ADDIS ABABA UNIVERSITY

COLLEGE OF NATURAL AND COMPUTAIONAL SCIENCES

DEPARTMENT OF MATHEMATICS

POST GRADUATE PROGRAM THESIS

ON

CONVEX QUADRATIC PROGRAMMING

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Permission

This is to certify that this project is compiled by Mr.Abebe Setargie in the Department of Mathematics, Addis Ababa University ,under my supervision .I here by also confirm that the project can be submitted for evaluation by examiners and eventual defense.

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Abstract

Quadratic programming is one of the types of . There is no single method available for solving quadratic programming problem. So a number of methods have been developed for solving quadratic programming problems. In this paper we consider and method, method and .   Method and penalty method are very important methods to solve equality, inequality and mixed constraint quadratic programming problems.

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CHAPTER ONE

Introduction

A quadratic program (QP) involves the minimization or maximization of a quadratic objective function subject to linear equality and inequality constraints on the variables.

The general form of quadratic programming problem is given by

Where

Quadratic programming problems (QPP) arise in many areas, including economics, applied science and engineering. Important applications include portfolio analysis, support vector machines, structural analysis and optimal control. Quadratic programming also forms a principal computational component of many sequential quadratic programming methods for nonlinear programming (for a recent survey, see Gilland Wong [13]).

The method under consideration defines a primal-dual search pair associated with the solution of an equality-constrained sub problem involving a working set of linearly independent constraints. Unlike existing quadratic programming methods, the working set may include constraints that need not be active at the current iterate. In this context, we reformulate a method for a general Quadratic Program that was first proposed by Fletcher [17], and modified subsequently by Gould [10]. In this reformulation, the primal-dual search directions satisfy a Karush Kuhn Tucker system of equations formed from the Hessian H and the gradients of the constraints in the working set. The working set is specified by an active-set strategy that controls the inertia (i.e., the number of positive, negative and zero Eigen values) of the Karush Kuhn Tucker matrix.

Not all active-set methods for a general Quadratic Program are inertia controlling see, for example, the methods of Bunch and Kaufman [6], Friedlander and Layer [9], and the quadratic programming methods in the GALAHAD software package of Gould, Orban, and

Toint [11]. A number of alternative methods have been proposed for strictly convex quadratic programming with a modest number of constraints and variables, see, e.g., Goldfarb and Idnani [3], Gill et al. [14], and Powell [8].

This paper focuses on further study about concepts, theories and solution methods of convex quadratic programming problems with linear constraints.

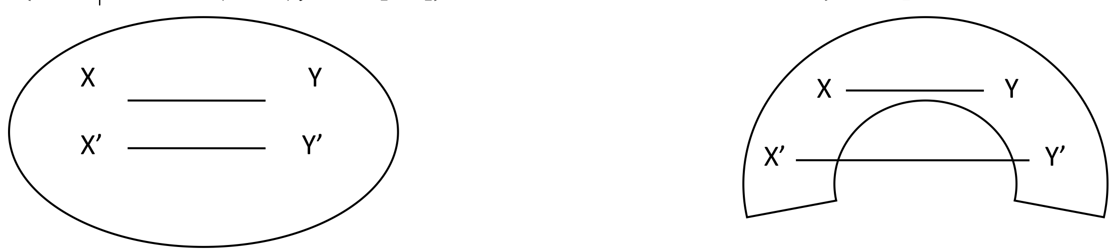
1.1. Preliminary concepts (background materials)

1.1.1 Convexity and minimization

Let are called convex combination of.

Geometrically expressed, a set S is convex, if for all $x, y \in S$, the whole segment

$[x, y] = \{z \in S | z = \lambda x + (1-\lambda) y, \lambda \in [0,1]\}$ is contained in S that is $[x, y \subseteq S]$.



Convex set   not convex set

Definition: - A set is called a convex set if for all and for all it holds that

A function where S is a non empty convex set, is convex function if

Definition: - let $M \subseteq R^n$ be a convex set and $f : M \to R$ then

A. The function f is said to be convex if and only if for all $x, y \in M; f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$ for all $\lambda \in [0,1]$

B. The function f is said to be strictly convex if and only if for all $x, y \in M; f(\lambda x + (1-\lambda) y) < \lambda f(x) + (1-\lambda) f(y)$ for all $\lambda \in [0,1]$

C. The function f is said to be concave if and only if for all $x, y \in M; f(\lambda x + (1-\lambda) y) \geq \lambda f(x) + (1-\lambda) f(y)$ for all $\lambda \in [0,1]$

Definition: - The Hessian matrix or Hessian is a square matrix of second order partial derivatives of a scalar valued function or scalar field .

Specifically , suppose is a function Taking as input a vector and outputting a scalar . if all second partial derivatives of exist and are continuous over the domain of the function , then the Hessian H of is a square matrix ,usually defined and arranged as follows.

Let be a convex set , let be twice differentiable and is the Hessian matrix of

A. if and is positive definite, then has a strict local minimum at
B. if and is negative definite, then has a strict local maximum at
C. if and has both negative and positive Eigen values , then does not have a local maximum and a local minimum at that is has a saddle point at .
D. If is possitive semi definite for all , then is convex and has a strict global minimum at any for which and is positive definite.
E. If is negative semi definite for all , then is concave and has a strict global maximum at any for which and is negative definite.
Since the determinant of a matrix is the product of its Eigen values we have the following the following special cases.

Let be a convex set , let be twice differentiable and is the Hessian matrix of (note that a,b,c are functions of).

A. , then has a strict local minimum at .
B. , then has a strict local maximum at .
C. and then has a saddle point at .
D. If and ,then is convex and has a strict global minimum at any for which
E. E. If and ,then is concave and has a strict global maximum at any for which

Definition: - Let the Hessian matrix H be symmetric matrix, then H is said to be;

a) Positive definite if

b) Positive semi definite if

c) Negative definite 0

d) Negative semi definite 0

e) Indefinite , if there exists

Example: let A be (m,n) matrix, $b \in R^m$ *and* $S = \{x \in R^n \,|\, Ax \le b\}$.

If $x, y \in S$ then $Ax \le b$ and $Ay \le b$, for $\lambda \in [0,1]$ we get

$\lambda[Ax] \le \lambda b$ And $(1-\lambda)\,[Ay] \le (1-\lambda)b$

Or

$A(\lambda x) \le \lambda b$ and $A(1-\lambda)y \le (1-\lambda)b$

When we add the last two inequalities we get

$A(\lambda x) + A(1-\lambda)y \le \lambda b + (1-\lambda)b$

$A(\lambda x + (1-\lambda)y) \le b$ From this we get $(\lambda x) + (1-\lambda)y \in S$. So, S is a convex set.

1.1 .2 Quadratic programming problems

Definition: - quadratic programming problem is a special type of optimization problem in which a quadratic objective function is minimized or maximized subject to linear constraints.

The general form of quadratic programming problem is given by

Minimize

Subject to

where .

.

is a convex quadratic objective function .

all functions, are differentiable on

Definition: - If and are partially differentiable on and let be minimum point of then we have the following necessary conditions

1

2

3

4

is called condition **.**

The Kuhn Tucker conditions are used to Test a point to determine whether or not it is a critical point in a constrained non linear program. They do not actually determine whether the point is a local optimum , just that it is a critical point (could be a local maximum , a local minimum ,or a saddle point).

The conditions are sufficient for a global minimum if

a. The Hessian matrix is positive definite.
b. The objective function is convex and smooth
c. The feasible space is convex and also the mixed constraints are affine (linear).

1.1.3 Concept of penalty functions

Methods using penalty functions transform a constrained problem in to a single unconstrained problem or in to a sequence of unconstrained problems. The constraints are placed in to the objective function via a penalty parameter in a way that penalizes any violation of the constraints. To motivate penalty functions, consider the following problem having the single constraint :

Minimize

Subject to

Suppose that this problem is replaced by the following unconstrained problem, where is a large number:

Minimize

Subject to

We can intuitively see that an optimal solution to the above problem must have close to zero. Otherwise, a large penalty will be incurred.

Now consider the following problem having single inequality constraint

Minimize

Subject to

It is clear that the form is not appropriate, since a penalty will be incurred whether . Needless to say, a penalty is desired only if the point is not feasible, that is, if

In general, a suitable penalty function must incur a positive penalty for infeasible points and no penalty for feasible points.

CHAPTER TWO

2. CONVEX QUADRATIC PROGRAMMING

Mathematically, optimization is the process of finding the conditions that give the maximum or the minimum value of a function. There is no single method available for solving all optimization problems efficiently. Hence, a number of methods have been developed for solving different types of problems. Quadratic programming is one of these methods.

Quadratic programming is an important class of non linear programming in which quadratic objective function is to be minimized or maximized subject to linear constraints. Based on the type of constraint quadratic programming problem can be classified as constrained and unconstrained quadratic programming problem.

Quadratic programming problem may be stated in many equivalent forms and we consider several different formulations. We define a Quadratic programming in the most general form to be

Minimize

Subject to

Where is the vector of decision variables.

is the symmetric hessian matrix

is the linear constraint matrix

is the linear objective vector

is column vector

The convexity of the quadratic objective function plays a defining role in the computational complexity of quadratic programming methods. We say that quadratic programming problem is convex if the objective function is a convex function.

The objective function is convex when the hessian matrix H is positive semi definite and strictly convex when H is positive definite.

When is convex a local minimum of the quadratic programming also a global minimum. When H is indefinite or negative definite is non convex and local optimal points may not be global optimal solutions. This paper concerns a method for solving convex quadratic programming problems with linear constraints when the hessian matrix H is positive semi definite.

Theorem:- Suppose S is a convex set and is a convex function. is a local minimum. then is a global minimum of over S.

Proof:- Suppose is not a local minimum that is there exists which f

Let =+(1-)y, which is a convex combination of and y for all .

() ∈ S f ∈ , note that () , as , from the convexity of or

This contradicts the hypothesis is convex function.

Therefore, is a local minimum and it is also a global minimum.

2.1 Penalty Method

A penalty method is an algorithm for solving a constrained problem by solving a sequence of unconstrained problems whose objectives contain a combination of the original objective and a penalty for violating the constraints. A penalty function is a function with the property that if satisfies the constraints. The additional argument to P indicates that the penalty function also depends on a penalty parameter we consider the penalty method applied to the quadratic programming problem.

Minimize

Subject to (

where

is a convex quadratic objective function and all functions, are differentiable on .

The penalty method to solving consists of replacing it by a sequence of unconstraint optimization problem of the form:

where is a positive parameter (increasing integer);

Definition :- function is called a penalty function for P if satisfies:

for each and

for each

A commonly used penalty function for is

Thus, the penalty method constructs a sequence of iterate points where is an optimal solution of for each

Eventually, an accumulation (limit) point of is a solution of That is,

is an optimal solution of

The penalty method generates a sequence of infeasible points whose limit is an optimal solution to the original problem.

Consider the constrained optimization problem

(2.2)

When we solve the above constraint nonlinear optimization problem by using penalty method the first task is to replace the constraint nonlinear optimization problem by unconstraint nonlinear optimization problem as follow

i.e. replace problem by

Where is a positive constant called penalty parameter

is called penalty function satisfying

I. for all not elements of
II. for all elements of

The connection between the constraint optimization problem and unconstraint optimization problem will be given by the following theorem

Theorem :- Let be continuous function on and .let be a minimum point of. Then each accumulation (limit) point of the sequence is a solution (minimum point) of.

Proof :- Let be an accumulation (limit) point of and be a sequence where

1. we prove that because is a solution of we have

Because are continuous we have

and

There fore

2.Now we will prove is a minimum point on

That is is minimum point of

2.2 Lagrangian method to solve convex quadratic programming

Let be differentiable. And consider a constrained nonlinear minimization problem:

min

Subject to

Where

To convert into unconstrained problem (or relatively easier problem), we try to find a functional on with the following.

differentiable, and

Then we consider an auxiliary functional L such that

L given by

Such L is called Lagrange functional (Lagrangian) for

Then, we consider the auxiliary (Lagrange) problem

min

note that is differentiable.

Theorem :- (Lagrange Lemma)

a.If is a global minimizer of and then is optimal solution of

b.If is a local minimizer of and , then is local optimal solution of). **Proof :** Since is given on , if is a (local/global) minimizer of ,

i.e., is a necessary condition for to be a minimizer of L on R^n.

If so then = 0 and are necessary conditions for to be a minimizer of) .

2.3 Lagrange method for mixed constraints

We consider optimization problem

min

Subject to

Where

Let T

Now, we define Lagrange function for (P) as follows

For any = (T +, i.e., $\lambda_{\_i} \geq 0$, ∀i, and

= , …,)T ∈ define

: , by

Thus, corresponding auxiliary problem

for any and

Theorem:- Let +, , , and , g() ⟩ = 0.

If is an optimal solution of (L $_{\lambda^*, \mu^*}$), then is an optimal solution of (2.4).

Hence, we have the following necessary condition for to be optimal solution of (2.4).

Karush-Kuhn-Tucker (KKT) conditions (for mixed constraints):

If is an optimal solution of (2.4), then for some , the following holds:

Dual feasibility

Primal feasibility

Complementary slackness

Remark: In the mixed constraints problem (above, if 0,i.e.,
then The Karush-Kuhn-Tucker (KKT) conditions become:

1 (

Example 2.1: let C= $\begin{pmatrix} -12 \\ -44 \end{pmatrix}, Q =$ then we consider the following quadratic optimization problem.

$$f(x_1, x_2) = C^T x + \frac{1}{2} x^T Q x \rightarrow \text{min}, \ x \in S$$

$$S = \{x \in R^n | Ax \le b, x \ge 0 \ \}$$

$$f(x_1, x_2) = (-12, -44)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \frac{1}{2}(x_1, x_2)$$

$$S = \{(x_1, x_2) \in R^2 | \quad \text{or}$$

$$f(x_1, x_2) = 5x_1^2 - 6x_1x_2 + 5x_2^2 - 12x_1 - 44x_2 \rightarrow \text{min}, x = (x_1, x_2) \in S$$

$$S = \{(x_1, x_2) \in R^2 | x_1 + x_2 \le 8, 4x_1 - 3x_2 \le 12, x_1 \ge 0, x_2 \ge 0 \ \}$$

Obviously, Q is positive definite that is the Hessian matrix
$H(x) =$ is positive semi-definite. So we have that f is a convex function. Since S is a convex set, we have that (p) is a convex optimization problem. The solution is $(x_1^*, x_2^*) = (3,5)$.

2.4 Wolfe's quadratic programming idea

To solve a quadratic programming problem we can rewrite the original constraints with slack and use the complementary slackness conditions to ensure the second and third Karush Kuhn Tucker conditions. For a quadratic programming problem, the main Karush Kuhn Tucker condition is obviously linear. We can therefore calculate the main Karush Kuhn Tucker condition(first condition for the given quadratic programming problem and use an extended simplex that ensures the complementary slackness conditions (Wolfe's quadratic programming problem simplex) to find the optimum.

Wolfe's method is an extended simplex procedure, which can be applied to quadratic programming problem in which all problem variables are non negative . the basis are the complementary slackness conditions for quadratic programming problems.

CHAPTER THREE

3. Solution methods for convex quadratic programming problem

We can solve any quadratic programming problem by one of the following methods.

I. Penalty method
II. method and Karush Kuhn Tucker condition
III. Algorithm of wolf

3.1 Penalty method:

3.1.1 Penalty Method for an Optimization Problem with Mixed Constraints

The idea of penalty method is to replace a given optimization problem which has constraints to unconstraint optimization

Constraint non-linear optimization problem

(

Step of penalty method

Step one: - choose the penalty function

Where

Step two:- Find the corresponding unconstraint non-linear optimization problem. i.e.

The corresponding unconstraint non-linear optimization problem

Step three: - solve the unconstrained optimization problem applying the first order necessary condition to obtain the optimal value

i.e.

by using Hessian matrix check the convexity characteristics of .if the Hessian matrix is positive semi definite , we have an optimal solution

Example :-Solve the following nonlinear optimization problems by using penalty method

Solution

First let as find the corresponding unconstraint optimization of the constraint optimization problem

When

Where,

If the point is not in our feasible region this implies that

Check whether convex function or not

The hessian matrix of is

The first principal minors is which is positive

The second principal minor is which is positive

This implies that since H is positive semi definite   is convex

Let be minimum point of .Thus, a necessary and sufficient condition for optimality is that:

When we solve these two equations simultaneously, we get)

Thus, the optimal solution of the penalty problem can be made arbitrarily close to the solution of the original problem by choosing sufficiently large.

The sequence of will converge to the following

and

Therefore   is optimal point and   is optimal value.

Else

Check whether convex function or not

The hessian matrix of is

The first principal minors is which is positive

The second principal minor is which is positive

This implies that is convex

Let be minimum point of

When we solve these two equations simultaneously, we get and

Thus, the optimal solution of the penalty problem can be made arbitrarily close to the solution of the original problem by choosing sufficiently large.

The sequence of will converge to the following

and

This implies

Therefore is the optimal point and is optimal value

3.2 The Lagrange method and Karush Kuhn Tucker condition

Consider a constrained optimization problem in

A set is the constraint set.

To transform the constraint optimization problem to unconstraint optimization problem we use the Lagrange function method.

The function

where is called the **Lagrange function** or simply the Lagrange of problem.

The vector and are called Lagrange multiplier of problem

Then we have now unconstrained optimization problem for

for any feasible point .That is for.

3.2.1 The Lagrange Method for Optimization Problem with Mixed Constraint

Consider the problem

for

Where

That is

Then the corresponding unconstraint optimization problem is

Where and

Then the Karush Kuhn Tucker condition is

1.
2.
3.
4.
5.

Example:- Solve the following non linear programming problem by using Lagrange method

Solution

The corresponding Lagrange function is

This implies that

is the corresponding unconstraint optimization problem

Now let as see the necessary optimality condition

The Karush-Kuhn-Tucker (KKT) condition

1.
2.
3.
4.
5.
6.
7.
8. ,

To solve this system of equations and inequalities we consider the following cases.

Case one:- **if** .Then from equation and equation we get

.but this is a contradiction to equation

Case two :- if

From equation we have the equation system

With the solution From equation we get

By the convexity of we have as a solution of with the minimal value

3.3Algorithm of wolf

We can formulate an algorithm to solve a quadratic optimization problem using algorithm of wolf. Let have the Lagrange function we consider the following four steps.

Step1. Formulate the Kuhn-Tucker condition for the given quadratic optimization problem.

That is The Karush-Kuhn-Tucker (KKT) condition

1

2.

5.

Step2. Introduce slack and surplus variable in order to have equations. Check whether there is a feasible variable. If there is a feasible basis, then the corresponding basic solution is optimal. If there is not a feasible basis, then go to step 3

Step3. Introduce artificial variables in order to get a feasible basis. Go to step 4

Step4. Iterating by means of the Big –M method until a feasible basic solution is found or until verifying that there is no feasible solution .for changing of basic variables there are restrictions:

A. Non-basic variables may become new basic variable only. If the corresponding slack (surplus) variable is not already a basic variable and conversely

B. Non-basic variables may become new basic variable only. If the corresponding slack variable is not already a basic variable and conversely

Example :- Solve the following non linear programming problem by using Wolf Algorithm.

Solution

The corresponding Lagrange function is

The corresponding unconstraint optimization problem is

Step1. The Karush-Kuhn-Tucker (KKT) condition

1.

3.

4.

5.

7.

8.

9.

Step2. Introduce slack and surplus variable

Step3. Introduce artificial variables

Step4. Iterating by means of the Big –M

BV														**RHS**
		0	**1**	**-4M**	**-4M**	**-2M**	**-M**	**M**	**M**	**0**	**0**	**0**	**0**	
M		1	0	10	-6	1	4	-1	0	0	0	1	0	12
M		2	0	-6	10	1	-3	0	-1	0	0	0	1	44
0		3	0	1	1	0	0	0	0	1	0	0	0	8
0		4	0	4	-3	0	0	0	0	0	1	0	0	12

The tabular of the above optimization problem is we take as entering variable, as leaving variable and 10 as pivot element since is not basic variable can be basic variable.

BV														RHS
		0	1	0	-32/5M	-8/5M	3/5M	3/5M	M	0	0	2/5M	0	
0		1	0	1	-3/5	1/10	2/5	-1/10	0	0	0	1/10	0	6/5
M		2	0	0	32/5	8/5	-3/5	-3/5	-1	0	0	3/5	1	256/5
0		3	0	0	8/5	-1/10	-2/5	1/10	0	1	0	-1/10	0	34/5
0		4	0	0	-3/5	-2/5	-8/5	2/5	0	0	1	-2/5	0	36/5

We take as entering variable, as leaving variable and 8/5 as pivot element since is not basic variable can be basic variable

BV													RHS

		0	1	0	0	-2M	-M	M	M	4M	0	0	0	
0		1	0	1	0	1/16	¼	-1/16	0	3/8	0	1/16	0	15/4
M		2	0	0	0	2	1	-1	-1	-4	0	1	1	24
0		3	0	0	1	-1/16	-1/4	1/16	0	5/8	0	-1/16	0	17/4
0		4	0	0	0	-7/16	-7/4	7/16	0	3/8	1	-7/16	0	39/4

We takeas entering variable, as leaving variable and 2 as pivot element since is not basic variablecan be basic variable.

BV														RHS
		0	1	0	0	0	0	0	0	0	0	M	M	
0		1	0	1	0	0	7/32	-1/32	1/32	½	0	1/32	-1/32	3
0		2	0	0	0	1	½	-1/2	-1/2	-2	0	½	½	12
0		3	0	0	1	0	-7/32	1/32	-1/32	½	0	-1/32	1/32	5
0		4	0	0	0	0	-49/32	7/32	-7/32	-1/2	1	-7/32	7/32	15

Since all are nonnegative we have optimal table

, , and

is optimal value which occures at an optimal solution of .

3.4.1 An alternative algorithm for Wolfe's modified simplex method

To find optimal of any Quadratic Programming Problems by an alternative method for Wolfe's modified simplex method , algorithm is given as follows

Step 1:-first convert inequality constraints in to equations by introducing slack variables in the constraints and the non negative restrictions by introducing slack variables in the restrictions.

Step 2:- construct the lagrangian function

Differentiating this lagrangian function with respect to the components of and equating the first order partial derivatives to zero, derive Karush khun Tucker condition from the resulting equation.

Step 3:- Introduce non negative artificial variables in the Karush khun Tucker condition.

For and construct an objective function

Step 4:- obtain the initial basic feasible solution to the linear programming problem:

min.

Subject to

and satisfying the slackness condition =0

Step 5:- solves this LPP by an alternative to phase method. Choose greatest coefficient of decision variables.

A. If the greatest coefficient is unique, then the variable corresponding to this column incoming variable .
B. if the greatest coefficient is not unique , then we use tie breaking technique.

Step 6:- compute the ratio with choose minimum ratio ,then the variable corresponding to this row is out going variable . The element corresponding to incoming variable and outgoing variable becomes pivotal (leading) element.

Step 7:- use usual simplex method for table and go to next step.

Step 8:- ignore corresponding row and column. Proceed to step 5 for remaining elements and repeat the same procedure either an optimal solution is obtain or there is an indication of an unbounded solution .

Step 9:- if all rows and columns are ignored , the current solution is an optimal solution .thus optimal solution is obtained and which is optimum solution of the given Quadratic programming problem.

Example:- solve the following quadratic programming problem.

Minimize

Subject to

Solution

first we convert the inequality constraints in equations by introducing slack variables respectively. Also the inequality constraints we convert them in to equations by introducing slack variables . so the problem becomes

Minimize

Subject to

Now ,construct the lagrangian function

By Karush khun Tucker condition ,we get

Where satisfying the complementary slackness conditions.

Now introducing the artificial variable the given quadratic programming problem is equivalent to

Minimize

Subject to

Where

Simplex table

CB	**BVS**												
1		4	2	-2	2	1	-1	0	1	0	0	0	
1		0	-2	**4**	1	-4	0	-1	0	1	0	0	
0		6	2	1	0	0	0	0	0	0	1	0	
0		0	1	-4	0	0	0	0	0	0	0	1	

From the above table we take as entering variable, as leaving variable and 4 as pivot element.

CB	**BVS**												
1		4	1	0	5/2	-1	-1	-1/2	1	1/2	0	0	
0		0	-1/2	1	1/4	-1	0	-1/4	0	1/4	0	0	
0		6	5/2	0	-1/4	1	0	1/4	0	1/4	1	0	
0		0	-1	0	1	-4	0	-1	0	1	0	1	

From the above table we take as entering variable, as leaving variable and 5/2 as pivot element.

CB	BVS												
1		8/5	0	0	13/5	-7/5	-1	-3/5	1	2/5	-2/5	0	
0		6/5	0	1	1/5	-4/5	0	-1/5	0	3/10	1/5	0	
0		12/5	1	0	-1/10	2/5	0	1/10	0	1/10	2/5	0	
0		12/5	0	0	9/10	-18/5	0	-9/10	0	11/10	2/5	1	

From the above table we take as entering variable, as leaving variable and 13/5 as pivot element.

CB	BVS												
0		8/13	0	0	1	-7/13	-5/13	-3/13	5/13	2/13	-2/13	0	
0		14/13	0	1	0	-9/13	1/13	-2/13	-1/13	7/26	3/13	0	
0		32/13	1	0	0	9/26	-1/26	1/13	1/26	3/26	5/13	0	
0		24/13	0	0	0	-81/26	9/26	-9/13	-9/26	25/26	7/13	1	

is the optimal solution of the given quadratic programming problem. And

is the minimum value of the given quadratic programming problem.

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