\langle F, A\rangle$ *is said to be a soft group over* G*.*

J.Ghosh [18] introduced a soft groupoid by defining a binary operation on the set of all nonempty soft elements of a given soft set using the concept of a soft element first proposed by Wardowski [38].

Definition 1.12. *[38] Let* $A \subseteq E$ *and* $F_A \in S(U)$*. We say that* $(x; \{y\})$ *is a nonempty soft element of* F_A *if* $x \in A$ *and* $y \in F(x)$ *: The pair* $(x; \emptyset)$*; where* $x \in A$*; will be called an empty soft element of* F_A*.*

The set of all nonempty soft elements of F_A is represented as $F_A^\bullet$.

Definition 1.13. *[18] If* $F_A^\bullet$ *is closed under the binary composition* $\bar{\star}$ *then the algebraic system* $(F_A^\bullet, \bar{\star})$ *is said to be a soft groupoid over* (E, U)*.*

Example 1.14. *[18] Let* $(E, \circ)$ *be the kleins 4-group and* $(U, \star)$ *the symmetric group* S_3*, where* $E = \{e_1, e_2, e_3, e_4\}$ *be the set of parameters and* $U = \{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$*. Define the binary composition* $\bar{\star}$ *on* $F_A^\bullet$ *is given by*

$$(e_i, \{\rho_k\})\bar{\star}(e_j, \{\rho_l)\}) = (e_i \circ e_j, \{\rho_k \star \rho_l\})$$

for all $(e_i, \{\rho_k\})$, $(e_j, \{\rho_l)\}) \bar{\in} F_A^\bullet$*. Then* $(F_A^\bullet, \bar{\star})$ *is a soft groupoid.*

Definition 1.15. *[18] Let $\langle E, \circ\rangle$, $\langle U, \star\rangle$, be two groups, $A \subseteq E$ and $F_A \in S(U)$. A soft group is defined as a soft groupoid $\langle F_A, \bar{\star}\rangle$ over $\langle E, U\rangle$ if*

(1) $\bar{\star}$ is associative;

(2) there is a soft element $(e, \{u\})\bar{\in}F_A^{\bullet}$ such that $(e, \{u\})\bar{\star}(e_i, \{u_j\}) = (e_i, \{u_j\})\bar{\star}(e, \{u\}) = (e_i, \{u_j\}$ for all $(e_i, \{u_j\}\bar{\in}F_A^{\bullet}$;

(3) for each soft element $(e_i, \{u_j\})\bar{\in}F_A^{\bullet}$ there exists $(e_i^{'}, \{u_j^{'}\})$ such that $(e_i, \{u_j\})\bar{\star}(e_i^{'}, \{u_j^{'}\}) = (e_i^{'}, \{u_j^{'}\})\bar{\star}(e_i, \{u_j\}) = (e, \{u\})$.

Here, $(e, \{u\})$ is the soft identity element, and the soft element $(e_i^{'}, \{u_j^{'}\})$ is the soft inverse of $(e_i, \{u_j\})$.

Gozde Yaylal [42] generalize soft group notions by redefining the binary operation based on the soft element [18] without assuming any further requirements.

Definition 1.16. *[42] An ordered pair $\langle F_A, \tilde{\star}\rangle$ is a soft group, where $F_A^{\bullet}$ is a non-null soft set and $\tilde{\star}$ is a binary operation on $F_A^{\bullet}$ such that the following characteristics hold:*

(1) the binary operation $\tilde{\star}$ is associative;

(2) the set $F_A^{\bullet}$ has an an identity element with respect to the binary operation $\tilde{\star}$

(3) for each $\beta\tilde{\in}F_A^{\bullet}$, there exists inverse element β^{-1} in $F_A^{\bullet}$.

1.3 Fuzzy Subsets

In this section, we collect important definition results from [3, 6, 8, 9, 22, 26, 33, 43].

Definition 1.17. *[43] A function from X into $[0, 1]$ is a fuzzy subset of X. The fuzzy power set of X is the collection of all fuzzy subsets of X.*

Definition 1.18. *[33] A fuzzy subset μ of G is referred to as a fuzzy subgroup of G if*

(1) $\mu(e) = 1$;

(2) For any $x, y \in G$, $\mu(xy) \geq \mu(x) \wedge \mu(y)$ and

(2) For every $x \in G$, $\mu(x^{-1}) \geq \mu(x)$.

Definition 1.19. *$FP(G)$ denotes the set of all fuzzy subsets of G. Define the binary operation $\circ$ on $FP(G)$ as follows: $\forall \mu, \nu \in FP(G)$ and $\forall x \in G$, $(\mu \circ \nu)(x) = \vee\{\mu(y) \wedge \nu(z) | y, z \in G, yz = x\}$. We call $\mu \circ \nu$ the product of μ and ν.*

Theorem 1.20. *[33] Suppose μ and ν are fuzzy subgroups of G. Then following conditions are equivalent:*

(1) $\mu(ab) = \mu(ba)$ for all $a, b \in G$. In this instance, μ is referred to as an Abelian fuzzy subset of G.

(2) For every $a, b \in G$, $\mu(aba^{-1}) = \mu(b)$.

(3) For every $a, b \in G$, $\mu(aba^{-1}) \geq \mu(b)$.

(4) For every $a, b \in G$, $\mu(aba^{-1}) \leq \mu(b)$

(5) $\mu \circ \nu = \nu \circ \mu$ for all $x \in G$. We call $\mu \circ \nu$ the product of μ and ν.

Definition 1.21. *[33] If μ satisfies one of the 5 conditions in the above theorem then it is referred to as a normal fuzzy subgroup of G.*

Definition 1.22. *[26] Let the set of all fuzzy sets on X be represented by $FP(X)$. A fuzzy soft set over X, is a pair $\langle \mu, A \rangle$, where μ is a mapping from A into $FP(X)$. In other words, $\mu(\alpha) : X \longrightarrow I$ is a fuzzy subset of X for every $\alpha \in A$. We write $\langle \alpha, x, t \rangle \in \mu$ to say that $\mu(\alpha)(x) = t$ where $t \in I$.*

Definition 1.23. *[25] Let $\langle \mu, A \rangle$ and $\langle v, A \rangle$ be fuzzy soft sets over X.*

(1) Let $r, s \in I$. For each $\alpha \in A$, $x \in X$, $\langle \alpha, x, r \rangle \in \mu$ and $\langle \alpha, x, s \rangle \in v$ and $r \leq s$ then we say $\langle \mu, A \rangle$ is contained in $\langle v, A \rangle$ and written as $\langle \mu, A \rangle \breve{\subseteq} \langle v, A \rangle$.

(2) $\langle \mu, A \rangle \breve{=} \langle v, A \rangle$ if and only if $\langle \mu, A \rangle \breve{\subseteq} \langle v, A \rangle$ and $\langle v, A \rangle \breve{\subseteq} \langle \mu, A \rangle$.

Definition 1.24. *[25] Let $\langle \mu, A \rangle$ and $\langle v, A \rangle$ be fuzzy soft sets over X. Then*

(1) their union denoted by $\langle \mu, A \rangle \breve{\cup} \langle v, A \rangle$, is a fuzzy soft set $\langle \eta, A \rangle$ over X defined as : for each $\alpha \in A$ and $x \in X$, $\langle \alpha, x, t \rangle \in \eta$ if and only if $\langle \alpha, x, r \rangle \in \mu$ or $\langle \alpha, x, s \rangle \in v$ where $t = max\{r, s\}$.

(2) *their intersection denoted by* $\langle\mu, A\rangle \breve{\cap} \langle v, A\rangle$, *is a fuzzy soft set* $\langle\eta, A\rangle$ *over* X *defined as : for each* $\alpha \in A$ *and* $x \in X$, $\langle\alpha, x, t\rangle \in \eta$ *if and only if* $\langle\alpha, x, r\rangle \in \mu$ *and* $\langle\alpha, x, s\rangle \in v$ *where* $t = min\{r, s\}$.

Definition 1.25. *[25] Let any non-empty family of fuzzy soft sets over* X *be represented by* $\{\langle\mu_i, A\rangle : i \in \Delta\}$. *Consequently, their union is the fuzzy soft set* $\langle\mu, A\rangle$ *over* X *given by*

$$\langle\alpha, x, t\rangle \in \mu \Leftrightarrow t = \bigvee_{i\in\Delta} t_i \quad where \ \langle\alpha, x, t_i\rangle \in \mu_i$$

and their intersection

$$\langle\alpha, x, t\rangle \in \mu \Leftrightarrow t = \bigwedge_{i\in\Delta} t_i \quad where \ \langle\alpha, x, t_i\rangle \in \mu_i.$$

1.4 Lattice

In this section, we collect important definition results from [11, 13, 32].

Definition 1.26. *[11] A poset* L *is called a lattice if any two of its elements have supremum and infimum. In this case, the infimum and supremum of* $x, y \in L$ *are denoted by* $x \wedge y$ *and* $x \vee y$ *respectively.*

Lattices can also be defined as an algebra with two binary operations as given in below.

Definition 1.27. *[11] Suppose* L *is a non-empty set and* $\vee$ *and* $\wedge$ *are two binary operations (read join and meet respectively) on* L. *Then the system* $(L, \vee, \wedge)$ *is called a lattice if it satisfies the following identities:*

(1) *commutative laws*

(a) $x \vee y = y \vee x$ (b) $x \wedge y = y \wedge x$

(2) *associative laws*

(a) $x \vee (y \vee z) = (x \vee y) \vee z$ (b) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$

(3) *absorption laws*

(a) $x = x \vee (x \wedge y)$ (b) $x = x \wedge (x \vee y)$

(4) *idempotent laws*

(a) $x \vee x = x$ (b) $x \wedge x = x$

Note: Observe that the idempotent laws follows from the absorption laws, as $x \vee x = x \vee [x \wedge (x \vee x)] = x$. Similarly, one can prove $x \wedge x = x$.

Definition 1.28. *[11] A lattice L is said to be complete if each of its subset S has both infimum and supremum in L.*

Definition 1.29. *Let L and L' be lattice. A map $f : L \longrightarrow L'$ is called a homomorphism of lattice if*

(a) $f(a \vee b) = f(a) \vee f(b)$;

(b) $f(a \wedge b) = f(a) \wedge f(b)\ \forall a, b \in L$.

Definition 1.30. *A map $f : L \longrightarrow L'$ is called*

(a) an order preserving map if $a \geq b$ in L imply $f(a) \geq f(b)$ in L'.

(b) an order inverting map if $a \geq b$ in L imply $f(a) \leq f(b)$ in L'.

Definition 1.31. *[35] A closure operator on set Y is a mapping $C : P(Y) \to P(Y)$ such that for $B, D \subseteq Y$, it satisfies:*

(1) $B \subseteq C(B)$,

(2) $C^2(B) = C(B)$,

(3) $B \subseteq D \Rightarrow C(B) \subseteq C(D)$.

Chapter 2

Soft groups based on soft binary operations

Dealing with uncertainties is a major problem in many areas such as economics, engineering, environmental science, medical science and social sciences. These kinds of problems cannot be dealt with by classical methods, because classical methods have inherent difficulties. To overcome these kinds of difficulties, Molodtsov proposed a completely new approach, which is called soft set theory, for modeling uncertainty. After that, Maji et al. presented a number of operations on soft sets. With idea of integrating soft computing techniques to the algebraic structures, in 2007 Aktas and Cagman initiated the study of soft groups. Further investigations can be found in [4, 18, 26, 29, 42].

In this chapter we introduce one of the main objects studied in this thesis, namely the soft groups based on soft binary operations. We begin by introducing the notion of soft groups and much of the associated terminology. We then consider soft subgroups and algebraic structures associated to soft groups which leads us to a definition for the normal soft subgroups. The structure of this normal soft subgroups is then studied in detail. Soft congruence relations are considered next, and we conclude by discussing the concepts of direct products in soft groups.

Throughout this thesis, unless and otherwise it is mentioned, G denotes the soft group $\langle G, *, A\rangle$ where A is the set of all convenient parameters for G.

2.1 Soft groups

This section presents the idea of soft groups, which are based on soft binary operations. It also provides examples and demonstrates some of the properties of soft groups.

Definition 2.1. *Let G be a nonempty set. By a soft binary operation on G we mean a soft mapping $\langle *, A\rangle$ from $G \times G$ to G, where A is a set of parameters.*

Definition 2.2. *A soft group is a triple $\langle G, *, A\rangle$ where G is nonempty set and $\langle *, A\rangle$ is a soft binary operation on G satisfying the following conditions:*

$\langle SG1\rangle$ Associativity: For all $\alpha \in A$ and $a, b, c, x, y, u, v \in G$, if $\langle \alpha, a, b, x\rangle \in *$, $\langle \alpha, x, c, y\rangle \in *$, $\langle \alpha, b, c, u\rangle \in *$ and $\langle \alpha, a, u, v\rangle \in *$ then $y = v$.

$\langle SG2\rangle$ *Existence of identity: For each $\alpha \in A$, there exists $e_\alpha \in G$ such that $\langle \alpha, a, e_\alpha, a\rangle \in *$ and $\langle \alpha, e_\alpha, a, a\rangle \in *$ for all $a \in G$.*

$\langle SG3\rangle$ *Existence of inverse: For each $\alpha \in A$ and all $a \in G$, there is an element of G denoted by $a^{-\alpha}$ such that $\langle \alpha, a, a^{-\alpha}, e_\alpha\rangle \in *$ and $\langle \alpha, a^{-\alpha}, a, e_\alpha\rangle \in *$.*

Example 2.3. *Let $G = \mathbb{R}$ and $A = \mathbb{Z}$. Define $* \subseteq \mathbb{Z} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ by*

$$\langle \alpha, a, b, c\rangle \in * \Leftrightarrow c = a + b + \alpha,$$

for all $\alpha \in \mathbb{Z}$, $a, b, c \in \mathbb{R}$ then $\langle \mathbb{R}, *, \mathbb{Z}\rangle$ is a soft group.

Proof. Let $\alpha \in A$ and $a, b, c, x, y, u, v \in G$ such that $\langle \alpha, a, b, x\rangle \in *$, $\langle \alpha, x, c, y\rangle \in *$, $\langle \alpha, b, c, u\rangle \in *$ and $\langle \alpha, a, u, v\rangle \in *$. We need to show that $y = v$. From the definition it follows that $x = a + b + \alpha$, $y = x + c + \alpha$, $u = b + c + \alpha$ and $v = a + u + \alpha$. This implies that $y = a + b + c + 2\alpha$ and $v = a + b + c + 2\alpha$. Thus $y = v$. Therefore $\langle *, A\rangle$ is associative. Let $\alpha \in A$ and $a \in G$ such that $\langle \alpha, a, e_\alpha, a\rangle \in *$ and $\langle \alpha, e_\alpha, a, a\rangle \in *$. Then we have $a = a + e_\alpha + \alpha$. Hence $e_\alpha = -\alpha$. Therefore $-\alpha$ is an identity element of $\langle G, *, A\rangle$ with respect to α. Let $\alpha \in A$ and $a \in G$ such that $\langle \alpha, a, a^{-\alpha}, e_\alpha\rangle \in *$ and $\langle \alpha, a^{-\alpha}, a, e_\alpha\rangle \in *$. Then we have $e_\alpha = \alpha + a + a^{-\alpha}$. This implies that $a^{-\alpha} = -2\alpha - a$. Therefore $-2\alpha - a$ is an inverse of a with respect to α. Hence $\langle G, *, A\rangle$ is a soft group. □

Example 2.4. *Let X be a non empty set. Put $G = P(X)$ the power set of X and $A = X$ be our set of parameters. Define a soft binary operation $\langle *, A\rangle$ on G by $* \subseteq A \times G \times G \times G$ such that $\langle x, B, C, D\rangle \in *$ if and only if*

$$D = \begin{cases} (B \oplus C) \cup \{x\} & if \quad x \notin B \oplus C \\ (B \oplus C) - \{x\} & if \quad x \in B \oplus C \end{cases}$$

where $\oplus$ is the symmetric difference of sets. Then $\langle G, *, A\rangle$ is a soft group.

Example 2.5. *Let $M_2(\mathbb{R})$ be the set of 2×2 Matrices over $\mathbb{R}$ given by:*

$$M_2(\mathbb{R}) = \left\{ \begin{bmatrix} 1 & a_1 \\ 0 & a_2 \end{bmatrix} : \quad a_1, a_2 \in \mathbb{R}, a_2 \neq 0 \right\}.$$

Put $A = \mathbb{R}\backslash\{0\}$ and define a soft binary operation $\langle *, A\rangle$ on $M_2(\mathbb{R})$ as follows : for $\alpha \in A$ and matrices $B = \begin{bmatrix} 1 & b_1 \\ 0 & b_2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & c_1 \\ 0 & c_2 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & d_1 \\ 0 & d_2 \end{bmatrix}$ in $M_2(\mathbb{R})$;

$$\langle \alpha, B, C, D\rangle \in * \text{ if and only if } D = \begin{bmatrix} 1 & c_1 + b_1 c_2 \alpha \\ 0 & b_2 c_2 \alpha \end{bmatrix}.$$

Then $\langle M_2(\mathbb{R}), *, A\rangle$ is a soft group.

Example 2.6. *Let X be any set with at least two distinct elements and $B(X)$ be the set of all bijective functions from X onto X. Put*

$$A = \{\langle x, y\rangle \in X \times X : x \neq y\},$$

and define a soft binary operation $\langle \circledast, A\rangle$ on $B(X)$ as follows. For any pair $\langle x, y\rangle \in A$ and $f, g, h \in B(X)$:

$$\langle\langle x, y\rangle, f, g, h\rangle \in \circledast \text{ if and only if } h(z) = \begin{cases} g(f(z)) & \text{if } z \notin \{x, y\} \\ g(f(y)) & \text{if } z = x \\ g(f(x)) & \text{if } z = y \end{cases} \quad for \ all \ z \in X.$$

Then $\langle B(X), \circledast, A\rangle$ is a soft group.

Lemma 2.7. *Let $\langle G, *, A\rangle$ be a soft group. Then the following conditions hold for all $\alpha \in A$ and $a, b, c \in G$:*

(1) $\langle \alpha, a, b, e_\alpha\rangle \in * \Leftrightarrow b = a^{-\alpha}$ *and* $a = b^{-\alpha}$*;*

(2) $\langle \alpha, a, b, c\rangle \in * \Leftrightarrow \langle \alpha, a^{-\alpha}, c, b\rangle \in *$*;*

(3) $\langle \alpha, a, b, c\rangle \in * \Leftrightarrow \langle \alpha, c, b^{-\alpha}, a\rangle \in *$*;*

(4) $\langle \alpha, a, b, c\rangle \in * \Leftrightarrow \langle \alpha, b^{-\alpha}, a^{-\alpha}, c^{-\alpha}\rangle \in *$*.*

Proof. (1) Suppose $\langle \alpha, a, b, e_\alpha\rangle \in *$. As $\langle \alpha, a^{-\alpha}, e_\alpha, a^{-\alpha}\rangle \in *$, $\langle \alpha, a^{-\alpha}, a, e_\alpha\rangle \in *$ and $\langle \alpha, e_\alpha, b, b\rangle \in *$ then from the associativity of $\langle *, A\rangle$ it follows that $b = a^{-\alpha}$. Similarly we get $a = b^{-\alpha}$. The converse directly follows from the definition.

(2) Suppose that $\langle \alpha, a, b, c\rangle \in *$. Let $d \in G$ such that $\langle \alpha, a^{-\alpha}, c, d\rangle \in *$. Then we have $\langle \alpha, a, b, c\rangle \in *$, $\langle \alpha, a^{-\alpha}, c, d\rangle \in *$, $\langle \alpha, a^{-\alpha}, a, e_\alpha\rangle \in *$ and $\langle \alpha, e_\alpha, b, b\rangle \in *$. By associativity it holds that $b = d$. Therefore $\langle \alpha, a^{-\alpha}, c, b\rangle \in *$. Conversely, suppose that $\langle \alpha, a^{-\alpha}, c, b\rangle \in *$. Let $x \in G$ such that $\langle \alpha, a, b, x\rangle \in *$. Then we have $\langle \alpha, a^{-\alpha}, c, b\rangle \in *$, $\langle \alpha, a, b, x\rangle \in *$, $\langle \alpha, a, a^{-\alpha}, e_\alpha\rangle \in *$ and $\langle \alpha, e_\alpha, c, c\rangle \in *$. So by associativity of $\langle *, A\rangle$ we get $x = c$. Therefore $\langle \alpha, a, b, c\rangle \in *$.

(3) The proof is similar to (2).

(4) Suppose that $\langle \alpha, a, b, c\rangle \in *$. We need to prove that $\langle \alpha, b^{-\alpha}, a^{-\alpha}, c^{-\alpha}\rangle \in *$. Let $d \in G$ such that $\langle \alpha, b^{-\alpha}, a^{-\alpha}, d\rangle \in *$. Then, it can be shown that $\langle \alpha, c, d, e_\alpha\rangle \in *$. Using the fact $\langle \alpha, c, c^{-\alpha}, e_\alpha\rangle \in *$, we get $d = c^{-\alpha}$. Therefore $\langle \alpha, b^{-\alpha}, a^{-\alpha}, c^{-\alpha}\rangle \in *$. Conversely suppose that $\langle \alpha, b^{-\alpha}, a^{-\alpha}, c^{-\alpha}\rangle \in *$ and let $x \in G$ such that $\langle \alpha, a, b, x\rangle \in *$. Then $\langle \alpha, c^{-\alpha}, x, e_\alpha\rangle \in *$. Also, as $\langle \alpha, c^{-\alpha}, c, e_\alpha\rangle \in *$, it holds that $x = c$. Thus, $\langle \alpha, a, b, c\rangle \in *$.

□

Theorem 2.8. *(Cancellation laws) Let $\langle G, *, A\rangle$ be a soft group, for each $\alpha \in A$ and $a, b, c \in G$. Then,*

(1) if $\langle \alpha, a, b, x\rangle \in *$ *and* $\langle \alpha, a, c, x\rangle \in *$ *then* $b = c$ *and*

(2) if $\langle \alpha, b, a, x\rangle \in *$ *and* $\langle \alpha, c, a, x\rangle \in *$ *then* $b = c$.

Proof. (1) Suppose $\langle \alpha, a, b, x\rangle \in *$ and $\langle \alpha, a, c, x\rangle \in *$. Let $y \in G$ such that $\langle \alpha, a^{-\alpha}, x, y\rangle \in *$. Then, by applying $\langle SG1\rangle$ it can be shown that $y = b$. Now we have $\langle \alpha, a, c, x\rangle \in *$, $\langle \alpha, a^{-\alpha}, x, b\rangle \in *$, $\langle \alpha, a^{-\alpha}, a, e_\alpha\rangle \in *$ and $\langle \alpha, e_\alpha, c, c\rangle \in *$. Again by applying $\langle SG1\rangle$ we get $b = c$.

(2) The proof is similar to (1).

□

Definition 2.9. *Suppose that G is a nonempty set and $\langle *, A\rangle$ is a soft binary operation. A soft semi-group is then defined as a triple $\langle G, *, A\rangle$ satisfying $\langle SG1\rangle$.*

Example 2.10. *Let $G = \mathbb{R}$ and $A = \mathbb{Z}$. Define $* \subseteq \mathbb{Z} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ by*

$$\langle \alpha, a, b, c\rangle \in * \Leftrightarrow c = \frac{\alpha ab}{5},$$

for all $\alpha \in \mathbb{Z}$ and $a, b, c \in \mathbb{R}$ then $\langle G, *, A\rangle$ is a soft semi-group.

Equivalent criteria for a soft semi-group to be a soft group are obtained in the next two theorems.

Theorem 2.11. *Let $\langle G, *, A\rangle$ be a soft semi-group. Then, $\langle G, *, A\rangle$ is a soft group if and only if the following conditions are satisfied:*

(1) For each $\alpha \in A$ and $a \in G$, there exists $e_\alpha \in G$ such that $\langle \alpha, a, e_\alpha, a\rangle \in *$;

(2) For each $\alpha \in A$ and $a \in G$, there exists $a^{-\alpha} \in G$ such that $\langle \alpha, a, a^{-\alpha}, e_\alpha\rangle \in *$.

Proof. By assumption $\langle G, *, A\rangle$ is a soft group. Then clearly (1) and (2) are satisfied. Conversely, assume that (1) and (2) are satisfied. By (1), there exists $e_\alpha \in G$ such that $\langle \alpha, a, e_\alpha, a\rangle \in *$ for all $a \in G$. Let a be any element in G. According to (2), $a^{-\alpha}$ exists in G such that $\langle \alpha, a, a^{-\alpha}, e_\alpha\rangle \in *$. Let $y \in G$ such that $\langle \alpha, a^{-\alpha}, a, y\rangle \in *$. Then by (2) again there is $x \in G$ such that $\langle \alpha, y, x, e_\alpha\rangle \in *$. Using the associative property of $\langle *, A\rangle$ one can easily verify that $\langle \alpha, y, y, y\rangle \in *$. Moreover, since $\langle \alpha, y, y, y\rangle \in *$, $\langle \alpha, y, x, e_\alpha\rangle \in *$ and $\langle \alpha, y, e_\alpha, y\rangle \in *$, it follows from the associativity of $\langle *, A\rangle$ that $y = e_\alpha$. This implies that $\langle \alpha, a^{-\alpha}, a, e_\alpha\rangle \in *$. Let $z \in G$ such that $\langle \alpha, e_\alpha, a, z\rangle \in *$. Since $\langle \alpha, a, a^{-\alpha}, e_\alpha\rangle \in *$, $\langle \alpha, a^{-\alpha}, a, e_\alpha\rangle \in *$, and $\langle \alpha, a, e_\alpha, a\rangle \in *$, it holds that $z = a$. i.e., $\langle \alpha, e_\alpha, a, a\rangle \in *$. Therefore $\langle G, *, A\rangle$ is a soft group. □

Theorem 2.12. *A soft semi-group $\langle G, *, A\rangle$ is a soft group if and only if the equations $\langle \alpha, a, x, b\rangle \in *$ and $\langle \alpha, y, a, b\rangle \in *$ are solvable in G for any elements a and b in G and all $\alpha \in A$. (In the sense that there are elements x and y in G satisfying these equations with respect to α).*

Proof. Suppose $\langle G, *, A\rangle$ is a soft group. Let $a, b, x \in G$ such that $\langle \alpha, a, x, b\rangle \in *$. As G is a soft group, there exists $a^{-\alpha} \in G$ such that $\langle \alpha, a, a^{-\alpha}, e_\alpha\rangle \in *$. Let $c \in G$ such that $\langle \alpha, a^{-\alpha}, b, c\rangle \in *$. Then by Lemma 2.7 it can be shown that $\langle \alpha, a, c, b\rangle \in *$ and hence c is a solution for the equation $\langle \alpha, a, x, b\rangle \in *$. Similarly, it can be verified that $\langle \alpha, y, a, b\rangle \in *$ has a solution in G. Conversely suppose that the equations $\langle \alpha, a, x, b\rangle \in *$ and $\langle \alpha, y, a, b\rangle \in *$ are solvable in G for all $a, b \in G$ and $\alpha \in A$. Then there exists $e_\alpha \in G$ such that $\langle \alpha, a, e_\alpha, a\rangle \in *$. Let b be any arbitrary element in G. We show that $\langle \alpha, b, e_\alpha, b\rangle \in *$. Given that equation $\langle \alpha, y, a, b\rangle \in *$ are solvable in G, we can select an element s in G such that $\langle \alpha, s, a, b\rangle \in *$. Let $m \in G$ such that $\langle \alpha, b, e_\alpha, m\rangle \in *$. So we have $\langle \alpha, b, e_\alpha, m\rangle \in *$, $\langle \alpha, a, e_\alpha, a\rangle \in *$ and $\langle \alpha, s, a, b\rangle \in *$. By associativity of $\langle *, A\rangle$ we get $m = b$. This implies that $\langle \alpha, b, e_\alpha, b\rangle \in *$. Moreover, as $\langle \alpha, a, x, e_\alpha\rangle \in *$ is solvable in G, we get that for any $a \in G$ there exists $a^{-\alpha} \in G$ such that $\langle \alpha, a, a^{-\alpha}, e_\alpha\rangle \in *$. Thus, by Theorem 2.11 $\langle G, *, A\rangle$ is a soft group. □

Definition 2.13. *By a soft element in G we mean a soft set $\langle F, A\rangle$ over G such that $Card(F(\alpha)) = 1$ for all $\alpha \in A$. That is, $F(\alpha)$ is a single element set for each $\alpha \in A$.*

We denote soft elements using lower case letters like $\langle \breve{a}, A\rangle$, $\langle \breve{b}, A\rangle$ etc. We say that a soft element $\breve{a}$ belongs to a soft set $\langle F, A\rangle$ provided that $a_\alpha \in F(\alpha)$ for all $\alpha \in A$, where a_α is the unique element of $\breve{a}(\alpha) = \{a_\alpha\}$. Let us denote by $SE_A(G)$ the collection of all soft elements of G with a set of parameter A.

Note : Every element $a \in G$ can be identified as a soft element denoted by $\langle \breve{a}, A\rangle$ over G in the following way $\breve{a}(\alpha) = \{a\}$ for all $\alpha \in A$.

Definition 2.14. *Given a soft binary operation $\langle *, A\rangle$ on G, define a binary operation $\bar{*}$ on $SE_A(G)$ by*

$$\breve{a}\,\bar{*}\,\breve{b} = \breve{c} \Leftrightarrow \langle \alpha, a_\alpha, b_\alpha, c_\alpha\rangle \in * \quad for\ all\ \alpha \in A.$$

Theorem 2.15. *Let $\langle *, A\rangle$ be a soft binary operation on G. Then $\langle G, *, A\rangle$ is a soft group if and only if $\langle SE_A(G), \bar{*}\rangle$ is a group.*

Proof. Suppose that $\langle G, *, A\rangle$ is a soft group. We first show that $\bar{*}$ is well-defined. Let $\breve{a}, \breve{b}, \breve{c}, \breve{x}, \breve{y}, \breve{z}$ be soft elements over G such that $\breve{a}\bar{*}\breve{b} = \breve{c}$, $\breve{x}\bar{*}\breve{y} = \breve{z}$ and $\breve{a} = \breve{x}$, $\breve{b} = \breve{y}$. We need to show that $\breve{c} = \breve{z}$. From $\breve{a} = \breve{x}$ and $\breve{b} = \breve{y}$, we have $\breve{a}(\alpha) = \breve{x}(\alpha)$ and $\breve{b}(\alpha) = \breve{y}(\alpha)$ for all $\alpha \in A$. Which implies that $a_\alpha = x_\alpha$ and $b_\alpha = y_\alpha$ for all $\alpha \in A$. Also since $\breve{a}\bar{*}\breve{b} = \breve{c}$ and $\breve{x}\bar{*}\breve{y} = \breve{z}$, we have $\langle \alpha, a_\alpha, b_\alpha, c_\alpha\rangle \in *$ and $\langle \alpha, x_\alpha, y_\alpha, z_\alpha\rangle \in *$ for all $\alpha \in A$. Moreover, as $*$ is well-defined it follows that $c_\alpha = z_\alpha$ for all $\alpha \in A$. This implies that $\breve{c} = \breve{z}$. Now $\breve{a}\bar{*}\breve{b} = \breve{x}$ and $\breve{x}\bar{*}\breve{c} = \breve{y}$. It follows that $\langle \alpha, a_\alpha, b_\alpha, x_\alpha\rangle \in *$ and $\langle \alpha, x_\alpha, c_\alpha, y_\alpha\rangle \in *$ for all $\alpha \in A$. Again if we put $\breve{b}\bar{*}\breve{c} = \breve{u}$ and $\breve{a}\bar{*}\breve{u} = \breve{v}$. Then $\langle \alpha, b_\alpha, c_\alpha, u_\alpha\rangle \in *$ and $\langle \alpha, a_\alpha, u_\alpha, v_\alpha\rangle \in *$ for all $\alpha \in A$. Since $\langle *, A\rangle$ is associative we get that $y_\alpha = v_\alpha$ for all $\alpha \in A$. This implies that $\breve{y} = \breve{v}$. That is $\bar{*}$ is associative. Next consider the soft element $\breve{e}$ in G defined by $\breve{e}(\alpha) = \{e_\alpha\}$ for all $\alpha \in A$ where each e_α is an identity element of $\langle G, *, A\rangle$ with respect to α. Then one can easily show that $\breve{e}$ is an identity element in $SE_A(G)$. Also for every soft element $\breve{a}$ over G, consider the soft element $\breve{a}^{-1}$ of G defined by $\breve{a}^{-1}(\alpha) = \{a_\alpha^{-\alpha}\}$ for all $\alpha \in A$. Then $\breve{a}^{-1}$ is the inverse of $\breve{a}$ in $SE_A(G)$. Thus $\langle SE_A(G), \bar{*}\rangle$ is a group. Conversely suppose that $\langle SE_A(G), \bar{*}\rangle$ is a group. We need to show that $\langle G, *, A\rangle$ is a soft group. We first to show that $\langle *, A\rangle$ is associative. Let $\alpha \in A$ and $a, b, c, x, y, u, v \in G$ such that $\langle \alpha, a, b, x\rangle \in *$, $\langle \alpha, x, c, y\rangle \in *$ and $\langle \alpha, b, c, u\rangle \in *$, $\langle \alpha, a, u, v\rangle \in *$. Our aim is to show that $y = v$. Consider the soft elements $\breve{a}, \breve{b}, \breve{c}, \breve{x}, \breve{y}, \breve{u}, \breve{v} \in SE_A(G)$ defined as follows $\breve{a}(\lambda) = \{a\}$, $\breve{b}(\lambda) = \{b\}$, $\breve{c}(\lambda) = \{c\}$ for all $\lambda \in A$.

$$\breve{x}(\lambda) = \begin{cases} \{x\} & if \;\; \lambda = \alpha \\ \{x_\lambda\} & if \;\; \lambda \neq \alpha \end{cases}$$

where $x_\lambda \in G$ with $\langle \lambda, a, b, x_\lambda\rangle \in *$, and

$$\breve{y}(\lambda) = \begin{cases} \{y\} & if \;\; \lambda = \alpha \\ \{y_\lambda\} & if \;\; \lambda \neq \alpha \end{cases}$$

where $y_\lambda \in G$ with $\langle \lambda, x_\lambda, c, y_\lambda\rangle \in *$. Similarly, define

$$\breve{u}(\lambda) = \begin{cases} \{u\} & if \;\; \lambda = \alpha \\ \{u_\lambda\} & if \;\; \lambda \neq \alpha \end{cases}$$

where $u_\lambda \in G$ with $\langle \lambda, b, c, u_\lambda \rangle \in *$, and

$$\breve{v}(\lambda) = \begin{cases} \{v\} & if \quad \lambda = \alpha \\ \{v_\lambda\} & if \quad \lambda \neq \alpha \end{cases}$$

where $v_\lambda \in G$ such that $\langle \lambda, a, u_\lambda, v_\lambda \rangle \in *$ for all $\lambda \neq \alpha$.

Then we get that $\breve{a} \bar{*} \breve{b} = \breve{x}$, $\breve{x} \bar{*} \breve{c} = \breve{y}$ and $\breve{b} \bar{*} \breve{c} = \breve{u}$, $\breve{a} \bar{*} \breve{u} = \breve{v}$. Since $\bar{*}$ is associative we get $\breve{y} = \breve{v}$. That is $\breve{y}(\lambda) = \breve{v}(\lambda)$ $\forall \lambda \in A$. In particular, for $\alpha \in A$ $\breve{y}(\alpha) = \breve{v}(\alpha)$. This implies that $y = v$. Thus $\langle *, A \rangle$ is associative. Secondly, as $\langle SE_A(G), \bar{*} \rangle$ is a group it has an identity element let say $\breve{e}$. Now let $a \in G$ and $\alpha \in A$. Then consider a soft element $\breve{a}$ given by $\breve{a}(\lambda) = \{a\}$ for all $\lambda \in A$. Then $\breve{a} \bar{*} \breve{e} = \breve{a}$. This implies that $\langle \lambda, \breve{a}(\lambda), \breve{e}(\lambda), \breve{a}(\lambda) \rangle \in *$ for all $\lambda \in A$. In particular, for $\alpha \in A$ $\langle \alpha, \breve{a}(\alpha), \breve{e}(\alpha), \breve{a}(\alpha) \rangle \in *$. Which implies that $\langle \alpha, a, e_\alpha, a \rangle \in *$. Similarly it can be shown that $\langle \alpha, e_\alpha, a, a \rangle \in *$. Finally, as $\langle SE_A(G), \bar{*} \rangle$ is a group every element $\breve{a}$ in $SE_A(G)$ has an inverse let say $\breve{a}^{-1}$. Now let $a \in G$ and $\alpha \in A$. Then consider a soft element $\breve{a}$ given by $\breve{a}(\lambda) = \{a\}$ for all $\lambda \in A$. Then $\breve{a} \bar{*} \breve{a}^{-1} = \breve{e}$. This implies that $\langle \lambda, \breve{a}(\lambda), \breve{a}^{-1}(\lambda), \breve{e}(\lambda) \rangle \in *$ for all $\lambda \in A$. In particular, for $\alpha \in A$ $\langle \alpha, \breve{a}(\alpha), \breve{a}^{-1}(\alpha), \breve{e}(\alpha) \rangle \in *$. It follows that $\langle \alpha, a, a^{-\alpha}, e_\alpha \rangle \in *$. Similarly it can be shown that $\langle \alpha, a^{-\alpha}, a, e_\alpha \rangle \in *$. Therefore $\langle G, *, A \rangle$ is a soft group. □

Note: The group $SE_A(G)$ obtained in the Theorem 2.15 is a model representing the soft group $\langle G, *, A \rangle$, and is useful to transfer most of the important properties of classical groups to soft groups.

2.2 Soft subgroups

Definition 2.16. *Let $\langle G, *, A \rangle$ be a soft group. A soft set $\langle H, A \rangle$ over G is said to be a soft subgroup of G if for each $\alpha \in A$ and $a, b, x \in G$:*

(1) $e_\alpha \in H(\alpha)$;

(2) If $a, b \in H(\alpha)$ and $\langle \alpha, a, b, x \rangle \in *$, then $x \in H(\alpha)$;

(3) $a^{-\alpha} \in H(\alpha)$ whenever $a \in H(\alpha)$.

Notation. The collection of all soft subgroups of $\langle G, *, A \rangle$ is represented by $SS_A(G)$.

Example 2.17. *(1) Given a soft group $\langle G, *, A \rangle$, the absolute soft set $\langle 1_G, A \rangle$ over G and the soft element $\langle \breve{e}, A \rangle$ of G are both soft subgroups of G.*

(2) Let $G = \mathbb{Z}$ and $A = \mathbb{N}$. Let $* \subseteq A \times G \times G \times G$ be given by

$$* = \{\langle \alpha, a, b, c\rangle : c = a + b - 6\alpha\}.$$

Define $H : A \to p(G)$ by

$$H(\alpha) = \{(2k)\alpha : k \in \mathbb{Z}\},$$

then $\langle H, A\rangle$ is a soft subgroup of G.

Theorem 2.18. *Let $\langle G, *, A\rangle$ be a soft group. A soft set $\langle H, A\rangle$ over G is a soft subgroup of G if and only if the following two conditions hold for each $\alpha \in A$ and all $a, b, x \in G$:*

(1) $e_\alpha \in H(\alpha)$;

(2) If $a, b \in H(\alpha)$ and $\langle \alpha, a, b^{-\alpha}, x\rangle \in *$, then $x \in H(\alpha)$.

Proof. Let $\langle H, A\rangle$ be a soft subgroup of $\langle G, *, A\rangle$. Let $\alpha \in A$ and $a, b \in H(\alpha)$. Then $b^{-\alpha} \in H(\alpha)$. If $x \in G$ such that $\langle \alpha, a, b^{-\alpha}, x\rangle \in *$, then it is immediate that $x \in H(\alpha)$. Conversely suppose that (1) and (2) hold. We need to show that $\langle H, A\rangle$ is a soft subgroup of G. By (1), $e_\alpha \in H(\alpha)$. Since $\langle \alpha, e_\alpha, b^{-\alpha}, b^{-\alpha}\rangle \in *$ it follows that $b^{-\alpha} \in H(\alpha)$. Let $x \in G$ such that $\langle \alpha, a, b, x\rangle \in *$. Then, as $b = (b^{-\alpha})^{-\alpha}$ we get that $x \in H(\alpha)$. Thus $\langle H, A\rangle$ is a soft subgroup of G. □

Definition 2.19. *For a soft subset $\langle H, A\rangle$ over G, define a subset $\widehat{H}$ of $SE_A(G)$ by:*

$$\widehat{H} = \{\breve{a} \in SE_A(G) : \breve{a}(\alpha) \subseteq H(\alpha) \text{ for all } \alpha \in A\}.$$

Theorem 2.20. *A soft set $\langle H, A\rangle$ over G is a soft subgroup of G if and only if $\widehat{H}$ is a subgroup of $SE_A(G)$.*

Proof. Suppose that $\langle H, A\rangle$ is a soft subgroup of G. Then for each $\alpha \in A$, $e_\alpha \in H(\alpha)$, so that the soft identity element $\breve{e}$ belongs to $\widehat{H}$. Let $\breve{a}, \breve{b}$ be any soft elements in G belonging to $\widehat{H}$. Then $\breve{a}(\alpha) \subseteq H(\alpha)$ and $\breve{b}(\alpha) \subseteq H(\alpha)$ for all $\alpha \in A$. Now let $\breve{c}$ be a soft element of G with $\breve{a} \bar{*} \breve{b}^{-1} = \breve{c}$. Then $\langle \alpha, \breve{a}(\alpha), \breve{b}^{-1}(\alpha), \breve{c}(\alpha)\rangle \in *$ for all $\alpha \in A$. Therefore $\widehat{H}$ is a subgroup of $SE_A(G)$. Conversely, suppose $\widehat{H}$ is a subgroup of $SE_A(G)$. Then identity soft element $\breve{e} \in \widehat{H}$ and so $\breve{e}(\alpha) = e_\alpha \in H(\alpha)$. Let $\alpha \in A$, $a, b, c \in G$ such that $\langle \alpha, a, b^{-\alpha}, c\rangle \in *$. We claim to show that $c \in H(\alpha)$. Define soft elements $\breve{a}, \breve{b}, \breve{c}$ of G by

$$\breve{a}(\lambda) = \begin{cases} \{a\} & if \quad \lambda = \alpha \\ \{e_\lambda\} & if \quad \lambda \neq \alpha, \end{cases}$$

$$\breve{b}(\lambda) = \begin{cases} \{b\} & if \quad \lambda = \alpha \\ \{e_\lambda\} & if \quad \lambda \neq \alpha \end{cases}$$

and

$$\breve{c}(\lambda) = \begin{cases} \{c\} & if \quad \lambda = \alpha \\ \{e_\lambda\} & if \quad \lambda \neq \alpha \end{cases}$$

for all $\lambda \in A$. Then $\breve{a}(\lambda) \in H(\lambda)$ and $\breve{b}(\lambda) \in H(\lambda)$ for all $\lambda \in A$. Which gives that $\breve{a}, \breve{b} \in \widehat{H}$. Since $\widehat{H}$ is a subgroup of $SE_A(G)$ it holds that $\breve{a} \bar{*} \breve{b}^{-1} \in \widehat{H}$. One can also verify that

$$\breve{b}^{-1}(\lambda) = \begin{cases} \{b^{-\alpha}\} & if \quad \lambda = \alpha \\ \{e_\lambda\} & if \quad \lambda \neq \alpha \end{cases}$$

and $\breve{a} \bar{*} \breve{b}^{-1} = \breve{c}$. That is $\breve{c} \in \widehat{H}$ so that $\breve{c}(\lambda) \subseteq H(\lambda)$ for all $\lambda \in A$. In particular, $\breve{c}(\alpha) = \{c\} \subseteq H(\alpha)$. Therefore $\langle H, A \rangle$ is a soft subgroup of G.

□

Lemma 2.21. *Arbitrary intersection of soft subgroups is a soft subgroup.*

Proof. (1) Let $\langle G, *, A \rangle$ be a soft group and $\alpha \in A$. Let $\{\langle H_i, A \rangle : i \in I\}$ be a family of soft subgroup of G. Since $e_\alpha \in H_i(\alpha)$ for all $i \in I$. This implies that $e_\alpha \in \bigcap_{i \in I} H_i(\alpha)$.

(2) Let $a, b \in \bigcap_{i \in I} H_i(\alpha)$ and $x \in G$ such that $\langle \alpha, a, b^{-\alpha}, x \rangle \in *$. This implies that $a, b \in H_i(\alpha)$ for all $i \in I$ and $\langle \alpha, a, b^{-\alpha}, x \rangle \in *$. So $x \in H_i(\alpha)$ for all $i \in I$. Hence $x \in \bigcap_{i \in I} H_i(\alpha)$. Therefore $\langle \breve{\bigcap}_{i \in I} H_i, A \rangle$ is a soft subgroup of G.

□

Remark 2.22. *The union of two soft subgroups may not be a soft subgroup. The example that follows verify this.*

Example 2.23. *Let* $G = \mathbb{Z}$ *and* $A = \mathbb{N}$. *Let* $* \subseteq A \times G \times G \times G$ *be given by*

$$* = \{\langle \alpha, a, b, c\rangle : c = a + b - 6\alpha\}.$$

Define $H : A \to p(G)$ *by*

$$H(\alpha) = \{(2k)\alpha : k \in \mathbb{Z}\}$$

and $K : A \to p(G)$ *by*

$$K(\alpha) = \{(3k)\alpha : k \in \mathbb{Z}\}$$

then $\langle H \cup K, A\rangle$ *is not a soft subgroup of* G.

Theorem 2.24. *Let* $\langle G, *, A\rangle$ *be a soft group and* $\langle H, A\rangle$ *and* $\langle K, A\rangle$ *are soft subgroups of* G. *If* $\langle H, A\rangle \breve{\subseteq} \langle K, A\rangle$ *or* $\langle K, A\rangle \breve{\subseteq} \langle H, A\rangle$ *then* $\langle H \breve{\cup} K, A\rangle$ *is a soft subgroup of* G. *However, the converse is not true.*

Example 2.25. *Let* $G = \mathbb{Z}$ *and* $A = \mathbb{N}$. *Let* $A \times G \times G \times G$ *be given by*

$$* = \{\langle \alpha, a, b, c\rangle : c = a + b - 6\alpha\}.$$

Define $H, K : A \to p(G)$ *by*

$$H(\alpha) = \begin{cases} \{2k\alpha : k \in \mathbb{Z}\} & if \quad \alpha = 1 \\ \mathbb{Z} & if \quad \alpha = 2 \\ \{6k\alpha : k \in \mathbb{Z}\} & if \quad \alpha \geq 3 \end{cases}$$

and

$$K(\alpha) = \begin{cases} \mathbb{Z} & if \quad \alpha = 1 \\ \{3k\alpha : k \in \mathbb{Z}\} & if \quad \alpha = 2 \\ \{6k\alpha : k \in \mathbb{Z}\} & if \quad \alpha \geq 3 \end{cases}$$

for all $\alpha \in A$. *Then* $\langle H \breve{\cup} K, A\rangle$ *is a soft subgroup of* G *but* $\langle H, A\rangle \breve{\nsubseteq} \langle K, A\rangle$ *and* $\langle K, A\rangle \breve{\nsubseteq} \langle H, A\rangle$.

Definition 2.26. *Let* $\langle H, A\rangle$ *and* $\langle K, A\rangle$ *be soft subgroups of* $\langle G, *, A\rangle$. *Define soft sets* $\langle HK, A\rangle$ *and* $\langle H^{-1}, A\rangle$ *over* G *respectively as follows:*

$$\begin{aligned} HK(\alpha) &= \{x \in G : \exists\ a \in H(\alpha) \text{ and } b \in K(\alpha) \text{ such that } \langle \alpha, a, b, x\rangle \in *\} \\ &\text{and} \\ H^{-1}(\alpha) &= \{x \in G : x^{-\alpha} \in H(\alpha)\}. \end{aligned}$$

Theorem 2.27. *The product $\langle HK, A\rangle$ is a soft subgroup of $\langle G, *, A\rangle$ if and only if $\langle HK, A\rangle \,\breve{=}\, \langle KH, A\rangle$*

Proof. Suppose that $\langle HK, A\rangle$ is a soft subgroup of $\langle G, *, A\rangle$. We show that $HK(\alpha) = KH(\alpha)$ for all $\alpha \in A$. Let $\alpha \in A$ and $a \in HK(\alpha)$ then $a^{-\alpha} \in HK(\alpha)$. This implies that there exist $h \in H(\alpha)$, $k \in K(\alpha)$ such that $\langle \alpha, h, k, a^{-\alpha}\rangle \in *$, and so $k^{-\alpha} \in k(\alpha)$, $h^{-\alpha} \in H(\alpha)$ and $\langle \alpha, k^{-\alpha}, h^{-\alpha}, a\rangle \in *$. Thus $a \in KH(\alpha)$ and hence $HK(\alpha) \subseteq KH(\alpha)$. Similarly, it can be shown that $KH(\alpha) \subseteq HK(\alpha)$ for all $\alpha \in A$ and hence the equality holds. Conversely suppose that $HK(\alpha) = KH(\alpha)$ for all $\alpha \in A$. Let $\alpha \in A$ be fixed. It is clear that $e_\alpha \in HK(\alpha)$. Let $a, b \in HK(\alpha)$ and $x \in G$ such that $\langle \alpha, a, b^{-\alpha}, x\rangle \in *$. Then there exist $h_1, h_2 \in H(\alpha)$ and $k_1, k_2 \in K(\alpha)$ such that $\langle \alpha, h_1, k_1, a\rangle \in *$ and $\langle \alpha, h_2, k_2, b\rangle \in *$. From $\langle \alpha, h_2, k_2, b\rangle \in *$ we have $\langle \alpha, k_2^{-\alpha}, h_2^{-\alpha}, b^{-\alpha}\rangle \in *$. Now let x_1, x_2 and $x_3 \in G$ such that $\langle \alpha, k_1, k_2^{-\alpha}, x_1\rangle \in *$, $\langle \alpha, x_1, h_2^{-\alpha}, x_2\rangle \in *$ and $\langle \alpha, h_1, x_2, x_3\rangle \in *$. Since k_1, $k_2 \in k(\alpha)$, $x_1 \in K(\alpha)$ and as $h_2 \in H(\alpha)$, $x_2 \in KH(\alpha) = HK(\alpha)$. Which implies that $x_2 \in HK(\alpha)$. Then there exists $y_1 \in H(\alpha)$ and $y_2 \in k(\alpha)$ such that $\langle \alpha, y_1, y_2, x_2\rangle \in *$. Let $z \in G$ such that $\langle \alpha, h_1, y_1, z\rangle \in *$. Then, $z \in H(\alpha)$. It follows that $\langle \alpha, z, y_2, x_3\rangle \in *$. Also by associative property of $\langle *, A\rangle$, we have $x = x_3$. That is $\langle \alpha, z, y_2, x\rangle \in *$, where $z \in H(\alpha)$ and $y_2 \in K(\alpha)$. Thus, $x \in HK(\alpha)$. Therefore $\langle HK, A\rangle$ is a soft subgroup of G. □

In the following theorem, we characterize soft subgroups using the product and inverse operations defined on the class of soft sets over G.

Theorem 2.28. *A soft set $\langle H, A\rangle$ over a soft group G is a soft subgroup of G if and only if*

(1) $e_\alpha \in H(\alpha)$ for all $\alpha \in A$;

(2) $\langle HH, A\rangle \,\breve{\subseteq}\, \langle H, A\rangle$;

(3) $\langle H^{-1}, A\rangle \,\breve{\subseteq}\, \langle H, A\rangle$.

Proof. Suppose that $\langle H, A\rangle$ is a soft subgroup of $\langle G, *, A\rangle$. Clearly $e_\alpha \in H(\alpha)$. Let $x \in HH(\alpha)$ then there exist $h_1, h_2 \in H(\alpha)$ such that $\langle \alpha, h_1, h_2, x\rangle \in *$. It follows that $x \in H(\alpha)$. Thus $\langle HH, A\rangle \breve{\subseteq} \langle H, A\rangle$. Let $x \in H^{-1}(\alpha)$ then $x^{-\alpha} \in H(\alpha)$. Which implies that $x \in H(\alpha)$. Therefore $\langle H^{-1}, A\rangle \breve{\subseteq} \langle H, A\rangle$. Conversely we show that $\langle H, A\rangle$ is a soft subgroup of G. By (1), $e_\alpha \in H(\alpha)$. Let $x \in H(\alpha)$. Which implies that $x^{-\alpha} \in H^{-1}(\alpha)$. By (3), we get $x^{-\alpha} \in H(\alpha)$. Let $a, b \in H(\alpha)$ and $y \in G$ such that $\langle \alpha, a, b, y\rangle \in *$. It follows that $y \in HH(\alpha)$. Hence by(2), $y \in H(\alpha)$. Therefore $\langle H, A\rangle$ is a soft subgroup of G.

□

Definition 2.29. *A soft group $\langle G, *, A\rangle$ is said to be abelian if for each $\alpha \in A$ and any $a, b, x \in G$, $\langle \alpha, a, b, x\rangle \in *$ if and only if $\langle \alpha, b, a, x\rangle \in *$.*

Example 2.30. *Soft groups given in Example 2.3 and 2.4 are abelian, whereas soft groups given in Example 2.5 and 2.6 are non-abelian.*

Lemma 2.31. *A soft group $\langle G, *, A\rangle$ is abelian if and only if for each $\alpha \in A$ and all $a, b, x, y \in G$ it holds that: $\langle \alpha, a, b, x\rangle \in *$ and $\langle \alpha, b, a, y\rangle \in *$ together imply $x = y$.*

Lemma 2.32. *A soft group $\langle G, *, A\rangle$ is abelian if and only if the classical group $SE_A(G)$ is an abelian group.*

Proof. The proof is similar to Theorem 2.15. □

Theorem 2.33. *Let $\langle H, A\rangle$ and $\langle K, A\rangle$ be soft subgroups of a soft group $\langle G, *, A\rangle$. If G is abelian, then $\langle HK, A\rangle$ is the least soft subgroup of G containing both $\langle H, A\rangle$ and $\langle K, A\rangle$.*

Proof. Since G is abelian soft group, then $\langle HK, A\rangle \breve{=} \langle KH, A\rangle$. So $\langle HK, A\rangle$ is a soft subgroup of G. Let $\langle M, A\rangle$ be a soft subgroup of G containing both $\langle H, A\rangle$ and $\langle K, A\rangle$. We show that $\langle HK, A\rangle \breve{\subseteq} \langle M, A\rangle$. Let $x \in HK(\alpha)$, there exists $h \in H(\alpha)$ and $k \in K(\alpha)$ such that $\langle \alpha, h, k, x\rangle \in *$. This implies that $x \in M(\alpha)$. Therefore $\langle HK, A\rangle$ is the least soft subgroup of G containing both $\langle H, A\rangle$ and $\langle K, A\rangle$.

□

Definition 2.34. *For any soft set $\langle F, A\rangle$ over a soft group $\langle G, *, A\rangle$, define a soft set $\langle C_F, A\rangle$ over G by*

$$C_F(\alpha) = \{a \in G : \langle \alpha, a, x, y_1\rangle \in * \text{ and } \langle \alpha, x, a, y_2\rangle \in * \Rightarrow y_1 = y_2, \text{ for all } x \in F(\alpha), y_1, y_2 \in G\}.$$

We call $\langle C_F, A\rangle$ a centralizer of $\langle F, A\rangle$.

Lemma 2.35. *Let $\langle F, A\rangle$ be a soft set over a soft group $\langle G, *, A\rangle$, $\alpha \in A$ and $a \in G$. Then $a \in C_F(\alpha)$ is equivalent to the condition $\langle \alpha, a, x, y\rangle \in *$ if and only if $\langle \alpha, x, a, y\rangle \in *$ for all $x \in F(\alpha)$ and all $y \in G$.*

Theorem 2.36. *For any soft set $\langle F, A\rangle$ over a soft group $\langle G, *, A\rangle$, $\langle C_F, A\rangle$ is a soft subgroup of G.*

Proof. For any $\alpha \in A$, all $x \in F(\alpha)$ and all $y \in G$, it is true that $\langle \alpha, e_\alpha, x, y\rangle \in *$ if and only if $\langle \alpha, x, e_\alpha, y\rangle \in *$. Which implies that $e_\alpha \in C_F(\alpha)$. Let $a \in C_F(\alpha)$. Then for any $x \in F(\alpha)$ and all $y \in G$ it is the case that $\langle \alpha, a, x, y\rangle \in *$ if and only if $\langle \alpha, x, a, y\rangle \in *$. We show that $a^{-\alpha} \in C_F(\alpha)$. Let $x \in F(\alpha)$ and $y, z \in G$ such that $\langle \alpha, a^{-\alpha}, x, y\rangle \in *$ and $\langle \alpha, x, a, z\rangle \in *$. Since $a \in C_F(\alpha)$ and $x \in F(\alpha)$ we have $\langle \alpha, a, x, z\rangle \in *$. Using the associativity of $\langle *, A\rangle$ we get $\langle \alpha, y, a, x\rangle \in *$. It follows that $\langle \alpha, x, a^{-\alpha}, y\rangle \in *$. Therefore $a^{-\alpha} \in C_F(\alpha)$. Next let $a, b \in C_F(\alpha)$ and $c \in G$ such that $\langle \alpha, a, b, c\rangle \in *$. Let $x \in F(\alpha)$ and $y, z, w \in G$ such that $\langle \alpha, c, x, y\rangle \in *$, $\langle \alpha, a, x, z\rangle \in *$ and $\langle \alpha, b, x, w\rangle \in *$. Since $b \in C_F(\alpha)$ it holds that $\langle \alpha, x, b, w\rangle \in *$. Which implies that $\langle \alpha, z, b, y\rangle \in *$. Again as $a \in C_F(\alpha)$ it holds that $\langle \alpha, x, a, z\rangle \in *$ implying $\langle \alpha, x, c, y\rangle \in *$. Thus, $c \in C_F(\alpha)$ and this is true for all $\alpha \in A$. Therefore, $\langle C_F(\alpha), A\rangle$ is a soft subgroup of G. □

Definition 2.37. *Let $\langle X, A\rangle$ be a soft set over a soft group $\langle G, *, A\rangle$. The smallest soft subgroup of G containing $\langle X, A\rangle$ is called the soft subgroup of G generated by $\langle X, A\rangle$ and is denoted by $Sg_A(X)$. That is*

$$\langle Sg_A(X), A\rangle = \breve{\bigcap}\{\langle H, A\rangle : \langle H, A\rangle$$

is a soft subgroup of G such that $\langle X, A\rangle \breve{\subseteq} \langle H, A\rangle\}$.

Theorem 2.38. *The soft subgroup of $\langle G, *, A\rangle$ generated by a soft set $\langle F, A\rangle$ over G can be described as follows : for $\alpha \in A$, if $F(\alpha) = \emptyset$, then $Sg_A(F)(\alpha) = \{e_\alpha\}$. If $F(\alpha) \neq \emptyset$, then $x \in Sg_A(F)(\alpha)$ if and only if there exist $n \in \mathbb{N}$ and sequences $\{a_i\}_{i=1}^n$ and $\{x_i\}_{i=1}^n$ of elements of G such that $x_1 = a_1, x_n = x$ and $\langle \alpha, x_i, a_{i+1}, x_{i+1}\rangle \in * \ \forall i = 1, 2, 3, ...n-1$ where for each i, either $a_i \in F(\alpha)$ or $a_i^{-\alpha} \in F(\alpha)$.*

Proof. Let $x, y \in Sg_A(F)(\alpha)$. Then there exist $n, m \in \mathbb{N}$ and sequences $\{a_i\}_{i=1}^n$, $\{b_j\}_{j=1}^m$ and $\{x_i\}_{i=1}^n$, $\{y_j\}_{j=1}^m$ of element of G such that $x_1 = a_1, x_n = x$ and $\langle \alpha, x_i, a_{i+1}, x_{i+1}\rangle \in *$, $y_1 =$

$b_1, y_m = y$ and $\langle \alpha, y_j, b_{j+1}, y_{j+1} \rangle \in *$ for all $i = 1, 2, 3, ...n-1$ and $j = 1, 2, 3, ...m-1$. Let $z \in G$ such that $\langle \alpha, x, y^{-\alpha}, z \rangle \in *$. We need to show that $z \in Sg_A(F)(\alpha)$. Consider the sequence $\{c_k\}_{k=1}^{n+1}$ of elements of G defined by $c_k = a_k$ for $1 \leqslant k \leqslant n$ and $c_{n+k} = b_{m-k+1}^{-\alpha}$ for $1 \leqslant k \leqslant m$. Then, for each $k = 1, 2, 3, ...n+m$ either $c_k \in F(\alpha)$ or $c_k^{-\alpha} \in F(\alpha)$. Moreover, if we define a sequence $\{z_k\}_{k=1}^{n+m}$ by $z_1 = c_1 = a_1 = x_1$ and $\langle \alpha, z_i, c_{i+1}, z_{i+1} \rangle \in * \; \forall i = 1, 2, 3, ...n+m-1$. Then, using associativity of $*$, it can be shown that $z_{n+m} = z$. Thus $z \in Sg_A(F)(\alpha)$. Therefore $\langle Sg_A(F), A \rangle$ is a soft subgroup of G. Next we show that $\langle F, A \rangle \,\breve{\subseteq}\, \langle Sg_A(F), A \rangle$. Let $\alpha \in A$ and $x \in F(\alpha)$. Now considering sequences $\{x_i\}_{i=1}^{n}$ and $\{a_i\}_{i=1}^{n}$ taking n=1, where $a_1 = x = x_1$. Then it is vacuously true that $\langle \alpha, x_i, a_{i+1}, x_{i+1} \rangle \in *$ for all $1 \leqslant i \leqslant n$ and hence $x \in Sg_A(F)(\alpha)$. Thus $\langle F, A \rangle \,\breve{\subseteq}\, \langle Sg_A(F), A \rangle$. Now let $\langle H, A \rangle$ be any other soft subgroup of G such that $\langle F, A \rangle \,\breve{\subseteq}\, \langle H, A \rangle$. Then $F(\alpha) \subseteq H(\alpha)$ for all $\alpha \in A$. We next show that $Sg_A(F)(\alpha) \subseteq H(\alpha)$ for all $\alpha \in A$. Let $\alpha \in A$ and $x \in Sg_A(F)(\alpha)$. Then there exist $n \in \mathbb{N}$ and sequences $\{a_i\}_{i=1}^{n}$ and $\{x_i\}_{i=1}^{n}$ of elements of G such that $x_1 = a_1, x_n = x$ and $\langle \alpha, x_i, a_{i+1}, x_{i+1} \rangle \in *$. $\forall i = 1, 2, 3, ...n-1$ where for each i,$1 \leqslant i \leqslant n$ either $a_i \in F(\alpha)$ or $a_i^{-\alpha} \in F(\alpha)$. Since $F(\alpha) \subseteq H(\alpha)$ and $\langle H, A \rangle$ is a soft subgroup of G. We get that $a_i \in H(\alpha)$, for all $1 \leqslant i \leqslant n$. As $x_1 = a_1$ and $\langle \alpha, x_1, a_2, x_2 \rangle \in *$ we get that $x_2 \in H(\alpha)$. Similarly, from $\langle \alpha, x_2, a_3, x_3 \rangle \in *$ we get that $x_3 \in H(\alpha)$. Continuing this process until all x_i are exhausted we get finally $x_n \in H(\alpha)$. That is $x \in H(\alpha)$. Thus $Sg_A(F)(\alpha) \subseteq H(\alpha)$. As $\alpha \in A$ is arbitrary it can be concluded that $\langle Sg_A(F), A \rangle \,\breve{\subseteq}\, \langle H, A \rangle$. Therefore $\langle Sg_A(F), A \rangle$ is the smallest soft subgroup of G containing $\langle F, A \rangle$.

□

Theorem 2.39. *The soft subgroup of $\langle G, *, A \rangle$ generated by the soft element $\langle \breve{a}, A \rangle$ can be described as follows: for $x \in G$, $x \in Sg_A(\breve{a})(\alpha)$ if and only if there is $n \in \mathbb{N}$ and a sequence $x_1, x_2, ...x_n$ in G such that $\langle \alpha, x_i, x_1, x_{i+1} \rangle \in * \; \forall i = 1, 2, 3, ...n-1$; where $x_1 \in \{a_\alpha, a_\alpha^{-\alpha}\}$ and $x_n = x$.*

Proof. The proof is similar to Theorem 2.38. □

Definition 2.40. *We call $\langle Sg_A(\breve{a}), A \rangle$ the cyclic soft subgroup of G generated by the soft element $\langle \breve{a}, A \rangle$ over G.*

Lemma 2.41. *For any soft sets $\langle F, A \rangle$ and $\langle E, A \rangle$ over G, the following hold:*

(1) $\langle F, A \rangle \,\breve{\subseteq}\, \langle Sg_A(F), A \rangle$;

(2) $\langle F, A \rangle \,\breve{\subseteq}\, \langle E, A \rangle \Rightarrow \langle Sg_A(F), A \rangle \,\breve{\subseteq}\, \langle Sg_A(E), A \rangle$;

(3) $\langle Sg_A(Sg_A(F)), A\rangle \,\breve{=}\, \langle Sg_A(F), A\rangle$.

The results in Lemma 2.41 justifies that the map $\langle F, A\rangle \mapsto \langle Sg_A(F), A\rangle$ forms a closure operator on the class of all soft sets over G with a fixed set A of parameters, and the closed elements with respect to this closure operator are precisely those soft subgroups of G.

Theorem 2.42. *Let $\langle G, *, A\rangle$ be a soft group and $SS_A(G)$ the class of all soft subgroups of G. Then, $\langle SS_A(G), \wedge, \vee\rangle$ is a complete lattice; where for soft subgroups $\langle H, A\rangle$ and $\langle K, A\rangle$ of G :*

$$\langle H, A\rangle \wedge \langle K, A\rangle = \langle H, A\rangle \,\breve{\cap}\, \langle K, A\rangle$$

and

$$\langle H, A\rangle \vee \langle K, A\rangle = \langle Sg_A(H \,\breve{\cup}\, K), A\rangle$$

Proof. It is given in Example 2.17 the absolute soft set $\langle 1_G, A\rangle$ over G is a soft subgroup of G and hence $SS_A(G)$ has the largest element. It is also proved in Lemma 2.21 that $SS_A(G)$ is closed under the arbitrary intersection of soft sets, and hence closed under arbitrary infima. Therefore it is a complete lattice. □

Theorem 2.43. *The lattice $SS_A(G)$ can be embedded into the lattice of all subgroups of the classical group $SE_A(G)$.*

Proof. It is enough to show that the map sending each soft group $\langle H, A\rangle$ of G to a subgroup $\widehat{H}$ of $SE_A(G)$ is an injective order homomorphism. □

2.3 Normal Soft Subgroups

Definition 2.44. *Let $\langle H, A\rangle$ be a soft subgroup of a soft group $\langle G, *, A\rangle$ and $a \in G$. Define a soft set $\langle {}^aH, A\rangle$ over G by:*

$${}^aH(\alpha) = \{x \in G : \langle \alpha, a, b, x\rangle \in * \text{ for some } b \in H(\alpha)\}.$$

We call $\langle {}^aH, A\rangle$ a left coset of $\langle H, A\rangle$ corresponding to a. Right cosets can be defined in a dual manner.

Theorem 2.45. *Let $\langle H, A\rangle$ be a soft subgroup of G and let $a, b \in G$.*

(1) $\langle {}^aH, A\rangle \,\breve{\subseteq}\, \langle {}^bH, A\rangle$ *if and only if for* $x \in G$, $\langle \alpha, b^{-\alpha}, a, x\rangle \in * \Rightarrow x \in H(\alpha)$;

(2) $\langle {}^{a}H, A\rangle \breve{=} \langle H, A\rangle$ *if and only if* $a \in H(\alpha)$ *for all* α *in* A*;*

(3) Either $\langle {}^{a}H, A\rangle \breve{=} \langle {}^{b}H, A\rangle$ *or* $\langle {}^{a}H, A\rangle$ *and* $\langle {}^{b}H, A\rangle$ *are weakly disjoint i.e* $\exists\ \alpha \in A$ *such that* $[{}^{a}H](\alpha) \cap [{}^{b}H](\alpha) = \emptyset$.

Proof. (1) Suppose that $\langle {}^{a}H, A\rangle \breve{\subseteq} \langle {}^{b}H, A\rangle$. Then, $[{}^{a}H](\alpha) \breve{\subseteq} [{}^{b}H](\alpha)$ for all $\alpha \in A$. Let $x \in G$ such that $\langle \alpha, b^{-\alpha}, a, x\rangle \in *$. Then,

$$\langle \alpha, b, x, a\rangle \in *. \tag{1}$$

Since $e_{\alpha} \in H(\alpha)$ and $\langle \alpha, a, e_{\alpha}, a\rangle \in *$, we have $a \in [{}^{a}H](\alpha) = [{}^{b}H](\alpha)$. Which implies that $a \in [{}^{b}H](\alpha)$. So there is some $y \in H(\alpha)$ such that

$$\langle \alpha, b, y, a\rangle \in *. \tag{2}$$

Applying the cancellation law on (1) and (2), we get $x = y \in H(\alpha)$. Conversely suppose that for any $x \in G$ it is the case that $\langle \alpha, b^{-\alpha}, a, x\rangle \in * \Rightarrow x \in H(\alpha)$. Let $\alpha \in A$ and $x \in [{}^{a}H](\alpha)$ then there exist $y \in H(\alpha)$ such that $\langle \alpha, a, y, x\rangle \in *$. Let $x_1, x_2, x_3 \in G$ such that $\langle \alpha, b^{-\alpha}, a, x_1\rangle \in *$, $\langle \alpha, x_1, y, x_2\rangle \in *$ and $\langle \alpha, b^{-\alpha}, x, x_3\rangle \in *$. Then by associative property of $*$, we get $x_2 = x_3$. Moreover it follows from our hypothesis that $x_1 \in H(\alpha)$. Since $x_1 \in H(\alpha)$ and $y \in H(\alpha)$. We get $x_2 \in H(\alpha)$. That is $x_3 \in H(\alpha)$. Further since $\langle \alpha, b^{-\alpha}, x, x_3\rangle \in *$, we have $\langle \alpha, b, x_3, x\rangle \in *$ and $x_3 \in H(\alpha)$. So that $x \in [{}^{b}H](\alpha)$. Therefore $[{}^{a}H](\alpha) \breve{\subseteq} [{}^{b}H](\alpha)$. Since α is arbitrary, we get $\langle {}^{a}H, A\rangle \breve{\subseteq} \langle {}^{b}H, A\rangle$.

(2) Let $\alpha \in A$ and $a \in G$. Since $e_{\alpha} \in H(\alpha)$ such that $\langle \alpha, a, e_{\alpha}, a\rangle \in *$. This implies that $a \in H(\alpha)$. Conversely suppose that $a \in H(\alpha)$. Let $\alpha \in A$ and $h \in H(\alpha)$ such that $\langle \alpha, a, a^{-\alpha}, h\rangle \in *$. This implies that $h \in {}^{a}H(\alpha)$. Therefore $H(\alpha) \subseteq {}^{a}H(\alpha)$. Let $x \in {}^{a}H(\alpha)$ then there exist $h \in H(\alpha)$ such that $\langle \alpha, a, h, x\rangle \in *$. This implies that $x \in H(\alpha)$. It follows that ${}^{a}H(\alpha) \subseteq H(\alpha)$. Thus, $H(\alpha) = {}^{a}H(\alpha)$. Therefore $\langle {}^{a}H, A\rangle \breve{=} \langle H, A\rangle$

(3) Suppose that $\langle {}^{a}H, A\rangle$ and $\langle {}^{b}H, A\rangle$ are not weakly disjoint. That is ${}^{a}H(\alpha) \breve{\cap} {}^{b}H(\alpha) \neq \emptyset$ for all $\alpha \in A$. This implies that there exist at least one $x \in {}^{a}H(\alpha) \cap {}^{b}H(\alpha)$. This implies that $x \in {}^{a}H(\alpha)$ and $x \in {}^{b}H(\alpha)$, for all $\alpha \in A$. It follows that there exist $h_1, h_2 \in H(\alpha)$ such that $\langle \alpha, a, h_1, x\rangle \in *$ and $\langle \alpha, b, h_2, x\rangle \in *$. Let $y \in G$ such that $\langle \alpha, x, h_1^{-\alpha}, y\rangle \in *$. Consider $\langle \alpha, h_1, h_1^{-\alpha}, e_{\alpha}\rangle \in *$, $\langle \alpha, a, e_{\alpha}, a\rangle \in *$ and $\langle \alpha, a, h_1, x\rangle \in *$, $\langle \alpha, x, h_1^{-\alpha}, y\rangle \in *$. Which implies

that $y = a$. (3)
Let $z \in G$ and $h_3 \in H(\alpha)$ such that $\langle \alpha, h_2, h_1^{-\alpha}, h_3 \rangle \in *$ and $\langle \alpha, b, h_3, z \rangle \in *$. Consider $\langle \alpha, b, h_2, x \rangle \in *$, $\langle \alpha, x, h_1^{-\alpha}, y \rangle \in *$, $\langle \alpha, h_2, h_1^{-\alpha}, h_3 \rangle \in *$ and $\langle \alpha, b, h_3, z \rangle \in *$. By $< SG1 >$ we get $y = z$. (4)
From (3) and (4) we have $a = z$. Let $x_1 \in {}^aH(\alpha)$. Which implies that there exist $h_4 \in H(\alpha)$ such that $\langle \alpha, a, h_4, x_1 \rangle \in *$. Let $h_5 \in G$ such that $\langle \alpha, h_3, h_4, h_5 \rangle \in *$. Since $\langle H, A \rangle$ is a normal soft subgroup, $h_5 \in H(\alpha)$. This implies that $\langle \alpha, b, h_5, x_1 \rangle \in *$. It follows that $x_1 \in {}^bH(\alpha)$. Therefore ${}^aH(\alpha) \subseteq {}^bH(\alpha)$. Similarly ${}^bH(\alpha) \subseteq {}^aH(\alpha)$. Hence ${}^aH(\alpha) = {}^bH(\alpha)$. This concludes our argument that $\langle {}^aH, A \rangle \,\breve{=}\, \langle {}^bH, A \rangle$.

□

Definition 2.46. *A soft subgroup $\langle N, A \rangle$ of a soft group $\langle G, *, A \rangle$ is called normal if $\langle {}^aN, A \rangle \,\breve{=}\, \langle N^a, A \rangle$ for all $a \in G$.*

Notation: We denote by $SN_A(G)$ the collection of all normal soft subgroups of G with the set of parameters A.

Theorem 2.47. *For a soft subgroup $\langle N, A \rangle$ of a soft group $\langle G, *, A \rangle$, the following are equivalent:*

(1) $\langle N, A \rangle$ is normal;

(2) For $\alpha \in A$, any a,x,y $\in G$ and $n \in N(\alpha)$, $\langle \alpha, a, n, x \rangle \in *$ and $\langle \alpha, x, a^{-\alpha}, y \rangle \in *$ together imply $y \in N(\alpha)$.

Proof. ($1 \Rightarrow 2$) Suppose that $\langle N, A \rangle$ is normal. That is ${}^aN(\alpha) = N^a(\alpha)$, for all $\alpha \in A$. Let $\alpha \in A$, $a, x, y \in G$ and $n \in N(\alpha)$ such that $\langle \alpha, a, n, x \rangle \in *$ and $\langle \alpha, x, a^{-\alpha}, y \rangle \in *$. Then $x \in {}^aN(\alpha)$ and $\langle \alpha, y, a, x \rangle \in *$. Since ${}^aN(\alpha) = N^a(\alpha)$ there is some $n_1 \in N(\alpha)$ such that $\langle \alpha, n_1, a, x \rangle \in *$. Then, by the cancellation law we get $y = n_1 \in N(\alpha)$.
($2 \Rightarrow 1$) Let $a, y \in G$ and $x \in {}^aN(\alpha)$. Then there exists $n \in N(\alpha)$ such that $\langle \alpha, a, n, x \rangle \in *$. By assumption $\langle \alpha, a, n, x \rangle \in *$ and $\langle \alpha, x, a^{-\alpha}, y \rangle \in *$ which implies that $y \in N(\alpha)$. As $\langle \alpha, x, a^{-\alpha}, y \rangle \in * \Rightarrow \langle \alpha, y, a, x \rangle \in *$. Since $y \in N(\alpha)$, $x \in N^a(\alpha)$. Therefore ${}^aN(\alpha) \subseteq N^a(\alpha)$. From the assumption $\langle \alpha, a, n, x \rangle \in *$ and $\langle \alpha, x, a^{-\alpha}, y \rangle \in *$ we get $x \in {}^aN(\alpha)$. Thus, $N^a(\alpha) \subseteq {}^aN(\alpha)$. Therefore $\langle N, A \rangle$ is normal soft subgroup of G.

□

Theorem 2.48. *A soft subgroup $\langle N, A\rangle$ of G is normal if and only if $\widehat{N}$ is a normal subgroup of $SE_A(G)$.*

Proof. The proof is similar to that of Theorem 2.20. □

Definition 2.49. *Consider the soft group $\langle G, *, A\rangle$. $\langle Z_A(G), A\rangle$ represents the center of G, which is a parameterized soft set over G defined by:*

$$Z_A(G)(\alpha) = \{a \in G : \langle \alpha, a, x, y\rangle \in * \quad \Leftrightarrow \quad \langle \alpha, x, a, y\rangle \in * \quad for\ all\ x, y \in G\}.$$

Theorem 2.50. *For any soft group $\langle G, *, A\rangle$ its center $\langle Z_A(G), A\rangle$ is a normal soft subgroup of G.*

Proof. We first show that $\langle Z_A(G), A\rangle$ is a soft subgroup of G. Let $\alpha \in A$ and $x, y \in G$. $\langle \alpha, e_\alpha, x, y\rangle \in *$. This implies that $y = x$ and we have $\langle \alpha, x, e_\alpha, x\rangle \in *$. Thus $e_\alpha \in Z_A(G)(\alpha)$. Now let $a, b \in Z_A(G)(\alpha)$ and $c \in G$ such that $\langle \alpha, a, b^{-\alpha}, c\rangle \in *$.
Claim: $c \in Z_A(G)(\alpha)$. We first show that $b^{-\alpha} \in Z_A(G)(\alpha)$. Let $x, y \in G$ such that $\langle \alpha, b^{-\alpha}, x, y\rangle \in *$. Then $\langle \alpha, x^{-\alpha}, b, y^{-\alpha}\rangle \in *$. Since $b \in Z_A(G)(\alpha)$, $\langle \alpha, b, x^{-\alpha}, y^{-\alpha}\rangle \in *$. It follows that $\langle \alpha, x, b^{-\alpha}, y\rangle \in *$. Similarly it can be shown that $\langle \alpha, b^{-\alpha}, x, y\rangle \in *$ and hence $b^{-\alpha} \in Z_A(G)(\alpha)$. Let $x, y, z \in G$ such that $\langle \alpha, c, x, y\rangle \in *$ and $\langle \alpha, b^{-\alpha}, x, z\rangle \in *$. Then using the associative property of $\langle *, A\rangle$ we get that $\langle \alpha, a, z, y\rangle \in *$. Since $a, b^{-\alpha} \in Z_A(G)(\alpha)$, $\langle \alpha, x, b^{-\alpha}, z\rangle \in *$ and $\langle \alpha, z, a, y\rangle \in *$. If $u \in G$ such that $\langle \alpha, b^{-\alpha}, a, u\rangle \in *$, then by the associativity of $*$ we get $\langle \alpha, x, u, y\rangle \in *$. Moreover, as $a \in Z_A(G)(\alpha)$ it holds that $\langle \alpha, a, b^{-\alpha}, u\rangle \in *$. Thus $u = c$ and hence $\langle \alpha, x, c, y\rangle \in *$. That is, the implication: $\langle \alpha, c, x, y\rangle \in * \Rightarrow \langle \alpha, x, c, y\rangle \in *$ holds $\forall x, y \in G$. By symmetry, the other side of the implication holds and then $c \in Z_A(G)(\alpha)$. Therefore $\langle Z_A(G), A\rangle$ is a soft subgroup of G. Next we show normality. Let $a \in Z_A(G)(\alpha)$ and $g, x, y \in G$ such that

$$\langle \alpha, g, a, x\rangle \in * \qquad (1)$$

and

$$\langle \alpha, x, g^{-\alpha}, y\rangle \in * \qquad (2)$$

Claim: $y \in Z_A(G)(\alpha)$. It is enough to show that $y = a$. As $a \in Z_A(G)(\alpha)$, it follows from (1) we get that

$$\langle \alpha, a, g, x\rangle \in * \qquad (3)$$

Then (2) and (3) together imply that $y = a \in Z_A(G)(\alpha)$ (using associative of $\langle *, A\rangle$). Therefore $\langle Z_A(G), A\rangle$ is a normal soft subgroup of G. $\square$

2.4 Soft Congruences

Definition 2.51. *A soft equivalence relation $\langle\theta, A\rangle$ on a soft group $\langle G, *, A\rangle$ is referred to as a soft congruence relation if, for each $\alpha \in A$ and any $a, b, c, d, x, y \in G$ with $\langle\alpha, a, c, x\rangle \in *$ and $\langle\alpha, b, d, y\rangle \in *$, $\langle a, b\rangle$, $\langle c, d\rangle \in \theta(\alpha)$ $\langle a, b\rangle$, $\langle c, d\rangle \in \theta(\alpha)$ imply $\langle x, y\rangle \in \theta(\alpha)$.*

Notation: $SCon_A(G)$ will represent the set of all soft congruence relations on G with the set of parameters A.

Let $\langle\theta, A\rangle$ be a soft congruence relation on G. Define the soft equivalence class $\theta_a : A \to P(G)$ of θ determined by $a \in G$ as follows:

$$\theta_a(\alpha) = \{x \in G : \langle a, x\rangle \in \theta(\alpha)\}.$$

Then we have the following properties.

Lemma 2.52. *Let $\langle\theta, A\rangle$ be a soft congruence relation on G and $a, b \in G$. Then,*

(1) $\langle\theta_a, A\rangle \overset{\smile}{=} \langle\theta_b, A\rangle$ if and only if $\langle a, b\rangle \in \theta(\alpha)$ for all $\alpha \in A$.

(2) Either $\langle\theta_a, A\rangle \overset{\smile}{=} \langle\theta_b, A\rangle$ or $\langle\theta_a, A\rangle$ and $\langle\theta_b, A\rangle$ are weakly disjoint.

Proof. (1) Suppose $\langle\theta_a, A\rangle \overset{\smile}{=} \langle\theta_b, A\rangle$. Then $\theta_a(\alpha) = \theta_b(\alpha)$ for all $\alpha \in A$. Since $e_\alpha \in \theta_a(\alpha)$ we have $\langle a, e_\alpha\rangle \in \theta(\alpha)$ and $e_\alpha \in \theta_b(\alpha)$ we have $\langle b, e_\alpha\rangle \in \theta(\alpha)$. As $\langle b, e_\alpha\rangle \in \theta(\alpha) \Rightarrow \langle e_\alpha, b\rangle \in \theta(\alpha)$. From $\langle a, e_\alpha\rangle \in \theta(\alpha)$ and $\langle e_\alpha, b\rangle \in \theta(\alpha)$, we get $\langle a, b\rangle \in \theta(\alpha)$ for all $\alpha \in A$. Conversely $\langle a, b\rangle \in \theta(\alpha)$ for all $\alpha \in A$. We need to show that $\langle\theta_a, A\rangle \overset{\smile}{=} \langle\theta_b, A\rangle$. Let $x \in \theta_a(\alpha)$. Then $\langle a, x\rangle \in \theta(\alpha)$. Since $\langle a, b\rangle \in \theta(\alpha)$, we get $\langle b, x\rangle \in \theta(\alpha)$ and hence $x \in \theta_b(\alpha)$. Therefore $\theta_a(\alpha) \subseteq \theta_b(\alpha)$. Similarly, it can be shown that $\theta_b(\alpha) \subseteq \theta_a(\alpha)$. Thus $\langle\theta_a, A\rangle \overset{\smile}{=} \langle\theta_b, A\rangle$.

(2) Suppose $\langle\theta_a, A\rangle$ and $\langle\theta_b, A\rangle$ are not weakly disjoint. This means that for all $\alpha \in A$, $\theta_a(\alpha)\cap\theta_b(\alpha) \neq \emptyset$. It follows that there exist at least one $x \in \theta_a(\alpha)\cap\theta_b(\alpha)$. This implies that $x \in \theta_a(\alpha)$ and $x \in \theta_b(\alpha)$. So $\langle a, x\rangle \in \theta(\alpha)$ and $\langle b, x\rangle \in \theta(\alpha)$ implying that $\langle a, b\rangle \in \theta(\alpha)$ for all $\alpha \in A$. Therefore, by (1) we get $\langle\theta_a, A\rangle \overset{\smile}{=} \langle\theta_b, A\rangle$.

$\square$

For a soft congruence $\langle \theta, A\rangle$ on G define a soft set $\langle \theta_o, A\rangle$ over G by

$$\theta_o(\alpha) = \{x \in G : \langle x, e_\alpha\rangle \in \theta(\alpha)\}.$$

Theorem 2.53. *$\langle \theta_o, A\rangle$ is a normal soft subgroup of a soft group $\langle G, *, A\rangle$.*

Proof. Clearly, $e_\alpha \in \theta_o(\alpha)$. Let $a, b \in \theta_0(\alpha)$. This implies that $\langle a, e_\alpha\rangle \in \theta(\alpha)$ and $\langle b, e_\alpha\rangle \in \theta(\alpha)$. As $\langle b, e_\alpha\rangle \in \theta(\alpha)$ which implies that $\langle b^{-\alpha}, e_\alpha\rangle \in \theta(\alpha)$. Let $x \in G$ such that $\langle \alpha, a, b^{-\alpha}, x\rangle \in *$. Since $\langle a, e_\alpha\rangle \in \theta(\alpha)$ and $\langle b^{-\alpha}, e_\alpha\rangle \in \theta(\alpha)$, we have $\langle x, e_\alpha\rangle \in \theta(\alpha)$ for all $\alpha \in A$. It follows that $x \in \theta_o(\alpha)$. Let $a, x, y \in G$ and $n \in \theta_o(\alpha)$. Suppose $\langle \alpha, a, n, x\rangle \in *$ and $\langle \alpha, x, a^{-\alpha}, y\rangle \in *$. As $n \in \theta_o(\alpha)$, then we have $\langle n, e_\alpha\rangle \in \theta(\alpha)$. Suppose $\langle x, a\rangle \in \theta(\alpha)$ and $\langle a^{-\alpha}, n\rangle \in \theta(\alpha)$. This implies that $\langle y, x\rangle \in \theta(\alpha)$. Since $\langle y, x\rangle \in \theta(\alpha)$ and $\langle x, e_\alpha\rangle \in \theta(\alpha)$, we have $\langle y, e_\alpha\rangle \in \theta(\alpha)$. It follows that $y \in \theta_o(\alpha)$. Therefore $\langle \theta_o, A\rangle$ is a normal soft subgroup of G.

□

Theorem 2.54. *Given a normal soft subgroup $\langle N, A\rangle$ of a soft group $\langle G, *, A\rangle$. Define a soft relation denoted by $\langle \theta^N, A\rangle$ over G by:*

$$\theta^N(\alpha) = \{\langle x, y\rangle \in G \times G : \quad \langle \alpha, y^{-\alpha}, x, z\rangle \in * \Rightarrow z \in N(\alpha)\}.$$

Then $\langle \theta^N, A\rangle$ is a soft congruence relation on G.

Proof. First we show that $\langle \theta^N, A\rangle$ a soft equivalence relation. Let $\alpha \in A$ and $x \in G$. Since $\langle \alpha, x^{-\alpha}, x, z\rangle \in *$, $z = e_\alpha \in N(\alpha)$. It follows that $\langle x, x\rangle \in \theta^N(\alpha)$. Therefore $\langle \theta^N, A\rangle$ is reflexive. Let $\alpha \in A$ and $x, y \in G$. Suppose $\langle x, y\rangle \in \theta^N(\alpha)$ such that $\langle \alpha, y^{-\alpha}, x, z\rangle \in *$, we have $z \in N(\alpha)$. Since $\langle \alpha, x^{-\alpha}, y, z^{-\alpha}\rangle \in *$ and $z^{-\alpha} \in N(\alpha)$, $\langle y, x\rangle \in \theta^N(\alpha)$. Therefore $\langle \theta^N, A\rangle$ is symmetric. Let $\alpha \in A$ and $x, y, z \in G$. Suppose $\langle x, y\rangle \in \theta^N(\alpha)$ and $\langle y, z\rangle \in \theta^N(\alpha)$. Which implies that $\langle \alpha, y^{-\alpha}, x, z_1\rangle \in *$, $\langle \alpha, z^{-\alpha}, y, z_2\rangle \in *$ for some $z_1, z_2 \in N(\alpha)$. Let $z_3 \in G$ such that $\langle \alpha, z_2, z_1, z_3\rangle \in *$. It follows that $z_3 \in N(\alpha)$. Since $\langle \alpha, z^{-\alpha}, x, z_3\rangle \in *$ and $z_3 \in N(\alpha)$, $\langle x, z\rangle \in \theta^N(\alpha)$. Therefore $\langle \theta^N, A\rangle$ is transitive. Thus, $\langle \theta^N, A\rangle$ a soft equivalence relation. Finally, let $\alpha \in A$ and $a, b, c, x, y \in G$ such that $\langle \alpha, a, c, x\rangle \in *$ and $\langle \alpha, b, d, y\rangle \in *$. Suppose $\langle a, b\rangle$, $\langle c, d\rangle \in \theta^N(\alpha)$. We need to prove that $\langle x, y\rangle \in \theta^N(\alpha)$. Let $z \in G$ with the property $\langle \alpha, y^{-\alpha}, x, z\rangle \in *$. As $\langle a, b\rangle \in \theta^N(\alpha)$ we have $\langle \alpha, b^{-\alpha}, a, z_1\rangle \in *$ for some $z_1 \in N(\alpha)$. Again as $\langle c, d\rangle \in \theta^N(\alpha)$, there is some $z_2 \in N(\alpha)$ such that $\langle \alpha, d^{-\alpha}, c, z_2\rangle \in *$. Let $z_3, z_4, z_5, w \in G$ such that $\langle \alpha, d^{-\alpha}, z_1, z_4\rangle \in *$, $\langle \alpha, z_4, d, z_5\rangle \in *$ and $\langle \alpha, z_5, z_2, w\rangle \in *$. Then, using the facts

$z_1, z_2 \in N(\alpha)$ and $\langle N, A\rangle$ is normal, one can easily check that $w \in N(\alpha)$. Moreover, by the associativity of $\langle *, A\rangle$, we have $z = w$ and hence $\langle x, y\rangle \in \theta^N(\alpha)$. Therefore $\langle \theta^N, A\rangle$ is a soft congruence relation on G. □

Theorem 2.55. *There is a lattice isomorphism between $SN_A(G)$ and $SCon_A(G)$.*

Proof. Let $g : SN_A(G) \longrightarrow SCon_A(G)$ and $h : SCon_A(G) \longrightarrow SN_A(G)$ be mapping defined by $g : \langle N, A\rangle \mapsto \langle \theta^N, A\rangle$ and $h : \langle \theta, A\rangle \mapsto \langle \theta_o, A\rangle$. We first show that g and h are mutually inverse to each other. That is, we show that $\langle [\theta^N]_o, A\rangle \breve{=} \langle N, A\rangle$ and $\langle \theta^{[\theta_o]}, A\rangle \breve{=} \langle \theta, A\rangle$ for all $\langle N, A\rangle \in SN_A(G)$ and $\langle \theta, A\rangle \in SCon_A(G)$.

(1) For each $\alpha \in A$,

$$\begin{aligned} [\theta^N]_o(\alpha) &= \{x \in G : \langle x, e_\alpha\rangle \in \theta^N(\alpha)\} \\ &= \{x \in G : \langle \alpha, e_\alpha^{-1}, x, z\rangle \in * \;\; for\ some\ z \in N(\alpha)\} \\ &= \{x \in G : x = z \in N(\alpha)\} = N(\alpha). \end{aligned}$$

Thus $\langle [\theta^N]_o, A\rangle \breve{=} \langle N, A\rangle$.

(2) For any $\alpha \in A$, $\langle x, y\rangle \in \theta^{[\theta_o]}(\alpha)$ if and only if $\langle \alpha, y^{-\alpha}, x, z\rangle *\quad for\ some\ z \in \theta_o(\alpha)$. Equivalently, $\langle x, y\rangle \in \theta(\alpha)$ showing that $\langle \theta^{[\theta_o]}, A\rangle \breve{=} \langle \theta, A\rangle$. Thus g and h are inverse to each other and hence both are one to one correspondences. It remains to show that both g and h are order homomorphisms. In other words, it is enough to show that $\langle N_1, A\rangle \breve{\subseteq} \langle N_2, A\rangle$ if and only if $\langle \theta^{N_1}, A\rangle \breve{\subseteq} \langle \theta^{N_2}, A\rangle$. Suppose that $N_1(\alpha) \subseteq N_2(\alpha)$ for all $\alpha \in A$. Let $\alpha \in A$ and $(x, y) \in \theta^{N_1}(\alpha)$. Then $\langle \alpha, x, y^{-\alpha}, z_1\rangle \in *$ for some $z_1 \in N_1(\alpha)$. That is, $z_1 \in N_2(\alpha)$ and $\langle \alpha, x, y^{-\alpha}, z_1\rangle *$. So $(x, y) \in \theta^{N_2}(\alpha)$. Since α is arbitrary in A, we get $\langle \theta^{N_1}, A\rangle \breve{\subseteq} \langle \theta^{N_2}, A\rangle$. Conversely, suppose $\theta^{N_1}(\alpha) \subseteq \theta^{N_2}(\alpha)$ for all $\alpha \in A$. Let $\alpha \in A$ and $x \in N_1(\alpha)$. Then it is clear that $\langle x, e_\alpha\rangle \in \theta^{N_1}(\alpha) \subseteq \theta^{N_2}(\alpha)$ and hence $\langle x, e_\alpha\rangle \in \theta^{N_2}(\alpha)$. Then, there exists some $z \in N_2(\alpha)$ such that $\langle \alpha, e_\alpha, x, z\rangle \in *$. Whence $x = z \in N_2(\alpha)$ and so $\langle N_1, A\rangle \breve{\subseteq} \langle N_2(\alpha), A\rangle$. Hence proved. □

2.5 Direct products

Lemma 2.56. *Let $\langle G_1, *_1, A\rangle, ...\langle G_n, *_n, A\rangle$ be soft groups with the same set of parameters A, and G be the set*

$$G = \prod_{i=1}^{n} G_i = \{\langle a_1, ..., a_n\rangle : a_i \in G_i \ \forall i = 1, 2, ..., n\}.$$

Define a soft binary operation $\langle *, A\rangle$ on G by: $* \subseteq A \times G \times G \times G$ given by $\langle \alpha, \bar{a}, \bar{b}, \bar{c}\rangle \in *$ if and only if $\langle \alpha, a_i, b_i, c_i\rangle \in *_i$ for all $1 \leqslant i \leqslant n$. Where $\bar{a} = \langle a_1, ..., a_n\rangle$, $\bar{b} = \langle b_1, ..., b_n\rangle$ and $\bar{c} = \langle c_1, ..., c_n\rangle$. Then $\langle G, *, A\rangle$ is a soft group.

Definition 2.57. *Given soft groups $\langle G_1, *_1, A\rangle, ..., \langle G_n, *_n, A\rangle$ with a fixed set A of parameters. The group $\langle G, *, A\rangle$ obtained in Lemma 2.56 is called the direct product of $\langle G_1, *_1, A\rangle, ..., \langle G_n, *_n, A\rangle$.*

In general, for any indexed family $\{\langle G_i, *_i, A\rangle\}_{i\in I}$ of soft groups, the underlying set for their direct product is given by:

$$\prod_{i\in I} G_i = \{a : I \longrightarrow \bigcup_{i\in I}(G_i) : a(i) \in G_i \ \forall i \in I\}.$$

Define a soft binary operation $\langle *, A\rangle$ on $\prod_{i\in I} G_i$ as follows $\langle \alpha, a, b, c\rangle \in *$ if and only if $\langle \alpha, a(i), b(i), c(i)\rangle \in *_i$ for all $i \in I$. Then $\langle \prod_{i\in I} G_i, *, A\rangle$ is a soft group.

Theorem 2.58. *Let $\langle G_1, *_1, A\rangle$ and $\langle G_2, *_2, A\rangle$ be soft groups. Let $\langle H_1, A\rangle$ and $\langle H_2, A\rangle$ be soft subgroups of G_1 and G_2 respectively. Then their product $\langle H_1 \times H_2, A\rangle$ is a soft subgroup of $G_1 \times G_2$. Moreover, any soft subgroup of $G_1 \times G_2$ is of the form $\langle H_1 \times H_2, A\rangle$ for some soft subgroups $\langle H_1, A\rangle$ and $\langle H_2, A\rangle$ of G_1 and G_2 respectively.*

Proof. Suppose that $\langle H_1, A\rangle$ is soft a subgroup of G_1 and $\langle H_2, A\rangle$ a soft subgroup of G_2. For $\alpha \in A$, we have $e^1_\alpha \in H_1(\alpha)$ and $e^2_\alpha \in H_2(\alpha)$. Which implies that $\langle e^1_\alpha, e^2_\alpha\rangle \in H_1(\alpha) \times H_2(\alpha)$. Let $\bar{a} = \langle a_1, a_2\rangle$, $\bar{b} = \langle b_1, b_2\rangle \in H_1(\alpha) \times H_2(\alpha)$ and $\bar{c} = \langle c_1, c_2\rangle \in G_1 \times G_2$ such that $\langle \alpha, \bar{a}, \bar{b}^{-\alpha}, \bar{c}\rangle \in *$. Then $\langle \alpha, a_1, b_1^{-\alpha}, c_1\rangle \in *_1$ and $\langle \alpha, a_2, b_2^{-\alpha}, c_2\rangle \in *_2$. Since $\langle H_1, A\rangle$ is a soft subgroup of G_1 and $\langle H_2, A\rangle$ is a soft subgroup of G_2, we get that $c_1 \in H_1(\alpha)$ and $c_2 \in H_2(\alpha)$. So that $\bar{c} = \langle c_1, c_2\rangle \in H_1(\alpha) \times H_2(\alpha)$. That is $\langle H_1 \times H_2, A\rangle$ is a soft subgroup of $G_1 \times G_2$. Conversely suppose that $\langle H, A\rangle$ is a soft subgroup of $G_1 \times G_2$. Define soft subsets $\langle H_1, A\rangle$ and $\langle H_2, A\rangle$ of G_1 and G_2 respectively by:

$$H_1(\alpha) = \{a \in G_1 : \langle a, e^2_\alpha\rangle \in H(\alpha)\}$$

and

$$H_2(\alpha) = \{a \in G_2 : \langle e^1_\alpha, a\rangle \in H(\alpha)\}.$$

Then we show that $\langle H_1, A\rangle$ is a soft subgroup of G_1. For each $\alpha \in A$, since $\langle e^1_\alpha, e^2_\alpha\rangle \in H(\alpha)$ we have $e^1_\alpha \in H_1(\alpha)$. Let $a, b \in H_1(\alpha)$. Then $\langle a, e^2_\alpha\rangle$, $\langle b, e^2_\alpha\rangle \in H(\alpha)$. Let $c \in G_1$ such that $\langle \alpha, a, b^{-\alpha}, c\rangle \in *_1$. Then $\langle \alpha, \langle a, e^2_\alpha\rangle, \langle b^{-\alpha}, e^2_\alpha\rangle, \langle c, e^2_\alpha\rangle\rangle \in *$ and since $\langle H, A\rangle$ is a soft subgroup of $G_1 \times G_2$, we get that $\langle c, e^2_\alpha\rangle \in H(\alpha)$, which implies that $c \in H_1(\alpha)$. Thus $\langle H_1, A\rangle$ is a soft subgroup of G_1. Similarly, it can be shown that $\langle H_2, A\rangle$ is a soft subgroup of G_2. Moreover, one can easily check that $\langle H_1 \times H_2, A\rangle \,\breve{=}\, \langle H, A\rangle$. Hence proved.

□

Chapter 3

Soft homomorphisms on Soft groups

One of the most significant steps for the theory of soft sets was to define relations and mappings on soft sets. The notions of soft set relation and function were introduced by Babitha and Sunil, who also covered a number of related ideas, including ordering on soft sets, partitioning of soft sets, and equivalency soft set relations. Babitha and Sunil continued their work on soft set relations and ordering by presenting the ideas of transitive closure of a soft set relation and anti-symmetric relation. Majumdar and Samanta investigated the concept of soft mappings and the representations of soft and crisp sets under soft mappings. While studying the characteristics of soft images and soft inverse images, Kharal and Ahmad also established the idea of mappings on the class of soft sets. A new concept of soft mappings and an investigation into its characteristics were recently proposed by Addis et al. Additional research is available in [2, 10, 21, 28].

In this chapter, soft mapping between two soft groups called soft homomorphisms are presented and discussed. We begin by studying soft homomorphisms, stating their definition and proving some basic results in the first Section . Given a soft mapping $\langle f, A\rangle$ from from G to G', we obtain an ordinary map $\breve{f}$ from the set $SE_A(G)$ of soft elements of G to the set $SE_A(G')$ of soft elements of G' and show that $\langle f, A\rangle$ is a soft homomorphism if and only if $\breve{f}$ is an ordinary group homomorphism . Section 2 is devoted to the study those class of soft isomorphisms by introducing the notions of the kernel, image and inverse image of soft homomorphisms. Moreover, we state and prove several soft isomorphism theorems on soft groups. In the last section we study the soft automorphisms of soft groups and particularly those soft inner automorphisms. We establish an embedding of the group of soft automorphisms of a soft group G into the group of automorphisms of $SE_A(G)$, where $SE_A(G)$ is the collection of all soft elements of G with a set A of parameters. Finally, it is shown that for every soft group G, the soft group of its inner automorphisms is soft isomorphic with the quotient of G by its center $Z_A(G)$.

3.1 Soft Homomorphisms

Definition 3.1. *Let $\langle G, *, A\rangle$ and $\langle G', \Delta, A\rangle$ be soft groups. A soft mapping $\langle f, A\rangle$ from G to G' is called a soft homomorphism, if for each $\alpha \in A$, $a, b, c \in G$ and $x, y, z \in G'$, $\langle \alpha, a, x\rangle \in f$, $\langle \alpha, b, y\rangle \in f$, $\langle \alpha, c, z\rangle \in f$ and $\langle \alpha, a, b, c\rangle \in *$ imply $\langle \alpha, x, y, z\rangle \in \Delta$.*

Example 3.2. *Define soft binary operations $\langle *, \mathbb{N}\rangle$ and $\langle \Delta, \mathbb{N}\rangle$ on $\mathbb{R}$ and $\mathbb{R}^+$ respectively as follows:*

$$\langle \alpha, a, b, c\rangle \in * \Leftrightarrow c = \alpha + a + b \quad \text{and} \quad \langle \alpha, a, b, c\rangle \in \Delta \Leftrightarrow c = (\alpha + 1)^{\alpha} ab.$$

$\langle \mathbb{R}, *, \mathbb{N}\rangle$ and $\langle \mathbb{R}^+, \Delta, \mathbb{N}\rangle$ are soft groups. Moreover, define

$$f = \{(\alpha, x, y) : y = (\alpha + 1)^x\}.$$

Then $\langle f, \mathbb{N}\rangle$ is a soft homomorphism from $\mathbb{R}$ to $\mathbb{R}^+$.

Example 3.3. With the notation of Example 3.2, define $f' = \{(\alpha, x, y) : y = \log_{\alpha+1} x\}$. Consequently, $\langle f', \mathbb{N}\rangle$ is a soft homomorphism from $\langle \mathbb{R}^+, \Delta, \mathbb{N}\rangle$ to $\langle \mathbb{R}, *, \mathbb{N}\rangle$.

Proposition 3.4. *Let $\langle f, A\rangle$ be a soft homomorphism from $\langle G, *, A\rangle$ to $\langle G', \triangle, A\rangle$. Then,*

(1) $\langle \alpha, e_\alpha, e'_\alpha\rangle \in f$, for all $\alpha \in A$ where e_α and e'_α are identity elements of G and G' respectively;

(2) $\langle \alpha, a^{-\alpha}, y^{-\alpha}\rangle \in f$ whenever $\langle \alpha, a, y\rangle \in f$ for all $\alpha \in A, a \in G$ and $y \in G'$.

Proof. (1) Let $x \in G'$ such that $\langle \alpha, e_\alpha, x\rangle \in f$. Since $\langle \alpha, x, e'_\alpha, x\rangle \in \Delta$ and $\langle \alpha, x, x, x\rangle \in \Delta$, by cancellation law it holds that $x = e'_\alpha$. Therefore $\langle \alpha, e_\alpha, e'_\alpha\rangle \in f$.

(2) Let $\alpha \in A$, $a \in G$ and $x, y \in G'$ such that $\langle \alpha, a, y\rangle \in f$ and $\langle \alpha, a^{-\alpha}, x\rangle \in f$. Since $\langle f, A\rangle$ is a soft homomorphism, it follows from (1) and the condition $\langle \alpha, a, a^{-\alpha}, e_\alpha\rangle \in *$ that

$\langle \alpha, y, x, e'_\alpha \rangle \in \Delta$. Again from the fact $\langle \alpha, y, y^{-\alpha}, e'_\alpha \rangle \in \Delta$ and cancellation law we get $x = y^{-\alpha}$. Therefore $\langle \alpha, a^{-\alpha}, y^{-\alpha} \rangle \in f$.

□

Using the idea of the composition of soft mappings given in [2], in following proposition the composition of two soft homomorphisms is a soft homomorphism.

Proposition 3.5. *Consider the soft homomorphisms $\langle f, A \rangle$ from G to G' and $\langle g, A \rangle$ from G' to G''. Then, $\langle g \circ f, A \rangle$ is a soft homomorphism.*

Proof. The proof is straight forward. □

Definition 3.6. *A soft homomorphism $\langle f, A \rangle$ from G to G' is said to be:*

(1) a soft monomorphism if it is injective.

(2) a soft epimorphism if it is surjective.

(3) a soft isomorphism if it is bijective.

Note: Given soft groups $\langle G, *, A \rangle$ and $\langle G', \Delta, A \rangle$. If there is a soft isomorphism from G to G', we say that G and G' are soft isomorphic and write $G \breve{\cong} G'$ symbolically. Note also that $\breve{\cong}$ forms an equivalence relation on the class of soft groups.

Recall from Theorem 2.15 that, if $\langle G, *, A \rangle$ is a soft group, then the set $SE_A(G)$ of all soft elements of G is an ordinary group together with the induced binary operation $\bar{*}$, that can be used as a model to represent soft groups. With the idea of extending soft homomorphisms to classical group homomorphism, we define the following.

Definition 3.7. *Let $\langle f, A \rangle$ be a soft mapping from $\langle G, *, A \rangle$ to $\langle G', \Delta, A \rangle$. Define a mapping $\breve{f} : SE_A(G) \to SE_A(G')$ as follows: for each $\breve{a} \in SE_A(G)$ and $\alpha \in A$:*

$$\breve{f}(\breve{a})(\alpha) = \{b\} \text{ if and only if } \langle \alpha, \breve{a}(\alpha), b \rangle \in f.$$

Theorem 3.8. *A soft mapping $\langle f, A \rangle$ from G to G' is a soft homomorphism if and only if $\breve{f}$ is a homomorphism.*

Proof. Suppose that $\langle f, A\rangle$ is a soft homomorphism. Let $\breve{a}, \breve{b} \in SE_A(G)$ and $\alpha \in A$ be arbitrary. Assume that $\breve{f}(\breve{a}\bar{*}\breve{b})(\alpha) = \{z\}$. Then, $\langle \alpha, (\breve{a}\bar{*}\breve{b})(\alpha), z\rangle \in f$. If $(\breve{a}\bar{*}\breve{b})(\alpha) = \{c\}$, then $\langle \alpha, \breve{a}(\alpha), \breve{b}(\alpha), c\rangle \in *$ and $\langle \alpha, c, z\rangle \in f$. On the other hand if $\langle \alpha, \breve{a}(\alpha), x\rangle \in f$ and $\langle \alpha, \breve{b}(\alpha), y\rangle \in f$, then as $\langle f, A\rangle$ is a soft homomorphism, we get $\langle \alpha, x, y, z\rangle \in \Delta$ and hence $\left(\breve{f}(\breve{a})\bar{\Delta}\breve{f}(\breve{b})\right)(\alpha) = \{z\}$. Therefore, $\breve{f}$ is a group homomorphism. Conversely, suppose that $\breve{f}$ is a group homomorphism. Let $\alpha \in A$, $a, b, c \in G$ and $x, y, z \in G'$ such that $\langle \alpha, a, x\rangle \in f$, $\langle \alpha, b, y\rangle \in f$, $\langle \alpha, c, z\rangle \in f$ and $\langle \alpha, a, b, c\rangle \in *$. Let $u \in G'$ such that $\langle \alpha, x, y, u\rangle \in \Delta$. We claim to show that $u = z$. Define soft elements $\breve{a}, \breve{b}, \breve{c}$ in G and $\breve{x}, \breve{y}, \breve{z}$ in G' as follows: for each $\lambda \in A$,

$$\breve{a}(\lambda) = \begin{cases} \{a\} & if \;\; \lambda = \alpha \\ \{e_\lambda\} & if \;\; \lambda \neq \alpha \end{cases}$$

$$\breve{b}(\lambda) = \begin{cases} \{b\} & if \;\; \lambda = \alpha \\ \{e_\lambda\} & if \;\; \lambda \neq \alpha \end{cases}$$

$$\breve{c}(\lambda) = \begin{cases} \{c\} & if \;\; \lambda = \alpha \\ \{e_\lambda\} & if \;\; \lambda \neq \alpha \end{cases}$$

Then, one can easily show that for each $\lambda \in A$:

$$\breve{f}(\breve{a}\bar{*}\breve{b})(\lambda) = \begin{cases} \{z\} & if \;\; \lambda = \alpha \\ \{e'_\lambda\} & if \;\; \lambda \neq \alpha \end{cases} \quad \text{and}$$

$$\left(\breve{f}(\breve{a})\bar{\Delta}\breve{f}(\breve{b})\right)(\lambda) = \begin{cases} \{u\} & if \;\; \lambda = \alpha \\ \{e'_\lambda\} & if \;\; \lambda \neq \alpha \end{cases}$$

Since $\breve{f}$ is a homomorphism, it should be the case that $\breve{f}(\breve{a}\bar{*}\breve{b})(\lambda) = \left(\breve{f}(\breve{a})\bar{\Delta}\breve{f}(\breve{b})\right)(\lambda)$ for all $\lambda \in A$. In particular, it works for $\lambda = \alpha$ and hence which would give $z = u$. Therefore, $\langle f, A\rangle$ is a soft homomorphism. □

Theorem 3.9. *A soft homomorphism $\langle f, A\rangle$ from G to G' is a soft isomorphism if and only if $\breve{f}$ is a group isomorphism.*

Proof. Suppose that $\langle f, A\rangle$ is a soft isomorphism. We first show that $\breve{f}$ is injective. Let

$\breve{a}, \breve{b} \in SE_A(G)$ such that $\breve{f}(\breve{a}) = \breve{f}(\breve{b})$. Then $\breve{f}(\breve{a})(\lambda) = \breve{f}(\breve{b})(\lambda)$ for all $\lambda \in A$. That is, for each $\lambda \in A$ and any $x \in G'$ we have:

$$\langle \lambda, \breve{a}(\lambda), x \rangle \in f \text{ if and only if } \langle \lambda, \breve{b}(\lambda), x \rangle \in f.$$

Since $\langle f, A \rangle$ is injective, it must be true that $\breve{a}(\lambda) = \breve{b}(\lambda)$ for all $\lambda \in A$. Thus, $\breve{a} = \breve{b}$ and hence $\breve{f}$ is injective. Next we show that $\breve{f}$ is surjective. Let $\breve{y}$ be any soft element in G'. Since $\langle f, A \rangle$ is a soft isomorphism, for each $\alpha \in A$ there is a unique element say $x_\alpha \in G$ such that $\langle \alpha, x_\alpha, \breve{y}(\alpha) \rangle \in f$ for all $\alpha \in A$. Now define a soft element $\breve{x}$ over G by $\breve{x}(\alpha) = \{x_\alpha\}$ for all $\alpha \in A$. Then we must have $\breve{f}(\breve{x}) = \breve{y}$. Thus, $\breve{f}$ is surjective and hence an isomorphism. The converse can be proved using similar procedure. □

Definition 3.10. *Suppose $\langle f, A \rangle$ is a soft homomorphism from $\langle G, *, A \rangle$ to $\langle G', \Delta, A \rangle$.*

(1) If $\langle H, A \rangle$ is a soft subgroup of G then the image of $\langle H, A \rangle$ under f is the soft set $\langle f(H), A \rangle$ over G' defined as follows:

$$f(H)(\alpha) = \{y \in G' : \langle \alpha, x, y \rangle \in f \ \ for \ some \ x \in H(\alpha)\}.$$

(2) If $\langle H', A \rangle$ is a soft subgroup of G', then the inverse image of $\langle H', A \rangle$ under f is the soft set $\langle f^{-1}(H'), A \rangle$ over G defined as follows:

$$f^{-1}(H')(\alpha) = \{x \in G : \langle \alpha, x, y \rangle \in f \ \ for \ some \ y \in H'(\alpha)\}.$$

Lemma 3.11. *Suppose $\langle f, A \rangle$ is a soft homomorphism from $\langle G, *, A \rangle$ to $\langle G', \Delta, A \rangle$. If $\langle H, A \rangle$ is a soft subgroup of G then $\langle f(H), A \rangle$ is a soft subgroup of G'.*

Proof. Let $\alpha \in A$. Since $e_\alpha \in H(\alpha)$ and $\langle \alpha, e_\alpha, e'_\alpha \rangle \in f$, we have $e'_\alpha \in f(H)(\alpha)$.
Let $x, y \in f(H)(\alpha)$ and $z \in G'$ such that $\langle \alpha, x, y^{-\alpha}, z \rangle \in \Delta$. We need to prove that $z \in f(H)(\alpha)$. As $x, y \in f(H)(\alpha)$ there exist $a, b \in H(\alpha)$ such that $\langle \alpha, a, x \rangle \in f$ and $\langle \alpha, b, y \rangle \in f$. Since $\langle \alpha, b, y \rangle \in f$ we have $\langle \alpha, b^{-\alpha}, y^{-\alpha} \rangle \in f$. Let $c \in G$ such that $\langle \alpha, a, b^{-\alpha}, c \rangle \in *$. Since $\langle f, A \rangle$ is a soft mapping there exist $w \in G'$ such that $\langle \alpha, c, w \rangle \in f$. As $\langle f, A \rangle$ is a soft homomorphism it follows that $w = z$. Since $\langle H, A \rangle$ is a soft subgroup of G, we have $c \in H(\alpha)$. Therefore $z \in f(H)(\alpha)$. □

Lemma 3.12. *Let $\langle f, A\rangle$ be a soft homomorphism from $\langle G, *, A\rangle$ to $\langle G', \Delta, A\rangle$. If $\langle H', A\rangle$ is a soft subgroup of G' then $\langle f^{-1}(H'), A\rangle$ is a soft subgroup of G.*

Proof. Let $\alpha \in A$. Since $e'_\alpha \in H'(\alpha)$ and $\langle \alpha, e_\alpha, e'_\alpha\rangle \in f$, we get $e_\alpha \in f^{-1}(H')(\alpha)$.
Let $a, b \in f^{-1}(H')(\alpha)$ and $c \in G$ such that $\langle \alpha, a, b^{-\alpha}, c\rangle \in *$. Since $\langle f, A\rangle$ is a soft mapping there exists $z \in G'$ such that $\langle \alpha, c, z\rangle \in f$. We need to prove that $c \in f^{-1}(H')(\alpha)$. As $a, b \in f^{-1}(H')(\alpha)$ there exist $x, y \in H'(\alpha)$ such that $\langle \alpha, a, x\rangle \in f$ and $\langle \alpha, b, y\rangle \in f$. As $\langle \alpha, b, y\rangle \in f$ it holds that $\langle \alpha, b^{-\alpha}, y^{-\alpha}\rangle \in f$. Since $\langle f, A\rangle$ is a soft homomorphism from G to G', $\langle \alpha, x, y^{-\alpha}, z\rangle \in \Delta$. Since $\langle H', A\rangle$ is a soft subgroup of G' we get $z \in H'(\alpha)$. Therefore $c \in f^{-1}(H')(\alpha)$. Hence $\langle f^{-1}(H'), A\rangle$ is a soft subgroup of G. □

Theorem 3.13. *Suppose $\langle f, A\rangle$ is a soft homomorphism from $\langle G, *, A\rangle$ to $\langle G', \Delta, A\rangle$.*

(1) If $\langle N, A\rangle$ is a normal soft subgroup of G and $\langle f, A\rangle$ is surjective then $\langle f(N), A\rangle$ is a normal soft subgroup of G'.

(2) If $\langle N', A\rangle$ is a normal soft subgroup of G' then $\langle f^{-1}(N'), A\rangle$ is a normal soft subgroup of G.

Proof. (1) By Lemma 3.11 $\langle f(N), A\rangle$ is a soft subgroup of G'. Let $a \in f(N)(\alpha)$. Then $\langle \alpha, x, a\rangle \in f$ for some $x \in N(\alpha)$. Let $b \in G'$. Since $\langle f, A\rangle$ is surjective, there exists $y \in G$ such that $\langle \alpha, y, b\rangle \in f$. Let $z \in G$ such that $\langle \alpha, y, x, z\rangle \in *$. Then there exists $c \in G'$ such that $\langle \alpha, z, c\rangle \in f$ because $\langle f, A\rangle$ is a soft homomorphism. Moreover, $\langle \alpha, b, a, c\rangle \in \Delta$. Again let $d \in G$ such that $\langle \alpha, z, y^{-\alpha}, d\rangle \in *$. Since $\langle f, A\rangle$ is a soft mapping, there exists $d' \in G'$ such that $\langle \alpha, d, d'\rangle \in f$. Since $\langle f, A\rangle$ is a soft homomorphism, $\langle \alpha, c, b^{-\alpha}, d'\rangle \in \Delta$. As $\langle N, A\rangle$ is a normal soft subgroup of G, $d \in N(\alpha)$. This implies that $d' \in f(N)(\alpha)$. Therefore $\langle f(N), A\rangle$ is a normal soft subgroup of G'.

(2) According to Lemma 3.12, $\langle f^{-1}(N'), A\rangle$ is a soft subgroup of G. Let $a \in f^{-1}(N')(\alpha)$. Then $\langle \alpha, a, x\rangle \in f$ for some $x \in N'(\alpha)$. Let $b, c \in G$ and $x_1, z_1 \in G'$ such that $\langle \alpha, b, x_1\rangle \in f$ and $\langle \alpha, c, z_1\rangle \in f$. Since $\langle f, A\rangle$ is a soft homomorphism from G to G' and $\langle \alpha, b, a, c\rangle \in *$, we have $\langle \alpha, x_1, x, z_1\rangle \in \Delta$. Let $d \in G$ and $z_2 \in G'$ such that $\langle \alpha, d, z_2\rangle \in f$. Since $\langle f, A\rangle$ is a soft homomorphism from G to G' and $\langle \alpha, c, b^{-\alpha}, d\rangle \in *$, $\langle \alpha, z_1, x_1^{-\alpha}, z_2\rangle \in \Delta$. It follows that $d \in f^{-1}(N')(\alpha)$. Therefore $\langle f^{-1}(N'), A\rangle$ is a normal soft subgroup of G'.

□

Definition 3.14. *Suppose $\langle f, A\rangle$ is a soft homomorphism from G to G'. The kernel of $\langle f, A\rangle$ is the soft set $\langle K_f, A\rangle$ over G which is defined as follows :*

$$K_f(\alpha) = \{x \in G : \langle \alpha, x, e'_\alpha\rangle \in f\}$$

for all $\alpha \in A$.

Example 3.15. *Let $\langle G, *, A\rangle$ and $\langle G', \triangle, A\rangle$ be soft groups.*

(1) Let $\langle f, A\rangle$ be a soft mapping from G to G' defined by:

$$f = \{(\alpha, x, e'_\alpha) : \alpha \in A, x \in G\}.$$

Then $\langle f, A\rangle$ is a soft homomorphism and $\langle K_f, A\rangle$ is the absolute soft set over G.

(2) Let $\langle f, A\rangle$ be a soft mapping from G to G defined by:

$$f = \{(\alpha, x, x) : \ \alpha \in A, x \in G\}.$$

Then $\langle f, A\rangle$ is a soft homomorphism and $\langle K_f, A\rangle$ is the trivial soft subgroup of G.

Lemma 3.16. *For any soft homomorphism $\langle f, A\rangle$ from $\langle G, *, A\rangle$ to $\langle G', \Delta, A\rangle$, the kernel $\langle K_f, A\rangle$ is a normal soft subgroup of G.*

Proof. For each $\alpha \in A$, we have

$$K_f(\alpha) = \{x \in G : \langle \alpha, x, e'_\alpha\rangle \in f\} = f^{-1}(\breve{e}')(\alpha).$$

Therefore the proof follows directly from Theorem 3.13. □

Lemma 3.17. *A soft homomorphism $\langle f, A\rangle$ from $\langle G, *, A\rangle$ to $\langle G', \Delta, A\rangle$ is a soft monomorphism if and only if $\langle K_f, A\rangle$ is the trivial soft subgroup of G.*

Proof. Suppose $\langle f, A\rangle$ is a soft monomorphism. We need to show that $K_f(\alpha) = \{e_\alpha\}$ for all $\alpha \in A$. Let $x \in K_f(\alpha)$. This implies that $\langle \alpha, x, e'_\alpha\rangle \in f$ for all $\alpha \in A$. Since $\langle \alpha, e_\alpha, e'_\alpha\rangle \in f$ for all $\alpha \in A$ we have $x = e_\alpha$. Therefore $K_f(\alpha) = \{e_\alpha\}$. Conversely, suppose that $\langle \alpha, x_1, y\rangle \in f$ and $\langle \alpha, x_2, y\rangle \in f$. Let $z \in G$ such that $\langle \alpha, x_1, x_2^{-\alpha}, z\rangle \in *$. Then $\langle \alpha, z, e'_\alpha\rangle \in f$. It follows that

$z \in K_f(\alpha)$. So $z = e_\alpha$. Therefore $\langle \alpha, x_1, x_2^{-\alpha}, e_\alpha \rangle \in *$. This implies that $\langle \alpha, e_\alpha, x_2, x_1 \rangle \in *$. Using the fact $\langle \alpha, e_\alpha, x_1, x_1 \rangle \in *$ and the cancellation law we get $x_1 = x_2$.

□

Proposition 3.18. *Let $\langle f, A \rangle$ and $\langle g, A \rangle$ be soft homomorphisms from $\langle G, *, A \rangle$ to $\langle G', \Delta, A \rangle$. Define a soft set $\langle H, A \rangle$ over G as follows for each $\alpha \in A$:*

$$H(\alpha) = \{a \in G : \langle \alpha, a, x \rangle \in f \Leftrightarrow \langle \alpha, a, x \rangle \in g \quad for\ some\ x \in G'\}.$$

Then $\langle H, A \rangle$ is a soft subgroup of G.

Proof. Since $\langle \alpha, e_\alpha, e'_\alpha \rangle \in f$ and $\langle \alpha, e_\alpha, e'_\alpha \rangle \in g$, we have $e_\alpha \in H(\alpha)$. Let $a, b \in H(\alpha)$ and $x \in G$ such that $\langle \alpha, a, b^{-\alpha}, x \rangle \in *$. Then by definition there exist $y_1, y_2 \in G'$ such that

$$\langle \alpha, a, y_1 \rangle \in f \Leftrightarrow \langle \alpha, a, y_1 \rangle \in g$$

and

$$\langle \alpha, b, y_2 \rangle \in f \Leftrightarrow \langle \alpha, b, y_2 \rangle \in g$$

This implies that $\langle \alpha, b^{-\alpha}, y_2^{-\alpha} \rangle \in f \Leftrightarrow \langle \alpha, b^{-\alpha}, y_2^{-\alpha} \rangle \in g$. Let $x \in G$ again $\langle f, A \rangle$ is a soft mapping then there exists $y_3 \in G'$ such that $\langle \alpha, x, y_3 \rangle \in f$. Since $\langle f, A \rangle$ is a soft homomorphism, $\langle \alpha, y_1, y_2^{-\alpha}, y_3 \rangle \in \Delta$. Again since $\langle g, A \rangle$ is a soft homomorphism and $\langle \alpha, x, y_3 \rangle \in g$, we have $x \in H(\alpha)$. Therefore $\langle H, A \rangle$ is a soft subgroup of G. □

The following theorem establishes a relationship between the soft kernel of $\langle f, A \rangle$ and the kernel of $\breve{f}$.

Theorem 3.19. *For any soft homomorphism $\langle f, A \rangle$ from G to G' we have*

$$ker(\breve{f}) = \widehat{K_f}.$$

Proof. We know that

$$\widehat{K_f} = \{\breve{a} \in SE_A(G) : \breve{a}(\alpha) \subseteq K_f(\alpha) \text{ for all } \alpha \in A\} \text{ and}$$

$$ker(\breve{f}) = \{\breve{a} \in SE_A(G) : \breve{f}(\breve{a})(\lambda) = \{e_\lambda\} \text{ for all } \lambda \in A\}.$$

Now we have the following

$$\begin{aligned} \breve{a} \in ker(\breve{f}) &\Leftrightarrow \breve{f}(\breve{a})(\lambda) = \{e_\lambda\} \text{ for all } \lambda \in A \\ &\Leftrightarrow \langle \alpha, \breve{a}(\lambda), e_\lambda \rangle \in f \\ &\Leftrightarrow \breve{a}(\lambda) \subseteq K_f(\lambda) \text{ for all } \lambda \in A \\ &\Leftrightarrow \breve{a} \in \widehat{K_f}. \end{aligned}$$

Therefore $ker(\breve{f}) = \widehat{K_f}$. □

3.2 Soft Isomorphism Theorems

In this Section, we will see a method for checking whether two soft groups defined in different ways are structurally the same or not.

Theorem 3.20. *Let $\langle H, A \rangle$ be a normal soft subgroup of a soft group $\langle G, *, A \rangle$. Put*

$$G/H = \{{}^aH : a \in G\}$$

where each ${}^aH$ is a left coset of $\langle H, A \rangle$. Define a soft coset multiplication $\circledast$ on G/H by:

$$\langle \alpha, {}^aH, {}^bH, {}^cH \rangle \in \circledast \Leftrightarrow \langle \alpha, a, b, x \rangle \in *$$

for some $x \in G$ with ${}^cH = {}^xH$. Then, $\langle G/H, \circledast, A \rangle$ is a soft group.

Proof. First we shall prove that $\circledast$ is well defined. Let $n, m, d \in G$ and $\alpha \in A$ such that ${}^aH = {}^dH$ and ${}^bH = {}^nH$. Suppose $\langle \alpha, {}^aH, {}^bH, {}^cH \rangle \in \circledast$ and $\langle \alpha, {}^dH, {}^nH, {}^mH \rangle \in \circledast$. This implies that $\langle \alpha, a, b, z_1 \rangle \in *$ and $\langle \alpha, d, n, z_2 \rangle \in *$ for some $z_1, z_2 \in G$ with ${}^cH = {}^{z_1}H$ and ${}^mH = {}^{z_2}H$. Let $k_1 \in G$ such that $\langle \alpha, c^{-\alpha}, z_1, k_1 \rangle \in *$. Since ${}^cH = {}^{z_1}H$, $k_1 \in H(\alpha)$. Let $k_2 \in G$ such that $\langle \alpha, m^{-\alpha}, z_2, k_2 \rangle \in *$. Since ${}^mH = {}^{z_2}H$, $k_2 \in H(\alpha)$. Let $x_1, x_2 \in G$ such that $\langle \alpha, a^{-\alpha}, d, x_1 \rangle \in *$ and $\langle \alpha, b^{-\alpha}, n, x_2 \rangle \in *$. It follows that $x_1, x_2 \in H(\alpha)$. Let $k_3, y \in G$ such that $\langle \alpha, n^{-\alpha}, x_1, y \rangle \in *$ and $\langle \alpha, y, n, k_3 \rangle \in *$. Since $\langle H, A \rangle$ is a normal soft subgroup of G we have $k_3 \in H(\alpha)$. Let $k_4, k_5, k_6 \in G$ such that $\langle \alpha, k_1, x_2, k_4 \rangle \in *$, $\langle \alpha, k_4, k_3, k_5 \rangle \in *$ and $\langle \alpha, k_5, k_2^{-\alpha}, k_6 \rangle \in *$. Since $\langle H, A \rangle$ is a soft subgroup of G we get $k_4, k_5, k_6 \in H(\alpha)$. Using the associative property of $\langle *, A \rangle$ we have $\langle \alpha, c^{-\alpha}, m, k_6 \rangle \in *$. This implies that ${}^cH = {}^mH$. Thus $\circledast$ is well defined. It remains to show that $\langle G/H, \circledast, A\rangle$ satisfies the soft group axioms which are straightforward.

$\square$

Definition 3.21. *If $\langle H, A\rangle$ is a normal soft subgroup of $\langle G, *, A\rangle$, then the soft group $\langle G/H, \circledast, A\rangle$ defined as in Theorem 3.20 is called the quotient soft group.*

Definition 3.22. *We say that a normal soft subgroup $\langle N, A\rangle$ has the property (Q) provided that for each $\alpha, \beta \in A$, $a \in N(\alpha)$ and $b \in G$;*

$$\langle \beta, a^{-\beta}, e_\alpha, b\rangle \in * \Rightarrow b \in N(\beta).$$

Lemma 3.23. *Let $\langle G/N, \circledast, A\rangle$ be a soft group and let $\langle N, A\rangle$ be a normal soft subgroup of $\langle G, *, A\rangle$ and define a soft mapping $\langle f, A\rangle$ from G to G/N by*

$$f = \{\langle \alpha, a, {}^{a}N\rangle : \alpha \in A, a \in G\}.$$

Then $\langle f, A\rangle$ is a soft epimorphism from G to G/N such that $\langle K_f, A\rangle \breve{\subseteq} \langle N, A\rangle$. Moreover, the equality holds, whenever $\langle N, A\rangle$ has the property (Q).

Proof. Define a soft mapping $\langle f, A\rangle$ from G to G/N by $\langle \alpha, a, {}^{a}N\rangle \in f$. Clearly, $\langle f, A\rangle$ is surjective. $\langle f, A\rangle$ is a soft homomorphism since $\langle N, A\rangle$ is normal in G. Now, for each $\alpha \in A$ and $a \in G$:

$$\begin{aligned} a \in K_f(\alpha) &\Rightarrow \langle \alpha, a, {}^{e_\alpha}N\rangle \in f \\ &\Rightarrow {}^{a}N = {}^{e_\alpha}N \\ &\Rightarrow {}^{a}N(\beta) = {}^{e_\alpha}N(\beta)\ \forall \beta \in A \\ &\Rightarrow {}^{a}N(\alpha) = {}^{e_\alpha}N(\alpha) \\ &\Rightarrow a \in N(\alpha). \end{aligned}$$

Therefore $\langle K_f, A\rangle \breve{\subseteq} \langle N, A\rangle$. Moreover, if $\langle N, A\rangle$ has the property (Q), then one can easily verify that $K_f(\alpha) = N(\alpha)$ for all $\alpha \in A$ and hence the equality holds. $\square$

Note: The soft mapping defined in Lemma 3.23 is called the soft canonical epimorphism.

Theorem 3.24. *(The first soft isomorphism theorem)*
Let $\langle G, *, A\rangle$ and $\langle G', \Delta, A\rangle$ be soft groups. If $\langle f, A\rangle$ is a soft epimorphism from G to G', then

$G/K_f \breve{\cong} G'$.

Proof. Define a soft mapping $\langle g, A\rangle$ from G/K_f to G' by $\langle \alpha, {}^aK_f, y\rangle \in g$ if and only if $\langle \alpha, a, y\rangle \in f$. We first show that $\langle g, A\rangle$ is well defined. Let $\alpha \in A$ and $a, b \in G$ such that ${}^aK_f = {}^bK_f$. Let $y_1, y_2 \in G'$ such that $\langle \alpha, {}^aK_f, y_1\rangle \in g$ and $\langle \alpha, {}^bK_f, y_2\rangle \in g$. Then $\langle \alpha, a, y_1\rangle \in f$ and $\langle \alpha, b, y_2\rangle \in f$. Since ${}^aK_f = {}^bK_f$, we have ${}^aK_f(\alpha) = {}^bK_f(\alpha)$ for all $\alpha \in A$. If $\langle \alpha, b^{-\alpha}, a, x_1\rangle \in *$ then $x_1 \in K_f(\alpha)$ for all $\alpha \in A$. This implies $\langle \alpha, x_1, e'_\alpha\rangle \in f$. Since $\langle f, A\rangle$ is a soft homomorphism, $\langle \alpha, y_2^{-\alpha}, y_1, e'_\alpha\rangle \in \Delta$. Then, we get $y_1 = y_2$. Therefore $\langle g, A\rangle$ is well defined. Given that $\langle f, A\rangle$ is an epimorphism for every $y \in G'$, there is $a \in G$ such that $\langle \alpha, a, y\rangle \in f$ and hence $\langle \alpha, {}^aK_f, y\rangle \in g$. Consequently, $\langle g, A\rangle$ is surjective. It remains to show that $\langle g, A\rangle$ is injective. Suppose that $\langle \alpha, {}^aK_f, y\rangle \in g$ and $\langle \alpha, {}^bK_f, y\rangle \in g$. Thus $\langle \alpha, a, y\rangle \in f$ and $\langle \alpha, b, y\rangle \in f$. Let $x \in G$ and $z \in G'$ such that $\langle \alpha, b^{-\alpha}, a, x\rangle \in *$ and $\langle \alpha, x, z\rangle \in f$. Then we have $\langle \alpha, y^{-\alpha}, y, z\rangle \in \triangle$. This implies $z = e_\alpha^{-\alpha}$. That is $\langle \alpha, x, e'_\alpha\rangle \in f \Rightarrow x \in K_f(\alpha)$. Since α is arbitrary we can see that $\langle \alpha, b^{-\alpha}, a, x\rangle \in * \Rightarrow x \in K_f(\alpha)$ for all $\alpha \in A$. Thus ${}^aK_f(\alpha) = {}^bK_f(\alpha)$. Therefore $\langle g, A\rangle$ is injective. Hence $G/K_f \breve{\cong} G'$. $\square$

Corollary 3.25. *Let $\langle f, A\rangle$ be a soft epimorphism from $\langle G, *, A\rangle$ to $\langle G', \Delta, A\rangle$ and $\langle M, A\rangle$ be a normal soft subgroup of G'. Then,*

$$G/f^{-1}(M) \breve{\cong} G'/M.$$

Proof. Define a soft mapping $\langle g, A\rangle$ from G to G'/M by : $\langle \alpha, a, x_M\rangle \in g$ if and only if $\langle \alpha, a, y\rangle \in f$ for some $y \in G'$ with ${}^xM = {}^yM$. We first show that $\langle g, A\rangle$ is a soft homomorphism. Let $\alpha \in A$, $a, b, c \in G$ and $x, y, z \in G'$ such that $\langle \alpha, a, x_M\rangle \in g$, $\langle \alpha, b, y_M\rangle \in g$, $\langle \alpha, c, z_M\rangle \in g$ and $\langle \alpha, a, b, c\rangle \in *$. Then, there exist $x', y', z' \in G'$ such that $x_M = x'_M$, $y_M = y'_M$, $z_M = z'_M$ and $\langle \alpha, a, x'\rangle \in f$, $\langle \alpha, b, y'\rangle \in f$, $\langle \alpha, c, z'\rangle \in f$. Since $\langle f, A\rangle$ is a soft homomorphism and $\langle \alpha, a, b, c\rangle \in *$, we get that $\langle \alpha, x', y', z'\rangle \in \Delta$ in G'. Then $\langle \alpha, x'_M, y'_M, z'_M\rangle \in \bar{\Delta}$ in G'/M. Thus $\langle g, A\rangle$ is a soft homomorphism from G to G'/M. As $\langle f, A\rangle$ is surjective it is straightforward that $\langle g, A\rangle$ is also surjective. Now for each $\alpha \in A$, consider the following:

$$\begin{aligned} K_g(\alpha) &= \{a \in G : \langle \alpha, a, e_{\alpha M} \rangle \in g\} \\ &= \{a \in G : \langle \alpha, a, x \rangle \in f \ for\ some\ x \in G' \ with\ x_M = e_{\alpha M}\} \\ &= \{a \in G : \langle \alpha, a, x \rangle \in f \ for\ some\ x \in M(\alpha)\} \\ &= f^{-1}(M)(\alpha). \end{aligned}$$

Thus $\langle K_g, A \rangle \breve{=} \langle f^{-1}(M), A \rangle$. Therefore by the first soft isomorphism theorem we get that

$$G/f^{-1}(M) \breve{\cong} G'/M.$$

□

Proposition 3.26. *Let $\langle G, *, A \rangle$ and $\langle G', \Delta, A \rangle$ be soft groups and $\langle \phi, A \rangle$ be a soft homomorphism from G to G'. Then, for any soft set $\langle F, A \rangle$ over G,*

$$\langle \phi(C_F), A \rangle \subseteq \langle C_{\phi(F)}, A \rangle,$$

where $\langle C_F, A \rangle$ is the centralizer of $\langle F, A \rangle$. Moreover, the equality holds if and only if $\langle \phi, A \rangle$ is a soft isomorphism.

Theorem 3.27. *(The Third Soft isomorphism theorem)*
Suppose that $\langle M, A \rangle$ and $\langle N, A \rangle$ are normal soft subgroups of G such that $\langle M, A \rangle \breve{\subseteq} \langle N, A \rangle$ and $\langle N, A \rangle$ has the property (Q). Define a soft set with $\langle N/M, A \rangle$ over G/M by

$$N/M(\alpha) = \{^aM : a \in N(\alpha)\}.$$

Then,

(1.) $\langle N/M, A \rangle$ is a normal soft subgroup of G/M.

(2.) $(G/M)/(N/M) \breve{\cong} G/N$.

Proof. First observe that $a_M \in N/M(\alpha)$ if and only if $a \in N(\alpha)$. It is clear from the definition of N/M that $a \in N(\alpha)$ implies $a_M \in N/M(\alpha)$. Let $a_M \in N/M(\alpha)$. Then $a_M = b_M$ for some $b \in N(\alpha)$. It follows that for each $\beta \in A$ and $x \in G$, if $\langle \beta, b^{-\beta}, a, x \rangle \in *$, then $x \in M(\beta) \subseteq N(\beta)$.

In particular, $x \in N(\alpha)$ and since $b \in N(\alpha)$ we get that $a \in N(\alpha)$. Define a soft mapping $\langle g, A\rangle$ from G/M to G/N by

$$g = \{\langle \alpha, a_M, a_N\rangle : \ \alpha \in A, a \in G\}.$$

Then we first show that $\langle g, A\rangle$ is well defined. Suppose ${}^aM = {}^bM$. Thus ${}^aM(\alpha) = {}^bM(\alpha)$, for all $\alpha \in A$. Since $\langle M, A\rangle$ is a normal soft subgroup, if $\langle \alpha, b^{-\alpha}, a, x\rangle \in *$ then $x \in M(\alpha)$. Since $M(\alpha) \subseteq N(\alpha)$, $x \in N(\alpha)$. This implies that ${}^aN(\alpha) = {}^bN(\alpha)$, for all $\alpha \in A$. Therefore ${}^aN = {}^bN$. Hence $\langle g, A\rangle$ is well-defined. Next we show that $\langle g, A\rangle$ is a soft homomorphism. Let $\alpha \in A$, $a, b, c, x, y, z \in G$ such that $\langle \alpha, a_M, x_N\rangle \in g$, $\langle \alpha, b_M, y_N\rangle \in g$, $\langle \alpha, c_M, z_N\rangle \in g$ and $\langle \alpha, a_M, b_M, c_M\rangle \in \circledast$ in G/M. Then $x_N = a_N$, $y_N = b_N$ and $c_N = z_N$ and $\langle \alpha, a, b, c\rangle \in *$ in G. Thus $\langle \alpha, a_N, b_N, c_N\rangle \in \circledast$ in the quotient G/N. Hence $\langle g, A\rangle$ is a soft homomorphism and it is clear that $\langle g, A\rangle$ is a soft epimorphism. Now consider its kernel:

$$\begin{aligned} K_g(\alpha) &= \{a_M \in G/M : \ \langle \alpha, a_M, e_{\alpha N}\rangle \in g\} \\ &= \{a_M \in G/M : \ {}^aN = {}^{e_\alpha}N\} \\ &= \{a_M \in G/M : \ {}^aN(\beta) = {}^{e_\alpha}N(\beta)\ \forall \beta \in A\} \\ &= \{a_M \in G/M : \ a \in N(\alpha)\} \\ &= N/M(\alpha). \end{aligned}$$

Therefore $\langle K_g, A\rangle \,\breve{=}\, \langle N/M, A\rangle$. By the first soft isomorphism theorem $(G/M)/(N/M) \,\breve{\cong}\, G/N$. ∎

Lemma 3.28. *Let $\langle M, A\rangle$,$\langle N_1, A\rangle$ and $\langle N_2, A\rangle$ be normal soft subgroups of G such that*

$$\langle M, A\rangle \,\breve{\subseteq}\, \langle N_1, A\rangle \,\breve{\cap}\, \langle N_2, A\rangle.$$

Then $\langle N_1/M, A\rangle \,\breve{\subseteq}\, \langle N_2/M, A\rangle$ if and only if $\langle N_1, A\rangle \,\breve{\subseteq}\, \langle N_2, A\rangle$.

Proof. Suppose that $\langle N_1, A\rangle \,\breve{\subseteq}\, \langle N_2, A\rangle$. Then $N_1(\alpha) \subseteq N_2(\alpha)$ for all $\alpha \in A$. Now for any $\alpha \in A$, consider $a_M \in N_1/M(\alpha)$. This implies that $a \in N_1(\alpha) \subseteq N_2(\alpha)$. It follows that $a_M \in N_2/M(\alpha)$. Thus $\langle N_1/M, A\rangle \,\breve{\subseteq}\, \langle N_2/M, A\rangle$. Conversely, suppose that $\langle N_1/M, A\rangle \,\breve{\subseteq}\, \langle N_2/M, A\rangle$. Then $N_1/M(\alpha) \subseteq N_2/M(\alpha)$ for all $\alpha \in A$. Now for any $\alpha \in A$, $a \in N_1(\alpha)$. It follows that $a_M \in N_1/M(\alpha)$. Hence $a_M \in N_2/M(\alpha)$. Therefore $a \in N_2(\alpha)$. Thus $N_1(\alpha) \subseteq N_2(\alpha)$ and hence $\langle N_1, A\rangle \,\breve{\subseteq}\, \langle N_2, A\rangle$.

□

Corollary 3.29. *Under the assumption of Lemma 3.28,* $\langle N_1/M, A\rangle \simeq \langle N_2/M, A\rangle$ *if and only if* $\langle N_1, A\rangle \simeq \langle N_2, A\rangle$.

Lemma 3.30. *If* $\langle G, *, A\rangle$ *is soft isomorphic to* $\langle G', \Delta, A\rangle$ *then* G *is abelian if and only if* G' *is abelian.*

Proof. Let $c, d, y \in G'$ such that $\langle \alpha, c, d, y\rangle \in \Delta$. Since $\langle f, A\rangle$ is surjective there exist $a, b \in G$ such that $\langle \alpha, a, c\rangle \in f$ and $\langle \alpha, b, d\rangle \in f$. Let $x \in G$ such that $\langle \alpha, a, b, x\rangle \in *$. Since $\langle f, A\rangle$ is soft homomorphism and $\langle \alpha, c, d, y\rangle \in \Delta$, $\langle \alpha, x, y\rangle \in f$. Since G is abelian we have $\langle \alpha, b, a, x\rangle \in *$. Again since $\langle f, A\rangle$ is a soft homomorphism, $\langle \alpha, d, c, y\rangle \in \Delta$. Therefore G' is abelian. Conversely, let $a, b, c \in G$ such that $\langle \alpha, a, b, c\rangle \in *$. Since $\langle f, A\rangle$ is a soft mapping, $\langle \alpha, a, c\rangle \in f$ and $\langle \alpha, b, d\rangle \in f$ for some $c, d \in G'$. Let $y \in G'$ such that $\langle \alpha, c, d, y\rangle \in \Delta$. From $\langle f, A\rangle$ is a soft homomorphism and $\langle \alpha, c, d, y\rangle \in \Delta$ then we have $\langle \alpha, c, y\rangle \in f$. Let $y_1 \in G'$ such that $\langle \alpha, d, c, y_1\rangle \in \Delta$. Since G' is abelian, $y = y_1$. Let $x \in G$ such that $\langle \alpha, b, a, x\rangle \in *$. Again from $\langle f, A\rangle$ is a soft homomorphism and $\langle \alpha, d, c, y\rangle \in \Delta$ we get $\langle \alpha, x, y\rangle \in f$. Since $\langle f, A\rangle$ is injective, $c = x$. Therefore G is abelian. □

Theorem 3.31. *(The correspondence theorem)*
Let $\langle M, A\rangle$ *be a normal soft subgroup of* G*. Then there is a one to one correspondence between the set of all normal soft subgroups of* G *containing* $\langle M, A\rangle$ *and the set of all normal soft subgroups of* G/M.

Proof. Define $[\langle M, A\rangle, G]$ to be the set of all normal soft subgroups of G containing $\langle M, A\rangle$. i.e

$$[M, G]_A \;=\; \{\langle N, A\rangle \in SN_A(G) : \langle M, A\rangle \subseteq \langle N, A\rangle\}$$

and let $SN_A(G/M)$ be the set of all normal soft subgroups of G/M. Define $g : [M, G]_A \longrightarrow SN_A(G/M)$ by $g(\langle N, A\rangle) = \langle N/M, A\rangle \vee \langle N, A\rangle \in [M, G]_A$. Then it is clear that g is well-defined. Moreover, it follows from Lemma 3.28 that g is one to one. It remains to show that g is onto. Let $\langle J, A\rangle$ be a normal soft subgroup of G/M. For each $\alpha \in A$ define $N(\alpha) = \{a \in G : \; a_M \in J(\alpha)\}$. Then one can easily verify that $\langle N, A\rangle$ is a normal soft subgroup of G containing $\langle M, A\rangle$ such that $\langle N/M, A\rangle \simeq \langle J, A\rangle$. Therefore g is onto and hence it is the required one to one correspondence. □

Theorem 3.32. *Suppose that $\langle N, A\rangle$ is a normal soft subgroup of G having the property (Q). If $\langle f, A\rangle$ is a soft epimorphism from G to G' with $\langle K_f, A\rangle \,\breve{\subseteq}\, \langle N, A\rangle$, then*

$$G/N \,\breve{\cong}\, G'/f(N).$$

Proof. The quotient soft group $G'/f(N)$ is defined since $\langle N, A\rangle$ is a normal soft subgroup of G and $\langle f, A\rangle$ is a soft epimorphism. This implies that $\langle f(N), A\rangle$ is a normal soft subgroup of G'. Then, define a soft mapping $\langle g, A\rangle$ from G to $G'/f(N)$ by $\langle \alpha, x, {}^{a}f(N)\rangle \in g$ where $\langle \alpha, x, a\rangle \in f$ for all $\alpha \in A$, $x \in G$ and $a \in G'$. Let $x, y, z \in G$ and $a, b, c \in G'$, $\alpha \in A$ such that $\langle \alpha, x, {}^{a}f(N)\rangle \in g$, $\langle \alpha, y, {}^{b}f(N)\rangle \in g$, $\langle \alpha, z, {}^{c}f(N)\rangle \in g$ and $\langle \alpha, x, y, z\rangle \in *$ where $\langle \alpha, x, a\rangle \in f$, $\langle \alpha, y, b\rangle \in f$ and $\langle \alpha, z, c\rangle \in f$. Since $\langle f, A\rangle$ be a soft homomorphism, $\langle \alpha, a, b, c\rangle \in \Delta$ in G'. It follows that $\langle \alpha, {}^{a}f(N), {}^{b}f(N), {}^{c}f(N)\rangle \in \bar{\Delta}$ in $G'/f(N)$. Therefore $\langle g, A\rangle$ is a soft homomorphism. Let $z \in G'$. Since $\langle f, A\rangle$ is a soft epimorphism, there exists $a \in G$ such that $\langle \alpha, a, z\rangle \in f$. Thus, $\langle \alpha, a, {}^{z}f(N)\rangle \in g$. Therefore $\langle g, A\rangle$ is a soft epimorphism. Further, for any $\alpha \in A$:

$$\begin{aligned}
K_g(\alpha) &= \{a \in G : \langle \alpha, a, {}^{e'_\alpha}f(N)\rangle \in g\} \\
&= \{a \in G : {}^{x}f(N) = {}^{e'_\alpha}f(N) \;\; for \;\; some \;\; x \in G' \; with \; \langle \alpha, a, x\rangle \in f\} \\
&= \{a \in G : x \in f(N)(\alpha) \; for \; some \; x \in G' \; with \; \langle \alpha, a, x\rangle \in f\} \\
&= \{a \in G : \langle \alpha, a, x\rangle \in f \; and \; \langle \alpha, b, x\rangle \in f \; for \; some \; b \in N(\alpha)\} \\
&= \{a \in G : \langle \alpha, c, e'_\alpha\rangle \in f \; and \; \langle \alpha, b^{-\alpha}, a, c\rangle \in * \; for \; some \; c \in G\} \\
&= \{a \in G : c \in K_f(\alpha) \subseteq N(\alpha) \;\; and \;\; b \in N(\alpha) \;\}\\
&= N(\alpha).
\end{aligned}$$

Therefore $\langle K_f, A\rangle \,\breve{=}\, \langle N, A\rangle$. Thus by the first soft isomorphism theorem,

$$G/N \,\breve{\cong}\, G'/f(N).$$

$\square$

Theorem 3.33. *Suppose that $\langle N, A\rangle$ and $\langle M, A\rangle$ are normal soft subgroups of G satisfying the* *property (Q) such that $\langle MN, A\rangle$ is absolute soft set over G. Then,*

$$G/M \cap N \breve{\cong} G/N \times G/M.$$

Proof. Consider a soft mapping $\langle f, A\rangle$ from G to $G/N \times G/M$ defined by $\langle \alpha, a, x\rangle \in f$ where $x = ({}^aN, {}^aM)$ for $a \in G$. Let $\alpha \in A$, $a, b, c \in G$ and $x_1, x_2, x_3 \in G/N \times G/M$ such that $\langle \alpha, a, x_1\rangle \in f$, $\langle \alpha, b, x_2\rangle \in f$, $\langle \alpha, c, x_3\rangle \in f$ and $\langle \alpha, a, b, c\rangle \in *$ in G where $x_1 = ({}^aN, {}^aM)$, $x_2 = ({}^bN, {}^bM)$ and $x_3 = ({}^cN, {}^cM)$. Since $\langle N, A\rangle$ and $\langle M, A\rangle$ are normal soft subgroups of G, $\langle \alpha, {}^aN, {}^bN, {}^cN\rangle \in \Delta$ in G/N and $\langle \alpha, {}^aM, {}^bM, {}^cM\rangle \in \bar{\Delta}$ in G/M. From the definition of direct product of a soft group and $\langle \alpha, c, x_3\rangle \in f$ then we have $\langle \alpha, x_1, x_2, x_3\rangle \in \circledast$ in $G/N \times G/M$ and hence $\langle f, A\rangle$ is a soft homomorphism. Next we show that $\langle f, A\rangle$ is surjective. Let $z = ({}^xN, {}^yM) \in G/N \times G/M$. where $x, y \in G$. Since $\langle NM, A\rangle$ is an absolute soft set, $[NM](\alpha) = G$ for all $\alpha \in A$. So if $x, y \in [NM](\alpha)$, then $\langle \alpha, r, s, x\rangle \in *$ for some $r \in N(\alpha)$ and $s \in M(\alpha)$ with $\langle \alpha, t, u, y\rangle \in *$ for some $t \in N(\alpha)$ and $u \in M(\alpha)$. Now, let $a \in G$ such that $\langle \alpha, s, t, a\rangle \in *$. Then, $\langle \alpha, a^{-\alpha}, x, n\rangle \in *$ for some $n \in N(\alpha)$. Hence ${}^aN(\alpha) = {}^xN(\alpha)$ for all $\alpha \in A$. Also, since $s, u, \in M(\alpha)$, we have $\langle \alpha, a^{-\alpha}, y, m\rangle \in *$ for some $m \in M(\alpha)$ and hence ${}^aM(\alpha) = {}^yM(\alpha)$. Therefore $\langle \alpha, a, x\rangle \in f$ where $x = ({}^aN, {}^aM)$. This implies $\langle \alpha, a, z\rangle \in f$. Thus $\langle f, A\rangle$ is a soft epimorphism. Now,

$$\begin{aligned}
K_f(\alpha) &= \{a \in G : \langle \alpha, a, z\rangle \in f \;\; \textit{where} \;\; z = ({}^{e_\alpha}N, {}^{e_\alpha}M)\} \\
&= \{a \in G : {}^aN = {}^{e_\alpha}N \;\; \textit{and} \;\; {}^aM = {}^{e_\alpha}M \;\} \\
&= \{a \in G : {}^aN(\beta) = {}^{e_\alpha}N(\beta) \;\; \textit{and} \;\; {}^aM(\beta) = {}^{e_\alpha}M(\beta) \; \forall \beta \in A\} \\
&= \{a \in G : a \in N(\alpha) \;\; \textit{and} \;\; a \in M(\alpha) \;\; \} \\
&= (M \cap N)(\alpha).
\end{aligned}$$

Thus $\langle K_f, A\rangle \breve{\cong} \langle M \cap N, A\rangle$. Therefore, by the first soft isomorphism theorem

$$G/M \cap N \breve{\cong} G/N \times G/M.$$

□

Corollary 3.34. *Any soft homomorphism of soft groups can be expressed as a composition of soft epimorphism and monomorphism.*

Proof. Since $\langle f, A\rangle$ is a soft homomorphism from G to G', by Theorem 3.24, there is a soft isomorphism $\langle g, A\rangle$ from G/k_f to $f(1_G)$ such that $\langle \alpha, {}^aK_f, y\rangle \in g$ if and only if $\langle \alpha, a, y\rangle \in f$ for all $\alpha \in A$ and $a \in G$. Since $\langle f(1_G), A\rangle$ is a soft subgroup of G', $\langle g, A\rangle$ can be defined as a soft monomorphism of G/k_f in to G'. Also, let $\langle h, A\rangle$ be a soft homomorphism from G to G/k_f defined by $\langle \alpha, a, {}^aK_f\rangle \in h$ for all $a \in G$. Then clearly $\langle h, A\rangle$ be a soft epimorphism. Now, for any $\alpha \in A$ and $a \in G$, $\langle \alpha, a, x\rangle \in g \circ h \Leftrightarrow \langle \alpha, {}^aK_f, x\rangle \in g \Leftrightarrow \langle \alpha, a, x\rangle \in f$ for some $x \in G'$. Thus, $\langle f, A\rangle$ composition of soft epimorphism $\langle h, A\rangle$ and soft monomorphism $\langle g, A\rangle$. □

Theorem 3.35. *Consider the following soft groups: $\langle G, *, A\rangle$, $\langle G_1, \circ, A\rangle$ and $\langle G_2, \triangle, A\rangle$. $G \breve{\cong} G_1 \times G_2$ if and only if there are normal soft subgroups $\langle N_1, A\rangle$ and $\langle N_2, A\rangle$ of G such that $\langle N_1N_2, A\rangle$ is the absolute soft set over G, $\langle N_1 \cap N_2, A\rangle$ is a trivial soft subgroup of G, $G_1 \breve{\cong} G/N_1$ and $G_2 \breve{\cong} G/N_2$.*

Proof. For $\alpha \in A$, suppose that e_α, e'_α and e''_α are identities in G, G_1 and G_2 respectively. Assume that $\langle f, A\rangle$ is a soft isomorphism from G to $G_1 \times G_2$ and $G \breve{\cong} G_1 \times G_2$. Put $\langle M, A\rangle \breve{=} \langle \breve{e}'_\alpha \times 1_{G_2}, A\rangle$ and $\langle N, A\rangle \breve{=} \langle 1_{G_1} \times \breve{e}''_\alpha, A\rangle$. Then $\langle M, A\rangle$ and $\langle N, A\rangle$ are normal soft subgroups of $G_1 \times G_2$. Then, put $N_1(\alpha) = f^{-1}(M)(\alpha)$ and $N_2(\alpha) = f^{-1}(N)(\alpha)$. Which implies that $\langle N_1, A\rangle$ and $\langle N_2, A\rangle$ are normal soft subgroups of G. For any $x \in G$, $\langle \alpha, x, a\rangle \in f$ with $a = (a_1, a_2) \in G_1 \times G_2$. Consider x_1 and x_2 in G such that $\langle \alpha, x_1, a'\rangle \in f$ and $\langle \alpha, x_2, a''\rangle \in f$ where $a' = (e'_\alpha, a_2)$, $a'' = (a_1, e''_\alpha)$. Then $x_1 \in f^{-1}(M)(\alpha)$, $x_2 \in f^{-1}(N)(\alpha)$ and $\langle \alpha, x, y\rangle \in f \Leftrightarrow \langle \alpha, a', a'', y\rangle \in *$. Since $\langle f, A\rangle$ is a soft isomorphism, we have $\langle \alpha, x_1, x_2, x\rangle \in *$ and it follows that $x \in (N_1N_2)(\alpha)$. Hence $(N_1N_2)(\alpha) = G$ for all $\alpha \in A$. Thus $\langle N_1N_2, A\rangle$ is the absolute soft set. Let $x \in (N_1 \cap N_2)(\alpha)$. This implies $x \in f^{-1}(M)(\alpha)$, $x \in f^{-1}(N)(\alpha)$. Consequently, $a \in M(\alpha)$ and $a \in N(\alpha)$. Then $a \in M(\alpha) \cap N(\alpha) = (M \cap N)(\alpha)$. Since $\langle f, A\rangle$ is soft homomorphism then we have $a = (e'_\alpha, e''_\alpha)$ and $\langle \alpha, e_\alpha, a\rangle \in f$. This implies $x = e_\alpha$. Therefore, $(N_1 \cap N_2)(\alpha) = \{e_\alpha\}$ for all $\alpha \in A$. Thus $\langle N_1 \cap N_2, A\rangle$ is the trivial soft subgroup of G. Next, define a soft mapping $\langle f_1, A\rangle$ from G to G_1 and $\langle f_2, A\rangle$ from G to G_2 by $\langle \alpha, x, a_1\rangle \in f_1$ and $\langle \alpha, x, a_2\rangle \in f_2$, if $\langle \alpha, x, a\rangle \in f$ where $a = (a_1, a_2)$. Thus $\langle f_1, A\rangle$ and $\langle f_2, A\rangle$ are soft epimorphism and $\langle K_{f_1}, A\rangle \breve{=} \langle N_1, A\rangle$ and $\langle K_{f_2}, A\rangle \breve{=} \langle N_2, A\rangle$. For

$$x \in N_1(\alpha) \Leftrightarrow a \in M(\alpha) \Leftrightarrow a = a' \Leftrightarrow a_1 = e'_\alpha$$

and

$$x \in N_2(\alpha) \Leftrightarrow a \in M(\alpha) \Leftrightarrow a = a'' \Leftrightarrow a_2 = e''_\alpha$$

where $a' = (e'_\alpha, a_2)$ and $a'' = (a', e''_\alpha)$. Therefore, $G_1 \breve{\cong} G/N_1$ and $G_2 \breve{\cong} G/N_2$. Conversely, let $\langle N_1, A\rangle$ and $\langle N_2, A\rangle$ be normal soft subgroups of G such that $(N_1N_2)(\alpha) = G$, $(N_1 \cap N_2)(\alpha) = \{e_\alpha\}$ for all $\in \alpha \in A$ and let $\langle g_1, A\rangle$ be a soft isomorphism from G/N_1 to G_1 and let $\langle g_2, A\rangle$ be a soft isomorphism from G/N_2 to G_2 and $\langle h_1, A\rangle$ be a soft homomorphism from G to G/N_1 and $\langle h_2, A\rangle$ be a soft homomorphism from G to G/N_2. Now define a soft mapping $\langle \theta, A\rangle$ from G to $G_1 \times G_2$ as follows: for $\alpha \in A$, $a \in G, z_1 \in G_1$ and $z_2 \in G_2$:

$$\langle \alpha, a, \langle z_1, z_2\rangle\rangle \in \theta \text{ if and only if } \langle \alpha, {}^aN_1, z_1\rangle \in g_1 \text{ and } \langle \alpha, {}^aN_2, z_2\rangle \in g_2$$

Since $\langle g_1, A\rangle$ and $\langle g_2, A\rangle$ are soft isomorphisms, $\langle \theta, A\rangle$ will also be a soft epimorphism. Put $e_\alpha = (e'_\alpha, e''_\alpha)$. Then for any $a \in G$, consider the following:

$$\begin{aligned} \langle \alpha, a, e_\alpha\rangle \in \theta \;&\Leftrightarrow\; \langle \alpha, {}^aN_1, e'_\alpha\rangle \in g_1 \;\; \textit{and} \;\; \langle \alpha, {}^aN_2, e''_\alpha\rangle \in g_2 \\ &\Leftrightarrow\; a \in (N_1 \cap N_2)(\alpha) = \{e_\alpha\}. \end{aligned}$$

Therefore $K_\theta(\alpha) = \{e_\alpha\}$ for all $\alpha \in A$. Thus $\langle \theta, A\rangle$ is a soft monomorphism and hence a soft isomorphism. Therefore, $G \breve{\cong} G_1 \times G_2$. □

3.3 Soft Automorphisms

This section is devoted to present some fundamental results on soft automorphisms on soft groups.

Definition 3.36. *For any soft group $\langle G, *, A\rangle$ a bijective soft homomorphism of G onto itself is called a soft automorphism of G. We denote by $SAut(G)$ the collection of all soft automorphisms of G.*

Example 3.37. *Let $\langle *, \mathbb{N}\rangle$ be a soft binary operation on $\mathbb{R}^+$ and defined by*

$$\langle \alpha, a, b, c\rangle \in * \Leftrightarrow c = \alpha ab.$$

Then it is clear that $\langle \mathbb{R}^+, *, \mathbb{N}\rangle$ is a soft group. Consider.

$$f = \{(\alpha, x, y) \in \mathbb{N} \times \mathbb{R}^+ \times \mathbb{R}^+ : y = \sqrt{\frac{x}{\alpha}}\}.$$

Then $\langle f, N\rangle$ *is a soft automorphism of* $\mathbb{R}^+$.

The following lemmas can be verified easily.

Lemma 3.38. *Let* $\langle f, A\rangle$ *be a soft isomorphism from* $\langle G, *, A\rangle$ *to* $\langle G', \Delta, A\rangle$. *Then* $\langle f^{-1}, A\rangle$ *is also a soft isomorphism from* $\langle G', \Delta, A\rangle$ *to* $\langle G, *, A\rangle$.

Lemma 3.39. *For any soft group* G, $\langle SAut(G), \circ\rangle$ *is a group; where* $\circ$ *is the composition of soft mappings.*

Theorem 3.40. $\langle SAut(G), \circ\rangle$ *is isomorphic to a subgroup of* $\langle Aut(SE_A(G)), \circ\rangle$.

Proof. We show that the map $f \longmapsto \breve{f}$ is an embedding of $SAut(G)$ into $Aut(SE_A(G))$ where for each soft map $\langle f, A\rangle$ from to G to G, $\breve{f}$ is a mapping from $SE_A(G)$ to $SE_A(G)$ given in Definition 3.7. It is proved in Theorem 3.9 that if $\langle f, A\rangle$ is a soft automorphism of G, then $\breve{f}$ is an automorphism of $SE_A(G)$.

Claim 1: $f \longmapsto \breve{f}$ is one to one. Let $f, g \in SAut(G)$ such that $\breve{f} = \breve{g}$. Then $\breve{f}(\breve{a}) = \breve{g}(\breve{a})$ for all $\breve{a} \in SE_A(G)$. Then $\breve{f}(\breve{a})(\alpha) = \breve{g}(\breve{a})(\alpha)$ for all $\alpha \in A$. That is, for any $x \in G$ it holds that $\langle \alpha, \breve{a}(\alpha), x\rangle \in f$ if and only if $\langle \alpha, \breve{a}(\alpha), x\rangle \in g$. Implying that, $\langle f, A\rangle \,\breve{\cong}\, \langle g, A\rangle$.

Claim 2: $\breve{(f \circ g)} = \breve{f} \circ \breve{g}$

Let $\breve{a} \in SE_A(G)$, $\alpha \in A$ and $x \in G$ such that $\breve{(f \circ g)}(\breve{a})(\alpha) = \{x\}$. Then, $\langle \alpha, \breve{a}(\alpha), x\rangle \in f \circ g$, and so there is some $b \in G$ such that $\langle \alpha, \breve{a}(\alpha), b\rangle \in g$, and $\langle \alpha, b, x\rangle \in f$. Thus $\breve{g}(\breve{a})(\alpha) = \{b\}$ and $\breve{f}(\breve{g}(\breve{a}))(\alpha) = \{x\}$. Conversely, if we are assuming that $\breve{f}(\breve{g}(\breve{a}))(\alpha) = \{x\}$, then it can be shown that $\breve{(f \circ g)}(\breve{a})(\alpha) = \{x\}$. Thus $\breve{(f \circ g)}(\breve{a})(\alpha) = \breve{f}(\breve{g}(\breve{a})(\alpha)$ for all $\alpha \in A$. Therefore, $\breve{f \circ g}(\breve{a}) = \breve{f}(\breve{g}(\breve{a})$ for all $\breve{a} \in SE_A(G)$. Hence $\breve{f \circ g} = \breve{f} \circ \breve{g}$. Therefore the map $f \longmapsto \breve{f}$ is a monomorphism and hence an embedding.

□

Theorem 3.41. *Let* $\langle G, *, A\rangle$ *and* $\langle G', \Delta, A\rangle$ *be soft groups. If* $G \,\breve{\cong}\, G'$, *then* $SAut(G) \,\breve{\cong}\, SAut(G')$.

Proof. Suppose $G \,\breve{\cong}\, G'$ and $\langle \phi, A\rangle$ be a soft isomorphism from G to G'. For any soft automorphism $\langle f, A\rangle$ of G, one can easily check that $\langle \phi \circ f \circ \phi^{-1}, A\rangle$ is a soft automorphism of G'. Moreover, the map $f \mapsto \phi \circ f \circ \phi^{-1}$ is an ordinary group isomorphism of $SAut(G)$ onto $SAut(G')$. □

Lemma 3.42. *Let $\langle G, *, A\rangle$ be a soft group. Define a soft mapping $\langle f, A\rangle$ from G to G by $\langle \alpha, a, a^{-\alpha}\rangle \in f$ for any $a \in G$. Then $\langle f, A\rangle$ is a soft automorphism of G if and only if $\langle G, *, A\rangle$ is an abelian soft group.*

Proof. Suppose $\langle f, A\rangle$ is a soft automorphism of G. Let $a, b, x \in G$ and $\alpha \in A$ such that $\langle \alpha, a, b, x\rangle \in *$. Since $\langle f, A\rangle$ is a soft homomorphism of G, $\langle \alpha, a^{-\alpha}, b^{-\alpha}, x^{-\alpha}\rangle \in *$. Then we have $\langle \alpha, b, a, x\rangle \in *$. Therefore $\langle G, *, A\rangle$ is abelian soft group. Conversely, suppose that $\langle G, *, A\rangle$ is abelian soft group. We first show that $\langle f, A\rangle$ is a soft homomorphism. Let $a, b, c \in G$ and $\alpha \in A$ such that $\langle \alpha, a, a^{-\alpha}\rangle \in f$, $\langle \alpha, b, b^{-\alpha}\rangle \in f$, $\langle \alpha, c, c^{-\alpha}\rangle \in f$ and $\langle \alpha, a, b, c\rangle \in *$. It follows that $\langle \alpha, b^{-\alpha}, a^{-\alpha}, c^{-\alpha}\rangle \in *$. Since G is abelian soft group, $\langle \alpha, a^{-\alpha}, b^{-\alpha}, c^{-\alpha}\rangle \in *$. Therefore $\langle f, A\rangle$ is a soft homomorphism. Next we show that $\langle f, A\rangle$ is injective. Let $x_1, x_2, y \in G$ such that $\langle \alpha, x_1, y\rangle \in f$ and $\langle \alpha, x_2, y\rangle \in f$. This implies that $y = x_1^{-\alpha}$ and $y = x_2^{-\alpha}$. It follows that $x_1 = x_2$. Therefore $\langle f, A\rangle$ is injective. Finlay it remains to show that $\langle f, A\rangle$ is surjective. Let $h \in G$ then $g = h^{-\alpha} \in G$. It follows that $\langle \alpha, g, h\rangle \in f$. Thus $\langle f, A\rangle$ is surjective. Hence $\langle f, A\rangle$ is a soft automorphism. □

Proposition 3.43. *Let $\langle f, A\rangle$ be a soft automorphism of G. If $\langle H, A\rangle$ is a soft set over G as described by*

$$H(\alpha) = \{g \in G : \langle \alpha, g, g\rangle \in f \circ f\}$$

, then $\langle H, A\rangle$ is a soft subgroup of G.

Proof. Since $\langle \alpha, e_\alpha, e_\alpha\rangle \in f \circ f$, $e_\alpha \in H(\alpha)$. Let $g, h \in H(\alpha)$ then $\langle \alpha, g, g\rangle \in f \circ f$ and $\langle \alpha, h, h\rangle \in f \circ f$. Let $x, z \in G$ such that $\langle \alpha, g, h, x\rangle \in *$ and $\langle \alpha, x, z\rangle \in f \circ f$. Since $\langle f, A\rangle$ is a soft automorphism, $\langle \alpha, g, h, z\rangle \in *$. This implies that $z = x$. Therefore $x \in H(\alpha)$. Let $g \in H(\alpha)$, $y \in G$ such that $\langle \alpha, g, g\rangle \in f \circ f$ and $\langle \alpha, g^{-\alpha}, y\rangle \in f \circ f$. As $\langle \alpha, g, g\rangle \in f \circ f$ it follows that $\langle \alpha, g^{-\alpha}, g^{-\alpha}\rangle \in f \circ f$. So $y = g^{-\alpha}$. Therefore $g^{-\alpha} \in H(\alpha)$. Hence $\langle H, A\rangle$ is a soft subgroup of G. □

Proposition 3.44. *Let $\langle G, *, A\rangle$ be a soft group and let $\langle H, A\rangle$ be a soft set over G defined by*

$$H(\alpha) = \{a \in G : \langle \alpha, a, a\rangle \in f, \quad \textit{for all } f \in SAut(G)\}$$

for all $\alpha \in A$. Then $\langle H, A\rangle$ is a normal soft subgroup of G.

Proof. Since $\langle \alpha, e_\alpha, e_\alpha\rangle \in f$, $e_\alpha \in H(\alpha)$. Let $a, b \in H(\alpha)$ and $x \in G$ such that $\langle \alpha, a, b^{-\alpha}, x\rangle \in *$. Then we have $\langle \alpha, a, a\rangle \in f$ and $\langle \alpha, b, b\rangle \in f$. Let $y \in G$ such that $\langle \alpha, x, y\rangle \in f$. Since $\langle f, A\rangle$

is a soft homomorphism, $\langle \alpha, a, b^{-\alpha}, y\rangle \in *$ and so $y = x$. Therefore $\langle \alpha, x, x\rangle \in f$. Thus $x \in H(\alpha)$. Hence $\langle H, A\rangle$ is a soft subgroup of G. Let $\alpha \in A, a, x, y \in G$ and $n \in H(\alpha)$. Suppose $\langle \alpha, a, n, x\rangle \in *$ and $\langle \alpha, x, a^{-\alpha}, y\rangle \in *$. We shall prove that $y \in H(\alpha)$. Let $y_1, y_2, z \in G$ such that $\langle \alpha, a, y_1\rangle \in f$, $\langle \alpha, y, z\rangle \in f$ and $\langle \alpha, x, y_2\rangle \in f$. Since $\langle f, A\rangle$ is a soft homomorphism and $\langle \alpha, a, y, x\rangle \in *$, it holds that $\langle \alpha, y_1, z, y_2\rangle \in *$. As $n \in H(\alpha)$ it follows that $\langle \alpha, n, n\rangle \in f$. Again since $\langle f, A\rangle$ is a soft homomorphism and $\langle \alpha, a, n, x\rangle \in *$, $\langle \alpha, y_1, n, y_2\rangle \in *$. It follows from the the cancellation law that $z = n$. As $\langle f, A\rangle$ is injective we get $y = n$. Then by transitivity we get $y = z$. Therefore $\langle \alpha, y, y\rangle \in f$. Thus $y \in H(\alpha)$. Hence $\langle H, A\rangle$ is a normal soft subgroup of G. □

Theorem 3.45. *Let $\langle G, *, A\rangle$ be a soft group and $a \in G$. Define a soft mapping $\langle T_a, A\rangle$ from G to G as follows, for each $\alpha \in A : \langle \alpha, x, z\rangle \in T_a$ if and only if there is some $y \in G$ such that $\langle \alpha, a, x, y\rangle \in *$ and $\langle \alpha, y, a^{-\alpha}, z\rangle \in *$. Then $\langle T_a, A\rangle$ is a soft automorphism on G.*

Proof. We first show that $\langle T_a, A\rangle$ is a soft homomorphism. Let $\alpha \in A, x_1, x_2, x_3, z_1, z_2, z_3 \in G$ such that $\langle \alpha, x_1, z_1\rangle \in T_a$, $\langle \alpha, x_2, z_2\rangle \in T_a$, $\langle \alpha, x_3, z_3\rangle \in T_a$ and $\langle \alpha, x_1, x_2, x_3\rangle \in *$. Then there exist $y_1, y_2, y_3 \in G$ such that $\langle \alpha, a, x_1, y_1\rangle \in *$ and $\langle \alpha, y_1, a^{-\alpha}, z_1\rangle \in *$, $\langle \alpha, a, x_2, y_2\rangle \in *$ and $\langle \alpha, y_2, a^{-\alpha}, z_2\rangle \in *$, $\langle \alpha, a, x_3, y_3\rangle \in *$ and $\langle \alpha, y_3, a^{-\alpha}, z_3\rangle \in *$. Using the fact that $\langle \alpha, a^{-\alpha}, a, e_\alpha\rangle \in *$ and by associative property of $\langle *, A\rangle$ we get that $\langle \alpha, z_1, z_2, z_3\rangle \in *$. Thus $\langle T_a, A\rangle$ is a soft homomorphism. In addition for any $\alpha \in A$ and any $z \in G$, let $x, y \in G$ such that $\langle \alpha, a^{-\alpha}, z, y\rangle \in *$ and $\langle \alpha, y, a, x\rangle \in *$. If $u \in G$ such that $\langle \alpha, a, x, u\rangle \in *$, then it can be verified that $\langle \alpha, z, a, u\rangle \in *$. It follows that $\langle \alpha, u, a^{-\alpha}, z\rangle \in *$. Therefore, u is an element of G such that $\langle \alpha, a, x, u\rangle \in *$, and $\langle \alpha, u, a^{-\alpha}, z\rangle \in *$. Thus $\langle \alpha, x, z\rangle \in T_\alpha$ and hence $\langle T_a, A\rangle$ is surjective. Finally, it remains to show that $\langle T_a, A\rangle$ is injective. Let $\alpha \in A$ and $x_1, x_2, z \in G$ such that $\langle \alpha, x_1, z\rangle \in T_a$ and $\langle \alpha, x_2, z\rangle \in T_a$. Then there exist y_1 and $y_2 \in G$ such that $\langle \alpha, a, x_1, y_1\rangle \in *$, $\langle \alpha, y_1, a^{-\alpha}, z\rangle \in *$, $\langle \alpha, a, x_2, y_2\rangle \in *$ and $\langle \alpha, y_2, a^{-\alpha}, z\rangle \in *$. Applying the cancellation law on equations $\langle \alpha, y_1, a^{-\alpha}, z\rangle \in *$ and $\langle \alpha, y_2, a^{-\alpha}, z\rangle \in *$ we have $y_1 = y_2$. Again using $y_1 = y_2$ and applying the cancellation law on equations $\langle \alpha, a, x_1, y_1\rangle \in *$ and $\langle \alpha, a, x_2, y_2\rangle \in *$ gives us $x_1 = x_2$. Thus $\langle T_a, A\rangle$ is injective and therefore it is a soft automorphism. □

Definition 3.46. *Let $\langle G, *, A\rangle$ be a soft group and $a \in G$. The soft automorphism $\langle T_a, A\rangle$ of G given in the above theorem is called the inner soft automorphism of G corresponding to a. We denote by $SI_A(G)$ the set of all inner soft isomorphisms of G with the given set of parameters A.*

Define a soft binary operation $\langle \circ, A\rangle$ on $SI_A(G)$ by; for $\alpha \in A$, $a, b, c \in G$

$$\langle \alpha, T_a, T_b, T_c\rangle \in \circ \text{ if and only if } \langle \alpha, a, b, d\rangle \in *$$

for some $d \in G$ with $T_c = T_d$. Then we have the following theorem.

Theorem 3.47. *For any soft group $\langle G, *, A\rangle$, $\langle SI_A(G), \circ, A\rangle$ is a soft group.*

Proof. We first show that $\langle \circ, A\rangle$ is well defined. Let $a, b, c, d, e, f \in G$ such that

$$\langle T_a, A\rangle \breve{=} \langle T_d, A\rangle,\ \langle T_b, A\rangle \breve{=} \langle T_e, A\rangle$$

$$\langle \alpha, T_a, T_b, T_c\rangle \in \circ \textit{ and } \langle \alpha, T_d, T_e, T_f\rangle \in \circ.$$

Then we have for any $x, z \in G$ and $\alpha \in A$, $\langle \alpha, x, z\rangle \in T_a$ if and only if $\langle \alpha, x, z\rangle \in T_d$. Similarly $\langle \alpha, x, z\rangle \in T_b$ if and only if $\langle \alpha, x, z\rangle \in T_e$. Again then we have $\langle \alpha, a, b, c\rangle \in *$ and $\langle \alpha, d, e, f\rangle \in *$.
Claim: $\langle T_c, A\rangle \breve{=} \langle T_f, A\rangle$.
For any $\alpha \in A$, and $x, z \in G$. Suppose that $\langle \alpha, x, z\rangle \in T_c$. Then there exists $y \in G$ such that $\langle \alpha, c, x, y\rangle \in *$ and $\langle \alpha, y, c^{-\alpha}, z\rangle \in *$. Since $\langle \alpha, a, b, c\rangle \in *$ there exist $u, w \in G$ such that $\langle \alpha, b, x, u\rangle \in *$ and $\langle \alpha, u, b^{-\alpha}, w\rangle \in *$ together implying $\langle \alpha, x, w\rangle \in T_b = T_e$ and there is also $v \in G$ such that $\langle \alpha, a, w, v\rangle \in *$ and $\langle \alpha, v, a^{-\alpha}, z\rangle \in *$ implying that $\langle \alpha, w, z\rangle \in T_a = T_d$. Therefore we can find $u_1, w_1, v_1 \in G$ such that $\langle \alpha, d, w_1, v_1\rangle \in *$, $\langle \alpha, v_1, d^{-\alpha}, z\rangle \in *$, $\langle \alpha, e, x, u_1\rangle \in *$ and $\langle \alpha, u_1, e^{-\alpha}, w_1\rangle \in *$. Since $\langle \alpha, d, e, f\rangle \in *$ we get that for some $g \in G$ $\langle \alpha, f, x, g\rangle \in *$ and $\langle \alpha, g, f^{-\alpha}, z\rangle \in *$ and therefore $\langle \alpha, x, z\rangle \in T_f$ that is $T_c(\alpha) \subseteq T_f(\alpha)$. Since α is arbitrary we get $\langle T_c, A\rangle \breve{\subseteq} \langle T_f, A\rangle$. Similarly it can be shown that $\langle T_f, A\rangle \breve{\subseteq} \langle T_c, A\rangle$ and hence the equality holds. Therefore the soft binary operation $\langle \circ, A\rangle$ is well-defined. It remains to show that $\langle SI_A(G), \circ, A\rangle$ satisfies the soft group axioms which are straightforward. □

Corollary 3.48. *For any abelian soft group G, an identity soft mapping is the only soft inner automorphism of G.*

Proof. Let $\langle G, *, A\rangle$ be an abelian soft group and $\langle T_a, A\rangle$ be a soft inner automorphism on G. Let $\alpha \in A$, $x, y, z \in G$ such that

$$\langle \alpha, x, z\rangle \in T_a \Leftrightarrow \langle \alpha, a, x, y\rangle \in * \textit{ and } \langle \alpha, y, a^{-\alpha}, z\rangle \in *$$

for all $\alpha \in G$. Since G is abelian, $\langle \alpha, x, a, y \rangle \in *$. Moreover $\langle *, A \rangle$ is associative and $\langle \alpha, a, a^{-\alpha}, e_\alpha \rangle \in *$ we get $z = x$. Therefore $\langle \alpha, x, x \rangle \in T_a$. □

We conclude this chapter by showing that $SI_A(G)$ is soft isomorphic with the quotient soft group $G/Z_A(G)$ in the following theorem.

Theorem 3.49. *For any soft group* $\langle G, *, A \rangle$ *we have* $G/Z_A(G) \breve{\cong} SI_A(G)$.

Proof. Define a soft mapping $\langle \pi, A \rangle$ from G to $SI_A(G)$ by:

$$\pi = \{\langle \alpha, a, T_a \rangle : \alpha \in A \text{ and } a \in G\}.$$

Then it is clear that $\langle \pi, A \rangle$ is a soft epimorphism.

Claim: $\langle K_\pi, A \rangle \breve{\cong} \langle Z_A(G), A \rangle$ for any $\alpha \in A$. Let $a \in K_\pi(\alpha)$. Then $T_a = T_{e_\alpha}$ i.e for $x, y \in G$ it holds that $\langle \alpha, x, z \rangle \in T_a$ if and only if $\langle \alpha, x, z \rangle \in T_{e_\alpha}$ equivalently, if $y \in G$ such that $\langle \alpha, a, x, y \rangle \in *$ and $\langle \alpha, y, a^{-\alpha}, z \rangle \in *$ then $z = x$ and $\langle \alpha, x, a, y \rangle \in *$ and hence $a \in Z_A(G)(\alpha)$. Since $\alpha \in A$ is arbitrary we have $\langle K_\pi, A \rangle \breve{\subseteq} \langle Z_A(G), A \rangle$. Conversely, for $\alpha \in A$, suppose that $a \in Z_A(G)(\alpha)$. Then for all $x, y \in G$ it is the case that $\langle \alpha, a, x, y \rangle \in *$ if and only if $\langle \alpha, x, a, y \rangle \in *$. For any $x \in G$ and $y \in G$ if $\langle \alpha, a, x, y \rangle \in *$, then $\langle \alpha, y, a^{-\alpha}, x \rangle \in *$. Now we show that $T_a = T_{e_\alpha}$ for $x, y \in G$. Let $\langle \alpha, x, z \rangle \in T_a$ which implies that there exists $y \in G$ such that $\langle \alpha, a, x, y \rangle \in *$, and $\langle \alpha, y, a^{-\alpha}, z \rangle \in *$. Then it follows that $z = x$, and hence $\langle \alpha, x, x \rangle \in T_{e_\alpha}$ i.e., $T_a \subseteq T_{e_\alpha}$. Also if $\langle \alpha, x, z \rangle \in T_{e_\alpha}$ we get that $z = x$. So that for any $y \in G$, if $\langle \alpha, a, x, y \rangle \in *$, then $\langle \alpha, y, a^{-\alpha}, x \rangle \in *$. So that $\langle \alpha, x, x \rangle \in T_a$ and hence $T_{e_\alpha} \subseteq T_a$. Therefore $T_a = T_{e_\alpha}$. Thus, $a \in K_\pi(\alpha)$. Therefore, $\langle K_\pi, A \rangle \breve{\cong} \langle Z_A(G), A \rangle$. Hence by the first soft isomorphism theorem $G/Z_A(G) \breve{\cong} SI_A(G)$.

□

Chapter 4

Action of a soft group on a set

The results of soft group based on soft binary operations, including the action of a soft group on a set, soft orbits, and soft stabilizers, are examined in this chapter along with their characteristics. Furthermore, we obtain an ordinary map $\widehat{\mu}$ from the set $SE_A(G) \times SE_A(X)$ to the set $SE_A(X)$ given a soft mapping $\langle \mu, A \rangle$ from $G \times X$ to X. We prove that $\langle \mu, A \rangle$ is a soft action if and only if $\widehat{\mu}$ is an ordinary action. In addition we establish relation between soft orbits and soft stabilizers of an element x in X with orbits and stabilizers of the soft elements $\breve{x}$.

4.1 Soft actions

Definition 4.1. *Let $\langle G, *, A \rangle$ be a soft group and X be a non empty set. By a soft action of G on X we mean a soft mapping $\langle \mu, A \rangle$ from $G \times X$ to X satisfying the following two conditions:*

$\langle SA1 \rangle$ $\langle \alpha, e_\alpha, x, x \rangle \in \mu$ *for every* $\alpha \in A$ *and* $x \in X$*;*

$\langle SA2 \rangle$ *For any* $\alpha \in A$, $a, b, c \in G$ *and* $x, y, z, u \in X$,

$\langle \alpha, a, b, c \rangle \in *$, $\langle \alpha, c, x, u \rangle \in \mu$, $\langle \alpha, b, x, y \rangle \in \mu$ *and* $\langle \alpha, a, y, z \rangle \in \mu$ *all together imply* $u = z$.

Note: If there is a soft action $\langle \mu, A \rangle$ of a soft group $\langle G, *, A \rangle$ on a set X, then we say that G acts on X softly and we call X a soft G-set.

Example 4.2. *Let $\langle G, *, A \rangle$ be a soft group and X be non empty set. Define a soft mapping $\langle \mu, A \rangle$ from $G \times X$ to X by:*

$$\mu = \{\langle \alpha, a, x, x \rangle : \quad \alpha \in A, a \in G, x \in X\}.$$

Then, $\langle \mu, A\rangle$ is the trivial soft action of G on X.

Example 4.3. *Let $\langle G, *, A\rangle$ be a soft group. Define a soft mapping $\langle \mu, A\rangle$ from $G \times G$ to G as follows: for each $\alpha \in A$ and $a, x, y \in G$. Then,*

$$\langle \alpha, a, x, y\rangle \in \mu \Leftrightarrow \langle \alpha, a, x, y\rangle \in *.$$

We call $\langle \mu, A\rangle$ is a soft action of G on itself by left translation.

Example 4.4. *Suppose $\langle G, *, A\rangle$ is a soft group. For every $\alpha \in A$ and $a, x, y \in G$, define a soft mapping $\langle \mu, A\rangle$ from $G \times G$ to G as follows:*

$$\langle \alpha, a, x, y\rangle \in \mu \Leftrightarrow \langle \alpha, x, a^{-\alpha}, y\rangle \in *.$$

This is known as the soft action of G on itself by right translation

Example 4.5. *Suppose $\langle G, *, A\rangle$ is a soft group. Define a soft mapping $\langle \mu, A\rangle$ from $G \times G$ to G as follows:*

$$\langle \alpha, a, x, y\rangle \in \mu \Leftrightarrow \langle \alpha, a, x, z\rangle \in * \text{ and } \langle \alpha, z, a^{-\alpha}, y\rangle \in *,$$

for all $\alpha \in A$ and $a, x, y, z \in G$. This is known as the soft action of G on itself by conjugation.

Lemma 4.6. *Suppose that $\langle G, *, A\rangle$ is a soft group and $\langle H, A\rangle$ is a normal soft subgroup of G. Let X be the set G/H. Define a soft mapping $\langle \mu, A\rangle$ from $G \times X$ to X by:*

$$\langle \alpha, a, {}^{b}H, {}^{c}H\rangle \in \mu \Leftrightarrow \langle \alpha, a, b, y\rangle \in *,$$

for some $y \in G$ with ${}^{c}H = {}^{y}H$. Then, $\langle \mu, A\rangle$ is a soft action of G on X.

Proof. By Theorem 3.20, clearly $\langle \mu, A\rangle$ is well-defined. Let $\langle \alpha, e_\alpha, {}^{b}H, {}^{c}H\rangle \in \mu$. This implies that $\langle \alpha, e_\alpha, b, y\rangle \in *$ for some $y \in G$ with ${}^{c}H = {}^{y}H$. It follows that $b = y$. Therefore $\langle \alpha, e_\alpha, {}^{b}H, {}^{b}H\rangle \in \mu$. Let $\alpha \in A$, $a, b, c \in G$ and ${}^{x}H, {}^{y}H, {}^{z}H, {}^{u}H \in X$ such that $\langle \alpha, a, b, c\rangle \in *$, $\langle \alpha, c, {}^{x}H, {}^{u}H\rangle \in \mu$, $\langle \alpha, b, {}^{x}H, {}^{y}H\rangle \in \mu$ and $\langle \alpha, a, {}^{y}H, {}^{z}H\rangle \in \mu$. We need to show that ${}^{u}H = {}^{z}H$. From $\langle \alpha, c, {}^{x}H, {}^{u}H\rangle \in \mu$ we have $\langle \alpha, c, x, m\rangle \in *$ for some $m \in G$ with ${}^{u}H = {}^{m}H$. From $\langle \alpha, b, {}^{x}H, {}^{y}H\rangle \in \mu$, we have $\langle \alpha, b, x, n\rangle \in *$ for some $n \in G$ with ${}^{y}H = {}^{n}H$. Again from $\langle \alpha, a, {}^{y}H, {}^{z}H\rangle \in \mu$ we have $\langle \alpha, a, {}^{n}H, {}^{z}H\rangle \in \mu$ and hence $\langle \alpha, a, n, w\rangle \in *$ for some $w \in G$ with ${}^{z}H = {}^{w}H$. It follows from associative $\langle *, A\rangle$ that $m = w$. Thus ${}^{u}H = {}^{z}H$. Therefore $\langle \mu, A\rangle$ is a soft action of G on G/H. $\square$

Lemma 4.7. *Let G acts on X softly.*

(a) $\langle \alpha, g, x, y\rangle \in \mu \Rightarrow \langle \alpha, g^{-\alpha}, y, x\rangle \in \mu$ *for any* $\alpha \in A$, $x, y \in X$ *and* $g \in G$.

(b) For each $\alpha \in A$, $x, x', y \in X$ *and* $g \in G$, *if* $\langle \alpha, g, x, y\rangle \in \mu$ *and* $\langle \alpha, g, x', y\rangle \in \mu$, *then* $x = x'$.

Proof. (a) Let $\alpha \in A$, $g \in G$ and $x, y, z \in X$ such that $\langle \alpha, g, x, y\rangle \in \mu$ and $\langle \alpha, g^{-\alpha}, y, z\rangle \in \mu$. Consider $\langle \alpha, g, x, y\rangle \in \mu$, $\langle \alpha, g^{-\alpha}, y, z\rangle \in \mu$, $\langle \alpha, g^{-\alpha}, g, e_\alpha\rangle \in *$ and $\langle \alpha, e_\alpha, x, x\rangle \in \mu$. Since $\langle \mu, A\rangle$ is a soft action, $z = x$. Hence $\langle \alpha, g^{-\alpha}, y, x\rangle \in \mu$.

(b) Let $\alpha \in A$, $g \in G$ and $x, y, x' \in X$ such that $\langle \alpha, g, x, y\rangle \in \mu$ and $\langle \alpha, g, x', y\rangle \in \mu$. We need to show that $x = x'$. From $\langle \alpha, g, x, y\rangle \in \mu$ and $\langle \alpha, g, x', y\rangle \in \mu$ by (a) we have $\langle \alpha, g^{-\alpha}, y, x\rangle \in \mu$ and $\langle \alpha, g^{-\alpha}, y, x'\rangle \in \mu$. Therefore $x = x'$.

□

Theorem 4.8. *Suppose $\langle \mu, A\rangle$ is a soft action of G on X. Define a soft relation $\langle \theta, A\rangle$ on X by:*

$$\langle \alpha, x, y\rangle \in \theta \Leftrightarrow \langle \alpha, g, x, y\rangle \in \mu$$

for some $g \in G$. Then $\langle \theta, A\rangle$ is a soft equivalence relation on X.

Proof. Suppose $x, y, z \in X$ and $\alpha \in A$.

(i) As $\langle \alpha, e_\alpha, x, x\rangle \in \mu$, we have $\langle \alpha, x, x\rangle \in \theta$.

(ii) Let $\langle \alpha, x, y\rangle \in \theta$. Consequently, for some $g \in G$, $\langle \alpha, g, x, y\rangle \in \mu$. Then by Lemma 4.7 $\langle \alpha, g^{-\alpha}, y, x\rangle \in \mu$. Therefore $\langle \alpha, y, x\rangle \in \theta$.

(iii) Let $\langle \alpha, x, y\rangle \in \theta$ and $\langle \alpha, y, z\rangle \in \theta$. Then, $a, b \in G$ exists such that $\langle \alpha, a, x, y\rangle \in \mu$ and $\langle \alpha, b, y, z\rangle \in \mu$. Let $c \in G$ and $v \in X$ such that $\langle \alpha, b, a, c\rangle \in *$ and $\langle \alpha, c, x, v\rangle \in \mu$. Considering that $\langle \mu, A\rangle$ is a soft action of G on X, $v = z$. Hence $\langle \alpha, c, x, z\rangle \in \mu$. Therefore $\langle \alpha, x, z\rangle \in \theta$. Thus $\langle \theta, A\rangle$ is a soft equivalence relation on X.

□

Lemma 4.9. *Given a soft group G, define a soft mapping $\langle \mu, A\rangle$ from $G \times G$ to G by:*

$$\langle \alpha, a, x, y\rangle \in \mu \Leftrightarrow \langle \alpha, x, a, y\rangle \in *$$

for all $\alpha \in A$, $a, x, y \in G$. Then, $\langle \mu, A\rangle$ is a soft action of G on itself if and only if G is abelian.

Proof. Suppose $\langle \mu, A\rangle$ is a soft action of G on G. Let $a, b \in G$. Suppose $\langle \alpha, a, b, c\rangle \in *$ and $\langle \alpha, b, a, d\rangle \in *$ for some $c, d \in G$. We need to show that $c = d$. Now, let $x, y, z, w \in G$ such that $\langle \alpha, c, x, y\rangle \in \mu$, $\langle \alpha, b, x, z\rangle \in \mu$, $\langle \alpha, a, z, w\rangle \in \mu$. Thus $y = w$ because $\langle \mu, A\rangle$ is a soft action. From $\langle \alpha, b, x, z\rangle \in \mu$, we have $\langle \alpha, x, b, z\rangle \in *$ and from $\langle \alpha, a, z, w\rangle \in \mu$, we have $\langle \alpha, z, a, w\rangle \in *$. By associativity, $\langle \alpha, x, d, w\rangle \in *$. Moreover, for $\langle \alpha, c, x, y\rangle \in \mu$, we get $\langle \alpha, x, c, y\rangle \in *$. Since $y = w$ and by left cancellation, $c = d$. Consequently, G is Abelian. Conversely, assume that G is abelian. For any $x \in G$, since $\langle \alpha, x, e_\alpha, x\rangle \in *$, we have $\langle \alpha, e_\alpha, x, x\rangle \in \mu$. Now let $\alpha \in A$ and $a, b, c, x, y, z, u \in G$ such that $\langle \alpha, a, b, c\rangle \in *$, $\langle \alpha, c, x, u\rangle \in \mu$, $\langle \alpha, b, x, y\rangle \in \mu$ and $\langle \alpha, a, y, z\rangle \in \mu$. From $\langle \alpha, c, x, u\rangle \in \mu$ we have $\langle \alpha, x, c, u\rangle \in *$. Since G is abelian, $\langle \alpha, c, x, u\rangle \in *$. From $\langle \alpha, b, x, y\rangle \in \mu$ and $\langle \alpha, a, y, z\rangle \in \mu$ we get $\langle \alpha, x, b, y\rangle \in *$ and $\langle \alpha, y, a, z\rangle \in *$. Since G is abelian, $\langle \alpha, b, x, y\rangle \in *$ and $\langle \alpha, a, y, z\rangle \in *$. Consider $\langle \alpha, a, b, c\rangle \in *$, $\langle \alpha, c, x, u\rangle \in *$, $\langle \alpha, b, x, y\rangle \in *$ and $\langle \alpha, a, y, z\rangle \in *$. Since $\langle *, A\rangle$ is associative, $u = z$. Hence $\langle \mu, A\rangle$ is a soft action of G on itself. □

The second chapter notes that if $\langle G, *, A\rangle$ is a soft group, then the set $SE_A(G)$ of all soft elements of G is an ordinary group along with the induced binary operation $\bar{*}$, that can be used as a model to represent soft groups. We define the following in order to extend soft action to classical group action.

Definition 4.10. *Let $\langle \mu, A\rangle$ be a soft action of G on X. Define a map $\widehat{\mu} : SE_A(G) \times SE_A(X) \to SE_A(X)$ by*

$$\widehat{\mu}(\breve{a}, \breve{x}) = \breve{y} \text{ if and only if } \langle \alpha, \breve{a}(\alpha), \breve{x}(\alpha), \breve{y}(\alpha)\rangle \in \mu \quad for\ all\ \alpha \in A.$$

Theorem 4.11. *A soft mapping $\langle \mu, A\rangle$ from $G \times X$ to X is a soft action of G on X if and only if $\widehat{\mu}$ is an action of $SE_A(G)$ on $SE_A(X)$.*

Proof. Suppose $\langle \mu, A\rangle$ is a soft action of G on X. We need to show that $\widehat{\mu}$ is an action of $SE_A(G)$ on $SE_A(X)$. Let $\breve{x} \in SE_A(X)$. Since $\langle \alpha, \breve{e}(\alpha), \breve{x}(\alpha), \breve{x}(\alpha)\rangle \in \mu$ for each $\alpha \in A$, we have $\widehat{\mu}(\breve{e}, \breve{x}) = \breve{x}$. Let $\breve{a}$, $\breve{b}$, $\breve{c} \in SE_A(G)$, $\breve{x}$, $\breve{y}$, $\breve{z}, \breve{u} \in SE_A(X)$ such that $\breve{a} \bar{*} \breve{b} = \breve{c}$, $\widehat{\mu}(\breve{c}, \breve{x}) = \breve{u}$, $\widehat{\mu}(\breve{b}, \breve{x}) = \breve{y}$ and $\widehat{\mu}(\breve{a}, \breve{y}) = \breve{z}$. Thus, for each $\alpha \in A$, $\langle \alpha, \breve{a}(\alpha), \breve{b}(\alpha), \breve{c}(\alpha)\rangle \in *$, $\langle \alpha, \breve{c}(\alpha), \breve{x}(\alpha), \breve{u}(\alpha)\rangle \in \mu$, $\langle \alpha, \breve{b}(\alpha), \breve{x}(\alpha), \breve{y}(\alpha)\rangle \in \mu$ and $\langle \alpha, \breve{a}(\alpha), \breve{y}(\alpha), \breve{z}(\alpha)\rangle \in \mu$. Since $\langle \mu, A\rangle$ is a soft action of G on X, $\breve{u}(\alpha) = \breve{z}(\alpha)$ for all $\alpha \in A$. Hence $\breve{u} = \breve{z}$. Conversely, define the soft elements $\breve{x} \in SE_A(X)$ by for each $\lambda \in A$:

$$\breve{x}(\lambda) = \begin{cases} \{x\} & if \ \ \lambda = \alpha \\ \{x_\lambda\} & if \ \ \lambda \neq \alpha. \end{cases}$$

Since $\widehat{\mu}(\breve{e}, \breve{x}) = \breve{x}$, $\langle \lambda, \breve{e}(\lambda), \breve{x}(\lambda), \breve{x}(\lambda) \rangle \in \mu$ for all $\lambda \in A$. In particular, it works for $\lambda = \alpha$ and hence $\langle \alpha, \breve{e}(\alpha), x, x \rangle \in \mu$. Let $a, b, c \in G$, $x, y, z, u \in X$ and $\alpha \in A$ such that $\langle \alpha, a, b, c \rangle \in *$, $\langle \alpha, c, x, u \rangle \in \mu$, $\langle \alpha, b, x, y \rangle \in \mu$ and $\langle \alpha, a, y, z \rangle \in \mu$. We need to show that $z = u$. Consider soft elements $\breve{a}, \breve{b}, \breve{c}$ in $SE_A(G)$ defined as follows: $\breve{a}(\lambda) = \{a\}$, $\breve{b}(\lambda) = \{b\}$, $\breve{x}(\lambda) = \{x\}$ for all $\lambda \in A$ and $\breve{x}, \breve{c}, \breve{u}, \breve{u}$, in $SE_A(X)$.

$$\breve{c}(\lambda) = \begin{cases} \{c\} & if \ \ \lambda = \alpha \\ \{c_\lambda\} & if \ \ \lambda \neq \alpha \end{cases}$$

where $c_\lambda \in G$ with $\langle \lambda, a, b, c_\lambda \rangle \in *$.

$$\breve{u}(\lambda) = \begin{cases} \{u\} & if \ \ \lambda = \alpha \\ \{u_\lambda\} & if \ \ \lambda \neq \alpha \end{cases}$$

where $u_\lambda \in G$ with $\langle \lambda, c_\lambda, x, u_\lambda \rangle \in \mu$. Similarly, define

$$\breve{y}(\lambda) = \begin{cases} \{y\} & if \ \ \lambda = \alpha \\ \{y_\lambda\} & if \ \ \lambda \neq \alpha \end{cases}$$

where $y_\lambda \in G$ with $\langle \lambda, b, x_\lambda, y_\lambda \rangle \in \mu$.

$$\breve{z}(\lambda) = \begin{cases} \{z\} & if \ \ \lambda = \alpha \\ \{z_\lambda\} & if \ \ \lambda \neq \alpha \end{cases}$$

where $z_\lambda \in G$ with $\langle \lambda, a, y_\lambda, z_\lambda \rangle \in \mu$ for all $\lambda \neq \alpha$. Then we get $\breve{a} \breve{*} \breve{b} = \breve{c}$, $\widehat{\mu}(\breve{c}, \breve{x}) = \breve{u}$, $\widehat{\mu}(\breve{b}, \breve{x}) = \breve{y}$ and $\widehat{\mu}(\breve{a}, \breve{y}) = \breve{z}$. Since $\widehat{\mu}$ is an action of $SE_A(G)$ on $SE_A(X)$, $\breve{u} = \breve{z}$. This implies $\breve{u}(\lambda) = \breve{z}(\lambda)$ for all $\lambda \in A$. In particular it works for $\alpha = \lambda$ and hence which would give $z = u$. Consequently, $\langle \mu, A \rangle$ is a soft action of G on X. □

Lemma 4.12. *Let $\langle G, *, A \rangle$ be a soft group and X_1 and X_2 be a non empty sets, $\langle \mu_1, A \rangle$ and*

$\langle \mu_2, A\rangle$ be soft actions of G on X_1 and X_2 respectively. Define a soft mapping $\langle \mu_1 \times \mu_2, A\rangle$ from $G \times (X_1 \times X_2)$ to $X_1 \times X_2$ by:

$$\mu_1 \times \mu_2 = \{\langle \alpha, a, x, y\rangle : \alpha \in A, a \in G \text{ and } x, y \in X_1 \times X_2\}.$$

where $x = (x_1, x_2), y = (y_1, y_2)$, $\langle \alpha, a, x_1, y_1\rangle \in \mu_1$ and $\langle \alpha, a, x_2, y_2\rangle \in \mu_2$. Then $\langle \mu_1 \times \mu_2, A\rangle$ is a soft action of G on $X_1 \times X_2$.

4.2 Soft Orbits and Soft Stabilizers

Definition 4.13. *Let $\langle \mu, A\rangle$ be a soft action of G on X and $x \in X$. For any $\alpha \in A$, the soft orbit of x is a soft set $\langle SO_x, A\rangle$ over X defined by*

$$SO_x(\alpha) = \{y \in X : \langle \alpha, g, x, y\rangle \in \mu \; for \; some \; g \in G\}.$$

Definition 4.14. *Let $x \in X$ and $\langle \mu, A\rangle$ be soft actions of G on X. For any $\alpha \in A$, the soft stabilizer of x is a soft set $\langle SSt_x, A\rangle$ over G defined by*

$$SSt_x(\alpha) = \{a \in G : \langle \alpha, a, x, x\rangle \in \mu\}.$$

Example 4.15. *Let $\langle \mu, A\rangle$ be a trivial soft action of G on X. So that for any $x \in X$, we get $SO_x(\alpha) = \{x\}$ for all α in A, and the soft stabilizer is an absolute soft set over G.*

Example 4.16. *Consider a soft action of G on itself by conjugation. For $x \in G$, the soft orbit $\langle SO_x, A\rangle$ of x in G is*

$$\begin{aligned} SO_x(\alpha) &= \{y \in X : \langle \alpha, a, x, y\rangle \in \mu \; for \; some \; a \in G\} \\ &= \{y \in X : \langle \alpha, a, x, z\rangle \in * \;\; and \;\; \langle \alpha, z, a^{-\alpha}, y\rangle \in * \quad for \; some \; a \in G\} \end{aligned}$$

and the soft stabilizer is

$$\begin{aligned} SSt_x(\alpha) &= \{a \in G : \langle \alpha, a, x, x\rangle \in \mu\} \\ &= \{a \in G : \langle \alpha, a, x, z\rangle \in * \;\; and \;\; \langle \alpha, z, a^{-\alpha}, x\rangle \in * \; for \; some \; z \in G\} \\ &= \{a \in G : \langle \alpha, a, x, z\rangle \in * \;\; if \; and \; only \; if \;\; \langle \alpha, x, a, z\rangle \in * \; for \; some \; z \in G\}. \end{aligned}$$

Thus the soft stabilizer $\langle SSt_x, A\rangle$ is called the centralizer of x in G.

Theorem 4.17. *$\langle SSt_x, A\rangle$ is a soft subgroup of G.*

Proof. Let $\alpha \in A$ and $x \in X$. Since $\langle \alpha, e_\alpha, x, x\rangle \in \mu$, $e_\alpha \in SSt_x(\alpha)$. Let $a, b \in SSt_x(\alpha)$. Then we have $\langle \alpha, a, x, x\rangle \in \mu$ and $\langle \alpha, b, x, x\rangle \in \mu$. Let $c \in G$ and $y \in X$ such that $\langle \alpha, a, b, c\rangle \in *$ and $\langle \alpha, c, x, y\rangle \in \mu$. Since $\langle \mu, A\rangle$ is a soft action of G on X, $y = x$. Thus $\langle \alpha, c, x, x\rangle \in \mu$. Hence $c \in SSt_x(\alpha)$. Let $a \in SSt_x(\alpha)$. Then we have $\langle \alpha, a, x, x\rangle \in \mu$. By Lemma 4.7, it follows that $\langle \alpha, a^{-\alpha}, x, x\rangle \in \mu$. Hence $a^{-\alpha} \in SSt_x(\alpha)$. Therefore $\langle SSt_x, A\rangle$ is a soft subgroup of G.

□

Corollary 4.18. *Let Y be a subset of X and let G be a soft group acts on X. Suppose $\langle SSt_Y, A\rangle$ is a soft set over G defined as follows*

$$SSt_Y(\alpha) = \{a \in G : \langle \alpha, a, y, y\rangle \in \mu \ for\ all\ y \in Y\}$$

for each $\alpha \in A$. Then $\langle SSt_Y, A\rangle$ is a soft subgroup of G.

Proof. Since $\langle \alpha, e_\alpha, y, y\rangle \in \mu$ for all $y \in Y$, $e_\alpha \in SSt_Y(\alpha)$. Let $\alpha \in A$, $a, b \in SSt_Y(\alpha)$ and $c \in G$ such that $\langle \alpha, a^{-\alpha}, b, c\rangle \in *$. This implies that $\langle \alpha, a, y, y\rangle \in \mu$ and $\langle \alpha, b, y, y\rangle \in \mu$ for all $y \in Y$. From $\langle \alpha, a, y, y\rangle \in \mu$ we have $\langle \alpha, a^{-\alpha}, y, y\rangle \in \mu$. Since G acts on X softly, $\langle \alpha, c, y, y\rangle \in \mu$. Hence $c \in SSt_Y(\alpha)$. Therefore $\langle SSt_Y, A\rangle$ is a soft subgroup of G. □

Theorem 4.19. *If a soft group G acts on a set X, and $x, y \in X$, then either $\langle SO_x, A\rangle \breve{=} \langle SO_y, A\rangle$ or $\langle SO_x, A\rangle$ and $\langle SO_y, A\rangle$ are weakly disjoint. Moreover, the union of all soft orbits is an absulute soft set over X.*

Proof. Assume that $x, y \in X$ and the soft orbits $\langle SO_x, A\rangle$ and $\langle SO_y, A\rangle$ are not weakly disjoint. Then, for all $\alpha \in A$, $SO_x(\alpha) \cap SO_y(\alpha) \neq \emptyset$. Let $\alpha \in A$ and $z \in SO_x(\alpha) \cap SO_y(\alpha)$. Thus $\langle \alpha, a, x, z\rangle \in \mu$ and $\langle \alpha, b, y, z\rangle \in \mu$ for some $a, b \in G$. Let $c \in G$ such that $\langle \alpha, a^{-\alpha}, b, c\rangle \in *$. Then, it follows that $\langle \alpha, c, y, x\rangle \in \mu$ and $\langle \alpha, c^{-\alpha}, x, y\rangle \in \mu$. So $SO_x(\alpha) \subseteq SO_y(\alpha)$ and $SO_y(\alpha) \subseteq SO_x(\alpha)$. Therefore $SO_x(\alpha) = SO_y(\alpha)$ for all $\alpha \in A$. Thus $\langle SO_x, A\rangle \breve{=} \langle SO_y, A\rangle$. Clearly $SO_x(\alpha)$ is a subset of X. Let $x \in X$. Since $\langle \alpha, e_\alpha, x, x\rangle \in \mu$, $x \in SO_x(\alpha)$. Therefore $X = \bigcup_{x \in X} SO_x(\alpha)$ for all $x \in X$. Hence the union of all soft orbits is an absulute soft set over X.

□

Theorem 4.20. *Let $\langle \mu, A\rangle$ be a soft action of G on a set X, and let $a \in G$. Define a soft mapping $\langle f_\mu(a), A\rangle$ from X to X as follows :*

$$\langle \alpha, x, y\rangle \in f_\mu(a) \Leftrightarrow \langle \alpha, a, x, y\rangle \in \mu$$

for each $\alpha \in A$ and $x, y \in X$. Then, $\langle f_\mu(a), A\rangle$ is a bijection on X.

Proof. Clearly $\langle f_\mu(a), A\rangle$ is a soft mapping from X to X. Let $\alpha \in A$, $x_1, x_2, y \in X$ such that $\langle \alpha, x_1, y\rangle \in f_\mu(a)$ and $\langle \alpha, x_2, y\rangle \in f_\mu(a)$. It follows that $\langle \alpha, a, x_1, y\rangle \in \mu$ and $\langle \alpha, a, x_2, y\rangle \in \mu$. From $\langle \alpha, a, x_1, y\rangle \in \mu$ and $\langle \alpha, a, x_2, y\rangle \in \mu$ we have $\langle \alpha, a^{-\alpha}, y, x_1\rangle \in \mu$ and $\langle \alpha, a^{-\alpha}, y, x_2\rangle \in \mu$. Thus $x_1 = x_2$. Therefore $\langle f_\mu(a), A\rangle$ is injective. Also for any $y \in X$, we have $\langle \alpha, a^{-\alpha}, y, y_1\rangle \in \mu$ for some $y_1 \in X$. Thus by Lemma 4.7, $\langle \alpha, a, y_1, y\rangle \in \mu$. Thus $\langle \alpha, y_1, y\rangle \in f_\mu(a)$. Hence $\langle f_\mu(a), A\rangle$ is surjective. Therefore $\langle f_\mu(a), A\rangle$ is a bijection of X onto itself.

□

Theorem 4.21. *Let X be a non-empty set and G be a soft group.*

(a) A soft homomorphism from G to S_X, where $\langle S_X, \circ, A\rangle$ is a soft group and S_X is the set of all bijective maps from X to X, is induced if G acts on X softly.

(b) Any soft homomorphism from G to S_X induces a soft action of G on X.

Proof. (a) Let $\langle \mu, A\rangle$ be a soft action of G on X. Define a soft mapping $\langle \phi_\mu, A\rangle$ from G to S_X by:

$$\langle \alpha, a, f\rangle \in \phi_\mu \Leftrightarrow f = f_\mu(a)(\alpha).$$

Let $\alpha \in A$, $a, b, c \in G$ and $f_1, f_2, f_3 \in S_X$ Such that $\langle \alpha, a, f_1\rangle \in \phi_\mu$, $\langle \alpha, b, f_2\rangle \in \phi_\mu$, $\langle \alpha, c, f_3\rangle \in \phi_\mu$ and $\langle \alpha, a, b, c\rangle \in *$. It follows that for any $x \in X$.

$$f_1(x) = f_\mu(a)(\alpha)(x) = \mu(a, x)(\alpha)$$
$$f_2(x) = f_\mu(b)(\alpha)(x) = \mu(b, x)(\alpha)$$
$$f_3(x) = f_\mu(c)(\alpha)(x) = \mu(c, x)(\alpha).$$

Now, $f_3(x) = f_\mu(c)(\alpha)(x) = \mu(c, x)(\alpha) = \mu(a, \mu(b, x))(\alpha) = \mu(a, f_\mu(b)(x))(\alpha) = [f_\mu(a)(\alpha) \circ (f_\mu(b)(\alpha)](x)$. It follows that $\langle \alpha, f_1, f_2, f_3\rangle \in \circ$. Therefore $\langle \phi_\mu, A\rangle$ is a soft homomorphism from G to S_X.

(b) Let $\langle \phi, A\rangle$ be a soft homomorphism from G to S_x. Define a soft mapping $\langle \mu_\phi, A\rangle$ from $G \times X$ to X by:

$$\langle \alpha, a, x, y\rangle \in \mu_\phi \Leftrightarrow \phi(a)(x)(\alpha) = y.$$

We need to show that $\langle \mu_\phi, A\rangle$ is a soft action of G on X. Let $\alpha \in A$, $e_\alpha \in G$ and $x, y \in X$. Suppose $\langle \alpha, e_\alpha, x, y\rangle \in \mu_\phi$ this implies that $\phi(e_\alpha)(x)(\alpha) = y$. Since $\langle \phi, A\rangle$ is a soft homomorphism, $y = x$. Therefore $\langle \alpha, e_\alpha, x, x\rangle \in \mu$. Let $a, b, c \in G$ and $x, y, u, v \in X$ such that $\langle \alpha, a, b, c\rangle \in *$, $\langle \alpha, c, x, y\rangle \in \mu_\phi$, $\langle \alpha, b, x, u\rangle \in \mu_\phi$ and $\langle \alpha, a, u, v\rangle \in \mu_\phi$. It follows that $\phi(c)(x)(\alpha) = y$, $\phi(b)(x)(\alpha) = u$ and $\phi(a)(u)(\alpha) = v$. Now, $v = \phi(a)(u)(\alpha) = \phi(a)(\phi(b)(x)(\alpha)) = \phi(c)(x)(\alpha) = y$. Therefore $\langle \mu_\phi, A\rangle$ is a soft action of G on X.

□

Definition 4.22. *The soft homomorphism $\langle \phi_\mu, A\rangle$ from G to S_X described by Theorem 4.21 is known as the soft homomorphism associated with the soft action $\langle \mu, A\rangle$ of a soft group G on a set X. If $\langle \phi_\mu, A\rangle$ is injective, then $\langle \mu, A\rangle$ is referred to as an effective soft action of G on X.*

Note: kernel of the soft action $\langle \mu, A\rangle$ we mean the kernel of soft homomorphism $\langle \phi_\mu, A\rangle$, denoted by $\langle K_\mu, A\rangle$.

Lemma 4.23. *For any $\alpha \in A$, the soft action $\langle \mu, A\rangle$ of G on X is effective if and only if $K_\mu(\alpha) = \{e_\alpha\}$.*

Proof. $\langle \mu, A\rangle$ is effective if and only if $\langle \phi_\mu, A\rangle$ is soft monomorphism if and only if $K_{\phi_\mu}(\alpha) = \{e_\alpha\}$ for all $\alpha \in A$ if and only if $K_\mu(\alpha) = \{e_\alpha\}$ for all $\alpha \in A$. Therefore, a soft action $\langle \mu, A\rangle$ is effective if and only if $K_\mu(\alpha) = \{e_\alpha\}$ for all $\alpha \in A$. In other words, e_α is the only element in G such that, for any $\alpha \in A$ and $x \in X$, $\langle \alpha, e_\alpha, x, x\rangle \in \mu$. □

Corollary 4.24. *If $\langle \mu, A\rangle$ is the soft action of G on itself by left translation then $\langle \mu, A\rangle$ is soft effective.*

Proof. Let $\alpha \in A$. Now,

$$\begin{aligned} K_\mu(\alpha) &= \{a \in G : \langle \alpha, a, id_X\rangle \in \phi_\mu\} \\ &= \{a \in G : id_X(x) = f_\mu(a)(x)(\alpha) \ \ \forall x \in G\} \\ &= \{a \in G : \langle \alpha, a, x, x\rangle \in \mu \ \ \forall x \in G\} \\ &= \{a \in G : \langle \alpha, a, x, x\rangle \in * \ \ \forall x \in G\} \\ &= \{e_\alpha\}. \end{aligned}$$

That is, $K_\mu(\alpha) = \{e_\alpha\}$ for $\alpha \in A$. Therefore $\langle \mu, A\rangle$ is soft effective.

□

Corollary 4.25. *Let $\langle \mu, A\rangle$ is the soft action of G on itself by conjugation. Then $\langle \mu, A\rangle$ is effective if and only if the center of G contained only an identity element of G.*

Proof. Let $\alpha \in A$ *and* $a, x, z \in G$.

$$\begin{aligned} Z_A(G)(\alpha) &= \{a \in G : \langle \alpha, a, x, z\rangle \in * \Leftrightarrow \quad \langle \alpha, x, a, z\rangle \in * \ \ \forall x \in G\} \\ &= \{a \in G : \langle \alpha, a, x, z\rangle \in * \ and \ \langle \alpha, z, a^{-\alpha}, x\rangle \in * \ \forall x \in G\} \\ &= \{a \in G : \langle \alpha, a, x, x\rangle \in \mu\} \\ &= K_\mu(\alpha). \end{aligned}$$

Therefore, $\langle \mu, A\rangle$ is soft effective if and only if the center of $Z_A(G) = \{e_\alpha\}$ for all $\alpha \in A$.

□

Theorem 4.26. *Let $\langle \mu, A\rangle$ be a soft action of G on a set X and let $Y \subseteq X$. Define the soft mapping $\langle \mu', A\rangle$ from $G \times 2^x$ to 2^x by:*

$$\langle \alpha, a, Y, Z\rangle \in \mu' \Leftrightarrow Z = \{z \in X : \langle \alpha, a, x, z\rangle \in \mu \ \ for\ some\ x \in Y\}$$

then, $\langle \mu', A\rangle$ is a soft action of G on the power set of X and $\langle K_\mu, A\rangle \breve{\cong} \langle K_{\mu'}, A\rangle$.

Proof. Let $\alpha \in A$ and $Y \subseteq X$. If $\langle \alpha, e_\alpha, Y, Z\rangle \in \mu'$, then

$$\begin{aligned} Z &= \{z \in X : \langle \alpha, e_\alpha, x, z\rangle \in \mu \ \ x \in Y\} \\ &= \{z \in X : z = x \ \ x \in Y\} \\ &= Y. \end{aligned}$$

Hence $\langle \alpha, e_\alpha, Y, Y\rangle \in \mu'$, for any $Y \subseteq X$. Let $\alpha \in A$ and $a, b, c \in G$ such that $\langle \alpha, a, b, c\rangle \in *$. Suppose $\langle \alpha, a, Z_1, W\rangle \in \mu'$, $\langle \alpha, b, Y, Z_1\rangle \in \mu'$ and $\langle \alpha, c, Y, Z\rangle \in \mu'$. We need to show that $W = Z$. Let $z \in W$. Thus, $\langle \alpha, a, x_1, z\rangle \in \mu$ for some $x_1 \in Z_1$. But, $\langle \alpha, b, x, x_1\rangle \in \mu$, for some $x \in Y$. Since $\langle \mu, A\rangle$ is a soft action of G on X and $\langle \alpha, a, b, c\rangle \in *$ we have $\langle \alpha, c, x, z\rangle \in \mu$. This implies $z \in Z$. Hence $W \subseteq Z$. Let $z \in Z$. Thus, $\langle \alpha, c, x, z\rangle \in \mu$, for some $x \in Y$. Let $\langle \alpha, b, x, u\rangle \in \mu$. Thus, $u \in Z_1$. Let $\langle \alpha, a, u, v\rangle \in \mu$. Thus $v \in W$. But, since $\langle \mu, A\rangle$ is a soft action of G on X

and $\langle \alpha, a, b, c\rangle \in *$, we have $z = v$. It follows that $z \in W$. Hence, $Z \subseteq W$. Therefore $Z = W$. Consequently, $\langle \mu^{'}, A\rangle$ is a soft action of G on 2^x. In addition, for every $\alpha \in A$,

$$\begin{aligned} K_\mu(\alpha) &= \{a \in G : \langle \alpha, a, id_X\rangle \in \phi_\mu\} \\ &= \{a \in G : id_X(x) = f_\mu(a)(x)(\alpha) \ \ \forall\, x \in X\} \\ &= \{a \in G : \langle \alpha, a, x, x\rangle \in \mu \ \ \forall\, x \in X\} \\ &= \{a \in G : \langle \alpha, a, Y, Y\rangle \in \mu^{'} \ \ \forall\, Y \subseteq X\} \\ &= K_{\mu'}(\alpha). \end{aligned}$$

Therefore $\langle K_\mu, A\rangle \cong \langle K_{\mu'}, A\rangle$. In particular $\langle \mu^{'}, A\rangle$ is effective if and only if $\langle \mu, A\rangle$ is effective.

$\square$

Corollary 4.27. *The soft action of a soft group G on a set X is effective if and only if for any $a, b \in G$, $x \in X$, $\langle \alpha, a, x, y\rangle \in \mu$ and $\langle \alpha, b, x, y\rangle \in \mu$ imply $a = b$.*

Proof. It follows directly from Lemma 4.7. $\square$

Example 4.28. *Let X be the set of left cosets of $\langle H, A\rangle$ in G and let $\langle H, A\rangle$ be a soft subgroup of G. For every $\alpha \in A$, $a, b, c, d \in G$, define a soft mapping $\langle \mu, A\rangle$ from $G \times X$ to X as follows:*

$$\langle \alpha, a, {}^bH, {}^dH\rangle \in \mu \Leftrightarrow {}^dH = {}^yH,$$

where ${}^bH, {}^dH, {}^yH \in X$, $\langle \alpha, a, b, c\rangle \in *$ and $\langle \alpha, c, a^{-\alpha}, y\rangle \in *$. Then $\langle \mu, A\rangle$ is a soft action of G on X but $\langle \mu, A\rangle$ is not effective.

Lemma 4.29. *Let $\langle \mu, A\rangle$ represent a soft action of G on X. Consider the action $\widehat{\mu} : SE_A(G) \times SE_A(X) \to SE_A(X)$. For $x \in X$, consider the soft element $\breve{x} \in SE_A(X)$ defined by $\breve{x}(\alpha) = \{x\}$. Then we have the following :*

(1) $St_{\breve{x}} = \widehat{SSt_x}$.

(2) $O_{\breve{x}} = \widehat{SO_x}$.

Proof. We know that

$$St_{\breve{x}} = \{\breve{a} \in SE_A(G) : \widehat{\mu}(\breve{a}, \breve{x}) = \breve{x}\}$$

and

$$\widehat{SSt_x} = \{\breve{a} \in SE_A(G) : \breve{a}(\alpha) \subseteq SSt_x(\alpha) \ \ for\ all\ \alpha \in A\}.$$

Claim I: $St_{\breve{x}} = \widehat{SSt_x}$.

$$\begin{aligned}
(\subseteq)\ \ Let\ \ \breve{a} \in St_{\breve{x}} \ &\Rightarrow\ \widehat{\mu}(\breve{a}, \breve{x}) = \breve{x} \\
&\Rightarrow\ \langle \alpha, \breve{a}(\alpha), \breve{x}(\alpha), \breve{x}(\alpha)\rangle \in \mu \quad for\ all\ \alpha \in A \\
&\Rightarrow\ \langle \alpha, \breve{a}(\alpha), x, x\rangle \in \mu \quad for\ all\ \alpha \in A \\
&\Rightarrow\ \breve{a}(\alpha) \subseteq SSt_x(\alpha) \quad for\ all\ \alpha \in A \\
&\Rightarrow\ \breve{a} \in \widehat{SSt_x}.
\end{aligned}$$

$$\begin{aligned}
(\supseteq)\ \ Let\ \ \breve{a} \in \widehat{SSt_x} \ &\Rightarrow\ \breve{a}(\alpha) \subseteq SSt_x(\alpha) \quad for\ all\ \alpha \in A \\
&\Rightarrow\ \langle \alpha, \breve{a}(\alpha), x, x\rangle \in \mu \quad for\ all\ \alpha \in A \\
&\Rightarrow\ \langle \alpha, \breve{a}(\alpha), \breve{x}(\alpha), \breve{x}(\alpha)\rangle \in \mu \quad for\ all\ \alpha \in A \\
&\Rightarrow\ \widehat{\mu}(\breve{a}, \breve{x}) = \breve{x} \\
&\Rightarrow\ \breve{a} \in St_{\breve{x}}.
\end{aligned}$$

Therefore $St_{\breve{x}} = \widehat{SSt_x}$. We also know that

$$O_{\breve{x}} = \{\breve{y} \in SE_A(X) : \widehat{\mu}(\breve{a}, \breve{x}) = \breve{y} \ \ for\ some\ \breve{a} \in SE_A(G)\}$$

and

$$\widehat{SO_x} = \{\breve{y} \in SE_A(X) : \breve{y}(\alpha) \subseteq SO_x(\alpha) \ \ for\ all\ \alpha \in A\}.$$

Claim II: $O_{\breve{x}} = \widehat{SO_x}$.

$$
\begin{aligned}
(\subseteq)\ \ Let\ \ \breve{y} \in O_{\breve{x}} \ &\Rightarrow\ \widehat{\mu}(\breve{a}, \breve{x}) = \breve{y}\ \ for\ some\ \breve{a} \in SE_A(G) \\
&\Rightarrow\ \langle \alpha, \breve{a}(\alpha), \breve{x}(\alpha), \breve{y}(\alpha)\rangle \in \mu \quad for\ all\ \alpha \in A \\
&\Rightarrow\ \langle \alpha, \breve{a}(\alpha), x, \breve{y}(\alpha)\rangle \in \mu \quad for\ all\ \alpha \in A \\
&\Rightarrow\ \breve{y}(\alpha) \subseteq SO_x(\alpha)\ \ for\ all\ \alpha \in A \\
&\Rightarrow\ \breve{y} \in \widehat{SO_x}.
\end{aligned}
$$

($\supseteq$) Let $\breve{y} \in \widehat{SO_x}$. Then $\breve{y}(\alpha) \subseteq SO_x(\alpha)$ *for all* $\alpha \in A$. It follows that for each $\alpha \in A$, $\langle \alpha, a_\alpha, x, \breve{y}(\alpha)\rangle \in \mu$ for some $a_\alpha \in G$. Consider the soft elements $\breve{a} \in SE_A(G)$ defined by $\breve{a}(\alpha) = \{a_\alpha\}$. Now, $\widehat{\mu}(\breve{a}, \breve{x}) = \breve{y}$. This implies that $\breve{y} \in O_{\breve{x}}$. Therefore $O_{\breve{x}} = \widehat{SO_x}$.

□

Lemma 4.30. *Suppose that $\langle \mu, A\rangle$ is a soft action of G on X. $\langle \mu, A\rangle$ is an effective soft action of G on X if and only if $\widehat{\mu}$ is an effective action of $SE_A(G)$ on $SE_A(X)$.*

Proof. $\langle \mu, A\rangle$ is effective if and only if $\langle \phi_\mu, A\rangle$ is injective if and only if $\phi_{\widehat{\mu}}$ is injective if and only if $\widehat{\mu}$ is effective. □

4.3 Transitive Soft Action

Definition 4.31. *If there is just one soft orbit in X, the soft action of G on a set X is said to be transitive. That is, $SO_x(\alpha) = X$ for every $\alpha \in A$, $x \in X$. In other words, for every $x, y \in X$, $\langle SO_x, A\rangle \,\breve{=}\, \langle SO_y, A\rangle$. In this case, we say that G acts transitively on X.*

Lemma 4.32. *For any pair of elements $x, y \in X$, a soft action $\langle \mu, A\rangle$ of G on X is transitive if and only if for every $\alpha \in A$ there is an element a in G such that $\langle \alpha, a, x, y\rangle \in \mu$.*

Proof. Suppose $\langle \mu, A\rangle$ is transitive. Let $x, y \in X$ and $\alpha \in A$. From $SO_x(\alpha) = X$, $y \in SO_x(\alpha)$. It follows that $a \in G$ such that $\langle \alpha, a, x, y\rangle \in \mu$. Conversely, let $x \in X$ and $\alpha \in A$. Suppose $y \in X$. We need to show that $y \in SO_x(\alpha)$. According to the hypothesis, $a \in G$ exists such that $\langle \alpha, a, x, y\rangle \in \mu$. It follows that $y \in SO_x(\alpha)$. Thus $X \subseteq SO_x(\alpha)$ and hence $SO_x(\alpha) = X$ for all $\alpha \in A$. Therefore $\langle \mu, A\rangle$ is transitive. □

Lemma 4.33. *If $\langle \mu, A\rangle$ is a soft action of G on itself by left(right) translation then $\langle \mu, A\rangle$ is transitive.*

Proof. Let $\alpha \in A$ and $x, y \in G$. Then $\langle \alpha, y, x^{-\alpha}, z\rangle \in *$ for some $z \in G$. It follows that $\langle \alpha, z, x, y\rangle \in *$. By left translation this implies that $\langle \alpha, z, x, y\rangle \in \mu$. Therefore by Lemma 4.32, $\langle \mu, A\rangle$ is transitive. Similarly if $\langle \alpha, y^{-\alpha}, x, z\rangle \in *$ for some $z \in G$. Hence $\langle \alpha, x, z^{-\alpha}, y\rangle \in *$. By right translation it follows that $\langle \alpha, z, x, y\rangle \in \mu$. Therefore again by the Lemma 4.32, $\langle \mu, A\rangle$ is transitive. □

Definition 4.34. *Let $\langle G, *, A$ be a soft group and let $\langle \mu, A\rangle$ and $\langle \mu^{'}, A\rangle$ be represented soft actions of G on X_1 and X_2 respectively. Then these two soft actions are said to be equivalent if there exists a bijective soft mapping $\langle f, A\rangle$ from X_1 to X_2 such that for each $\alpha \in A$, $a \in G$, $x_1, y_1 \in X$ and $x_2, y_2, z_2 \in X_2$,*

$$\langle \alpha, x_1, x_2\rangle \in f,\ \langle \alpha, a, x_2, z_2\rangle \in \mu',\ \langle \alpha, a, x_1, y_1\rangle \in \mu \text{ and } \langle \alpha, y_1, y_2\rangle \in f \text{ imply } y_2 = z_2.$$

Lemma 4.35. *Let $\langle \mu, A\rangle$ be a soft action of G on itself by left translation, and $\langle \mu^{'}, A\rangle$ be a soft action of G on itself by right translation.Then, $\langle \mu, A\rangle$ and $\langle \mu^{'}, A\rangle$ are equivalent soft actions.*

Proof. Suppose $\langle \mu, A\rangle$ and $\langle \mu^{'}, A\rangle$ are soft actions of G on itself by left translation and right translation respectively. Let $\alpha \in A$ and $a, x \in G$. Define a soft mapping $\langle f, A\rangle$ from G to G by $\langle \alpha, x, x^{-\alpha}\rangle \in f$. Consequently, it is evident that $\langle f, A\rangle$ is a bijective on G. Let $y, z \in G$ such that $\langle \alpha, a, x, y\rangle \in \mu$ and $\langle \alpha, y, z\rangle \in f$. Now, $\langle \alpha, y, z\rangle \in f \Leftrightarrow z = y^{-\alpha}$. From $\langle \alpha, a, x, y\rangle \in *$ we have $\langle \alpha, x^{-\alpha}, a^{-\alpha}, z\rangle \in *$. This implies that $\langle \alpha, a, x^{-\alpha}, z\rangle \in \mu^{'}$. Therefore the soft actions $\langle \mu, A\rangle$ and $\langle \mu^{'}, A\rangle$ are equivalent.

□

Theorem 4.36. *Given a soft group G acts transitively on a set X, $x \in X$ and $\langle H, A\rangle \,\breve{\cong}\, \langle SSt_x, A\rangle$. Then the soft action of G on X is equivalent to the soft action of G on the set of left cosets of $\langle H, A\rangle$ in G by left translation.*

Proof. Let Y be the collection of left cosets of $\langle H, A\rangle$ in G. and $\langle \mu, A\rangle$ be a specified transitive action on X. Let $\langle \mu^{'}, A\rangle$ represent the left translation soft action of G on Y. Then $\langle \mu^{'}, A\rangle$ is a soft mapping from $G \times Y$ to Y defined by $\langle \alpha, a, {}^{b}H, {}^{y}H\rangle \in \mu^{'}$ where $\langle \alpha, a, b, y\rangle \in *$ for some $y \in G$, for all $a, b \in G$ and ${}^{b}H, {}^{y}H \in Y$. Since the soft action $\langle \mu, A\rangle$ is transitive, we get that the soft orbits $SO_x(\alpha) = X$ for all $\alpha \in A$, $x \in X$. Define a soft mapping $\langle f, A\rangle$ from X to Y by $\langle \alpha, z_1, {}^{a}H\rangle \in f$ where $\langle \alpha, a, x, z_1\rangle \in \mu$. Now, for any $a \in G$ and $y, y_1 \in X$ with $\langle \alpha, a, y, y_1\rangle \in \mu$, choose $b \in G$ such that $\langle \alpha, b, x, y\rangle \in \mu$. Then we have

$$\begin{aligned}\langle \alpha, y_1, z\rangle \in f \ and \ \langle \alpha, a, y, y_1\rangle \in \mu \ &\Leftrightarrow \ z = {}^{d}H \ \ where \ \langle \alpha, a, b, d\rangle \in * \ \ for\ some \ \ d \in G \\ &\Leftrightarrow \ \langle \alpha, a, {}^{b}H, z\rangle \in \mu^{'} \\ &\Leftrightarrow \ \langle \alpha, a, z_1, z\rangle \in \mu^{'} \ \ and \ \ \langle \alpha, y, z_1\rangle \in f \ \ for\ some \ \ z_1 \in Y.\end{aligned}$$

Thus the soft actions $\langle \mu, A\rangle$ and $\langle \mu^{'}, A\rangle$ are equivalent.

□

Definition 4.37. *Let $\langle \mu, A\rangle$ be a soft action of G on X. A soft equivalence relation $\langle \psi, A\rangle$ on X is said to be compatible with the soft action $\langle \mu, A\rangle$ if for any $\alpha \in A$, x_1,x_2,y_1,$y_2 \in X$ and $a \in G$ with $\langle \alpha, a, x_1, y_1\rangle \in \mu$ and $\langle \alpha, a, x_2, y_2\rangle \in \mu$,*

$$\langle \alpha, x_1, x_2\rangle \in \psi \Rightarrow \langle \alpha, y_1, y_2\rangle \in \psi.$$

Example 4.38. *The absolute soft relation on X and the soft diagonal $\langle \Delta, A\rangle$ (i.e for every $\alpha \in A$, and $x, y \in X$, $\langle \alpha, x, y\rangle \in \Delta \Leftrightarrow x = y$) are soft equivalence relation on X that are compatible with every soft action of G on X.*

Definition 4.39. *A soft action $\langle \mu, A\rangle$ of G on X is referred to as primitive if the only soft equivalence relations on X that are compatible with the soft action $\langle \mu, A\rangle$ are $\langle 1_{X\times X}, A\rangle$ and $\langle \Delta, A\rangle$. The term imprimitive refers to a soft action that is not primitive.*

Theorem 4.40. *Let $\langle \mu, A\rangle$ and $\langle \mu^{'}, A\rangle$ be equivalent soft actions of G on X_1 and X_2 respectively. In such case, $\langle \mu, A\rangle$ is primitive if and only if $\langle \mu^{'}, A\rangle$ is primitive.*

Proof. Let the equivalent actions of G be $\langle \mu, A\rangle$ on X_1 and $\langle \mu^{'}, A\rangle$ on X_1 then there is a bijective soft mapping from X_1 to X_2 such that for each $\alpha \in A$, $a \in G$, $x_1, y_1 \in X$ and $x_2, y_2, z_2 \in X_2$; $\langle \alpha, x_1, x_2\rangle \in f$, $\langle \alpha, a, x_2, z_2\rangle \in \mu'$, $\langle \alpha, a, x_1, y_1\rangle \in \mu$ and $\langle \alpha, y_1, y_2\rangle \in f \Rightarrow y_2 = z_2$. Let $\langle \psi_1, A\rangle$ be a soft equivalence relation on X_1. Define a soft relation $\langle \psi_2, A\rangle$ on X_2 by

$$\langle \alpha, x_2, y_2\rangle \in \psi_2 \Leftrightarrow \langle \alpha, x_1, y_1\rangle \in \psi_1,$$

where $x_1, y_1 \in X_1$ and $x_2, y_2 \in X_2$ with $\langle \alpha, x_1, x_2\rangle \in f$ and $\langle \alpha, y_1, y_2\rangle \in f$.

Claim: $\langle \psi_2, A\rangle$ is a soft equivalence relation on X_2.

(i) Let $x_1 \in X_1$. Then there exists $x_2 \in X_2$ such that $\langle \alpha, x_1, x_2 \rangle \in f$. Since $\langle \psi_1, A \rangle$ is an equivalence relation on X_1, $\langle \alpha, x_1, x_1 \rangle \in \psi_1$. This implies $\langle \alpha, x_2, x_2 \rangle \in \psi_2$. Therefore $\langle \psi_2, A \rangle$ is reflective.

(ii) Suppose $x_2, y_2 \in X_2$ such that $\langle \alpha, x_2, y_2 \rangle \in \psi_2$.

$$\begin{aligned} &\Rightarrow \langle \alpha, x_1, y_1 \rangle \in \psi_1 \\ &\Rightarrow \langle \alpha, y_1, x_1 \rangle \in \psi_1 \\ &\Rightarrow \langle \alpha, y_2, x_2 \rangle \in \psi_2. \end{aligned}$$

Therefore $\langle \psi_2, A \rangle$ is symmetric.

(iii) Suppose $x_2, y_2, z_2 \in X_2$ such that $\langle \alpha, x_2, y_2 \rangle \in \psi_2$ and $\langle \alpha, y_2, z_2 \rangle \in \psi_2$

$$\begin{aligned} &\Rightarrow \langle \alpha, x_1, y_1 \rangle \in \psi_1 \textit{ and } \langle \alpha, y_1, z_1 \rangle \in \psi_1 \\ &\Rightarrow \langle \alpha, x_1, z_1 \rangle \in \psi_1 \\ &\Rightarrow \langle \alpha, x_2, z_2 \rangle \in \psi_2. \end{aligned}$$

Therefore $\langle \psi_2, A \rangle$ is transitive. Hence $\langle \psi_2, A \rangle$ is a soft equivalence relation on X_2. Using the soft inverse of $\langle f, A \rangle$ we can show that there is a one to one correspondence between soft equivalence relation on X_1 and soft equivalence relation on X_2. Suppose $\langle \psi_1, A \rangle$ soft equivalence relation on X_1 which is compatible with $\langle \mu, A \rangle$. Let $\langle \psi_2, A \rangle$ be the corresponding soft equivalence relation on X_2. Suppose $\langle \psi_1, A \rangle \breve{=} \langle 1_{x_1 \times x_1}, A \rangle$. Then there exist $x_1, y_1 \in X_1$ such that $\langle \alpha, x_1, x_2 \rangle \in f$ and $\langle \alpha, y_1, y_2 \rangle \in f$. Since $\langle \alpha, x_1, y_1 \rangle \in \psi_1$, we have $\langle \alpha, x_2, y_2 \rangle \in \psi_2$. Therefore $\langle \psi_2, A \rangle \breve{=} \langle 1_{x_2 \times x_2}, A \rangle$. Suppose $\langle \psi_1, A \rangle \breve{=} \langle \Delta, A \rangle$. Let $x_2, y_2 \in X_2$ such that $\langle \alpha, x_2, y_2 \rangle \in \psi_2$. Then, $\langle \alpha, x_1, y_1 \rangle \in \psi_1$. This implies $\langle \alpha, x_1, y_1 \rangle \in \Delta$. It follows that $x_1 = y_1$. Hence $x_2 = y_2$. Therefore $\langle \psi_2, A \rangle \breve{=} \langle \Delta, A \rangle$. Then there is one to one correspondence between compatible soft equivalent relation. Hence, the theorem.

□

Theorem 4.41. *Let $\langle \mu, A \rangle$ represent a soft group G acts on a set X. Then, $\langle \mu, A \rangle$ is soft imprimitive if and only if $|Y| > 1$ is a proper subset Y of X such that, for any $a \in G$,*

$\langle \alpha, a, Y, Z\rangle \in \mu$, *either* $Z = Y$ *or* $Z \cap Y = \emptyset$ *where* $Z = \{z_1 \in X : \langle \alpha, a, y, z_1\rangle \in \mu \; for \; some \; y \in Y\}$.

Proof. Assume that $\langle \mu, A\rangle$ is imprimitive. Then, for some $\alpha \in A$, there exists a soft equivalence relation $\langle \psi, A\rangle$ on X that is compatible with $\langle \mu, A\rangle$ such that $\psi(\alpha) \neq \Delta(\alpha)$ and $\psi(\alpha) \neq 1_{X\times X}(\alpha)$. If $x \neq y \in X$ then $\langle \alpha, x, y\rangle \in \psi$. Let $Y = \{z \in X : \langle \alpha, x, z\rangle \in \psi\}$. Considering that $x \neq y \in Y$, $|Y| > 1$. Additionally, given that $\psi(\alpha) \neq 1_{X\times X}(\alpha)$ for some $\alpha \in A$, Y is a proper subset of X. Now let $a \in G$ such that $Z \cap Y \neq \emptyset$ with $\langle \alpha, a, Y, Z\rangle \in \mu$. Then choose an element $z_1 \in Y$ such that $\langle \alpha, a, z_1, z_2\rangle \in \mu$ for some $z_2 \in Y$. Thus $\langle \alpha, x, z_2\rangle \in \psi$. Let $\langle \alpha, a, x, u\rangle \in \mu$ for some $u \in X$. Since $\langle \alpha, x, z_1\rangle \in \psi$ and $\langle \psi, A\rangle$ is soft compatible with $\langle \mu, A\rangle$, we get $\langle \alpha, u, z_2\rangle \in \psi$ which implies that $\langle \alpha, x, u\rangle \in \psi$. Currently, it is simple to verified that $\langle \alpha, a, Y, Y\rangle \in \mu$. Conversely, let Y be a proper subset of X such that $|Y| > 1$ and that $Z \cap Y = \emptyset$ or $Z = Y$, where $\langle \alpha, a, Y, Z\rangle \in \mu$. Then for any $a, b \in G$ either $Z = Z_1$ or $Z \cap Z_1 = \emptyset$ where $\langle \alpha, b, Y, Z_1\rangle \in \mu$, $\langle \alpha, a, Y, Z\rangle \in \mu$. Put $M = X - (\bigcup_{a\in G} Z)$ where $\langle \alpha, a, Y, Z\rangle \in \mu$. Consequently, the soft action $\langle \mu, A\rangle$ is compatible with the equivalent soft equivalence relation (ψ, A) on X. Since $\langle \alpha, e_\alpha, Y, Y\rangle \in \mu$ and $Y \neq X$, $\psi(\alpha) \neq 1_{X\times X}(\alpha)$ for some $\alpha \in A$. Also since $|Y| > 1$, $\psi(\alpha) \neq \Delta(\alpha)$ for some $\alpha \in A$. Thus $\langle \mu, A\rangle$ is imprimitive.

□

Lemma 4.42. *Let* $\langle \mu, A\rangle$ *be a soft action of* G *on a set* X. *Define a soft set* $\langle \psi_\mu, A\rangle$ *over* X *as follows :*

$$\psi_\mu(\alpha) = \{\langle x, y\rangle \in 1_{X\times X}(\alpha) : \langle \alpha, a, x, y\rangle \in \mu \; for \; some \; a \in G\}.$$

Consequently, $\langle \psi_\mu, A\rangle$ *is a soft equivalence relation on* X *that is compatible with the soft action* $\langle \mu, A\rangle$.

Proof. Observe that

$$\langle x, y\rangle \in \psi_\mu(\alpha) \Leftrightarrow y \in SO_x(\alpha), \; the \; soft \; orbit \; of \; x.$$

$$\Leftrightarrow x \in SO_y(\alpha), \; the \; soft \; orbit \; of \; y.$$

Therefore $\langle \psi_\mu, A\rangle$ is the soft equivalence relation consisting the soft orbits of elements of X.

Then, for every $\alpha \in A$, $a \in G$ and $x, y, x_1, y_1 \in X$:

$$\begin{aligned}
\langle x, y\rangle \in \psi_\mu(\alpha) &\Leftrightarrow \langle \alpha, b, x, y\rangle \in \mu \;\; for \;\; some \;\; b \in G \\
&\Leftrightarrow \langle \alpha, b, x_1, y\rangle \in \mu \;\; and \;\; \langle \alpha, a, x, x_1\rangle \in \mu \\
&\Leftrightarrow \langle \alpha, c, x, y\rangle \in \mu \;\; and \;\; \langle \alpha, a, b, c\rangle \in * \\
&\Leftrightarrow \langle \alpha, a, y, y\rangle \in \mu \;\; for \;\; some \;\; a \in G \\
&\Leftrightarrow \langle x_1, y_1\rangle \in \psi_\mu(\alpha) \;\; where \; \langle \alpha, a, x, x_1\rangle \in \mu \; and \; \langle \alpha, a, y, y_1\rangle \in \mu.
\end{aligned}$$

Therefore $\langle \psi_\mu, A\rangle$ is a soft compatible with the soft action $\langle \mu, A\rangle$.

□

Corollary 4.43. *If G on X has a primitive soft action $\langle \mu, A\rangle$, then either $\langle \mu, A\rangle$ is transitive or $\langle \mu, A\rangle$ is trivial (i.e. $\langle \alpha, a, x, x\rangle \in \mu$ for all $\alpha \in A$, $a \in G$ and $x \in X$).*

Proof. Since $\langle \mu, A\rangle$ is primitive, $\psi_\mu(\alpha) = \Delta(\alpha)$ or $\psi_\mu(\alpha) = 1_{X\times X}(\alpha)$ for all $\alpha \in A$. Then there is just one soft orbit or all soft orbits are singleton sets, this implies that either $\langle \mu, A\rangle$ is transitive or trivial. □

Note : A nontrivial primitive soft action must be transitive in particular, hence the class of nontrivial primitive soft actions of a soft group G on a set X is a subclass of the transitive soft actions of G on X. However, a transitive soft action does not always have to be primitive.This is verified by the example that follows.

Example 4.44. *Let $\langle H, A\rangle$ be a nontrivial proper soft subgroup of G, and let $\langle \mu, A\rangle$ be a soft action of G on itself by left translation. Define a soft relation $\langle \phi, A\rangle$ on G by $\langle \alpha, x, y\rangle \in \phi \Leftrightarrow z \in H(\alpha)$ where $\langle \alpha, x^{-\alpha}, y, z\rangle \in *$. Then, $\langle \mu, A\rangle$ is transitive but not primitive.*

Definition 4.45. *A soft group G acts on a set X is said to be doubly transitive if, for each $\alpha \in A$ and $x_1, x_2, y_1, y_2 \in X$, there exists $a \in G$ such that $\langle \alpha, a, x_1, y_1\rangle \in \mu$ and $\langle \alpha, a, x_2, y_2\rangle \in \mu$.*

Theorem 4.46. *Any doubly transitive soft action is primitive.*

Proof. Let the soft action $\langle \mu, A\rangle$ of G on X is doubly transitive and $\langle \phi, A\rangle$ be soft equivalence relation on X which is compatible with $\langle \mu, A\rangle$. If $\phi(\alpha) \neq \Delta(\alpha)$ for some $\alpha \in A$, then there exists $x \neq y \in X$ such that $\langle \alpha, x, y\rangle \in \phi$ and now, for any $z \in X$, there exists $a \in G$ such that $\langle \alpha, a, x, x\rangle \in \mu$ and $\langle \alpha, a, y, z\rangle \in \mu$ and hence $\langle \alpha, x, z\rangle \in \phi$. This proves that $\phi(\alpha) = 1_{X\times X}(\alpha)$ for all $\alpha \in A$. Thus, the soft action is primitive.

□

For each $z \in X$, there exists $a \in G$ such that $\langle \alpha, a, x, x \rangle \in \mu$ and $\langle \alpha, a, y, z \rangle \in \mu$. If $\phi(\alpha) \neq \Delta(\alpha)$ for some $\alpha \in A$, then there exists $x \neq y \in X$ such that $\langle \alpha, x, x, y \rangle \in \phi$.

Remark 4.47. *The converse of Theorem 4.46, is not true. This is verified by the example that follows.*

Example 4.48. *Consider the soft binary operation $\langle *, \mathbf{N} \rangle$ on G, which is defined as follows:*

$$\langle \alpha, a, b, c \rangle \in * \Leftrightarrow c = \alpha + a + b,$$

and Take $G = \mathbf{Z}$ and $H = \mathbf{pZ}$ for some prime p. For example, let $\langle H, A \rangle$ be a proper nontrivial normal soft subgroup of a group G such that $\langle H, A \rangle$ is a maximal soft subgroup of G. Let $\langle \mu, A \rangle$ represent the left translation soft action of G on G/H. Then $\langle \mu, A \rangle$ is primitive, it is not doubly transitive.

Chapter 5

Fuzzy Soft Subgroups of a Soft Group

Uncertainty is a common phenomenon in our daily life. To exceed these uncertainties, many hybrid structures involving soft sets were proposed. Some of them are the following: fuzzy soft sets (Maji et al. 2001), soft fuzzy set (Xu et al. 2010), fuzzy N-soft sets (Akram and Adeel 2018), intuitionistic fuzzy soft set (Xu et al. 2010), hesitant fuzzy soft sets (Wang et al. 2014), rough soft sets (Roy and Bera 2015) and so on. Numerous scholars have been working to expand the ideas of abstract algebra within the context of fuzzy environments. Rosenfeld developed the idea of fuzzy subgroups of a group and presented fuzzy sets in the framework of group theory in 1971 [33]. All of these theories, however, have intrinsic problems, as Molodtsov noted in 1999 [29].

In this chapter, we present fuzzy soft subgroups of a soft group, which are a generalization of soft groups that are discussed in [39, 40], and examine their different characteristics. Furthermore, we demonstrate that a fuzzy soft subgroup's image and inverse image under soft homomorphisms are also fuzzy soft subgroups. Lastly, the concept of normal fuzzy soft subgroups of a soft group is presented.

Throughout this chapter, unless and otherwise it is mentioned G denotes the soft group $\langle G, *, A\rangle$ and I denotes the closed interval [0, 1].

5.1 Fuzzy Soft Subgroups

Definition 5.1. *Let $\langle G, *, A\rangle$ be a soft group. A fuzzy soft set $\langle \mu, A\rangle$ over G is said to be a fuzzy soft subgroup of G if for each $\alpha \in A$ and $x, y, z \in G$ and $r, s, t \in I$:*

$\langle FSG1\rangle$ $\langle \alpha, e_\alpha, 1\rangle \in \mu$;

$\langle FSG2\rangle$ $\langle \alpha, x, r\rangle \in \mu$ *and* $\langle \alpha, x^{-\alpha}, s\rangle \in \mu$ *imply* $s \geq r$ *and*

$\langle FSG3\rangle$ $\langle \alpha, x, y, z\rangle \in *$, $\langle \alpha, x, r\rangle \in \mu$, $\langle \alpha, y, s\rangle \in \mu$ *and* $\langle \alpha, z, t\rangle \in \mu$ *all together imply* $t \geq min \{r, s\}$.

Example 5.2. *Let $G = \{1, -1, i, -i\}$ and $A = \{1, -1\}$. Define $* \subseteq A \times G \times G \times G$ by*

$$\langle \alpha, a, b, c\rangle \in * \Leftrightarrow c = \alpha ab.$$

The fuzzy soft set $\langle \mu, A\rangle$ from G to I is again defined by $\langle \alpha, 1, 1\rangle \in \mu$, $\langle \alpha, -1, 1\rangle \in \mu$, $\langle \alpha, i, 0.3\rangle \in \mu$ and $\langle \alpha, -i, 0.3\rangle \in \mu$. In such case, $\langle \mu, A\rangle$ is a fuzzy soft subgroup of G.

Lemma 5.3. *Suppose that G is a soft group and $\langle \mu, A\rangle$ is a fuzzy soft subgroup of G. For every $\alpha \in A$, $x \in G$ and $r, s \in I$, $\langle \alpha, x, r\rangle \in \mu$ and $\langle \alpha, x^{-\alpha}, s\rangle \in \mu$ imply $r = s$.*

Proof. Let $\alpha \in A$, $x \in G$ and $r, s \in I$. $\langle \alpha, x, r\rangle \in \mu$ if and only if $\langle \alpha, (x^{-\alpha})^{-\alpha}, r\rangle \in \mu$. Moreover, $\langle \alpha, x^{-\alpha}, s\rangle \in \mu$. Hence $r \geq s$. Therefore $r = s$.

□

Lemma 5.4. *Let $\langle \mu, A\rangle$ be a fuzzy soft set over a soft group G. $\langle \mu, A\rangle$ is a fuzzy soft subgroup of G if and only if for each $\alpha \in A$, $x, y, z \in G$ and $r, s, t \in I$, $\langle \alpha, x, y^{-\alpha}, z\rangle \in *$, $\langle \alpha, x, r\rangle \in \mu$, $\langle \alpha, y, s\rangle \in \mu$, $\langle \alpha, z, t\rangle \in \mu$ all together imply $t \geq min\{r, s\}$ and $\langle \alpha, e_\alpha, 1\rangle \in \mu$.*

Proof. Let $\langle G, *, A\rangle$ be a soft group and $\langle \mu, A\rangle$ be a fuzzy soft set over G. Let $\alpha \in A$ and $x, y, z \in G$ and $r, s, t \in I$ such that $\langle \alpha, x, y^{-\alpha}, z\rangle \in *$, $\langle \alpha, x, r\rangle \in \mu$, $\langle \alpha, y, s\rangle \in \mu$, $\langle \alpha, z, t\rangle \in \mu$. From $\langle \alpha, y, s\rangle \in \mu$ we have $\langle \alpha, y^{-\alpha}, s\rangle \in \mu$. Since $\langle \mu, A\rangle$ is a fuzzy soft subgroup of G, $t \geq min\{r, s\}$. Conversely, suppose that the conditions hold. Assume that $\langle \alpha, y^{-\alpha}, t_1\rangle \in \mu$. Since $\langle \alpha, e_\alpha, y^{-\alpha}, y^{-\alpha}\rangle \in *$, $t_1 \geq min\{s, 1\} = s$. Let $z_1 \in G$ and $t_2 \in I$ such that $\langle \alpha, x, y, z_1\rangle \in *$ and $\langle \alpha, z_1, t_2\rangle \in \mu$. Hence $\langle \alpha, x, (y^{-\alpha})^{-\alpha}, z_1\rangle \in *$. It follows that $t_2 \geq min\{r, t_1\} \geq min\{r, s\}$. Therefore, $\langle \mu, A\rangle$ is a fuzzy soft subgroup of G.

□

Definition 5.5. *Let $\langle \mu, A\rangle$ be a fuzzy soft set over a soft group G. For $t \in I$, define the soft set $\langle \mu_t, A\rangle$ over G as :*

$$\mu_t(\alpha) = \{x \in G : \langle \alpha, x, r\rangle \in \mu \;\; with \;\; r \geq t\}$$

for each $\alpha \in A$. The soft set $\langle \mu_t, A\rangle$ is called a soft t-level of the fuzzy soft set $\langle \mu, A\rangle$.

Theorem 5.6. *Let a soft group be $\langle G, *, A\rangle$. Then $\langle \mu, A\rangle$ is a fuzzy soft subgroup of G if and only if $\langle \mu_t, A\rangle$ is a soft subgroup of G for all $t \in I$.*

Proof. Let a soft group be $\langle G, *, A\rangle$ and let $\langle \mu, A\rangle$ be a fuzzy soft subgroup G. From $\langle \alpha, e_\alpha, 1\rangle \in \mu$, we get $e_\alpha \in \mu_t(\alpha)$. Let $x, y \in \mu_t(\alpha)$, $z \in G$ and $r, s_1, s_2 \in I$ such that $\langle \alpha, x, y^{-\alpha}, z\rangle \in *$, $\langle \alpha, x, r\rangle \in \mu$, $\langle \alpha, y, s_1\rangle \in \mu$ and $\langle \alpha, z, s_2\rangle \in \mu$. We have $r \geq t$ and $s_1 \geq t$. Since $\langle \mu, A\rangle$ is a fuzzy soft subgroup of G, $s_2 \geq \; min\{r, s_1\}$. Thus $s_2 \geq t$. It follows that $z \in \mu_t(\alpha)$. This means that $\langle \mu_t, A\rangle$ is a soft subgroup of G. Conversely assume that for every $t \in I$, $\langle \mu_t, A\rangle$ is a soft subgroup of G. Given that $\langle \mu_1, A\rangle$ is a soft subgroup of G, $e_\alpha \in \mu_1(\alpha)$ for every $\alpha \in A$. Hence $\langle \alpha, e_\alpha, 1\rangle \in \mu$. Let $\alpha \in A$, $x, y, z \in G$ and $r, s, t_1 \in I$ such that $\langle \alpha, x, y^{-\alpha}, z\rangle \in *$, $\langle \alpha, x, r\rangle \in \mu$, $\langle \alpha, y, s\rangle \in \mu$ and $\langle \alpha, z, t_1\rangle \in \mu$. Let $t = min\{r, s\}$. We get $x \in \mu_t(\alpha)$ and $y \in \mu_t(\alpha)$. Then we have $z \in \mu_t(\alpha)$ because $\langle \mu_t, A\rangle$ is a soft subgroup of G. Consequently, $t_1 \geq t = min\{r, s\}$. By Lemma 5.4, $\langle \mu, A\rangle$ is a fuzzy soft subgroup of G.

□

Definition 5.7. *Consider a fuzzy soft set $\langle \mu, A\rangle$ over a soft group G. For each $\alpha \in A$, the soft set $\langle \mu^*, A\rangle$ over G is defined as follows:*

$$\mu^*(\alpha) = \{x \in G : \langle \alpha, x, t\rangle \in \mu \; where \; t > 0\}.$$

Definition 5.8. *Let $\langle \mu, A\rangle$ be a fuzzy soft subgroup of G. The soft set $\langle \mu_*, A\rangle$ over G defined as*

$$\mu_*(\alpha) = \{x \in G : \langle \alpha, x, 1\rangle \in \mu\}.$$

Corollary 5.9. *If $\langle \mu, A\rangle$ is a fuzzy soft subgroup of a soft group G, then $\langle \mu_*, A\rangle$ is a soft subgroup of G.*

Proof. For a soft group G, let $\langle \mu, A\rangle$ be a fuzzy soft subgroup of G. It follows that $\langle \mu_*, A\rangle \; \breve{\subseteq} \; \langle \mu_1, A\rangle$. By Theorem 5.6, $\langle \mu_*, A\rangle$ is a soft subgroup of G.

□

Lemma 5.10. *If $\langle \mu, A \rangle$ is a fuzzy soft subgroup of G then $\langle \mu^*, A \rangle$ is a soft subgroup of G.*

Proof. Suppose $\langle \mu, A \rangle$ is a fuzzy soft subgroup of a soft group G. For each $\alpha \in A$, clearly $e_\alpha \in \mu^*(\alpha)$. Let $a, b \in \mu^*(\alpha)$, $\langle \alpha, x, t \rangle \in \mu$ and $x \in G$ such that $\langle \alpha, a, b^{-\alpha}, x \rangle \in *$. Then $\langle \alpha, a, r \rangle \in \mu$ and $\langle \alpha, b, s \rangle \in \mu$ for $r, s \in I$ with $r, s > 0$. Suppose that $\langle \alpha, x, t_1 \rangle \in \mu$. It follows that $t_1 \geq t = min\{r, s\}$. Thus $t_1 > 0$. Hence $x \in \mu^*(\alpha)$. Consequently, $\langle \mu^*, A \rangle$ is a soft subgroup of G. □

Lemma 5.11. *Suppose that $x \in G$ and $\langle \mu, A \rangle$ is fuzzy soft subgroups of a soft group G. Then for every $\alpha \in A$, the following statements are equivalent:*

(1) For $y, z \in G$, if $\langle \alpha, x, y, z \rangle \in *$, $\langle \alpha, y, s \rangle \in \mu$ and $\langle \alpha, z, t \rangle \in \mu$, then $s = t$.

(2) $\langle \alpha, x, 1 \rangle \in \mu$.

Proof. Let $x \in G$ and $\langle \mu, A \rangle$ be fuzzy soft subgroups of a soft group G. Suppose (1) is true. We have $\langle \alpha, x, e_\alpha, x \rangle \in *$ and $\langle \alpha, e_\alpha, 1 \rangle \in \mu$. Hence $t = 1$. Thus $\langle \alpha, x, 1 \rangle \in \mu$. Conversely, assume that $\langle \alpha, x, 1 \rangle \in \mu$. Suppose $y, z \in G$ and $s, t \in I$ such that $\langle \alpha, x, y, z \rangle \in *$, $\langle \alpha, y, s \rangle \in \mu$ and $\langle \alpha, z, t \rangle \in \mu$. Since $\langle \alpha, x, 1 \rangle \in \mu$ and $\langle \mu, A \rangle$ is fuzzy soft subgroup of a soft group G, $t \geq s$. Now it remains to show that $s \geq t$. From $\langle \alpha, x, y, z \rangle \in *$ we get $\langle \alpha, x^{-\alpha}, z, y \rangle \in *$. Consider $\langle \alpha, x^{-\alpha}, z, y \rangle \in *$, $\langle \alpha, x^{-\alpha}, 1 \rangle \in \mu$, $\langle \alpha, z, t \rangle \in \mu$ and $\langle \alpha, y, s \rangle \in \mu$. We have $s \geq min\{t, 1\} = t$. Therefore $s = t$. Hence the Lemma follows.

□

Proposition 5.12. *Suppose $\langle \mu, A \rangle$ is a fuzzy soft subgroup of a soft group G.*

(1) If soft t-level subgroups $\langle \mu_{t_1}, A \rangle$, $\langle \mu_{t_2}, A \rangle$ with $t_1 < t_2$ of G are equal, then for each $\alpha \in A$ there is no $x \in G$ such that $\langle \alpha, x, s \rangle \in \mu$ and $t_1 < s < t_2$.

(2) If for each $\alpha \in A$ there is no $x \in G$ such that $\langle \alpha, x, s \rangle \in \mu$ and $t_1 \leq s < t_2$ then the soft t-level subgroups $\langle \mu_{t_1}, A \rangle$ and $\langle \mu_{t_2}, A \rangle$ with $t_1 < t_2$ of G are equal.

Proof. Let $\langle G, *, A \rangle$ be a soft group and $\langle \mu, A \rangle$ be a fuzzy soft subgroup of G.

(1) Assume that the soft t-level subgroups $\langle \mu_{t_1}, A \rangle$ and $\langle \mu_{t_2}, A \rangle$ with $t_1 < t_2$ of G are equal.

Suppose there exist $\alpha \in A$ and $x \in G$ such that $\langle \alpha, x, s \rangle \in \mu$ and $t_1 < s < t_2$. Then $\mu_{t_2}(\alpha) \subsetneqq \mu_{t_1}(\alpha)$ because $x \in \mu_{t_1}(\alpha)$, but not in $x \notin \mu_{t_2}(\alpha)$ which contradicts the hypothesis.

(2) Suppose there is no $x \in G$ such that $t_1 \leq s < t_2$ with $\langle \alpha, x, s\rangle \in \mu$ for each $\alpha \in A$. We show that $\langle \mu_{t_1}, A\rangle \mathrel{\breve{=}} \langle \mu_{t_2}, A\rangle$. Let $x \in \mu_{t_1}(\alpha)$. Then $\langle \alpha, x, s\rangle \in \mu$ with $s \geq t_1$. If $s \geq t_1$, then $s \geq t_2$. Hence $x \in \mu_{t_2}(\alpha)$. Thus $\mu_{t_1}(\alpha) \subseteq \mu_{t_2}(\alpha)$. From $t_1 < t_2$ we have $\mu_{t_2}(\alpha) \subseteq \mu_{t_1}(\alpha)$. Hence $\langle \mu_{t_1}, A\rangle \mathrel{\breve{=}} \langle \mu_{t_2}, A\rangle$.

□

Proposition 5.13. *Suppose G is a soft group and $\langle \mu, A\rangle$ is a fuzzy soft subgroup of G. If $\langle \alpha, x, x, y\rangle \in *$, $\langle \alpha, x, r\rangle \in \mu$ and $\langle \alpha, y, s\rangle \in \mu$ for each $\alpha \in A$, $r, s \in I$ and $x, y \in G$, then $s \geq r$.*

Proof. Immediate □

Theorem 5.14. *For some $t \in I$, any soft subgroup of a soft group can be realized as a t-level soft subgroup of some fuzzy soft subgroup.*

Proof. Consider the soft subgroup $\langle H, A\rangle$ of the soft group $\langle G, *, A\rangle$. Let $\langle \mu, A\rangle$ be a fuzzy soft set over G that is characterised by

$$\langle \alpha, x, s\rangle \in \mu \Leftrightarrow s = \begin{cases} 1 & if \;\; x = e_\alpha \\ t_0 & if \;\; x \in H(\alpha) - \{e_\alpha\} \\ 0 & if \;\; x \notin H(\alpha) \qquad\qquad where\ 0 < t_0 < 1. \end{cases}$$

First we show that $\mu_{t_0}(\alpha) = H(\alpha)$ for‘each $\alpha \in A$. Let $x \in \mu_{t_0}(\alpha)$. Then $\langle \alpha, x, s\rangle \in \mu$ where $s \geq t_0$ and hence $s > 0$. We have $x \in H(\alpha)$. Thus $\mu_{t_0}(\alpha) \subseteq H(\alpha)$. Suppose $x \in H(\alpha)$. If $x = e_\alpha$, then $x \in \mu_{t_0}(\alpha)$. If $x \in H(\alpha) - \{e_\alpha\}$, then $\langle \alpha, x, t_0\rangle \in \mu$ and hence $x \in \mu_{t_0}(\alpha)$. It follows that $H(\alpha) \subseteq \mu_{t_0}(\alpha)$ and consequently, $H(\alpha) = \mu_{t_0}(\alpha)$ for each $\alpha \in A$. Next, we show that $\langle \mu, A\rangle$ is a fuzzy soft subgroup of G. Let $x, y, z \in G$ and $r, s, t_1 \in I$ such that $\langle \alpha, x, y, z\rangle \in *$, $\langle \alpha, x, r\rangle \in \mu$, $\langle \alpha, y, s\rangle \in \mu$ and $\langle \alpha, z, t_1\rangle \in \mu$. If $x = e_\alpha$ or $y = e_\alpha$, then $z = y$ or $z = x$. Hence the result follows. If $x, y \in H(\alpha) - \{e_\alpha\}$, then $r = s = t_0$. Moreover, $z \in H(\alpha)$. Hence $t_1 = 1$ or $t_1 = t_0$. Therefore $t_1 \geq min\{r, s\}$. Suppose $x \in H(\alpha) - \{e_\alpha\}$ and $y \notin H(\alpha)$. Then $\langle \alpha, y, 0\rangle \in \mu$ and hence $t_1 \geq min\{r, 0\} = 0$. Suppose $x, y \notin H(\alpha)$. Then $r = s = 0$. Hence $t_1 \geq min\{r, s\}$. Similarly $\langle \alpha, x^{-\alpha}, r\rangle \in \mu$. Therefore $\langle \mu, A\rangle$ is a fuzzy soft subgroup of G.

□

Lemma 5.15. *Arbitrary intersection of fuzzy soft subgroups is a fuzzy soft subgroup.*

Proof. Let $\{\langle \mu_i, A\rangle : i \in \Delta\}$ be a nonempty family of fuzzy soft subgroups of G. Then their intersection is a fuzzy soft set over G. Let $\langle \mu, A\rangle$ be their intersection. Since $\langle \alpha, e_\alpha, 1\rangle \in \mu_i$ for each $i \in \Delta$, $\langle \alpha, e_\alpha, 1\rangle \in \mu$. Suppose that $\langle \alpha, x, r\rangle \in \mu$. Then $\langle \alpha, x, r_i\rangle \in \mu_i$ for each $i \in \Delta$ with $r = \bigwedge_{i\in\Delta} r_i$. Then $\langle \alpha, x^{-\alpha}, r_i\rangle \in \mu_i$ for each $i \in \Delta$. Hence $\langle \alpha, x^{-\alpha}, r\rangle \in \mu$. Suppose $\langle \alpha, x, r\rangle \in \mu$, $\langle \alpha, y, s\rangle \in \mu$, $\langle \alpha, z, t\rangle \in \mu$ and $\langle \alpha, x, y, z\rangle \in *$. Then $\langle \alpha, x, r_i\rangle \in \mu_i$, $\langle \alpha, y, s_i\rangle \in \mu_i$, $\langle \alpha, z, t_i\rangle \in \mu_i$, and $\langle \alpha, x, y, z\rangle \in *$. Given that every μ_i is a fuzzy soft subgroup of G, $t_i \geq min\{r_i, s_i\}$. Thus, $t \geq min\{r, s\}$. Consequently, $\langle \mu, A\rangle$ is a fuzzy soft subgroup of G.

□

Definition 5.16. *Let $\langle \mu, A\rangle$ be a fuzzy soft set over G. Define a mapping $\widehat{\mu} : SE_A(G) \to I$ as for each $\breve{a} \in SE_A(G)$ and $\alpha \in A$:*

$$\widehat{\mu}(\breve{a}) = \wedge\{t_\alpha : \langle \alpha, a_\alpha, t_\alpha\rangle \in \mu \;\; where \;\; \breve{a}(\alpha) = \{a_\alpha\}\}.$$

Lemma 5.17. *Let $\langle \mu, A\rangle$ be fuzzy soft set over a soft group G. Then*

$$\widehat{\mu}_t = (\widehat{\mu})_t \qquad for \; each \; t \in I.$$

Proof. Let $\langle \mu, A\rangle$ be a fuzzy soft set over a soft group G. Let $t \in I$. Suppose $\breve{a} \in (\widehat{\mu})_t$. Then $\widehat{\mu}(\breve{a}) \geq t$. Hence $\wedge\{s_\alpha : \langle \alpha, a_\alpha, s_\alpha\rangle \in \mu \;\; and \; \alpha \in A\} \geq t$ where $\breve{a}(\alpha) = \{a_\alpha\}$ for each $\alpha \in A$. Thus $\langle \alpha, a_\alpha, s_\alpha\rangle \in \mu$ with $s_\alpha \geq t$ for each $\alpha \in A$. It follows that $a_\alpha \in \mu_t(\alpha)$ for each $\alpha \in A$ and hence $\{a_\alpha\} \subseteq \mu_t(\alpha)$. Therefore, $\breve{a} \in \widehat{\mu}_t$. Now, assume that $\breve{a} \in \widehat{\mu}_t$. Then $\breve{a}(\alpha) \subseteq \mu_t(\alpha)$ for each $\alpha \in A$. Thus $\langle \alpha, a_\alpha, s_\alpha\rangle \in \mu$ with $s_\alpha \geq t$ for each $\alpha \in A$. It follows that $\wedge\{s_\alpha : \langle \alpha, a_\alpha, s_\alpha\rangle \in \mu\} \geq t$. Consequently, $\widehat{\mu}(\breve{a}) \geq t$. Hence $\breve{a} \in (\widehat{\mu})_t$ and the lemma follows. □

Theorem 5.18. *Let $\langle \mu, A\rangle$ be a fuzzy soft set over a soft group G. Then the following statements are equivalent:*

(1) $\langle \mu, A\rangle$ is a fuzzy soft subgroup of G.

(2) $\widehat{\mu}$ is a fuzzy subgroup of $SE_A(G)$.

(3) $\widehat{\mu}_t$ is a subgroup of $SE_A(G)$ for all $t \in I$.

(4) $\langle \mu_t, A\rangle$ is a soft subgroup of G for all $t \in I$.

Proof. Let $\langle\mu, A\rangle$ be a fuzzy soft set over a soft group G.

(1) $\Rightarrow$ (2) : Suppose $\langle\mu, A\rangle$ is a fuzzy soft subgroup of G. For each $\alpha \in A$, $\langle\alpha, e_\alpha, 1\rangle \in \mu$ implies $\widehat{\mu}(\breve{e}) = 1$ where $\breve{e}(\alpha) = \{e_\alpha\}$. Suppose $\widehat{\mu}(\breve{a}) = r$ and $\widehat{\mu}(\breve{a}^{-1}) = s$. We show that $s = r$. Let $\breve{a}(\alpha) = \{a_\alpha\}$ for each $\alpha \in A$. Then $\breve{a}^{-1}(\alpha) = \{a_\alpha^{-\alpha}\}$. Hence $\wedge\{r_\alpha : \langle\alpha, a_\alpha, r_\alpha\rangle \in \mu\} = r$ and $\wedge\{s_\alpha : \langle\alpha, a_\alpha^{-\alpha}, s_\alpha\rangle \in \mu\} = s$. Since $r_\alpha = s_\alpha$ for each $\alpha \in A$, $r = s$. Let $\widehat{\mu}(\breve{a} \bar{*} \breve{b}) = \breve{c}$. Then $\langle\alpha, a_\alpha, b_\alpha, c_\alpha\rangle \in *$ where $\breve{a}(\alpha) = \{a_\alpha\}$, $\breve{b}(\alpha) = \{b_\alpha\}$ and $\breve{c}(\alpha) = \{c_\alpha\}$ for each $\alpha \in A$. Suppose that $\widehat{\mu}(\breve{a}) = r$, $\widehat{\mu}(\breve{b}) = s$, and $\widehat{\mu}(\breve{c}) = t$. From $\widehat{\mu}(\breve{a}) = r$ we get $\wedge\{r_\alpha : \langle\alpha, a_\alpha, r_\alpha\rangle \in \mu\} = r$. Hence $\langle\alpha, a_\alpha, r_\alpha\rangle \in \mu$ with $r_\alpha \geq r$ for each $\alpha \in A$. From $\widehat{\mu}(\breve{b}) = s$ we have $\wedge\{s_\alpha : \langle\alpha, b_\alpha, s_\alpha\rangle \in \mu\} = s$. We get $\langle\alpha, b_\alpha, s_\alpha\rangle \in \mu$ with $s_\alpha \geq s$ for each $\alpha \in A$. Moreover $\widehat{\mu}(\breve{c}) = t$ implies $\wedge\{t_\alpha : \langle\alpha, c_\alpha, t_\alpha\rangle \in \mu\} = t$ for each $\alpha \in A$. Consider $\langle\alpha, a_\alpha, b_\alpha, c_\alpha\rangle \in *$, $\langle\alpha, a_\alpha, r_\alpha\rangle \in \mu$, $\langle\alpha, b_\alpha, s_\alpha\rangle \in \mu$ and $\langle\alpha, c_\alpha, t_\alpha\rangle \in \mu$. Since $\langle\mu, A\rangle$ is a fuzzy soft subgroup of G, $t_\alpha \geq min\{r_\alpha, s_\alpha\} \geq min\{r, s\}$ for each $\alpha \in A$. Hence $\bigwedge t_\alpha \geq min\{r, s\}$. Thus $t \geq min\{r, s\}$. Therefore $\widehat{\mu}$ is a fuzzy subgroup of $SE_A(G)$.

(2) $\Rightarrow$ (3) : Suppose $\widehat{\mu}$ is a fuzzy subgroup of $SE_A(G)$. Since $\widehat{\mu_t} = (\widehat{\mu})_t$ for all $t \in I$, $\widehat{\mu_t}$ is a subgroup of $SE_A(G)$.

(3) $\Rightarrow$ (4) : Suppose that $\widehat{\mu_t}$ is a subgroup of $SE_A(G)$ for all $t \in I$. We show that $\langle\mu_t, A\rangle$ is a soft subgroup of G for all $t \in I$. From $\breve{e} \in \widehat{\mu_t}$ we get $\breve{e}(\alpha) = \{e_\alpha\} \subseteq \widehat{\mu_t}(\alpha)$ for each $\alpha \in A$. Hence $e_\alpha \in \widehat{\mu_t}(\alpha)$ for all $\alpha \in A$. Let $\alpha \in A$, $a, b \in \widehat{\mu_t}(\alpha)$ and $c \in G$ such that $\langle\alpha, a, b^{-\alpha}, c\rangle \in *$. We show that $c \in \widehat{\mu_t}(\alpha)$. Define the soft elements $\breve{a}$, $\breve{b}$ and $\breve{c}$ of G by

$$\breve{a}(\lambda) = \begin{cases} \{a\} & if \;\; \lambda = \alpha \\ \{e_\lambda\} & if \;\; \lambda \neq \alpha \end{cases}$$

$$\breve{b}(\lambda) = \begin{cases} \{b\} & if \;\; \lambda = \alpha \\ \{e_\lambda\} & if \;\; \lambda \neq \alpha \end{cases}$$

$$\breve{c}(\lambda) = \begin{cases} \{c\} & if \;\; \lambda = \alpha \\ \{e_\lambda\} & if \;\; \lambda \neq \alpha \end{cases}$$

for all $\lambda \in A$. Since $a \in \widehat{\mu_t}(\alpha)$, $\langle\alpha, a, r\rangle \in \mu$ with $r \geq t$. Moreover, $b \in \widehat{\mu_t}(\alpha)$ implies $\langle\alpha, b, s\rangle \in \mu$ with $s \geq t$. Hence $\breve{a}$, $\breve{b} \in (\widehat{\mu})_t$ and consequently $\breve{a}$, $\breve{b} \in \widehat{\mu_t}$. Since $\widehat{\mu_t}$ is a subgroup of $SE_A(G)$

for all $t \in I$, $\breve{a} \breve{*} \breve{b}^{-1} \in \widehat{\mu}_t$. Moreover,

$$\breve{b}^{-1}(\lambda) = \begin{cases} \{b^{-\alpha}\} & if \quad \lambda = \alpha \\ \{e_\lambda\} & if \quad \lambda \neq \alpha \end{cases}.$$

Now we show that $\breve{a} \breve{*} \breve{b}^{-1} = \breve{c}$. If $\lambda \neq \alpha$, then $\langle \lambda, a_\lambda, b_\lambda^{-\lambda}, c_\lambda \rangle \in * = \langle \lambda, e_\lambda, e_\lambda^{-\lambda}, e_\lambda \rangle \in *$. If $\lambda = \alpha$, then $\langle \lambda, a_\lambda, b_\lambda^{-\lambda}, c_\lambda \rangle \in * = \langle \alpha, a, b^{-\alpha}, c \rangle \in *$. Therefore, $\breve{a} \breve{*} \breve{b}^{-1} = \breve{c}$ and hence $\breve{c} \in \mu_t(\alpha)$. Clearly $\breve{c}(\alpha) \subseteq \mu_t(\alpha)$ for each $\alpha \in A$ from which it follows that $c \in \mu_t(\alpha)$ for every $\alpha \in A$. Hence $\langle \mu_t, A \rangle$ is a soft subgroup of G for all $t \in I$.

(4) $\Rightarrow$ (1) : By Theorem 5.6.

□

Definition 5.19. *Let $\langle f, A \rangle$ be a soft mapping from X to Y. Let $\langle \mu, A \rangle$ and $\langle v, A \rangle$ be fuzzy soft sets over X and Y respectively.*

(a) The inverse image of $\langle v, A \rangle$ under $\langle f, A \rangle$, represented by $\langle f^{-1}(v), A \rangle$, is the fuzzy soft set over X described by, for each $x \in X$, $\alpha \in A$ and $t \in I$ there exist some $y \in Y$ such that

$$\langle \alpha, x, t \rangle \in f^{-1}(v) \Leftrightarrow \langle \alpha, x, y \rangle \in f \text{ and } \langle \alpha, y, t \rangle \in v.$$

(b) The image of $\langle \mu, A \rangle$ under $\langle f, A \rangle$, represented by $\langle f(\mu), A \rangle$ is the fuzzy soft set over Y described by

$$\langle \alpha, y, t \rangle \in f(\mu) \Leftrightarrow t = \vee\{r : \quad x \in X, \langle \alpha, x, y \rangle \in f \text{ and } \langle \alpha, x, r \rangle \in \mu\}.$$

Theorem 5.20. *Let $\langle G, *, A \rangle$ and $\langle G', \Delta, A \rangle$ be soft groups and $\langle v, A \rangle$ be a fuzzy soft subgroup of G'. If $\langle f, A \rangle$ is a soft homomorphism from G to G', then $\langle f^{-1}(v), A \rangle$ is a fuzzy soft subgroup of G.*

Proof. Let e_α and e'_α be identity elements of G and G' for each $\alpha \in A$ respectively. From $\langle \alpha, e_\alpha, e'_\alpha \rangle \in f$ and $\langle \alpha, e'_\alpha, 1 \rangle \in v$ we get $\langle \alpha, e_\alpha, 1 \rangle \in f^{-1}(v)$. Suppose $\langle \alpha, x, r \rangle \in f^{-1}(v)$ and $\langle \alpha, x^{-\alpha}, s \rangle \in f^{-1}(v)$. We show that $s = r$. Since $\langle \alpha, x, r \rangle \in f^{-1}(v)$, there exists $y \in G'$ such that $\langle \alpha, x, y \rangle \in f$ and $\langle \alpha, y, r \rangle \in v$. From $\langle \alpha, y, r \rangle \in v$ we have $\langle \alpha, y^{-\alpha}, r \rangle \in v$. Moreover, $\langle \alpha, x, y \rangle \in f$ implies $\langle \alpha, x^{-\alpha}, y^{-\alpha} \rangle \in f$. Since $\langle \alpha, x^{-\alpha}, y^{-\alpha} \rangle \in f$ and $\langle \alpha, y^{-\alpha}, r \rangle \in v$, $\langle \alpha, x^{-\alpha}, r \rangle \in f^{-1}(v)$.

Hence $s = r$. Let $\alpha \in A$, $x, y, z \in G$ and $r, s, t \in I$ such that $\langle \alpha, x, y, z\rangle \in *$, $\langle \alpha, x, r\rangle \in f^{-1}(v)$, $\langle \alpha, y, s\rangle \in f^{-1}(v)$ and $\langle \alpha, z, t\rangle \in f^{-1}(v)$. We prove that $t \geq min\{r, s\}$. $\langle \alpha, x, r\rangle \in f^{-1}(v)$ implies there exists $x' \in G'$ such that $\langle \alpha, x, x'\rangle \in f$ and $\langle \alpha, x', r\rangle \in v$. Similarly, $\langle \alpha, y, s\rangle \in f^{-1}(v)$ means there is $y' \in G'$ such that $\langle \alpha, y, y'\rangle \in f$ and $\langle \alpha, y', s\rangle \in v$. Additionally, $\langle \alpha, z, t\rangle \in f^{-1}(v)$ implies there exists $z' \in G'$ such that $\langle \alpha, z, z'\rangle \in f$ and $\langle \alpha, z', t\rangle \in v$. Consider $\langle \alpha, x, y, z\rangle \in *$, $\langle \alpha, x, x'\rangle \in f$, $\langle \alpha, y, y'\rangle \in f$ and $\langle \alpha, z, z'\rangle \in f$. Then $\langle \alpha, x', y', z'\rangle \in \Delta$ because $\langle f, A\rangle$ is a soft homomorphism. Now we have $\langle \alpha, x', y', z'\rangle \in \Delta$, $\langle \alpha, x', r\rangle \in v$, $\langle \alpha, y', s\rangle \in v$ and $\langle \alpha, z', t\rangle \in v$. Since $\langle v, A\rangle$ be a fuzzy soft subgroup of G', $t \geq min\{r, s\}$. Hence the theorem follows. □

Theorem 5.21. *Let $\langle G, *, A\rangle$ and $\langle G', \Delta, A\rangle$ be soft groups and let $\langle \mu, A\rangle$ be a fuzzy soft subgroup of G. If $\langle f, A\rangle$ is a soft homomorphism from G to G', then $\langle f(\mu), A\rangle$ is a fuzzy soft subgroup of G'.*

Proof. For each $\alpha \in A$, let e_α and e'_α be an identity elements of G and G' respectively. Then $\langle \alpha, e_\alpha, e'_\alpha\rangle \in f$. Moreover, $\langle \alpha, e_\alpha, 1\rangle \in \mu$. Hence $\langle \alpha, e'_\alpha, 1\rangle \in f(\mu)$. Suppose $\langle \alpha, y, t_1\rangle \in f(\mu)$ and $\langle \alpha, y^{-\alpha}, t_2\rangle \in f(\mu)$. If $\langle \alpha, x, y\rangle \in f$ then $\langle \alpha, x^{-\alpha}, y^{-\alpha}\rangle \in f$. If $\langle \alpha, x, r\rangle \in \mu$ then $\langle \alpha, x^{-\alpha}, r\rangle \in \mu$. Thus $t_2 = t_1$. Let $u, v, w \in G'$ and $r, s, t \in I$ such that $\langle \alpha, u, v, w\rangle \in \Delta$, $\langle \alpha, u, r\rangle \in f(\mu)$, $\langle \alpha, v, s\rangle \in f(\mu)$ and $\langle \alpha, w, t\rangle \in f(\mu)$. We show that $t \geq min\{r, s\}$. If there is no $x \in G$ such that $\langle \alpha, x, u\rangle \in f$, then $\langle \alpha, u, 0\rangle \in f(\mu)$. Hence $t \geq min\{s, 0\} = 0$. Similarly if there is no $x \in G$ such that $\langle \alpha, x, v\rangle \in f$, then $\langle \alpha, v, 0\rangle \in f(\mu)$. Hence $t \geq min\{0, r\} = 0$. Now, there is $x, y \in G$ such that $\langle \alpha, x, u\rangle \in f$ and $\langle \alpha, y, v\rangle \in f$. Let $z \in G$ such that $\langle \alpha, x, y, z\rangle \in *$. Let $w_1 \in G'$ such that $\langle \alpha, z, w_1\rangle \in f$. Since $\langle f, A\rangle$ is a soft homomorphism, $\langle \alpha, u, v, w_1\rangle \in \Delta$. Thus $w_1 = w$. Therefore $\langle \alpha, z, w\rangle \in f$. Now,

$$\begin{aligned}
\langle \alpha, w, t\rangle \in f(\mu) \Leftrightarrow t &= \vee\{l : \exists z_1 \in G : \langle \alpha, z_1, l\rangle \in \mu \text{ and } \langle \alpha, z_1, w\rangle \in f\}\\
\langle \alpha, u, r\rangle \in f(\mu) \Leftrightarrow r &= \vee\{m : \exists x_1 \in G : \langle \alpha, x_1, m\rangle \in \mu \text{ and } \langle \alpha, x_1, u\rangle \in f\}\\
\langle \alpha, v, s\rangle \in f(\mu) \Leftrightarrow s &= \vee\{n : \exists y_1 \in G : \langle \alpha, y_1, n\rangle \in \mu \text{ and } \langle \alpha, y_1, v\rangle \in f\}.
\end{aligned}$$

Thus

$$\begin{aligned} r \wedge s &= \vee\{m : \exists x_1 \in G : \langle \alpha, x_1, m\rangle \in \mu \ and \ \langle \alpha, x_1, u\rangle \in f\} \\ &\wedge \vee\{n : \exists y_1 \in G : \langle \alpha, y_1, n\rangle \in \mu \ and \ \langle \alpha, y_1, v\rangle \in f\} \\ &= \vee\{min\{m,n\} : \exists x_1, y_1 \in G : \langle \alpha, x_1, m\rangle \in \mu, \langle \alpha, y_1, n\rangle \in \mu, \langle \alpha, x_1, u\rangle \in f \ and \ \langle \alpha, y_1, v\rangle \in f\} \\ &\leq \vee\{l : \exists z_1 \in G \ \ such \ that \langle \alpha, z_1, l\rangle \in \mu \ and \ \langle \alpha, z_1, w\rangle \in f\} \\ &= t. \end{aligned}$$

Hence $t \geq min\{r,s\}$. Therefore, $\langle f(\mu), A\rangle$ is a fuzzy soft subgroup of G'. □

Definition 5.22. *Let $\langle \mu, A\rangle$ be a fuzzy soft subgroup of a soft group G. Define the soft set $\langle N(\mu), A\rangle$ over G by: for each $\alpha \in A$, $x, y \in G$, $x \in N(\mu)(\alpha) \Leftrightarrow \langle \alpha, x, y, z\rangle \in *, \langle \alpha, y, x, w\rangle \in *, \langle \alpha, z, r\rangle \in \mu$ and $\langle \alpha, w, s\rangle \in \mu$ all together imply $r = s$.*

Theorem 5.23. *$\langle N(\mu), A\rangle$ is a soft subgroup of G.*

Proof. Since G is a soft group, $\langle \alpha, e_\alpha, y, y\rangle \in *$ and $\langle \alpha, y, e_\alpha, y\rangle \in *$ for all $y \in G$ and $\alpha \in A$. Hence $e_\alpha \in N(\mu)(\alpha)$ for each $\alpha \in A$. Let $x \in N(\mu)(\alpha)$. We show that $x^{-\alpha} \in N(\mu)(\alpha)$. Let $y, z, w \in G$ and $r, s \in I$ such that $\langle \alpha, x^{-\alpha}, y, z\rangle \in *$, $\langle \alpha, y, x^{-\alpha}, w\rangle \in *$, $\langle \alpha, z, r\rangle \in \mu$ and $\langle \alpha, w, s\rangle \in \mu$. From $\langle \alpha, z, r\rangle \in \mu$ and $\langle \alpha, w, s\rangle \in \mu$ we have $\langle \alpha, z^{-\alpha}, r\rangle \in \mu$ and $\langle \alpha, w^{-\alpha}, s\rangle \in \mu$. Moreover, $\langle \alpha, x^{-\alpha}, y, z\rangle \in *$ if and only if $\langle \alpha, y^{-\alpha}, x, z^{-\alpha}\rangle \in *$ and $\langle \alpha, y, x^{-\alpha}, w\rangle \in *$ if and only if $\langle \alpha, x, y^{-\alpha}, w^{-\alpha}\rangle \in *$. Consider $\langle \alpha, y^{-\alpha}, x, z^{-\alpha}\rangle \in *$, $\langle \alpha, x, y^{-\alpha}, w^{-\alpha}\rangle \in *$, $\langle \alpha, z^{-\alpha}, r\rangle \in \mu$ and $\langle \alpha, w^{-\alpha}, s\rangle \in \mu$. Since $x \in N(\mu)(\alpha)$, $r = s$. Hence $x^{-\alpha} \in N(\mu)(\alpha)$. Let $x, y \in N(\mu)(\alpha)$ and $z_1 \in G$ such that $\langle \alpha, x, y, z_1\rangle \in *$. We show that $z_1 \in N(\mu)(\alpha)$. Let $z, w, n \in G$ such that $\langle \alpha, z_1, z, w\rangle \in *$, $\langle \alpha, z, z_1, n\rangle \in *$. Suppose that $\langle \alpha, w, t_1\rangle \in \mu$ and $\langle \alpha, n, t_2\rangle \in \mu$. We need to show that $t_1 = t_2$. Let $z_2, k \in G$ such that $\langle \alpha, y, z, z_2\rangle \in *$ and $\langle \alpha, x, z_2, k\rangle \in *$. From associativity of $\langle *, A\rangle$, we get $w = k$. Hence $\langle \alpha, x, z_2, w\rangle \in *$. Let $z_3 \in G$ and $t_3 \in I$ such that $\langle \alpha, z_2, x, z_3\rangle \in *$ and $\langle \alpha, z_3, t_3\rangle \in \mu$. From $\langle \alpha, z_2, x, z_3\rangle \in *$, $\langle \alpha, x, z_2, w\rangle \in *$, $\langle \alpha, z_3, t_3\rangle \in \mu$ and $\langle \alpha, w, t_1\rangle \in \mu$ we get $t_1 = t_3$ because $x \in N(\mu)(\alpha)$. Thus $\langle \alpha, z_3, t_1\rangle \in \mu$. Hence $\langle \alpha, z_3^{-\alpha}, t_1\rangle \in \mu$. Let $z_4, p \in G$ such that $\langle \alpha, z_1, x, z_4\rangle \in *$ and $\langle \alpha, y, z_4, p^{-\alpha}\rangle \in *$. Consider $\langle \alpha, y, z, z_2\rangle \in *$, $\langle \alpha, z_2, x, z_3\rangle \in *$, $\langle \alpha, z, x, z_4\rangle \in \mu$ and $\langle \alpha, y, z_4, p^{-\alpha}\rangle \in \mu$. From associative $\langle *, A\rangle$, we have $z_3 = p^{-\alpha}$. Hence $\langle \alpha, y, z_4, z_3\rangle \in *$. Let $z_5 \in G$ and $t_4 \in I$ such that $\langle \alpha, z_4, y, z_5^{-\alpha}\rangle \in *$ and $\langle \alpha, z_5, t_4\rangle \in \mu$. Since $y \in N(\mu)(\alpha)$, $\langle \alpha, z_4, y, z_5^{-\alpha}\rangle \in *$, $\langle \alpha, y, z_4, z_3\rangle \in *$, $\langle \alpha, z_5^{-\alpha}, t_4\rangle \in \mu$ and $\langle \alpha, z_3, t_1\rangle \in \mu$, $t_4 = t_1$. Thus $\langle \alpha, z_5^{-\alpha}, t_1\rangle \in \mu$. Consider $\langle \alpha, z, x, z_4\rangle \in \mu$, $\langle \alpha, z_4, y, z_5^{-\alpha}\rangle \in *$, $\langle \alpha, x, y, z_1\rangle \in *$

and $\langle\alpha, z, z_1, n\rangle \in *$. Since $\langle *, A\rangle$ is associative, $z_5^{-\alpha} = n$. Hence $t_1 = t_2$. Thus $z_1 \in N(\mu)(\alpha)$. Therefore $\langle N(\mu), A\rangle$ is a soft subgroup of G.

□

Definition 5.24. *Let $\langle\mu, A\rangle$ and $\langle v, A\rangle$ be fuzzy soft sets over G. Define $\mu \circ v$ by*

$$\langle\alpha, z, t\rangle \in \mu\circ v \Leftrightarrow t = \vee\{r\wedge s : \langle\alpha, x, r\rangle \in \mu, \langle\alpha, y, s\rangle \in v \; for\ some\ x, y \in G\ with\ \langle\alpha, x, y, z\rangle \in *\}$$

and

$$\langle\alpha, x, t\rangle \in \mu^{-1} \Leftrightarrow \langle\alpha, x^{-\alpha}, t\rangle \in \mu.$$

Remark 5.25. *We call μ^{-1} is the inverse of μ and $\mu \circ \nu$ is the product of μ and ν and where $\circ$ is a binary operation on the collection of all fuzzy soft sets over G and -1 is a unary operation. Moreover $\circ$ is an associative binary operation on the collection of all fuzzy soft sets over G.*

Lemma 5.26. *For fuzzy soft sets $\langle\mu, A\rangle$ and $\langle v, A\rangle$ over a soft group G.*

(1) If $\langle\alpha, e_\alpha, 1\rangle \in \mu$ and $\langle\alpha, e_\alpha, 1\rangle \in v$ for all $\alpha \in A$, then $\langle\alpha, e_\alpha, 1\rangle \in \mu \circ v$.

(2) $\langle(\mu \circ v)^{-1}, A\rangle \breve{\equiv} \langle v^{-1} \circ \mu^{-1}, A\rangle$.

(3) If $\langle\mu, A\rangle \breve{\subseteq} \langle v, A\rangle$ then $\langle\mu \circ \eta, A\rangle \breve{\subseteq} \langle v \circ \eta, A\rangle$ for any fuzzy soft $\langle\eta, A\rangle$.

(4) $\langle(\mu \cup v)^{-1}, A\rangle \breve{\equiv} \langle\mu^{-1} \cup v^{-1}, A\rangle$.

(5) $\langle(\mu \cap v)^{-1}, A\rangle \breve{\equiv} \langle\mu^{-1} \cap v^{-1}, A\rangle$.

Proof. Suppose that $\langle\mu, A\rangle$ and $\langle v, A\rangle$ are fuzzy soft sets over a soft group $\langle G, *, A\rangle$. Let $\alpha \in A$, $x, y, z \in G$ and $r, s, t \in I$.

(1) $\langle\alpha, e_\alpha, 1\rangle \in \mu \circ v$ because $\langle\alpha, e_\alpha, e_\alpha, e_\alpha\rangle \in *$, $\langle\alpha, e_\alpha, 1\rangle \in \mu$ and $\langle\alpha, e_\alpha, 1\rangle \in v$.

(2) $\langle\alpha, x, t\rangle \in (\mu \circ v)^{-1}$

$$\begin{aligned}
&\Leftrightarrow \langle\alpha, x^{-\alpha}, t\rangle \in \mu \circ v\\
&\Leftrightarrow t = \vee\{r \wedge s : \langle\alpha, y, r\rangle \in \mu, \langle\alpha, z, s\rangle \in v \; for\ some\ y, z \in G\ with\ \langle\alpha, y, z, x^{-\alpha}\rangle \in *\}\\
&\Leftrightarrow t = \vee\{r \wedge s : \langle\alpha, y^{-\alpha}, r\rangle \in \mu^{-1}, \langle\alpha, z^{-\alpha}, s\rangle \in v^{-1}\ with\ \langle\alpha, z^{-\alpha}, y^{-\alpha}, x\rangle \in *\}\\
&\Leftrightarrow t = \vee\{s \wedge r : \langle\alpha, z^{-\alpha}, s\rangle \in v^{-1}, \langle\alpha, y^{-\alpha}, r\rangle \in \mu^{-1}\ with\ \langle\alpha, z^{-\alpha}, y^{-\alpha}, x\rangle \in *\}\\
&\Leftrightarrow \langle\alpha, x, t\rangle \in v^{-1} \circ \mu^{-1}.
\end{aligned}$$

Hence, $\langle(\mu \circ v)^{-1}, A\rangle \breve{=} \langle v^{-1} \circ \mu^{-1}, A\rangle$.

(3) Suppose that $\langle\mu, A\rangle \breve{\subseteq} \langle v, A\rangle$ and $\langle\eta, A\rangle$ be a fuzzy soft set over G.

$$\begin{aligned}\langle\alpha, z, t\rangle \in \mu \circ \eta \;\; &\Leftrightarrow \\ t \;\; &= \;\; \vee\{r \wedge s : \langle\alpha, x, r\rangle \in \mu, \langle\alpha, y, s\rangle \in \eta \; for\ some\ x, y \in G\ with\ \langle\alpha, x, y, z\rangle \in *\} \\ &\leq \;\; \vee\{r \wedge s : \langle\alpha, x, r\rangle \in v, \langle\alpha, y, s\rangle \in \eta \; for\ some\ x, y \in G\ with\ \langle\alpha, x, y, z\rangle \in *\}.\end{aligned}$$

Hence $\langle\alpha, z, t_1\rangle \in v \circ \eta$ with $t \leq t_1$. Therefore, $\langle\mu \circ \eta, A\rangle \breve{\subseteq} \langle v \circ \eta, A\rangle$.

(4) $\langle\alpha, y, t\rangle \in (\mu \cup v)^{-1}$

$$\begin{aligned} &\Leftrightarrow \langle\alpha, y^{-\alpha}, t\rangle \in \mu \cup v \\ &\Leftrightarrow \langle\alpha, y^{-\alpha}, t\rangle \in \mu \; or \; \langle\alpha, y^{-\alpha}, t\rangle \in v \\ &\Leftrightarrow \langle\alpha, y, t\rangle \in \mu^{-1} \; or \; \langle\alpha, y, t\rangle \in v^{-1} \\ &\Leftrightarrow \langle\alpha, y, t\rangle \in \mu^{-1} \cup v^{-1}.\end{aligned}$$

Thus, $\langle(\mu \cup v)^{-1}, A\rangle \breve{=} \langle\mu^{-1} \cup v^{-1}, A\rangle$.

(5) $\langle\alpha, y, t\rangle \in (\mu \cap v)^{-1}$

$$\begin{aligned} &\Leftrightarrow \langle\alpha, y^{-\alpha}, t\rangle \in \mu \cap v \\ &\Leftrightarrow \langle\alpha, y^{-\alpha}, t\rangle \in \mu \; and \; \langle\alpha, y^{-\alpha}, t\rangle \in v \\ &\Leftrightarrow \langle\alpha, y, t\rangle \in \mu^{-1} \; and \; \langle\alpha, y, t\rangle \in v^{-1} \\ &\Leftrightarrow \langle\alpha, y, t\rangle \in \mu^{-1} \cap v^{-1}.\end{aligned}$$

Therefore, $\langle(\mu \cap v)^{-1}, A\rangle \breve{=} \langle\mu^{-1} \cap v^{-1}, A\rangle$.

□

In the following Lemma, we characterize fuzzy soft subgroups using the product and inverse operations defined on the class of fuzzy soft sets over G.

Lemma 5.27. *Let $\langle\mu, A\rangle$ be a fuzzy soft set over G. Then $\langle\mu, A\rangle$ is a fuzzy soft subgroup of G if and only if $\langle\mu, A\rangle$ satisfies the following conditions:*

(1) $\langle \alpha, e_\alpha, 1\rangle \in \mu$;

(2) $\langle \mu \circ \mu, A\rangle \breve{\subseteq} \langle \mu, A\rangle$ *and*

(3) $\langle \mu^{-1}, A\rangle \breve{=} \langle \mu, A\rangle$.

Proof. Let $\langle \mu, A\rangle$ be a fuzzy soft set over G. Suppose $\langle \mu, A\rangle$ is a fuzzy soft subgroup of G. Clearly, $\langle \alpha, e_\alpha, 1\rangle \in \mu$ for each $\alpha \in A$. Let $\alpha \in A$, $z \in G$ and $r, s, t, t_1 \in I$.

$$\begin{aligned} \langle \alpha, z, t\rangle \in \mu \circ \mu \Leftrightarrow t &= \vee\{r \wedge s : \langle \alpha, x, r\rangle \in \mu, \langle \alpha, y, s\rangle \in \mu \ for\ some\ x, y \in G\ with\ \langle \alpha, x, y, z\rangle \in *\} \\ &\leq \vee\{t_1 : \langle \alpha, z, t_1\rangle \in \mu, \langle \alpha, x, y, z\rangle \in *\}. \end{aligned}$$

Hence $\langle \mu\circ\mu, A\rangle \breve{\subseteq} \langle \mu, A\rangle$. Now $\langle \alpha, x, t\rangle \in \mu^{-1}$ if and only if $\langle \alpha, x^{-\alpha}, t\rangle \in \mu$. Hence $\langle \alpha, x, t\rangle \in \mu$. It follows that $\langle \mu^{-1}, A\rangle \breve{=} \langle \mu, A\rangle$. Conversely, suppose that (1), (2) and (3) are satisfied. Suppose $\langle \alpha, x, t\rangle \in \mu$. Since $\langle \mu^{-1}, A\rangle \breve{=} \langle \mu, A\rangle$, $\langle \alpha, x, t\rangle \in \mu^{-1}$. Thus $\langle \alpha, x^{-\alpha}, t\rangle \in \mu$. Let $\langle \alpha, x, r\rangle \in \mu$, $\langle \alpha, y, s\rangle \in \mu$, $\langle \alpha, z, t\rangle \in \mu$ and $\langle \alpha, x, y, z\rangle \in *$. Suppose that $\langle \alpha, z, t_1\rangle \in \mu \circ \mu$. Then $t_1 = \vee\{r_1 \wedge s_1 : \langle \alpha, x_1, r_1\rangle \in \mu, \langle \alpha, y_1, s_1\rangle \in \mu \ for\ some\ x_1, y_1 \in G\ with\ \langle \alpha, x_1, y_1, z\rangle \in *\} \geq$min $\{r, s\}$. It follows that $t_1 \geq min\{r, s\}$. Moreover, from $\langle \mu \circ \mu, A\rangle \breve{\subseteq} \langle \mu, A\rangle$ and $\langle \alpha, z, t\rangle \in \mu$ we get $t_1 \leq t$. Thus $t \geq min\{r, s\}$. Hence the theorem follows.

□

Corollary 5.28. *Suppose that* $\langle \mu, A\rangle$ *and* $\langle v, A\rangle$ *are fuzzy soft subgroups of a soft group* G. $\langle \mu \circ v, A\rangle$ *is a fuzzy soft subgroup of* G *if and only if* $\langle \mu \circ v, A\rangle \breve{=} \langle v \circ \mu, A\rangle$.

Proof. Let $\langle \mu, A\rangle$ and $\langle v, A\rangle$ be fuzzy soft subgroups of a soft group G. Suppose $\langle \mu \circ v, A\rangle$ is a fuzzy soft subgroup of G. Then, $\langle \mu \circ v, A\rangle \breve{=} \langle (\mu \circ v)^{-1}, A\rangle \breve{=} \langle v^{-1} \circ \mu^{-1}, A\rangle \breve{=} \langle v \circ \mu, A\rangle$. Hence $\langle \mu \circ v, A\rangle \breve{=} \langle v \circ \mu, A\rangle$. Conversely, suppose that $\langle \mu \circ v, A\rangle \breve{=} \langle v \circ \mu, A\rangle$. From $\langle \alpha, e_\alpha, e_\alpha, e_\alpha\rangle \in *$, $\langle \alpha, e_\alpha, 1\rangle \in \mu$ and $\langle \alpha, e_\alpha, 1\rangle \in v$ we get $\langle \alpha, e_\alpha, 1\rangle \in \mu \circ v$.

$$\begin{aligned} &\langle (\mu \circ v) \circ (\mu \circ v), A\rangle \\ &\qquad \breve{=} \langle (\mu \circ (v \circ \mu) \circ v), A\rangle \\ &\qquad \breve{=} \langle (\mu \circ (\mu \circ v) \circ v), A\rangle \\ &\qquad \breve{=} \langle (\mu \circ \mu) \circ (v \circ v), A) \\ &\qquad \breve{\subseteq} \langle \mu \circ v, A\rangle. \end{aligned}$$

Hence $\langle(\mu \circ v) \circ (\mu \circ v), A\rangle \breve{\subseteq} \langle(\mu \circ v), A\rangle$. Moreover,

$$\begin{aligned} &\langle(\mu \circ v)^{-1}, A\rangle \\ &\quad \breve{=} \langle(v \circ \mu)^{-1}, A\rangle \\ &\quad \breve{=} \langle \mu^{-1} \circ v^{-1}, A\rangle \\ &\quad \breve{=} \langle \mu \circ v, A\rangle. \end{aligned}$$

Therefore $\langle(\mu \circ v)^{-1}, A\rangle \breve{=} \langle \mu \circ v, A\rangle$. Thus by Lemma 5.27 $\langle \mu \circ v, A\rangle$ is a fuzzy soft subgroup of G.

□

Corollary 5.29. *Suppose $\langle \mu, A\rangle$ is a fuzzy soft set over G. Then $\langle \mu, A\rangle$ is a fuzzy soft subgroup of G if and only if $\langle \mu \circ \mu^{-1}, A\rangle \breve{=} \langle \mu, A\rangle$ and $\langle \alpha, e_\alpha, 1\rangle \in \mu$ for each $\alpha \in A$.*

Proof. Consider a fuzzy soft set $\langle \mu, A\rangle$ over a soft group $\langle G, *, A\rangle$. Suppose $\langle \mu, A\rangle$ is a fuzzy soft subgroup of G. Then $\langle \mu^{-1}, A\rangle \breve{=} \langle \mu, A\rangle$. Since $\langle \mu \circ \mu, A\rangle \breve{\subseteq} \langle \mu, A\rangle$, $\langle \mu \circ \mu^{-1}, A\rangle \breve{\subseteq} \langle \mu, A\rangle$. Let $\alpha \in A$ and $z \in G$ and $t \in I$.

$$\begin{aligned} \langle \alpha, z, t\rangle \in \mu \circ \mu^{-1} &\Leftrightarrow t = \vee\{r \wedge s : \langle \alpha, x, r\rangle \in \mu, \langle \alpha, y, s\rangle \in \mu^{-1} \text{ for some x,y} \in G \text{ with} \langle \alpha, x, y, z\rangle \in *\} \\ &\Leftrightarrow t = \vee\{r \wedge s : \langle \alpha, x, r\rangle \in \mu, \langle \alpha, y^{-\alpha}, s\rangle \in \mu \text{ for some x,y} \in G \text{ with} \langle \alpha, x, y, z\rangle \in *\} \\ &\Leftrightarrow t = \vee\{r \wedge s : \langle \alpha, x, r\rangle \in \mu, \langle \alpha, y, s\rangle \in \mu \text{ for some x,y} \in G \text{ with} \langle \alpha, x, y, z\rangle \in *\}. \end{aligned}$$

Hence $t \geq t_1$ where $\langle \alpha, z, t_1\rangle \in \mu$. Thus $\langle \mu, A\rangle \breve{\subseteq} \langle \mu \circ \mu^{-1}, A\rangle$. Therefore $\langle \mu \circ \mu^{-1}, A\rangle \breve{=} \langle \mu, A\rangle$. Conversely, $\langle \mu \circ \mu^{-1}, A\rangle \breve{=} \langle \mu, A\rangle$ and $\langle \alpha, e_\alpha, 1\rangle \in \mu$ for each $\alpha \in A$. Suppose $\langle \alpha, x, r\rangle \in \mu$ and $\langle \alpha, x^{-\alpha}, s\rangle \in \mu$. Now,

$$\begin{aligned} \langle \alpha, x^{-\alpha}, s\rangle \in \mu \circ \mu^{-1} \Leftrightarrow s &= \vee\{r_1 \wedge r_2 : \langle \alpha, y, r_1\rangle \in \mu \textit{ and } \langle \alpha, z, r_2\rangle \in \mu^{-1} \textit{ with } \langle \alpha, y, z, x^{-\alpha}\rangle \in *\} \\ &= \vee\{r_1 \wedge r_2 : \langle \alpha, y, r_1\rangle \in \mu \textit{ and } \langle \alpha, z^{-\alpha}, r_2\rangle \in \mu \textit{ with } \langle \alpha, y, z, x^{-\alpha}\rangle \in *\} \\ &\geq r \;\; \textit{where } \langle \alpha, e_\alpha, 1\rangle \in \mu \textit{ and } \langle \alpha, (x^{-\alpha})^{-\alpha}, r\rangle \in \mu. \end{aligned}$$

Hence $s \geq r$. Therefore $\langle \mu, A\rangle$ is a fuzzy soft subgroup of G.

□

Definition 5.30. *Let $\langle \mu_1, A\rangle$, $\langle \mu_2, A\rangle$, be fuzzy soft sets over soft groups G_1 and G_2 respectively.*

Then $\langle \mu_1 \times \mu_2, A\rangle$ is a fuzzy soft sets of $G_1 \times G_2$ given by $\langle \alpha, (x_1, x_2), t\rangle \in \mu_1 \times \mu_2$ if and only if there exist $t_1, t_2 \in I$ with $\langle \alpha, x_1, t_1\rangle \in \mu_1$ and $\langle \alpha, x_2, t_2\rangle \in \mu_2$ where $t = min\{t_1, t_2\}$.

Lemma 5.31. *Let G_1 and G_2 be soft groups. Let $\langle \mu_1, A\rangle$ and $\langle \mu_2, A\rangle$ be fuzzy soft subgroups of G_1 and G_2 respectively. Then their product $\langle \mu_1 \times \mu_2, A\rangle$ is a fuzzy soft subgroup of $G_1 \times G_2$.*

Proof. Suppose that $\langle \mu_1, A\rangle$ and $\mu_2, A\rangle$ are fuzzy soft subgroups of G_1 and G_2 respectively. Then $G_1 \times G_2$ is a soft group. Let $\alpha \in A$. Then $\langle \alpha, e^1_\alpha, 1\rangle \in \mu_1$ and $\langle \alpha, e^2_\alpha, 1\rangle \in \mu_2$. Hence $\langle \alpha, (e^1_\alpha, e^2_\alpha), 1\rangle \in \mu_1 \times \mu_2$. Let $\bar{x} = (x_1, x_2)$, $\bar{y} = (y_1, y_2)$, $\bar{z} = (z_1, z_2)$. Then $\bar{x}, \bar{y}, \bar{z} \in G_1 \times G_2$. Let $r, s, t \in I$ such that $\langle \alpha, \bar{x}, r\rangle \in \mu_1 \times \mu_2$, $\langle \alpha, \bar{y}, s\rangle \in \mu_1 \times \mu_2$, $\langle \alpha, \bar{z}, t\rangle \in \mu_1 \times \mu_2$ and $\langle \alpha, \bar{x}, \bar{y}, \bar{z}\rangle \in *$. Since $\langle \alpha, \bar{x}, \bar{y}, \bar{z}\rangle \in *$, $\langle \alpha, x_1, y_1, z_1\rangle \in *_1$ and $\langle \alpha, x_2, y_2, z_2\rangle \in *_2$. Since $\langle \alpha, \bar{x}, r\rangle \in \mu_1 \times \mu_2$, there exist $r_1, r_2 \in I$ such that $\langle \alpha, x_1, r_1\rangle \in \mu_1$ and $\langle \alpha, x_2, r_2\rangle \in \mu_2$ with $r = min\{r_1, r_2\}$. Similarly $\langle \alpha, \bar{y}, s\rangle \in \mu_1 \times \mu_2$ there exist $s_1, s_2 \in I$ such that $\langle \alpha, y_1, s_1\rangle \in \mu_1$ and $\langle \alpha, y_2, s_2\rangle \in \mu_2$ with $s = min\{s_1, s_2\}$. Moreover $\langle \alpha, \bar{z}, t\rangle \in \mu_1 \times \mu_2$ there exist $t_1, t_2 \in I$ such that $\langle \alpha, z_1, t_1\rangle \in \mu_1$ and $\langle \alpha, z_2, t_2\rangle \in \mu_2$ with $t = min\{t_1, t_2\}$. Since $\langle \mu_1, A\rangle$ is a fuzzy soft subgroup of G_1, $t_1 \geq min\{r_1, s_1\}$. Similarly $t_2 \geq min\{r_2, s_2\}$.

$$\begin{aligned} t = min\{t_1, t_2\} &\geq min\{min\{r_1, s_1\}, min\{r_2, s_2\}\} \\ &= min\{min\{r_1, r_2\}, min\{s_1, s_2\}\} \\ &= min\{r, s\}. \end{aligned}$$

Hence $t \geq min\{r, s\}$. Suppose $\langle \alpha, \bar{x}, r\rangle \in \mu_1 \times \mu_2$. Let $\bar{x} = (x_1, x_2)$ such that $\langle \alpha, x_1, r_1\rangle \in \mu_1$ and $\langle \alpha, x_2, r_2\rangle \in \mu_2$ with $r = min\{r_1, r_2\}$. It follows that $\langle \alpha, x_1^{-\alpha}, r_1\rangle \in \mu_1$ and $\langle \alpha, x_2^{-\alpha}, r_2\rangle \in \mu_2$. Hence $\langle \alpha, \bar{x}^{-\alpha}, r\rangle \in \mu_1 \times \mu_2$ with $r = min\{r_1, r_2\}$. The theorem follows. □

5.2 Normal Fuzzy Soft Subgroups

Theorem 5.32. *Let $\langle \mu, A\rangle$ be a fuzzy soft subgroup of a soft group G. Then, for each $\alpha \in A$, the following statements are equivalent.*

(1) For $x, y, z_1, z_2 \in G$ and $t_1, t_2 \in I$ with $\langle \alpha, y, x, z_1\rangle \in *$, $\langle \alpha, z_1, y^{-\alpha}, z_2\rangle \in *$, $\langle \alpha, x, t_1\rangle \in \mu$ and $\langle \alpha, z_2, t_2\rangle \in \mu$ together imply $t_2 \geq t_1$.

(2) For $x, y, z_1, z_2 \in G$ and $t_1, t_2 \in I$ with $\langle \alpha, y, x, z_1\rangle \in *$, $\langle \alpha, z_1, y^{-\alpha}, z_2\rangle \in *$, $\langle \alpha, x, t_1\rangle \in \mu$ and $\langle \alpha, z_2, t_2\rangle \in \mu$ together imply $t_1 = t_2$.

(3) Let $x, y \in G$ and $t_1, t_2 \in I$ with $\langle \alpha, x, y, z_1 \rangle \in *$, $\langle \alpha, y, x, z_2 \rangle \in *$, $\langle \alpha, z_1, t_1 \rangle \in \mu$ and $\langle \alpha, z_2, t_2 \rangle \in \mu$ together imply $t_1 = t_2$.

Proof. (1) $\Rightarrow$ (2) : Suppose that $\langle \alpha, y, x, z_1 \rangle \in *$, $\langle \alpha, z_1, y^{-\alpha}, z_2 \rangle \in *$, $\langle \alpha, x, t_1 \rangle \in \mu$ and $\langle \alpha, z_2, t_2 \rangle \in \mu$. We show that $t_1 = t_2$. From $\langle \alpha, y, x, z_1 \rangle \in *$ we have $\langle \alpha, y^{-\alpha}, z_1, x \rangle \in *$. Moreover, from $\langle \alpha, z_1, y^{-\alpha}, z_2 \rangle \in *$, we get $\langle \alpha, z_1^{-\alpha}, z_2, y^{-\alpha} \rangle \in *$. Consider $\langle \alpha, z_1^{-\alpha}, z_2, y^{-\alpha} \rangle \in *$, $\langle \alpha, y^{-\alpha}, z_1, x \rangle \in *$, $\langle \alpha, x, t_1 \rangle \in \mu$ and $\langle \alpha, z_2, t_2 \rangle \in \mu$. By (1) we have $t_1 \geq t_2$. Hence $t_1 = t_2$.
(2) $\Rightarrow$ (3) : Suppose $\langle \alpha, x, y, z_1 \rangle \in *$, $\langle \alpha, y, x, z_2 \rangle \in *$, $\langle \alpha, z_1, t_1 \rangle \in \mu$ and $\langle \alpha, z_2, t_2 \rangle \in \mu$. $\langle \alpha, x, y, z_1 \rangle \in *$ implies $\langle \alpha, x^{-\alpha}, z_1, y \rangle \in *$. Consider $\langle \alpha, x^{-\alpha}, z_1, y \rangle \in *$, $\langle \alpha, y, (x^{-\alpha})^{-\alpha}, z_2 \rangle \in *$, $\langle \alpha, z_1, t_1 \rangle \in \mu$ and $\langle \alpha, z_2, t_2 \rangle \in \mu$. By (2) $t_1 = t_2$.
(3) $\Rightarrow$ (1) : Suppose $\langle \alpha, y, x, z_1 \rangle \in *$, $\langle \alpha, z_1, y^{-\alpha}, z_2 \rangle \in *$, $\langle \alpha, x, t_1 \rangle \in \mu$ and $\langle \alpha, z_2, t_2 \rangle \in \mu$. $\langle \alpha, y, x, z_1 \rangle \in *$ implies $\langle \alpha, y^{-\alpha}, z_1, x \rangle \in *$. From $\langle \alpha, y^{-\alpha}, z_1, x \rangle \in *$, $\langle \alpha, z_1, y^{-\alpha}, z_2 \rangle \in *$, $\langle \alpha, x, t_1 \rangle \in \mu$ and $\langle \alpha, z_2, t_2 \rangle \in \mu$. We have $t_1 = t_2$. Hence the theorem follows. □

Definition 5.33. *A normal fuzzy soft subgroup of G is defined as a fuzzy soft subgroup $\langle \mu, A \rangle$ over a soft group G that satisfies the equivalent conditions of Theorem 5.32.*

Remark 5.34. *If $\langle \mu, A \rangle$ be a fuzzy soft subgroup of an abelian soft group G then $\langle \mu, A \rangle$ is a normal fuzzy soft subgroup of G.*

Theorem 5.35. *Given a fuzzy soft set over G, $\langle \mu, A \rangle$ is a normal fuzzy soft subgroup of G if and only if $\langle \mu_t, A \rangle$ is a normal soft subgroup of G for all $t \in I$.*

Proof. Suppose that $\langle \mu, A \rangle$ is a normal fuzzy soft subgroup of G. Given that $\langle \mu, A \rangle$ is a fuzzy soft subgroup of G, $\langle \mu_t, A \rangle$ is a soft subgroup of G for every $t \in I$. Let $\alpha \in A$, $a, x, y \in G$ and $n \in \mu_t(\alpha)$ with $\langle \alpha, a, n, x \rangle \in *$ and $\langle \alpha, x, a^{-\alpha}, y \rangle \in *$. We show that $y \in \mu_t(\alpha)$. From $n \in \mu_t(\alpha)$ we get $\langle \alpha, n, r \rangle \in \mu$ with $r \geq t$. Consider $\langle \alpha, a, n, x \rangle \in *$, $\langle \alpha, x, a^{-\alpha}, y \rangle \in *$ and $\langle \alpha, n, r \rangle \in \mu$. We have $\langle \alpha, y, r \rangle \in \mu$. Hence $y \in \mu_t(\alpha)$. Conversely, suppose that $\langle \mu_t, A \rangle$ is a normal soft subgroup of a soft group G for all $t \in I$. By Theorem 5.6, $\langle \mu, A \rangle$ is a fuzzy soft subgroup of G. Suppose $\langle \alpha, y, x, z_1 \rangle \in *$, $\langle \alpha, z_1, y^{-\alpha}, z_2 \rangle \in *$, $\langle \alpha, x, t_1 \rangle \in \mu$ and $\langle \alpha, z_2, t_2 \rangle \in \mu$. We show that $t_2 \geq t_1$. Since $\langle \alpha, x, t_1 \rangle \in \mu$, $x \in \mu_{t_1}(\alpha)$. From $x \in \mu_{t_1}(\alpha)$, $\langle \alpha, y, x, z_1 \rangle \in *$ and $\langle \alpha, z_1, y^{-\alpha}, z_2 \rangle \in *$ we have $t_2 \geq t_1$ because $\langle \mu_t, A \rangle$ is a normal soft subgroup of a soft group G for all $t \in I$.

□

Theorem 5.36. *Let $\langle \mu, A \rangle$ be a normal fuzzy soft subgroup of a soft group G. Then $\langle \mu_*, A \rangle$ and $\langle \mu^*, A \rangle$ are normal soft subgroups of G.*

Proof. Since $\langle \mu_*, A\rangle \,\breve{=}\, \langle \mu_1, A\rangle$, by Theorem 5.35 $\langle \mu_*, A\rangle$ is a normal soft subgroup of G. Now we show that $\langle \mu^*, A\rangle$ is a normal soft subgroup of G. From $\langle \mu, A\rangle$ is a fuzzy soft subgroup of G. We get $\langle \mu^*, A\rangle$ is a soft subgroup of G. Let $\alpha \in A$, $a, x, y \in G$, $t \in I$ and $n \in \mu^*(\alpha)$, $\langle \alpha, a, n, x\rangle \in *$ and $\langle \alpha, x, a^{-\alpha}, y\rangle \in *$. We show that $y \in \mu^*(\alpha)$. $n \in \mu^*(\alpha)$ implies $\langle \alpha, n, t\rangle \in \mu$ for some $t > 0$. $\langle \alpha, a, n, x\rangle \in *$, $\langle \alpha, x, a^{-\alpha}, y\rangle \in *$. Then we have $\langle \alpha, y, t\rangle \in \mu$. Therefore $y \in \mu^*(\alpha)$.

□

Remark 5.37. *The converse of Theorem 5.36 does not hold. The example that follows verified this.*

Example 5.38. *Let $\langle H, A\rangle$ be a soft subgroup of G which is not normal. Let a fuzzy soft set $\langle \mu, A\rangle$ on G can be defined as follows: $\langle \alpha, e_\alpha, 1\rangle \in \mu$, $\langle \alpha, x, \frac{1}{2}\rangle \in \mu$ if $x \in H(\alpha) - \{e_\alpha\}$ and $\langle \alpha, x, \frac{1}{4}\rangle \in \mu$ if $x \notin H(\alpha)$. Then $\langle \mu, A\rangle$ is a fuzzy soft subgroup of G. But $\langle \mu_{\frac{1}{2}}, A\rangle \,\breve{=}\, \langle H, A\rangle$ which is not a normal soft subgroup of G. Hence $\langle \mu, A\rangle$ is not normal fuzzy soft subgroup of G.*

Theorem 5.39. *For a fuzzy soft set over a soft group G the following are equivalent:*

(1) $\langle \mu, A\rangle$ is a normal fuzzy soft subgroup of G'.

(2) $\widehat{\mu}$ is a normal fuzzy subgroup of $SE_A(G)$.

(3) $\widehat{\mu}_t$ is a normal subgroup of $SE_A(G)$ for all $t \in I$.

(4) $\langle \mu_t, A\rangle$ is a normal soft subgroup over G for all $t \in I$.

Proof. By Lemma 5.17 we know that $\widehat{\mu}_t = (\widehat{\mu})_t$. Hence (2) ⇔ (3). Moreover from Theorem 5.35, (1) ⇔ (4). From Theorem 2.48 we have (3) ⇔ (4). Thus the theorem holds. □

Theorem 5.40. *Let $\langle G, *, A\rangle$ and $\langle G', \Delta, A\rangle$ be soft groups and let $\langle v, A\rangle$ be a normal fuzzy soft subgroup of G'. If $\langle f, A\rangle$ be a soft homomorphism from G to G', then $\langle f^{-1}(v), A\rangle$ is a normal fuzzy soft subgroup of G.*

Proof. Let $\langle G, *, A\rangle$ and $\langle G', \Delta, A\rangle$ be soft groups and let $\langle v, A\rangle$ be a normal fuzzy soft subgroup of G'. Let $\langle f, A\rangle$ be a soft homomorphism from G to G'. By Theorem 5.20, $\langle f^{-1}(v), A\rangle$ is a fuzzy soft subgroup of G. Let $x, y, z_1, z_2 \in G$ and $\alpha \in A$. Suppose that $\langle \alpha, y, x, z_1\rangle \in *$, $\langle \alpha, z_1, y^{-\alpha}, z_2\rangle \in *$, $\langle \alpha, x, t_1\rangle \in f^{-1}(v)$ and $\langle \alpha, z_2, t_2\rangle \in f^{-1}(v)$. We show that $t_1 = t_2$. $\langle \alpha, x, t_1\rangle \in f^{-1}(v)$ implies that there exists $x_1 \in G$ such that $\langle \alpha, x, x_1\rangle \in f$ *and* $\langle \alpha, x_1, t_1\rangle \in v$. Moreover, from $\langle \alpha, z_2, t_2\rangle \in f^{-1}(v)$ we get that

$\langle \alpha, z_2, w\rangle \in f$ *and* $\langle \alpha, w, t_2\rangle \in v$ for some $w \in G'$. Let $y_1, z_3 \in G$ such that $\langle \alpha, y, y_1\rangle \in f$ and $\langle \alpha, z_1, z_3\rangle \in f$. Consider $\langle \alpha, y, x, z_1\rangle \in *$, $\langle \alpha, y, y_1\rangle \in f$, $\langle \alpha, x, x_1\rangle \in f$ and $\langle \alpha, z_1, z_3\rangle \in f$. Then $\langle \alpha, y_1, x_1, z_3\rangle \in \Delta$ because $\langle f, A\rangle$ is a soft homomorphism. From $\langle \alpha, y, y_1\rangle \in f$ we have $\langle \alpha, y^{-\alpha}, y_1^{-\alpha}\rangle \in f$. Since $\langle f, A\rangle$ is a soft homomorphism, $\langle \alpha, z_3, y_1^{-\alpha}, w\rangle \in \Delta$. Now we have $\langle \alpha, y_1, x_1, z_3\rangle \in \Delta$, $\langle \alpha, z_3, y_1^{-\alpha}, w\rangle \in \Delta$, $\langle \alpha, x_1, t_1\rangle \in v$ and $\langle \alpha, w, t_2\rangle \in v$. From $\langle v, A\rangle$ be a normal fuzzy soft subgroup of G' we have $t_1 = t_2$. Hence the theorem follows.

□

Theorem 5.41. *Suppose that $\langle G, *, A\rangle$ and $\langle G', \Delta, A\rangle$ are soft groups, and let $\langle \mu, A\rangle$ is a normal fuzzy soft subgroup of G'. Let $\langle f, A\rangle$ be a soft epimorphism from G to G'. Then, $\langle f(\mu), A\rangle$ is a normal fuzzy soft subgroup of G'.*

Proof. According to Theorem 5.21, $\langle f(\mu), A\rangle$ is a fuzzy soft subgroup of G'. Let $x, y, z_1, z_2 \in G'$ and $t_1, t_2 \in I$ such that $\langle \alpha, x, y, z_1\rangle \in \Delta$, $\langle \alpha, z_1, x^{-\alpha}, z_2\rangle \in \Delta$, $\langle \alpha, y, t_1\rangle \in f(\mu)$ and $\langle \alpha, z_2, t_2\rangle \in f(\mu)$. We show that $t_1 = t_2$. Let $u, v \in G$ such that $\langle \alpha, u, x\rangle \in f$ and $\langle \alpha, v, y\rangle \in f$. Let $v_1 \in G$ such that $\langle \alpha, u, v, v_1\rangle \in *$. Then $\langle \alpha, v_1, z_1\rangle \in f$ because $\langle f, A\rangle$ is a homomorphism. Similarly, let $v_2 \in G$ such that $\langle \alpha, v_1, u^{-\alpha}, v_2\rangle \in *$. From $\langle f, A\rangle$ is a soft homomorphism we get $\langle \alpha, v_2, z_2\rangle \in f$. Let $s_1, s_2 \in I$ such that $\langle \alpha, v, s_1\rangle \in \mu$ and $\langle \alpha, v_2, s_2\rangle \in \mu$. Since $\langle \mu, A\rangle$ is a normal fuzzy soft subgroup of G, $s_1 = s_2$.
$\langle \alpha, y, t_1\rangle \in f(\mu) \Leftrightarrow t_1 = \vee\{s_1 | v \in G, \langle \alpha, v, s_1\rangle \in \mu, \langle \alpha, v, y\rangle \in f\}$

$$\begin{aligned}\langle \alpha, z_2, t_2\rangle \in f(\mu) \quad \Leftrightarrow \quad t_2 &= \vee\{s_2 | v_2 \in G, \langle \alpha, v_2, s_2\rangle \in \mu, \langle \alpha, v_2, z_2\rangle \in f\} \\ &= \vee\{s_1 | v \in G, \langle \alpha, v, s_1\rangle \in \mu, \langle \alpha, v, y\rangle \in f\} \\ &= t_1.\end{aligned}$$

Hence $t_1 = t_2$. Therefore, $\langle f(\mu), A\rangle$ is a normal fuzzy soft subgroup of G'. □

Proposition 5.42. *$\langle \mu, A\rangle$ is normal fuzzy soft subgroup of G if and only if $\langle \mu \circ \mu^{-1}, A\rangle \,\breve{\cong}\, \langle \mu, A\rangle$ and $\langle \mu \circ v, A\rangle \,\breve{\cong}\, \langle v \circ \mu, A\rangle$ holds for any fuzzy soft set $\langle v, A\rangle$.*

Proof. Suppose $\langle \mu, A\rangle$ is normal fuzzy soft subgroup of G. By corollary 5.29, $\langle \mu \circ \mu^{-1}, A\rangle \,\breve{\cong}\, \langle \mu, A\rangle$. Now it remains to show that $\langle \mu \circ v, A\rangle \,\breve{\cong}\, \langle v \circ \mu, A\rangle$ for any fuzzy soft set $\langle v, A\rangle$.

$\langle \alpha, z, t\rangle \in \mu \circ v \Leftrightarrow \ t = \ \vee\{r \wedge s : \langle \alpha, x, r\rangle \in \mu, \langle \alpha, y, s\rangle \in v \ for\ some\ x, y \in G\ with\ \langle \alpha, x, y, z\rangle \in *\}.$

Let $z_1 \in G$ and $t_1 \in I$ such that $\langle \alpha, y^{-\alpha}, z, z_1 \rangle \in *$ and $\langle \alpha, z_1, t_1 \rangle \in \mu$. Since $\langle \mu, A \rangle$ is normal and $\langle \alpha, y^{-\alpha}, z, z_1 \rangle \in *$ and $\langle \alpha, z, y^{-\alpha}, x \rangle \in *$, $t_1 = r'$ where $\langle \alpha, x, r' \rangle \in \mu$. Hence,

$$\begin{aligned} \langle \alpha, z, t \rangle \in \mu \circ v \quad &\Leftrightarrow \quad t = \vee\{s \wedge r : \langle \alpha, y, s \rangle \in v, \langle \alpha, z_1, r \rangle \in \mu \; with \; \langle \alpha, y, z_1, z \rangle \in *\} \\ &\Leftrightarrow \quad \langle \alpha, z, t \rangle \in v \circ \mu. \end{aligned}$$

Hence $\langle \mu \circ v, A \rangle \breve{=} \langle v \circ \mu, A \rangle$. Conversely, suppose that the conditions hold. Assume that $\langle \mu \circ \mu^{-1}, A \rangle \breve{=} \langle \mu, A \rangle$ and $\langle \mu \circ v, A \rangle \breve{=} \langle v \circ \mu, A \rangle$ holds for any fuzzy soft set $\langle v, A \rangle$. Since $\langle \mu \circ \mu^{-1}, A \rangle \breve{=} \langle \mu, A \rangle$, $\langle \mu, A \rangle$ is fuzzy soft subgroup of G by Corollary 5.29 . Now we show that $\langle \mu, A \rangle$ is a normal fuzzy soft subgroup of G. Suppose that $\langle \alpha, x, y, z_1 \rangle \in *$, $\langle \alpha, y, x, z_2 \rangle \in *$, $\langle \alpha, z_1, t_1 \rangle \in \mu$ and $\langle \alpha, z_2, t_2 \rangle \in \mu$. We show that $t_1 = t_2$. Consider the fuzzy soft set $\langle v, A \rangle$ which is defined by $\langle \alpha, y^{-\alpha}, 1 \rangle \in v$ and $\langle \alpha, w, 0 \rangle \in v$ for any $w \neq y^{-\alpha}$. Suppose that $\langle \alpha, x, t' \rangle \in \mu \circ v$. Then $\langle \alpha, x, t' \rangle \in v \circ \mu$. $\langle \alpha, x, t' \rangle \in \mu \circ v \Leftrightarrow t' = \vee\{r \wedge s : \langle \alpha, x_1, r \rangle \in \mu, \langle \alpha, y_1, s \rangle \in v \; for \; some \; x_1, y_1 \in G \; with \; \langle \alpha, x_1, y_1, x \rangle \in *\}$. Hence $\langle \alpha, x, t' \rangle \in \mu \circ v \Leftrightarrow t' = t_1$. Moreover,$\langle \alpha, x, t' \rangle \in v \circ \mu \Leftrightarrow t' = \vee\{r \wedge s : \langle \alpha, x_1, r \rangle \in v, \langle \alpha, y_1, s \rangle \in \mu$ for some $x_1, y_1 \in G$ with $\langle \alpha, x_1, y_1, x \rangle \in *\}$. Hence $\langle \alpha, x, t' \rangle \in \mu \circ v \Leftrightarrow t' = t_2$. *Thus* $t_1 = t_2$. Then the Theorem follows.

□

Lemma 5.43. *Let $\langle G_1, *_1, A \rangle$ and $\langle G_2, *_2, A \rangle$ be soft groups. Let $\langle \mu_1, A \rangle$ and $\langle \mu_2, A \rangle$ be normal fuzzy soft subgroups over G_1 and G_2 respectively. Then their product $\langle \mu_1 \times \mu_2, A \rangle$ is a normal fuzzy soft subgroup of $G_1 \times G_2$.*

Proof. The proof is similar to Theorem 5.31. □

Definition 5.44. *Suppose that $\langle \mu, A \rangle$ is a fuzzy soft subgroup of G. For $x \in G$, define fuzzy soft sets $\langle {}^x\mu, A \rangle$ and $\langle \mu^x, A \rangle$ over G by:*

$$\langle \alpha, y, t \rangle \in {}^x\mu \; if \; and \; only \; if \; there \; is \; some \; z \in G \; such \; that \; \langle \alpha, x^{-\alpha}, y, z \rangle \in * \; and \; \langle \alpha, z, t \rangle \in \mu$$

and

$$\langle \alpha, y, t \rangle \in \mu^x \; if \; and \; only \; if \; there \; is \; some \; z \in G \; such \; that \; \langle \alpha, y, x^{-\alpha}, z \rangle \in * \; and \; \langle \alpha, z, t \rangle \in \mu.$$

Then $\langle {}^x\mu, A \rangle$ and $\langle \mu^x, A \rangle$ are called left coset and right coset of $\langle \mu, A \rangle$ w.r.t x respectively.

Lemma 5.45. *Let $\langle \mu, A \rangle$ be a fuzzy soft subgroup of a soft group G. Then $\langle \mu, A \rangle$ is normal if and only if $\langle {}^x\mu, A \rangle \breve{=} \langle \mu^x, A \rangle$ for all $x \in G$.*

Proof. Let $x, y, z_1, z_2 \in G$ and $t \in I$ such that $\langle \alpha, y, x^{-\alpha}, z_1 \rangle \in *$, $\langle \alpha, x^{-\alpha}, y, z_2 \rangle \in *$ and $\langle \alpha, z_1, t \rangle \in \mu$. Now, $\langle \alpha, y, t \rangle \in \mu^x \Leftrightarrow \langle \alpha, y, x^{-\alpha}, z_1 \rangle \in *$ and $\langle \alpha, z_1, t \rangle \in \mu$. Since $\langle \mu, A \rangle$ is normal, $\langle \alpha, z_2, t \rangle \in \mu$. This implies that $\langle \alpha, y, t \rangle \in {}^x\mu$. Thus $\langle {}^x\mu, A \rangle \stackrel{\smile}{=} \langle \mu^x, A \rangle$. Conversely, suppose $\langle {}^x\mu, A \rangle \stackrel{\smile}{=} \langle \mu^x, A \rangle$. Let $z_3, z_4 \in G$ such that $\langle \alpha, x, y, z_3 \rangle \in *$, $\langle \alpha, y, x, z_4 \rangle \in *$. Now $\langle \alpha, z_3, t \rangle \in \mu \Leftrightarrow \langle \alpha, y, t \rangle \in {}^{x^{-\alpha}\mu} \Leftrightarrow \langle \alpha, y, t \rangle \in \mu^{x^{-\alpha}} \Leftrightarrow \langle \alpha, z_4, t \rangle \in \mu$ for all $x, y \in G$. Hence $\langle \mu, A \rangle$ is normal fuzzy soft subgroup.

□

Lemma 5.46. *Suppose $\langle \mu, A \rangle$ is a fuzzy soft subgroup of G. Then $\langle {}^x\mu, A \rangle \stackrel{\smile}{=} \langle {}^y\mu, A \rangle$ if and only if $\langle {}^x\mu_*, A \rangle \stackrel{\smile}{=} \langle {}^y\mu_*, A \rangle$ for all $x, y \in G$.*

Proof. Suppose that $\langle {}^x\mu, A \rangle \stackrel{\smile}{=} \langle {}^y\mu, A \rangle$. Let $\alpha \in A$, $x, y \in G$ and $t \in I$. Now, $\langle \alpha, z, t \rangle \in {}^x\mu \Leftrightarrow \langle \alpha, z, t \rangle \in {}^y\mu$ for all $z \in G$. This implies that $\exists z_1, z_2 \in G$ such that $\langle \alpha, x^{-\alpha}, z, z_1 \rangle \in *$, $\langle \alpha, y^{-\alpha}, z, z_2 \rangle \in *$, $\langle \alpha, z_1, t \rangle \in \mu$ and $\langle \alpha, z_2, t \rangle \in \mu$. Let $z_3 \in G$ and $s \in I$ such that $\langle \alpha, x^{-\alpha}, y, z_3 \rangle \in *$ and $\langle \alpha, z_3, s \rangle \in \mu$. Choosing $z = y$ yields $s = 1$ and thus $z_3 \in \mu_*(\alpha)$. Therefore ${}^x\mu_*(\alpha) = {}^y\mu_*(\alpha)$ for all $\alpha \in A$. Conversely, suppose that $\langle {}^x\mu_*, A \rangle \stackrel{\smile}{=} \langle {}^y\mu_*, A \rangle$. Let $\alpha \in A$, $z_1, z_2, z_3, z_4, \in G$ and $r, s, t \in I$, such that $\langle \alpha, x^{-\alpha}, z, z_1 \rangle \in *$, $\langle \alpha, y^{-\alpha}, z, z_2 \rangle \in *$, $\langle \alpha, x^{-\alpha}, y, z_3 \rangle \in *$, $\langle \alpha, y^{-\alpha}, x, z_4 \rangle \in *$, $\langle \alpha, z_1, r \rangle \in \mu$, $\langle \alpha, z_2, s \rangle \in \mu$ and $\langle \alpha, z_3, t \rangle \in \mu$. Then $z_3, z_4 \in \mu_*(\alpha)$. Since $\langle \mu, A \rangle$ be a fuzzy soft subgroup of G, $r \geq min\{t, s\} = s$. Thus $r \geq s$ for all $z \in G$. Similarly $s \geq r$. It follows that $r = s$ for all $z \in G$. Hence $\langle {}^x\mu, A \rangle \stackrel{\smile}{=} \langle {}^y\mu, A \rangle$.

□

Corollary 5.47. *Let $x, y \in G$ and $\alpha \in A$. Suppose $\langle \mu, A \rangle$ is a normal fuzzy soft subgroup of G. If $\langle {}^x\mu, A \rangle \stackrel{\smile}{=} \langle {}^y\mu, A \rangle$ and $\langle \alpha, x, r \rangle \in \mu$ and $\langle \alpha, y, s \rangle \in \mu$, then $r = s$.*

Proof. Suppose that $\langle {}^x\mu, A \rangle \stackrel{\smile}{=} \langle {}^y\mu, A \rangle$. Let $\alpha \in A$ and $x, y, z_1, z_2 \in G$ such that $\langle \alpha, y^{-\alpha}, x, z_1 \rangle \in *$, $\langle \alpha, z_1, y, z_2 \rangle \in *$, $\langle \alpha, x, r \rangle \in \mu$, $\langle \alpha, y, s \rangle \in \mu$, $\langle \alpha, z_1, t \rangle \in \mu$ and $\langle \alpha, z_2, t_1 \rangle \in \mu$. Since $\langle \mu, A \rangle$ is a normal fuzzy soft subgroup of G, $r = t_1$. Then $t_1 \geq \ min\{t, s\}$. By Lemma 5.46, $z_1 \in \mu_*(\alpha)$. It follows that $t = 1$. Thus $r \geq s$. Similarly $s \geq r$. Hence $r = s$.

□

Theorem 5.48. *Let $\langle \mu, A \rangle$ be a fuzzy normal soft subgroup of a soft group $\langle G, *, A \rangle$. Put*

$$G/\mu = \{{}^a\mu : a \in G\}.$$

Define a soft binary operation $\circledast$ on G/μ by: $\langle \alpha, {}^a\mu, {}^b\mu, {}^c\mu\rangle \in \circledast \Leftrightarrow \langle \alpha, a, b, x\rangle \in * \; for \; some \; x \in G$ such that ${}^c\mu = {}^x\mu$. Then

(1) $\langle G/\mu, \circledast, A\rangle$ is a soft group.

(2) $G/\mu \;\breve{\cong}\; G/\mu_*$.

Proof. (1) First we shall prove that $\circledast$ is well defined. Let $n, m, d \in G$ and $\alpha \in A$ such that ${}^a\mu = {}^d\mu$ and ${}^b\mu = {}^n\mu$ with $\langle \alpha, {}^a\mu, {}^b\mu, {}^c\mu\rangle \in \circledast$ and $\langle \alpha, {}^d\mu, {}^n\mu, {}^m\mu\rangle \in \circledast$. We show that ${}^c\mu = {}^m\mu$. $\langle \alpha, {}^a\mu, {}^b\mu, {}^c\mu\rangle \in \circledast$ implies $\langle \alpha, a, b, z_1\rangle \in *$ with ${}^c\mu = {}^{z_1}\mu$ and from $\langle \alpha, {}^d\mu, {}^n\mu, {}^m\mu\rangle \in \circledast$ implies $\langle \alpha, d, n, z_2\rangle \in *$ with ${}^m\mu = {}^{z_2}\mu$. Let $k_1 \in G$ such that $\langle \alpha, c^{-\alpha}, z_1, k_1\rangle \in *$ and $\langle \alpha, k_1, t_1\rangle \in \mu$ for $t_1 \in I$. Hence $\langle \alpha, z_1, t_1\rangle \in {}^c\mu$. Since ${}^c\mu = {}^{z_1}\mu$, $\langle \alpha, z_1, t_1\rangle \in {}^{z_1}\mu$. Thus $t_1 = 1$. Therefore $\langle \alpha, k_1, 1\rangle \in \mu$. Let $k_2 \in G$ such that $\langle \alpha, m^{-\alpha}, z_2, k_2\rangle \in *$. Similarly $\langle \alpha, k_2, 1\rangle \in \mu$. Let $x_1, x_2 \in G$ such that $\langle \alpha, a^{-\alpha}, d, x_1\rangle \in *$ and $\langle \alpha, b^{-\alpha}, n, x_2\rangle \in *$ and $\langle \alpha, x_1, s_1\rangle \in \mu$ and $\langle \alpha, x_2, s_2\rangle \in \mu$. Thus $\langle \alpha, d, s_1\rangle \in {}^a\mu$. Hence $s_1 = 1$. So we have $\langle \alpha, x_1, 1\rangle \in \mu$. It follows that $\langle \alpha, n, s_2\rangle \in {}^b\mu$. So $\langle \alpha, x_2, 1\rangle \in \mu$. Let $y, k_3 \in G$ such that $\langle \alpha, n^{-\alpha}, x_1, y\rangle \in *$. and $\langle \alpha, y, n, k_3\rangle \in *$. Since $\langle \mu, A\rangle$ is a normal fuzzy soft subgroup of the soft group G, $\langle \alpha, k_3, 1\rangle \in \mu$. Similarly, let $k_4, k_5, k_6 \in G$ such that $\langle \alpha, k_1, x_2, k_4\rangle \in *$, $\langle \alpha, k_4, k_3, k_5\rangle \in *$ and $\langle \alpha, k_5, k_2^{-\alpha}, k_6\rangle \in *$. Since $\langle \alpha, k_1, 1\rangle \in \mu$, $\langle \alpha, x_2, 1\rangle \in \mu$ and $\langle \alpha, k_3, 1\rangle \in \mu$ we have $\langle \alpha, k_4, 1\rangle \in \mu$, $\langle \alpha, k_5, 1\rangle \in \mu$ and $\langle \alpha, k_6, 1\rangle \in \mu$. It follows that $\langle \alpha, c^{-\alpha}, m, k_6\rangle \in *$. This implies that ${}^c\mu = {}^m\mu$. Thus $\circledast$ is well defined. It remains to show that $\langle G/\mu, \circledast, A\rangle$ satisfies the soft group axioms which are straightforward.

(2) Since $\langle \mu, A\rangle$ be a fuzzy normal soft subgroup of G, $\langle \mu_*, A\rangle$ is a normal subgroup of G. Hence G/μ_* is a soft group. Define a soft mapping $\langle f, A\rangle$ from G/μ to G/μ_* given by ${}^x\mu \to {}^x\mu_*$. So by Lemma 5.46 $\langle f, A\rangle$ is a soft isomorphism.

□

Definition 5.49. *The soft group $\langle G/\mu, \circledast, A\rangle$ established in Theorem 5.48 is referred to as the quotient soft group of G relative to the normal fuzzy soft subgroup of $\langle \mu, A\rangle$.*

Proposition 5.50. *Let $\langle \mu, A\rangle$ and $\langle \eta, A\rangle$ be normal fuzzy subgroups of G. Define a fuzzy soft set $\langle \bar{\eta}, A\rangle$ over G/μ by for each $\alpha \in A$, ${}^x\mu \in G/\mu$ and $t \in I$, $\langle \alpha, {}^x\mu, t\rangle \in \bar{\eta} \Leftrightarrow \langle \alpha, a, t\rangle \in \eta$ for some $a \in G$ with ${}^x\mu = {}^a\mu$. Then $\langle \bar{\eta}, A\rangle$ is a normal fuzzy soft subgroup of G/μ.*

Proof. Let $\alpha \in A$, ${}^x\mu \in G/\mu$ and $t \in I$. Suppose $\langle \alpha, {}^{x^{-\alpha}}\mu, t\rangle \in \bar{\eta}$. It follows that $\langle \alpha, a^{-\alpha}, t\rangle \in \eta$ for some $a^{-\alpha} \in G$ with ${}^{x^{-\alpha}}\mu = {}^{a^{-\alpha}}\mu$. Since $\langle \eta, A\rangle$ is a normal fuzzy soft subgroup of G, $\langle \alpha, a, t\rangle \in$

η for some $a \in G$ with ${}^{x}\mu = {}^{a}\mu$. Let ${}^{x}\mu, {}^{y}\mu, {}^{z}\mu \in G/\mu$ and $r, s, t \in I$ such that $\langle \alpha, {}^{x}\mu, {}^{y}\mu, {}^{z}\mu \rangle \in \circledast$, $\langle \alpha, {}^{x}\mu, r \rangle \in \bar{\eta}$, $\langle \alpha, {}^{y}\mu, s \rangle \in \bar{\eta}$ and $\langle \alpha, {}^{z}\mu, t \rangle \in \bar{\eta}$. This implies that $\langle \alpha, x, y, c \rangle \in *$, $\langle \alpha, a, r \rangle \in \eta$, $\langle \alpha, b, s \rangle \in \eta$ and $\langle \alpha, c, t \rangle \in \eta$ for some $a, b, c \in G$ with ${}^{x}\mu = {}^{a}\mu$, ${}^{y}\mu = {}^{b}\mu$, ${}^{z}\mu = {}^{c}\mu$. It follows that $t \geq min\{r, s\}$. Thus $\langle \bar{\eta}, A \rangle$ is a fuzzy soft subgroup of G/μ. Further, let ${}^{x}\mu, {}^{y}\mu, {}^{z}\mu, {}^{k}\mu \in G/\mu$ and $t \in I$ such that $\langle \alpha, x, y, k \rangle \in *$, $\langle \alpha, {}^{x}\mu, {}^{y}\mu, {}^{z}\mu \rangle \in \circledast$, $\langle \alpha, {}^{y}\mu, {}^{x}\mu, {}^{k}\mu \rangle \in \circledast$. This implies $\langle \alpha, x, y, c \rangle \in *$, $\langle \alpha, y, x, d \rangle \in *$ for some a, b, c, d with ${}^{x}\mu = {}^{a}\mu$, ${}^{y}\mu = {}^{b}\mu$, ${}^{z}\mu = {}^{c}\mu$ and ${}^{k}\mu = {}^{d}\mu$. Now, $\langle \alpha, {}^{z}\mu, t \rangle \in \bar{\eta}$. It implies that $\langle \alpha, c, t \rangle \in \eta$ for some $c \in G$ with ${}^{c}\mu = {}^{z}\mu$. Since $\langle \eta, A \rangle$ is a normal fuzzy soft subgroup of G, $\langle \alpha, d, t \rangle \in \eta$ for some $d \in G$ with ${}^{d}\mu = {}^{k}\mu$. Therefore $\langle \bar{\eta}, A \rangle$ is a normal fuzzy soft subgroup of G/μ.

□

Conclusion and future directions

This thesis presents a new approach to soft groups based on soft binary operations aiming to incorporate the concept of ”softness” into the realm of algebraic structures. By proposing a novel definition for soft groups and utilizing soft binary operations parameterized by suitable parameters, we have successfully introduced a framework to model and analyze algebraic structures that capture uncertainty and imprecision.
Moreover, we construct an ordinary group model that represents our soft group. This model is important to describe and characterize the overall internal structure of soft groups through the existing classical group theories. The study of soft subgroups and normal soft subgroups further enhances our understanding of the internal structure of soft groups. We believe that our research will contribute to the growing field of soft computing and pave the way for new applications in various domains.

We suggest the following possible future works from our studies:

(1) Further investigate soft groups especially structure soft groups.

(2) Study the concept of soft rings, soft modules and soft lattice.

(3) It is under investigation by the author to extend the categorical structure of soft groups and fuzzy soft rings.

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