

COLLEGE OF NATURAL SCIENCE
DEPARTMENT OF MATHEMATICS



GRADUATE PROJECT REPORT ON
DISTRIBUTIONAL SOLUTION OF CONFLUENT
HYPERGEOMETRIC DIFFERENTIAL EQUATION
Submitted in Partial fulfilment of the Requirements for
the Degree of Master of Science in Mathematics

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Contents

Acknowledgement	iii
Notation	iv
Abstract	v
Introduction	vi
1 Preliminaries	1
1.1 The Dirac Delta Function	1
1.2 Test Functions ($D(\Omega)$)	3
1.3 Distributions ($D'(\Omega)$)	5
1.3.1 Properties of Distributions($D'(\Omega)$)	7
1.3.2 Properties of Derivative of Functions in $D'(\Omega)$	8
1.4 Fourier Transform	10
1.5 Regular Singular Points	11
2 Formal distributional solutions to the confluent hypergeometric equation	13
2.1 Further Analysis Of Theorem 2.1: The Case Where $p, m \in \mathbb{N}$ and $m \leq p$	19
2.2 Further analysis of theorem 2.1: The case where $p \in \mathbb{N}$ and $m \in -\mathbb{N} \cup \{0\}$	26
3 The Hypergeometric equation	29
4 Summary	33
5 Bibliography	34

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The undersigned hereby certify that they have read and recommend to the school of graduate studies for acceptance of a project entitled **Distributional solution of Confluent Hypergeometric Differential equation** by Seyome Alemu in partial fulfilment of the requirements for the degree of master of Science.

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Notation

C^∞	Infinitely differentiable functions
N^n	n-tuple of Natural number
R^n	n-tuple of Real number
\mathbb{C}	complex number
\langle, \rangle	Inner product
$supp$	<i>Support</i>
$D(\Omega)$	The set of test functions
$D'(\Omega)$	<i>Distribution</i>
$\delta(x)$	Dirac delta function
$\delta^{(n)}$	n th derivative of the Dirac delta function
\mathcal{F}	Fourier transform
\mathcal{F}^{-1}	Inverse of Fourier transform
$H(x)$	Heaviside function
$(p)_n$	Pochhammer's symbol
${}_pF_q$	Hypergeometric function

Abstract

This project deals with distributional solutions

$$y(x) = \sum_{n=0}^{\infty} a_n \delta^{(n)}(x)$$

of the confluent hypergeometric differential equation. Such differential equations also have solutions of the form

$${}_1F_1(p, m; x) = 1 + \sum_{n=1}^{\infty} \frac{(p)_n x^n}{(m)_n n!}$$

where $(p)_n$ is the pochhammer's symbol.

Introduction

It is well known that the normal linear homogeneous systems of ordinary differential equations with infinitely smooth coefficients have no distributional solutions other than the classical ones. However, distributional solutions may appear in the case of the equations whose coefficients have singularities.[9] Recently there has been considerable interest in problems concerning the existence of solutions to differential equations in various spaces of generalized functions. Many important areas in theoretical and mathematical physics, theory of partial differential equations, quantum electrodynamics, operational calculus, and functional analysis use the methods of distributions theory. Yet for ordinary differential equations (ODE) research in this direction is insufficiently developed and remains restricted to isolated results for some second- order equations or special higher-order systems. A simple example is the first order ordinary differential equation .

$$x^2 \frac{dy}{dx} - 2y = 0 \quad (0.1)$$

The point $x = 0$ is an essential(irregular) singularity of this equation. It is readily verified that the infinite series

$$y = \sum_{n=0}^{\infty} \frac{2^{n+1} \delta^{(n)}(x)}{n!(n+1)!}$$

formally satisfies equation (0.1).Here $\delta(x)$ is the Dirac delta function and the superscript n stands for n th order differentiation.

Wiener[1] has studied various differential equations with singular coefficients and has obtained their distributional solutions. Wiener and Saha [2] surveyed the work in this field and have exhibited a unified approach in the study of both distributional and entire solutions to some classes of linear ordinary differential equations. A brief introduction to these concepts is also available in reference[3].

My aim is to present solutions of the form

$$\sum_{n=0}^{\infty} a_n \delta^{(n)}(x) \tag{0.2}$$

for the hypergeometric equation. The main part of the paper is devoted to the confluent hypergeometric differential equation. At the end I present some glimpses into the general hypergeometric differential equation as well. I also want to display their uses and exhibit their interplay with related results in the theory of ordinary differential equations. For instance, I find that with the help of these distributional solutions I can sum up some hypergeometric series ${}_1F_1$'s. Take, for example, the case when p, m are positive integers, where N stands for the positive integers, then I find that

$${}_1F_1(p; m; x) = \sum_{n=0}^{\infty} \frac{(p)_n}{(m)_n} \frac{x^n}{n!}$$

where $(p)_n$ is the pochhammer's symbol:

$$(p)_n = p(p+1)(p+2) \cdots (p+n-1), n > 0 \text{ and } (p)_0 = 1 \text{ and } {}_1F_1$$

is the hypergeometric function. Another motivation for studying solutions of the type (0.2) to ordinary differential equations comes from work of Morton and Krall [4], Krall [5], and Littlejohn [6]. These researchers have collectively shown that weight distributions for a certain class of orthogonal polynomials have the form (0.2) and simultaneously satisfy a system of ordinary differential equations.

Chapter 1

Preliminaries

1.1 The Dirac Delta Function

The Heaviside function, $H(x)$ is defined to be equal to zero for every negative value of x and unity for every positive value of x ; that is,

$$H(x) = \begin{cases} 1 & , x > 0 \\ 0 & , x \leq 0 \end{cases}$$

In physical problems one often encounters idealized concepts such as a force concentrated at a point ξ or an impulsive force that acts instantaneously. These forces are described by the Dirac delta function $\delta(x - \xi)$. Dirac delta function has several significant properties :

The Dirac delta function is defined by the properties

$$\delta(x) = \begin{cases} 0 & , if x \neq 0 \\ \infty & , if x = 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x) dx = 1 \quad (1.1)$$

The second property comes from the fact that $\delta(x)$ represents the derivative of $H(x)$.

The derivative of the Heaviside function is the Dirac delta function $\delta(x)$.

Properties of Dirac delta function

- The delta function satisfies the following scaling property for a non zero scalar c :

$$\int_{-\infty}^{\infty} \delta(cx) dx = \int_{-\infty}^{\infty} \delta(u) \frac{du}{|c|} = \frac{1}{|c|} \text{ and so, } \delta(cx) = \frac{\delta(x)}{|c|}$$

- dirac delta function is symmetric. i.e $\delta(-x) = \delta(x)$

- Suppose $f(x)$ is a sufficiently smooth function continuous at the origin. Then
 $\langle f(x), \delta(x) \rangle = f(0)$.
 This formula is called the **Sifting property** of the delta function.

Some sequences converge to the Dirac delta function. For example as $n \rightarrow \infty$ the sequence $\left\{ \frac{\sin(nx)}{\pi x} \right\} \rightarrow \delta(x)$.

Verification: We know that the integral properties of $\delta(x)$ function,

i.e. $\int_{-\infty}^{\infty} \delta(x) dx = 1$,

Hence we have to prove that

$$\begin{aligned} \int_{-\infty}^{\infty} \lim_{n \rightarrow \infty} \frac{\sin(nx)}{\pi x} dx &= \lim_{n \rightarrow \infty} \left(\frac{2}{\pi} \right) \int_0^{\infty} \frac{\sin(t)}{t} dt \\ &= \left(\frac{2}{\pi} \right) \left(\frac{\pi}{2} \right), \text{ since } \int_0^{\infty} \frac{\sin(t)}{t} dt = \frac{\pi}{2} \\ &= 1 = \int_{-\infty}^{\infty} \delta(x) dx \\ \therefore \lim_{n \rightarrow \infty} \frac{\sin(nx)}{\pi x} &= \delta(x) \end{aligned}$$

Theorem 1.1. Let $\delta^{(n)}(x)$ be the n th derivative of the Dirac delta function and x^n a monomial function then

$$x^n \delta^{(m)}(x) = \begin{cases} 0 & , m < n \\ (-1)^n \frac{m!}{(m-n)!} \delta^{(m-n)}(x) & , m \geq n. \end{cases}$$

Proof: We know that $x\delta(x) = 0$. Now differentiating this equation with respect to x repeatedly up to m times yields the following:

$$\begin{aligned} x\delta(x) &= 0 \\ x\delta^{(1)}(x) &= -\delta(x) \\ x\delta^{(2)}(x) &= -2\delta^{(1)}(x) \\ x\delta^{(3)}(x) &= -3\delta^{(2)}(x) \\ &\dots \\ x\delta^{(m)}(x) &= -m\delta^{(m-1)}(x) \end{aligned}$$

The last result can be expressed as follows:

$$x\delta^{(m)}(x) = \begin{cases} 0 & , m < 1 \\ (-1)^1 \frac{m!}{(m-1)!} \delta^{(m-1)}(x) & , m \geq 1. \end{cases}$$

Now using the identity $x^2\delta(x) = 0$ and differentiating this equation with respect to x repeatedly up to m times and using the results obtained above yields the following:

$$\begin{aligned} x^2\delta(x) &= 0 \\ x^2\delta^{(1)}(x) &= 0 \\ x^2\delta^{(2)}(x) &= 2\delta(x) \\ x^2\delta^{(3)}(x) &= 6\delta^{(1)}(x) \\ x^2\delta^{(4)}(x) &= 12\delta^{(2)}(x) \\ &\dots \\ x^2\delta^{(m)}(x) &= m(m-1)\delta^{(m-2)}(x) \end{aligned}$$

The last equation can be expressed easily as follows:

$$x^2\delta^{(m)}(x) = \begin{cases} 0 & , m < 2 \\ (-1)^2 \frac{m!}{(m-2)!} \delta^{(m-2)}(x) & , m \geq 2. \end{cases}$$

Proceeding in this way we can easily write the identity $x^3\delta(x) = 0$ as

$$x^3\delta^{(m)}(x) = \begin{cases} 0 & , m < 3 \\ (-1)^3 \frac{m!}{(m-3)!} \delta^{(m-3)}(x) & , m \geq 3. \end{cases}$$

Thus generalizing the result using the identity $x^n\delta(x) = 0$ and after repeated differentiation with respect to x, n times yields

$$x^n\delta^{(m)}(x) = \begin{cases} 0 & , m < n \\ (-1)^n \frac{m!}{(m-n)!} \delta^{(m-n)}(x) & , m \geq n. \end{cases}$$

This completes the proof of this important identity. □

1.2 Test Functions ($D(\Omega)$)

Let $\Omega \subset R^n$ be an open set. We recall that if f is a continuous function on Ω , the support of f is the set

$$\text{supp}(f) = \overline{\{x \in \Omega : f(x) \neq 0\}}.$$

Definition 1.1. The spaces of infinitely differentiable functions with compact support in Ω is defined as

$$D(\Omega) := \{f : \Omega \rightarrow \mathbb{C}; f \in C^\infty(\Omega), \text{ and } \text{supp}(f) \text{ is compact in } \Omega\} = C_c^\infty(\Omega).$$

This space is called a space of test functions. The set of test functions, the supports of which are contained in the given region Ω , is denoted by $D(\Omega)$;

Properties of Test Functions in $D(\Omega)$

1. If ϕ_1 , and ϕ_2 are in D , then so is $c_1\phi_1 + c_2\phi_2$, where c_1 and c_2 are real numbers. Thus D is a linear space.
2. If $\phi \in D$, then so is $D^k\phi$
3. For C^∞ function $f(x)$ and for $\phi \in D$, $f\phi \in D$.
4. Multiplication by a function $a \in C^\infty$:
Let $\varphi(x) \in D$ then $a(x)\varphi(x)$, are continuous from D into D .

Example 1.1.

$$f(x) = \begin{cases} \exp\frac{1}{1-|x|^2}, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

Show that f is a test function over R .

Verification : If $K \subset\subset V \subset\subset \bar{V} \subseteq R$ where K is compact and V is an open, then $\exists \varphi \in D(\Omega)$ such that $\varphi = 1$ on K and $\text{supp}(\varphi) \subseteq V$.

$$\text{supp}(f) = \overline{\{x : f(x) \neq 0\}} = [-1, 1]$$

since $[-1, 1]$ is closed and bounded. Thus $f(x)$ has a compact support in $[-1, 1]$ and $f(x)$ is infinitely differentiable in this region. Hence $f(x)$ is a test function in $[-1, 1]$.

1.3 Distributions ($D'(\Omega)$)

Definition 1.2. *Distribution in $D'(\Omega)$ is a class of continuous linear functional that maps a set of test function in $D(\Omega)$ into the (complex) numbers.*

i.e.: A functional $f : D(\Omega) \rightarrow \mathbb{C}$ such that

1. $\langle f, \varphi \rangle \in \mathbb{C}$
2. $\langle f, c_1\varphi_1 + c_2\varphi_2 \rangle = \langle f, c_1\varphi_1 \rangle + \langle f, c_2\varphi_2 \rangle$
3. $\lim_{n \rightarrow \infty} \langle f, \varphi_n \rangle = \langle f, \lim_{n \rightarrow \infty} \varphi_n \rangle$. where φ_1, φ_2 and $\varphi_n \in D(\Omega)$ and c_1, c_2 constants is called a distribution (generalized function).

The linear space of distributions denoted $D'(\Omega)$. The distribution f is a linear functional over $D(\Omega)$; if $\varphi, \psi \in D(\Omega)$ and λ and μ are complex numbers, then

$$\langle f, \lambda\varphi + \mu\psi \rangle = \lambda\langle f, \varphi \rangle + \mu\langle f, \psi \rangle$$

The distribution $f \in D'(\Omega)$ is a continuous functional over $D'(\Omega)$; if $\varphi_n \rightarrow \varphi$ as $n \rightarrow \infty$ in $D'(\Omega)$, then

$$\langle f, \varphi_n \rangle \rightarrow \langle f, \varphi \rangle$$

The set $D'(\Omega)$ is linear if the linear combination $\lambda f + \mu g$ of distribution f and g is defined as a functional acting in accordance with the formula

$$\langle \lambda f + \mu g, \varphi \rangle = \lambda\langle f, \varphi \rangle + \mu\langle g, \varphi \rangle, \forall \varphi \in D(\Omega), \forall \mu, \lambda \in \mathbb{C}$$

Verification: The functional $\lambda f + \mu g$ is linear and continuous over $D(\Omega)$, that is, it belongs to $D'(\Omega)$. If $\varphi, \psi \in D(\Omega)$, and $\alpha, \beta \in \mathbb{C}$, then, according to the definition,

$$\begin{aligned} \langle \lambda f + \mu g, \alpha\varphi + \beta\psi \rangle &= \lambda\langle f, \alpha\varphi + \beta\psi \rangle + \mu\langle g, \alpha\varphi + \beta\psi \rangle \\ &= \alpha[\lambda\langle f, \varphi \rangle + \mu\langle g, \varphi \rangle] + \beta[\lambda\langle f, \psi \rangle + \mu\langle g, \psi \rangle] \\ &= \alpha\langle \lambda f + \mu g, \varphi \rangle + \beta\langle \lambda f + \mu g, \psi \rangle \end{aligned}$$

and so this functional is linear. Its continuity follows from the continuity of the functionals f and g :

If $\varphi_n \rightarrow \varphi$ as $n \rightarrow \infty$, then

$$\begin{aligned} \langle \lambda f + \mu g, \varphi_n \rangle &= \lambda\langle f, \varphi_n \rangle + \mu\langle g, \varphi_n \rangle \\ &\Rightarrow \lambda\langle f, \varphi \rangle + \mu\langle g, \varphi \rangle \\ &= \langle \lambda f + \mu g, \varphi \rangle \end{aligned}$$

Definition 1.3. The sequence of distribution $f_1, f_2 \dots f_n$ belonging to $D'(\Omega)$ converges to the distribution $f \in D'(\Omega)$, if for any $\varphi \in D(\Omega)$

$$\langle f_n, \varphi \rangle \rightarrow \langle f, \varphi \rangle, \text{ as } n \rightarrow \infty.$$

We can write as $f_n \rightarrow f$.

Definition 1.4. Let Ω be an open set in R^n . Then a function $f : \Omega \rightarrow C$ such that

$$\int_{\Omega} |f\varphi| dx < \infty,$$

for each test function $\varphi \in D(\Omega)$ is called locally integrable. The set of such functions is denoted by $L^1_{loc}(\Omega)$

Definition 1.5 (Functions as distribution). Suppose that $f \in L^1_{loc}(\Omega)$, then the corresponding distribution $T_f : D(\Omega) \rightarrow C$ is defined by

$$\langle f, \varphi \rangle = \int_{\Omega} f(x)\varphi(x)dx, \forall \varphi \in D(\Omega).$$

The distribution T_f becomes zero in the region Ω if

$$\langle f, \varphi \rangle = 0, \forall \varphi \in D(\Omega).$$

Remark 1.1. $\text{supp}(T_f)$ is closed set.

If $\text{supp}(f)$ is a bounded set, then distribution f is said to have a compact support.

Definition 1.6. The linear and continuous map $T_f : D(\Omega) \rightarrow R$ defined by

$$T_f(\varphi) = \langle f, \varphi \rangle = \int_{\Omega} f\varphi dx \text{ where } f \in L^1_{loc}(\Omega) \text{ and } \varphi \in D(\Omega).$$

is called a regular distribution.

A distributions that is not Regular is called singular distribution.

Definition 1.7. The distribution T_f and T_g are said to be equal if $f - g = 0$, for $x \in \Omega$ a.e on Ω . Specifically, the distribution f and g are said to be equal a.e if for all $\varphi \in D(\Omega)$,

$$\langle f, \varphi \rangle = \langle g, \varphi \rangle.$$

1.3.1 Properties of Distributions($D'(\Omega)$)

1. Multiplication of distribution

Let $f(x)$ be a function locally integrable in R^n and $a(x) \in C^\infty(R^n)$. Then for any $\varphi \in D(\Omega)$ the equation

$$\langle af, \varphi \rangle = \langle f, a\varphi \rangle, \varphi \in D(\Omega). \quad (1.2)$$

2. Differentiation :

Any distribution is infinitely differentiable, and converging series of distribution can be differentiated term by term an infinite number of times.

Let $f \in C^k(\Omega)$. then whenever $\alpha, |\alpha| \leq k$, and $\varphi \in D(\Omega)$ the formula for integration by parts,

$$\begin{aligned} \langle D^\alpha f, \varphi \rangle &= \int D^\alpha f(x)\varphi(x)dx = (-1)^{|\alpha|} \int f(x)D^\alpha \varphi(x)dx \\ &= (-1)^{|\alpha|} \langle f(x), D^\alpha \varphi(x) \rangle. \end{aligned}$$

is valid

Definition 1.8. If $f \in D'(\Omega)$ is a distribution then the partial derivative of f with respect to the coordinate α_k is define by:

$$\langle D^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle, \forall \varphi \in D(\Omega).$$

We shall check that $D^\alpha f \in D'(\Omega)$. since $f \in D'(\Omega)$, the functional, $D^\alpha f$, is linear:

$$\begin{aligned} \langle D^\alpha f, \lambda\varphi + \mu\psi \rangle &= (-1)^{|\alpha|} \langle f, D^\alpha \langle \lambda\varphi + \mu\psi \rangle \rangle \\ &= (-1)^{|\alpha|} \langle f, \lambda D^\alpha \varphi + \mu D^\alpha \psi \rangle \\ &= \lambda(-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle + \mu(-1)^{|\alpha|} \langle f, D^\alpha \psi \rangle \\ &= \lambda \langle D^\alpha f, \varphi \rangle + \mu \langle D^\alpha f, \psi \rangle \end{aligned}$$

and continuous :

$$\begin{aligned} \langle D^\alpha f, \varphi_n \rangle &= (-1)^{|\alpha|} \langle f, D^\alpha \varphi_n \rangle \\ &\longrightarrow (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle = \langle D^\alpha f, \varphi \rangle. \end{aligned}$$

for if $\varphi_n \rightarrow \varphi$ as $n \rightarrow \infty$ in $D(\Omega)$. then also $D^\alpha \varphi_n \Rightarrow D^\alpha \varphi$ as $n \rightarrow \infty$ in $D(\Omega)$

1.3.2 Properties of Derivative of Functions in $D'(\Omega)$

1. The result of differentiation does not depend on the order of differentiation;

For example:

$$D^{\alpha_1}(D^{\alpha_2}f) = D^{\alpha_2}(D^{\alpha_1}f) = D^{\alpha_1+\alpha_2}f \quad (1.3)$$

In general,

$$D^{\alpha+\beta}f = D^\alpha(D^\beta f) = D^\beta(D^\alpha f) \quad (1.4)$$

2. If $f \in D'(\Omega)$ and $a \in C^\infty(R^n)$, then Leibnitz' formula for differentiation of the product af is valid.

For example: $D^{\alpha_1}(af) = fD^{\alpha_1}(a) + aD^{\alpha_1}(f)$ if φ is any basic function, then

$$\begin{aligned} \langle D^{\alpha_1}(af), \varphi \rangle &= -\langle af, D^{\alpha_1}\varphi \rangle \\ &= -\langle f, aD^{\alpha_1}\varphi \rangle \\ &= -\langle f, D^{\alpha_1}(a\varphi) - \varphi D^{\alpha_1}a \rangle \\ &= -\langle f, D^{\alpha_1}(a\varphi) \rangle + \langle f, \varphi D^{\alpha_1}a \rangle \\ &= \langle D^{\alpha_1}f, a\varphi \rangle + \langle fD^{\alpha_1}a, \varphi \rangle \\ &= \langle aD^{\alpha_1}f, \varphi \rangle + \langle fD^{\alpha_1}a, \varphi \rangle \\ &= \langle aD^{\alpha_1}f + fD^{\alpha_1}a, \varphi \rangle \\ &\implies D^{\alpha_1}(af) = fD^{\alpha_1}(a) + aD^{\alpha_1}(f) \end{aligned}$$

3. If the distribution $f = 0$ for $x \in \Omega$, then also $D^\alpha f = 0$ for $x \in \Omega$.

So that $\text{supp}(D^\alpha f) \subset \text{supp}(f)$, if $\varphi \in D(\Omega)$ then $D^\alpha f \in D(\Omega)$ and

$$\begin{aligned} \langle D^\alpha f, \varphi \rangle &= (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle = 0 \\ &\implies D^\alpha f = 0. \end{aligned}$$

4. The operation of differentiation is continuous from $D'(\Omega)$ into $D'(\Omega)$.

That is, if $f_n \rightarrow f$ as $n \rightarrow \infty$ in $D'(\Omega)$, then $D^\alpha f_n \rightarrow D^\alpha f$ as $n \rightarrow \infty$ in $D'(\Omega)$. Indeed, according to the definition of convergence in the space $D'(\Omega)$, $\forall \varphi \in D(\Omega)$. we have,

$$\begin{aligned} \langle D^\alpha f_n, \varphi \rangle &= (-1)^{|\alpha|} \langle f_n, D^\alpha \varphi \rangle \\ &\implies (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle \text{ as } n \rightarrow \infty \\ &= \langle D^\alpha f, \varphi \rangle \end{aligned}$$

which shows that $D^\alpha f_n \rightarrow D^\alpha f$, $n \rightarrow \infty$ in $D'(\Omega)$

Example 1.2. The Dirac delta function has a distributional derivative.

Defined by : $\langle \delta', \varphi \rangle = -\langle \delta, \varphi' \rangle$,

where $\langle \delta, \varphi \rangle = \varphi(0)$.

Verification: Let $\varphi \in D(\mathbb{R})$

$$H(x) = \begin{cases} 1, & x > 0 \\ 0, & x \leq 0. \end{cases} \quad \text{then}$$

$$\begin{aligned} \langle H', \varphi \rangle &= -\langle H, \varphi' \rangle \\ &= -\int_0^{\infty} H(x)\varphi'(x)dx. \\ &\implies \langle H, \varphi' \rangle \\ &= -\int_0^{\infty} \varphi' dx \\ &= \varphi(0). \\ &= \langle \delta, \varphi \rangle \end{aligned}$$

So $H' = \delta$. Hence $\langle \delta', \varphi \rangle = -\langle \delta, \varphi' \rangle = \varphi'(0)$ and Since $\varphi \in D(\mathbb{R})$
 $\implies \varphi' \in D(\mathbb{R})$ and $\varphi'(0)$ satisfies linearity and continuity.

In this case we say that the function g is the weak or distributional derivative of the function f and we write $g(x) = D^\alpha f$.

Theorem 1.2. Let $\delta^{(n)}$ be the n th derivative of the Dirac delta function and $f(x)$ smooth function, then $\int_{-\infty}^{\infty} f(x)\delta^{(n)}(x)dx = (-1)^n f^{(n)}(0)$

Proof: The proof will be given by applying integration by parts repeatedly.

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\delta^{(n)}(x)dx &= -\int_{-\infty}^{\infty} f'(x)\delta^{(n-1)}(x)dx \\ &= (-1)^2 \int_{-\infty}^{\infty} f''(x)\delta^{(n-2)}(x)dx \\ &= (-1)^3 \int_{-\infty}^{\infty} f'''(x)\delta^{(n-3)}(x)dx \\ &= \dots \\ &= (-1)^n \int_{-\infty}^{\infty} f^{(n)}(x)\delta(x)dx \\ &= (-1)^n \int_{-\infty}^{\infty} f^{(n)}(0)\delta(x)dx \\ &= (-1)^n f^{(n)}(0) \end{aligned}$$

□

Theorem 1.3. *If $\phi(x)$ is any test function, then $\int_{-\infty}^{\infty} \delta^{(n)}(x)\phi(x)dx = (-1)^n\phi^n(0)$*

$$\begin{aligned}
 \text{Proof : } \int_{-\infty}^{\infty} \delta^{(n)}(x)\phi(x)dx &= \langle \delta^{(n)}(x), \phi(x) \rangle \\
 &= (-1)^n \langle \delta(x), \phi^{(n)}(x) \rangle \\
 &= (-1)^n \phi^n(0)
 \end{aligned}$$

1.4 Fourier Transform

Let us first recall the definition of the Fourier transform(FT) and inverse Fourier transform(IFT) that will be used in this paper. Given a function $f(t)$, its Fourier transform $\mathcal{F}(\omega)$ is defined as

$$\mathcal{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt$$

And given a Fourier transform $\mathcal{F}(\omega)$, then its inverse Fourier transform is given by

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)e^{i\omega t} d\omega$$

Properties of Fourier transform and inverse of Fourier transform

In what follows, we assume that the functions $f(t)$ and $g(t)$ are differentiable as often as necessary, and that all necessary integrals exist. (This implies that $f(t) \rightarrow 0$ as $|t| \rightarrow \infty$.) . Regarding notation: $f^{(n)}$ denotes the n th derivative of f with respect to t ; $F^{(n)}(\omega)$ denotes the n th derivative of F with respect to ω .

1. Linearity of the Fourier transform operator and the inverse Fourier transform operator:

$$\begin{aligned}
 \mathcal{F}(f + g) &= \mathcal{F}(f) + \mathcal{F}(g) \\
 \mathcal{F}(cf) &= c\mathcal{F}(f), c \in \mathcal{C}(or\mathfrak{R}) \\
 \mathcal{F}^{-1}(f + g) &= \mathcal{F}^{-1}(f) + \mathcal{F}^{-1}(g) \\
 \mathcal{F}^{-1}(cf) &= c\mathcal{F}^{-1}(f), c \in \mathcal{C}(or\mathfrak{R})
 \end{aligned}$$

These follow naturally from the integral definition of the Fourier transform.

2. Fourier transform of a product of f with t^n

$$\mathcal{F}(t^n f(t)) = (i)^n F^{(n)}(\omega)$$

3. Inverse Fourier transform of a product of F with ω^n

$$\mathcal{F}^{-1}(\omega^n F(\omega)) = (-i)^n f^{(n)}(t)$$

4. Fourier transform of an n th derivative:

$$\mathcal{F}(f^{(n)}(t)) = (i\omega)^n F(\omega)$$

5. Inverse Fourier transform of an n th derivative:

$$\mathcal{F}^{-1}(F^{(n)}(\omega)) = (-it)^n f(t)$$

6. Fourier transform of the Dirac delta function $\delta(t)$

$$\mathcal{F}[\delta(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(t) e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}}$$

7. Fourier transform of an n th derivative of Dirac delta function $\delta(t)$

$$\mathcal{F}[\delta^{(n)}(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta^{(n)}(t) e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}} (i\omega)^n$$

1.5 Regular Singular Points

We have studied solutions to the linear second order differential equations of the form

$$P(x)y'' + Q(x)y' + R(x)y = 0 \quad (1.5)$$

in the cases $P(x), Q(x), R(x)$ real analytic in a neighborhood of a point $x = a$ and $P(a) \neq 0$. If $P(a) = 0$, then a is called a singular point of the equation. We say that the singular point $x = a$ is a regular singular point of equation (1) if the equation can be written as

$$y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0 \quad (1.6)$$

where

$$\lim_{x \rightarrow a} \frac{(x-a)Q(x)}{P(x)} \quad \text{and} \quad \lim_{x \rightarrow a} \frac{(x-a)^2 R(x)}{P(x)}$$

both exist and are finite.

A singular point which is not regular is called irregular.
 For demonstration , consider the differential equation

$$(1 - x^2)y'' - 2xy' + 6y = 0$$

determine whether points are regular or irregular singular points.
 The singular points of the ordinary differential equation are the zeros of

$$(1 - x^2) = (1 - x)(1 + x)$$

and clearly, $x = 1$ and $x = -1$. To test whether 1 is regular, we divide by the function $(1 - x^2)$ obtaining

$$y'' - \frac{2x}{(1 - x)(1 + x)}y' + \frac{6}{(1 - x)(1 + x)}y = 0$$

Thus

$$\lim_{x \rightarrow 1} \frac{2x(x - 1)}{(1 - x)(1 + x)} = -1 \quad \text{and} \quad \lim_{x \rightarrow 1} \frac{6(1 - x^2)}{(1 - x)(1 + x)} = 0$$

both exist and finite. Hence, $x = 1$ is a regular singular point.
 A similar test shows that $x = -1$ also a regular singular point.
 A simple example is the first order differential equation

$$x^2 \frac{dy}{dx} - 2y = 0 \tag{1.7}$$

Here the point $x = 0$ is an essential(irregular) singularity of this equation because $\lim_{x \rightarrow 0} \frac{-2x}{x^2}$ doesn't exist.

Chapter 2

Formal distributional solutions to the confluent hypergeometric equation

The second order differential equation

$$xy'' + (m - x)y' - py = 0, \quad m, p \in \mathbb{C} \quad (2.1)$$

is Known as the **confluent Hypergeometric differential equation**.

One of its classical solution of confluent Hypergeometric differential equation is well known and given by

$${}_1F_1(p, m; x) = \sum_{n=0}^{\infty} \frac{(p)_n x^n}{(m)_n n!}$$

Where $(p)_n$ is the pochhammer's symbol: $(p)_n = p(p+1)(p+2) \cdots (p+n-1)$, $n > 0$, $(p)_0 = 1$ and ${}_1F_1$ is the hypergeometric function. If we substitute

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

into the differential equation (2.1) we obtain

$$\sum_{n=0}^{\infty} [n(n+1)a_{n+1} + m(n+1)a_{n+1} - n a_n - p a_n] x^n = 0$$

$$a_{n+1}((n+1)(m+n)) - a_n(n+p) = 0$$

$$a_{n+1} = \frac{a_n(n+p)}{(n+1)(m+n)}, \quad n = 0, 1, 2, \dots$$

Thus, $a_n = \frac{(p)_n}{(m)_n n!}$

Hence, its one of the classical solution is $y = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{(p)_n x^n}{(m)_n n!}$

Example 2.1. *The classical solution of the equation $xy'' + (1-x)y' - y = 0$ is given by*

$$y(x) = \sum_{n=0}^{\infty} \frac{(1)_n x^n}{(1)_n n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

Thus, $y(x) = e^x$ is the classical solution of $xy'' + (1-x)y' - y = 0$

My aim is to find distributional solutions to (2.1) of the form

$$y(x) = \sum_{n=0}^{\infty} a_n \delta^{(n)}(x), \quad (2.2)$$

where $\delta^{(n)}$ is the n th derivative of the Dirac delta function.

A few reasons for deriving the solution of the above form are explained in the introduction. We can also motivate the study of solutions of the form (2.1) in another way.

Consider the following two examples:

Example 2.2. *$y(x) = e^x$ is easily seen to be a solution of*

$$xy'' + (1-x)y' - y = 0$$

If we identify e^x with the distribution defined by:

$$\langle e^x, \phi(x) \rangle = \int_{-\infty}^{\infty} e^x \phi(x) dx, \phi \in D$$

(D consists of infinitely differentiable functions of compact support), then $\langle e^x, \phi(x) \rangle$ is a distributional solution to $xy'' + (1-x)y' - y = 0$. But, Is $\int_{-\infty}^0 e^x \phi(x) dx$ a distributional solution to $xy'' + (1-x)y' - y = 0$? One would suspect not, since we would expect "boundary" conditions involving $\phi(0), \phi'(0)$ etc. to appear when $\int_{-\infty}^0 e^x \phi(x) dx$ was substituted into $xy'' + (1-x)y' - y = 0$. However, $\int_{-\infty}^0 e^x \phi(x) dx$ is a distributional solution to $xy'' + (1-x)y' - y = 0$. we will find out the reason for this in example of next section.

Example 2.3. $y(x) = e^x$ is also a solution to

$$xy'' + (2 - x)y' - 2y = 0$$

Again, by identification,

$$\langle e^x, \phi(x) \rangle = \int_{-\infty}^{\infty} e^x \phi(x) dx$$

is a distributional solution. Is $\int_{-\infty}^0 e^x \phi(x) dx$ a distributional solution to $xy'' + (2 - x)y' - 2y = 0$? In this case, the answer is no. We will find the reason in the next section .

Theorem 2.1. Suppose $y(x) = \sum_{n=0}^{\infty} a_n \delta^{(n)}(x)$ is the formal distributional solution to the confluent Hypergeometric differential equation

$$xy'' + (m - x)y' - py = 0.$$

Then: (i) if $p \notin \mathbb{N}$ but $m \in \mathbb{N}$, then $y(x) = 0$

(ii) if $p \in \mathbb{N}$ but $m \notin \mathbb{N}$, then

$$y(x) = \sum_{n=0}^{\infty} \frac{(p - m + 1)_n \delta^{(p+n-1)}(x)}{n!}$$

(iii)(a) If $p, m \in \mathbb{N}$, then the distributional solution of $xy'' + (m - x)y' - py = 0$ is given by

$$y(x) = \sum_{n=p-1}^{\infty} \binom{n - m + 1}{n - p + 1} \delta^{(n)}(x) \text{ if } m \leq p$$

(b) If $p, m \in \mathbb{N}$, then

$$y(x) = \sum_{n=p-1}^{m-2} (-1)^{n-p+1} \binom{m - p - 1}{n - p + 1} \delta^{(n)}(x) \text{ if } m > p$$

Proof. Suppose $y(x) = \sum_{n=0}^{\infty} a_n \delta^{(n)}(x)$, then $y' = \sum_{n=1}^{\infty} a_{n-1} \delta^{(n)}$ and $y'' = \sum_{n=2}^{\infty} a_{n-2} \delta^{(n)}$.

Now substituting y, y' and y'' into equation(2.1) and use the property of

$$x^n \delta^{(m)}(x) = \begin{cases} 0 & , m < n \\ (-1)^n \frac{m!}{(m-n)!} \delta^{(m-n)}(x) & , m \geq n. \end{cases} \text{ gives}$$

$$\sum_{n=0}^{\infty} ((-n-1+m)a_{n-1} \delta^{(n)}(x) + (n+1-p)a_n) \delta^{(n)}(x) = 0$$

$$a_{n-1}(-n-1+m) + a_n(n+1-p) = 0$$

$$a_n = \frac{(n-m+1)a_{n-1}}{(n-p+1)}, n = 1, 2, \dots$$

Thus,

$$a_n = \binom{n-m+1}{n-p+1}$$

(i) If $p \notin \mathbb{N}$ but $m \in \mathbb{N}$, then $\binom{n-m+1}{n-p+1} = 0$

because $n-m+1 < n-p+1$, so $y(x) = 0$

(ii) If $p \in \mathbb{N}$ but $m \notin \mathbb{N}$, then

$$y(x) = \sum_{n=0}^{\infty} \binom{n-m+1}{n-p+1} \delta^{(n)}(x)$$

Now replacing n by $n+p-1$, then

$$y(x) = \sum_{n=0}^{\infty} \binom{n-m+p}{n} \delta^{(n+p-1)}(x) = \sum_{n=0}^{\infty} \frac{(p-m+1)_n \delta^{(p+n-1)}(x)}{n!}$$

(iii)(a) If $p, m \in \mathbb{N}$ and $m \leq p$, then

$$y(x) = \sum_{n=p-1}^{\infty} \binom{n-m+1}{n-p+1} \delta^{(n)}(x)$$

(b) Also If $p, m \in \mathbb{N}$ and $m > p$, then $n-m+1 < 0$.

From the identity $\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$

we obtain

$$\binom{n-m+1}{n-p+1} = (-1)^{n-p+1} \binom{m-p-1}{n-p+1}$$

As we see $m - p - 1 \geq n - p + 1$ so $n \leq m - 2$

Thus

$$y(x) = \sum_{n=p-1}^{m-2} (-1)^{n-p+1} \binom{m-p-1}{n-p+1} \delta^{(n)}(x) \text{ if } m > p$$

□

Theorem 2.2. (Laplace Method) If $p, m \in \mathbb{N}$, then the distributional solution of $xy'' + (m-x)y' - py = 0$ is given by

$$w(x) = \sum_{n=p-1}^{\infty} \binom{n-m+1}{n-p+1} \delta^{(n)}(x) \text{ if } m \leq p$$

Proof. Formally applying the distribution two sided laplace transformation $L[x] = F(s)$ to $xy'' + (m-x)y' - py = 0$ yields the equation

$$-(s^2 F)' + msF + (sF)' - pF = 0$$

$$\text{i.e., } (s^2 - s)F' = [(m-2)s - (p-1)]F,$$

with the general solution

$$F(s) = Cs^{p-1}(s-1)^{m-p-1}, C = \text{constant}$$

The particular solution $F_0(s)$ corresponding to $C = (-1)^{p-m+1}$ is:

$$F_0(s) = s^{p-1}(1-s)^{-1-p+m}$$

Differentiating the series $(1-s)^{-1} = \sum_{n=0}^{\infty} s^n$, $|s| < 1$ $p-m$ times and multiplying by s^{p-1} give

$$F_0(s) = \sum_{n=p-m}^{\infty} \binom{n}{p-m} s^{n+m-1},$$

which changes to

$$F_0(s) = \sum_{n=p-1}^{\infty} \binom{n-m+1}{n-p+1} s^n$$

by the substitution $n+m-1 \rightarrow n$ and $L[\delta^{(n)}(x)] = s^n$. The distributional solution become

$$w(x) = \sum_{n=p-1}^{\infty} \binom{n-m+1}{n-p+1} \delta^{(n)}(x)$$

□

Corollary 2.1. For $p, m \in \mathbb{N}$ and $m > p$, The solution of $xy'' + (m - x)y' - py = 0$ is given by:

$$y(x) = \sum_{n=p-1}^{m-2} (-1)^{n-p+1} \binom{m-p-1}{n-p+1} \delta^{(n)}(x)$$

$$= \frac{(-1)^{p+m-1} d^{p-1} \left(\left(\frac{d}{dx} - 1 \right)^{m-p-1} \delta(x) \right)}{dx^{p-1}}$$

This corollary is in agreement with the result of Wiener in[1].

Example 2.4. For $xy'' + (2 - x)y' - y = 0$ its distributional solution is

$$y(x) = \delta(x)$$

Example 2.5. For $xy'' + (10 - x)y' - 3y = 0$, its distributional solution is

$$y(x) = \sum_{n=2}^8 (-1)^{n-2} \binom{6}{n-2} \delta^{(n)}(x)$$

Example 2.6. For $xy'' + (4 - x)y' - 7y = 0$, its distributional solution is

$$y(x) = \sum_{n=6}^{\infty} \binom{n-3}{n-6} \delta^{(n)}(x)$$

Example 2.7. For $xy'' + (3 - x)y' - y = 0$, its distributional solution is

$$y(x) = \delta(x) - \delta'(x)$$

Example 2.8. For $xy'' + (-1 - x)y' - y = 0$, its distributional solution is

$$y(x) = \sum_{n=0}^{\infty} \binom{n+2}{n} \delta^{(n)}(x)$$

2.1 Further Analysis Of Theorem 2.1: The Case Where $p, m \in \mathbb{N}$ and $m \leq p$

It is easy to see that the solution $y(x)$ in *Theorem 2.1* is a continuous linear functional on D , the space of all infinitely differentiable functions; i.e. $y \in C^\infty$. By considering certain test functions, it is easy to see that $y(x)$, in the other cases of *Theorem 2.2*, is not in D' . It is natural to ask if $y(x)$ has any extensions to S' (S is the test function space of infinitely differentiable functions of rapid decay) or D' . Also, by using the Fourier transform, can we get a better representation of $y(x)$? We discuss these questions in this section and succeeding sections.

From here on in this section, assume, $p, m \in \mathbb{N}$ and $m \leq p$. Let

$$\begin{aligned} y(x) &= \sum_{n=p-1}^{\infty} \binom{n-m+1}{n-p+1} \delta^{(n)}(x) \\ &= \sum_{j=0}^{\infty} \binom{p-m+j}{j} \delta^{(p+j-1)}(x) \end{aligned}$$

We seek to get a better representation of $y(x)$.

Theorem 2.3.

$$\begin{aligned} &\sum_{n=p-1}^{\infty} \binom{n-m+1}{n-p+1} \delta^{(n)}(x) \\ &= \sum_{k=0}^{p-m} \frac{\binom{p-1}{k} x^{p-m-k} (e^x - e^x H(x))}{(p-m-k)!} - \sum_{k=0}^{m-2} \binom{p-k-2}{m-k-2} \delta^{(k)}(x) \end{aligned}$$

where

$$H(x) = \begin{cases} 1 & , x > 0 \\ 0 & , x \leq 0 \end{cases}$$

is the Heaviside function. To prove this theorem, we first must prove a number of lemmas. They are

Lemma 2.1. $\sum_{j=0}^{\infty} \binom{n+j}{j} x^j = \left(\frac{d^n \frac{x^n}{n!(1-x)}}{dx^n} \right)$, $|x| < 1$

Proof. $\frac{d^n \frac{x^n}{n!(1-x)}}{dx^n} = \frac{d^n \sum_{j=0}^{\infty} \frac{x^{n+j}}{n!}}{dx^n} = \sum_{j=0}^{\infty} \frac{p(n+j, n) x^j}{n!} = \sum_{j=0}^{\infty} \binom{n+j}{j} x^j \quad \square$

Let $\hat{f}(t)$ denote the Fourier transform of the function or distribution f .

Lemma 2.2.
$$\hat{w}(t) = \frac{(it)^{p-1}}{\sqrt{2\pi}(p-m)!} \left(\frac{d^{p-m} \left(\frac{(it)^{p-m}}{1-it} \right)}{dt^{p-m}} i^{p-m} \right)$$

Proof. The fourier transform of $\delta^{(j)}$ is $\frac{(it)^j}{\sqrt{2\pi}}$. Hence,

$$\begin{aligned} \hat{w}(t) &= \sum_{j=0}^{\infty} \binom{p-m+j}{j} \frac{(it)^{p+j-1}}{\sqrt{2\pi}} \\ &= \frac{(it)^{p-1}}{\sqrt{2\pi}} \sum_{j=0}^{\infty} \binom{p-m+j}{j} (it)^j \\ &= \frac{(it)^{p-1}}{\sqrt{2\pi}} \left(\frac{d^{p-m} \left(\frac{(it)^{p-m}}{(p-m)!(1-it)} \right)}{dt^{p-m}} i^{p-m} \right) \end{aligned}$$

□

Lemma 2.3. *Let*

$$f(x) = e^x - e^x H(x) = \begin{cases} 0 & , x > 0 \\ e^x & , x \leq 0 \end{cases}$$

Then the r^{th} distributional derivative of $f(x)$ is given by

$$f^{(r)}(x) = f(x) - \sum_{j=0}^{r-1} \delta^{(j)}(x), \quad r \geq 0.$$

Its proof follows by induction.

Lemma 2.4. (a) $f(x) = e^x - e^x H(x)$.

Then

$$\widehat{f^{(p-m)}}(x)(t) = \frac{(it)^{p-m}}{\sqrt{2\pi}(1-it)}$$

(b) $x^{p-m} f^{(p-m)}(x) = x^{p-m} f(x)$ whenever $P - m \geq 0, p, m \in \mathbb{N}$

(c) Let $g(x) = \left(\frac{i^{-(p-m)} x^{(p-m)} f^{(p-m)}(x)}{(p-m)!} \right)$. Then

$$\hat{g}(t) = \frac{1}{(p-m)!} \left(\frac{d^{p-m} \left(\frac{(it)^{p-m}}{\sqrt{2\pi}(1-it)} \right)}{dt^{p-m}} \right)$$

Let $g(x)$ be as in (c). Then

$$g^{\widehat{(p-1)(x)}}(t) = \frac{(it)^{p-1}}{\sqrt{2\pi}(p-m)!} \left(\frac{d^{p-m} \left(\frac{(it)^{p-m}}{1-it} \right)}{dx^{p-m}} \right)$$

Proof. (a) From Lemma 2.3 , $f^{(p-m)}(x) = f(x) - \sum_{j=0}^{p-m-1} \delta^{(j)}(x)$.
Hence

$$f^{\widehat{(p-m)}(x)}(t) = \widehat{f(x)} - \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{p-m-1} (it)^j.$$

$$\text{But } \widehat{f(t)} = \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{\infty} (it)^j.$$

Hence

$$f^{\widehat{(p-m)}(x)}(t) = \frac{1}{\sqrt{2\pi}} \sum_{j=p-m}^{\infty} (it)^j = \frac{(it)^{p-m}}{\sqrt{2\pi}(1-it)}$$

(b) This follows from Lemma 3.3 and the fact that $x^j \delta^{(k)}(x) = 0$ if $j > k$.
(c) In general , If h is a function or a distribution, then

$$x^r h(x)(t) = i^r \frac{d^r \widehat{h(t)}}{dt^r}.$$

Hence

$$\begin{aligned} \widehat{g(t)} &= \frac{i^{-(p-m)} i^{(p-m)}}{(p-m)!} \frac{d^{p-m} (f^{\widehat{(p-m)}(x)}(t))}{dt^{p-m}} \\ &= \frac{1}{(p-m)!} \left(\frac{d^{p-m} \left(\frac{(it)^{p-m}}{\sqrt{2\pi}(1-it)} \right)}{dt^{p-m}} \right) \text{ from (a)} \end{aligned}$$

(d) from (c) ,

$$g^{(p-1)}(x) = \frac{i^{-(p-m)}}{(p-m)!} \frac{d^{p-1} (x^{(p-1)} f(x))}{dx^{p-1}}.$$

Hence,

$$g^{\widehat{(p-1)}(x)}(t) = \frac{i^{-(p-m)}}{(p-m)!} \left[\frac{d^{p-1} (\widehat{x^{(p-1)} f(x)})}{dx^{p-1}} \right] (t)$$

From (c)'

$$\begin{aligned}\widehat{x^{p-m}f(x)}(t) &= i^{p-m}(p-m)!\hat{g}(t) \\ &= i^{p-m}\left(\frac{d^{p-m}\left(\frac{(it)^{p-m}}{\sqrt{2\pi}(1-it)}\right)}{dt^{p-m}}\right)\end{aligned}$$

Now, if h is any function or distribution, then

$$\frac{d^r\widehat{h(x)}}{dx^r}(t) = i^r t^r \hat{h}(t).$$

Hence,

$$\begin{aligned}\frac{d^{p-1}\widehat{(x^{p-m}f(x))}(t)}{dx^{p-1}} &= (it)^{p-1}\widehat{x^{p-m}f(x)}(t) \\ &= (it)^{p-1}i^{p-m}\left(\frac{d^{p-m}\left(\frac{(it)^{p-m}}{\sqrt{2\pi}(1-it)}\right)}{dt^{p-m}}\right)\end{aligned}$$

Thus ,

$$\widehat{g^{(p-1)}(x)}(t) = \frac{(it)^{p-1}}{(p-m)!}\left(\frac{d^{p-m}\left(\frac{(it)^{p-m}}{\sqrt{2\pi}(1-it)}\right)}{dt^{p-m}}\right)$$

□

Corollary 2.2. $\hat{w}(t) = i^{p-m}\widehat{g^{(p-1)}(x)}(t)$

Corollary 2.3. $y(x) = i^{p-m}g^{(p-1)}(x)$

Lemma 2.5. Let $h(x) = x^r f(x)$ where $f(x) = e^x - e^x H(x)$.

Then

$$(a) \quad h^{(j)}(x) = \sum_{k=0}^j P(r, k) \binom{j}{k} x^{r-k} (f(x)) \text{ if } j \leq r.$$

$$(b) \quad h^{(r+j)}(x) = \sum_{k=0}^r P(r, k) \binom{r+j}{k} x^{r-k} f(x) - r! \sum_{k=0}^{j-1} \binom{r+j-k+1}{j-k-1} \delta^{(k)}(x) \text{ if } j \geq 0.$$

where the last sum is interpreted to be zero if $j = 0$.

Proof. (by Induction). (a) obviously, the formula holds for $j = 0$.

Suppose $h^{(l)}(x) = \sum_{k=0}^l P(r, k) \binom{l}{k} x^{r-k} f(x)$ for all $l \leq j < r$, and

suppose $j + 1 \leq r$. Then

$$h^{(j+1)}(x) = \sum_{k=0}^j P(r, k) \binom{j}{k} (x^{r-k} f(x))'$$

Now

$$\langle (x^{r-k} f(x))', \phi \rangle = - \langle x^{r-k} f(x), \phi' \rangle = - \int_{-\infty}^0 x^{r-k} e^x \phi'(x) dx$$

$$\langle (r-k)x^{r-k-1} f(x) + x^{r-k} f(x), \phi \rangle, \phi \in D.$$

Hence,

$$(x^{r-k} f(x))' = (r-k)x^{r-k-1} f(x) + x^{r-k} f(x)$$

Substituting this into(3.2) yield:

$$\begin{aligned} h^{(j+1)}(x) &= \sum_{k=0}^j P(r, k) \binom{j}{k} ((r-k)x^{r-k-1} f(x) + x^{r-k} f(x)) \\ &= \sum_{k=0}^{j+1} P(r, k) \binom{j+1}{k} x^{r-k} f(x). \end{aligned}$$

Hence the formula is true for $j < r$. In Particular,

$$h^{(r-1)}(x) = \sum_{k=0}^{r-1} P(r, k) \binom{r-1}{k} x^{r-k} f(x)$$

Thus,

$$\begin{aligned} h^{(r)}(x) &= \sum_{k=0}^{r-1} P(r, k) \binom{r-1}{k} ((r-k)x^{r-k-1} f(x) + x^{r-k} f(x)) \\ &= \sum_{k=0}^r P(r, k) \binom{r}{k} x^{r-k} f(x) \end{aligned}$$

(b) The above equation shows the statement in (b) to be true for $j = 0$.

Suppose

$$r^{(r+j)}(x) = \sum_{k=0}^r P(r, k) \binom{r+j}{k} x^{r-k} f(x) - r! \sum_{k=0}^{j-1} \binom{r+j-k-1}{j-k-1} \delta^{(k)}(x)$$

$$= \sum_{k=0}^{r-1} P(r, k) \binom{r+j}{k} x^{r-k} f(x) + p(k, r) \binom{r+j}{r} f(x) - r! \sum_{k=0}^{j-1} \binom{r+j-k-1}{j-k-1} \delta^{(k)}(x).$$

Then

$$\begin{aligned} h^{(r+j+1)}(x) &= \sum_{k=0}^{r-1} P(r, k) \binom{r+j}{k} ((r-k)x^{r-k-1}f(x) + x^{r-k}f(x)) \\ &+ p(r, r) \binom{r+j}{r} (f(x) - \delta(x)) - r! \sum_{k=0}^{j-1} \binom{r+j-k-1}{j-k-1} \delta^{(k+1)} \text{ see lemma 3} \\ &= \sum_{k=0}^r P(r, k) \binom{r+j+1}{k} x^{r-k} f(x) - r! \sum_{k=0}^j \binom{r+j-k}{j-k} \delta^{(k)}(x) \end{aligned}$$

where we have used the facts:

$$\binom{r+j}{k-1} + \binom{r+j}{k} = \binom{r+j+1}{k} \text{ and } p(r, k)(r-k) = p(r, k+1).$$

This establishes the lemma. \square

Corollary 2.4.

$$\frac{d^{p-1}(x^{p-m}f(x))}{dx^{p-1}} = \sum_{k=0}^{p-m} p(p-m, k) \binom{p-1}{k} x^{p-m-k} f(x) - (p-m)! \sum_{k=0}^{m-2} \binom{p-k-2}{m-k-2} \delta^{(k)}(x)$$

Proof. Let $r = p - m$ and $j = m - 2 (\geq 0)$ in the lemma. \square

Corollary 2.5.

$$g^{(p-1)}(x) = \frac{i^{-(p-m)}}{(p-m)!} \left(\sum_{k=0}^{p-m} p(p-m, k) \binom{p-1}{k} x^{p-m-k} f(x) - (p-m)! \sum_{k=0}^{m-2} \binom{p-k-2}{m-k-2} \delta^{(k)}(x) \right)$$

Theorem 2.4. if $p, m \in \mathbb{N}$ and $m \leq p$, then

$$\sum_{n=p-1}^{\infty} \binom{n-m+1}{n-p+1} \delta^{(n)}(x) = \sum_{k=0}^{p-m} \left(\frac{\binom{p-1}{k} x^{p-m-k} (e^x - e^x H(x))}{(p-m-k)!} \right) - \sum_{k=0}^{m-2} \binom{p-k-2}{m-k-2} \delta^{(k)}(x)$$

Proof. From Corolary2 , we know $y(x) = i^{p-m} g^{(p-1)}(x)$.

This theorem now follows since

$$\frac{p(p-m, k)}{(p-m)!} \binom{p-1}{k} = \frac{\binom{p-1}{k}}{(p-m-k)!}.$$

\square

Example 2.9.

$$\sum_{j=0}^{\infty} \delta^{(j)}(x) = e^x - e^x H(x) = \begin{cases} 0 & x > 0 \\ e^x & x \leq 0 \end{cases} \quad (p = m = 1)$$

Hence $e^x - e^x H(x)$ is a distributional solution to $xy'' + (1-x)y' - y = 0$.
That is

$$\langle (e^x)_-, \phi \rangle = \int_{-\infty}^0 e^x \phi(x) dx$$

is a solution.

Example 2.10.

$$\sum_{j=1}^{\infty} \delta^{(j)}(x) = e^x - e^x H(x) - \delta(x) \quad (p = m = 2)$$

Hence a distributional solution to

$$xy'' + (2-x)y' - 2y = 0 \text{ is}$$

$$\langle (e^x)_-, \phi \rangle - \langle \delta(x), \phi \rangle = \int_{-\infty}^0 e^x \phi(x) dx - \phi(0).$$

These two examples were both mentioned in §2

Example 2.11. From Theorem 2.1, we see that $\sum_{j=0}^{\infty} (j+1)\delta^{(j+1)}(x)$ is a solution to $xy'' + (1-x)y' - 2y = 0$. From Theorem 3.2 we see that

$$\sum_{j=0}^{\infty} (j+1)\delta^{(j+1)}(x) = (x+1)(e^x - e^x H(x)).$$

Example 2.12. From Theorem 3.2, a distributional solution to

$$xy'' + (4-x)y' - 7y = 0 \text{ is}$$

$$\left(\frac{x^3}{6} + 3x^2 + 15x + 20\right)(e^x - e^x H(x)) - 10\delta(x) - 4\delta'(x) - \delta''(x).$$

An Important Observation

In Example 2.11 and 2.12, it is easy to verify that $(x+1)e^x$ and $\left(\frac{x^3}{6} + 3x^2 + 15x + 20\right)e^x$ are classical solutions to the respective equations.

2.2 Further analysis of theorem 2.1:

The case where $p \in \mathbb{N}$ and $m \in -\mathbb{N} \cup \{0\}$

In this section , we shall assume $p \in \mathbb{N}$ and $m \in -\mathbb{N} \cup \{0\}$

From Theorem 2.1 . we know that

$$y(x) = \sum_{n=0}^{\infty} \frac{(p-m+1)_n \delta^{(p+n-1)}(x)}{n!} \quad (2.3)$$

is a solution to $xy'' + (m-x)y' - py = 0$. Obsreve that $\frac{(p-m+1)_n}{n!} = \binom{n+p-m}{n}$ so that

$$y(x) = \sum_{n=p-1}^{\infty} \binom{n-m+1}{n-p+1} \delta^{(n)}(x) \quad (2.4)$$

From (2.3), we see that

$$\begin{aligned} \hat{y}(t) &= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(p-m+1)_n (it)^{(p+n-1)}}{n!} \\ &= \frac{(it)^{p-1}}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(p-m+1)_n (it)^{(n)}}{n!} \\ &= \frac{(it)^{p-1}}{\sqrt{2\pi}} \{ {}_1F_0(p-m+1; it) \} \\ &= \frac{(it)^{p-1}}{\sqrt{2\pi}(1-it)^{p-m+1}} \end{aligned}$$

Recall that if f is a function or distribution, then

$$\mathcal{F}\left(\frac{d^k f(x)}{dx^k}\right) = (it)^k \mathcal{F}(f(x)) \quad (2.5)$$

where \mathcal{F} denotes the Fourier transform of f. Also,

$$\mathcal{F}((-ix)^k f(x)) = \frac{d^k \mathcal{F}(f(x))}{dt^k} \quad (2.6)$$

since $\mathcal{F}(e^x - e^x H(x)) = \frac{1}{\sqrt{2\pi}(1-it)}$, it is easy to verify that

$$\frac{d^k \left(\frac{1}{\sqrt{2\pi}(1-it)} \right)}{dt^k} = \frac{i^k k!}{\sqrt{2\pi}(1-it)^{k+1}}$$

From(2.6), it follows that

$$\mathcal{F}((-1)^{p-m}i^{p-m}x^{p-m}(e^x - e^x H(x))) = \frac{i^{p-m}(p-m)!}{\sqrt{2\pi}(1-it)^{p-m+1}}$$

Hence

$$\mathcal{F}\left(\frac{(-1)^{p-m}x^{p-m}}{(p-m)!}(e^x - e^x H(x))\right) = \frac{1}{\sqrt{2\pi}(1-it)^{p-m+1}} \quad (2.7)$$

From(2.5), it follows that

$$\frac{\mathcal{F}(d^{p-1} \frac{(-1)^{p-m}x^{p-m}(e^x - e^x H(x))}{(p-m)!})}{dx^{p-1}} = \frac{(it)^{p-1}}{\sqrt{2\pi}(1-it)^{p-m+1}}$$

But,from(a) of lemma 2.5 ,(a) $h^{(j)}(x) = \sum_{k=0}^j P(r, k) \binom{j}{k} x^{r-k}(f(x))$ if $j \leq r$.

, we know

$$\begin{aligned} \frac{d^{p-1} \frac{(-1)^{p-m}x^{p-m}(e^x - e^x H(x))}{(p-m)!}}{dx^{p-1}} &= (-1)^{p-m} \sum_{k=0}^{p-1} \frac{p(p-m, k)}{(p-m)!} \binom{p-1}{k} x^{p-m-k}(e^x - e^x H(x)) \\ &= (-1)^{p-m} \sum_{k=0}^{p-1} \frac{\binom{p-1}{k}}{(p-m-k)!} x^{p-m-k}(e^x - e^x H(x)) \end{aligned}$$

Theorem 2.5. Suppose $p \in \mathbb{N}$ and $m \in -\mathbb{N} \cup \{0\}$ Then:

$$\sum_{n=p-1}^{\infty} \binom{n-m+1}{n-p+1} \delta^{(n)}(x) = (-1)^{p-m} \sum_{k=0}^{p-1} \frac{\binom{p-1}{k}}{(p-m-k)!} x^{p-m-k}(e^x - e^x H(x))$$

Example 2.13. A distributional solution to $xy'' + (-3 - x)y' - 4y = 0$ is

$$\sum_{k=0}^3 \left(\frac{\binom{3}{k}}{(7-k)!} \right) x^{7-k} (e^x - e^x H(x)) = \left(\frac{x^7}{5040} + \frac{x^6}{240} + \frac{x^5}{40} + \frac{x^4}{24} \right) (e^x - e^x H(x)).$$

Example 2.14. A distributional solution to $xy'' + (-1 - x)y' - y = 0$ is

$$\sum_{n=0}^{\infty} \binom{n+2}{n} \delta^{(n)}(x) = \frac{x^2}{2} (e^x - e^x H(x))$$

These examples are easily verified.

Chapter 3

The Hypergeometric equation

In this section we shall discuss certain aspects of the hypergeometric equation

$$x(1-x)y''(x) + [c - (a+b+1)x]y'(x) - aby(x) = 0. \quad (3.1)$$

Theorem 3.1. *The hypergeometric equation (3.1) has a distributional*

solution $y(x) = \sum_{n=0}^m a_n \delta^{(n)}(x)$ of order m if and only if

(i) $c = m + 2$ and

(ii) a or b is an integer $j = 1, \dots, m + 1$

This solution written in terms of the variables x and y is given by the formula

$$y(x) = \sum_{n=a-1}^m \frac{\binom{m+1-a}{m-n}}{\Gamma(n+2-b)} \delta^{(n)}(x)$$

where a is either single integer or greater of the two integers satisfying (ii).

Proof. Substituting $y(x) = \sum_{n=0}^m a_n \delta^{(n)}(x)$ in equation (3.1) written in terms of x and y leads to the relations

$$(m+2-c)a_m = 0$$
$$(n+1-a)(n+1-b)a_n = (m+1-n)a_{n-1}, \quad n = 0, \dots, m; \quad a_{-1} = 0 \quad (3.2)$$

since $a_m \neq 0$, hypothesis (i) is a necessary condition for the existence of a_n m - order solution $y(x) = \sum_{n=0}^m a_n \delta^{(n)}(x)$ to (3.1). The equation for a_0 is

$$(1-a)(1-b)a_0 = 0,$$

and if $a \neq 0, b \neq 0$, then $a_0 = 0$. Furthermore, if (ii) is not satisfied (3.2) implies that $a_n = 0$, for all $k = 0, \dots, m$. On the other hand, if a is an integer such that $1 \leq a \leq m + 1$ and b is not, then $a_m = 0$, $n = 0, \dots, a - 2$ and

$$a_n = \frac{(m+1-n)a_{n-1}}{(n+1-a)(n+1-b)} \quad \text{for } n = 0, \dots, m; a_{-1} = 0$$

Thus the recurrence formula

$$a_n = \frac{\binom{m+1-a}{m-n}}{\Gamma(n+2-b)} \delta^{(n)}(x)$$

Hence

$$y(x) = \sum_{n=a-1}^m \frac{\binom{m+1-a}{m-n}}{\Gamma(n+2-b)} \delta^{(n)}(x)$$

is a distributional solution for (3.1) with support $x = 0$. This result remains valid also when both a and b are integers such that $1 \leq a, b \leq m+1, a \geq b$. \square

Theorem 3.2. *The hypergeometric equation (3.1) has a distributional*

solution $y(x) = \sum_{n=0}^m a_n \delta^{(n)}(x)$ of order m with support $x = 1$ if and only if

(i) $a + b - c = m + 1$ and

(ii) a or b is an integer $j = 1, \dots, m + 1$

This solution written in terms of the variables x and y is given by the formula

$$y(x) = \sum_{n=a-1}^m (-1)^n \frac{\binom{m+1-a}{m-n}}{\Gamma(n+2-b)} \delta^{(n)}(x-1)$$

where a is either single integer or greater of the two integers satisfying (ii).

Proof. The proof similar procedure to the above theorem 3.1 but in this case $\delta^{(n)}(1-x) = (-1)^n \delta^{(n)}(x-1)$. \square

Theorem 3.3. *The classical solution of the hypergeometric differential equation (3.1) is given by*

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 \frac{(a)_n (b)_n x^n}{(c)_n n!}$$

Proof. It is clear that $x = 0$ and $x = 1$ are the only singular points of this differential equation. Also $x = 0$, is a regular singular point of (3.1). Note that we have

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitution into the differential equation (3.1) leads to

$$\sum_{n=0}^{\infty} \{ \{n(n+1) + c(n+1)\} a_{n+1} - \{ [n(n+1) + (a+b+1)n + ab] \} a_n \} x^n = 0$$

This leads to the recurrence relation

$$(n+1)(n+c)a_{n+1} = (n+a)(n+b)a_n, \quad n = 0, 1, 2, \dots$$

for the coefficient $\{a_n\}_{n=0}^{\infty}$. The solution can be written as

$$a_n = \frac{(a)_n (b)_n x^n a_0}{(c)_n n!}, \quad n = 0, 1, 2, \dots$$

$$\text{Hence, } y = \sum_{n=0}^{\infty} a_n x^n = a_0 \frac{(a)_n (b)_n x^n}{(c)_n n!} \quad \square$$

Example 3.1. Let $a=b=c=1$, so that (3.1) reduces to

$$x(1-x)y''(x) + [1-3x]y'(x) - y(x) = 0. \quad (3.3)$$

then it is easily verified that

$$y(x) = \sum_{n=0}^{\infty} a_n \delta^{(n)}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{\delta^{(n)}(x)}{x} = \delta(x-1) \quad (3.4)$$

satisfies (3.3). It is interesting to observe that

$${}_2F_1(x) = \sum_{n=0}^{\infty} \frac{(1)_n (1)_n x^n}{(1)_n n!} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad (3.5)$$

is the classical solution of (3.3). I observe the intriguing similarities. The function $\frac{1}{1-x}$ has a pole of order 1 and the distributional solution $\delta(x-1)$ also is a simple pole.

Example 3.2. Let $a=b=2, c=1$. Then (3.1) reduces to

$$x(1-x)y''(x) + [1-5x]y'(x) - 4y(x) = 0. \quad (3.6)$$

Its distributional solution is

$$y(x) = \sum_{n=0}^{\infty} a_n \delta^{(n)}(x) = \delta'(x) + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} n \delta^{(n)}(x)}{(n-1)!} = \delta'(x-1) - \delta''(x-1) \quad (3.7)$$

The classical solution of(3.6) is

$${}_2F_1(x) = \sum_{n=0}^{\infty} \frac{(2)_n (2)_n x^n}{(1)_n n!} = \sum_{n=0}^{\infty} (n+1)^2 x^n = \frac{1+x}{(1-x)^3} \quad (3.8)$$

Again when we compare(3.7) and (3.8) we find that both these solutions have a pole of order 3.

Chapter 4

Summary

In this work ,we have discussed the classical solution and distributional solution of the confluent hypergeometric differential equation and hypergeometric differential equation.Further more distributional solution of the confluent hypergeometric differential equation can be expressed using fourier transform as the sum of classical solution and distributional solution the case where $m, p \in N$ and $m \leq P$.The classical solution of confluent hypergeometric differential equation form ${}_1F_1$ which is a new function called hypergeometric series.

Chapter 5

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