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Graduate Seminar Report

On

Finite Difference Method

(Submitted in Partial fulfillment of M.Sc. Degree)

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1. Introduction

Numerical Analysis is a branch of applied mathematics that devises methods, study their efficiency for solving problems by purely numerical computations.

Many problems in science and engineering for example, the motion of a swinging pendulum under certain simplifying assumptions $\frac{d^2\theta}{dt^2}+\frac{g}{L}\sin\theta=0$ where L is the length and θ is the angle the pendulum makes with the vertical can be reduced to the problem of solving differential equation satisfying certain given conditions. The analytical methods of solution can be applied to solve only a selected type (class) of differential equations. Those equations which govern physical systems which are in general non-linear in nature do not posses exact solutions and hence we must look for numerical methods to solve such differential equations.

There are various numerical methods for solving differential equations.

These are Taylor's series

Euler's method

Picard's method of successive approximation

Runge-kutta methods

Finite difference method, etc.

1.1 Errors

In practical applications, one would finally require results in numerical form. For example, a set of tabulated data is given and inferences have to be drawn from it, or a system of linear algebraic equations is given and one is required to solve them. The aim of numerical analysis is to provide efficient methods for obtaining numerical answers to such problems. When solving problems one usually starts with some initial data and then computes to get results. During such process errors will be introduced due to the finite word length of the computing machine, the numerical data used may be approximations to two or three or more significant figures or sometimes the methods used may also be approximate. Hence, the errors in the computed result may be due to the errors in the data or the errors in the method or both.

Definition: Error is defined as the quantity true value – Approximate value. That is if x is the true value and $\bar{x}$ is its approximate value of a quantity, then its error equal to $|x-\bar{x}|$

1.2 Sources of error

In numerical analysis, most computations are inexact, either due to the given data being approximate or due to the limitations of the computing aids such as mathematical tables, desk calculators, the digital computers or due to the method used and so on.

1.3 Types of errors and minimization mechanisms

In Numerical analysis, we can group errors into two

i) **Inherent errors**: Most numerical computations are in exact either due to the given date being approximate or due to the limitations of the computing aids such as mathematical tables, desk calculators or the digital computers. Due to such limitations, numbers have to be rounded-off, causing what are called rounding errors.

In computations inherent errors can be minimized by obtaining better data, by correcting obvious errors in the data and by using computing aids of higher precision and in hand computation, the round – off error can be reduced by carrying the computation to more significant figures at each step of the computation.

ii) **Truncation Error** – These errors are caused by using approximate formulae in computations such as the one that arises when a function f(x) is evaluated from an infinite series for x after 'truncating' it at a certain stage. The error committed in a series approximation can be evaluated by using the remainder after n +1 terms. Taylor's series for f(x) at x = a is given by

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots + \frac{(x-a)^n}{n!} f^{(n)}(a) + R_n(x) \qquad \text{Where}$$

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi), \; a < \xi < x$$

For a series which is convergent, the remainder tends to zero as $n \to \infty$. If we approximate f(x) by the 1st n – terms of a series, then the maximum error committed in this approximation is given by the remainder term. Conversely, if the accuracy required is specified in advance it would be possible to find n (number of terms) such that the finite series yields the required accuracy.

1.4 Absolute, relative and percentage errors

1. Absolute error is the numerical difference between the true value of the quantity and its approximate value. That is if x is the true value of a quantity and $\bar{x}$ is its approximate value, then the absolute error E_A is given by $E_A = \left|x - \bar{x}\right| = \Delta x$
2. The relative error E_R is defined by
   $$E_R = \frac{E_A}{x} = \left|\frac{x-\bar{x}}{x}\right| = \frac{|Error|}{|true\ value|} = \frac{\Delta x}{x} \text{ and}$$
3. The percentage Error E_p is defined as $E_p = 100\ E_R$

   Example: if x=0.3100 $\times 10$ and x^{*} =0.3000$\times 10$ then, $E_A = \left|x - \bar{x}\right| = \Delta x = 0.01$, the relative error is 0.333. . . $\times 10^{-1}$ and its percentage error is 0.333. . . $\times$ 10 = 3.333…%

2. Numerical solutions of Ordinary Differential Equations

There are various types of Numerical methods for solving ordinary differential equations. Consider the general 1^{st} order Initial value problem

$$\frac{dy}{dx} = f(x,y) \text{ , with the initial condition } y(x_0) = y_0 \qquad (2.1)$$

The methods can in general be applied to the solution of systems of 1^{st} order equations and will yield the solution in one of the two forms

a) A series for y in terms of powers of x, from which the value of y can be obtained by direct substitution.

   Example the method of Taylor and Picards

b) A set of tabulated values of x and y

   Example Euler, Runge- kutta, Adams Bashforth etc. these methods are called step by step (marching) methods as the values of y are computed by short steps ahead for equal intervals h of the independent variable

It is known that a differential equation of the n^{th} order will have n arbitrary constants in its general solution. In order to compute the numerical solution of such an equation, we need n-conditions.

Definition: Problems in which all the conditions are specified at the initial point only are called initial value problems (IVP).

Example $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$

On the other hand, in problems involving second and higher order differential equations we may prescribe the conditions at two or more points. Such problems are called boundary value problems (BVP).

Example $y''(x) + p(x)\, y'(x) + q(x)\, y(x) = r(x)$, $a \le x \le b$ with the boundary conditions $y(a)=\alpha$ and $y(b)=\beta$

There exist many methods for solving second order boundary value problems of the type $y''(x) + p(x)\, y'(x) + q(x)\, y(x) = r(x)$ with the given boundary conditions of which finite – difference method is the most popular one.

We need to introduce some terms before discussing the finite difference method.

I) Finite difference operators

Let the mesh points $x_0, x_1, x_2, \ldots, x_n$ be equally spaced, that is

$x_i = x_0 + ih$, $i = 0,1,2,\ldots, n$. we now define the following operators:

Notation: $f_i=f(x_i)$, $f_{i+1}= f(x_i + h)$, $f_{i-1}=f(x_i - h)$ and so on

a) The shift operator: $E\, f(x_i) = f(x_i + h)$ or $Ef_i = f_{i+1}$

b) The forward difference operator:

$$\Delta f(x_i) = f(x_i + h) - f(x_i) \quad \text{Or}$$

$$\Delta f_i = f_{i+1} - f_i$$

c) The backward difference operator:

$$\nabla f(x_i) = f(x_i) - f(x_i - h)$$

$$\nabla f_i = f_i - f_{i-1}$$

d) The central difference operator:

$$\delta\, f(x_i) = f\left(x_i + \frac{h}{2}\right) - f\left(x_i - \frac{h}{2}\right)$$

$$\delta\, f_i = f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}$$

e) The average operator:

$$\mu\, f(x_i) = \frac{1}{2}\left[f\left(x_i + \frac{h}{2}\right) + f\left(x_i - \frac{h}{2}\right)\right]$$

$$\mu\, f_i = \frac{1}{2}\left[f_{i+1/2} + f_{i-1/2}\right]$$

II) <u>Higher order differences</u>

Repeated applications of the difference operators lead to the following higher order differences.

$$E^n f_i = f_{i+n} \text{ i.e. } E^n\ (f(x_i) = f(x_i + nh)$$

$$\Delta^n f(x_i) = \Delta^{n-1} f(x_i + h) - \Delta^{n-1} f(x_i)$$

$\Delta^n f_i = \Delta^{n-1} f_{i+1} - \Delta^{n-1} f_i$, Compactly this can be written as

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} f_{i+n-k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f_{i+n-k}$$

$\nabla^n f(x_i) = \nabla^{n-1} f_i - \nabla^{n-1} f_{i-1}$ Compactly this can be written as

$$\nabla^n f(x_i) = \nabla^n f_i = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} f_{i-k}$$

$$\Rightarrow \nabla^n f_i = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} f_{i-k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f_{i-k}$$

III) Interpolating polynomials

It is a way of approximating a given function by a class of simpler functions called polynomials.

Definition: a polynomials p(x) is called interpolating polynomial if the values of P(x) and/ or its certain order derivatives coincide with those of f(x) and/ or its same order derivatives at one or more tabular points.

In general, if there are n+ 1 distinct points

$$a = x_0 < x_1 < x_2 < x_3 < \dots < x_n = b$$

then the problem of interpolating is to obtain p(x) satisfying the conditions

i) $p(x_i) = f(x_i)$ for $i = 0, 1, 2, 3, \dots, n$ or

ii) $p(x_i) = f(x_i)$

$p'(x_i) = f'(x_i)$ for $i = 0, 1, 2, 3, \dots, n$

The condition in (i) give rise to Lagrange/Newton interpolating polynomial and the derivative conditions in (ii) may be replaced by more general derivatives conditions involving higher order derivatives and give rise to Hermite interpolating polynomials.

X	**f(x)**	**Δf**	**Δ^2f**	**Δ^3f**	**Δ^4f**
x_0	f_0				
		Δf_0			
x_1	f_1		$\Delta^2 f_0$		
		Δf_1		$\Delta^3 f_0$	
x_2	f_2		$\Delta^2 f_1$		$\Delta^4 f_0$
		Δf_2		$\Delta^3 f_1$	
x_3	f_3		$\Delta^2 f_2$		
		Δf_3			
x_4	f_4				

Table 1 Forward difference table

Note that: In the forward difference table the differences $\Delta^k f_0$ lies on a straight line sloping downward to the right.

X	f(x)	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$
x_0	f_0				
		∇f_1			
			$\nabla^2 f_2$		
x_1	f_1				
				$\nabla^3 f_3$	
		∇f_2			$\nabla^4 f_4$
			$\nabla^2 f_3$		
x_2	f_2				
		∇f_3		$\nabla^3 f_4$	
x_3	f_3		$\nabla^2 f_4$		
		∇f_4			
x_4	f_4				

Table 2. Backward difference table

Note that: In the backward difference table the difference $\nabla^k f_4$ lies on a Straight line sloping upward to the right.

Example: For the following data, calculate the differences and obtain the forward and backward difference polynomials and interpolate at x = 0.25

x	0.1	0.2	0.3	0.4	0.5
f(x)	1.40	1.56	1.76	2.00	2.28

Solution: The difference table is as follows

0.1	1.40				
		0.16			
0.2	1.56		0.04		
		0.20		0.0	
0.3	1.76		0.04		0.0
		0.24		0.0	
0.4	2.00		0.04		
		0.28			
0.5	2.28				

a) The forward difference interpolating polynomial is given by

$$P(x)=f_o+\frac{(x-x_0)}{1!}\frac{\Delta f_0}{h}+\frac{(x-x_0)(x-x_1)}{2!}\frac{\Delta^2 f_0}{h^2}+\frac{(x-x_o)(x-x_1)(x-x_2)}{3!}\frac{\Delta^3 f_o}{h^3}$$

$$+\ldots+\frac{(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{n-1})}{n!}\frac{\Delta^n f_0}{h^n}$$

Now, $f_o = 1.40$, $h = 0.1$, $\Delta f_0=0.16$, $\Delta^2 f_0=0.04$, $\Delta^3 f_0=0$, $\Delta^4 f_0=0$

$\therefore$ $P(x) = 1.40 + (x-0.1)\ \dfrac{0.16}{0.1}+\dfrac{(x-0.1)(x-0.2)}{2}\ \dfrac{0.04}{(0.1)^2}+0$

$= 1.40 + 1.6\ (x-0.1) + 2(x-0.1)\ (x-0.2)$

$= 1.40 + 1.6x - 0.16 + 2(x^2 - 0.3x + 0.02)$

$= 1.40 - 0.16 + 0 - 04 + 1.6x + 2x^2 - 0.6x$

$= 1.28 + x + 2x^2$

$f(x) \approx p(x) = 2x^2 + x + 1.28$

$\therefore$ $f(0.25) \approx P(0.25) = 2(0.25)^2 + 0.25 + 1.28$

$= 1.655$

Hence $f(0.25) \approx 1.655$

b) The backward difference interpolating polynomial is given by

$$p(x)=f_n+\frac{(x-x_n)}{1!}\frac{\nabla f_n}{h}+\frac{(x-x_n)(x-x_{n-1})}{2!}\frac{\nabla^2 f_n}{h^2}+\text{- - -}+$$

$$\frac{(x-x_n)(x-x_{n-1})---(x-x_1)}{n!}\frac{\nabla^n f_n}{h^n}$$

$P(x) = 2.28 + \dfrac{(x-0.5)}{1!}\dfrac{0.28}{0.1}+\dfrac{(x-0.5)(x-0.4)}{2!}\ \dfrac{0.04}{(0.1)^2}+0$

$= 2.28 + 2.8\ (x-0.5) + 2(x^2 - 0.9\ x + 0.20)$

$= 2.28 + 2.8\ x - 1.4 + 2x^2 - 1.8x + 0.40$

$= 2.28 - 1.40 + 0.40 + 2.8x - 1.8x + 2x^2$

$= 2x^2 + x + 1.28$

Since both the polynomials are identical, we obtain

$f(0.25) \approx p(0.25) = 1.655$

2.1 Derivatives and Finite differences

a) Consider the forward difference $\Delta f(x) = f(x+h) - f(x)$

Using Taylor's Expansion we have,

$$\Delta f(x) = f(x+h) - f(x)$$

$$= \left[f(x) + hf'(x) + \frac{h^2}{2} f''(x) + --- \right] - f(x)$$

$$= hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + ---$$

Now neglecting terms of order two and above we have,

$\Delta f(x) = h f'(x) + o(h^2)$ where, $o(h^2)$ means terms of order two and above are truncated. Hence,

$$f'(x) = \frac{\Delta f(x)}{h} + o(h)$$

$$\Rightarrow f'(x) \approx \frac{\Delta f(x)}{h}$$

Note: In forward difference the error is of $o(h)$ which means the error is less than or equal to some constant times h.

b) Consider the backward difference $\nabla f(x) = f(x) - f(x-h)$

Using Taylor's Expansion we have,

$$\nabla f(x) = f(x) - f(x-h)$$

$$= f(x) - \left[f(x) - hf'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{3} f'''(x) + --- \right]$$

$$= f(x) - f(x) + hf'(x) - \frac{h^2}{2} f''(x) + \frac{h^3}{3} f'''(x) ---$$

$$= hf'(x) - \frac{h^2}{2} f''(x) + \frac{h^3}{3} f'''(x) + ---$$

$$= hf'(x) + o(h^2)$$

$$f'(x) = \frac{\nabla f(x)}{h} + o(h)$$

$$\Rightarrow f'(x) \approx \frac{\nabla f(x)}{h}$$

Note: The error in the backward difference is also of order h.

For second order derivative,

I) In forward difference we have,

$$\Delta^2 f(x) = \sum_{k=0}^{2} (-1)^k \binom{n}{k} f_{i+n-k} = \sum_{0}^{2} (-1)^k \binom{2}{k} f_{i+2-k}$$

$$= f_{i+2} - 2f_{i+1} + f_i$$

$$= f(x+2h) - 2f(x+h) + f(x)$$

$$= \left[f(x) + 2hf'(x) + \frac{4h^2}{2!} f''(x) + --- \right] - 2\left[f(x) + hf''(x) + \frac{h^2 f''(x)}{2!} \right.$$

$$\left. + \frac{h^3 f'''(x)}{3!} + --- \right] + f(x)$$

$$= h^2 f''(x) + h^3 f'''(x) + ---$$

$$\Rightarrow f''(x) = \frac{\Delta^2 f(x)}{h^2} + o(h)$$

Hence,

$$f''(x) \approx \frac{\Delta^2 f(x)}{h^2}$$

Note: the error is of o(h)

II) In Backward difference we have,

$$\nabla^2 f(x) = \sum_{k=0}^{2} (-1)^k \binom{2}{k} f_{i-k} = f(x) - 2f(x-h) + f(x-2h)$$

$$= f(x) - 2\left[f(x) - hf'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f'''(x) + --- \right]$$

$$+ f(x) - 2hf'(x) + \frac{4h^2}{2!} f''(x) - \frac{8h}{3!} f'''(x) + ---$$

$$= h^2 f''(x) + o(h^3)$$

$$f''(x) = \frac{\nabla^2 f(x)}{h^2} + o(h)$$

$$\Rightarrow f''(x) \approx \frac{\nabla^2 f(x)}{h^2}$$

The error is of o(h)

Similarly we can derive for higher order derivatives

3. Finite difference method

Finite difference method is one of the most popular numerical methods to find a solution to boundary value problems. The method mainly consists in replacing the derivatives occurring in the differential equation and in the boundary conditions as well by means of their finite-difference approximations and solving the resulting system of equations by a standard procedure.

The particular difference quotient and step size h generally chosen so that a certain order of truncation error is maintained. However, h cannot be chosen too small because of the instability of the derivative approximation.

In the linear second order boundary value problem,

$$y''(x) + p(x)\, y'(x) + q(x)\, y(x) = r(x) \text{ where, } a \leq x \leq b,\ y(a) = \alpha,\ y(b) = \beta$$

Both, y' and y" should be replaced by their finite difference approximation.

To accomplish this task, we select an integer $N > 0$ and divide the interval [a, b] into N+1 equal subintervals, whose end points are the mesh points, $x_i = a + ih$ for $i = 0, 1, 2, 3, ---$, N+1 where $h = \frac{b-a}{N+1}$

- At the interior mesh points x_i, the differential equation to be approximated is

$$Y''(x_i) + P(x_i)\, y'(x_i) + q(x_i)y(x_i) = r(x_i)$$

$$y''_i + p_i y'_i + q_i y_i = r_i \quad \text{where } y_i = y(x_i),\ p_i = p(x_i) \text{ and so on}$$

3.1 Finite-difference approximation to derivatives

To obtain the appropriate finite-difference approximation to derivatives, we proceed as follows:

i) Expanding y in Taylor polynomial about x_i and evaluated at x_{i+1}, we have

$$y(x_i+h) = y(x_i) + hy'(x_i) + \frac{h^2 y''(x_i)}{2!} + \frac{h^3 y'''(x_i)}{3!} + ------- (3.1.1)$$

$$\Rightarrow hy'(x_i) = y(x_i + h) - y(x_i) - \frac{h^2 y''(x_i)}{2!} - \frac{h^3 y'''}{3!}(x_i) ---$$

Neglecting terms of order greater than one we have

$$hy'(x_i) \approx y(x_i + h) - y(x_i)$$

$$\Rightarrow y'(x_i) \approx \frac{y(x_i + h) - y(x_i)}{h} \approx \frac{1}{h}\Delta y(x_i) \text{-----------------} (3.1.2)$$

This is called forward difference quotient approximation for $y'(x_i)$.

ii) Expanding y in Taylor polynomial about x_i and evaluated at x_{i-1} we have,

$$y(x_i - h) = y(x_i) - hy'(x_i) + \frac{1}{2}h^2 y''(x_i) - \frac{h^3}{6}y'''(x_i) + \text{-------} (3.1.3)$$

$$\Rightarrow hy'(x_i) = y(x_i) - y(x_i - h) + \frac{h^2}{2}y''(x_i) - \frac{h^3}{6}y'''(x_i) + \text{---}$$

$$\Rightarrow y'(x_i) = \frac{y(x_i) - y(x_i - h)}{h} + \frac{h}{2}y''(x_i) - \frac{h^2}{6}y'''(x_i) + \text{---}$$

Neglecting terms of order greater than one we have,

$$y'(x_i) \approx \frac{y(x_i) - y(x_i - h)}{h} \approx \frac{1}{h}\nabla y(x_i) \text{---------------} (3.1.4)$$

This is called backward difference approximation for $y'(x_i)$.

iii) Subtracting equation (3.1.3) from (3.1.1) i.e.

$$-\begin{cases} y(x_i + h) = y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(x_i) + \frac{h^3}{6}y'''(x_i) + \text{---} \\ y(x_i - h) = y(x_i) - hy'(x_i) + \frac{h^2}{2}y''(x_i) - \frac{h^3}{6}y'''(x_i) + \text{---} \end{cases}$$

$y(x_i + h) - y(x_i - h) = 2hy'(x_i) + \frac{2h^3}{6}y'''(x_i) + \text{---}$, and neglecting terms of order three and above, we get

$y(x_i + h) - y(x_i - h) = 2h\, y'(x_i) + o(h^3)$

$\Rightarrow 2hy'(x_i) = y(x_i + h) - y(x_i - h)$

$\Rightarrow y'(x_i) = \frac{y(x_i + h) - y(x_i - h)}{2h} + o(h^2)$, and we can write this as:

$$\Rightarrow y'_i \approx \frac{y_{i+1} - y_{i-1}}{2h} \text{----------------------------} (3.1.5)$$

Equation (3.1.5) is called central difference approximation for y'(x) which is a better approximation to y'(x) than the forward and backward difference approximation.

iv) Adding equations (3.1.1) and (3.1.3) we get an approximation for $y''(x)$. That is,

$$+\begin{cases} y(x_i+h) = y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(x_i) + \frac{h^3}{6}y'''(x_i) + --- \\ y(x_i-h) = y(x_i) - hy'(x_i) + \frac{h^2}{2}y''(x_i) - \frac{h^3}{6}y'''(x_i) + --- \end{cases}$$

$$\Rightarrow y(x_i+h) + y(x_i-h) = 2y(x_i) + \frac{2h^2y''(x_i)}{2!} + \frac{2h^4y^{(4)}(x_i)}{4!} + ---$$

Neglecting terms of order three and above, we have:

$$y(x_i + h) + y(x_i - h) = 2y(x_i) + h^2y''(x_i) + 0(h^4)$$

$$h^2y''(x_i) = y(x_i + h) + y(x_i - h) - 2y(x_i) + 0(h^4)$$

$$\Rightarrow y''(x_i) = \frac{y(x_i+h) - 2y(x_i) + y(x_i-h)}{h^2} + 0(h^2)$$

$$\Rightarrow y''_i \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \text{---------------------} (3.1.6)$$

In this approximation of $y''(x_i)$ the error is $o(h^2)$

Note: In a similar manner, it is possible to derive finite-difference approximation to higher derivatives.

3.2 Finite – difference Method for linear problems

The finite difference method for the linear second order boundary value problem(BVP), $y'' = p(x)y' + q(x)\, y + r(x)$ $a \le x \le b$, $y(a) = \alpha$, $y(b) = \beta$ requires the difference quotient approximations be used to approximate both y' and y''.

To accomplish this task, we select an integer $N > 0$ and divide the interval [a, b] into $N + 1$ equal subintervals whose endpoints are the mesh points

$x_i = a + ih$ for $i = 0, 1, 2, 3, \ldots, N + 1$, where $h = \frac{b-a}{N+1}$ choosing the step size h in this manner facilitates the application of a matrix algorithm which solves a linear system involving an N x N matrix.

At the interior mesh points x_i, for i = 1, 2, 3, . . . , N the differential equation to be approximated is

$$y''(x_i) = p(x_i)y'(x_i) + g(x_i)y(x_i) + r(x_i)$$

$$y_i'' = p_i y_i' + q_i y_i + r_i$$

Now, substituting $y_i' = \dfrac{y_{i+1} - y_{i-1}}{2h}$ *and* $y_i'' = \dfrac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$ in the BVP

y"(x) = p(x) y'(x) + q(x)y(x) + r(x), we have

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = P_i\left[\frac{y_{i+1} - y_{i-1}}{2h}\right] + q_i y_i + r_i$$

Multiplying both sides by h^2 we have,

$$y_{i-1} - 2y_i + y_{i+1} = \frac{h}{2}p_i\left(y_{i+1} - y_{i-1}\right) + h^2 q_i y_i + h^2 r_i$$

$$y_{i-1} - 2y_i + y_{i+1} = \frac{h}{2}p_i\, y_{i+1} - \frac{h}{2}p_i y_{i-1} + h^2 q_i y_i + h^2 r_i$$

$$y_{i-1} + \frac{h}{2}p_i\, y_{i-1} - 2y_i - h^2 q_i y_i + y_{i+1} - \frac{h}{2}p_i y_{i+1} = h^2 r_i$$

$\left(1 + \dfrac{h}{2}p_i\right)y_{i-1} - (2 + h^2 q_i)y_i + \left(1 - \dfrac{h}{2}p_i\right)y_{i+1} = h^2 r_i$ Multiplying both sides by (-1)

we get,

$$\Rightarrow -\left(1 + \frac{h}{2}p_i\right)y_{i-1} + (2 + h^2 q_i) - (1 - \frac{h}{2}p_i)y_{i+1} = -h^2 r_i$$

For i = 1, we have $(2 + h^2 q(x_1))y_1 - (1 - \frac{h}{2}p(x_1))y_2 = -h^2 r(x_1) + \left(1 + \frac{h}{2}p(x_1)\right)\alpha$

For i = 2, $-\left(1 + \frac{h}{2}p(x_2)\right)y_1 + (2 + h^2 q(x_2))y_2 - \left(1 - \frac{h}{2}p(x_2)\right)y_3 = -h^2 r(x_2)$

For i = 3, $-\left(1 + \frac{h}{2}p(x_3)\right)y_2 + (2 + h^2 q(x_3))y_3 - \left(1 - \frac{h}{2}p(x_3)\right)y_4 = -h^2 r(x_3)$

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For i = n-1, $-\left(1+\frac{h}{2}p(x_{n-1})\right)y_{n-2}+(2+h^2q(x_{n-1}))y_{n-1}-\left(1-\frac{h}{2}p(x_{n-1})\right)y_n=-h^2r(x_n)$

For i = n, $-\left(1+\frac{h}{2}p(x_n)\right)y_{n-1}+(2+h^2q(x_n))y_n=-h^2r(x_n)+\left(1-\frac{h}{2}p(x_n)\right)\beta$

This leads to the following system

$$2+h^2q(x_1)y_1-\left(1-\frac{h}{2}p(x_1)\right)y_2 = -h^2r(x_1)+\left(1+\frac{h}{2}p(x_1)\right)\alpha$$

$$-\left(1+\frac{h}{2}p(x_2)\right)y_1+\left(2+h^2q(x_2)\right)y_2-\left(1-\frac{h}{2}p(x_2)\right)y_3 = -h^2r(x_2)$$

$$-\left(1+\frac{h}{2}p(x_3)\right)y_2+\left(2+h^2q(x_3)\right)y_3-\left(1-\frac{h}{2}p(x_3)\right)y_4 = -h^2r(x_3)$$

$$\vdots$$

$$-\left(1+\frac{h}{2}p(x_{n-1})\right)y_{n-2}+\left(2+h^2q(x_{n-1})\right)y_{n-1}-\left(1-\frac{h}{2}p(x_{n-1})\right)y_n=-h^2r(x_{n-1})$$

$$-\left(1+\frac{h}{2}p(x_n)\right)y_{n-1}+\left(2+h^2q(x_n)\right)y_n=-h^2r(x_n)+\left(1-\frac{h}{2}p(x_n)\right)\beta$$

These system of equations give an equation of the form AY = B where, A is a tri-diagonal matrix of order NXN. That is

$$A=\begin{bmatrix} 2+h^2q(x_1) & -1+\frac{h}{2}P(x_1) & 0 & 0 & 0\cdots0 \\ -1-\frac{h}{2}p(x_2) & 2+h^2q(x_2) & -1+\frac{h}{2}p(x_2) & 0 & 0\cdots0 \\ 0 & -1-\frac{h}{2}p(x_3) & 2+h^2q(x_3) & -1+\frac{h}{2}p(x_3) & 0\cdots0 \\ \vdots & \vdots & & & \\ 0 & 0\ \ldots 0-1-\frac{h}{2}p(x_{n-1}) & & 2+h^2q(x_{n-1}) & -1+\frac{h}{2}p(x_{n-1}) \\ 0 & 0 & --- & -1-\frac{h}{2}p(x_n) & 2+h^2q(x_n) \end{bmatrix}$$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} \quad and \quad B = \begin{bmatrix} -h^2 r(x_1) + & (1+\frac{h}{2}p(x_1))\alpha \\ -h^2 r(x_2) & \\ -h^2 r(x_3) & \\ \vdots & \\ -h^2 r(x_{n-1}) & \\ -h^2 r(x_n) + & (1-\frac{h}{2}p(x_n))\beta \end{bmatrix}$$

Solving AY=B by Gauss elimination gives $y_1, y_2, y_3, \dots y_n$. Then the approximate solution of the differential equation is obtained by interpolating the points

$(x_o, y_o), (x_1,y_1),(x_2,y_2),(x_3,y_3),\dots, (x_n, y_n),(x_{n+1},y_{n+1})$. That is $Y(x) = \sum_{k=0}^{n+1} L_k(x) y_k$ where $L_k(x)$ is Lagrange Fundamental polynomial.

The following theorem gives conditions under which the tri-diagonal Linear system AY=B has a unique solution.

Theorem: suppose that p, q and r, are continuous on [a, b]. If $q(x) \geq 0$ on [a, b], then the tri-diagonal linear system AY = B has a unique solution provided that $h < 2/L$ where

$L = \max\limits_{a \leq x \leq b} |P(x)|$

Note: the hypothesis of this theorem guarantee a unique solution to the BVP but it doesn't guarantee that $Y \varepsilon C^4$ [a, b], we need to establish that $y^{(4)}$ is continuous on [a, b] to ensure that the truncation error has order $o(h^2)$.

Example solve $y'' = \frac{-2}{x}(y'-2),\ y(1/2) = 3, y(1) = 3$

Using finite difference method for n = 4

Solution $h = \frac{b-a}{n+1} = \frac{1-\frac{1}{2}}{5} = \frac{1}{10} = 0.1$

$$y'' = \frac{-2}{x}(y'-2)$$

$\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}=\frac{-2}{x_i}\left[\frac{y_{i+1}-y_{i-1}}{2h}-2\right]$, multiplying both sides of this equation

by h^2x_i we have,

$$x_iy_{i-1}-2x_iy_i+x_iy_{i+1}=-h[y_{i+1}-y_{i-1}]+4h^2$$

$$x_iy_{i-1}-hy_{i-1}-2x_iy_i+x_iy_{i+1}+hy_{i+1}=4h^2$$

$$(x_i-h)y_{i-1}-2x_iy_i+(x_i+h)y_{i+1}=4h^2$$

$x_{i-1}y_{i-1}-2x_iy_i+x_{i+1}y_{i+1}=4h^2$ for i = 1, 2, 3, 4 Then we have the following linear system of equations

For i = 1, $x_oy_o-2x_1y_1+x_2y_2=4h^2$

$$\Rightarrow -2x_1y_1+x_2y_2=4h^2-x_oy_o=4h^2-3/2$$

For i = 2, $x_1y_1-2x_2y_2+x_3y_3=4h^2$

For i = 3, $x_2y_2-2x_3y_3+x_4y_4=4h^2$

For i = 4, $x_3y_3-2x_4y_4+x_5y_5=4h^2$ but since $x_5=1$, $y_5=3$

$\Rightarrow x_3y_3-2x_4y_4=4h^2-x_5y_5=4h^2-3$. This leads to the following system of equation

$$\begin{cases} -2x_1y_1+x_2y_2 & =4h^2-3/2 \\ x_1y_1-2x_2y_2+x_3y_3 & =4h^2 \\ x_2y_2-2x_3y_3+x_4y_4 & =4h^2 \\ x_3y_3-2x_4y_4 & =4h^2-3 \end{cases}$$ which leads to an equation of the form

$$AY=B \quad \text{that is} \quad \begin{bmatrix} -2x_1 & x_2 & 0 & 0 \\ x_1 & -2x_2 & x_3 & 0 \\ 0 & x_2 & -2x_3 & x_4 \\ 0 & 0 & x_3 & -2x_4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}=\begin{bmatrix} 4h^2-\frac{3}{2} \\ 4h^2 \\ 4h^2 \\ 4h^2-3 \end{bmatrix}$$

Since h = 0.1, $x_0=0.5$, $x_1=0.6$, $x_2=0.7$, $x_3=0.8$, $x_4=0.9$ and $x_5=1.0$, Then

$$\begin{bmatrix} -1.2 & 0.7 & 0 & 0 \\ 0.6 & -1.4 & 0.8 & 0 \\ 0 & 0.7 & -1.6 & 0.9 \\ 0 & 0 & 0.8 & -1.8 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}=\begin{bmatrix} -1.46 \\ 0.04 \\ 0.04 \\ -2.96 \end{bmatrix}$$

Solving the system by using Gaussian elimination/ Thomas/ we have $y_1 \approx 2.8667$, $y_2 \approx 2.8286$, $y_3 = 2.850$, $y_4 \approx 2.91$ then the approximate solution y(x) is obtained by interpolating the six points (0.5, 3), (0.6, 2.8667), (0.7, 2.8286), (0.8, 2.850), (0.9, 2.91),(1,3) that is $y(x) \approx P_{n+1}(x) = P_5(x)$

3.3 Finite – difference Methods for non-linear problems

For the general non-linear boundary value problem

$y'' = f(x, y, y'), a \le x \le b, y(a) = \alpha, y(b) = \beta$, the difference method is similar to the method applied to linear problems, here however, the system of equations will not be linear, so an iterative process is required to solve it.

For the development of the procedure, we assume throughout that f satisfies the following conditions

- f and the partial derivatives f_y and $f_{y'}$ are all continuous on

  $D = \{(x,y,y')\}\ a \le x \le b, -\infty < y < \infty, -\infty < y' < \infty\}$
- $f_y(x, y, y') \ge \delta$ on D, for some $\delta > 0$
- Constants k & L exist, with $k = \max_{(x,y,y') \in D} |f_y(x, y, y')|$ and

  $L = \max_{(x,y,y') \in D} |f_{y'}(x,y,y')|$

This ensures that $y'' = f(x,y,y')$ has unique solution. As in the linear case, we divide [a, b] into (N+1) equal subintervals whose endpoints are at $x_i = a + ih$ for $i = 0, 1, 2, \ldots, N + 1$. Assuming that the exact solution has abounded fourth derivative allows us to replace

$y''(x_i)$ and $y'(x_i)$ in each of the equations

$y''(x_i) = f(x_i, y(x_i), y'(x_i))$ by the appropriate centered difference formula. This gives, for each $i = 1, 2, 3, \ldots, N$.

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = f\left(x_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6} y''(\eta_i\right) +$$

$$\frac{h^2}{12} y^{(4)}(\xi_i) \textit{ for some } \eta_i \textit{ and } \xi_i \text{ in the interval } (x_{i+1}, x_{i+1})$$

As in the linear case, the difference method results when the error terms are deleted and the boundary conditions are employed $y_o = \alpha$, $y_{N+1} = \beta$

$$and \ \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = f\left(x_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right)$$ Multiplying both sides by h^2,

we have

$$y_{i+1} - 2y_i + y_{i-1} - h^2 f\left(x_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right) = 0$$

$$For \ \ i = 1, -2y_1 + y_2 - h^2 f\left(x_1, y_1, \frac{y_2 - y_0}{2h}\right) - y_0 = 0$$

$$2y_1 - y_2 + h^2 f\left(x_1, y_1, \frac{y_2 - \alpha}{2h}\right) - \alpha = 0$$

$$For \ i = 2, -y_1 + 2y_2 - y_3 + h^2 f\left(x_2, y_2, \frac{y_3 - y_1}{2h}\right) = 0$$

$$For \ i = 3, -y_2 + 2y_3 - y_4 + h^2 f\left(x_3, y_3, \frac{y_4 - y_2}{2h}\right) = 0$$

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$$For \ i = n-1, -y_{n-2} + 2y_{n-1} - y_n + h^2 f\left(x_{n-1}, y_{n-1}, \frac{y_n - y_{n-2}}{2h}\right) = 0$$

$$For \ \ i = n, -y_{n-1} + 2y_n + h^2 f(x_n, y_n, \frac{y_{n+1} - y_{n-1}}{2h}) - y_{n+1} = 0$$

$$\Rightarrow -y_{n-1} + 2y_n + h^2 f(x_n, y_n, \frac{\beta - y_{n-1}}{2h}) - \beta = 0$$

The system

$$\begin{cases} 2y_1 - y_2 + h^2 f\left(x_1, y_1, \dfrac{y_2-\alpha}{2h}\right) - \alpha & = 0 \\ -y_1 + 2y_2 - y_3 + h^2 f\left(x_2, y_2, \dfrac{y_3-y_1}{2h}\right) & = 0 \\ -y_2 + 2y_3 - y_4 + h^2 f\left(x_3, y_3, \dfrac{y_4-y_2}{2h}\right) & = 0 \end{cases}$$

$$-y_{n-2} + 2y_{n-1} - y_n + h^2 f\left(x_{n-1}, y_{n-1}, \frac{y_n - y_{n-2}}{2h}\right) = 0$$

$$-y_{n-1} + 2y_n + h^2 f\left(x_n, y_n, \frac{\beta - y_{n-1}}{2h}\right) - \beta = 0$$

has unique solution provided that $h < 2/L$

Where $L = \max\limits_{(x,y,y')\,\varepsilon\, D} |f_{y'}(x,y,y')|$

3.4 The Error in the numerical solution of finite difference method

To estimate the error in the numerical solution we define the local truncation error, τ by

$$\tau = \left[\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} - y_i''\right] + p_i\left[\frac{y_{i+1} - y_{i+1}}{2h} - y'_i\right] \text{ ------------------------ (3.4.1)}$$

Expanding y_{i-1} and y_{i+1} by Taylor's about x_i we get

$$y_{i-1} = y_i - \frac{h}{1!}y'_i + \frac{h^2}{2!}y''_i - \frac{h^3}{3!}y_i''' + \frac{h^4}{4!}y_i^{(4)} + \cdots$$

$$y_{i+1} = y_i + \frac{h}{1!}y'_i + \frac{h^2}{2!}y''_i + \frac{h^3}{3!}y_i''' + \frac{h^4}{4!}y_i^{(4)} + \cdots$$

i) $\dfrac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = \dfrac{y_{i-1} + y_{i+1} - 2y_i}{h^2}$

$$= \frac{2y_i + \frac{2h^2}{2!}y''_i + \frac{2h^4}{4!}y_i^{(4)} + \ldots - 2y_i}{h^2}$$

$$= \frac{2y_i}{h^2} + \frac{h^2 y''_i}{h^2} + \frac{h^4 y_i^{(4)}}{12h^2} + \ldots - \frac{2y_i}{h^2}$$

$$= y_i'' + \frac{h^2}{12} y_i^{(4)} + 0(h^4) \text{-------------------------}(*)$$

ii) $$\frac{y_{i+1} - y_{i-1}}{2h} = \frac{2hy'_i + \frac{2h^3 h_i'''}{3!} + ---}{2h}$$

$$= y'_i + \frac{h^2 y_i'''}{6} + 0(h^4) \text{- - - -----------}(**)$$

Substituting (*) and (**) in the equation (3.4.1) and simplifying we get

$$\tau = \frac{h^2}{12}\left(y_i^{(4)} + 2p_i y_i'''\right) + 0\,(h^4)$$

$$= \frac{h^2}{12}\left(y^{(4)}(\varepsilon_1) + 2p_i y'''(\varepsilon_2)\right) where, \varepsilon_1, \varepsilon_2 \in \left(x_{i-1}, x_{i+1}\right)$$

Thus, the finite-difference approximation

$$-\left(1 + \frac{h}{2} p_i\right) y_{i-1} + (2 + q_i h^2)\, y_i - \left(1 - \frac{h}{2} p_i\right) y_{i+1} = r_i h^2$$

has second-order accuracy for functions with continuous fourth derivatives on $[x_0, x_n]$. Further, it follows that $\tau \to 0$ as $h \to 0$, implying the greater accuracy in the result can be achieved by using a smaller value of h. But, in such a case, of course, more computational effort would be required since the number of equations became larger. But h should not be too small not to cause rounding error.

4. Numerical solutions to partial differential equations

Partial differential equations occur in many branches of applied mathematics, such as, in hydrodynamics, elasticity, quantum mechanics and electromagnetic theory. The analytical treatment of these equations is rather involved process and requires application of advanced mathematical methods. On the other hand, it is generally easier to produce sufficient approximate solutions by simple and efficient numerical methods. Several numerical methods have been proposed for the solution of partial differential equations but the finite difference method is the most popular one.

The general second order linear partial differential equation is of the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$ This can be written as

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \ldots \qquad (4.1)$$

Where A, B, C, . . . , G are all functions of x and y.

Equations of the form (4.1) can be classified with respect to the sign of the discriminant

$\Delta = B^2 - 4AC$

i) If $\Delta < 0$ at a point in the (x, y) plane, the equation is said to be of elliptic type

ii) If $\Delta = 0$ at the point in the (x, y) plane, the equation is said to be of parabolic type

iii) If $\Delta > 0$ at the point in the (x, y) plane, the equation is said to be of hyperbolic type.

Example: 1. Consider the Laplace equation $u_{xx} + u_{yy} = 0$

here A = 1, C = 1, B =D = E = F = G = 0 and

$\Delta = B^2 - 4AC = -4 < 0 \Rightarrow$ The Laplace equation is of elliptic type.

Similarly, the wave equation, $u_{xx} - \frac{1}{c^2}u_{tt} = 0$ is hyperbolic type and the heat equation $u_{xx} - u_t = 0$ where (x, y) are space coordinates and t is the time coordinate is of parabolic type

4.1 Finite – difference approximations to partial-derivatives

In partial differentiation, we consider any one variable at a time and treat the remaining variables as constants. Consider a function f(x,y) of two variables. let the values of the function f(x,y) be given at a set of points (x_i, y_j) in the (x,y) plane with spacing h and k in x and y directions respectively. Then we have $x_i = x_0 + ih$, $y_j = y_0 + jk$ for i, j = 1,2,3 . . .

Notation: $f_{i,j}=f(x_i, y_j)$, $f_{i+1,j}=f(x_i+h, y_j)$, $f_{i-1,j}=f(x_i-h, y_j)$, $f_{i,j+1}=f(x_i, y_{j+k})$, $f_{i,j-1}=f(x_i, y_j-k)$, To derive approximations for the partial derivatives we use Taylor's series expansion

1) Expanding $f_{i+1,j}$ i.e. $f(x_i + h, y_j)$ using Taylor's about the point (x_i,y_j), we have

$f(x_i+h,y_i) = f_{i+i,j}$

$$= f(x_i,y_j) + hf_x(x_i, y_i) + \frac{h^2}{2} f_{xx}(x_i)y_j) + \frac{h^3}{3!} f_{xxx}(x_i,y_j) + --- \qquad (4.1.1)$$

$$\Rightarrow f_{i+1,j} = f_{i,j} + hf_x(x_i,y_j) + \frac{h^2}{2} f_{xx}(x_i,y_j) + 0(h^3)$$

$$\Rightarrow f_{i+1,j} - f_{i,j} = hf_x(x_i,y_j) + 0(h^2)$$

$$\Rightarrow hf_x(x_i,y_j) = f_{i+1,j} - f_{i,j} + 0(h^2)$$

$$\Rightarrow f_x(x_i,y_j) = \frac{f_{i+1,j} - f_{i,j}}{h} + 0(h)$$

$$\Rightarrow \left(\frac{\partial f}{\partial x}\right)(x_i,y_j) = \frac{f_{i+1,j} - f_{i,j}}{h} + 0(h)...-----------(i)$$

2) Expanding $f_{i-1,j}$ i.e $f(x_i-h, y_j)$ using Taylor's about the point (x_i, y_j), we have

$$f(x_i - h,y_j) = f(x_i, y_j) - hf_x(x_i, y_j) + \frac{h^2}{2} f_{xx}\left(x_{i,}y_j\right) - \frac{h^3}{3!} f_{xxx}(x_i,y_j) + --- \qquad (4.1.2)$$

$$f_{i-1,j} = f_{i,j} - hf_x(x_i,y_j) + \frac{h^2}{2} f_{xx}(x_i,y_j) + 0(h^3)$$

$$\Rightarrow f_{i-1,j} - f_{i,j} - \frac{h^2}{2} f_{xx}(x_i,y_j) + 0(h^3) = -hf_x(x_i,y_j)$$

$$\Rightarrow f_x(x_i,y_j) = \frac{f_{i,j} - f_{i-1,j}}{h} + 0(h)$$

$$\boxed{\Rightarrow \left(\frac{\partial f}{\partial x}\right)(x_i,y_j) = \frac{f_{i,j} - f_{i-1,j}}{h} + 0(h)...-----------.(ii)}$$

3) Expanding $f_{i,j+1}$, i.e $f(x_i, y_j + k)$ using Taylor's about the point (x_i , y_j), we have,

$$f(x_i,y_j+k) = f_{i,j+1} = f(x_i,y_j) + k\, f_y(x_i,y_j) + \frac{k^2}{2} f_{yy}(x_i,y_j) + \frac{k3}{3!} f_{yyy}(x_i,y_j) + 0(k^4) ---- \quad (4.1.3)$$

$$f_{i,j+1} = f_{i,j} + k\ f_y(x_i,y_j) + 0(k^2)$$

$$\Rightarrow kf_y(x_i,y_j) = f_{i,j+1} - f_{i,j} + 0(k^2)$$

$$f_y(x_i,y_j) = \frac{f_{i,j+1} - f_{i,j}}{k} + 0(k)$$

$$\therefore \left(\frac{\partial f}{\partial y}\right)(x_i, y_j) = \frac{f_{i,j+1} - f_{i,j}}{k} + 0(k)....(iii)$$

4) Similarly expanding $f(x_i,y_j-k)$ using Taylor's about the point (x_i, y_j) we have:

$$f(x_i, y_j - k) = f(x_i, y_j) - k\, f_y(x_i, y_j) + \frac{k^2}{2} f_{yy}(x_i, y_j) - \frac{k^3}{3!} f_{yyy}(x_i, y_j) + 0(k^4)...(4.1.4)$$

$$\Rightarrow f_{i,j-1} = f_{i,j} - kf_y(x_i, y_j) + 0(k^2)$$

$$\Rightarrow f_y(x_i, y_j) = \frac{f_{i,j} - f_{i-1,j}}{k} + 0(k)$$

$$\Rightarrow \left(\frac{\partial f}{\partial y}\right)(x_i, y_j) = \frac{f_{i,j} - f_{i,j-1}}{k} + o(k)$$

$$\therefore \left(\frac{\partial f}{\partial y}\right)(x_i, y_j) \approx \frac{f_{i,j} - f_{i,j-1}}{k} \quad . . . \text{(iv)}$$

5) Subtracting (4.1.2) from (4.1.1), that is

$$-\begin{cases} f_{i+1,j} = f_{i,j} + hf_x(x_i, y_j) + \dfrac{h^2}{2} f_{xx}(x_i, y_j) + 0(h^3) \\ f_{i-1,j} = f_{i,j} - hf_x(x_i, y_j) + \dfrac{h^2}{2} f_{xx}(x_i, y_j) + 0(h^3), \end{cases}$$

$$\Rightarrow f_{i+1,j} - f_{i-1,j} = 2hf_x(x_i, y_j) + 0(h^3)$$

$$\Rightarrow f_x(x_i, y_j) = \frac{f_{i+1,j} - f_{i-1,j}}{2h} + 0(h^2)$$

$$\therefore \left(\frac{\partial f}{\partial x}\right)(x_i, y_j) \approx \frac{f_{i+1,j} - f_{i-1,j}}{2h} \quad . . . \text{(v)}$$

This is called second order approximation for $\left(\frac{\partial f}{\partial x}\right)(x_i, y_j)$.

6) Subtracting (4.1.4) from (4.1.3) that is

$$-\begin{cases} f_{i,j+1} = f_{i,j} + kf_y(x_i, y_i) + \dfrac{k^2}{2} f_{yy}(x_i, y_j) + 0(k^3) \\ f_{i,j-1} = f_{i,j} - kf_y(x_i, y_j) + \dfrac{k^2}{2} f_{yy}(x_i, y_j) + 0(k^3) \end{cases}$$

$$\Rightarrow f_{i,j+1} - f_{i,j-1} = 2kf_y(x_i, y_j) + 0(k^3)$$

$$\Rightarrow f_y(x_i, y_j) = \frac{f_{i,j+1} - f_{i,j-1}}{2k} + 0(k^2)$$ Hence, the second order approximation for $\dfrac{\partial f}{\partial y}$

is

$$\boxed{\left(\frac{\partial f}{\partial y}\right)(x_i, y_j) \approx \frac{f_{i,j+1} - f_{i,j-1}}{2k} \quad . . . \text{(vi)}}$$

7) adding equations (4.1.1) and (4.1.2) that is

$$+\begin{cases} f_{i+1,j} = f_{i,j} + hf_x(x_i, y_j) + \dfrac{h^2}{2} f_{xx}(x_i, y_j) + \dfrac{h^3}{3!} f_{xxx}(x_i y_j) + o(h^4) \\ f_{i-1,j} = f_{i,j} - hf_x(x_i, y_j) + \dfrac{h^2}{2} f_{xx}(x_i, y_j) - \dfrac{h^3}{2!} f_{xxx}(x_i, y_j) + 0(h^4) \end{cases}$$

$$\Rightarrow f_{i+1,j} + f_{i-1,j} = 2f_{i,j} + h^2 f_{xx}(x_i, y_j) + 0(h^4)$$

$$\Rightarrow h^2 f_{xx}(x_i, y_j) = f_{i+1,j} - 2f_{i,j} + f_{i-1,j} + 0(h^4)$$

$$f_{xx}(x_i, y_j) = \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{h^2} + 0(h^2)$$

Hence

$$\boxed{\left(\frac{\partial^2 f}{\partial x^2}\right)(x_i, y_j) \approx \frac{f_{i,j+1} - 2f_{i,j} + f_{i-1,j}}{h^2} \quad . . . \text{(vii)}}$$

8) Adding equations (4.1.3) and (4.1.4) that is

$$+\begin{cases} f_{i,j+1} = f_{i,j} + kf_y(x_i, y_j) + \dfrac{k^2}{2!} f_{yy}(x_i, y_j) + \dfrac{k^3}{3!} f_{yyy}(x_i y_j) + - - - \\ f_{i,j-1} = f_{i,j} - kf_y(x_i, y_j) + \dfrac{k^2}{2!} f_{yy}(x_i, y_j) - \dfrac{k^3}{3!} f_{yyy}(x_i, y_j) + - - - \end{cases}$$

$$\Rightarrow\ f_{i,j+1} + f_{i,j-1} = 2f_{i,j} + k^2 f_{yy}(x_i, y_j) + 0(k^4)$$

$$\Rightarrow\ k^2 f_{yy}(x_i, y_j) = f_{i,j+1} - 2f_{i,j} + f_{i,j-1} + 0(k^4)$$

$$\Rightarrow\ f_{yy}(x_i, y_j) = \frac{f_{i,j-1} - 2f_{i,j} + f_{i,j+1}}{k^2} + 0(k^2)$$

$$\boxed{\therefore \left(\frac{\partial^2 f}{\partial y^2}\right)(x_i, y_j) \approx \frac{f_{i,j-1} - 2f_{i,j} + f_{i,j+1}}{k^2} \qquad \text{(viii)}}$$

Note: $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial}{\partial x}\left[\dfrac{f_{i,j+1} - f_{i,j-1}}{2k}\right]$

$$= \frac{1}{2k}\frac{\partial}{\partial x}\left[f_{i,j+1} - f_{i,j-1}\right]$$

$$= \frac{1}{2k}\left[\frac{f_{i+1,j+1} - f_{i-1,j+1}}{2h} - \frac{\left(f_{i+1,j-1} - f_{i-1,j-1}\right)}{2h}\right]$$

$$= \frac{1}{4kh}\left[f_{i+1,,j+1} - f_{i-1,j+1} - \left(f_{i+1,j-1} - f_{i-1,j-1}\right)\right]$$

$$= \frac{1}{4kh}\left[f_{i+1,j+1} - f_{i-1,j+1} - f_{i+1,j-1} + f_{i-1,j-1}\right]$$

And $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left[\dfrac{\partial f}{\partial x}\right] = \dfrac{\partial}{\partial y}\left[\dfrac{f_{i+1,j} - f_{i-1,j}}{2h}\right]$

$$= \frac{1}{2h}\left[\frac{f_{i+1,j+1} - f_{i+1,j-1}}{2k} - \frac{\left(f_{i-1,j+1} - f_{i-1,j-1}\right)}{2k}\right]$$

$$= \frac{1}{4kh}\left[f_{i+1,j+1} - f_{i-1,j-+} - f_{i+1,j-1} + f_{i-1,j-1}\right]$$

Hence $\left(\dfrac{\partial^2 f}{\partial x \partial y}\right)(x_i y_j) = \left(\dfrac{\partial^2 f}{\partial y \partial x}\right)(x_i, y_j)$

$$= \frac{1}{4kh}\left[f_{i+1,j+1} - f_{i-1,j+1} - f_{i+1,j-1} + f_{i-1,j-1}\right]$$

Example: 1 if $f(x, y) = x^2 + y^2 - x$. Find

a) $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ at the point (1,1) using first order formulas with h = k = 1

b) $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}$ at the point (1,1) using second order formulas with h = k = 1

Solution:

a) The first order formulas are:

i) $$\left(\frac{\partial f}{\partial x}\right)(x_i, y_j) = \begin{cases} \frac{f_{i+1,j} - f_{i,j}}{h} + 0(h) \\ \frac{f_{i,j} - f_{i-1,j}}{h} + 0(h) \end{cases}$$

$$\therefore \left(\frac{\partial f}{\partial x}\right)(1,1) \approx \frac{f(x_i + h, y_j) - f(x_i, y_j)}{h}$$

$$\approx \frac{f(2,1) - f(1,1)}{1} = 3 - 1 = 2$$

$$\left(\frac{\partial f}{\partial x}\right)(1,1) \approx \frac{f(x_i y_j) - f(x_i, y_j)}{h} = \frac{f(1,1) - f(0,1)}{1} \approx \frac{1-1}{1} = 0$$

Similarly, we can find $\left(\frac{\partial f}{\partial y}\right)(x_i, y_j)$ using

ii) $$\left(\frac{\partial f}{\partial y}\right)(x_i, y_j) = \begin{cases} \frac{f_{i,j+1} - f_{i,j}}{k} + 0(k) \\ \frac{f_{i,j} - f_{i,j-1}}{k} + 0(k) \end{cases}$$

b) The second order formulas are

$$\left(\frac{\partial f}{\partial x}\right)(x_i, y_j) = \frac{f_{i+1,j} - f_{i-1,j}}{2h} + 0(h^2)$$

$$\left(\frac{\partial f}{\partial x}\right)(x_i, y_j) \approx \frac{f(x_i + h, y_j) - f(x_i - h, y_j)}{2h}$$

$$\approx \frac{f(2,1) - f(0,1)}{2} = \frac{3-1}{2} = \frac{2}{2} = 1$$

$$\left(\frac{\partial f}{\partial y}\right)(x_i, y_j) \approx \frac{f_{i,j+1} - f_{i,j-1}}{2k} + 0(k^2)$$

$$\left(\frac{\partial f}{\partial y}\right)(x_i, y_j) \approx \frac{f(x_i, y_j + k) - f(x_i, y_j - k)}{2k}$$

$$\left(\frac{\partial f}{\partial y}\right)(1,1) \approx \frac{f(1,2) - f(1,0)}{2} = \frac{4-0}{2} = 2$$

$$\left(\frac{\partial^2 f}{\partial x^2}\right)(x_i, y_j) \approx \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{h^2} + 0(h^2)$$

$$\left(\frac{\partial^2 f}{\partial x^2}\right)(1,1) \approx \frac{f(1+h,1) - 2f(1,1) + f(1-h,1)}{1}$$

$$\approx f(2,1) - 2f(1,1) + f(0,1)$$

$$\approx 3 - 2 + 1 = 2$$

$$\left(\frac{\partial^2 f}{\partial y^2}\right)(x_i, y_j) \approx \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{k^2}$$

$$\approx \frac{f(x_i, y_{j+k}) - 2f(x_i, y_j) + f(x_i, y_j - k)}{k^2}$$

$$\left(\frac{\partial^2 f}{\partial y^2}\right)(1,1) \approx f(1,2) - 2f(1,1) + f(1,0) \approx 4 - 2 + 0 = 2$$

Example 2 The following table of values are given for the function f(x,y)

y/x	0.1	0.2	0.3
0.1	2.0200	2.0351	2.0403
0.2	2.0351	2.0801	2.1153
0.3	2.0403	2.1153	2.1803

Table. 3

i) Estimate the value $\left(\frac{\partial f}{\partial x}\right)$ at (0.2, 0.1) by 1st and 2nd order formulas

ii) Estimate the value $\left(\frac{\partial^2 f}{\partial x \partial y}\right)$ at (0.2, 0.2) using second order formula.

Solution i) from the table above, we have h = k = 0.1 and the first order formula is

$$\left(\frac{\partial f}{\partial x}\right)(x_i\ y_j) = \frac{f_{i+1,j} - f_{i,j}}{h} = \frac{f(x_i + h, y_j) - f(x_i, y_j)}{h}$$

$$\left(\frac{\partial f}{\partial x}\right)(0.2,\ 0.1) = \frac{f(0.2 + 0.1, 0.1) - f(0.2, 0.1)}{h}$$

$$= \frac{f(0.3, 0.1) - f(0.2, 0.1)}{h}$$

$$= \frac{2.0403 - 2.0351}{0.1} = 0.0520$$

The second order formula is

$$\left(\frac{\partial f}{\partial x}\right)(x_i\ y_j) = \frac{f_{i+1,j} - f_{i-1,j}}{2h} + 0(h^2)$$

$$\left(\frac{\partial f}{\partial x}\right)(x_i\ y_j) = \frac{f(x_i + h, y_j) - f(x_i - h, y_j)}{2h}$$

$$\left(\frac{\partial f}{\partial x}\right)(0.2, 0.1) \approx \frac{f_{(0.2+0.1, 0.1)} - f_{(0.1, 0.1)}}{0.2} = \frac{f_{(0.3, 0.1)} - f_{(0.1, 0.1)}}{0.2}$$

$$\approx \frac{2.0403 - 2.0200}{0.2} = \frac{0.203}{0.2} = 0.1015$$

ii) $\left(\frac{\partial^2 f}{\partial x \partial y}\right)(x_i,\ y_j) = \frac{1}{4kh}\left[f_{i+1,j+1} - f_{i-1,j+1} - f_{i+1,j-1} + f_{i-1,j-1}\right]$ is the second order formula

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)(x_i\ y_j) = \frac{1}{4kh}\left[f(x_i + h, y_j + k) - f(x_i - h, y_j + k) - f(x_i + h, y_j - k) + f(x_i - h, y_j - k)\right]$$

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)(0.2, 0.2) = \frac{1}{4(0.1)(0.1)}\left[f_{(0.3, 0.3)} - f_{(0.1, 0.3)} - f_{(0.3, 0.1)} + f_{(0.1, 0.1)}\right]$$

$$\approx \frac{1}{0.04}\left[2.1803 - 2.0403 - 2.0403 + 2.0200\right]$$

$$\approx 25\ [4.2003 - 4.0806] = 25\ (0.1196)$$

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)(0.2, 0.2) \approx 2.9925$$

4.2. Solving partial differential equation using finite difference method

In this topic, we restrict ourselves to the study of those partial differential equations which can be replaced by their finite difference analogues. These analogues or difference equations are then used to the concerned partial derivatives.

The general procedure is to replace the partial differential equation by a finite difference analogue and then obtain the solution at the mesh points.

Consider the Laplace equation in two dimensions, $u_{xx} + u_{yy} = 0$ then, its finite difference analogue will be $\dfrac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \dfrac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} = 0$ multiplying both sides by h^2,

we have ,

$u_{i-1,j} - 2u_{i,j} + u_{i+1,j} + \left(\dfrac{h}{k}\right)^2 \left(u_{i,j-1} - 2u_{i,j} + u_{i,j+1}\right) = 0$ Rearranging the terms we get,

$$\Rightarrow u_{j,j} = \frac{u_{i-1,j} + u_{i+1,j} + \left(\frac{h}{k}\right)^2 \left(u_{i,j-1} + u_{,j+1}\right)}{2\left(\left(\frac{h}{k}\right)^2 + 1\right)}$$

For simplicity consider the case when $\Delta x = \Delta y \Rightarrow h = k$. Then,

$$u_{i,j} = \frac{1}{4}\left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right] \text{.........} \quad \text{...}(4.2.1)$$

This shows that the value of u at any point is the mean of its values at the four neighboring points. This is called the standard 5 – point formula see fig 4.2a below

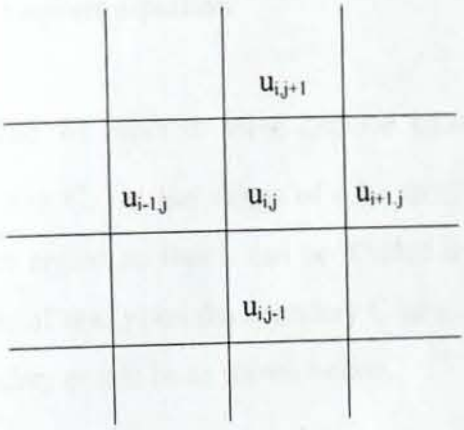


Fig 4.2 a

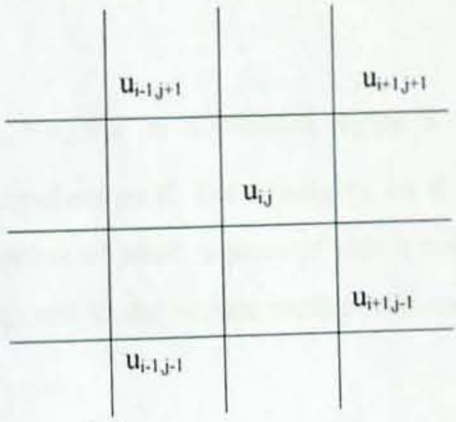


Fig. 4.2b

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i,j-1} + u_{i+1,j} + u_{i,j+1}\right]$$

$$\Rightarrow u_{i-1,j} + u_{i,j-1} + u_{i+1,j} + u_{i,j+1} - 4u_{i,j} = 0 \text{ - - - - - } (4.2.2)$$

By expanding the terms on the right side of eqn. (4.2.1)by Taylor's series. It can be shown that

$$u_{i-1,j} + u_{i,j-1} + u_{i+1,j} + u_{i,j+1} - 4u_{i,j} = h^2\left(u_{xx} + u_{yy}\right) - \frac{1}{6}h^4 u_{xxyy} + o(h^6)$$

$$= -\frac{1}{6}h^4 u_{xxyy} + o(h^6) \text{.................................} (4.2.3)$$

Instead of using the standard 5- point formula. We may use the formula

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j+1}\right] \text{.........................} (4.2.4)$$ which uses the function values at the diagonal points as shown in fig (4.2b) above.

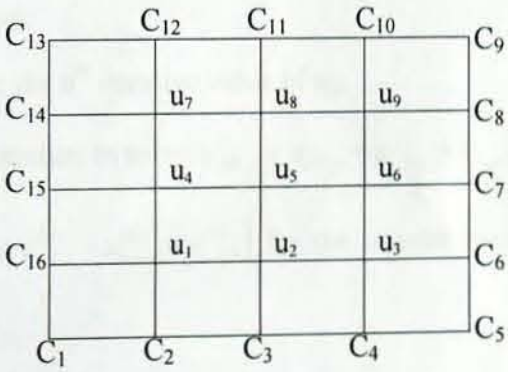
Expanding the terms on the right side of (4.2.4) by Taylor's series, it can be shown that

$$u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j+1} - 4u_{i,j} = \frac{2}{3}h^4 u_{xxyy} + o(h^6) \text{........................} (4.2.5)$$

Neglecting terms of order h^6, it follows from (4.2.3) and (4.2.5) that the error in the diagonal formula is four times that of the standard 5-point formula. Hence, we should prefer to use the standard 5-point formula whenever possible.

4.2.1 Laplace equation

Suppose we wish to solve Laplace equation $u_{xx}+u_{yy}=0$ in a bounded region R with boundary C. Let the values of u be specified everywhere on C. For simplicity, let R be a square region so that it can be divided into a network of small squares of side h and the values of u(x, y) on the boundary C be given by C_i and let the interior mesh points and the boundary points be as shown below.



Now, the equation $u_{xx} + u_{yy} = 0$ can be replaced by the standard five point formula or the diagonal 5 – point formula. The approximate function values at the interior mesh points can now be computed according to the following scheme: first we use the diagonal five point formula to compute u_5, u_7, u_9, u_1 and u_3 respectively, thus we obtain.

$$u_5 = \frac{1}{4}[c_1 + c_9 + c_5 + c_{13}]$$

$$u_7 = \frac{1}{4}[u_5 + c_{13} + c_{11} + c_{15}]$$

$$u_9 = \frac{1}{4}[u_5 + c_9 + c_7 + c_{11}]$$

$$u_1 = \frac{1}{4}[c_1 + c_3 + u_5 + c_{15}]$$

$$u_3 = \frac{1}{4}[c_3 + c_5 + c_7 + u_5]$$

we then compute, the remaining quantities in the order u_8, u_4, u_6 and u_2 by the standard

5- point formula

$$u_8 = \frac{1}{4}\left[u_5 + u_9 + c_{11} + u_7\right],\ u_4 = \left[u_1 + u_5 + u_7 + c_{15}\right]$$

$$u_6 = \frac{1}{4}\left[u_3 + c_7 + u_9 + u_5\right],\ u_2 = \frac{1}{4}\left[c_3 + u_3 + u_5 + u_1\right]$$

Once all the u_i ($i = 1, 2, 3 \ldots 9$) are computed their accuracy can be improved by any one of Jacobi's Method, Gauss-Seidel method or S.O.R method.

Jacobi's method

Let $u_{i,j}^{(n)}$ denote the n^{th} iterative value of $u_{i,j}$.

An iterative procedure to solve $u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0$ is

$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n)}\right]$ for the interior mesh points. This is called the point Jacobi method

Gauss-Seidel Method

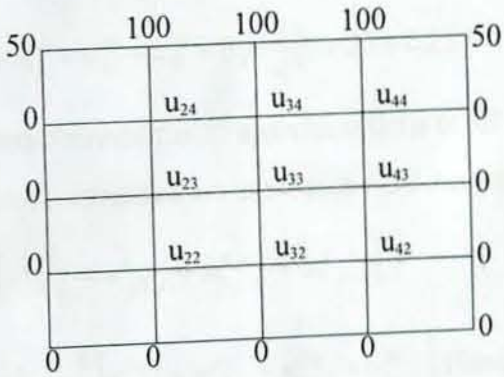
The method uses the latest iterative values available and scans the mesh points systematically from left to right along successive rows.

The iterative formula is

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n+1)} + u_{i,j+1}^{(n)}\right]$$

It can be shown that the Gauss-Seidel scheme converges twice as fast as the Jacobi scheme.

Example : Solve Laplace's equation as given in the figure below



Solution

Now, we have given the boundary values, we need to compute the interior mesh points. We can find $u_{23}^{(1)}, u_{24}^{(1)}, u_{44}^{(1)}, u_{22}^{(1)}$ *and* $u_{42}^{(1)}$ *by* diagonal 5-point formula as follows

$$u_{33}^{(1)} = \frac{1}{4}\left[0+50+0+50\right] = 25.00$$

$$u_{24}^{(1)} = \frac{1}{4}\left[50+25+100+0\right] = 43.75$$

$$u_{44}^{(1)} = \frac{1}{4}\left[u_{33}^{(1)}+50+0+100\right]$$

$$= \frac{1}{4}\left[25+150\right] = 43.75$$

$$u_{22}^{(1)} = \frac{1}{4}\left[u_{33}^{(1)}+0+0+0\right] = \frac{1}{4}(25) = 6.25$$

$$u_{42}^{(1)} = \frac{1}{4}\left[u_{33}^{(1)}+0+0+0\right] = \frac{1}{4}(25) = 6.25$$

Next, we will find $u_{34}^{(1)}, u_{23}^{(1)}, u_{43}^{(1)},$ *and* $u_{32}^{(1)}$ respectively using the standard 5 – point formula. That is

$$u_{34}^{(1)} = \frac{1}{4}\left[u_{33}^{(1)}+u_{24}^{(1)}+u_{44}^{(1)}+100\right] = \frac{1}{4}\left[25+43.75+43.75+100\right]$$

$$= \frac{1}{4}(212.50) = 53.125$$

$$u_{23}^{(1)} = \frac{1}{4}\left[u_{33}^{(1)}+u_{22}^{(1)}+u_{24}^{(1)}+0\right] = \frac{1}{4}\left[25+6.25+43.75\right] = 18.75$$

$$u_{43}^{(1)} = \frac{1}{4}\left[u_{42}^{(1)}+u_{33}^{(1)}+u_{44}^{(1)}+0\right] = \frac{1}{4}\left[6.25+25+43.75+0\right] = 18.75$$

$$u_{32}^{(1)} = \frac{1}{4}\left[0+u_{33}^{(1)}+u_{22}^{(1)}+u_{42}^{(1)}+0\right] = \frac{1}{4}\left[0+25+6.25+6.25\right] = 9.38$$

We have thus obtained the 1st approximations of all the nine mesh points and now to find 2nd, 3rd, 4th, . . . iterations we use Gauss-Seidel method

$$u_{i,j}^{k+1} = \frac{1}{4}\left[u_{i-1,j}^{k+1}+u_{i+1,j}^{k}+u_{i,j-1}^{k+1}+u_{i,j+1}^{k}\right]$$

For k = 1 $u_{i,j}^{2} = \frac{1}{4}\left[u_{i-1,j}^{(2)}+u_{i+1,j}^{(1)}+u_{i,j-1}^{(2)}+u_{i,j+1}^{(1)}\right]$ Hence

$$u_{2,2}^{(2)} = \frac{1}{4}\left[u_{1,2}^{(2)} + u_{3,2}^{(1)} + u_{2,1}^{(2)} + u_{2,3}^{(1)}\right] = \frac{1}{4}\left[0 + 9.38 + 0 + 18.75\right] \approx 7.03$$

$$u_{3,2}^{(2)} = \frac{1}{4}\left[u_{2,2}^{(2)} + u_{4,2}^{(1)} + u_{3,1}^{(2)} + u_{3,3}^{(1)}\right] = \frac{1}{4}\left[7.03 + 6.25 + 0 + 25\right] \approx 9.57$$

$$u_{4,2}^{(2)} = \frac{1}{4}\left[u_{3,2}^{(2)} + u_{5,2}^{(1)} + u_{4,1}^{(2)} + u_{4,3}^{(1)}\right] = \frac{1}{4}\left[9.57 + 0 + 0 + 18.75\right] \approx 7.08$$

$$u_{2,3}^{(2)} = \frac{1}{4}\left[u_{1,3}^{(2)} + u_{3,3}^{(1)} + u_{2,2}^{(2)} + u_{2,4}^{(1)}\right] = \frac{1}{4}\left[0 + 25 + 7.03 + 43.75\right] = 18.94$$

$$u_{3,3}^{(2)} = \frac{1}{4}\left[u_{2,3}^{(2)} + u_{4,3}^{(1)} + u_{3,2}^{(2)} + u_{3,4}^{(1)}\right] = \frac{1}{4}\left[18.94 + 18.75 + 9.57 + 53.125\right] = 25.10$$

$$u_{4,3}^{(2)} = \frac{1}{4}\left[u_{3,3}^{(2)} + u_{5,3}^{(1)} + u_{4,2}^{(2)} + u_{4,4}^{(1)}\right] = \frac{1}{4}\left[25.10 + 0 + 7.08 + 43.75\right] = 18.98$$

$$u_{2,4}^{(2)} = \frac{1}{4}\left[u_{1,4}^{(2)} + u_{3,4}^{(1)} + u_{2,3}^{(2)} + u_{2,5}^{(1)}\right] = \frac{1}{4}\left[0 + 53.125 + 18.94 + 100\right] = 43.02$$

$$u_{3,4}^{(2)} = \frac{1}{4}\left[u_{2,4}^{(2)} + u_{4,4}^{(1)} + u_{3,3}^{(2)} + u_{3,5}^{(1)}\right] = \frac{1}{4}\left[43.02 + 43.75 + 25.10 + 100\right] = 52.97$$

$$u_{4,4}^{(2)} = \frac{1}{4}\left[u_{3,4}^{(2)} + u_{5,4}^{(1)} + u_{4,3}^{(2)} + u_{4,5}^{(1)}\right] = \frac{1}{4}\left[52.97 + 0 + 18.98 + 100\right] = 42.99$$

We will continue in this manner for k=2, 3, 4… until we get to successive iterates which are very close to each other.

The following table gives the first four iterates obtained by using the Gauss-Seidel formula for the above problem.

U_{22}	U_{32}	U_{42}	U_{23}	U_{33}	U_{43}	U_{24}	U_{34}	U_{44}
7.03	9.57	7.08	18.94	25.10	18.98	43.02	52.97	42.99
7.13	9.83	7.20	18.81	25.15	18.84	42.94	52.77	42.90
7.16	9.88	7.18	18.81	25.08	18.79	42.89	52.72	42.88
7.17	9.86	7.16	18.78	25.04	18.77	42.88	52.70	42.87

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