Two-Mode Coherent and Subharmonic Light Beams

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Abstract

Applying c-number Langevin equations, we have calculated the Q function for the two-mode coherent and subharmonic light. We have then determined the Q function for the superposition of these two light beams. With the aid of the pertinent Q functions, we have obtained the mean and variance of the photon number sum and difference for the two-mode coherent light, the two-mode subharmonic light, and the superposition of the two-mode coherent and subharmonic light beams.

We have found that the signal and idler modes are separately in chaotic states. We have also found that the mean photon number for the superposition of two-mode coherent and subharmonic light turns out to the sum of the mean photon number of the individual light beams.

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Chapter 1

Introduction

Quantum optics deals mainly with the quantum properties of the light generated by various optical systems such as lasers and with the effect of light on the dynamics of atoms. The quantum properties of light are largely determined by the state of the light mode and the well known quantum states of light are the number state, the coherent state, the chaotic state, and the squeezed state. A considerable interest is particularly shown in squeezed state [1, 6, 8, 10, 13]. Squeezed light can be generated from light in a coherent state by means of certain nonlinear optical interactions. Some of the quantum optical processes which can generate squeezed light are subharmonic generation [1-10], second harmonic generation [1, 3], and four-wave mixing [3, 5, 6,].

A subharmonic generator has been considered as a typical source of squeezed light. It is one of the most interesting and well characterized optical device in quantum optics. In this device a pump photon interacts with a nonlinear crystal inside a cavity and is down converted into two highly correlated photons. If these photons have the same frequency the device is called a one-mode subharmonic generator, otherwise it is called a two-mode subharmonic generator.

In this thesis, with the aid of the pertinent master equation we obtain c-number Langevin equations for a two-mode coherent light and the light produced by a two-mode subharmonic generator with the cavity modes coupled to a two-mode vacuum reservoir via a single port mirror. With the aid of the resulting equations, we determine the Q function for these light beams. Applying the pertinent Q function, we calculate the mean and variance of the photon number sum and difference. In addition, we also determine the Q function for the superposition of the two-mode coherent and subharmonic light beams and employing the resulting Q function, we calculate the mean and variance of the photon number.

Chapter 2

Two-Mode Coherent Light

2.1 c-number Langevin equation

We consider here a cavity mode driven by a two-mode coherent light and coupled to a two-mode vacuum reservoir. A two-mode cavity light is driven by two-mode coherent light and coupled to a two-mode vacuum reservoir via a single port mirror. The interaction between the two-mode cavity light and the two-mode coherent light, can be described by the Hamiltonian

$$\hat{H} = i\varepsilon(\hat{a}^\dagger - \hat{a} + \hat{b}^\dagger - \hat{b}), \tag{2.1.1}$$

where $\hat{a}(\hat{b})$ is the annihilation operator for the two-mode cavity light and ε is proportional to the amplitude of the driving light modes. Using (2.1.1) and taking into account the interaction of the two-mode cavity light with a two-mode vacuum reservoir via a single port mirror, the equation of evolution of the density operator for the two-mode cavity light can be written as [1]

$$\begin{aligned} \frac{d\hat{\rho}}{dt} &= -\varepsilon(\hat{a}\hat{\rho} - \hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a} + \hat{\rho}\hat{a}^\dagger + \hat{b}\hat{\rho} - \hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{b} + \hat{\rho}\hat{b}^\dagger) \\ &+ \frac{\kappa}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) + \frac{\kappa}{2}(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}). \end{aligned} \tag{2.1.2}$$

Using the relation

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr(\frac{d\hat{\rho}}{dt}\hat{A}), \tag{2.1.3}$$

along with Eq. (2.1.2), we have

$$\begin{aligned} \frac{d}{dt}\langle\hat{a}(t)\rangle \quad &= -\varepsilon Tr(\hat{a}\hat{\rho}\hat{a} - \hat{a}^{\dagger}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^{2} + \hat{\rho}\hat{a}^{\dagger}\hat{a} + \hat{b}\hat{\rho}\hat{a} - \hat{b}^{\dagger}\hat{\rho}\hat{a} - \hat{\rho}\hat{b}\hat{a} + \hat{\rho}\hat{b}^{\dagger}\hat{a}) \\ &+\frac{\kappa}{2}Tr(2\hat{a}\hat{\rho}\hat{a}^{\dagger}\hat{a} - \hat{a}^{\dagger}\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^{\dagger}\hat{a}^{2}) + \frac{\kappa}{2}Tr(2\hat{b}\hat{\rho}\hat{b}^{\dagger}\hat{a} - \hat{b}^{\dagger}\hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{b}^{\dagger}\hat{b}\hat{a}). \end{aligned} \tag{2.1.4}$$

Applying the cyclic property of the trace operation together with the commutation relations

$$[\hat{a}, \hat{a}^{\dagger}] = 1, \tag{2.1.5}$$

and

$$[\hat{a}, \hat{b}] = [\hat{a}, \hat{b}^{\dagger}] = [\hat{a}^{\dagger}, \hat{b}] = [\hat{a}^{\dagger}, \hat{b}^{\dagger}] = 0, \tag{2.1.6}$$

we readily find

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = -\frac{\kappa}{2}\langle\hat{a}(t)\rangle + \varepsilon. \tag{2.1.7}$$

It can be shown in a similar procedure that

$$\frac{d}{dt}\langle\hat{b}(t)\rangle = -\frac{\kappa}{2}\langle\hat{b}(t)\rangle + \varepsilon, \tag{2.1.8}$$

$$\frac{d}{dt}\langle\hat{a}(t)\hat{b}(t)\rangle = -\kappa\langle\hat{a}(t)\hat{b}(t)\rangle + \varepsilon(\langle\hat{a}(t)\rangle + \langle\hat{b}(t)\rangle), \tag{2.1.9}$$

$$\frac{d}{dt}\langle\hat{a}^{2}(t)\rangle = -\kappa\langle\hat{a}^{2}(t)\rangle + 2\varepsilon\langle\hat{a}(t)\rangle, \tag{2.1.10}$$

$$\frac{d}{dt}\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle = -\kappa\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle + \varepsilon(\langle\hat{a}^{\dagger}(t)\rangle + \langle\hat{a}(t)\rangle), \tag{2.1.11}$$

$$\frac{d}{dt}\langle\hat{a}^{\dagger}(t)\hat{b}(t)\rangle = -\kappa\langle\hat{a}^{\dagger}(t)\hat{b}(t)\rangle + \varepsilon(\langle\hat{a}^{\dagger}(t)\rangle + \langle\hat{b}(t)\rangle). \tag{2.1.12}$$

The normal-ordering c-number equations corresponding to Eqs. (2.1.7), (2.1.8), (2.1.9), (2.1.10), (2.1.11), and (2.1.12) are

$$\frac{d}{dt}\langle\alpha(t)\rangle = -\frac{\kappa}{2}\langle\alpha(t)\rangle + \varepsilon, \tag{2.1.13}$$

$$\frac{d}{dt}\langle\beta(t)\rangle = -\frac{\kappa}{2}\langle\beta(t)\rangle + \varepsilon, \tag{2.1.14}$$

$$\frac{d}{dt}\langle\alpha(t)\beta(t)\rangle = -\kappa\langle\alpha(t)\beta(t)\rangle + \varepsilon(\langle\alpha(t)\rangle + \langle\beta(t)\rangle), \tag{2.1.15}$$

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = -\kappa\langle\alpha^2(t)\rangle + 2\varepsilon\langle\alpha(t)\rangle, \tag{2.1.16}$$

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = -\kappa\langle\alpha^*(t)\alpha(t)\rangle + \varepsilon(\langle\alpha^*(t)\rangle + \langle\alpha(t)\rangle), \tag{2.1.17}$$

$$\frac{d}{dt}\langle\alpha^*(t)\beta(t)\rangle = -\kappa\langle\alpha^*(t)\beta(t)\rangle + \varepsilon(\langle\alpha^*(t)\rangle + \langle\beta(t)\rangle). \tag{2.1.18}$$

On the basis of Eqs. (2.1.13) and (2.1.14), one can write

$$\frac{d}{dt}\alpha(t) = -\frac{\kappa}{2}\alpha(t) + \varepsilon + f_\alpha(t) \tag{2.1.19}$$

and

$$\frac{d}{dt}\beta(t) = -\frac{\kappa}{2}\beta(t) + \varepsilon + f_\beta(t), \tag{2.1.20}$$

where $f_\alpha(t)$ and $f_\beta(t)$ are noise forces associated with the normal ordering. We next seek to determine the properties of the noise forces $f_\alpha(t)$ and $f_\beta(t)$. We note that Eq. (2.1.13) and the expectation value of Eq. (2.1.19) as well as Eq. (2.1.14) and the expectation value of Eq. (2.1.20) will have the same form if

$$\langle f_\alpha(t)\rangle = \langle f_\beta(t)\rangle = 0. \tag{2.1.21}$$

Moreover, using Eqs. (2.1.19) and (2.1.20) together with the relation

$$\frac{d}{dt}\langle\alpha(t)\beta(t)\rangle = \langle\frac{d\alpha(t)}{dt}\beta(t)\rangle + \langle\alpha(t)\frac{d\beta(t)}{dt}\rangle, \tag{2.1.22}$$

we find

$$\frac{d}{dt}\langle\alpha(t)\beta(t)\rangle = -\kappa\langle\alpha(t)\beta(t)\rangle + \varepsilon(\langle\alpha(t)\rangle + \langle\beta(t)\rangle) + \langle\alpha(t)f_\beta(t)\rangle + \langle\beta(t)f_\alpha(t)\rangle. \tag{2.1.23}$$

Comparison of Eqs. (2.1.15) and (2.1.23) indicates that

$$\langle\alpha(t)f_\beta(t)\rangle + \langle\beta(t)f_\alpha(t)\rangle = 0. \tag{2.1.24}$$

The formal solution of Eqs. (2.1.19) and (2.1.20) can be written as

$$\alpha(t) = \alpha(0)e^{-\kappa t/2} + \int_0^t e^{-\kappa(t-t')/2}[\varepsilon + f_\alpha(t')]dt', \tag{2.1.25}$$

and

$$\beta(t) = \beta(0)e^{-\kappa t/2} + \int_0^t e^{-\kappa(t-t')/2}[\varepsilon + f_\beta(t')]dt'. \tag{2.1.26}$$

Then, on account of (2.1.25) and (2.1.26), one readily obtains

$$\langle\alpha(t)f_\beta(t)\rangle = \langle\alpha(0)f_\beta(t)\rangle e^{-\kappa t/2} + \int_0^t e^{-\kappa(t-t')/2}[\varepsilon\langle f_\beta(t)\rangle + \langle f_\beta(t)f_\alpha(t')\rangle]dt', \tag{2.1.27}$$

and

$$\langle\beta(t)f_\alpha(t)\rangle = \langle\beta(0)f_\alpha(t)\rangle e^{-\kappa t/2} + \int_0^t e^{-\kappa(t-t')/2}[\varepsilon\langle f_\alpha(t)\rangle + \langle f_\alpha(t)f_\beta(t')\rangle]dt'. \tag{2.1.28}$$

Since a noise operator at a certain time should not affect the system operator at an earlier time, we note that

$$\langle\alpha(0)f_\beta(t)\rangle = \langle\alpha(0)\rangle\langle f_\beta(t)\rangle = 0, \tag{2.1.29}$$

$$\langle\beta(0)f_\alpha(t)\rangle = \langle\beta(0)\rangle\langle f_\alpha(t)\rangle = 0. \tag{2.1.30}$$

Thus on account of Eqs. (2.1.21), (2.1.27), (2.1.28), (2.1.29), and (2.1.30), we obtain

$$\langle\alpha(t)f_\beta(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle f_\beta(t)f_\alpha(t')\rangle dt', \tag{2.1.31}$$

$$\langle\beta(t)f_\alpha(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle f_\alpha(t)f_\beta(t')\rangle dt'. \tag{2.1.32}$$

Therefore, in view of Eqs. (2.1.24), (2.1.31), and (2.1.32) and assuming

$$\langle f_\beta(t)f_\alpha(t')\rangle = \langle f_\alpha(t)f_\beta(t')\rangle, \tag{2.1.33}$$

we arrive at

$$\int_0^t e^{-\kappa(t-t')/2}\langle f_\alpha(t)f_\beta(t')\rangle dt' = 0. \tag{2.1.34}$$

Now on the basis of the relation [1]

$$\int_0^t e^{-a(t-t')/2}\langle f(t)g(t')\rangle dt' = b, \tag{2.1.35}$$

we assert that

$$\langle f(t)g(t')\rangle = 2b\delta(t-t'), \tag{2.1.36}$$

where a is a constant and b is a constant or some function of time t. We then see that

$$\langle f_\beta(t) f_\alpha(t')\rangle = \langle f_\alpha(t) f_\beta(t')\rangle = 0. \tag{2.1.37}$$

Furthermore, using Eq. (2.1.19) along with the relation

$$\frac{d}{dt}\langle \alpha(t)\alpha(t)\rangle = \langle \frac{d\alpha(t)}{dt}\alpha(t)\rangle + \langle \alpha(t)\frac{d\alpha(t)}{dt}\rangle, \tag{2.1.38}$$

we obtain

$$\frac{d}{dt}\langle \alpha^2(t)\rangle = -\kappa\langle \alpha^2(t)\rangle + 2\varepsilon\langle \alpha(t)\rangle + 2\langle \alpha(t) f_\alpha(t)\rangle. \tag{2.1.39}$$

Comparison of Eqs. (2.1.16) and (2.1.39) shows that

$$\langle \alpha(t) f_\alpha(t)\rangle = 0, \tag{2.1.40}$$

in view of (2.1.25), one readily obtains

$$\int_0^t e^{-\kappa(t-t')/2}\langle f_\alpha(t) f_\alpha(t')\rangle dt' = 0, \tag{2.1.41}$$

from which follows

$$\langle f_\alpha(t) f_\alpha(t')\rangle = 0. \tag{2.1.42}$$

It can be established in a similar procedure that

$$\langle f_\beta(t) f_\beta(t')\rangle = 0. \tag{2.1.43}$$

Moreover, employing Eq. (2.1.19) and its complex conjugate along with the relation

$$\frac{d}{dt}\langle \alpha^*(t)\alpha(t)\rangle = \langle \frac{d\alpha^*(t)}{dt}\alpha(t)\rangle + \langle \alpha^*(t)\frac{d\alpha(t)}{dt}\rangle, \tag{2.1.44}$$

we find

$$\frac{d}{dt}\langle \alpha^*(t)\alpha(t)\rangle = -\kappa\langle \alpha^*(t)\alpha(t)\rangle + \varepsilon(\langle \alpha^*(t)\rangle + \langle \alpha(t)\rangle) + \langle \alpha(t) f_\alpha^*(t)\rangle + \langle \alpha^*(t) f_\alpha(t)\rangle. \tag{2.1.45}$$

Comparison of Eqs. (2.1.17) and (2.1.45) indicates that

$$\langle \alpha(t) f_\alpha^*(t)\rangle + \langle \alpha^*(t) f_\alpha(t)\rangle = 0. \tag{2.1.46}$$

So that using (2.1.25) and its complex conjugate, and assuming

$$\langle f_\alpha(t) f^*_\alpha(t')\rangle = \langle f^*_\alpha(t) f_\alpha(t')\rangle, \tag{2.1.47}$$

we get

$$\int_0^t e^{-\kappa(t-t')/2}\langle f_\alpha(t) f^*_\alpha(t')\rangle dt' = 0. \tag{2.1.48}$$

Hence on account of (2.1.35) and (2.1.36), we note that

$$\langle f_\alpha(t) f^*_\alpha(t')\rangle = \langle f^*_\alpha(t) f_\alpha(t')\rangle = 0. \tag{2.1.49}$$

It can also be established in a similar procedure that

$$\langle f_\beta(t) f^*_\beta(t')\rangle = \langle f^*_\beta(t) f_\beta(t')\rangle = 0. \tag{2.1.50}$$

Finally, using (2.1.20) and the complex conjugate of (2.1.19) along with the relation

$$\frac{d}{dt}\langle \alpha^*(t)\beta(t)\rangle = \langle \frac{d\alpha^*(t)}{dt}\beta(t)\rangle + \langle \alpha^*(t)\frac{d\beta(t)}{dt}\rangle, \tag{2.1.51}$$

we obtain

$$\frac{d}{dt}\langle \alpha^*(t)\beta(t)\rangle = -\kappa\langle \alpha^*(t)\beta(t)\rangle + \varepsilon(\langle \alpha^*(t)\rangle + \langle \beta(t)\rangle) + \langle \alpha^*(t) f_\beta(t)\rangle + \langle \beta(t) f^*_\alpha(t)\rangle. \tag{2.1.52}$$

Comparison of (2.1.18) and (2.1.52), indicates that

$$\langle \alpha^*(t) f_\beta(t)\rangle + \langle \beta(t) f^*_\alpha(t)\rangle = 0. \tag{2.1.53}$$

In view of (2.1.26) and the complex conjugate of (2.1.25), and assuming

$$\langle f_\beta(t) f^*_\alpha(t')\rangle = \langle f^*_\alpha(t) f_\beta(t')\rangle, \tag{2.1.54}$$

we get

$$\int_0^t e^{-\kappa(t-t')/2}\langle f_\beta(t) f^*_\alpha(t')\rangle dt' = 0, \tag{2.1.55}$$

from which follows

$$\langle f_\beta(t) f^*_\alpha(t')\rangle = \langle f^*_\alpha(t) f_\beta(t')\rangle = 0. \tag{2.1.56}$$

It can also be established in a similar fashion that

$$\langle f_\alpha(t) f^*_\beta(t')\rangle = \langle f^*_\beta(t) f_\alpha(t')\rangle = 0. \tag{2.1.57}$$

We would like to point out that (2.2.21), (2.1.37), (2.1.42), (2.1.43), (2.1.49), (2.1.50), (2.1.56), and (2.1.57), describes the correlation properties of the noise forces $f_\alpha(t)$ and $f_\beta(t)$ associated with the normal ordering.

Now we proceed to find the solutions of Eqs. (2.1.19) and (2.1.20), to this end we introduce a new variable defined by

$$\gamma_\pm(t) = \alpha(t) \pm \beta^*(t). \tag{2.1.58}$$

Applying Eq. (2.1.19) and the complex conjugate of Eq. (2.1.20), we obtain

$$\frac{d}{dt}\gamma_\pm(t) = -\frac{\kappa}{2}\gamma_\pm(t) + \varepsilon \pm \varepsilon + f_\alpha(t) \pm f^*_\beta(t). \tag{2.1.59}$$

Thus, the solution of this equation can be written as

$$\gamma_\pm(t) = \gamma_\pm(0) e^{-\kappa t/2} + \int_0^t e^{-\kappa(t-t')/2}[\varepsilon \pm \varepsilon + f_\alpha(t') \pm f^*_\beta(t')]dt', \tag{2.1.60}$$

from which we obtain

$$\alpha(t) = p(t)\alpha(0) + q(t) + F_+(t) + F_-(t), \tag{2.1.61}$$

$$\beta(t) = p(t)\beta(0) + q(t) + F^*_+(t) - F^*_-(t), \tag{2.1.62}$$

where

$$p(t) = e^{-\kappa t/2}, \tag{2.1.63}$$

$$q(t) = \frac{2\varepsilon}{\kappa}(1 - e^{-\kappa t/2}), \tag{2.1.64}$$

and

$$F_\pm(t) = \frac{1}{2}\int_0^t e^{-\kappa(t-t')/2}[f_\alpha(t') \pm f^*_\beta(t')]dt'. \tag{2.1.65}$$

Now Eqs. (2.1.61) and (2.1.62) can be rewritten as

$$\alpha(t) = \alpha'(t) + q(t), \tag{2.1.66}$$

$$\beta(t) = \beta'(t) + q(t), \tag{2.1.67}$$

with

$$\alpha'(t) = p(t)\alpha(0) + \mu_\alpha(t), \tag{2.1.68}$$

$$\beta'(t) = p(t)\beta(0) + \mu_\beta(t), \tag{2.1.69}$$

where

$$\mu_\alpha(t) = \frac{1}{2}\int_0^t e^{-\kappa(t-t')/2}[f_\alpha(t') + f_\beta^*(t')]dt' + \frac{1}{2}\int_0^t e^{-\kappa(t-t')/2}[f_\alpha(t') - f_\beta^*(t')]dt'. \tag{2.1.70}$$

$$\mu_\beta(t) = \frac{1}{2}\int_0^t e^{-\kappa(t-t')/2}[f_\alpha^*(t') + f_\beta(t')]dt' - \frac{1}{2}\int_0^t e^{-\kappa(t-t')/2}[f_\alpha^*(t') - f_\beta(t')]dt'. \tag{2.1.71}$$

2.2 The Q function for a coherently driven two-mode cavity light

We now proceed to obtain the Q function for a coherently driven two-mode cavity light. The Q function for a two-mode coherent light is expressed as

$$Q(\alpha,\beta,t) = \frac{1}{\pi^4}\int d^2z d^2\eta \Phi_a(z,\eta,t) exp(z^*\alpha + \eta^*\beta - z\alpha^* - \eta\beta^*), \tag{2.2.1}$$

with the antinormally-ordered characteristic function $\Phi_a(z,\eta,t)$ is defined in the Heisenberg picture by

$$\Phi_a(z,\eta,t) = Tr\big[\hat{\rho}(0)e^{-z^*\hat{a}(t)}e^{z\hat{a}^\dagger(t)}e^{-\eta^*\hat{b}(t)}e^{\eta\hat{b}^\dagger(t)}\big]. \tag{2.2.2}$$

Employing the identity [1]

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{B}}e^{\hat{A}}e^{[A,B]}, \tag{2.2.3}$$

the characteristic function can be expressed as

$$\Phi_a(z,\eta,t) = e^{-z^*z-\eta^*\eta}Tr\big[\hat{\rho}(0)e^{z\hat{a}^\dagger(t)}e^{-z^*\hat{a}(t)}e^{\eta\hat{b}^\dagger(t)}e^{-\eta^*\hat{b}(t)}\big]. \tag{2.2.4}$$

Then this function can be written in terms of c-number variables associated with the normal-order as

$$\Phi_a(z,\eta,t) = e^{-z^*z-\eta^*\eta}\langle exp[z\alpha^* - z^*\alpha + \eta\beta^* - \eta^*\beta]\rangle. \tag{2.2.5}$$

Therefore using Eqs. (2.1.66) and (2.1.67), we have

$$\begin{aligned}\Phi_a(z,\eta,t) &= exp[(-z^*z - \eta^*\eta) + (z - z^* + \eta - \eta^*)q] \\ &\times\langle exp[z\alpha'^* - z^*\alpha' + \eta\beta'^* - \eta^*\beta']\rangle.\end{aligned} \tag{2.2.6}$$

Now we seek to show α' and β' are Gaussian variables. To this end, taking the expectation values of Eqs. (2.1.68), and (2.1.69), and differentiating with respect to t yields

$$\frac{d}{dt}\langle\alpha'(t)\rangle = -\frac{\kappa}{2}\langle\alpha'(t)\rangle, \tag{2.2.7}$$

and

$$\frac{d}{dt}\langle\beta'(t)\rangle = -\frac{\kappa}{2}\langle\beta'(t)\rangle. \tag{2.2.8}$$

which are linear equations. Which shows $\alpha'(t)$ and $\beta'(t)$ are Gaussian variables. In addition, using (2.1.68), and (2.1.69), and assuming the cavity mode is initially in a two-mode vacuum state, one easily gets

$$\langle\alpha'(t)\rangle = \langle\beta'(t)\rangle = 0 \tag{2.2.9}$$

So that $\alpha'(t)$ and $\beta'(t)$ are Gaussian variables with vanishing means. Thus employing the relation [1]

$$\langle exp(\alpha+\beta)\rangle = exp\big[\frac{1}{2}\langle(\alpha+\beta)^2\rangle\big], \tag{2.2.10}$$

where α and β are Gaussian variables with vanishing mean. The characteristic function can be put in the form

$$\begin{aligned}\Phi_a(z,\eta,t) &= exp[(-z^*z - \eta^*\eta) + (z - z^* + \eta - \eta^*)q] \\ &\times exp[\frac{1}{2}\langle(z\alpha'^* - z^*\alpha' + \eta\beta'^* - \eta^*\beta')^2\rangle],\end{aligned} \tag{2.2.11}$$

which can be rewritten as

$$\begin{aligned}\Phi_a(z,\eta,t) \quad &= exp\Big[(-z^*z-\eta^*\eta)+(z-z^*+\eta-\eta^*)q]\\ &\times exp[\frac{z^2}{2}\langle\alpha'^{*2}\rangle+\frac{z^{*2}}{2}\langle\alpha'^2\rangle+\frac{\eta^2}{2}\langle\beta'^{*2}\rangle+\frac{\eta^{*2}}{2}\langle\beta'^2\rangle\\ &-z^*z\langle\alpha'\alpha'^*\rangle+z\eta\langle\alpha'^*\beta'^*\rangle-z^*\eta\langle\alpha'\beta'^*\rangle\\ &-z\eta^*\langle\alpha'^*\beta'\rangle+z^*\eta^*\langle\alpha'\beta'\rangle-\eta^*\eta\langle\beta'\beta'^*\rangle\Big]\end{aligned} \tag{2.2.12}$$

Now using Eqs. (2.1.70) and (2.1.71), and the assumption that the cavity mode is initially in a two-mode vacuum state, we obtain

$$\langle\alpha'^{*2}\rangle=\langle\alpha'^2\rangle=0, \tag{2.2.13}$$

$$\langle\beta'^{*2}\rangle=\langle\beta'^2\rangle=0, \tag{2.2.14}$$

$$\langle\alpha'\alpha'^*\rangle=\langle\beta'\beta'^*\rangle=0, \tag{2.2.15}$$

$$\langle\alpha'^*\beta'^*\rangle=\langle\alpha'\beta'\rangle=0, \tag{2.2.16}$$

$$\langle\alpha'\beta'^*\rangle=\langle\alpha'^*\beta'\rangle=0. \tag{2.2.17}$$

Using the above relations the characteristic function becomes

$$\Phi_a(z,\eta,t)=exp\big[-z^*z-\eta^*\eta+(z-z^*+\eta-\eta^*)q\big]. \tag{2.2.18}$$

Then, introducing (2.2.18) into (2.2.1), the Q function becomes

$$\begin{aligned}Q(\alpha,\beta,t) \quad &= \frac{1}{\pi^2}\int\frac{d^2zd^2\eta}{\pi^2}exp\big[-z^*z-\eta^*\eta+(z-z^*+\eta-\eta^*)q\big]\\ &\times exp\big[z^*\alpha-z\alpha^*+\eta^*\beta-\eta\beta^*\big],\end{aligned} \tag{2.2.19}$$

which can be rewritten as

$$Q(\alpha,\beta,t)=\frac{1}{\pi^2}\int\frac{d^2zd^2\eta}{\pi^2}exp\big[-z^*z-\eta^*\eta+az+bz^*+u\eta+v\eta^*],\tag{2.2.20}$$

where

$$a=q-\alpha^*, \tag{2.2.21}$$

$$b = -(q-\alpha), \tag{2.2.22}$$

$$u = q-\beta^*, \tag{2.2.23}$$

$$v = -(q-\beta). \tag{2.2.24}$$

Furthermore, using the relation given [1]

$$\begin{aligned}&\int \frac{d^2z}{\pi} exp\big[-az^*z+bz+cz^*+Az^2+Bz^{*2}\big]\\ &= \big[\frac{1}{a^2-4AB}\big]^{\frac{1}{2}} exp\bigg(\frac{abc+Ac^2+Bb^2}{a^2-4AB}\bigg),\ a>0,\end{aligned} \tag{2.2.25}$$

and performing the integration, one readily obtains

$$Q(\alpha,\beta,t) = \frac{1}{\pi^2} exp\big[-\alpha^*\alpha-\beta^*\beta+(\alpha+\beta+\alpha^*+\beta^*)q-2q^2\big]. \tag{2.2.26}$$

2.3 Photon statistics

The statistical properties of a light beam is described in terms of the mean and variance of the photon number. Here we wish to calculate the mean and variance of the photon number for modes a and b, employing the Q function.

2.3.1 The mean of the photon number sum and difference

Here we seek to study, employing the Q function, the statistical properties of the two-mode coherent light. To this end, upon integrating Eq. (2.2.26) over β, we get

$$\int d^2\beta Q(\alpha,\beta,t) = \int \frac{d^2\beta}{\pi^2} exp\big[-\alpha^*\alpha-\beta^*\beta+(\alpha+\beta+\alpha^*+\beta^*)q-2q^2\big], \tag{2.3.1}$$

from which follows

$$Q(\alpha,t) = \frac{1}{\pi} exp\big[-\alpha^*\alpha+q\alpha+q\alpha^*-q^2\big]. \tag{2.3.2}$$

We define the photon number sum and difference by

$$\hat{n}_\pm = \hat{n}_a \pm \hat{n}_b, \tag{2.3.3}$$

where $\hat{n}_a = \hat{a}^\dagger\hat{a}$ and $\hat{n}_b = \hat{b}^\dagger\hat{b}$ are the photon number operators for mode a and mode b. The mean of the photon number sum and difference can be expressed in terms of the Q function as

$$\bar{n}_\pm = \bar{n}_a \pm \bar{n}_b = \int d^2\alpha Q(\alpha^*, \alpha, t) n_a(\alpha) \pm \int d^2\beta Q(\beta^*, \beta, t) n_b(\beta), \tag{2.3.4}$$

where $n_a(\alpha) = \alpha^*\alpha - 1$ and $n_b(\beta) = \beta^*\beta - 1$, are the c-number variables corresponding to the operators $\hat{n}_a$ and $\hat{n}_b$ in the antinormal-order. The mean photon number for mode a is given by

$$\bar{n}_a = \int \frac{d^2\alpha}{\pi} exp\big[-\alpha^*\alpha + q\alpha + q\alpha^* - q^2\big]\alpha^*\alpha - 1, \tag{2.3.5}$$

which can be rewritten in the form

$$\bar{n}_a = -\frac{d}{da}\bigg[\int \frac{d^2\alpha}{\pi} exp\big[-a\alpha^*\alpha + q\alpha + q\alpha^* - q^2\big]\bigg]_{a=1} - 1, \tag{2.3.6}$$

so that performing the integration employing Eq. (2.2.25), one readily obtains

$$\bar{n}_a = e^{-q^2}\frac{d}{da}\big[-\frac{1}{a}e^{\frac{q^2}{2}}\big]_{a=1} - 1. \tag{2.3.7}$$

Thus carrying out the differentiation and applying the condition $a = 1$, we readily obtain

$$\bar{n}_a = q^2, \tag{2.3.8}$$

so that on account of (2.1.64), there follows

$$\bar{n}_a = \frac{4\varepsilon^2}{\kappa^2}(1 - e^{-\frac{\kappa t}{2}})^2. \tag{2.3.9}$$

We observe that at steady state the mean photon number reduces to

$$\bar{n}_a = \frac{4\varepsilon^2}{\kappa^2}. \tag{2.3.10}$$

Following the same procedure, the mean photon number for mode b is also obtained as

$$\bar{n}_b = \frac{4\varepsilon^2}{\kappa^2}(1 - e^{-\frac{\kappa t}{2}})^2, \tag{2.3.11}$$

and at steady state, we see that

$$\bar{n}_b = \frac{4\varepsilon^2}{\kappa^2}. \tag{2.3.12}$$

Therefore, in view of Eqs. (2.3.9) and (2.3.11), Eq. (2.3.3) can be written in the form

$$\bar{n}_\pm = \frac{4\varepsilon^2}{\kappa^2}(1 - e^{-\frac{\kappa t}{2}})^2 \pm \frac{4\varepsilon^2}{\kappa^2}(1 - e^{-\frac{\kappa t}{2}})^2, \tag{2.3.13}$$

from which the sum of the mean photon number is found to be

$$\bar{n}_+ = \frac{8\varepsilon^2}{\kappa^2}(1 - e^{-\frac{\kappa t}{2}})^2. \tag{2.3.14}$$

We see that at steady state the mean of the photon number sum reduces to

$$\bar{n}_+ = \frac{8\varepsilon^2}{\kappa^2}. \tag{2.3.15}$$

And the mean of the photon number difference turns out to be

$$\bar{n}_- = 0. \tag{2.3.16}$$

2.3.2 The variance of the photon number sum and difference

We next proceed to obtain the variance of the photon number sum and difference. The variance of the photon number sum and difference can be expressed as

$$(\Delta n_\pm)^2 = \langle \hat{n}_\pm^2 \rangle - \langle \hat{n}_\pm \rangle^2. \tag{2.3.17}$$

Applying Eq. (2.3.3), the variance of the photon number sum and difference can be put in the form

$$(\Delta n_\pm)^2 = (\Delta n_a)^2 + (\Delta n_b)^2 \pm 2\big(\langle \hat{n}_a \hat{n}_b \rangle - \bar{n}_a \bar{n}_b\big). \tag{2.3.18}$$

On the other hand, the photon number variance for mode a can be written in the form

$$(\Delta n_a)^2 = \langle \hat{n}_a^2 \rangle - \bar{n}_a^2. \tag{2.3.19}$$

Using the Q function (2.3.2), we find

$$\langle\hat{n}_a^2\rangle = \int \frac{d^2\alpha}{\pi} exp\big[-\alpha^*\alpha + q\alpha + q\alpha^* - q^2\big]\big(\alpha^{*2}\alpha^2 - 3\alpha^*\alpha + 1\big), \tag{2.3.20}$$

where

$$\alpha^{*2}\alpha^2 - 3\alpha^*\alpha + 1,$$

is the c-number function corresponding to the operator $\hat{n}_a^2$ in the antinormal-order. Now (2.3.20) can be written as

$$\begin{aligned}\langle\hat{n}_a^2\rangle \quad &= e^{-q^2}\Bigg[\frac{d^2}{da^2}\int \frac{d^2\alpha}{\pi} exp\big(-a\alpha^*\alpha + q\alpha + q\alpha^*\big) \\ &+3\frac{d}{da}\int \frac{d^2\alpha}{\pi} exp\big(-a\alpha^*\alpha + q\alpha + q\alpha^*\big)\Bigg]_{a=1} + 1.\end{aligned} \tag{2.3.21}$$

Thus performing the integration using the relation given by (2.2.25), one gets

$$\langle\hat{n}_a^2\rangle = e^{-q^2}\Bigg[\frac{d^2}{da^2}\big(\frac{1}{a}e^{\frac{q^2}{a}}\big) + \frac{d}{da}\big(\frac{3}{a}e^{\frac{q^2}{a}}\big)\Bigg]_{a=1} + 1, \tag{2.3.22}$$

so that carrying out the differentiation and applying the condition $a = 1$, one readily obtains

$$\langle\hat{n}_a^2\rangle = q^2(q^2 + 1). \tag{2.3.23}$$

Now with the aid of Eqs. (2.3.8) and (2.3.23), Eq. (2.3.19) can be put in the form

$$(\Delta n_a)^2 = q^2, \tag{2.3.24}$$

so that on account of (2.1.64), the variance of the photon number for mode a can be put in the form

$$(\Delta n_a)^2 = \frac{4\varepsilon^2}{\kappa^2}(1 - e^{-\kappa t/2})^2. \tag{2.3.25}$$

This takes at steady state the form

$$(\Delta n_a)^2 = \frac{4\varepsilon^2}{\kappa^2}. \tag{2.3.26}$$

Following a similar procedure, one can readily establish that the variance of the photon number for mode b can be put in the form

$$(\Delta n_b)^2 = \frac{4\varepsilon^2}{\kappa^2}(1 - e^{-\kappa t/2})^2, \tag{2.3.27}$$

and at steady state, it takes the form

$$(\Delta n_b)^2 = \frac{4\varepsilon^2}{\kappa^2}. \tag{2.3.28}$$

In view of (2.3.10),(2.3.12),(2.3.26), and (2.3.28), we observe that

$$(\Delta n_a)^2 = \bar{n}_a, \tag{2.3.29}$$

and

$$(\Delta n_b)^2 = \bar{n}_b. \tag{2.3.30}$$

In addition, one can also write

$$\langle \hat{n}_a \hat{n}_b \rangle = \int d^2\alpha d^2\beta Q(\alpha, \beta, t) n_a(\alpha) n_b(\beta), \tag{2.3.31}$$

with

$$n_a(\alpha) = \alpha^*\alpha - 1,$$

and

$$n(\beta) = \beta^*\beta - 1,$$

are the c-number functions corresponding to the operators $\hat{n}_a$ and $\hat{n}_b$ in the antinormal-order. Thus on account of the Q function (2.2.26), one can write

$$\begin{aligned}\langle \hat{n}_a \hat{n}_b \rangle \quad &= \int \frac{d^2\alpha d^2\beta}{\pi^2} exp\big[-\alpha^*\alpha - \beta^*\beta + q\alpha + q\beta + q\alpha^* + q\beta^* - 2q^2\big] \\ &(\alpha^*\alpha\beta^*\beta - \alpha^*\alpha - \beta^*\beta + 1),\end{aligned} \tag{2.3.32}$$

which can be rewritten as

$$\begin{aligned}\langle \hat{n}_a \hat{n}_b \rangle \quad &= e^{-2q^2}\Bigg[\frac{d}{da}\frac{d}{db}\int \frac{d^2\alpha d^2\beta}{\pi^2} exp[-a\alpha^*\alpha - b\beta^*\beta + q\alpha^* + q\alpha + q\beta^* + q\beta] \\ &+\frac{d}{da}\int \frac{d^2\alpha d^2\beta}{\pi^2} exp[-a\alpha^*\alpha - b\beta^*\beta + q\alpha^* + q\alpha + q\beta^* + q\beta] \\ &+\frac{d}{db}\int \frac{d^2\alpha d^2\beta}{\pi^2} exp[-a\alpha^*\alpha - b\beta^*\beta + q\alpha^* + q\alpha + q\beta^* + q\beta]\Bigg]_{a=b=1} \\ &+1,\end{aligned} \tag{2.3.33}$$

thus carrying out the integration using the relation (2.2.25), one obtains

$$\begin{aligned}\langle\hat{n}_a\hat{n}_b\rangle \quad &= e^{-2q^2}\Big[\frac{d}{da}\frac{d}{db}\big(\frac{1}{ab}e^{q^2(\frac{1}{a}+\frac{1}{b})}\big)+\frac{d}{da}\big(\frac{1}{ab}e^{q^2(\frac{1}{a}+\frac{1}{b})}\big)\\ &+\frac{d}{db}\big(\frac{1}{ab}e^{q^2(\frac{1}{a}+\frac{1}{b})}\big)\Big]_{a=b=1}+1. \end{aligned} \tag{2.3.34}$$

Differentiating and applying the condition $a=b=1$, we get

$$\langle\hat{n}_a\hat{n}_b\rangle = q^4. \tag{2.3.35}$$

On account of (2.1.64), one readily obtains

$$\langle\hat{n}_a\hat{n}_b\rangle = \frac{16\varepsilon^4}{\kappa^4}\big(1-e^{-\frac{\kappa t}{2}}\big)^4. \tag{2.3.36}$$

Introducing (2.3.9), (2.3.11), (2.3.25), (2.3.27), and (2.3.36) into (2.3.18), we obtain

$$(\Delta n_\pm)^2 = \frac{8\varepsilon^2}{\kappa^2}\big(1-e^{-\frac{\kappa t}{2}}\big)^2, \tag{2.3.37}$$

from which we observe the variance of the photon number sum and difference takes at steady state the form

$$(\Delta n_\pm)^2 = \frac{8\varepsilon^2}{\kappa^2}. \tag{2.3.38}$$

Chapter 3

Two-Mode Subharmonic Light

3.1 c-number Langevin equation

Here we first obtain c-number Langevin equations associated with the normal ordering, for the signal-idler modes produced in two-mode subharmonic generation. Then using the solutions of the resulting equations, we determine the Q function for these modes. In a two-mode subharmonic generator a pump photon of frequency ω_c is down converted into highly correlated signal and idler photons with frequency ω_a and ω_b, such that $\omega_c = \omega_a + \omega_b$.

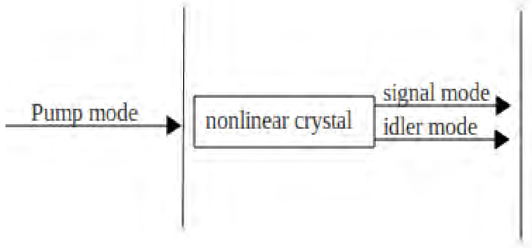

Figure 3.1: Two-mode subharmonic generator

The process of two-mode subharmonic generation is described by the Hamiltonian

$$\hat{H} = i\lambda(\hat{a}\hat{b}\hat{c}^{\dagger} - \hat{a}^{\dagger}\hat{b}^{\dagger}\hat{c}), \tag{3.1.1}$$

where $\hat{a}$ and $\hat{b}$ are the annihilation operators for the signal and idler modes respectively and $\hat{c}$ is the annihilation operator for the pump mode, with λ being the coupling constant.

With the pump mode represented by a real and constant c-number μ, the Hamiltonian can be rewritten as

$$\hat{H} = i\gamma(\hat{a}\hat{b} - \hat{a}^\dagger\hat{b}^\dagger), \tag{3.1.2}$$

with

$$\gamma = \lambda\mu \tag{3.1.3}$$

Applying Eq. (3.1.2) and taking into account the interaction of the signal-idler modes with a two-mode vacuum reservoir via a single port mirror, the equation of evolution for the reduced density operator (in short the master equation) for the cavity modes can be written as [1]

$$\frac{d\hat{\rho}}{dt} = \gamma(\hat{a}\hat{b}\hat{\rho} - \hat{\rho}\hat{a}\hat{b} + \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho}) + \frac{\kappa}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) + \frac{\kappa}{2}(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}), \tag{3.1.4}$$

in which the cavity damping constant κ is assumed to be the same for both the signal and idler modes. Now employing the relation

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr\big(\frac{d\hat{\rho}}{dt}\hat{A}\big), \tag{3.1.5}$$

we readily obtain

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}(t)\rangle \quad &= \gamma Tr(\hat{a}\hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{b}\hat{a} + \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{\rho}\hat{a} - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho}\hat{a}) \\ &+ \frac{\kappa}{2}Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2) + \frac{\kappa}{2}Tr(2\hat{b}\hat{\rho}\hat{b}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{b}\hat{a}).\end{aligned} \tag{3.1.6}$$

Applying the cyclic property of the trace operation together with the commutation relations

$$[\hat{a}, \hat{a}^\dagger] = 1, \tag{3.1.7}$$

and

$$[\hat{a}, \hat{b}] = [\hat{a}, \hat{b}^\dagger] = [\hat{a}^\dagger, \hat{b}] = [\hat{a}^\dagger, \hat{b}^\dagger] = 0, \tag{3.1.8}$$

one finds

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = -\frac{\kappa}{2}\langle\hat{a}(t)\rangle - \gamma\langle\hat{b}^\dagger\rangle. \tag{3.1.9}$$

It can be shown in a similar manner that

$$\frac{d}{dt}\langle\hat{b}(t)\rangle = -\frac{\kappa}{2}\langle\hat{b}(t)\rangle - \gamma\langle\hat{a}^{\dagger}\rangle, \tag{3.1.10}$$

$$\frac{d}{dt}\langle\hat{a}(t)\hat{b}(t)\rangle = -\kappa\langle\hat{a}(t)\hat{b}(t)\rangle - \gamma\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle - \gamma\langle\hat{b}^{\dagger}(t)\hat{b}\rangle - \gamma \tag{3.1.11}$$

$$\frac{d}{dt}\langle\hat{a}(t)\hat{a}(t)\rangle = -\kappa\langle\hat{a}(t)\hat{a}(t)\rangle - 2\gamma\langle\hat{a}(t)\hat{b}^{\dagger}(t)\rangle, \tag{3.1.12}$$

$$\frac{d}{dt}\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle = -\kappa\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle - \gamma\langle\hat{a}(t)\hat{b}(t)\rangle - \gamma\langle\hat{a}^{\dagger}(t)\hat{b}^{\dagger}(t)\rangle, \tag{3.1.13}$$

$$\frac{d}{dt}\langle\hat{a}^{\dagger}(t)\hat{b}(t)\rangle = -\kappa\langle\hat{a}^{\dagger}(t)\hat{b}(t)\rangle - \gamma\langle\hat{a}^{\dagger 2}(t)\rangle - \gamma\langle\hat{b}^{2}(t)\rangle). \tag{3.1.14}$$

We see that the normal-ordering c-number equations corresponding to Eqs. (3.1.9), (3.1.10), (3.1.11), (3.1.12), (3.1.13), and (3.1.14) are

$$\frac{d}{dt}\langle\alpha(t)\rangle = -\frac{\kappa}{2}\langle\alpha(t)\rangle - \gamma\langle\beta^{*}(t)\rangle, \tag{3.1.15}$$

$$\frac{d}{dt}\langle\beta(t)\rangle = -\frac{\kappa}{2}\langle\beta(t)\rangle - \gamma\langle\alpha^{*}(t)\rangle, \tag{3.1.16}$$

$$\frac{d}{dt}\langle\alpha(t)\beta(t)\rangle = -\kappa\langle\alpha(t)\beta(t)\rangle - \gamma\langle\alpha^{*}(t)\alpha(t)\rangle - \gamma\langle\beta^{*}(t)\beta(t)\rangle - \gamma, \tag{3.1.17}$$

$$\frac{d}{dt}\langle\alpha^{2}(t)\rangle = -\kappa\langle\alpha^{2}(t)\rangle - 2\gamma\langle\alpha(t)\beta^{*}(t)\rangle, \tag{3.1.18}$$

$$\frac{d}{dt}\langle\alpha^{*}(t)\alpha(t)\rangle = -\kappa\langle\alpha^{*}(t)\alpha(t)\rangle - \gamma\langle\alpha(t)\beta(t)\rangle - \gamma\langle\alpha^{*}(t)\beta^{*}(t)\rangle, \tag{3.1.19}$$

$$\frac{d}{dt}\langle\alpha^{*}(t)\beta(t)\rangle = -\kappa\langle\alpha^{*}(t)\beta(t)\rangle - \gamma\langle\alpha^{*2}(t)\rangle - \gamma\langle\beta^{2}(t)\rangle). \tag{3.1.20}$$

On the basis of Eqs. (3.1.15) and (3.1.16), one can write

$$\frac{d}{dt}\alpha(t) = -\frac{\kappa}{2}\alpha(t) - \gamma\beta^{*}(t) + f_{\alpha}(t) \tag{3.1.21}$$

and

$$\frac{d}{dt}\beta(t) = -\frac{\kappa}{2}\beta(t) - \gamma\alpha^{*}(t) + f_{\beta}(t), \tag{3.1.22}$$

where $f_{\alpha}(t)$ and $f_{\beta}(t)$ are noise forces associated with the normal ordering. We next seek to determine the properties of the noise forces $f_{\alpha}(t)$ and $f_{\beta}(t)$. We note that Eq. (3.1.15) and the expectation value of Eq. (3.1.21) as well as Eq. (3.1.16) and the expectation value of Eq. (3.1.22) will have the same form if

$$\langle f_\alpha(t)\rangle = \langle f_\beta(t)\rangle = 0 \tag{3.1.23}$$

Moreover, using Eqs. (3.1.21) and (3.1.22) together with the relation

$$\frac{d}{dt}\langle\alpha(t)\beta(t)\rangle = \langle\frac{d\alpha(t)}{dt}\beta(t)\rangle + \langle\alpha(t)\frac{d\beta(t)}{dt}\rangle, \tag{3.1.24}$$

we find

$$\begin{aligned}\frac{d}{dt}\langle\alpha(t)\beta(t)\rangle \quad &= -\kappa\langle\alpha(t)\beta(t)\rangle - \gamma\langle\alpha^*(t)\alpha(t)\rangle - \gamma\langle\beta^*(t)\beta(t)\rangle \\ &+\langle\alpha(t)f_\beta(t)\rangle + \langle\beta(t)f_\alpha(t)\rangle.\end{aligned} \tag{3.1.25}$$

Comparison of Eqs. (3.1.17) and (3.1.25) indicates that

$$\langle\alpha(t)f_\beta(t)\rangle + \langle\beta(t)f_\alpha(t)\rangle = -\gamma. \tag{3.1.26}$$

The formal solution of Eqs. (3.1.21) and (3.1.22) can be written as

$$\alpha(t) = \alpha(0)e^{-\kappa t/2} + \int_0^t e^{-\kappa(t-t')/2}[f_\alpha(t') - \gamma\beta^*(t')]dt', \tag{3.1.27}$$

and

$$\beta(t) = \beta(0)e^{-\kappa t/2} + \int_0^t e^{-\kappa(t-t')/2}[f_\beta(t') - \gamma\alpha^*(t')]dt'. \tag{3.1.28}$$

On account of (3.1.27) and (3.1.28), one readily obtains

$$\langle\alpha(t)f_\beta(t)\rangle = \langle\alpha(0)f_\beta(t)\rangle e^{-\kappa t/2} + \int_0^t e^{-\kappa(t-t')/2}\langle f_\beta(t)f_\alpha(t')\rangle - \gamma\langle\beta^*(t')f_\beta(t)\rangle]dt', \tag{3.1.29}$$

$$\langle\beta(t)f_\alpha(t)\rangle = \langle\beta(0)f_\alpha(t)\rangle e^{-\kappa t/2} + \int_0^t e^{-\kappa(t-t')/2}[\langle f_\alpha(t)f_\beta(t')\rangle - \gamma\langle\alpha^*(t')f_\alpha(t)\rangle]dt'. \tag{3.1.30}$$

Since a noise operator at a certain time should not affect the system operator at an earlier time, we note that

$$\langle\alpha(0)f_\beta(t)\rangle = \langle\alpha(0)\rangle\langle f_\beta(t)\rangle = 0, \tag{3.1.31}$$

$$\langle\beta(0)f_\alpha(t)\rangle = \langle\beta(0)\rangle\langle f_\alpha(t)\rangle = 0, \tag{3.1.32}$$

$$\langle\alpha^*(t')f_\alpha(t)\rangle = \langle\alpha^*(t')\rangle\langle f_\alpha(t)\rangle = 0, \tag{3.1.33}$$

$$\langle\beta^*(t')f_\alpha(t)\rangle = \langle\beta^*(t')\rangle\langle f_\beta(t)\rangle = 0. \tag{3.1.34}$$

Thus on account of Eqs. (3.1.23), (3.1.31), (3.1.32), (3.1.33), and (3.1.34), Eqs. (3.1.29) and (3.1.30) reduces to

$$\langle\alpha(t)f_\beta(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle f_\beta(t)f_\alpha(t')\rangle dt', \tag{3.1.35}$$

$$\langle\beta(t)f_\alpha(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle f_\alpha(t)f_\beta(t')\rangle dt'. \tag{3.1.36}$$

Therefore, in view of Eqs. (3.1.26), (3.1.35), and (3.1.36) and assuming

$$\langle f_\beta(t)f_\alpha(t')\rangle = \langle f_\alpha(t)f_\beta(t')\rangle, \tag{3.1.37}$$

we arrive at

$$\int_0^t e^{-\kappa(t-t')/2}\langle f_\alpha(t)f_\beta(t')\rangle dt' = \int_0^t e^{-\kappa(t-t')/2}\langle f_\beta(t)f_\alpha(t')\rangle dt' = -\frac{\gamma}{2}. \tag{3.1.38}$$

Now on the basis of the relation (2.1.35) and (2.1.36), we assert that

$$\langle f_\beta(t)f_\alpha(t')\rangle = \langle f_\alpha(t)f_\beta(t')\rangle = -\gamma\delta(t-t') \tag{3.1.39}$$

Furthermore, using Eq. (3.1.21) along with the relation

$$\frac{d}{dt}\langle\alpha(t)\alpha(t)\rangle = \langle\frac{d\alpha(t)}{dt}\alpha(t)\rangle + \langle\alpha(t)\frac{d\alpha(t)}{dt}\rangle, \tag{3.1.40}$$

we obtain

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = -\kappa\langle\alpha^2(t)\rangle - 2\gamma\langle\alpha(t)\beta^*(t)\rangle + 2\langle\alpha(t)f_\alpha(t)\rangle. \tag{3.1.41}$$

Comparison of Eqs. (3.1.18) and (3.1.41) shows that

$$\langle\alpha(t)f_\alpha(t)\rangle = 0, \tag{3.1.42}$$

in view of (3.1.27), one readily obtains

$$\int_0^t e^{-\kappa(t-t')/2}\langle f_\alpha(t)f_\alpha(t')\rangle dt' = 0, \tag{3.1.43}$$

from which follows that

$$\langle f_\alpha(t) f_\alpha(t')\rangle = 0. \tag{3.1.44}$$

It can be established in a similar procedure that

$$\langle f_\beta(t) f_\beta(t')\rangle = 0. \tag{3.1.45}$$

Moreover, employing Eq. (3.1.21) and its complex conjugate along with the relation

$$\frac{d}{dt}\langle \alpha^*(t)\alpha(t)\rangle = \langle \frac{d\alpha^*(t)}{dt}\alpha(t)\rangle + \langle \alpha^*(t)\frac{d\alpha(t)}{dt}\rangle, \tag{3.1.46}$$

we find

$$\begin{aligned} \frac{d}{dt}\langle \alpha^*(t)\alpha(t)\rangle \quad &= -\kappa\langle \alpha^*(t)\alpha(t)\rangle - \gamma\langle \alpha(t)\beta(t)\rangle - \gamma\langle \alpha^*(t)\beta^*(t)\rangle) \\ &+\langle \alpha(t) f^*_\alpha(t)\rangle + \langle \alpha^*(t) f_\alpha(t)\rangle. \end{aligned} \tag{3.1.47}$$

Comparison of Eqs. (3.1.19) and (3.1.47) indicates that

$$\langle \alpha(t) f^*_\alpha(t)\rangle + \langle \alpha^*(t) f_\alpha(t)\rangle = 0, \tag{3.1.48}$$

with the aid of (3.1.27) and its complex conjugate, we obtain

$$\langle \alpha(t) f^*_\alpha(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle f^*_\alpha(t) f_\alpha(t')\rangle dt', \tag{3.1.49}$$

$$\langle \alpha^*(t) f_\alpha(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle f_\alpha(t) f^*_\alpha(t')\rangle dt'. \tag{3.1.50}$$

Now in view of (3.1.48), (3.1.49), (3.1.50) and assuming

$$\langle f_\alpha(t) f^*_\alpha(t')\rangle = \langle f^*_\alpha(t) f_\alpha(t')\rangle, \tag{3.1.51}$$

we get

$$\int_0^t e^{-\kappa(t-t')/2}\langle f_\alpha(t) f^*_\alpha(t')\rangle dt' = 0, \tag{3.1.52}$$

from which one readily obtains

$$\langle f_\alpha(t) f^*_\alpha(t')\rangle = \langle f^*_\alpha(t) f_\alpha(t')\rangle = 0. \tag{3.1.53}$$

It can also be established in a similar procedure that

$$\langle f_\beta(t) f^*_\beta(t')\rangle = \langle f^*_\beta(t) f_\beta(t')\rangle = 0. \tag{3.1.54}$$

Finally, introducing (3.1.22) and the complex conjugate of (3.1.21) along with the relation

$$\frac{d}{dt}\langle \alpha^*(t)\beta(t)\rangle = \langle \frac{d\alpha^*(t)}{dt}\beta(t)\rangle + \langle \alpha^*(t)\frac{d\beta(t)}{dt}\rangle, \tag{3.1.55}$$

we obtain

$$\frac{d}{dt}\langle \alpha^*(t)\beta(t)\rangle = -\kappa\langle \alpha^*(t)\beta(t)\rangle - \gamma\langle \alpha^{*2}(t)\rangle - \gamma\langle \beta^2(t)\rangle) + \langle \alpha^*(t) f_\beta(t)\rangle + \langle \beta(t) f^*_\alpha(t)\rangle. \tag{3.1.56}$$

Comparison of (3.1.20) and (3.1.56), indicates that

$$\langle \alpha^*(t) f_\beta(t)\rangle + \langle \beta(t) f^*_\alpha(t)\rangle = 0, \tag{3.1.57}$$

with the aid of (3.1.28) and the complex conjugate of (3.1.27), we obtain

$$\langle \alpha^*(t) f_\beta(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle f_\beta(t) f^*_\alpha(t')\rangle dt', \tag{3.1.58}$$

and

$$\langle \beta(t) f^*_\alpha(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle f^*_\alpha(t) f_\beta(t')\rangle dt'. \tag{3.1.59}$$

Now taking into account Eqs. (3.1.57), (3.1.58), (3.1.59), and assuming

$$\langle f_\beta(t) f^*_\alpha(t')\rangle = \langle f^*_\alpha(t) f_\beta(t')\rangle, \tag{3.1.60}$$

we arrive at

$$\int_0^t e^{-\kappa(t-t')/2}\langle f_\beta(t) f^*_\alpha(t')\rangle dt' = 0 \tag{3.1.61}$$

Therefore, in view of the relations (2.1.35) and (2.1.36), one obtains

$$\langle f_\beta(t) f^*_\alpha(t')\rangle = \langle f^*_\alpha(t) f_\beta(t')\rangle = 0. \tag{3.1.62}$$

It can also be established in a similar manner that

$$\langle f_\alpha(t) f^*_\beta(t')\rangle = \langle f^*_\beta(t) f_\alpha(t')\rangle = 0. \tag{3.1.63}$$

We would like to point out that (3.1.23), (3.1.39), (3.1.44), (3.1.45), (3.1.53), (3.1.54), (3.1.62), and (3.1.63) describe the correlation properties of the noise forces $f_\alpha(t)$ and $f_\beta(t)$ associated with the normal-ordering.

In order to obtain the solutions of Eqs. (3.1.21) and (3.1.22), we introduce a new variable defined by

$$z_\pm(t) = \alpha(t) \pm \beta^*(t). \tag{3.1.64}$$

Applying Eq. (3.1.21) along with the complex conjugate of Eq. (3.1.22), we readily obtain

$$\frac{d}{dt} z_\pm = -\frac{1}{2}\lambda_\pm z_\pm + f_\alpha(t) \pm f^*_\beta(t), \tag{3.1.65}$$

where

$$\lambda_\pm = \kappa \pm 2\gamma. \tag{3.1.66}$$

According to Eq. (3.1.65) together with (3.1.66), the solution of $z_-(t)$ does not have a well-behaved solution for $\kappa < 2\gamma$. We then identify $\kappa = 2\gamma$ as the threshold condition. For $2\gamma<\kappa$, the solution of Eq. (3.1.65) can be written as

$$z_\pm(t) = z_\pm(0)e^{-\frac{1}{2}\lambda_\pm t} + \int_0^t e^{-\frac{1}{2}\lambda_\pm(t-t')}[f_\alpha(t') \pm f_\beta{}^*(t')]dt'. \tag{3.1.67}$$

Now with the aid of (3.1.64) and (3.1.67), we readily obtain

$$\alpha(t) = A_+(t)\alpha(0) + A_-(t)\beta^*(0) + F_+(t) + F_-(t), \tag{3.1.68}$$

$$\beta(t) = A_+(t)\beta(0) + A_-(t)\alpha^*(0) + F^*_+(t) - F^*_-(t), \tag{3.1.69}$$

where

$$A_\pm(t) = \frac{1}{2}\big(e^{-\frac{1}{2}\lambda_+ t} \pm e^{-\frac{1}{2}\lambda_- t}\big), \tag{3.1.70}$$

$$F_\pm(t) = \frac{1}{2}\int_0^t e^{-\frac{1}{2}\lambda_\pm(t-t')}[f_\alpha(t') \pm f^*_\beta(t')]dt'. \tag{3.1.71}$$

3.2 The Q function for a two-mode subharmonic light

We next seek to obtain the Q function for the signal-idler modes. The Q function for a two-mode light is expressible as

$$Q(\alpha,\beta,t)=\frac{1}{\pi^4}\int d^2zd^2\chi\Phi_a(z,\chi,t)exp(z^*\alpha-z\alpha^*+\chi^*\beta-\chi\beta^*), \tag{3.2.1}$$

where the antinormally ordered characteristic function is defined in the Heisenberg picture by

$$\Phi_a(z,\chi,t)=Tr\big(\hat\rho(0)e^{-z^*\hat a(t)}e^{z\hat a^\dagger(t)}e^{-\chi^*\hat b(t)}e^{\chi\hat b^\dagger(t)}\big). \tag{3.2.2}$$

This can be put employing the identity [1]

$$e^{\hat A}e^{\hat B}=e^{\hat B}e^{\hat A}e^{[\hat A,\hat B]}, \tag{3.2.3}$$

in the form

$$\Phi_a(z,\chi,t)=exp[-z^*z-\chi^*\chi]Tr\big[\hat\rho(0)e^{z\hat a^\dagger(t)}e^{-z^*\hat a(t)}e^{\chi\hat b^\dagger(t)}e^{-\chi^*\hat b(t)}\big], \tag{3.2.4}$$

the characteristic function can be expressed in terms of c-number variables associated with the normal ordering in the form

$$\Phi_a(z,\chi,t)=exp[-z^*z-\chi^*\chi]\langle exp\big(z\alpha^*-z^*\alpha+\chi\beta^*-\chi^*\beta\big)\rangle. \tag{3.2.5}$$

Moreover, Eqs. (3.1.68) and (3.1.69) can be written as

$$\alpha(t)=A_+(t)\alpha(0)+A_-(t)\beta^*(0)+\eta_\alpha(t), \tag{3.2.6}$$

$$\beta(t)=A_+(t)\beta(0)+A_-(t)\alpha^*(0)+\eta_\beta(t), \tag{3.2.7}$$

where

$$\eta_\alpha(t)=\frac{1}{2}\Big[\int_0^t e^{-\frac{1}{2}\lambda_+(t-t')}\big(f_\alpha(t')+f^*_\beta(t')\big)dt'+\int_0^t e^{-\frac{1}{2}\lambda_-(t-t')}\big(f_\alpha(t')-f^*_\beta(t')\big)dt'\Big], \tag{3.2.8}$$

and

$$\eta_\beta(t)=\frac{1}{2}\Big[\int_0^t e^{-\frac{1}{2}\lambda_+(t-t')}\big(f^*_\alpha(t')+f_\beta(t')\big)dt'-\int_0^t e^{-\frac{1}{2}\lambda_-(t-t')}\big(f^*_\alpha(t')-f_\beta(t')\big)dt'\Big]. \tag{3.2.9}$$

Now in view of (3.2.6) and (3.2.7), one readily obtains

$$\begin{aligned}\Phi_a(z,\chi,t) \quad &= exp[-z^*z-\chi^*\chi]\langle exp\big(zA_+\alpha^*(0)+zA_-\beta(0)\\ &+z\eta^*_\alpha - z^*A_+\alpha(0)-z^*A_-\beta^*(0)-z^*\eta_\alpha+\chi A_+\beta^*(0)\\ &+\chi A_-\alpha(0)+\chi\eta^*_\beta-\chi^*A_+\beta(0)-\chi^*A_-\alpha^*(0)-\chi^*\eta_\beta\big)\rangle, \qquad (3.2.10)\end{aligned}$$

it then follows that

$$\begin{aligned}\Phi_a(z,\chi,t) \quad &= exp[-z^*z-\chi^*\chi]\langle exp\big[(\chi A_- - z^*A_+)\alpha(0)+(zA_+ - \chi^*A_-)\alpha^*(0)\\ &+(zA_- - \chi^*A_+)\beta(0)+(\chi A_+ - z^*A_-)\beta^*(0)\big]\rangle\\ &\times\langle exp\big[z\eta^*_\alpha - z^*\eta_\alpha+\chi\eta^*_{\beta-\chi^*\eta_\beta}\big]\rangle. \qquad (3.2.11)\end{aligned}$$

Considering the cavity radiation to be initially in a two-mode vacuum state, we obtain

$$\begin{aligned}&\langle exp\big[(\chi A_- - z^*A_+)\alpha(0)+(zA_+ - \chi^*A_-)\alpha^*(0)\\ &+(zA_- - \chi^*A_+)\beta(0)+(\chi A_+ - z^*A_-)\beta^*(0)\big]\rangle = 1. \qquad (3.2.12)\end{aligned}$$

In view of (3.2.12), Eq. (3.2.11) takes the form

$$\Phi_a(z,\chi,t) = exp[-z^*z-\chi^*\chi]\times\langle exp\big[z\eta^*_\alpha - z^*\eta_\alpha+\chi\eta^*_\beta-\chi^*\eta_\beta\big]\rangle. \qquad (3.2.13)$$

We recall that Gaussian variables with vanishing mean satisfy the relation

$$\langle exp(z\alpha^*-z^*\alpha)\rangle = exp\big[\frac{1}{2}\big\langle(z\alpha^*-z^*\alpha)^2\big\rangle. \qquad (3.2.14)$$

Next we proceed to show that $\eta_\alpha(t)$ and $\eta_\beta(t)$ are Gaussian variables. One can rewrite Eq. (3.2.8) as

$$\eta_\alpha = \eta_+ + \eta_-, \qquad (3.2.15)$$

in which

$$\eta_+ = \frac{1}{2}\int_0^t e^{-\frac{1}{2}\lambda_+(t-t')}\big(f_\alpha(t')+f^*_\beta(t')\big)dt', \qquad (3.2.16)$$

and

$$\eta_- = \frac{1}{2}\int_0^t e^{-\frac{1}{2}\lambda_-(t-t')}\big(f_\alpha(t')-f^*_\beta(t')\big)dt'. \qquad (3.2.17)$$

Employing Eqs. (3.2.16) and (3.2.17), the time evolution for the expectation values of η_+ and η_- can be expressed as

$$\frac{d}{dt}\langle\eta_+\rangle = -\frac{\lambda_+}{4}\langle\eta_+\rangle, \tag{3.2.18}$$

and

$$\frac{d}{dt}\langle\eta_-\rangle = -\frac{\lambda_-}{4}\langle\eta_-\rangle. \tag{3.2.19}$$

On account of (3.2.18) and (3.2.19), indicates that η_+ and η_- are Gaussian variables. Thus, in view of (3.2.15), we see that η_α is also Gaussian variable. Similarly we rewrite Eq. (3.2.9) as

$$\eta_\beta = \sigma_+ - \sigma_-, \tag{3.2.20}$$

where

$$\sigma_+ = \frac{1}{2}\int_0^t e^{-\frac{1}{2}\lambda_+(t-t')}\big(f_\alpha^*(t') + f_\beta(t')\big)dt', \tag{3.2.21}$$

and

$$\sigma_- = \frac{1}{2}\int_0^t e^{-\frac{1}{2}\lambda_-(t-t')}\big(f_\alpha^*(t') - f_\beta(t')\big)dt'. \tag{3.2.22}$$

Now the time evolution for the expectation values of σ_+ and σ_- can be expressed as

$$\frac{d}{dt}\langle\sigma_+\rangle = -\frac{\lambda_+}{4}\langle\sigma_+\rangle, \tag{3.2.23}$$

and

$$\frac{d}{dt}\langle\sigma_-\rangle = -\frac{\lambda_-}{4}\langle\sigma_-\rangle. \tag{3.2.24}$$

In view of (3.2.20), (3.2.23), and (3.2.24), we see that η_β is Gaussian variable. In addition, using (3.2.8), and (3.2.9), one easily gets

$$\langle\eta_\alpha(t)\rangle = \langle\eta_\beta(t)\rangle = 0. \tag{3.2.25}$$

Therefore, we have that η_α and η_β are Gaussian variables with vanishing mean. Then the characteristics function can be put in the form

$$\Phi_a(z,\chi,t) = exp[-z^*z - \chi^*\chi]exp\Big[\frac{1}{2}\langle\big(z\eta_\alpha^* - z^*\eta_\alpha + \chi\eta_\beta^* - \chi^*\eta_\beta\big)^2\rangle\Big], \tag{3.2.26}$$

which is equivalent to

$$\begin{aligned}\Phi_a(z,\chi,t) \quad &= exp[-z^*z-\chi^*\chi]exp\big[\frac{z^2}{2}\langle\eta_\alpha^{*2}\rangle+\frac{z^{*2}}{2}\langle\eta_\alpha^2\rangle+\frac{\chi^2}{2}\langle\eta_\beta^{*2}\rangle\\ &+\frac{\chi^{*2}}{2}\langle\eta_\beta^2\rangle-zz^*\langle\eta_\alpha\eta_\alpha^*\rangle+z\chi\langle\eta_\alpha^*\eta_\beta^*\rangle-z^*\chi\langle\eta_\alpha\eta_\beta^*\rangle\\ &-z\chi^*\langle\eta_\alpha^*\eta_\beta\rangle+z^*\chi^*\langle\eta_\alpha\eta_\beta\rangle-\chi\chi^*\langle\eta_\beta\eta_\beta^*\rangle. \qquad (3.2.27)\end{aligned}$$

On account of Eqs. (3.2.8) and (3.2.9), one readily obtains

$$\langle\eta_\alpha^2\rangle=\langle\eta_\alpha^{*2}\rangle=\langle\eta_\beta^2\rangle=\langle\eta_\beta^{*2}\rangle=0, \qquad (3.2.28)$$

$$\langle\eta_\alpha^*\eta_\beta\rangle=\langle\eta_\alpha\eta_\beta^*\rangle=0, \qquad (3.2.29)$$

$$\langle\eta_\alpha\eta_\alpha^*\rangle=\langle\eta_\beta\eta_\beta^*\rangle=-\frac{\gamma}{2}(Q-R), \qquad (3.2.30)$$

$$\langle\eta_\alpha\eta_\beta\rangle=\langle\eta_\alpha^*\eta_\beta^*\rangle=-\frac{\gamma}{2}(Q+R), \qquad (3.2.31)$$

where

$$Q=\frac{1}{\lambda_+}(1-e^{-\lambda_+t}), \qquad (3.2.32)$$

$$R=\frac{1}{\lambda_-}(1-e^{-\lambda_-t}). \qquad (3.2.33)$$

Thus in view of (3.2.28), (3.2.29), (3.2.30), and (3.2.31) expression (3.2.27), goes over into

$$\Phi_a(z,\chi,t)=exp\big[-\big(1-\frac{\gamma}{2}(Q-R)\big)\big(z^*z+\chi^*\chi\big)-\frac{\gamma}{2}(Q+R)(\chi z+\chi^*z^*)\big]. \qquad (3.2.34)$$

Finally, we can also put the characteristics function in the form

$$\Phi_a(z,\chi,t)=exp\big[-a(z^*z+\chi^*\chi)-b(\chi z+\chi^*z^*)\big], \qquad (3.2.35)$$

in which

$$a=1-\frac{\gamma}{2}(Q-R)=1-\frac{\gamma}{2\lambda_+}\big(1-e^{-\lambda_+t}\big)+\frac{\gamma}{2\lambda_-}\big(1-e^{-\lambda_-t}\big), \qquad (3.2.36)$$

$$b=\frac{\gamma}{2}(Q+R)=\frac{\gamma}{2\lambda_+}\big(1-e^{-\lambda_+t}\big)+\frac{\gamma}{2\lambda_-}\big(1-e^{-\lambda_-t}\big). \qquad (3.2.37)$$

Now upon introducing (3.2.35) into (3.2.1), we obtain

$$\begin{aligned} Q(\alpha,\beta,t) &= \frac{1}{\pi^2}\int \frac{d^2zd^2\chi}{\pi^2} exp\big[-a(z^*z+\chi^*\chi)-b(\chi z+\chi^*z^*) \\ &+\alpha z^*-\alpha^*z+\beta\chi^*-\beta^*\chi\big]. \end{aligned} \tag{3.2.38}$$

Thus on performing the integration employing Eq. (2.2.25), the Q function for the signal-idler modes turns out to be

$$Q(\alpha,\beta,t) = \frac{1}{\pi^2}(u^2-v^2)exp\big[-u(\alpha^*\alpha+\beta^*\beta)-v(\alpha\beta+\alpha^*\beta^*)\big], \tag{3.2.39}$$

in which

$$u = \frac{a}{a^2-b^2}, \tag{3.2.40}$$

$$v = \frac{b}{a^2-b^2}. \tag{3.2.41}$$

3.3 Photon statistics

Here we seek to study, employing the Q function, the statistical properties of the signal and idler modes.

3.3.1 The mean of the photon number sum and difference

Now we proceed to determine the mean and variance of the photon number sum and difference for the signal-idler modes. To this end, upon integrating Eq. (3.2.39) over β, we get

$$Q(\alpha,t) = \big[\frac{u^2-v^2}{\pi u}\big]exp\big[-(\frac{u^2-v^2}{u})\alpha^*\alpha\big]. \tag{3.3.1}$$

Now the mean photon number of the signal mode in terms of the Q function is expressible as

$$\bar{n}_a = \int d^2\alpha Q(\alpha^*,\alpha,t)n_a(\alpha), \tag{3.3.2}$$

where

$$n_a(\alpha) = \alpha^*\alpha - 1, \tag{3.3.3}$$

is the c-number variable corresponding to $\hat{n}_a(\hat{a}^\dagger, \hat{a})$ in the antinormal-order. In view of (3.3.2), and (3.3.3), the mean photon number of the signal mode is given by

$$\bar{n}_a = [\frac{u^2 - v^2}{\pi u}] \int exp\big[-(\frac{u^2 - v^2}{u})\alpha^*\alpha\big]\alpha^*\alpha - 1, \tag{3.3.4}$$

from which we obtain

$$\bar{n}_a = \frac{u}{u^2 - v^2} - 1. \tag{3.3.5}$$

Hence with the aid of Eqs. (3.2.40) and (3.2.41), one readily obtains

$$\bar{n}_a = a - 1, \tag{3.3.6}$$

so that on account of (3.2.36), there follows

$$\bar{n}_a = \frac{\gamma}{2\lambda_-}\big(1 - e^{-\lambda_- t}\big) - \frac{\gamma}{2\lambda_+}\big(1 - e^{-\lambda_+ t}\big). \tag{3.3.7}$$

We observe that at steady state the mean photon number reduces to

$$\bar{n}_a = \frac{\gamma}{2\lambda_-} - \frac{\gamma}{2\lambda_+}, \tag{3.3.8}$$

and in view of (3.1.66), we see that

$$\bar{n}_a = \frac{2\gamma^2}{\kappa^2 - 4\gamma^2}. \tag{3.3.9}$$

One can also establish in a similar manner that the mean photon number of the idler mode is found to be

$$\bar{n}_b = \frac{\gamma}{2\lambda_-}\big(1 - e^{-\lambda_- t}\big) - \frac{\gamma}{2\lambda_+}\big(1 - e^{-\lambda_+ t}\big). \tag{3.3.10}$$

Thus at steady state, introducing the value for $\lambda_\pm$, we see that

$$\bar{n}_b = \frac{2\gamma^2}{\kappa^2 - 4\gamma^2}. \tag{3.3.11}$$

Furthermore, in view of Eqs. (3.3.7) and (3.3.10), the mean of the photon number sum and difference can be written as

$$\begin{aligned} \bar{n}_\pm \quad &= \bar{n}_a \pm \bar{n}_b \\ &= \big[\frac{\gamma}{2\lambda_-}(1 - e^{-\lambda_- t}) - \frac{\gamma}{2\lambda_+}(1 - e^{-\lambda_+ t})\big] \\ &\pm \big[\frac{\gamma}{2\lambda_-}(1 - e^{-\lambda_- t}) - \frac{\gamma}{2\lambda_+}(1 - e^{-\lambda_+ t})\big]. \end{aligned} \tag{3.3.12}$$

Then the mean of the photon number sum can be written as

$$\bar{n}_+ = \frac{\gamma}{\lambda_-}(1 - e^{-\lambda_- t}) - \frac{\gamma}{\lambda_+}(1 - e^{-\lambda_+ t}). \tag{3.3.13}$$

This can be put at steady state in the form

$$\bar{n}_+ = \frac{\gamma}{\lambda_-} - \frac{\gamma}{\lambda_+}. \tag{3.3.14}$$

Introducing Eq. (3.1.66), we have

$$\bar{n}_+ = \frac{4\gamma^2}{\kappa^2 - 4\gamma^2}. \tag{3.3.15}$$

On the other hand, the mean of the photon number difference is given by

$$\bar{n}_- = \bar{n}_a - \bar{n}_b, \tag{3.3.16}$$

in view of Eqs.(3.3.9) and (3.3.11), we have

$$\bar{n}_a = \bar{n}_b. \tag{3.3.17}$$

Then, the mean of the photon number difference turns out to be

$$\bar{n}_- = 0. \tag{3.3.18}$$

3.3.2 The variance of the photon number sum and difference

Here, we seek to obtain the variance of the photon number sum and difference. The variance of the photon number sum and difference can be expressed as

$$(\Delta n_\pm)^2 = \langle \hat{n}_\pm^2 \rangle \pm \langle \hat{n}_\pm \rangle^2, \tag{3.3.19}$$

Applying Eq. (3.3.1), the variance of the photon number sum and difference can be put in the form

$$(\Delta n_\pm)^2 = (\Delta n_a)^2 + (\Delta n_b)^2 \pm 2\big(\langle \hat{n}_a \hat{n}_b \rangle - \bar{n}_a \bar{n}_b\big). \tag{3.3.20}$$

On the other hand, the photon number variance for signal mode can be written in the form

$$\begin{aligned}(\Delta n_a)^2 \quad &= \langle \hat{n}_a^2 \rangle - \bar{n}_a^2 \\ &= \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle - \bar{n}_a^2 - 3\bar{n}_a - 2. \end{aligned} \tag{3.3.21}$$

Using the Q function (3.3.1), we readily find

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \frac{u^2 - v^2}{u} \int \frac{d^2\alpha}{\pi} exp\big[- (\frac{u^2 - v^2}{u})\alpha^* \alpha \big] \alpha^{*2} \alpha^2, \tag{3.3.22}$$

which can be rewritten as

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \frac{u^2 - v^2}{u} \frac{d}{d\chi} \frac{d}{dz} \bigg[\int \frac{d^2\alpha}{\pi} exp\big[- (\frac{u^2 - v^2}{u})\alpha^* \alpha + z\alpha^2 + \chi \alpha^{*2} \big] \bigg]_{z=\chi=0}, \tag{3.3.23}$$

so that performing the integration using Eq. (2.2.25), one readily obtains

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \frac{u^2 - v^2}{u} \frac{d}{d\chi} \frac{d}{dz} \bigg[\frac{1}{(\frac{u^2 - v^2}{u})^2 - 4\chi z} \bigg]_{z=\chi=0}. \tag{3.3.24}$$

Thus carrying out the differentiation and applying the condition $z = \chi = 0$, one obtains

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = 2a^2. \tag{3.3.25}$$

On account of (3.3.25) along with (3.3.6), we put Eq. (3.3.21) in the form

$$(\Delta n_a)^2 = a^2 - a, \tag{3.3.26}$$

which can be rewritten as

$$(\Delta n_a)^2 = \bar{n}_a^2 + \bar{n}_a. \tag{3.3.27}$$

Following the same procedure, one can readily establish that the variance of the photon number of the idler-mode is exactly the same as those of the signal-mode. From which follows

$$(\Delta n_b)^2 = \bar{n}_b^2 + \bar{n}_b. \tag{3.3.28}$$

This indicates that the signal-idler modes are in a chaotic states. Now on account of Eq. (3.2.36), we readily find

$$\begin{aligned}(\Delta n_a)^2 \quad &= \frac{\gamma^2}{4\lambda_+^2}\big(1-e^{-\lambda_+ t}\big)^2 + \frac{\gamma^2}{4\lambda_-^2}\big(1-e^{-\lambda_- t}\big) + \frac{\gamma}{2\lambda_-}\big(1-e^{-\lambda_- t}\big)\\ &-\frac{\gamma}{2\lambda_+}\big(1-e^{-\lambda_+ t}\big) - \frac{\gamma^2}{2\lambda_+\lambda_-}\big(1-e^{-\lambda_+ t}\big)\big(1-e^{-\lambda_- t}\big). \qquad (3.3.29)\end{aligned}$$

Thus at steady state it reduced to

$$(\Delta n_a)^2 = \big(\frac{2\gamma^2}{\kappa^2-4\gamma^2}\big)^2 + \frac{2\gamma^2}{\kappa^2-4\gamma^2}. \qquad (3.3.30)$$

Similarly at steady state, we also have

$$(\Delta n_b)^2 = \big(\frac{2\gamma^2}{\kappa^2-4\gamma^2}\big)^2 + \frac{2\gamma^2}{\kappa^2-4\gamma^2}. \qquad (3.3.31)$$

Furthermore, on account of Eq. (3.2.39), one finds

$$\begin{aligned}\langle\hat{n}_a\hat{n}_b\rangle \quad &= (u^2-v^2)\int\frac{d^2\alpha d^2\beta}{\pi^2}exp\big[-u(\alpha^*\alpha+\beta^*\beta)-v(\alpha\beta+\alpha^*\beta^*)\big]\\ &(\alpha^*\alpha-1)(\beta^*\beta-1), \qquad (3.3.32)\end{aligned}$$

where

$$\alpha^*\alpha-1,$$

and

$$\beta^*\beta-1,$$

are the c-number variables corresponding to the operator $\hat{n}_a$ and $\hat{n}_b$ in the antinormal order. Therefore, Eq. (3.3.32) can be rewritten as

$$\begin{aligned}\langle\hat{n}_a\hat{n}_b\rangle \quad &= (u^2-v^2)\bigg[\frac{1}{u^2}\frac{d}{da}\frac{d}{db}\int\frac{d^2\alpha d^2\beta}{\pi^2}exp\big[-au\alpha^*\alpha-bu\beta^*\beta-v(\alpha\beta+\alpha^*\beta^*)\\ &+\frac{1}{u}\frac{d}{da}\int\frac{d^2\alpha d^2\beta}{\pi^2}exp\big[-au\alpha^*\alpha-bu\beta^*\beta-v(\alpha\beta+\alpha^*\beta^*)\\ &\frac{1}{u}\frac{d}{db}\int\frac{d^2\alpha d^2\beta}{\pi^2}exp\big[-au\alpha^*\alpha-bu\beta^*\beta-v(\alpha\beta+\alpha^*\beta^*)\bigg]_{a=b=1}+1, \quad (3.3.33)\end{aligned}$$

so that performing the integration using Eq. (2.2.25), we obtain

$$\begin{aligned}\langle\hat{n}_a\hat{n}_b\rangle \quad &= (u^2-v^2)\bigg[\frac{1}{u^2}\frac{d}{da}\frac{d}{db}\Big(\frac{1}{abu^2-v^2}\Big)+\frac{1}{u}\frac{d}{da}\Big(\frac{1}{abu^2-v^2}\Big)\\ &+\frac{1}{u}\frac{d}{db}\Big(\frac{1}{abu^2-v^2}\Big)\bigg]_{a=b=1}+1. \end{aligned} \tag{3.3.34}$$

Thus carrying out the differentiation and applying the condition $a = b = 1$, one readily obtains

$$\langle\hat{n}_a\hat{n}_b\rangle = \frac{u^2+v^2}{\big(u^2-v^2\big)^2}-\frac{2u}{u^2-v^2}+1. \tag{3.3.35}$$

Introducing Eqs. (3.2.40), and (3.2.41), into (3.3.35), we readily obtain

$$\begin{aligned}\langle\hat{n}_a\hat{n}_b\rangle \quad &= a^2+b^2-2a+1\\ &= \bar{n}_a^2+b^2. \end{aligned} \tag{3.3.36}$$

Hence, the results of the mean and variance of the photon number for the signal mode are the same as those of the idler mode. Therefore the variance of the photon number sum and difference defined by Eq. (3.3.20), can be rewritten in the form

$$(\Delta n_{\pm})^2 = 2(\Delta n_a)^2 \pm (\langle\hat{n}_a\hat{n}_b\rangle - \bar{n}_a^2), \tag{3.3.37}$$

in view of Eqs. (3.3.26), and (3.3.36), we readily obtain

$$(\Delta n_{\pm})^2 = 2\big[a^2-a\pm b^2\big]. \tag{3.3.38}$$

Finally, applying Eqs. (3.2.36) and (3.2.37), the variance of the photon number sum is obtained as

$$\begin{aligned}(\Delta n_+)^2 \quad &= \frac{\gamma^2}{\lambda_+^2}\big(1-e^{-\lambda_+ t}\big)^2+\frac{\gamma^2}{\lambda_-^2}\big(1-e^{-\lambda_- t}\big)^2\\ &+\frac{\gamma}{\lambda_-}\big(1-e^{-\lambda_- t}\big)-\frac{\gamma}{\lambda_+}\big(1-e^{-\lambda_+ t}\big). \end{aligned} \tag{3.3.39}$$

The sum of the photon number variance takes at steady state the form

$$(\Delta n_+)^2 = \frac{2\gamma^2(3\kappa^2-4\gamma^2)}{\big(\kappa^2-4\gamma^2\big)^2}. \tag{3.3.40}$$

And the variance of the photon number difference is found to be

$$(\Delta n_-)^2 = \frac{\gamma}{\lambda_-}\big(1 - e^{-\lambda_- t}\big) - \frac{\gamma}{\lambda_+}\big(1 - e^{-\lambda_+ t}\big) - \frac{2\gamma^2}{\lambda_+\lambda_-}\big(1 - e^{-\lambda_+ t}\big)\big(1 - e^{-\lambda_- t}\big). \tag{3.3.41}$$

Thus at steady state, it takes the form

$$(\Delta n_-)^2 = \frac{2\gamma^2}{\kappa^2 - 4\gamma^2}. \tag{3.3.42}$$

Chapter 4

Superposition of Two-mode Coherent and Subharmonic Light

4.1 The Q function

We now seek to obtain the Q function for the supper position of two mode coherent and subharmonic light. To this end, we first obtain the Q function for the superposition of any two-mode light beams. Thus we expand the density operator for light beam-1 as a function of $\hat{a}$, $\hat{a}^\dagger$ and $\hat{b}$, $\hat{b}^\dagger$ in the antinormal order as.

$$\hat{\rho}_1 = \hat{\rho}_1(\hat{a}^\dagger, \hat{b}^\dagger, \hat{a}, \hat{b}) = \sum_{k\ell mn} C_{k\ell mn} \hat{a}^{\dagger k} \hat{b}^{\dagger \ell} \hat{a}^m \hat{b}^n. \tag{4.1.1}$$

Inserting the completeness relation for two-mode states, with

$$I = \frac{1}{\pi^2} \int d^2\eta_A d^2 z_B |\eta_A, z_B\rangle\langle z_B, \eta_A|, \tag{4.1.2}$$

we readily find

$$\hat{\rho}_1 = \int d^2\eta_A d^2 z_B \frac{1}{\pi^2} \sum_{k\ell mn} C_{k\ell mn} |\eta_A, z_B\rangle\langle z_B, \eta_A| \hat{a}^{\dagger k} \hat{b}^{\dagger \ell} \hat{a}^m \hat{b}^n. \tag{4.1.3}$$

Applying the relations [1]

$$|\alpha\rangle\langle\alpha|\hat{a}^\dagger = \alpha^* |\alpha\rangle\langle\alpha|, \tag{4.1.4}$$

and

$$|\alpha\rangle\langle\alpha|\hat{a} = (\alpha + \frac{\partial}{\partial\alpha^*})|\alpha\rangle\langle\alpha|, \tag{4.1.5}$$

one readily obtains

$$\hat{\rho}_1 = \int d^2\eta_A d^2 z_B \frac{1}{\pi^2} \sum_{k\ell mn} C_{k\ell mn} \eta_A^{*k} \big(\eta_A + \frac{\partial}{\partial \eta_A^*}\big)^m z_B^{*\ell} \big(z_B + \frac{\partial}{\partial z_B^*}\big)^n |\eta_A, z_B\rangle\langle z_B, \eta_A|, \quad (4.1.6)$$

which can be rewritten using the displacement operator as

$$\begin{aligned}\hat{\rho}_1 \quad &= \int d^2\eta_A d^2 z_B \frac{1}{\pi^2} \sum_{k\ell mn} C_{k\ell mn} \eta_A^{*k} z_B^{*\ell} \big(\eta_A + \frac{\partial}{\partial \eta_A^*}\big)^m \big(z_B + \frac{\partial}{\partial z_B^*}\big)^n \\ &\hat{D}(\eta_A)\hat{D}(z_B)\hat{\rho}_{0_A,0_B}\hat{D}(-z_B)\hat{D}(-\eta_A), \qquad (4.1.7)\end{aligned}$$

where

$$\hat{\rho}_{0_A,0_B} = |0_A, 0_B\rangle\langle 0_B, 0_A|.$$

Furthermore, using a similar procedure one can readily obtain the density operator for light beam-2 as

$$\hat{\rho}_2 = \hat{\rho}_2(\hat{a}^\dagger, \hat{b}^\dagger, \hat{a}, \hat{b}) = \sum_{k'\ell'm'n'} C_{k'\ell'm'n'} \hat{a}^{\dagger k'} \hat{b}^{\dagger \ell'} \hat{a}^{m'} \hat{b}^{n'}. \quad (4.1.8)$$

from which follows

$$\begin{aligned}\hat{\rho}_2 \quad &= \int d^2\lambda_A d^2\omega_B \frac{1}{\pi^2} \sum_{k'\ell'm'n'} C_{k'\ell'm'n'} \lambda_A^{*k'} \omega_B^{*\ell'} \big(\lambda_A + \frac{\partial}{\partial \lambda_A^*}\big)^{m'} \big(\omega_B + \frac{\partial}{\partial \omega_B^*}\big)^{n'} \\ &\hat{D}(\lambda_A)\hat{D}(\omega_B)\hat{\rho}_{0_A,0_B}\hat{D}(-\omega_B)\hat{D}(-\lambda_A). \qquad (4.1.9)\end{aligned}$$

Moreover, on account of Eqs. (4.1.7) and (4.1.9), the density operator for the super position of two mode light beams can be put in the form

$$\begin{aligned}\hat{\rho} \quad &= \int \frac{d^2\eta_A d^2 z_B d^2\lambda_A d^2\omega_B}{\pi^4} \sum_{k\ell mn} C_{k\ell mn} \eta_A^{*k} z_B^{*\ell} \big(\eta_A + \frac{\partial}{\partial \eta_A^*}\big)^m \big(z_B + \frac{\partial}{\partial z_B^*}\big)^n \\ &\sum_{k'\ell'm'n'} C_{k'\ell'm'n'} \lambda_A^{*k'} \omega_B^{*\ell'} \big(\lambda_A + \frac{\partial}{\partial \lambda_A^*}\big)^{m'} \big(\omega_B + \frac{\partial}{\partial \omega_B^*}\big)^{n'} \hat{D}(\lambda_A)\hat{D}(\omega_B)\hat{D}(\eta_A) \\ &\hat{D}(z_B)|0_A, 0_B\rangle\langle 0_B, 0_A|\hat{D}(-z_B)\hat{D}(-\eta_A)\hat{D}(-\omega_B)\hat{D}(-\lambda_A). \qquad (4.1.10)\end{aligned}$$

Using the results described in [1, 3]

$$\hat{D}(\alpha)|\beta\rangle\langle\beta|\hat{D}(-\alpha) = |\alpha + \beta\rangle\langle\alpha + \beta|, \quad (4.1.11)$$

the density operator for the superposition can be put in the form

$$\begin{aligned}\hat{\rho} \quad &= \int \frac{d^2\eta_A d^2 z_B d^2\lambda_A d^2\omega_B}{\pi^4} \sum_{k\ell mn} C_{k\ell mn}\eta_A^{*k} z_B^{*\ell}\big(\eta_A + \frac{\partial}{\partial \eta_A^*}\big)^m \big(z_B + \frac{\partial}{\partial z_B^*}\big)^n \\ &\sum_{k'\ell'm'n'} C_{k'\ell'm'n'}\lambda_A^{*k'}\omega_B^{*\ell'}\big(\lambda_A + \frac{\partial}{\partial \lambda_A^*}\big)^{m'}\big(\omega_B + \frac{\partial}{\partial \omega_B^*}\big)^{n'} \\ &|\lambda_A + \eta_A, \omega_B + z_B\rangle\langle \omega_B + z_B, \lambda_A + \eta_A|. \qquad (4.1.12)\end{aligned}$$

We now proceed to obtain the Q function for the superposition of two mode light beams, which is defined by

$$Q(\alpha, \beta) = \frac{1}{\pi^2}\langle \alpha, \beta|\hat{\rho}|\beta, \alpha\rangle, \qquad (4.1.13)$$

is the c-number function corresponding to the normally ordered density operator divided by π^2. Introducing Eq. (4.1.12) into (4.1.13), we readily find

$$\begin{aligned}Q(\alpha,\beta) \quad &= \int \frac{d^2\eta_A d^2 z_B d^2\lambda_A d^2\omega_B}{\pi^6} \sum_{k\ell mn} C_{k\ell mn}\eta_A^{*k} z_B^{*\ell}\big(\eta_A + \frac{\partial}{\partial \eta_A^*}\big)^m \big(z_B + \frac{\partial}{\partial z_B^*}\big)^n \\ &\sum_{k'\ell'm'n'} C_{k'\ell'm'n'}\lambda_A^{*k'}\omega_B^{*\ell'}\big(\lambda_A + \frac{\partial}{\partial \lambda_A^*}\big)^{m'}\big(\omega_B + \frac{\partial}{\partial \omega_B^*}\big)^{n'} \\ &\langle \alpha, \beta|\lambda_A + \eta_A, \omega_B + z_B\rangle\langle \omega_B + z_B, \lambda_A + \eta_A|\beta, \alpha\rangle. \qquad (4.1.14)\end{aligned}$$

Now using the results described in [1, 3]

$$\mid \langle \alpha|\beta\rangle \mid^2 = exp[- \mid \alpha - \beta \mid^2], \qquad (4.1.15)$$

we find the results

$$\begin{aligned}&\langle \alpha, \beta|\lambda_A + \eta_A, \omega_B + z_B\rangle\langle \omega_B + z_B, \lambda_A + \eta_A|\beta, \alpha\rangle = \big|\langle \alpha|\lambda_A + \eta_A\rangle\big|^2\big|\langle \beta|\omega_B + z_B\rangle\big|^2 \\ &= exp\big[- \alpha^*\alpha - \beta^*\beta + \alpha^*\lambda_A + \alpha^*\eta_A + \beta^* z_B + \beta^*\omega_B + \eta_A^*(\alpha - \lambda_A - \eta_A) \\ &+ z_B^*(\beta - \omega_B - z_B) + \lambda_A^*(\alpha - \lambda_A - \eta_A) + \omega_B^*(\beta - \omega_B - z_B)\big]. \qquad (4.1.16)\end{aligned}$$

In view of (4.1.16), Eq. (4.1.14) can be rewritten as

$$\begin{aligned}Q(\alpha,\beta) \quad &= \tau_1 \int \frac{d^2\eta_A d^2 z_B d^2\lambda_A d^2\omega_B}{\pi^6} \tau_2 \sum_{k\ell mn} C_{k\ell mn}\eta_A^{*k} z_B^{*\ell}\big(\eta_A + \frac{\partial}{\partial \eta_A^*}\big)^m \tau_3 \big(z_B + \frac{\partial}{\partial z_B^*}\big)^n \\ &\tau_4 \sum_{k'\ell'm'n'} C_{k'\ell'm'n'}\lambda_A^{*k'}\omega_B^{*\ell'}\big(\lambda_A + \frac{\partial}{\partial \lambda_A^*}\big)^{m'} \tau_5 \big(\omega_B + \frac{\partial}{\partial \omega_B^*}\big)^{n'} \tau_6, \qquad (4.1.17)\end{aligned}$$

where

$$\tau_1 = exp[-\alpha^*\alpha - \beta^*\beta], \tag{4.1.18}$$

$$\tau_2 = exp[\alpha^*\lambda_A + \alpha^*\eta_A + \beta^* z_B + \beta^*\omega_B], \tag{4.1.19}$$

$$\tau_3 = exp[\eta_A^*(\alpha - \lambda_A - \eta_A)], \tag{4.1.20}$$

$$\tau_4 = exp[z_B^*(\beta - \omega_B - z_B)], \tag{4.1.21}$$

$$\tau_5 = exp[\lambda_A^*(\alpha - \lambda_A - \eta_A)], \tag{4.1.22}$$

$$\tau_6 = exp[\omega_B^*(\beta - \omega_B - z_B)]. \tag{4.1.23}$$

On the other hand, using the binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^{n-k} y^k, \tag{4.1.24}$$

one finds

$$\begin{aligned}(\omega_B + \frac{\partial}{\partial\omega_B^*})^{n'} exp[\omega_B^*(\beta - \omega_B - z_B)] &= \sum_{k=0}^{n'} \frac{n'!}{k!(n'-k)!} \omega_B^{n'-k} \frac{\partial^k}{\partial\omega_B^{*k}} exp[\omega_B^*(\beta - \omega_B - z_B)] \\ &= (\beta - z_B)^{n'} exp[\omega_B^*(\beta - \omega_B - z_B)]. \qquad (4.1.25)\end{aligned}$$

In a similar fashion, one readily finds

$$(\omega_B + \frac{\partial}{\partial\omega_B^*})^{n'} \tau_6 = (\beta - z_B)^{n'} \tau_6, \tag{4.1.26}$$

$$(\lambda_A + \frac{\partial}{\partial\lambda_A^*})^{n'} \tau_5 = (\alpha - \eta_A)^{m'} \tau_5, \tag{4.1.27}$$

$$(z_B + \frac{\partial}{\partial z_B^*})^{n} \tau_4 = (\beta - \omega_B)^{n} \tau_4, \tag{4.1.28}$$

$$(\eta_A + \frac{\partial}{\partial\eta_A^*})^{m} \tau_3 = (\alpha - \lambda_A)^{m} \tau_3. \tag{4.1.29}$$

Inserting Eqs. (4.1.26), (4.1.27), (4.1.28), and (4.1.29) into (4.1.17), the Q function for the superposition of two-mode light beams is written in the form

$$Q(\alpha,\beta) = \int \frac{d^2\eta_A d^2z_B d^2\lambda_A d^2\omega_B}{\pi^2} Q(\eta_A^*, z_B^*, \alpha-\lambda_A, \beta-\omega_B)\times$$

$$Q(\lambda_A^*, \omega_B^*, \alpha-\eta_A, \beta-z_B) exp\big[-\big|\alpha-\eta_A-\lambda_A\big|^2 - \big|\beta-z_B-\omega_B\big|^2\big]. \quad (4.1.30)$$

We now proceed to obtain the Q function for the superposition of the two-mode coherent and subharmonic light beams. To this end, in view of (2.2.26) and (3.2.39), we have

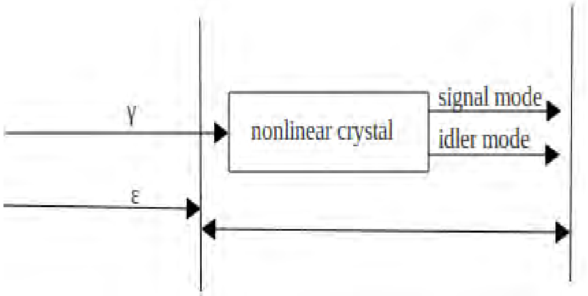

Figure 4.1: Two-mode coherent and subharmonic light beams

$$\begin{aligned} Q(\eta_A^*, z_B^*, \alpha-\lambda_A, \beta-\omega_B) \quad &= \frac{1}{\pi^2} exp\big[-\eta_A^*\alpha + \eta_A^*\lambda_A - z_B^*\beta + z_B^*\omega_B \\ &+ q(\alpha-\lambda_A+\eta_A^*+\beta-\omega_B+z_B^*) - 2q^2\big], \quad (4.1.31) \end{aligned}$$

and

$$\begin{aligned} Q(\lambda_A^*, \omega_B^*, \alpha-\eta_A, \beta-z_B) \quad &= \frac{u^2-v^2}{\pi^2} exp\big[-u\lambda_A^*\alpha + u\lambda_A^*\eta_A - u\omega_B^*\beta + u\omega_B^* z_B \\ &- v\alpha\beta + v\alpha z_B + v\eta_A\beta - v\eta_A z_B - v\lambda_A^*\omega_B^*\big]. \quad (4.1.32) \end{aligned}$$

Introducing Eq, (4.1.31) and (4.1.32) into (4.1.30) the Q function for the superposition of two-mode coherent and subharmonic light beams is expressed as

$$\begin{aligned} Q(\alpha,\beta,t) \quad &= \frac{u^2-v^2}{\pi^2} \int \frac{d^2\eta_A d^2z_B d^2\lambda_A d^2\omega_B}{\pi^4} exp\big[-\eta_A^*\alpha + \eta_A^*\lambda_A - z_B^*\beta \\ &+ z_B^*\omega_B + q(\alpha-\lambda_A+\eta_A^*+\beta-\omega_B+z_B^*) - 2q^2\big] exp\big[-u\lambda_A^*\alpha \\ &+ u\lambda_A^*\eta_A - u\omega_B^*\beta + u\omega_B^* z_B - v\alpha\beta + v\alpha z_B + v\eta_A\beta - v\eta_A z_B \\ &- v\lambda_A^*\omega_B^*\big] exp\big[-\big|\alpha-\eta_A-\lambda_A\big|^2 - \big|\beta-z_B-\omega_B\big|^2\big], \quad (4.1.33) \end{aligned}$$

which can be rewritten as

$$\begin{aligned}Q(\alpha,\beta,t) \quad &= \frac{u^2-v^2}{\pi^2}\int\frac{d^2\eta_A d^2 z_B d^2\lambda_A d^2\omega_B}{\pi^4}exp\big[-\alpha^*\alpha-\beta^*\beta-\eta_A^*\eta_A-z_B^*z_B\\ &-\lambda_A^*\lambda_A-\omega_B^*\omega_B+\alpha\lambda_A^*+\eta_A\alpha^*-\eta_A\lambda_A^*+\lambda_A\alpha^*+\beta\omega_B^*+z_B\beta^*\\ &-z_B\omega_B^*+\omega_B\beta^*-u(\lambda_A^*\alpha-\lambda_A^*\eta_A-\omega_B^*z_B+\omega_B^*\beta)\\ &-v(\alpha\beta+\eta_A z_B+\lambda_A^*\omega_B^*-\alpha z_B-\eta_A\beta)\\ &+q(\alpha-\lambda_A+\eta_A^*+\beta-\omega_B+z_B^*)-2q^2\big]. \qquad (4.1.34)\end{aligned}$$

Thus, on performing the integration employing the relation (3.2.28), one readily obtains

$$\begin{aligned}Q(\alpha,\beta,t) \quad &= \frac{u^2-v^2}{\pi^2}exp\big[-u(\alpha^*\alpha+\beta^*\beta)-v(\alpha\beta+\alpha^*\beta^*)\\ &q(u+v)(\alpha+\alpha^*+\beta+\beta^*)-2q^2(u+v)\big]. \qquad (4.1.35)\end{aligned}$$

4.2 Photon statistics

4.2.1 The mean of the photon number sum and difference

We wish here to determine the mean of the photon number sum and difference for the superposition of two-mode coherent and subharmonic light. To this end, upon integrating Eq. (4.1.35) over β, we get

$$Q(\alpha,t)=\frac{u^2-v^2}{\pi u}exp\big[-(\frac{u^2-v^2}{u})\alpha^*\alpha+q(\frac{u^2-v^2}{u})(\alpha+\alpha^*)-q^2(\frac{u^2-v^2}{u})\big]. \qquad (4.2.1)$$

Now the mean photon number for mode a for the super position of two mode coherent and sub harmonic light is expressed as

$$\begin{aligned}\bar{n}_a \quad &= \frac{u^2-v^2}{u}exp[-q^2\frac{u^2-v^2}{u}]\int\frac{d^2\alpha}{\pi}exp\big[-\frac{u^2-v^2}{u}\alpha^*\alpha\\ &+q\frac{u^2-v^2}{u}(\alpha+\alpha^*)\big]\alpha^*\alpha-1, \qquad (4.2.2)\end{aligned}$$

where

$$\alpha^*\alpha - 1,$$

is the c-number function corresponding to the operator $\hat{n}_a$, in the antinormal order. Eq. (4.2.2) can be rewritten as

$$\begin{aligned}\bar{n}_a \quad &= \frac{u}{q^2(u^2-v^2)} exp[-q^2\frac{u^2-v^2}{u}]\frac{d}{da}\frac{d}{db}\Big[\int\frac{d^2\alpha}{\pi} \\ &\quad exp\big[-\frac{u^2-v^2}{u}\alpha^*\alpha + q\frac{u^2-v^2}{u}(a\alpha + b\alpha^*)\big]\Big]_{a=b=1} - 1,\end{aligned} \tag{4.2.3}$$

so that performing the integration using (2.2.25), we get

$$\bar{n}_a = \frac{u^2}{q^2(u^2-v^2)^2} exp[-q^2\frac{u^2-v^2}{u}]\frac{d}{da}\frac{d}{db}\Big[exp\big[q^2\frac{u^2-v^2}{u}ab\big]\Big]_{a=b=1} - 1. \tag{4.2.4}$$

Thus carrying out the differentiation and applying the condition $a = b = 1$, we readily obtain

$$\bar{n}_a = \frac{u}{u^2-v^2} + q^2 - 1. \tag{4.2.5}$$

so that on account of Eqs. (3.2.40) and (3.2.41), one gets

$$\bar{n}_a = a - 1 + q^2, \tag{4.2.6}$$

in view of (2.1.64) and (3.2.36), the mean photon number takes the form

$$\bar{n}_a = \frac{\gamma}{2\lambda_-}\big[1 - e^{-\lambda_- t}\big] - \frac{\gamma}{2\lambda_+}\big[1 - e^{-\lambda_+ t}\big] + \frac{4\varepsilon^2}{\kappa^2}\big[1 - e^{-\frac{\kappa t}{2}}\big]^2. \tag{4.2.7}$$

We observe that at steady state the mean photon number reduces to

$$\bar{n}_a = \frac{2\gamma^2}{\kappa^2 - 4\gamma^2} + \frac{4\varepsilon^2}{\kappa^2}. \tag{4.2.8}$$

Following the same procedure, the mean photon number for mode b is also found.

$$\bar{n}_b = \frac{\gamma}{2\lambda_-}\big[1 - e^{-\lambda_- t}\big] - \frac{\gamma}{2\lambda_+}\big[1 - e^{-\lambda_+ t}\big] + \frac{4\varepsilon^2}{\kappa^2}\big[1 - e^{-\frac{\kappa t}{2}}\big]^2. \tag{4.2.9}$$

And at steady state, we see that

$$\bar{n}_b = \frac{2\gamma^2}{\kappa^2 - 4\gamma^2} + \frac{4\varepsilon^2}{\kappa^2}. \tag{4.2.10}$$

From which we observe that the mean photon number for the superposition of two-mode coherent and subharmonic light is the sum of the mean photon number of the individual light beams.

Furthermore, in view of Eqs. (4.2.7) and (4.2.9), the mean of the photon number sum and difference can be found as

$$\bar{n}_{\pm} = \bar{n}_a \pm \bar{n}_b, \tag{4.2.11}$$

then the mean of the photon number sum can be written as

$$\bar{n}_{+} = \frac{\gamma}{\lambda_-}\big[1 - e^{-\lambda_- t}\big] - \frac{\gamma}{\lambda_+}\big[1 - e^{-\lambda_+ t}\big] + \frac{8\varepsilon^2}{\kappa^2}\big[1 - e^{-\frac{\kappa t}{2}}\big]^2. \tag{4.2.12}$$

This can put at steady state in the form

$$\bar{n}_{+} = \frac{4\gamma^2}{\kappa^2 - 4\gamma^2} + \frac{8\varepsilon^2}{\kappa^2}. \tag{4.2.13}$$

On the other hand the mean of the photon number difference is given by

$$\bar{n}_{-} = \bar{n}_a - \bar{n}_b. \tag{4.2.14}$$

In view of Eqs. (4.2.7) and (4.2.9), we have

$$\bar{n}_a = \bar{n}_b. \tag{4.2.15}$$

Therefore, the mean of the photon number difference turns out to be

$$\bar{n}_{-} = 0. \tag{4.2.16}$$

4.2.2 The variance of the photon number sum and difference

We next proceed to obtain the variance of the photon number sum and difference for the superposition of two mode coherent and subharmonic light, employing the Q function. The variance of the photon number sum and difference is defined by

$$(\Delta n_{\pm})^2 = (\Delta n_a)^2 + (\Delta n_b)^2 \pm 2\big(\langle \hat{n}_a \hat{n}_b \rangle - \bar{n}_a \bar{n}_b\big). \tag{4.2.17}$$

Furthermore, the variance of the photon number for mode a, given by

$$(\Delta n_a)^2 = \langle(\hat{a}^\dagger\hat{a})^2\rangle - \bar{n}_a^2, \tag{4.2.18}$$

from which follows

$$(\Delta n_a)^2 = \langle\hat{a}^2\hat{a}^{\dagger 2}\rangle - \bar{n}_a^2 - 3\bar{n}_a - 2. \tag{4.2.19}$$

Using the Q function (4.2.1), we readily find

$$\begin{aligned}\langle\hat{a}^2\hat{a}^{\dagger 2}\rangle \quad &= \frac{u^2-v^2}{\pi u} exp[-q^2\frac{u^2-v^2}{u}] \int d^2\alpha exp\big[-\frac{u^2-v^2}{u}\alpha^*\alpha \\ &+q\frac{u^2-v^2}{u}(\alpha+\alpha^*)\big]\alpha^{*2}\alpha^2,\end{aligned} \tag{4.2.20}$$

which can be rewritten as

$$\begin{aligned}\langle\hat{a}^2\hat{a}^{\dagger 2}\rangle \quad &= \frac{u^2-v^2}{u} exp[-q^2\frac{u^2-v^2}{u}]\frac{d}{d\eta}\frac{d}{d\gamma}\bigg[\int\frac{d^2\alpha}{\pi} exp\big[-\frac{u^2-v^2}{u}\alpha^*\alpha \\ &+q\frac{u^2-v^2}{u}(\alpha+\alpha^*)+\eta\alpha^2+\gamma\alpha^{*2}\big]\bigg]_{\eta=\gamma=0},\end{aligned} \tag{4.2.21}$$

so that performing the integration using the relation (2.2.25), yields

$$\begin{aligned}\langle\hat{a}^2\hat{a}^{\dagger 2}\rangle \quad &= \frac{u^2-v^2}{u} exp[-q^2\frac{u^2-v^2}{u}]\frac{d}{d\eta}\frac{d}{d\gamma}\bigg[\bigg(\frac{1}{\big(\frac{u^2-v^2}{u}\big)^2-4\eta\gamma}\bigg)^{\frac{1}{2}} \\ &exp\bigg(\frac{q^2\big(\frac{u^2-v^2}{u}\big)^2\big(\frac{u^2-v^2}{u}+\eta+\gamma\big)}{\big(\frac{u^2-v^2}{u}\big)^2-4\eta\gamma}\bigg)\bigg]_{\eta=\gamma=0}.\end{aligned} \tag{4.2.22}$$

Thus differentiating and applying the condition $\eta=\gamma=0$, we obtain

$$\langle\hat{a}^2\hat{a}^{\dagger 2}\rangle = 2\big(\frac{u}{u^2-v^2}\big)^2 + 4q^2\big(\frac{u}{u^2-v^2}\big) + q^4, \tag{4.2.23}$$

in view of (3.2.40) and (3.2.41), we obtain

$$\langle\hat{a}^2\hat{a}^{\dagger 2}\rangle = 2a^2 + 4aq^2 + q^4. \tag{4.2.24}$$

With the aid of (4.2.6), the above equation can be put in the form

$$\langle\hat{a}^2\hat{a}^{\dagger 2}\rangle = 2\bar{n}_a^2 + 4\bar{n}_a - q^4 + 2. \tag{4.2.25}$$

Now employing (4.2.6) along with (4.2.25), one can put Eq. (4.2.19), in the form

$$(\Delta n_a)^2 = \bar{n}_a^2 + \bar{n}_a - q^4. \tag{4.2.26}$$

Following a similar procedure, one can readily establish that the variance of the photon number for mode b is the same as that of mode a.

$$(\Delta n_b)^2 = \bar{n}_b^2 + \bar{n}_b - q^4. \tag{4.2.27}$$

On account of Eqs. (2.1.64) and (3.2.36), one finds

$$\begin{aligned}(\Delta n_a)^2 \quad &= \frac{\gamma^2}{4\lambda_+^2}\big(1-e^{-\lambda_+ t}\big)^2 + \frac{\gamma^2}{4\lambda_-^2}\big(1-e^{-\lambda_- t}\big)^2 - \frac{\gamma^2}{2\lambda_+\lambda_-}\big(1-e^{-\lambda_+ t}\big)\big(1-e^{-\lambda_- t}\big)\\ &+\frac{4\gamma\varepsilon^2}{\lambda_-\kappa^2}\big(1-e^{-\lambda_- t}\big)\big(1-e^{-\frac{\kappa t}{2}}\big)^2 - \frac{4\gamma\varepsilon^2}{\lambda_+\kappa^2}\big(1-e^{-\lambda_+ t}\big)\big(1-e^{-\frac{\kappa t}{2}}\big)^2\\ &+\frac{\gamma}{2\lambda_-}\big(1-e^{-\lambda_- t}\big) - \frac{\gamma}{2\lambda_+}\big(1-e^{-\lambda_+ t}\big) + \frac{4\varepsilon^2}{\kappa^2}\big(1-e^{-\frac{\kappa t}{2}}\big)^2.\end{aligned} \tag{4.2.28}$$

Thus at steady state, we observe that

$$(\Delta n_a)^2 = \frac{2\gamma^2}{\kappa^2-4\gamma^2}\big(\frac{2\gamma^2}{\kappa^2-4\gamma^2}+1\big) + \frac{4\varepsilon^2}{\kappa^2}\big(\frac{4\gamma^2}{\kappa^2-4\gamma^2}+1\big), \tag{4.2.29}$$

and

$$(\Delta n_b)^2 = \frac{2\gamma^2}{\kappa^2-4\gamma^2}\big(\frac{2\gamma^2}{\kappa^2-4\gamma^2}+1\big) + \frac{4\varepsilon^2}{\kappa^2}\big(\frac{4\gamma^2}{\kappa^2-4\gamma^2}+1\big). \tag{4.2.30}$$

On the other hand, employing the Q function (4.1.35), we readily find

$$\begin{aligned}\langle \hat{n}_a\hat{n}_b\rangle \quad &= (u^2-v^2)e^{-2q^2(u+v)}\int \frac{d^2\alpha d^2\beta}{\pi^2} exp\big[-u(\alpha^*\alpha+\beta^*\beta) - v(\alpha\beta+\alpha^*\beta^*)\\ &q(u+v)(\alpha+\alpha^*+\beta+\beta^*)\big]\big(\alpha^*\alpha\beta^*\beta - \alpha^*\alpha - \beta^*\beta + 1\big),\end{aligned} \tag{4.2.31}$$

which can be rewritten in the form

$$\begin{aligned}\langle \hat{n}_a\hat{n}_b\rangle \quad &= (u^2-v^2)e^{-2q^2(u+v)}\Bigg[\frac{1}{u^2}\frac{d}{da}\frac{d}{db}\int \frac{d^2\alpha d^2\beta}{\pi^2} exp\big[-au\alpha^*\alpha - bu\beta^*\beta\\ &-v(\alpha\beta+\alpha^*\beta^*) + q(u+v)(\alpha+\alpha^*+\beta+\beta^*)\big]\\ &+\frac{1}{u}\frac{d}{da}\int \frac{d^2\alpha d^2\beta}{\pi^2} exp\big[-au\alpha^*\alpha - bu\beta^*\beta - v(\alpha\beta+\alpha^*\beta^*)\\ &+q(u+v)(\alpha+\alpha^*+\beta+\beta^*)\big]\\ &+\frac{1}{u}\frac{d}{db}\int \frac{d^2\alpha d^2\beta}{\pi^2} exp\big[-au\alpha^*\alpha - bu\beta^*\beta - v(\alpha\beta+\alpha^*\beta^*)\\ &+q(u+v)(\alpha+\alpha^*+\beta+\beta^*)\big]\Bigg]_{a=b=1} + 1,\end{aligned} \tag{4.2.32}$$

so that performing the integration employing Eq. (2.2.25), we obtain

$$\begin{aligned}\langle\hat{n}_a\hat{n}_b\rangle \quad &= (u^2-v^2)e^{-2q^2(u+v)}\Bigg[\frac{1}{u^2}\frac{d}{da}\frac{d}{db}\Big(\big[\frac{1}{abu^2-v^2}\big]exp\big[q^2(u+v)^2\frac{ua+ub-2v}{abu^2-v^2}\big]\Big)\\ &+\frac{1}{u}\frac{d}{da}\Big(\big[\frac{1}{abu^2-v^2}\big]exp\big[q^2(u+v)^2\frac{ua+ub-2v}{abu^2-v^2}\big]\Big)\\ &+\frac{1}{u}\frac{d}{db}\Big(\big[\frac{1}{abu^2-v^2}\big]exp\big[q^2(u+v)^2\frac{ua+ub-2v}{abu^2-v^2}\big]\Big)\Bigg]_{a=b=1}+1. \qquad (4.2.33)\end{aligned}$$

Thus carrying out the differentiation and applying the condition $a = b = 1$, we get

$$\langle\hat{n}_a\hat{n}_b\rangle = \frac{1}{u^2-v^2}\left[\frac{2u^2}{u^2-v^2}+2q^2(u-v)-2u-1\right]-2q^2+q^4+1. \qquad (4.2.34)$$

On account of Eqs. (3.2.40) and (3.2.41), one readily obtains

$$\begin{aligned}\langle\hat{n}_a\hat{n}_b\rangle \quad &= a^2+b^2+2q^2(a-b)-2q^2-2a+q^4+1\\ &= \bar{n}_a^2+b^2-2bq^2. \qquad (4.2.35)\end{aligned}$$

We have found that, the mean and the variance of the photon number for mode a are the same as those of mode b. Therefore, the variance of the photon number sum and difference defined by Eq. (4.2.17), can be put in the form

$$(\Delta n_\pm)^2 = 2(\Delta n_a)^2 \pm 2\big(\langle\hat{n}_a\hat{n}_b\rangle-\bar{n}_a^2\big). \qquad (4.2.36)$$

Introducing Eqs. (4.2.6), (4.2.26) and (4.2.35), into (4.2.36), we readily find

$$(\Delta n_+)^2 = 2[a^2+b^2+2q^2(a-b)-q^2-a], \qquad (4.2.37)$$

in view of (2.1.64), (3.2.36) and (3.2.37), we obtain

$$\begin{aligned}(\Delta n_+)^2 \quad &= \frac{\gamma^2}{\lambda_+^2}(1-e^{-\lambda_+t})^2+\frac{\gamma^2}{\lambda_-^2}(1-e^{-\lambda_-t})^2+\frac{\gamma}{\lambda_-}(1-e^{-\lambda_-t})\\ &-\frac{\gamma}{\lambda_+}(1-e^{-\lambda_+t})-\frac{16\gamma\varepsilon^2}{\lambda_+\kappa^2}(1-e^{-\lambda_+t})(1-e^{-\frac{\kappa t}{2}})^2\\ &+\frac{8\varepsilon^2}{\kappa^2}(1-e^{-\frac{\kappa t}{2}})^2. \qquad (4.2.38)\end{aligned}$$

Thus, at steady state we observe that

$$(\Delta n_+)^2 = \frac{2\gamma^2(3\kappa^2-4\gamma^2)}{(\kappa^2-4\gamma^2)^2}+\frac{8\varepsilon^2}{\kappa(\kappa+2\gamma)}. \qquad (4.2.39)$$

On the other hand, the variance of the photon number difference is found to be

$$(\Delta n_-)^2 = 2[a^2 - b^2 + 2q^2(a+b) - q^2 - a]. \tag{4.2.40}$$

On account of (2.1.64), (3.2.36) and (3.2.37), we obtain

$$\begin{aligned}(\Delta n_-)^2 \qquad &= \frac{\gamma}{\lambda_-}(1 - e^{-\lambda_- t}) - \frac{\gamma}{\lambda_+}(1 - e^{-\lambda_+ t}) - \frac{2\gamma^2}{\lambda_+\lambda_-}(1 - e^{-\lambda_+ t})(1 - e^{-\lambda_- t}) \\ &+ \frac{16\gamma\varepsilon^2}{\lambda_-\kappa^2}(1 - e^{-\lambda_- t})(1 - e^{-\frac{\kappa t}{2}})^2 + \frac{8\varepsilon^2}{\kappa^2}(1 - e^{-\frac{\kappa t}{2}})^2. \end{aligned} \tag{4.2.41}$$

We observe that at steady state, it takes the form

$$(\Delta n_-)^2 = \frac{2\gamma^2}{\kappa^2 - 4\gamma^2} + \frac{8\varepsilon^2}{\kappa(\kappa - 2\gamma)}. \tag{4.2.42}$$

Chapter 5

Conclusion

In this thesis we have obtained c-number Langevin equations for the two-mode coherent and subharmonic light beams employing the pertinent master equation following the procedure discussed in Ref. [1]. With the aid of the c-number Langevin equations we have obtained the antinormally ordered characteristics function and then we have determined the Q function for these light beams. Employing the pertinent Q functions we have determined the mean and variance of the photon number sum and difference. We have found that the signal and idler modes are separately in chaotic states.

Furthermore, we have determined the Q function for the superposition of the two-mode coherent and subharmonic light beams. With the aid of the resulting Q function we have calculated the mean and the variance of the photon number sum and difference. We have found that the mean of the photon number for the superposition of two-mode coherent and subharmonic light beams is the sum of the individual mean photon numbers and the variance of the photon number for modes a is the same as that of mode b. We have seen that the mean of the photon number difference vanishes.

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Declaration

This thesis is my original work, has not been presented for a degree in any other University and that all the sources of material used for the thesis have been dully acknowledged.

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