Addis Ababa University
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Sensitivity Analysis in Linear and Nonlinear Programming Models

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Contents

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The undersigned hereby certify that they have read and recommend to the school of graduate studies for acceptance of a project entitled **Sensitivity Analysis in Linear and Nonlinear programming Models** by Aragaw Sisay in partial fulfillment of the requirements for the degree of master of Science.

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Introduction

Linear and nonlinear optimization problems are used in many areas of Economics and Finance, but they also arise in many other disciplines, with Chemistry, Physics, and Biology being just a few of them. Sensitivity analysis consists of determining how and how much specific changes in the parameters of an optimization problem influence the optimal objective function value and the point or points where the optimum is attained. Linear and nonlinear programming provide an excellent opportunity to introduce the idea of "what-if" analysis, using post-optimality analysis developed for the linear and nonlinear programming models. The purpose of this project is to study the sensitivity of an optimal value without changing the optimal solutions. Finding the optimal solution to a linear and nonlinear programming model is important. Having obtained the optimal solution for a programming problem, it would be desirable to know what happens when data values are changed. This project tries to show a perturbation approach for performing sensitivity analysis of mathematical programming problems [7].

The first chapter talks about the basic concepts of Optimization problems. The second and third chapters deals with the sensitivity analysis in a linear and nonlinear optimization problems. It has a brief discussion of the effect of changes of the cost coefficients and the right-hand sides of the constraints in a linear programming problem. A similar analysis can be done for nonlinear problems. However, a calculation of the partial derivatives with respect to the parameters can be done without forcing the set of active constraints to remain active. All the variables, parameters, KarushKuhnTucker multipliers, and objective function values vary provided that optimality is maintained and the general structure of a feasible perturbation is obtained[6].

Chapter 1

Preliminaries

1.1 Linear Programming

Linear Programming introduced by George B. Dantzig in 1947, and is a method of optimizing an objective function by solving a system of linear equations where the solution is subject to a set of constraints. Any linear programming problem (LPP) consists of an objective function and a set of constraints. A solution set for the decision variables, where all of the constraints are satisfied, is called a feasible solution. Most solution algorithms start by finding a feasible solution, then move from one feasible solution to another until the objective function has been optimized maximized or minimized [7].

Definition 1.1.1. *(General form of an LPP) An LPP is an optimization problem of the general form*

$$Minimize\ cx = \sum_{j}^{n} c_j x_j$$

$$Subject\ to \sum_{j}^{n} a_{ij} x_j \geq b_i \quad for\ i = 1, 2, ..., m \tag{1.1}$$
$$x_j \geq 0 \quad for\ j = 1, 2, ..., n$$

The problem has n variables and m constraints.

Note that a problem where we would like to minimize the cost function instead of maximize it may be rewritten in standard form by negating the cost coefficients $c_j(c^T)$. Any vector x satisfying the constraints of the linear programming problem is called a feasible solution of the problem. Every linear programming problem falls into one of three categories:

- Infeasible. A linear programming problem is infeasible if a feasible solution to the problem does not exist; that is, there is no vector x for which all the constraints of the problem are satisfied.

- Unbounded. A linear programming problem is unbounded if the constraints do not sufficiently restrain the cost function so that for any given feasible solution, another feasible solution can be found that makes a further improvement to the cost function.

- Has an optimal solution. Linear programming problems that are not infeasible or unbounded have an optimal solution; that is, the cost function has a unique minimum. (or maximum) cost function value. This does not mean that the values of the variables that yield that optimal solution are unique. However, the basic algorithm most often used to solve linear programming problems is called the simplex method.

The primal and dual linear programming problems

Linear programs are usually stated and analyzed in the standard forms using vector terminology:

$$\begin{aligned} &min\ c^T x \\ &Ax = b \\ &x \geq 0 \end{aligned} \tag{1.2}$$

where c is vector in R^n, b is a vector in R^m, A is an mxn real matrix with rank(A)=m, and x in R^n is the unknown vector.We call this problem as the primal linear programming problem.

Every linear programming problem where we seek to maximize the objective function gives rise to a related problem, called the dual problem, where we seek to minimize the objective function. The two problems interact in an interesting way: every feasible solution to one problem gives rise to a bound on the optimal solution in the other problem. If one problem has an optimal solution, so does the other problem and the two objective function values are the same. The equations below show a problem in standard form with n variables and m constraints on the left, and its corresponding dual problem on the right[8].

$$(P) \qquad min\ c^T x \tag{1.3}$$
$$S.t\ Ax \geq b$$
$$x \geq 0,$$
$$(D) \qquad max\ b^T y \tag{1.4}$$
$$s.t\ A^T y \leq c$$
$$y \geq 0$$

If the original or primal problem has the optimal solution x^*, its dual problem has an optimal solution y^* and $c^T x^* = b^T y^*$. If the primal problem is infeasible or unbounded, then the dual problem is infeasible or unbounded.

The Simplex Method

The simplex method has two basic steps, often called phases. The first phase is to find a feasible solution to the problem. For small problems, or larger problems of certain forms, this is not at all difficult. Often, a trivial solution such as x = 0 is a feasible solution, as in the production planning problem described earlier. We will omit the details of solving the first phase to find a feasible solution for now. After a feasible solution to the problem is found, the simplex method works by iteratively improving the value of the cost function. This is accomplished by finding a variable in the problem that can be increased, at the expense of decreasing another variable, in such a way as to effect an overall improvement in the cost function [8].

Let A be an mxn real matrix ($m \leq n$) and the matrix A has m linearly independent columns. Let us define by I_B the set of indices of such m columns, and by A_B the corresponding mxm submatrix of A. In this case A_B is nonsingular, therefore invertible. The columns of A which indices are not in the I_B form an $m \times (n - m)$ submatrix A_N of A, the corresponding index set is denoted by I_N.

If a nonhomogeneous linear equation is given as $Ax = b$ then we may split the unknowns into two different vector according to the index sets I_B and I_N. The vector x_B contains those variables which indices are in I_B, so, the x_B is the corresponding basic vector, while x_N the nonbasic vector. Thus,

$x = (x_B, x_N)$. In the similarly way, $c = (c_B, c_N)$,in which c_B is the row vector of the basic cost coefficients.The nonhomogeneous linear equation can be expressed as

$$A_B X_B + A_N x_N = b \tag{1.5}$$

Using the invertibility of A_B we can solve the equation as

$$X_B = A_B^{-1} b - A_B^{-1} A_N x_N \tag{1.6}$$

If $x_N = 0$ and $x_B = A_B^{-1} b$ then the solution $x = (A_B^{-1} b, 0)$ is called a primal basic solution corresponding to the given basis A_B. If $A_B^{-1} \geq 0$ then the basic solution $x = (x_B, x_N)$ is called as basic feasible solution and the A_B is called as primal feasible basis.

1.1.1 Duality Theory

For every linear programming problem there is a companion problem, called the dual linear program, in which the roles of variables and constraints are reversed. That is, for every variable in the original or primal linear program there is a constraint in the dual problem, and for every constraint in the primal there is a variable in the dual. In an application, the variables in the primal problem might represent products, and the objective coefficients might represent the profits associated with manufacturing those products. Hence the objective in the primal indicates directly how an increase in production affects profit. The constraints in the primal problem might represent the availability of raw materials. An increase in the availability of raw materials might allow an increase in production, and hence an increase in profit, but this relationship is not as easy to deduce from the primal problem. One of the effects of duality theory is to make explicit the effect of changes in the constraints on the value of the objective. It is because of this interpretation that the variables in the dual problem are sometimes called shadow prices, since they measure the implicit costs associated with the constraints[3].
Duality can also be used to develop efficient linear programming methods. For example, at the current time, the most successful interior-point software relies on a combination of primal and dual information.If the primal problem has n variables and m constraints, then the dual problem will have m variables (one dual variable for each primal constraint) and n constraints (one dual constraint for each primal variable). The coefficients in the objective of the primal are the coefficients on the right-hand side of the dual, and vice versa. The constraint matrix in the dual is the transpose of the matrix in the primal. There are two major results relating the primal and dual problems. The first,

called weak duality, is easier to prove. It states that primal objective values provide bounds for dual objective values, and vice versa. This weak duality property can be extended to nonlinear optimization problems and other more general settings. The second, called strong duality, states that the optimal values of the primal and dual problems are equal, provided that they exist. For nonlinear problems there may not be a strong duality result[7].

1.1.2 Relationships between Primal and Dual

Primal

$$Maximize\ z = \sum_{j=1}^{n} c_j x_j,$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad for\ i = 1, 2, ..., m,$$

$$x_j \geq 0 \quad for\ j = 1, 2, ..., n,$$

Dual

$$Minimize\ v = \sum_{j=1}^{m} b_i y_i,$$

subject to

$$\sum_{j=1}^{n} a_{ij} y_i \geq c_j \quad for\ j = 1, 2, ..., n,$$

$$y_i \geq 0 \quad for\ i = 1, 2, ..., m,$$

The first property is referred to as weak duality and provides a bound on the optimal value of the objective function of either the primal or the dual.

Weak Duality Theorem

If $\bar{x}$, solution to the primal problem and $\bar{y}$,is a feasible solution to the dual problem, then Then

$$\sum_{j=1}^{n} c_j \bar{x}_j \leq \sum_{i=1}^{m} b_i \bar{y}_i$$

Proof: The weak duality property follows immediately from the respective feasibility of the two solutions. Primal feasibility implies:

$$\sum_{j=1}^{n} a_{ij}\bar{x}_j \leq b_i \quad for\ i = 1, 2, ..., m,$$

and

$$\bar{x}_j \geq 0 \quad for\ j = 1, 2, ..., n$$

while dual feasibility implies

$$\sum_{j=1}^{m} a_{ij}\bar{y}_i \geq c_i \quad for\ j = 1, 2, ..., n,$$

and

$$\bar{y}_i \geq 0 \quad for\ i = 1, 2, ..., m$$

Hence, multiplying the ith primal constraint by $\bar{y}_i$ and adding yields:

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\bar{x}_j\bar{y}_i \leq \sum_{i=1}^{m} b_i\bar{y}_i,$$

while multiplying the jth dual constraint by $\bar{x}_j$ and adding yields:

$$\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}\bar{y}_i\bar{x}_j \geq \sum_{j=1}^{n} c_j\bar{x}_j,$$

Since the left hand sides of these two inequalities are equal, together they imply the desired result that

$$\sum_{j=1}^{n} c_j\bar{x}_j \leq \sum_{i=1}^{m} b_i\bar{y}_i$$

Strong Duality Theorem

Consider a pair of primal and dual linear programming problems. If one of the problems has an optimal solution then so does the other, and the optimal objective values are equal. that is,$\hat{z} = \hat{v}$ where

$$\hat{z} = Max \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i,$$

$$x_j \geq 0;$$

and

$$\hat{v} = Min \sum_{i=1}^{m} b_i y_i$$

subject to

$$\sum_{i=1}^{m} a_{ij}y_i \geq c_j,$$

$$y_i \geq 0;$$

Proof: Let us see how to establish this property. We can convert the primal problem to the equivalent equality form by adding slack variables as follows:

$$Maximize\ z = \sum_{j=1}^{n} c_j x_j,$$

subject to

$$\sum_{j=1}^{n} a_{ij}x_j + x_{n+i} = b_i, \quad for\ i = 1, 2, ..., m,$$

$$x_j \geq 0 \quad for\ j = j = 1, 2, ..., n+m.$$

Suppose that we have applied the simplex method to the linear program and $\hat{x_j}$, $j = 1, 2, ..., n$, is the resulting optimal solution. Let $\hat{y_i}$, $i = 1, 2, ..., m$, be the shadow prices associated with the optimal solution. The shadow prices associated with the original constraints are the multiples of those constraints which, when subtracted from the original form of the objective function, yield the form of the objective function in the final tableau. Thus the following condition holds:

$$-z + c_j - \sum_{i=1}^{m} b_i \hat{y_i}, \tag{1.7}$$

where, due to the optimality criterion of the simplex method, the reduced costs satisfy:

$$\bar{c_j} = c_j - \sum_{i=1}^{m} a_{ij}\hat{y_i} \leq 0, \quad for\ j = 1, 2, ..., n \tag{1.8}$$

and

$$\bar{c_j} = 0 - \hat{y_i} \leq 0, \quad for \ j = n+1, n+2, ..., n+m \tag{1.9}$$

Conditions (1.8) and (1.9) imply that $\hat{y_i}$, for $i = 1, 2, ..., m$, constitutes a feasible solution to the dual problem. When x_j is replaced by the optimal value $\hat{x_j}$ in expression (1.7), the term

$$\sum_{j=1}^{n} \bar{c_j}\hat{x_j}$$

is equal to zero, since $\bar{c_j} = 0$ when $\hat{x_j}$ is basic, and $\hat{x_j} = 0$ when $\hat{x_j}$ is nonbasic. Therefore, the maximum value of z, say $\hat{z}$, is given by:

$$-\hat{z} = -\sum_{i=1}^{m} b_i\hat{y_i}.$$

Moreover, since $\hat{x}_j$ is an optimal solution to the primal problem,

$$\sum_{j=1}^{n} c_j\hat{x_j} = \sum_{i=1}^{m} b_i\hat{y_i}.$$

1.2 Nonlinear Programming

Definition 1.2.1. *Let* $f : R^n \longrightarrow R, \quad g_i : R^n \longrightarrow R \quad for\ i \in I = (1, 2, ..., k), \quad h_j : R^n \longrightarrow R, \quad for\ j \in J = (1, 2, ..., l)$. *The standard form of an NLPP is:*

$$min\ f(x) \tag{1.10}$$

$$s.t \quad g_i(x) \le 0, \quad for\ each\ i \in I$$

$$h_j(x) = 0, \quad for\ each\ j \in J$$

or,

$$min\ f(x)$$

$$s.t\ x \in S$$

.

where, $S := (x \in R^n / g_i(x) \le 0,\ h_j(x) = 0)$. Such problem is called unconstrained if $S = R^n$. A first important question is related to the existence of optimal solutions for this problem.

Optimality Conditions for constrained problems

Optimality condition is a condition that has to be satisfied for a feasible point x^* to be an optimal solution.

1.2.1 Lagrange Methods for Constrained nonlinear programming Problems

Lagrange multipliers have long been used in optimality conditions for problems with constraints, but recently, their role has come to be understood from many different angles. The theory of Lagrange multipliers has been one of the major research areas in nonlinear optimization and there has been a variety of different approaches. Lagrange multipliers were originally introduced for problems with equality constraints. Converting constrained problem to an unconstrained problem with help of certain unspecified parameters known as Lagrange Multipliers. Inequality-constrained problems were addressed considerably later. Lagrangian multipliers require equalities. So a conversion of inequalities is necessary. The Lagrange multipliers have an important economic interpretation as shadow prices of the constraints, and their optimal values are very useful in sensitivity analysis [1]. Let

$$f : R^n \longrightarrow R, \text{be differentiable, and}$$

$$S \subseteq R^n,$$

consider a constrained nonlinear minimization problem:

$$(P) \quad min\ f(x)$$

$$s.t\ x \in S,$$

To convert (P) into unconstrained problem,we try to find a functional f_0 on R^n with the following property:

$$f_0 : R^n \longrightarrow R, \text{be differentiable, and}$$

$$f_0(x) = 0,$$

for all $x \in S$ Then we consider an auxiliary functional L such that

$$L : R^n \longrightarrow R, \quad given\ by$$

$$L(x) = f(x) + f_0(x)$$

Such L is called Lagrange functional (Lagrangian) for (P).Then, we consider the auxiliary (Lagrange) problem is

$$(P_L) \quad min\ L(x) \tag{1.11}$$

$$s.t\ x \in R^n$$

The relation between (P) and (PL) is given in the following Theorem.

Theorem 1.2.1. *(Lagrange Lemma)*
a) If x^* is a (global) minimizer of L and $x^* \in S$, then x^* is optimal solution of (P). b) If x^* is a local minimizer of L and $x^* \in S$, then x^* is local optimal solution of (P) .

Lagrange Method for Equality Constraints

Let

$$f : R^n \longrightarrow R, \text{be differentiable,and}$$

$$h_j : R^n \longrightarrow R, \quad for\ j \in J = (1, 2, ..., k)$$

we consider the problem

$$min\ f(x)$$

$$s.t\ x \in S,$$

If

$$f_0(x) = \sum_{j=1}^{k} \lambda_j h_j(x)$$

where $\lambda_j \in R$, then $f_0(x) = 0 \quad \forall x \in S$. The Lagrangian function for the equality constraint is given by

$$L : R^n \times R^n \mapsto R,$$

$$L(x, \lambda) = f(x) + \sum_{j=1}^{k} \lambda_j h_j(x)$$

$\lambda = (\lambda_1, \mu_2, ..., \lambda_k)^T$ are Lagrange multipliers
The following necessary condition is used for x^* to be an optimal solution. It is called Ksrush-Kuhn-Tucker (KKT) conditions for equality constraint.

- $\nabla f(x^*) + \sum_{j=1}^{k} \lambda_j^* \nabla h_j(x^*) = 0$ (Dual Feasibility)
- $h_j(x*) = 0,\ j = 1, 2, ..., k.$ (Primal feasibility).

Lagrange Method for Inequality Constraints

Let

$$f : R^n \longrightarrow R, \text{be differentiable, and}$$
$$g_j : R^n \longrightarrow R, \quad for\ i \in i = (1, 2, ..., m)$$

we consider the problem

$$(P) \quad min\ f(x) \tag{1.12}$$

$$s.t\ x \in S := \{x \in U/g(x) \leq 0\}$$

The KKT condition for in inequality constraint problem is:

- $\nabla f(x^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(x^*)$ (Dual Feasibility)
- $\mu_i^* \geq 0.$ (Dual Feasibility)

- $g_i(x^*) \leq 0$ (Primal feasibility)
- $\mu_i^* g_i(x^*) = 0$ (Complimentary slackness)

Then, from the Lagrange Lemma, we have the following result:If x^* is an optimal solution of (L) for some $\mu^* \in R^k$ and $x^* \in S$, then x^* is an optimal solution of (P). Hence x^* is an optimal solution of (L_μ^*),for some $\mu^* \in R^k$, implies

$$\nabla L_x(x^*, \mu^*) = 0 \tag{1.13}$$

for some $\mu^* \in R^k$, and $x^* \in$ implies, $g_j(x^*) = 0$,for all $j = 1, 2, ..., k$. Thus, we have the following necessary condition for x^* to be an optimal solution of (P).

Lagrange Method for Mixed Constraints

$$g(x) = (g_1(x), g_2(x), ..., g_m(x))^T.$$

we consider the problem

$$(P) \quad min\ f(x) \tag{1.14}$$

$$s.t\ x \in S := \{x \in U/g(x) \leq 0, h(X) = 0\}$$

Now, the Lagrange function for (P)is defined as:

$$L(x, \lambda, \mu) = f(x) + < \lambda, g(x) > + < \mu, h(x) > \tag{1.15}$$

Hence the KKT condition for mixed constraints are:

- $\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{j=1}^{k} \mu_j^* \nabla h_j(x^*) = 0$ (Dual Feasibility)
- $\lambda_j^* \geq 0.$ (Dual Feasibility)
- $h_j(x^*) = 0$ (Primal feasibility)
- $g_i(x^*) \leq 0$ (Primal feasibility)
- $\lambda_i g_i(x^*) = 0$ (Complimentary slackness)

Sufficiency of the KKT conditions

This section shows that necessary conditions of optimality become sufficient as well, under the main assumption of convexity of all functions involved. Since optimality conditions are formulated in terms of first derivatives, the appropriate notion of convexity is the one in which first derivatives appear: A function $f : KRn \mapsto R$ is convex if K is a convex set of vectors and $f(y) \geq f(x) + \nabla f(x)(yx)$, $y, x \in K, t \in [0, 1]$.

Theorem 1.2.2. *Assume that f and g are convex differentiable functions.If the pair (x, μ) satisfy the KKT conditions above, x is an optimal solution of the problem. If in addition, f is strictly convex, x is the only solution of the problem.*

Chapter 2

Sensitivity Analysis in Linear Programming Model

2.1 The Fundamental Principle of the Sensitivity Analysis

This section is aimed at showing the sensitivity of an optimal value without changing the optimal solutions with the issue of uncertainty in the data elements a_{ij} and b_i.
Assume, a linear programming problem is given in standard form (P). From(1.6), we know, if a basic solution $x = (A_B^{-1}b, 0)$ satisfied the following two conditions, then it is a optimal solution for the problem(P).

$$A_B^{-1}b \geq 0 \quad (feasibility) \tag{2.1}$$

$$S_N = c_N - A_N^T A_B^{-T} c_B \leq 0 \quad (Optimality) \tag{2.2}$$

If the data in the pivot tableau satisfied the previous conditions, then the basic solution is optimal, and the correspond basis A_B is an optimal basis. Sometimes we have to modify the data in the problem, so it is useful to find out how the optimal solutions and the optimal basis vary with the changes in data. For example, the change in b can change, whether the $A_B^{-1}b \geq 0$ is satisfied or not. The change in c has an effect, whether the optimality condition $s_N \leq 0$ is satisfied still, and the change in data in A can influence on both of the two conditions.
In order to analysis how these modified data affect on the optimal solutions, we do the job of sensitivity analysis. The analysis bases on the obtained solution and basis, this method is also called post optimality analysis [7].
In practical problem, the following data or condition modify frequently

- the coefficient of the objective function, c_j
- the right-hand-side constant, b_i

2.2 The sensitivity analysis of coefficients of the objective function

From the pivot tableau using simplex method, we can observe the change of coefficients in the objective function may vary the s_j ,it may effect, whether the optimality condition is satisfied. There are two possibilities in this area. First, when the corresponding variable is in basis, and second, when the corresponding variable is nonbasic.
Suppose x_r is a variable that is not in the optimal basis A_B. Assuming that all the other cost coefficients except c_r remain fixed at their specified values, determine the interval of values of c_r, within which A_B remains an optimal basis. If the cost coefficient of nonbasic variable x_r is changed from c_r into $c_r + \nabla c_r$, then the basis A_B remains an optimal basis as long as

$$c_r^{'} + \nabla c_r - c_B^T A_B^{-T} \bar{a}_r \leq 0 \tag{2.3}$$

consequently,

$$\nabla c_r \geq -\bar{c}_r \tag{2.4}$$

The reduced cost coefficients of all the other variables are independent of the value of c_r, hence they remain nonnegative. If the new value ofc_r is not in the closed interval $[\bar{c}_r, +\infty]$, x_r is the only nonbasic variable that has a negative reduced cost coefficient in the modified problem. We shall bring x_r into the basis and continue the algorithm until a terminal basis for the modified problem is obtained.

2.3 The sensitivity analysis of constant on the right hand side

In the linear programming problem (P) suppose we want to determine the interval of values of one of the right-hand-side constants, assume b_r, for which the basisA_B remains optimal.
Due to $x_B = A_B^{-1}b$ and $z = c_B A_B^{-1}b$, the changing of bi influences the primal feasibility of the original optimal solution and the optimal value. Hence, A_B

remains dual feasible.

Suppose the right-hand-side constant b_r changes into $b_r^{'} = b_r + \triangle b_r$ and assume the other data in the problem stay fixed, then the solution changes according into

$$x_B^{'} = A_B^{-1}(b + \triangle b) \tag{2.5}$$

in which,

$$b = (b_1, b_2, ..b_r.., b_m)^T \ and \ \ \triangle b = (0, ..., \triangle b_r, ..., 0)^T. \tag{2.6}$$

In this case

$$x_B^{'} = A_B^{-1}(b + \triangle b) = A_B^{-1}b + A_B^{-1}\triangle b = A_B^{-1}b + A_B^{-1}\begin{pmatrix} 0 \\ . \\ . \\ . \\ \triangle b_r \\ . \\ . \\ . \\ 0 \end{pmatrix} = \tag{2.7}$$

$$\begin{pmatrix} \bar{b_1} + a_{1r}^{-1}\Delta b_r \\ . \\ . \\ . \\ \bar{b_i} + a_{ir}^{-1}1\Delta b_r \\ . \\ . \\ . \\ \bar{b_m} + a_{mr}^{-1}\Delta b_r \end{pmatrix} \tag{2.8}$$

in which $(\bar{a_{1r}}^{-1}, \bar{a_{2r}}^{-1}, ..., \bar{a_{mr}}^{-1})$ is $r^t h$ row in inverse matrix A_B^{-1}. If the optimal basis A_B remains optimal, then the feasibility condition $X_B^{'} \geq 0$ must to be satisfied.
in other words,

$$\bar{b_i}^{-1} + a_{ir}^{-1}\Delta b_r \geq 0 \tag{2.9}$$

From this we can conclude, if $\bar{a_{ir}}^{-1} \geq 0$, then $\triangle b_r \geq -\bar{b_i}/a_{ir}^{-1}$ and if $a_{ir}^{-1} <$ 0,then $\triangle b_r \leq -\bar{b_i}/a_{ir}^{-1}$. Therefore,

$$max \ \left\{-\bar{b_i}/a_{ir}^{-1} | a_{ir}^{-1} > 0\right\} \leq min \left\{-\bar{b_i}/a_{ir}^{-1} | a_{ir}^{-1} < 0\right\} \tag{2.10}$$

After b changed into $b + \triangle b$, if the optimal basis is not changed, then the optimal value is:

$$z^{'} = c_B^T A_B^{-1}(b + \triangle b) = z^* + c_B^T A_B^{-1} \Delta b \tag{2.11}$$

If the changed data is not in the interval any more, then we shall continue the dual simplex algorithm, until a new optimal basis is obtained.

2.4 Example

Consider the linear program:

$$Maximize \quad z = 5x_1 + 4.5x_2 + 6x_3 \tag{2.12}$$

Subject to:

$$6x_1 + 5x_2 + 8x_3 \leq 60 \tag{2.13}$$

$$10x_1 + 20x_2 + 10x_3 \leq 150 \tag{2.14}$$

$$x_1 \qquad\qquad \leq 8 \tag{2.15}$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

If we add one slack variable in each of the less-than-or-equal-to constraints, the problem will be in the following canonical form for performing the simplex method:

$$5x_1 + 4.5x_2 + 6x_3 \qquad - z = 0 \tag{2.16}$$

$$6x_1 + 5x_2 + 8x_3 + x_4 \qquad = 60 \tag{2.17}$$

$$10x_1 + 20x_2 + 10x_3 \quad + x_5 \quad = 150 \tag{2.18}$$

$$x_1 + \qquad\qquad + x_6 \quad = 8 \tag{2.19}$$

This example is aimed at showing the sensitivity of an optimal value without changing the optimal solutions with the issue of uncertainty in the data elements and to revise the final set of equations in tableau form to identify a new solution and to test the new solution for feasibility and optimality.

The corresponding initial tableau is shown in the following Tableau.

Basic variable	Current values	x_1	x_2	x_3	x_4	x_5	x_6
(-z)	0	5	4.5	6	0	0	0
x_4	60	6	5	8	1	0	0
x_5	150	10	20	10	0	0	0
x_6	8	1	0	0	0	0	1

After applying the simplex method, we obtain the final tableau.

Basic variable	Current values	x_1	x_2	x_3	x_4	x_5	x_6
(-z)	-360/7	0	0	-4/7	-11/14	-1/35	0
x_2	30/7	0	1	-2/7	-1/7	3/35	0
x_6	11/7	0	0	-11/7	-2/7	1/14	1
x_1	45/7	1	0	11/7	2/7	-1/14	0

The basic variables associated with this final tableau are x_1, x_2 and x_6; therefore, we have the optimal solution, consisting of $x_1 = 45/7$, $x_2 = 30/7$,and $x_6 = 11/7$. which has an objective function value of $z = 360/7$.

2.4.1 Shadow price and Reduced costs

Definition 2.4.1. *The shadow price associated with a particular constraint is the change in the optimal value of the objective function per unit increase in the righthand-side value for that constraint, all other problem data remaining unchanged.*

The shadow price for a particular constraint is merely the negative of the coefficient of the appropriate slack (or artificial) variable in the objective function of the final tableau.The nonnegativity constraints also have a shadow price, which, in linear-programming terminology, is given the special name of reduced cost.

Definition 2.4.2. *The reduced cost associated with the nonnegativity constraint for each variable is the shadow price of that constraint (i.e., the corresponding change in the objective function per unit increase in the lower bound of the variable).*

2.4.2 Variations in the Objective Coefficients

Now let us consider the question of how much the objective-function coefficients can vary without changing the values of the decision variables in the optimal solution. We will make the changes one at a time, holding all other coefficients and righthand-side values constant.

We return to consideration of the objective coefficient of x_3, a nonbasic variable in this example. Suppose that we increase the objective function coefficient of x_3 in the original problem formulation by $\triangle c_3$, giving us:

$$5x_1 + 4.5x_2 + (6 + \triangle c_3)\, x_3 - z = 0 \tag{2.20}$$

$$(6 + \triangle c_3) \Rightarrow c_3^{new}$$

In applying the simplex method, multiples of the rows were subtracted from the objective function to yield the final system of equations. Therefore, the objective function in the final tableau will remain unchanged except for the addition of $\triangle c_3 x_3$. The modified final tableau is given in the following Tableau.

Basic variable	Current values	x_1	x_2	x_3	x_4	x_5	x_6
(-z)	-360/7	0	0	$-4/7 + \triangle c_3$	-11/14	-1/35	0
x_2	30/7	0	1	-2/7	-1/7	3/35	0
x_6	11/7	0	0	-11/7	-2/7	1/14	1
x_1	45/7	1	0	11/7	2/7	-1/14	0

Now x_3 will become a candidate to enter the optimal solution at a positive level, i.e., to enter the basis, only when its objective-function coefficient is positive. The optimal solution remains unchanged so long as:

$$-4/7 + \triangle c_3 \leq 0 \;\; or \;\; \triangle c_3 \leq 4/7 \tag{2.21}$$

Equivalently, we know that the range on the original objective-function coefficient of x_3, say c_3^{new} , must satisfy

$$-\infty < c_3^{new} \leq 46/7 \tag{2.22}$$

if the optimal solution is to remain unchanged.
Next, let us consider what happens when the objective-function coefficient of a basic variable is varied. let us add $\triangle c_1$ to the objective-function coefficient of x_1 in the original problem formulation to yield the following modified objective function:

$$(5 + \triangle c_1)\, x_1 + 4.5x_2 + 6x_3 - z = 0 \tag{2.23}$$

If we apply the same logic as in the case of the nonbasic variable, the result is:

Basic variable	Current values	x_1	x_2	x_3	x_4	x_5	x_6
(-z)	-360/7	$\triangle c_1$	0	-4/7	-11/14	-1/35	0
x_2	30/7	0	1	-2/7	-1/7	3/35	0
x_6	11/7	0	0	-11/7	-2/7	1/14	1
x_1	45/7	1	0	11/7	2/7	-1/14	0

However, the simplex method requires that the final system of equations be in canonical form with respect to the basic variables. Since the basis is to be unchanged, in order to make the coefficient of x_1 zero in the final tableau we must subtract $\triangle c_1$ times row 3 from row 4 in above Tableau.The result is presented in the following Tableau.

Basic variable	Current values	x1	x2	x3	x4	x5	x6
(-z)	-360/7-45/7△c1	0	0	-4/7-11/7△c1	-11/14-2/7△c1	-1/35+1/14△c1	0
x2	30/7	0	1	-2/7	-1/7	3/35	0
x6	11/7	0	0	-11/7	-2/7	1/14	1
x1	45/7	1	0	11/7	2/7	-1/14	0

By the simplex optimality criterion, all the objective-function coefficients in the final tableau must be non positive in order to have the current solution remain unchanged. Hence, we must have:

$$-4/7 - 11/7\triangle c_1 \leq 0 \qquad (that\ is, \triangle c_1 \geq -4/11)\,,$$

$$-11/14 - 2/7\triangle c_1 \leq 0 \qquad (that\ is, \triangle c_1 \geq -11/4)\,,$$

$$-1/35 + 1/14\triangle c_1 \leq 0 \qquad (that\ is, \triangle c_1 \leq 2/5)\,,$$

and, taking the most limiting inequalities, the bounds on $\triangle c_1$ are:

$$-4/11 \leq \triangle c_1 \leq 2/5,$$

If we let $c_1^{new} = 5 + \triangle c_1$ be the objective-function coefficient of x_1 in the initial tableau, then:

$$51/11 \leq c_1^{new} \leq 27/5,$$

It is easy to determine which variables will enter and leave the basis when the new cost coefficient reaches either of the extreme values of the range. When $c_1^{new} = 27/5$, the objective coefficient of x_5 in the final tableau becomes 0; thus x_5 enters the basis for any further increase of c_1^{new}. By the usual ratio test of the simplex method,

$$Min\left\{\bar{b_i}/\bar{a_{is}}|a_{is} \geq 0\right\} = Min\,(50, 22)\}$$

and the variable x6, which is basic in row 2, leaves the basis when x_5 is introduced. Similarly, when $c_1^{new} = 51/11$, the objective coefficient of x_3 in the final tableau becomes 0, and x3 is the entering variable. In this case, the ratio test shows that x_1 leaves the basis.

2.4.3 Variation in the right-hand side of a constraint

Now let us turn to the questions related to the righthand-side ranges. We already have noted that a right hand side range is the interval over which an individual righthand-side value can be varied, all the other problem data being held constant, such that variables that constitute the basis remain the same. Over these ranges, the values of the decision variables are clearly modified. Of what use are these righthand-side ranges? Any change in the righthand-side values that keep the current basis, and therefore the canonical form, unchanged has no effect upon the objective-function coefficients. Consequently, the righthand-side ranges are such that the shadow prices (which are the negative of the coefficients of the slack or artificial variables in the final tableau) and the reduced costs remain unchanged for variations of a single value within the stated range[7].

Since this constraint is not binding, the shadow price associated with it is zero and it is simple to determine the appropriate range. If we add an amount $\triangle b_3$ to the righthand side of this constraint (2.18), the constraint changes to:

$$x_1 + x_6 = 8 + \triangle b_3$$

In the original problem formulation, it should be clear that, since x_6, the slack in this constraint is a basic variable in the final system of equations,

x_6 is merely increased or decreased by $\triangle b_3$. In order to keep the current solution feasible, x6 must remain greater than or equal to zero. From the final tableau, we see that the current value of $x_6 = 11/7$; therefore x_6 remains in the basis if the following condition is satisfied:

$$x_6 = 11/7 + \triangle b_3 \geq 0.$$

This implies that:

$$\triangle b_3 \geq -11/7$$

or, equivalently:

$$b_3^{new} = 8 + \triangle b_3 \geq 45/7.$$

Now let us consider changing the righthand-side value of constraint (2.17) by adding $\triangle b_2$, this constraint changes to:

$$10x_1 + 20x_2 + 10x_3 + x_5 = 150 + \triangle b_2$$

In the original problem formulation, as was previously remarked, changing the righthand-side value is essentially equivalent to decreasing the value of the slack variable x_5 of the corresponding constraint by $\triangle b_2$; that is, substituting $x_5 \triangle b_2$ for x_5 in the original problem formulation. In this case, x_5, which is zero in the final solution, is changed to $x_5 = \triangle b_2$. We can analyze the implications of this increase in the righthand-side value by using the relationships among the variables represented by the final tableau. Since we are allowing only one value to change at a time, we will maintain the remaining nonbasic variables, x_3 and x_4, at zero level, and we let $x_5 = \triangle b_2$. Making these substitutions in the final tableau provides the following relationships:

$$\begin{aligned}
x_2 - 3/35\triangle b_2 &= 30/7 \\
x_6 - 1/14\triangle b_2 &= 11/7 \\
x_1 + 1/14\triangle b_2 &= 45/7
\end{aligned}$$

In order for the current basis to remain optimal, it need only remain feasible, since the reduced costs will be unchanged by any such variation in the righthand-side value. Thus,

$$\begin{aligned}
x_2 = 30/7 + 3/35\triangle b_2 \geq 0 &\quad (that\ is, \triangle b_2 \geq -50) \\
x_6 = 11/7 + 1/14\triangle b_2 \geq 0 &\quad (that\ is, \triangle b_2 \geq -22) \\
x_1 = 45/7 - 1/14\triangle b_2 \geq 0 &\quad (that\ is, \triangle b_2 \geq \leq 90)
\end{aligned}$$

which implies:

$$-22 \leq \triangle b_2 \leq 90$$

The General rule

We can change the coefficient b_k of the k^{th} righthand side in the initial tableau by $\triangle b_k$ with all the other data held fixed, simply by substituting

$$x_{n+k} - \triangle b_k$$

for x_{n+k} in the original tableau. To see how this change affects the updated righthand-side coefficients, we make the same substitution in the final tableau. Only the terms $\beta_{ik} x_{n+k}$ for $i = 1, 2, ..., m$ change in the final tableau.

To see how this change affects the updated righthand-side coefficients, we make the same substitution in the final tableau. Only the terms $\beta_{ik} x_{n+k}$ for $i = 1, 2, ..., m$ change in the final tableau. They become $\beta_{ik}(x_{n+k} - \Delta b_k) = \beta_{ik} x - n + k \beta_{ik} \Delta b_k$. Since $\beta_{ik} \Delta b_k$ is a constant, we move it to the righthand side to give modified righthand-side values:

$$\bar{b_i} + \beta_{ik} \Delta b_k \qquad \text{i=1,2,...,m} \tag{2.24}$$

As long as all of these values are nonnegative, the basis specified by the final tableau remains optimal, since the reduced costs have not been changed. Consequently, the current basis is optimal whenever $\bar{b_i} + \beta_{ik} \Delta b_k \geq 0$ for i = 1, 2, . . . ,m or, equivalently,

$$Max \left\{ \frac{-\bar{b_i}}{\beta_{ik}} | \beta_{ik} > 0 \right\} \leq \Delta b_k \leq Min \left\{ \frac{-\bar{b_i}}{\beta_{ik}} | \beta_{ik} < 0 \right\} \tag{2.25}$$

The lower bound disappears if all $\beta_{ik} < 0$, and the upper bound disappears if all $\beta_{ik} > 0$[8].

Chapter 3

Sensitivity Analysis in Non Linear programming

In this section, we analyze the sensitivity of the optimal solution of a nonlinear programming problem to changes in the data values.in the second chapter of this paper(Sensitivity Analysis in linear programming problem) discussed the effect of changes of (i) the cost coefficients, (ii) the right-hand sides of the constraints. A similar analysis has been done for nonlinear problems.
It is relevant to note that, to perform a local sensitivity analysis, the objective function and the nonlinear constraints in of the general problem below can be replaced by the corresponding quadratic approximations that share tangent hyper planes and Hessian matrices at the solution point[2].

3.1 Constrained nonlinear optimization problem

Consider the following Constrained nonlinear programming problem:

$$min \quad z = f(x, a), \tag{3.1}$$

$$s.t. \quad h(x, a) = 0, \tag{3.2}$$

$$g(x, a) \leq 0, \tag{3.3}$$

where $f : R^n \times R^p \longrightarrow R$, $h : R^n \times R^p \longrightarrow R^l$,$g : R^n \times R^p \longrightarrow R^m$, with $h(x, a) = (h_1(x, a), h_2(x, a), ..., h_l(x, a))^T$ and $g(x, a) = (g_1(x, a), g_2(x, a), ..., g_m(x, a))^T$, are functions over the feasible region

$$S(a) = \{x|h(x, a) = 0, g(x, a) \leq 0\} \tag{3.4}$$

and $f, h, g \in C^2$

Let x^* be a local solution problem and a regular point of the constraints. If J is the set of indices j for which $g_j(x^*, a) = 0)$,a local solution x^* is a regular point of the constraints $h(x, a) = 0$ and $g(x, a) \leq 0)$ if the gradient vectors $\nabla_x h_k(x^*, a)$, and $\nabla_x g_j(x^*, a)$ are linearly independent. Optimality condition is a condition that has to be satisfied for a feasible point x^* to be an optimal solution.

In order to find a local solution for the problem, the so-called first order necessary conditions have to be satisfied. First, the Lagrangian function is defined as:
Let $f : R^n \longrightarrow R, \quad g_i : R^n \longrightarrow R \quad for \ i \in I = (1, 2, ..., k), \quad h_j : R^n \longrightarrow R, \quad for \ j \in J = (1, 2, ..., l).$

we consider the problem

$$min \ f(x) \qquad (P) \tag{3.5}$$

$$s.t \ x \in S(a) := \{x \in U / g(x, a) \leq 0, h(X, a) = 0\} \tag{3.6}$$

Now, the Lagrange function for (P)is defined as:

$$L(x, \mu, \lambda) = f(x, a) + < \mu, g(x, a) > + < \lambda, h(x, a) > \tag{3.7}$$

where λ is the vector of Lagrange multipliers for the nonlinear equality constraints and μ is the vector of Lagrange multipliers for the nonlinear inequality constraints.
Let x^* be a local solution for the problem with the vectors of Lagrange multipliers λ^* and μ^* at the minimum point. Hence the necessary conditions are:

$$\nabla_x f(x^*, a) + \sum_{k=1}^{l} \lambda_k \nabla_x h_k(x^*, a) + \sum_{j=1}^{m} \mu_j^* \nabla_x g_j(x^*, a) = 0 \tag{3.8}$$

$$\mu^* \geq 0_m. \qquad (Dual Feasibility) \tag{3.9}$$

$$h_k(x^*, a) = 0, \quad \text{k=1,2,...,l} \ \ (Primal feasibility) \tag{3.10}$$

$$g_j(x^*, a) \leq 0 \quad \text{j=1,2,...,m} \quad (Primal feasibility) \tag{3.11}$$

$$\mu_j g_j(x^*, a) = 0 \quad \text{j=1,2,...,m} \quad (Complimentary \ \ slackness) \tag{3.12}$$

the complementarity conditions that imply which nonlinear inequality constraint is active. The minimum point is acquired by satisfying the conditions. As we know, the vectors λ^* and μ_* are called the KKT multipliers. Furthermore, the Hessian of the Lagrangian at x^*, μ^*, λ^*,

$$\nabla_2 f(x^*, a) + \sum_{k=1}^{l} \lambda_k \nabla_2 h_k(x^*, a) + \sum_{j=1}^{m} \mu_j^* \nabla_2 g_j(x^*, a) \tag{3.13}$$

is assumed to be positive definite on the subspace orthogonal to the subspace spanned by the gradients of the constraint functions. Then,for λ and μ $geq0$ near λ^* and μ^*,the dual function is defined as

$$\Phi(\lambda, \mu) = min_x[f(x, a) + \sum_{k=1}^{l} \lambda_k h_k(x, a) + \sum_{j=1}^{m} \mu_j g_j(x, a)], \tag{3.14}$$

where the minimum is taken locally near x^* and the dual problem is

$$max_{\lambda,\mu\geq 0} \;\; \Phi(\lambda, \mu) \tag{3.15}$$

whose solution is λ^*, μ^*. We aim at determining the sensitivity of the optimal solution $(x^*, \lambda^*, \mu^*, z^*)$ to change in the parameters;that is, we perturb or modify $x^*, \lambda^*, \mu^*, z^*$ in such a way that the KKT conditions still hold. Thus, to obtain the sensitivity equations, we differentiate (3.1) and (3.7) to (3.11) as

$$(\nabla_x f(x^*, a))^T d_x + (\nabla_a f(x^*, a))^T d_a - d_z = 0 \tag{3.16}$$

$$\begin{gathered}\left(\nabla_2 f(x^*, a) + \sum_{k=1}^{l} \lambda_k^* \nabla_2 h_k(x^*, a) + \sum_{j=1}^{m} \mu_j \nabla_2 g_j(x^*, a)\right) d_x \\ + \\ \left(\nabla_{xa} f(x^*, a) + \sum_{k=1}^{l} \lambda_k^* \nabla_{xa} h_k(x^*, a) + \sum_{j=1}^{m} \mu_j \nabla_{xa} g_j(x^*, a)\right) d_a \\ + \\ \nabla_x h(x^*, a) d\lambda + \nabla_x g(x^*, a) d_\mu = 0_n,\end{gathered} \tag{3.17}$$

$$(\nabla_x h(x^*, a))^T dx + (\nabla_a h(x^*, a))^T da = 0_l, \tag{3.18}$$

$$(\nabla_x g_j(x^*, a))^T dx + (\nabla_a g_j(x^*, a))^T da = 0, \quad if \quad \mu_j^* \neq 0, j \in J \tag{3.19}$$

$$(\nabla_x g_j(x^*,a))^T dx + (\nabla_a g_j(x^*,a))^T da \leq 0, \quad if \;\; \mu_j^* = 0, j \in J \tag{3.20}$$

$$-d\mu_j \leq 0, \qquad if \;\; \mu_j^* = 0, j \in J \tag{3.21}$$

$$d\mu_j \left[(\nabla_x g_j(x^*,a))^T dx + (\nabla_a g_j(x^*,a))^T da\right] = 0, \quad if \;\; \mu_j^* = 0, j \in J \tag{3.22}$$

where all the matrices are evaluated at the optimal solution and where redundant constraints have been removed. More precisely, the constraints (3.19)to(3.22) are simplifications of the constraints that result directly from differentiating (3.9),(3.11) and (3.12) , i.e., from

$$(\nabla_x g_j(x^*,a))^T dx + (\nabla_a g_j(x^*,a))^T da \leq 0, \quad j \in J \tag{3.23}$$

$$(\mu_j^* + d\mu_j)(g_j(x^*,a)) = \mu_j^* g_j(x^*,a) + d\mu_j(g_j(x^*,a) + dg_j(x^*,a)), \quad j \in J \tag{3.24}$$

Since all these inequality constraints are active, we have

$$g_j(x^*,a) = 0, \quad \forall_j \in J \tag{3.25}$$

then, (3.24) results in (3.14) for $\mu_j^* \neq 0$ and in (3.22) for $\mu_j^* = 0$,finally since (3.19) implies (3.23), for $\mu_j^* \neq 0$,(3.22) must be written only for $\mu_j^* = 0$, that is (3.20)

Note that:

- Constraint (3.19) forces the constraints $g_j(x^*,a) = 0$ whose multipliers are different from zero ($\mu_j^* \neq 0$) to remain active.

- The constraint (3.20) allows the optimal point to move inside the feasible region, the constraint (3.21) forces the Lagrange multipliers to be greater or equal to zero.

- The constraint (3.22) forces the new point to hold the complementary slackness condition for $\mu_j^* = 0$.

- The last constraint is a second-order constraint that implies that one of the constraints (3.20) or (3.21) has to be equality constraint.

3.2 Price Interpretation of Lagrange Multipliers

Lagrange multipliers can be viewed as the equilibrium prices of an optimization problem. This interpretation forms an important link between mathematics and theoretical economics.
To illustrate this interpretation, we consider an inequality-constrained problem,

$$min\ f(x) \tag{3.26}$$

$$g_j(x) \leq 0 \qquad for\ j = 1, ..., r$$

and assume that the functions f, g_j are smooth and convex over R^n, and that the optimal value of this problem is finite. The Lagrange multiplier condition for this problem is that, under appropriate assumptions, at a given global minimum x^*, there exist nonnegative multipliers $\mu_1^*, ..., \mu_r^*$ such that

$$\nabla f(x^*) + \sum_{j=1}^{r} \mu_j^* \nabla g_j(x^*) = 0, \tag{3.27}$$

where the μ_j^* satisfy the complementary slackness condition:

$$\mu_j^* g_j(x^*) = 0, \quad for\ all\ \ j = 1, ..., r \tag{3.28}$$

We next consider a perturbed version of problem (3.26) for some $u = (u_1, ..., u_r)$

$$min\ f(x) \tag{3.29}$$

$$g_j(x) \leq u_j, \qquad for\ j = 1, ..., r \tag{3.30}$$

We denote the optimal value of the perturbed problem by $p(u)$. Considering vector $u = (u_1, ..., u_r)$ as perturbations of the constraint functions, we call the function p as the perturbation function or the primal function.
We interpret the value $f(x)$ as the cost of choosing the decision vector x. Thus, in the original problem, our objective is to minimize the cost subject to certain constraints.

We also consider another scenario in which we are allowed to relax the constraints to our advantage by buying perturbations u.The price being j per unit of perturbation variable. Then, for any perturbation u, the minimum cost we can achieve in the perturbed problem (3.26),plus the perturbation cost, is given by

$$p(u) + \sum_{j=1}^{r} \mu_j u_j \tag{3.31}$$

and we have

$$inf_{u\in R^r}\left\{p(u)+\sum_{j=1}^{r}\mu_j u_j\right\}\leq f(x^*) \tag{3.32}$$

i.e., the minimum cost that can be achieved by a perturbation is at most as high as the optimal cost of the original unperturbed problem. A perturbation is worth buying if we have strict inequality in the preceding relation[5].

3.3 Optimal solution and Optimal value bounds

In order to find optimal solution and optimal value bounds,we describe a procedure originally proposed by Facco [2], for calculating piecewise linear continuous global upper and lower parametric bounds can be improved in a systematic manner until a desired accuracy, as measured by the maximal deviation from the optimal value over the interval of parameter values, is achieved. We first derive bounds on the optimal solution of an unperturbed problem using a uniform quadratic underestimation of the objective function. Then we combine, these result with the optimal value bounds to derive computable parametric solution for several classes of perturbed convex programs. In addition, a parametric feasible solution is readily available, as well as a systematic procedure for refining the solution bounds[5].

Let us consider a general parametric nonlinear programming problem $P(\varepsilon)$ The optimal value function of $P(\varepsilon)$ is defined by

$$f^*(\varepsilon)=min_x\left\{f(x,\varepsilon):x\in R(\varepsilon)\right\} \tag{3.33}$$

Where $R(\varepsilon)$ is the feasible set of the problem $P(\varepsilon)$ given by

$$R(\varepsilon)=\{x\in R(\varepsilon)\mid g(x,\varepsilon)\geq 0, h(x,\varepsilon)=0\} \tag{3.34}$$

Assume that $P(\varepsilon)$ is a jointly convex program, that is f convex in (x,ε),the components of g are concave in (x,ε), and the component of h are affine in (x,ε). Then the program's optimal value function, $f^*(\varepsilon)$, is convex. This will be the case even if we generalize the assumptions on g and h by requiring only that the feasible point-to set map R is convex.
Suppose now that we have evaluated $f^*(\varepsilon)$ and its slope at two distinct values ε^1 and ε^2 of the parameter. The global definitional properties of convex functions immediately provide global parametric continuous, piecewise-linear bounds via linear supports and linear interpolation on the graph of f^* over the line segment $(\varepsilon^1,\varepsilon^2)$. Practical implementation of bounds calculations

requires only the information provided by most standard nonlinear programming algorithms. In particular, the solution of problem $P(\varepsilon)$ as well as the associated optimal Lagrangian multipliers must be determined for two distinct parameter values.
The Lagrangian multipliers can be used to compute the directional derivatives or the sub gradients of $f^*(\varepsilon)$ and therefore also the lower bounds on f^*.

To illustrate how bounds can be calculated when the assumption of the sensitivity analysis from the previous section results hold,suppose that $x^*(\varepsilon^1)$ and $x^*(\varepsilon^2)$ are optimal solution of $P(\varepsilon^1)$ and $P(\varepsilon^2)$, respectively.
Suppose also that the gradients if f^* at ε^1 and ε^2,$\nabla_\varepsilon f^*(\varepsilon^1)$ and $\nabla_\varepsilon f^*(\varepsilon^2)$ exist.

Under these assumption, the following relationships hold for $\alpha \in [0, 1]$.

$$L_0(\alpha) \leq f^*(\varepsilon(\alpha)) \leq U(\alpha) \tag{3.35}$$

where

$$\varepsilon(\alpha) = (1 - \alpha)\varepsilon^1 + \alpha\varepsilon^2$$

$$U(\alpha) = (1 - \alpha)f^*(\varepsilon^1) + \alpha f^*(\varepsilon^2)$$

$$L_0(\alpha) = max\, \{L_1(\alpha), L_2(\alpha)\}$$

$$L_1(\alpha) = f^*(\varepsilon^1) + \nabla_\varepsilon f^*(\varepsilon^1)^T(\epsilon(\alpha) - \varepsilon^1)$$

$$L_2(\alpha) = f^*(\varepsilon^1) + \nabla_\varepsilon f^*(\varepsilon^2)^T(\epsilon(\alpha) - \varepsilon^2)$$

The functions $U(\alpha)$ and $L_0(\alpha)$ are computable parametric linear and pairwise- lineae upper and lower bounds respectively on the convex optimal value function f^* on the interval $[\varepsilon^1, \varepsilon^2]$.

A-by-product of this approach is the observation that since the feasible map R is assumed convex and $x^*(\varepsilon^1) \in R(\varepsilon^1)$ and $x^*(\varepsilon^2) \in R(\varepsilon^2)$, it follows that $\bar{x}(\varepsilon(\alpha)) = (1 - \alpha)x^*(\varepsilon^2) + \alpha x^*(\varepsilon^2) \in R(\varepsilon(\alpha))$ Thus, a parametric feasible solution solution $\bar{x}(\varepsilon(\alpha))$ of a problem $P(\varepsilon(\alpha))$ can be calculated whenever the feasible map R is convex, even if function f is not convex. As a consequence, we also obtain an upper bound on f^* given by $\bar{f}(\alpha) = f[\bar{x}(\varepsilon(\alpha)), \varepsilon(\alpha)]$.
Since the calculation of $\bar{x}(\varepsilon(\alpha))$ does not depend on f, this does not require f to be convex.

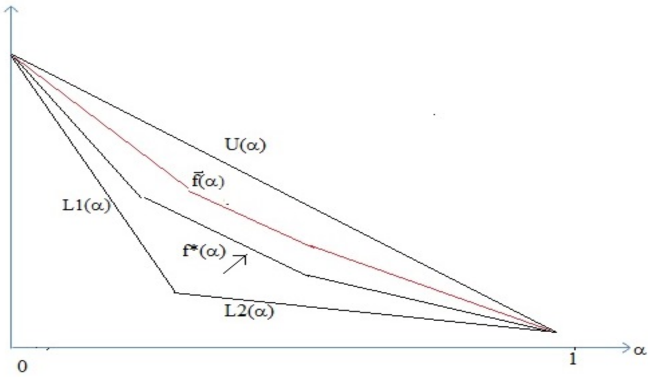

Figure: Parametric bounds on convex f^*

If f is convex in (x, ε),then $\bar{f}(\alpha)$ is also convex and provides a sharper parametric upper bound on f^*, since the following in equalities hold for $\alpha \in [0, 1]$:

$$f^*(\varepsilon(\alpha)) \leq f[\bar{x}(\varepsilon(\alpha)), \varepsilon(\alpha)] \leq U(\alpha) \tag{3.36}$$

The bounds given by in inequalities:(35) and (36) are illustrated by the above figure.[5]

Conclusion

Sensitivity analysis shows how the optimal solution and the value of its objective function change, given changes in various inputs to the problem. The task of sensitivity analysis is to find out the change limits of data, in other words the stable interval, so that optimal solution or optimal basis will remain within its range of optimality. The solution of the optimization problem is terminated when it reaches a minimum value subject to the given constraints. If the solution belongs to the boundary of some constraint, such constraints are called active and the impact of these active constraints on the solution could be found by acquiring the Lagrange multiplier associated with these constraints. Since the solution is optimal,the Lagrange multipliers associated with inequality constraints have nonnegative values [7]. To understand how sensitive is the optimal solution with respect to the small change in the right-hand side of the active constraints, the constraints are perturbed and the new so-obtained optimization problem is solved.

The sensitivity analysis provides information for both linear and nonlinear programming problems, including dual values (in both cases) and range information (for linear problems only). The dual values for (nonbasic) variables are called reduced Costs in the case of linear programming problems, and reduced gradients for nonlinear problems.The dual values for (binding) constraints are called Shadow Prices for linear programming problems, and Lagrange Multipliers for nonlinear problems. The active constraints are not assumed to remain active if the problem data are perturbed, nor the partial derivatives are assumed to exist. In other words, all the elements, variables, parameters, KKT multipliers, and objective function values may vary provided that optimality is maintained and the general structure of a feasible perturbation.

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