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SEMINAR REPORT

ON

BIMODULES AND AFFILIATED PRIME IDEALS

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Introduction

When we study the theory of structure of a ring we encounter theories which hold in commutative case rather than non-commutative case. For instance if P is a prime ideal of a ring R, then R/P is a domain in commutative case but it need not be a domain in the case of non-commutative. Thus a more relaxed definition for the concept of structures of a ring in non-commutative case is desirable. And since theory of modules over a ring in a commutative case is differ from that of non-commutative case, a more relaxed definition for the concept of modules over a ring in a non-commutative case is desirable.

In this seminar we shall see that modules over a ring R such as bimodules and prime ideals of a ring R such as affiliated prime ideals. (R may be commutative or non-commutative) .

Bimodules have become of increasing importance in the ideal theory of a noetherian ring R, particularly ideal factors $I//J$ where $I \supseteq J$ are ideals of R, and over rings $S \supseteq R$,viewed as (S,R)-bimodules. To make the notion more convenient in both case, we study bimodules in general. Thus we investigate the structure of bimodules over noetherian rings, particularly bimodules which are noetherian or artinian on at least one side, and we illustrate the results by indicating a number of applications, particularly to the relationships between the prime ideals of a ring and the prime ideals of a sub ring.

CHAPTER-1

PREREQUISITES AND PRELIMINARIES

1.1 Modules

Definition: Let R be a ring. A left R-module is an additive abelian group A together with a function (scalar multiplication $R \times A \longrightarrow A$ (the image of (r, a) being denoted by ra) such that for all $r, s \in R \text{ and } a, b \in A$:

i. $r(a + b) = ra + rb$
ii. $(r + s)a = ra + sa$
iii. $r(sa) = (rs)a$

Remark: A right R-module is defined similarly via a function

$A \times R \rightarrow A$ denoted $(a, r) \longmapsto ar$ and satisfying the obvious analogues of (i)-(iii)

❖ Examples:

- Every ring is a module over itself.
- A Z-module can be interpreted as an abelian group (and vice versa).
- An additive group consisting of 0 alone is a module over any ring.
- An ideal of a ring R is an R-module.
- Vector space over a field F is F-modules.

Definition: Let R be a ring, A be an R-module and B a nonempty subset of A. Then B is a submodule of A provided that B is additive subgroup of A and $rb \in B$ for all

$r \in R, b \in B$.

Let R be a ring. Then

I. A left R-module A (denoted $_RA$) is called

- **Noetherian;** if A satisfies ascending chain condition (ACC) on submodules.
- **Artinian;** if A satisfies descending chain condition (DCC) on submodules.
- **Simple;** if A has no proper submodules and $RA \neq 0$.
- **Torsion**; if each element of A is torsion element of A.
- **Torsion free**; if 0 is the only torsion element of A.
- **Divisible**; if $xA = A$ for all regular element x of R.

II. A submodule B of a left R-module A is called

- **Essential**; if it has nonzero intersection with every nonzero submodule of A. we write$B <_e A$
- **Socle**; if it is the sum of all simple submodule of A and denoted

  Soc (A).

Remark: the right R-module A (denoted A_R) defined similarly as (I) and (II) above.

➢ Let R be a ring and let A be a left R-module. Then

1. If B is a submodule of A, then the factor group A/B a left R-module called factor module.
2. A is said to be semisimple if Soc(A)= A .
3. If X is a subset of A, then the intersection of all submodules of A containing X (say B) is called the submodule generated by X. If X is finite and X generates the module B, is said to be finitely generated.

Note that: -Let A be a right module over a ring R. given any subset $X \subseteq A$,the annihilator of X is the set $ann(X) = \{r \in R | xr = 0 \; for \; all \; x \in X\}$ which is a right ideal of R. In case the ring R must be made explicit, we write $ann_R(X)$ and refer to the annihilator of X in R. Similarly to emphasize that we are taking an annihilator on the right side of X (because A is a right module), we may write $r.ann(X)$ for

$ann(X)$.When X consists of a single element x, we abbreviate $ann(\{x\})$ to $ann(x)$. We have already noted that $ann(X)$ is a right ideal of R; moreover, if X is a submodule of A, then $ann(X)$ is an ideal of R.

Remark:

i. Annihilators of subsets of a left R-module are defined analogously, and are left ideals of R.
ii. A right (left) annihilator in a ring R is any right (left) ideal of R which equals the right (left) annihilator of some subset of R.

Observation: Let R be a ring. Then

1. If an R-module A is noetherian(artinian), then A^n is also noetherian(artinian) for $n \in \mathbf{N}$
2. Let B be a submodule of a right R-module A. Then if $B <_e A$, then A/B is torsion.
3. If A is finitely generated and semisimple, then it is artinian.

1.2 The structures of rings

Let R be a ring. Then

i. A proper ideal I of R is called
    - **Prime**; if for any ideals J, J^l $of\ R, JJ^l \subseteq I$ implies either $J \subseteq I\ or$ $J' \subseteq I$
    - **Semiprime**; if it is any ideal of R which is an intersection of prime ideals.
    - **Minimal prime**; if it is any prime ideal of R that does not properly contain any other prime ideals
    - **Maximal;** if there exists no ideal J with $I \subsetneq J \subsetneq R$.
    - **Nilpotent;** if $I^n = 0$ for some n$\in$ **N**
    - **Left (right) primitive**; if it is the left (right) annihilator of a simple left(right)R-module.

- **Semiprimitive**; if it is any ideal of R which is an intersection of left (right) primitive ideals.

ii. R is called

- **Noetherian;** if R is noetherian as an R-module.
- **Artinian**; if R is artinian as an R-module.
- **Simple**; if 0 is maximal ideal.
- **Semisimple**; if $_{,R} R$ or R_R is semi simple.
- **Primitive**; if 0 is primitive ideals.
- **Semiprimitive**; if 0 is a semi primitive ideal.
- **Prime**; if 0 is a prime ideal.
- **Semiprime**; if 0 is a semiprime ideal.
- **Left (right) Quotient ring** with respect to a multiplicative set of regular elements $X \subseteq R$: if for any ring $S \supseteq R$ (a) Every element of X is invertible in S (b) Every element of S can be expressed in the form $ax^{-1}(x^{-1}a)$ for some$a \in R$ and $x \in X$.
- **Left (right) Goldie ring**; if (a) R satisfies the ACC on left(right) annihilators

  (b) Every independent set of left (right) ideal of R is finite.

➢ 1. Let R be a ring and let I be an ideal of R. Then R/I is said to be a factor ring if R/I is a factor R-module.

2. Let S be a commutative ring with identity and R a subring of S containing1_S. Then S is said to be an extension ring of R.

Observation:

1. If R is a prime left/right Goldie ring, then every nonzero ideal of R contain a regular element.

2. Let R be prime right Goldie ring. Then all submodules of torsion free right R-modules are torsion free.

3. Let R be a semiprime right Goldie ring with right Goldie Quotient ring Q. Then every torsion free divisible right R-module has a unique right Q-module structure compatible with its right R-module structure.

4. Let R be a noetherian ring.Then $J = \{a \in R |\ aI = 0\ for\ some\ I <_e R_R\}$ is nilpotent.

5. Let P be a minimal prime in a semi prime right Goldie ring R. then R/P torsion free as both a right R-module and a right R/P-module.

6. Let P be a prime ideal in a semiprime right Goldie ring R. Then TFAE

a) P is minimal c) P is right annihilator

b) P is left annihilator

7. If R is a semiprime left and right Goldie ring and B is finitely generated torsion free left R-module, then B can be embedded in a finitely generated free left R-module.

8. Every left/right noetherian ring is a left/right Goldie rin

1.3 Module composition series

A strictly descending chain

$$A = A_0 \supsetneq A_1 \supsetneq \cdots \supsetneq A_l = 0 \qquad (1)$$

Of submodules of A, starting in A and ending in 0, is called a normal series of A of length$l$. One calls (1) a composition series of A if each module A_{i-1}/A_i where $1 \leq i \leq l$ is simple.

-The length $l(A)$ of A is defined to be the supermom of all length l of chains of the form (1) we have $l(A) \in N_0 \cup \{\infty\}$

- A module A is said to have finite length if $l(A) < \infty$

Observation: 1. Let R be a ring and let A be an R-module. Then TFAE

a) A has finite length
b) A has a composition series
c) A is noetherian
d) A is artinian

2. Let R be a ring and let A be an R-module that possesses a composition series. Then all composition series of have the same length L, and $L = l(A)$. (Jordan Holder theorem)

1.4 prime Ideals

Let R be a ring. Then a left R-module $_{,R}A$ is called

- **Faithful;** if $ann_R(A) = 0$
- **Fully faithful**; if A and all nonzero submodules of A are faithful
- **Prime;** if A is nonzero R-module which is fully faithful as a module over $R/ann_R\,(A)$.

-Let A be a nonzero module over a ring R. Then

- **An annihilator prime for A** is any prime ideal P of R which equals the annihilator of some nonzero submodule of A. in this case, $ann_A(P)$ is clearly nonzero and is faithful (R/P)-module.
- **Associated prime of A** is any annihilator prime P which equals the annihilator of some prime submodule B of A.
- **A prime series for A** is a series of submodules of the form $A_0 = 0 < A_1 <\cdots < A_n = A$ Where for each $i = 1,\cdots,n$ the module A_i/A_{i-1} is a prime submodule of A/A_{i-1}

Observation: For any ring R we have the following properties

1. Let A be an R-module. Then A is faithful module over$R/ann_R\,(A)$.
2. If R is prime ring, then every nonzero right/left ideal of R faithful.

3. Let A and B be an R-module. Then if $B \subseteq A$, *then* $ann_R(A) \subseteq ann_R(B)$
4. Let A be an R-module and P be an ideal of R. Then $P \subseteq ann_R(ann_A(P))$
5. Let A be an R-module and I be an ideal of R $s.t\ I \subseteq ann_R(A)$. Then A is an R/I-module and $ann_R(A)/P = ann_{R/P}(A)$.

Theorem 1.1 let A be nonzero module over a ring R. suppose that there exists an ideal P maximal among the annihilators of nonzero submodule of A. then P is prime ideal of R, and $ann_A(P)$ is a fully faithful (R/P)-module.

Proof: For specificity, suppose that A is a right R-module

Since P is the annihilators of nonzero submodule of A, there is a nonzero submodule B of A such that $P = ann_R(B)$ and $P \neq R$ because $B \neq 0$

1. We show that P is a prime ideal of R: let I *and* J be ideals of R. Then we need to show that if $IJ \subseteq P$, *then* $I \subseteq P$ *or* $J \subseteq P$ (or by contra positive)
   if $P \subset I$ *and* $P \subset J$, *then* $IJ \nsubseteq P$

Now to show $IJ \nsubseteq P$, suppose $IJ \subseteq P$

Since$B \neq 0$, consider a nonzero submodule BI of A

Let $s \in J$.Then $rs \in IJ \subseteq P$ *for* $r \in I$

$\Rightarrow rs \in P \Rightarrow b(rs) = 0$ *for all* $r \in B \Rightarrow (br)s = 0$ Since B is a right R-module

$\Rightarrow s \in ann_R(BI)$

Hence $J \subseteq ann_R(BI)$

Thus $P \subset J \subseteq ann_R(BI)$ contradicting the maximality of P

Therefore $IJ \nsubseteq P$

Hence P is a prime ideal of R

2. We show that $ann_A(P)$ is a fully faithful (R/P)-module :

Let $C := ann_A(P)$ .Then C is a submodule of A and

$$ann_R(C) = ann_R(ann_A(P)) \supseteq P$$

$$\Rightarrow P \subseteq ann_R(C)$$

Hence by maximality of P, $P = ann_R(C)$ (1)

Since C is faithful over $R/ann_R\,(C)$, by (1) C is faithful over R/P

Given any nonzero submodule $D \subseteq C$, we have $ann_R(C) \subseteq ann_R(D)$

By (1), $P \subseteq ann_R(D)$.It follows by maximality of P, $P = ann_R(D)$

Since D is faithful over $R/ann_R\,(D)$, it is faithful over R/P

Therefore C is fully faithful as a right (R/P)-module ■

In general, given an R-module A, the family of annihilators of nonzero submodule of A need not have any maximal elements. If R is either right/left noetherian, the existence of maximal annihilators is automatic.

Lemma 1.2: If A is a prime module over a ring R, then show that $ann_R(A)$ is a prime ideal of R.

Proof: let$P := ann_R(A)$. Then by hypothesis, A is fully faithful over R/P –module

We claim that R/P is a prime ring.$i.e$ we need to show that $\{P\}$ is a prime ideal of R/P

Suppose not $i.e\{P\}$ is not a prime ideal of R/P .then there are ideals I/P and J/P of R/P such that $(I/P\,)\,(J/P) = \{P\}$ with $I/P\,, J/P \neq \{P\}$

Since $I \sqsupseteq P$,$I = ann_R(B)$ for some $0 \neq$ B submodule of A

$$\Rightarrow I/P = ann_R(B)/P = ann_{R/P}(B)$$

Since A is fully faithful as R/P –module , $I/P = ann_{R/P}(B) = \{P\} \rightarrow\leftarrow$

Therefore $\{P\}$ is a prime ideal of R/P. It follows R/P is a prime ring.

Therefore P is a prime ideal of R ■

Theorem 1.3: let A be a nonzero right module over a right noetherian ring R. if A is finitely generated, then A has a prime series. If $A_0 = 0 < A_1 <\cdots< A_n = A$ is a prime series for A and $P_i = ann_R(A_i/A_{i-1})\ for\ i = 1,\cdots,n$, then each P_i is prime ideal of R and A_i/A_{i-1} is a fully faithful right (R/P_i)-module.

Proof: 1. we show that A has a prime series: By hypothesis we have A is a right noetheroan R-module

$\Rightarrow$ A satisfies ACC on submodules $i.e$ we have

$A_0 = 0 < A_1 <\cdots< A_n = A$ where A_i is a submodule of A for each

$i = 1,\cdots,n$

Also we have an ideal which is maximal among the annihilators of nonzero submodule of A

$\Rightarrow$ By theorem 1.1, A contains a prime submodule, say A_1, if $A \neq A_1$, A/A_1 contains a prime submodule, say A_2/A_1 . If $A/A_1 \neq A_2/A_1$, A/A_2 contains a prime submodule say, A_3/A_2

Continuing this process we get prime submodules A_i/A_{i-1} $of\ A/A_{i-1}$ for each $i = 1,\cdots,n$

Hence $A_0 = 0 < A_1 <\cdots< A_n = A$ a prime series for A

2. We show that P_i is prime ideal of R for each $i = 1,\cdots,n$: by hypothesis, A_i/A_{i-1} a prime module over R for each $i = 1,\cdots,n$

$\Rightarrow$ By lemma 1.2, $P_i = ann_R(A_i/A_{i-1})$ is prime ideal of R

3. We show that A_i/A_{i-1} is a fully faithful right (R/P_i)-module: since A_i/A_{i-1} a prime module over R for each $i = 1,\cdots,n$, by definition of prime module it is a fully faithful over $R/ann_R(A_i/A_{i-1})$ for each $i = 1,\cdots,n$

Hence A_i/A_{i-1} is a fully faithful right (R/P_i)-module ■

Theorem 1.4: let A be a module over a ring R, assume that A has a prime series

$A_0 = 0 < A_1 <\cdots< A_n = A$ and set $P_i = ann_R(A_i/A_{i-1})$ for each $i = 1,\cdots,n$. if P is any prime minimal over $ann_R(A)$, there is an index i *such that* $P = P_i$

Proof: assume that A is a right R-module. By theorem 1.3, A_i/A_{i-1} is a fully faithful right (R/P_i)-module for each $i = 1,\cdots,n$

$i.e$ For $i = 1, A_1$ is a faithful right (R/P_1)-module

$\Rightarrow ann_{R/P_1}(A_1) = \{P_1\} \Rightarrow A_1P_1 = 0$ (1)

For$i = 2$, A_2/A_1 is a fully faithful right (R/P_2)-module

$\Rightarrow ann_{R/P_2}(A_2/A_1) = \{P_2\}$

$\Rightarrow \{r + P_2 \in R/P_2 \,|(a_2 + A_1)(r + P_2) = A_1 \text{ for all } a_2 \in A_2\} = \{P_2\}$

$\Rightarrow (a_2 + A_1)P_2 = A_1 \text{ for all } a_2 \in A_2 \Rightarrow a_2P_2 \in A_1 \text{ for all } a_2 \in A_2$

$\Rightarrow A_2P_2 \subseteq A_1 \Rightarrow (A_2P_2)P_1 \subseteq A_1P_1 = 0$ by (1)

$\Rightarrow A_2P_2P_1 = 0$

$\Rightarrow$ by induction on n,we have $AP_nP_{n-1}\cdots P_1 = 0$

Let$x \in P_nP_{n-1}\cdots P_1$, then $ax \in AP_nP_{n-1}\cdots P_1 \text{ for all } a \in A$

$\Rightarrow ax = 0 \text{ for all } a \in A$

$\Rightarrow x \in ann_R(A)$ (because $x \in R$)

Hence $P_nP_{n-1}\cdots P_1 \subseteq ann_R(A) \subseteq P$

Since P is prime, $\exists i \ni P_i \subseteq P$ (2)

On the other hand let $r \in ann_R(A)$. Then $ar = 0 \; for \; all \; a \in A$

$\Rightarrow a_i r = 0 \; for \; all \; a_i \in A_i$

Since $a_i r = 0 \in A_{i-1}$, $(a_i + A_{i-1})r = A_{i-1} \; for \; all \; a_i \in A_i$

$\Rightarrow r \in ann_R(A_i/A_{i-1}) = P_i$

Hence $ann_R(A) \subseteq P_i$ for each $i = 1, \cdots, n$

Since P is prime minimal over $ann_R(A)$, $P \subseteq P_i$ (3)

From (2) and (3), $P = P_i$ ■

CHAPTER TWO

BIMODULE AND AFFILIATED PRIME IDEALS

2.1 Definition and examples of Bimodules

Definition: Let R and S be rings. An (R,S)-Bimodule is an abelian group A equipped with a left R-module structure and a right S-module structure (both utilizing the given addition) such that $r(as) = (ra)s$ for all $r \in R, a \in A, s \in S$. The symbol $_RA_S$ is used to denote this situation.

-An (R,S)-sub- bimodule of A (or just a sub- bimodule,if R and S are clear from the context) is any sub group of A which is both a left R-submodule and a right S-submodule.

Note that: - if B is a sub bimodule of A ,the factor group A/B is a bimodule in the obvious manner.

-An (R,R)-bimodule is known as an R-bimodule.

Examples - If R is a ring, then R itself is an R-bimodule and so is R^n (the n-fold direct product of R)

-Any two-sided ideal of a ring R is an R-bimodule.

-Any module over a commutative ring R is automatically an R-bimodule.

-If A is a left R-module, then A is an (R,Z)-bimodule where Z is the ring of integers .Similarly ,right R-module may be interpreted as (Z,R)-bimodule.and indeed an abelian group may be treated as a (Z,Z)-bimodule.-If R is a subring of S,then S is an R-bimodule.It is also an (R,S) and an (S,R)-bimodule.

2.2 Noetherian Bimodules

By a "Noetherian bimodule " is usually meant a bimodule ${}_RA_S$ which not only has the ascending chain condition (ACC) on sub- bimodules , but also is noetherian as left R-module and as a right S-module.

Lemma 2.1 Let ${}_RA_S$ be a bimodule and B a right S-submodule of A such that ${}_R(RB)$ is finitely generated. (The latter hypothesis is automatically satisfied if ${}_RA$ is noetherian.) If $I = r.ann_S(B)$,there exists $n \in \boldsymbol{N}$ such that S/I is isomorphic to a (right) submodule of B^n.

<u>Proof:</u> given that $RB = \{\sum_{finite} rb : r \in R, b \in B\}$ is finitely generated.

Let $\{x_1,...x_n\}$ be generated set of RB .Then

$RB = \sum_{i=1}^{n} Rx_i = Rx_1 + \cdots + Rx_n$

$= R\sum_{finite} rb + \cdots + R\sum_{finite} rb \qquad since\ x_i \in RB,\ i = 1, . . . n$

$= \sum_{finite} Rrb + \cdots + \sum_{finite} Rrb$

$= \sum_{finite} Rb + \cdots + \sum_{finite} Rb$ Since RB is a left R-module

$= R\sum_{finite} b + \cdots + R\sum_{finite} b$

$= Rb_1 + \cdots + Rb_n \qquad since\ (\sum_{finite} b) \in B$

$= B$

Hence B is a left R-submodule of A generated by $b_1, \cdots, b_n$ and hence B is an (R,S)-sub bimodule of A and so is B^n

Also RB is a direct sum of $Rb_1, \cdots, Rb_n$

-Since RB has finite summands,$Rb_1 + \cdots + Rb_n = Rb_1 \times \cdots \times Rb_n$

Now $I = r.ann_S(R) = r \cdot ann_S(RB) = r \cdot ann_S(Rb_1 \times \cdots \times Rb_n)$

$=\{s \in S | (rb_1 \cdots rb_n)s = o\ for\ all\ (rb_1, \cdots, rb_n) \in Rb_1 \times \cdots \times Rb_n\} \quad o \neq r \in R$

$= \{s \in S | ((rb_1)s, \cdots, (rb_n)s) = o\}$

$= \{s \in S | (r(b_1 s), \cdots, r(b_n s)) = o\}$ Since B is an (R,S)-bimodule

$= \{s \in S \mid b_1 s = \cdots = b_n s = 0\}$ Since $r \neq o$

$= r.ann_S(b_1) \cap \cdots \cap r.ann_S(b_n)$

Define a map (from a right S-module S to a right S-module B^n)

$f: S \longrightarrow B^n$ by $s \longmapsto (b_1 s, \cdots, b_n s)$ Then

1.*We show that f is well defined: Let* $s_1, s_2 \in S$ *with* $s_1 = s_2$.

Then$f(s_1) = (b_1 s_1, \cdots, b_n s_1) = (b_1 s_2, \cdots, b_n s_2) = f(s_2)$

$\Rightarrow f$ Is well-defined

2. We show that f is a right S-module homomorphism: For any $s_1, s_2 \in S$, $f(s_1 + s_2) = (b_1(s_1 + s_2), \cdots b_n(s_1 + s_2)) = (b_1 s_1 + b_1 s_2, \cdots, b_n s_1 + b_n s_2)$

$= (b_1 s_1, \cdots b_n s_1) + (b_1 s_2, \cdots b_n s_2) = f(s_1) + f(s_2)$

And for an scalar $\alpha \in S, f(s\alpha) = (b_1(s\alpha), \cdots, b_n(s\alpha))$

$= ((b_1 s)\alpha, \cdots, (b_n s)\alpha) = (b_1 s, \cdots, b_n s)\alpha = f(s)\alpha)$

$\Rightarrow f$ is right S-module homomorphism

3.We show that $\ker(f) = I$: $\ker(f) = \{s \in S | f(s) = o = (o, \cdots, o) \in B^n\} = \{s \in S | (b_1 s, \cdots, b_n s) = (o, \cdots, o)\} = \{s \in S | b_1 s = \cdots, = b_n s = o\} = r.ann_S(b_1) \cap \cdots \cap r.ann_S(b_n) = I$

4.We show that $im(f) = B^n$:
$im(f) = \{y \in B^n | y = f(s) \text{ for some } s \in S\} = \{y \in B^n | y = (b_1 s, \cdots, b_n s) \in B \times \cdots B\} = \{y \in B^n | y = (b_1, \cdots, b_n)s\} = \left\{y \in B^n \middle| s = \frac{y}{(b_1, \cdots, b_n)}\right\} = B^n$

Since for arbitrary $y \in B^n$ *we have* $s \in S$ *(i.e f is onto)*

Finally by FTH, we have $S/I \cong B^n$ as a right S-module ■

Theorem 2.2: Le $_RA_S$ be a bimodule and B a right S-submodule of A such that $_R\,(RB)$ is finitely greeted.Set $I = r.ann_S(B)$. *If* B_S is artinian(noetherian), then S/I is right artinian(noetherian).

Proof: By lemma 2.1 $\exists n \in N$ *s.t* $S/I \cong B^n$ as a right S-module

Since B_s is artinian(noetherian),we have ${B^n}_S$ is artinian(noetherian).

Hence S/I is right artinian (noetherian). ■

Lemma 2.3: Let $_RA_S$ be a bimodule such that $_R\,A$ is noetherian and S is a prime right Goldie ring. Let C be a sub bimodule of A and B a right S-submodule containing C. If B/C is torsion as a right S-module, there is a nonzero ideal I of S such that $BI \subseteq C$.

Proof: $_R\,A$ is noetherian $\Rightarrow$ $_R(A/C)$ is noetherian . .. .(1)

Since B is a right S-submodule of A, B/C is a right S-submodule ofA/C (2)

Consider $I = r.ann_S(B/C)$.

As (1) and (2), by lemma 2.1, we see that there is $n \in \boldsymbol{N}$ such that, $S/I \cong (B/C)^n$ as a right S-module. (3)

Observe that $(B/C)^n$ *is* right S-submodule ofB/C

Since $(B/C)_S$ is torsion and S is prime right Goldie ring, ${(B/C)_S}^n$ is torsion (see chapter -1).

Hence S/I is torsion as right S-module by (3)

We need to show $I \neq (o)$.

Suppose $I = (o)$.then S is torsion as right S-module

$\Rightarrow sc = (o)$ *for all* $s \in S$ *and for some regular element* $c \in S \Rightarrow sc = (o)$

$\Rightarrow \langle c \rangle = (o) \Rightarrow c = o \rightarrow\leftarrow$ .Thus $I \neq (o)$

Finally for any $o \neq r \in I, we\ have\ (b + C)r = C for\ all\ b \in B$

$\Longrightarrow br \in C \mapsto Br \in C$

$\Longrightarrow BI \subseteq C$ ∎

Theorem 2.4 Let $_RA_S$ be a bimodule , where R is a prime left noetherian ring and S is a prime right noetherian ring ,and suppose that the module $_R\ A\ and\ A_S$ are both finitely generated and torsion free. Let B and C be sub-bimodules of A with$B \supseteq C$ .Then the following condition are equivalent;

a) B/C is $torsion$ as a right S-module
b) B/C is $torsion$ as a left R-module
c) There is a nonzero ideal I of S such that$BI \subseteq C$.$(that\ is, B/C$ is an unfaithful right S-module)
d) There is a non zero ideal J of R such that $JB \subseteq C$. $(that\ is, B/C$ is an unfaithful left R-module)
e) C is essential as a right S-submodule of B
f) C is essential as a left R-submodule of B

Proof $(a \Longrightarrow c)$ suppose B/C is $torsion$ as a right S-module

From R is noetheria and $_R\ A\ finitely\ generated$, we have $_R\ A$ is noetherian.

Since every right noetherian ring is right Goldie, S is a right Goldie ring whence

S is a prime right Goldie ring.

Thus by lemma 2.3 there is a nonzero ideal I of S such that$BI \subseteq C$.$(that\ is, B/C$ Is an unfaithful right S-module.

$(c \Longrightarrow f)$ Suppose there is a nonzero ideal I of S such that $BI \subseteq C$

Since I is non zero ideal in the prime right Goldie ring S, it containing a regular element c .Then by hypothesis $Bc \subseteq C$ (1)

SinceA_S *is torsion free*, B is *torsion free* as a right S-module(because S is a semi prime right Goldie ring (see chapter-1)

Hence $for\ any\ o \neq b \in B\ , bc \neq o$ for all regular element c of S

Thus we have a monomorphism $f: B \rightarrow B\ by\ b \mapsto bc$ where $b \in B$, c regular element c of S

Since R is a prime left Goldie ring, S is a prime right Goldiering *and* B_S is finitely generated torsion free, we have B has finite rank and hence $f(B) \leq_{e,R} B$ (see chapter-1) $Bc \leq_{e,R} B$ whence $,_R C \leq_{e,R} B$ by(1)
$(f \Rightarrow b)$ suppose,$_R C \leq_{e,R} B$ .Then since C is a submodule of a right S-module B and,$_R C \leq_{e,R} B$, B/C is *torsion* as a left R-module

Finally $(b \Rightarrow d), (d \Rightarrow e) and\ (e \Rightarrow a)$ are holds by symmetry ■

2.3 Affiliated prime ideals

If $_RA_S$ nonzero noetherian bimodule ,then the right module A_S and the left module $_R A$ have prime series, by theorem 1.3

Such a "right prime series" and a "left prime series" that coincide, but unless we choose them carefully, such series need not even consist of sub-bimodules of A.

When such series consists of sub-bimodules of A?

A_S has prime series implies A_i/A_{i-1} prime right S-submodule of A/A_{i-1} for each i=1, . . .,n. particularly ,for i=1, A_1 *is* prime right S-submodule of A.

By lemma 1.2 there is prime ideal $P_1$of S such that $P_1 = ann_S(A_1)$

Also by theorem 1.1 $ann_A(P_1)$ is prime right S-submodule of A.

Moreover $ann_A(P_1)$ is a sub-bimodule of A.

Hence $A/ann_A(P_1)$ is non zero noetherian bimodule

Thus $A/ann_A(P_1)$ has prime series for A_S by theorem 1.3

Let $B := ann_A(P_1)$. Then $({}^{A_i}/_B)\big/({}^{A_{i-1}}/_B)$ is prime right S-submodule of $({}^{A}/_B)\big/({}^{A_{i-1}}/_B)$.

Again by lemma 1.2, there is a prime ideal P of S such that $P = ann_S({}^{A_i}/_B)\big/({}^{A_{i-1}}/_B)$

$\Longrightarrow$ by third isomorphism theorem$P = ann_S(A_i/A_{i-1})$

Hence $ann_{A/A_{i-1}}(P)$ is prime right S- submodule $of\ A/A_{i-1}$. Morover,$ann_{A/A_{i-1}}(P)$Is a sub bimodule of A/A_{i-1}.

Thus A_i/A_{i-1} is a sub-bimodule of A/A_{i-1}

Hence A_i Is a sub-bimodule of A for each i=1, . . . n

Therefore A_S consists a sub- bimodule of A.

Definition: 1.Let A be a nonzero module over a ring R. An affiliated submodule of A is any submodule of the form $ann_A(P)$, where P is an ideal of R maximal among the annihilators of nonzero submodule of A.

We note that such an ideal P is associated prime of A by theorem 1.1 and that P equals the annihilator in R of the affiliated submodule $ann_A(P)$

2. An affiliated series for A is a series of submodules of the form

$$A_0 = 0 < A_1 < \cdots < A_n = A$$

Where for each i=1. . . n the module A_i/A_{i-1} is an affiliated submodule $of\ A/A_{i-1}$.

If $P_i = ann_R(A_i/A_{i-1})$, then the list $P_1,\cdots,P_n$ is the list of affiliated primes of A corresponding to the given affiliated series. In general an affiliated prime of A is a prime ideal of R which appears in the list of affiliated primes corresponding to some affiliated series of A.

Theorem 2.5: If A is a nonzero finitely generated right module over a right noetherian ring R, then A has an affiliated series. If $A_0 = 0 < A_1 < \cdots < A_n = A$ is such an affiliated series, and $P_1, \cdots, P_n$ are the corresponding affiliated primes, then each A_i/A_{i-1} is a fully faithful right (R/P_i)-module, and each $A_i = ann_A(P_i P_{i-1} \cdots P_1)$.In particular, any affiliated series for A is a prime series .

Proof: 1.we show that A has an affiliated series: by theorem 1.3, we have

$A_0 = 0 < A_1 < \cdots < A_n = A$ With A_i/A_{i-1} is prime submodule of A/A_{i-1} ,

$P_i = ann_R(A_i/A_{i-1})$ Is a prime ideal of R and $A_i$is submodule of A for each i= 1, . . .n

-since R is noetherian , P_i is maximal among annihilater of nonzero submodule of A/A_{i-1} for each i

Hence by theorem 1.1 , $A_i/A_{i-1} = ann_{A/A_{i-1}}(P_i)$ *. it follows* A_i/A_{i-1} is affiliated submodule of A/A_{i-1} for each i

Therefore A has an affiliated series

2. We show that A_i/A_{i-1} is fully faithful: by theorem 1.3 clearly A_i/A_{i-1} is fully faithful

3. We show that each $A_i = ann_A(P_i P_{i-1} \cdots P_1)$: Given that $P_i = ann_R(A_i/A_{i-1})$ .by definition of an affiliated submodule ,we have

$A_1/A_0 = A_1 = ann_A(P_1), \quad A_2/A_1 = ann_{A/A_1}(P_2),$

Let a be arbitrary element ofA_2. Then $a + A_1 \in A_2/A_1 \Rightarrow (a + A_1)P_2 = A_1$ *for all* $p_2 \in P_2$

$\Rightarrow aP_2 + A_1 = A_1 \Rightarrow aP_2 \in A_1$ *for all* $p_2 \in P_2$

$\Rightarrow aP_2 \subseteq A_1 = ann_A(P_1)$

Hence $A_2 = \{a \in A | aP_2 \subseteq ann_A(P_1)\}$ (1)

Now, we claim that $A_2 = ann_A(P_2 P_1)$

Let $a \in A_2$. Then $aP_2 \subseteq A_1 \Rightarrow ann_R(A_1) \subseteq ann_R(aP_2)$

$\Rightarrow P_1 \subseteq ann_R(aP_2) \Rightarrow p_1 \in ann_R(aP_2)$ *for all* $p_1 \in P_1$

$\Rightarrow (ap_2)p_1 = 0$*for all* $ap_2 \in aP_2$

$\Rightarrow (ap_2)p_1 = 0 \; for \; all \; p_1p_2 \in P_1P_2$

$\Rightarrow a \in ann_A(P_2P_1)$

Hence $A_2 \subseteq ann_A(P_2P_1)$ (2)

Let a$\in ann_A(P_2P_1)$.then $a \in ann_A(P_2)$ since $P_2P_1 \subseteq P_2$

$\Rightarrow ap_2 = 0 \; for \; all \; p_2 \in P_2 \Rightarrow aP_2 = 0$

Since $P_1 \neq R, ann_A(P_1) \neq (o)$

Hence $aP_2 \subseteq ann_A(P_1) \Rightarrow a \in A_2$

Thus $ann_A(P_2P_1) \subseteq A_2$ (3)

From (2)and (3) , $A_2 = ann_A(P_2P_1)$

Continuing by induction, we conclude that

$A_i = ann_A(P_iP_{i-1} \cdots P_1)$ for each i ∎

Theorem 2.6: Let R be right notherian ring and A be a nonzero finitely generated right R-module, and let $P_1, \cdots, P_n$ be affiliated primes corresponding to some affiliated series for A .Then ,if P is a prime minimal over $ann_R(A)$,there is an index i such that $P = P_i$

Proof : Since any affiliated series for A is also prime series this result follows from theorem 1.4. ∎

Remark -Associated primes are always annihilator primes

-All annihilator primes are affiliated primes

-All affiliated primes need not be annihilators, unless for a finitely generated module over a commutative notherian ring.

- Annihilator primes need not be associated

We next consider affiliated submodules and affiliated primes in the context of bimodules .

Definition: Let $_RA_S$ be a bimodule .a right affiliated sub-bimodule of A is any affiliated sub-module for A_S, a right affiliated series for A is any affiliated series $A_0 = 0 < A_1 < \cdots < A_m = A$ for A_S,and the corresponding affiliated prime ideals $P_i = r.ann_S(A_i/A_{i-1})$, are called right affiliated primes of A (in case of need ,one can view a zero bimodule as having a right affiliated series with length zero and no affiliated primes.)

Left affiliated sub-bimodule, series and primes are defined analogously.

Theorem 2.7 Let $_RA_S$ be a nonzero bimodule such that $_R A$ is noetherian and S is right notherian.then there exists a right affiliated series

$A_0 = 0 < A_1 < \cdots < A_m = A$ for A. If $P_1, \cdots, P_m$ are the corresponding right affiliated primes ,then each A_i/A_{i-1} is a torsion free right (S/P_i)-module.

Proof: 1. we show that A has a right affiliated series: As $_{,R} A$ is noetherian, by lemma 2.1 we see that $S/I \cong A^n$ is a right S-module $\Rightarrow (S/I)_S$ is noetherian (because S is right noetherian)

$\Rightarrow A^n$ is noetherian as a right S-module$\Rightarrow A_S$ is noetherian

Hence A_S is finitely generated

Thus by theorem 2.5 A_S has an affiliated series

Hence A has a right affiliated series

2. We show that A_i/A_{i-1} is a torsion free right (S/P_i)-module: it suffices to show that A_1 is a torsion free right (S/P_1)-module

Now $since$ A_1 is faith full as a right (S/P_i)-module, A_1 is an $(R, S/P_1) - bi$module

Let B be the torsion submodule of $A_1 as\ a$ right (S/P_i)-module.Then since A_1 is a right S-module and B⊆ A_1, B is a right S-module and RB is left R-submodule of A_1

$\Rightarrow_{,R} (RB)$ Is finitely generated

$\Rightarrow_{,R} (RB) =_{,R} B$ .Hence B is a $(R, S/P_1)$ −sub bimodule of A_1

Observe that -P_1 is prime ideal$\Rightarrow S/P_1$ is prime ring

-S is right noetherian$\Longrightarrow S/P_1$ is right noetherian

Thus S/P_1 is prime right noetherian ring and hence S/P_1 is prime right Goldie ring

We need to show that $B = (0)$

Suppose $B \neq (0)$.Then as $B/(0)$ is torsion as a right (S/P_1)-module, by lemma 2.3 there is an ideal $(0) \neq I$ *of* $^S/_{P_1}$ *such that* BI =(0)

$\Rightarrow I = J/P_1$ Such that J *is an ideal of* S *and* $P_1 \subseteq J$ *and* $bx = 0$ *for all* $b \in B$ and $x \in I$

$\Rightarrow b(y + P_1) = 0$*for all* $b \in B$ *where* $x = y + P_1$ *for some* $y \in I$

$\Rightarrow y + P_1 \in ann_{S/P_1}(B)$

Observe; A_1 is fully faithful as a right (S/P_1)-module and B is a sub-module of A_1 as a right (S/P_1)-module $\Rightarrow ann_{S/P_1}(B) = \{P_1\}$

Thus $y + P_1 \in ann_{S/P_1}(B) = \{P_1\}$

$\Rightarrow y \in P_1$

Hence $P_1 = J$, *it follows* $I = \{P_1\} \rightarrow\leftarrow$

Thus $B = (0)$

Therefore A_1 is a torsion free right (S/P_1)-module. ■

Corollary 2.8: let ${}_RA_S$ be a nonzero bimodule such that ${}_R A$ and A_S are finitely generated. If R is left noetherian and S is right noetherian, there exists a chain of sub bimodules

$$A_0 = 0 < A_1 < \cdots < A_m = A \qquad (\alpha)$$

Such that the ideal $Q_i = l.ann_R(A_i/A_{i-1})$ and $P_i = r.ann_S(A_i/A_{i-1})$ are prime, and A_i/A_{i-1} is torsion free both as a left (R/Q_i)-module and as a right (S/P_i)-module, for each I = 1, . . . , m

Inparticular, the chain (α) is both a prime series for A_S and a prime series for ${}_RA$

Proof: it suffices to show that A contains a nonzero sub- bimodule B such that $l.ann_R(B)$ and $r.ann_S(B)$ are prime ideals and B is torsion free on the left over $R/l.ann_R(B)$ and on the right over $S/r.ann_S(B)$.

Since by theorem 2.7 A has right affiliated series, there exists a right affiliated sub-bimodule of A (say C) and since C is an (R, S)-bimodule ,C has left affiliated series .it follows there exists a left affiliated sub-bimodule of C (say B)

Moreover, if $P = r.ann_S(C)$ and $Q = l.ann_R(B)$, then

I. P is maximal among right annihilator of nonzero sub- module of A and Q is maximal among left annihilator of nonzero sub- module of C
II. P and Q are prime ideals of S and R respectively
III. C is a torsion free right (S/P)-module and B is a torsion free left (R/Q)-module

In addition $B_S \subseteq C_S \Rightarrow P = r.ann_S(C) \subseteq r.ann_S(B)$

Hence by maximality of P, $r.ann_S(B) = P$

Thus B is a torsion free right (S/P)-module ■

Note that : comparing theorem 2.7 with theorem 2.5 ,we see that the advantage of working with noetherian bimodule as opposed to a noetherian module is that in bimodule case we obtain sub-modules which are torsion free, rather than just fully faithful ,modules over prime factor rigs

Theorem 2.9: ${}_RA_S$ be a nonzero bimodule, where R is a left noetherian ring and S is a prime right noetherian ring, and suppose that the module ${}_RA$ and A_S are both finitely generated and that A_S is torsion free.

Let $A_0 = 0 < A_1 < \cdots < A_m = A$ be a left affiliated series for A .then each factor A/A_{i-1} ($i = 1,\cdots,m$) is a torsion free right S-module.In particular $r.ann_S(A_i/A_{i-1}) = 0$

Proof: It suffices to show that $(A/A_1)_S$) is torsion free. Set $P_1 = l.ann_R(A_1)$ so that $A_1 = r.ann_A(P_1)$ .If $a \in A \; and \; ax \in A_1$ for some regular element $x \in S, then \; P_1ax = 0$. since A_S is torsion free, $P_1a = 0$,and so $a \in A_1 \Rightarrow A/A_1 = \{A_1\}$

Hence A/A_1 is torsion free as a right S-module ■

2.4 Artinian Bimodules

Like noetherian bimodule , artinian bimodule is usually meant a bimodule $_R A_S$ which not only has the DCC on sub-bimodules but also is artinian as a left R-module and as a right S-module .

Lemma 2.10: Let $_R A_S$ be a bimodule such that S is a semiprime right Goldie ring and A_S is torsion free .let Q be the right Goldie quotient ring of S

a) If $_R A$ has finite length ,then A_S is divisible
b) If A_S is divisible ,its right S-module structure extends to a right Q- module structure and A becomes an (R,Q)-bimodule.

Proof: a) By corollary of Goldie's regular element lemma S contain regular element, say c

Consider the map $f: A \longrightarrow A \; by \; a \mapsto ac$

Clearly f is well-defined

Ker(f) $= \{a \in A | f(a) = 0\} = \{a \in A | ac = 0\} = \{0\}$ since c is regular element

$\Rightarrow f$ is injective .it follows $A \cong f(A) = Ac$ (1)

Also f is left R-module endomorphism of A (because A is a bimodule)

⇒Ac is a left R-sub-module of A (2)

Since $_R$A has finite length, by (1) Ac has the same length as A (3)

From (2) and (3), $Ac = A$

Since c is arbitrary $Ac = A \; for \; all \; regular \; element \; c \; of \; S$

Therefore by definition A_S is divisible.

b) from chapter-1 A has a unique right Q-module structure compatible with its right S-module structure

$\Rightarrow A_S$ *extends to* A_Q .Next we need to show A is an (R, Q)-bimodule

Given $r \in R, a \in A$ $q \in Q$.we can write $q = sc^{-1}$ for some $s, c \in S$ with c regular

Now $[r(aq)]c = r[(aq)c]$ since A is a right Q-module and also an (R,S)-bimodule

$= r[aqc]$

$= r[asc^{-1}c] = r[as] = (ra)s \Rightarrow r(aq) = [(ra)s]c^{-1} = (ra)sc^{-1} = (ra)q$

Therefore A is an (R, Q)-bimodule. ∎

Lemma 2.11: let ${}_RA_S$ be a bimodule such that ${}_RA$ and A_S are finitely generated. Assume that S is a semiprime right noetherian ring and that A is a torsion free,faithful, divisible right S-module.then S is semisimple ring and A_S is semisimple module with finite length.

<u>Proof:</u> Given that S= semiprime right noetherian ring (1)

${}_RA$ = finitely generated (2)

A_S= finitely generated (3)

A_S= torsion free (4)

A_S= faithful (5)

A_S= divisible (6)

From (1) we have S is semiprime right Goldie ring (7)

⇒ S has right Goldie quotient ring

Let Q be the right Goldie quotient ring of S. Then by lemma 2.10 A is an (R, Q)-bimodule

We claim that A_Q is faithful

Since by (5), $r.ann_S(A) = 0$ *for any* $s \in S, as = 0$ *for all* $a \in A \Rightarrow s = 0$ (8)

Let $q \in r.ann_Q(A)$. *Then* $aq = 0$ *for all* $a \in A$

$\Rightarrow a(sc^{-1}) = 0\ for\ all\ a \in A \Rightarrow (as)c^{-1} = 0\ for\ all\ a \in A$

$\Rightarrow as = 0\ for\ all\ a \in A$

From (8), we have $s = 0$

Since q is arbitrary, $r.ann_Q(A) = 0$

Hence A_Q is faithful (9)

As $_{,R}A$ noetherian, by lemma 2.1 we see that $(Q/I)_Q \cong A_Q{}^n for\ some\ n \in N$

But from (9),$I = r.ann_Q(A) = 0$

Hence $(Q)_Q \cong A_Q{}^n$ (10)

From (1) and (3), A_S is noetherian $\Rightarrow Q_S$ is noetherian

Now given any regular element $z \in S, the\ chain\ S \leq z^{-1}S \leq z^{-2}S \leq \cdots$ of right S-submodule of Q must terminate i.e $\exists m \in N\ such\ that\ z^{-m}S = z^{-(m+1)}S \Rightarrow z^{-m}S = z^{-m-1}S \Rightarrow S = z^{-1}S\ (multiplying\ by z^m) \Rightarrow z^{-1} \in S$

Thus all regular elements of S are invertible, whence $Q = S$ (11)

Since Q has identity element, every right ideal of Q is regular (12)

Also by (7), there are only finitely many distinct maximal ideal in Q and there intersection is 0 (see chapter-1) (13)

From (12) and (13) , the intersection of all regular maximal right ideal of Q is 0

Hence by definition Q is semisimple ring

$\Rightarrow Q_Q$ is simisimple module it follows by (10) , $A_Q{}^n$ is simisimple module

$\Rightarrow A_Q$ is simisimple module

Also by (11), A_S is simisimple module

Therefore S is semisimple ring

Finally since S is nonzero unitary semisimple right S-module, S is semisimple right artinian (see chapter-1)

$\Rightarrow by\ (3)\ A_S$ is artinian

Thus A_S is both noetherian and artinian

Therefore A_S has finite length ∎

Theorem 2.12: (Lenagan) let ${}_RA_S$ be a bimodule such that ${}_RA$ has finite length and A_S is noetherian. Then A_S has finite length.

Proof: we may assume that $A \neq 0$, and we may replace A by B from theorem 2.2 we have S/I is right noetherian where $I = r.ann_S(A)$

$\Rightarrow S$ is right noetherian (1)

Also by theorem 2.7, there exists a right affiliated series

$A_0 = 0 < A_1 < \cdots < A_m = A$ for A and for each i=1. . . m , the ideal $P_i = r.ann_S(A_i/A_{i-1})$ is a prime ideal of S, A_i/A_{i-1} is torsion free and faithful right S/P_i-module (2)

${}_R(A_i)$ is both noetherian and artinian since ${}_RA$ has finite length

$\Rightarrow_{,R} (A_i/A_{i-1})$ is both noetherian and artinian .it follows ${}_{,R}(A_i/A_{i-1})$ has finite length (3)

Since P_i is prime ideal of S, S/P_i is prime ring.

$\Rightarrow\ S/P_i$ is semiprime ring. (4)

From (2) and (3), A_i/A_{i-1} is left R-module and right S/P_i –module. Also for $r \in R, a_i \in A_i and\ s \in S, [r(a_i) + A_{i-1}](s + P_i) = [(ra_i) + A_{i-1}](s + P_i)$

$= (ra_i)s + A_{i-1}P_i \quad = r(a_is) + A_{i-1}P_i$ since A is an (R,S)-module

$= r[a_is + A_{i-1}P_i] \ = r[(a_is + A_{i-1})(s + P_i)]$

Thus A_i/A_{i-1} is a $(R, S/P_i)$ –bimodule (5)

From (1) and (2), S/P_i is semiprime right noetherian ring (6)

Also from (3), we have $_R\,(A_i/A_{i-1})$ is finitely generated (7)

And since $P_i = r.ann_S(A_i/A_{i-1})$, by lemma 2.1 $\exists n \in N$ *such that* $(S/P_i)_S \cong (A_i/A_{i-1})_S{}^n$ (8)

Observe that: let $T = S/ann_S\,(A)$. then the T-submodules of A are precisely the same as the S-submodule of A.

Since A_i/A_{i-1} is both the (S/P_i) *and the* $S -$ *submodule of* A_i/A_{i-1},
$(A_i/A_{i-1})_{S/P_i} = (A_i/A_{i-1})_S$

Thus (8) implies $(S/P_i)_{S/P_i} \cong (A_i/A_{i-1})_{S/P_i}{}^n$

Since $(S/P_i)_{S/P_i}$ is noetherian, $(A_i/A_{i-1})_{S/P_i}{}^n$ is noetherian.

$\Rightarrow (A_i/A_{i-1})_{S/P_i}$ is finitely greeted.

As (3) we see by lemma 2.10, we have $(A_i/A_{i-1})_{S/P_i}$ is divisible (10)

Also as (2),(5),(6),(7),(9) and (10) we see by lemma 2.11,we have A_i/A_{i-1} has finite length.it follows by (8), $(A_i/A_{i-1})_S$ has finite length.for i=1, . . . m

For i=1, $(A_1)_S$ has finite length

For i=2, $(A_2/A_1)_S$ has finite length.it follows as $(A_1)_S$ has finite length,

$(A_2)_S$ has finite length.

Therefore by induction on" m" $(A_m)_S = (A)_S$ has finite length. ∎

Corollary 2.13: $_R A_S$ be a bimodule which is noetherian on each side.Then $_R\,A$ is artinian if and only if A_S artinian .

Proof: ($\Longrightarrow$) suppose $_R$A is artinian.then we need to show A_S is artinian .

Since $_R$A is noetherian, by assumption $_R$A has finite length and since A_S is noetherian, by theorem 2.12, A_S has finite length

Hence A_S is artinian.

($\Leftarrow$) Suppose A_S is artinian .Then we need to show $_RA$ is artinian

Observe that: theorem 2.12 holds true for interchanging left R-module to right S-module and vice versa

Now since A_S is noetherian, by assumption A_S has finite length and since $_RA$ is noetherian, by theorem 2.12, $_RA$ has finite length .

Hence $_RA$ is artinian. ■

Corollary 2.14: let I be an ideal in a noetherian ring R.Then $_R$ is artinian if and only if I_R is artinian.

Proof: I is an ideal of R$\Longrightarrow$ I is an (R, R)-bimodule and it is both left and right R-submodule of an R-module R

As R is noetherian , both $,_R I$ and I_R *are* noetherian

By Corollary 2.13, we have $,_R I$ is artinian if and only if I_R is artinian. ■

Corollary 2.15: $_RA_S$ be a bimodule which is noetherian on each side.There exists a unique sub-bimodule $B \leq A$ such that B is artinian on each side and B contain all artinian right or left submodule of A.

Proof: let B be the sum of all the artinian right S-submodule of A and

C be the sum of all the artinian left R-submodule of A .Then B is right S-submodule of A and C is left R-submodule of A

As A_S is noetherian,B_S is finitely generated

$\Rightarrow B$ is the sum of finitely many artinian right S-submodule of A (1)

$\Rightarrow B_S$ is artinian

Similarly $_RC$ is artinian

For any $x \in R, (xB)_S$ is an epimorphic image of B_S

$\Rightarrow (xB)_S$ is artinian

$\Rightarrow xB \subseteq B$ Since B is maximal right S-submodule of A (2)

Since $xb \in xB$ $for\ any\ x \in R$, by (2),$xb \in B$

Thus B is left R-module (3)

From (1) and (3),B is a sub-bimodule of A.it follows as A is noetherian on each side and so is$B$.

Similarly C is a sub-bimodule of A which is noetherian on each side

By corollary 2.13 ,$_R B$ and C_S are artinian .it follows B and C have finite length

$\Rightarrow B = C$

Particularly ,$_R B$ is artinian and B contain all the artinian left R-submodule of A

To show uniqueness, let ,$_R D_S$ be sub bimodule of A such that D artinian on each side .Then ,$_R D_S$ has finite length

Thus since $B, D \subseteq A$, $B = D$ ∎

Definition: The sub-bimodule B in Corollary 2.15 is called the artinian radical of A

Theorem 2.16: (Ginn –Moss) let R be anoetherian ring .if $soc(R_R)$ is essential as either a right or a left ideal of R, and then R is an artinian ring.

Proof: if $I = soc(R_R)$, I_R is semisimple

Since I_R finitely generated and semisimple,it is artinian (see chapter-1)

By corollary 2.14 ,$_R I$ is artinian

We now have symmetric hypothesis. Namely, we have an ideal I of R which is artinian on both sides and essential on (at least) one side .Hence it is enough to consider the case that $I_R \leq_e R_R$

If $N = r.ann_R(I)$, then, as $_{,R} I$ is artinian, we see by lemma 2.2 that R/N is left artinian (artinian as left R-module)

On the other hand, $NI = 0$ since $NI \leq I, N \neq R$ and I is simple ,

Since $NI = 0$ and $I_R \leq_e R_R$,N is nilpotent (see chapter-1)

For $i = o, 1, \cdots$ note that $N^{i+1} \leq N^i$

N is left ideal (or left R-submodule) of R $\Rightarrow N^i/N^{i+1}$ is left R module and left R/N −module

Since R/N is noetherian, N^i/N^{i+1} is finitely generated as R/N −module and since R/N is artinian, $_{,R/N} (N^i/N^{i+1})$ is artinian

$\Rightarrow_{,R} (N^i/N^{i+1})$ is artinian

Since N is nilpotent,$N^k = 0$ *for some* $k \in N$ i.e we can put $k = i + 1,\ i = 0,1,\cdots$

$\Rightarrow_{,R} (N^i/N^k) =_{,R} (N^i)$ is artinian

$\Rightarrow_{,R} N$ is artinian

Thus R/N *and* N are left artinian

$\Rightarrow R$ is left artinian

By Corollary 2.14 ,R is right artinian as well

Therefore R is an artinian ring ■

2.5 prime ideals in finite ring extensions

In this section, we will study the relations between prime ideals in rings R and S,where S is an extension ring of R. Suppose that R is noetherian ring and R is a subring of a ring S such that the modules $_{,R} S$ and S_R are finitely generated; this immediately implies that S is noetherian .we can,of course regard S itself as an (R,S)-bimodule or as an (S,R)-bimodule , but there are other bimodules arising from these which give even more information about the relation between the ideal theories of R and S.

If P is a prime ideal of S, then (unlike the commutative case) $P \cap R$ need not be a prime (or even a semi prime) ideal of R .for instance , let $R = \begin{bmatrix} Q & Q \\ 0 & Q \end{bmatrix}$ $and\ S = M_2(Q)$, while $P = 0$.

Lemma 2.17: let R be noetherian ring which is a subring of a ring S such that ${}_RS$ and S_R are finitely generated , and let P be a prime ideal of S .For any prime ideal Q of R which is minimal over $P \cap R$,there exist nonzero bimodules ${}_{R/Q} A_{S/P}$ and ${}_{S/P} B_{R/Q}$ which are finitely generated and torsion free on each side .Similarly if $Q_1 and\ Q_2$ are prime ideals of R which are both minimal over $P \cap R$, there exist nonzero bimodules ${}_{R/Q_1} C_{R/Q_2}$ and ${}_{R/Q_2} D_{R/Q_1}$ which are finitely generated and torsion free on each side .

<u>Proof:</u> without loss of generality we may assume that $P = 0$

-since $f: R \longrightarrow S$ is a ring homomorphism,$f^{-1}(P) = f^{-1}(0) = 0$ is a prime ideal of R

$\Rightarrow P \cap R = 0 \subseteq Q \Longrightarrow Q$ is minimal prime ideal of R over $P \cap R$

$\Rightarrow$ Our assumption is true that is 0 is a prime ideal of S .it follows S is prime ring.

Given a minimal prime Q of R , the first bimodule we need to find is an $(R/Q, S)$-bimodule which is finitely generated and torsion free as a left (R/Q)-module and as a right S-module.We first regard S as (R,S)-module and choose a left affiliated series $S_0 = 0 < S_1 < \cdots < S_n = S$ for this bimodule

As S is noetherian, S is semi prime right Goldie ring. It follows S_S is torsion free (see chapter-1)

By corollary 2.8 and theorem 2.9, each factor S_i/S_{i-1} is a torsion free left (R/L_i)-module, where $L_i = r.ann_R\ (S_i/S_{i-1})$, and a torsion free right S-module.

Q is minimal implies Q is left annihilator i.e Q is minimal over $l.ann_R(S)$.

Hence by theorem 2.6 we must have $Q = L_j$ for some index j .

Thus we obtain the desired bimodule A by choosing $A = S_j/S_{j-1}$.

The bimodule B is obtained by a symmetric argument.

Now suppose that $Q_1 and\ Q_2$ are minimal primes of R .

Let ${}_{,R/Q_1}A_S$ be the bimodule obtained in the previous paragraph, and now regard A as a $(R/Q_1, R)$-bimodule. Since A is faithful right S-module (by theorem 2.9), it is also a faithful right R-module.

Let $A_0 = 0 < A_1 < \cdots < A_m = A$ be a right affiliated series for this bimodule

Since R/Q_1 is semi prime right Goldie ring, ${}_{,R/Q_1}(R/Q_1)$ torsion free

Thus by corollary 2.8 theorem 2.9 again each factor A_i/A_{i-1} is torsion free as a left (R/Q_1) - module and as a right (R/P_j) –module where $P_j = r.ann_R(A_i/A_{i-1})$.

Since Q_2 is minimal, $Q_2$is right annihilator (see chapter-1)

$\Rightarrow Q_2$ is minimal over $r.ann_R(A)$, we again apply theorem 2.6 to see that $Q_2 = P_k$ for some index k.

Therefore we may choose $C = A_k/A_{k-1}$. the corresponding argument (beginning with B as an $(R, R/Q_1)$-bimodule) . ∎

Theorem 2.18 :(Jategaonkar,Letzer) let R and S be prime noetherian rings and ${}_RB_S$ a nonzero bimodule which is finitely generated and torsion free on each side .Then

a) If R is semi primitive , so is S
b) If R is right primitive, so is S

Proof: First observe that, since B is finitely generated and torsion free as a left R-module, ${}_{,R}B$ embeds in a free left R-module (see chapter-1) i.e ${}_{,R}B \cong {}_{,R}R$.

Hence $Hom_R({}_{,R}B, {}_{,R}R) \neq 0$. Next

Let $T = \sum\{f(B) | f \in Hom_R({}_{,R}B, {}_{,R}R)\}$ and observe that T is a nonzero ideal of R, we claim that $B/IB = 0$ only if $(R/I)T = 0$

Given any $f \in Hom_R({}_{,R}B, {}_{,R}R)$, it follow from $B/IB = 0$ $(i.e\ B = IB)$ that $f(B) = f(IB) = If(B) \leq I$

Hence $T \leq I$, and so let xt $such\ that\ x \in R/I\ and\ t \in T\ (i.e\ xt \in (R/I)T)$.

Then $x = r + I$ for some $r \in R$ and $xt = (r + I)t = rt + I = I$

Since xt is arbitrary element of $(R/I)T)$, $(R/I)T) = 0$ as claimed .

a) Let $J = J(S) = \cap_i P_i$ with P_i is a primitive ideal of S ($i.e\ J$ is semi primitive ideal of S)

We need to show that $J = 0$

Suppose not $i.e\ J \neq 0$. Then by theorem 2.2(applied to the sub bimodules $B \supseteq BJ$), there is a nonzero ideal $K\ of\ R\ such\ that\ \ KB \subseteq BJ$ (1)

Next 1. we need to show that: $T \cap K \neq 0$

Suppose not $i.e\ T \cap K = 0$. Then $TK \subseteq T \cap K = 0$

$\Rightarrow$ 0 is primitive ideal of R since R is semi primitive

$\Rightarrow$ 0 is prime ideal of R. It follows R is a prime ring

Now $TK \subseteq T \cap K = 0 \Rightarrow TK = 0 \Rightarrow T = 0\ or\ K = 0 \rightarrow\leftarrow$ with $T, K \neq 0$

Thus $T \cap K \neq 0$

2. we need to show (find) a maximal right ideal $I\ of\ R$ such that I does not contain $T \cap K$.

Suppose not $i.e\ T \cap K \subseteq I$

0 is right primitive ideal of R implies $r.ann_R(A) = 0$ where A is simple right R-module (see chapter-1)

Also R/I is simple right R-module where $I$is maximal right ideal $of\ R$ implies R/I is faithful simple module

$\Rightarrow ann_R(R/I) = 0$

$\Rightarrow \{r \in R | r(a + I) = I\ for\ all\ a \in R\} = 0$

$\Rightarrow \{r \in R | ra \in I\quad for\ all\ a \in R\} = 0$

Hence $ra \in I \Rightarrow r = 0\ for\ all\ a \in R$

Let $r \in K \cap T$.Then $r \in K$ *and* $r \in T \Rightarrow ra \in K$ *and* $ra \in T$ *for all* $a \in R$

$\Rightarrow ra \in K \cap T \subseteq I \Rightarrow ra \in I$ *for all* $a \in R \Rightarrow r = 0$

Hence $K \cap T = 0 \rightarrow\leftarrow$

Thus $K \cap T \nsubseteq I$ .Also $K \nsubseteq I$ (because if $K \subseteq I$, *then* $K \cap T \subseteq K \subseteq I$).It follows immediately that $K + I = R$

$\Rightarrow KB + IB = RB \Rightarrow KB + IB = B$ Since RB is finitely generated (2)

On the other hand ,$(R/I)\,T \neq 0$ (as $I \neq R$).By the claimed proved, above

$B/IB \neq 0$ ($i.e$ $B \neq IB$)

$B \neq IB$ implies we can find a maximal right submodule C in B such that $IB \subseteq C$.

Since B/C is simple right S-module, $(B/C)J = 0$ (3)

Let bj such that $b \in B$ *and* $j \in J$ ($i.e$ $bj \in BJ$). Then

$(b + C)j \in (B/C) = C$ by (3)

$\Rightarrow bj + C = C \Rightarrow bj \in C$

Hence $BJ \subseteq C$ (4)

Combining (1), (2) and (4),we get $B \subseteq C$

Hence $C = B \rightarrow\leftarrow$

Thus $J = 0$. *Therefore S is semi primitive*

b) Since there exists a maximal right ideal I in R such that R/I is a faithful simple module,$(R/I)\,T \neq 0$, and the claim above shows that $B/IB \neq 0$

Let C be maximal right submodule of B containing IB

We claim that the simple right S-module B/C is faithful

Suppose not $i.e$ B/C is not faithful. Then $r.ann_S(B/C) \neq 0$

$\Rightarrow \{s \in S | (b + C)s = C \text{ for all } b \in B\} \neq 0$

$\Rightarrow \exists\, 0 \neq s \in S \text{ s.t } (b + C) = C \text{ for all } b \in B$

$\Rightarrow bs \in C \text{ for all } b \in B$

$\Rightarrow BJ \subseteq C$ for some nonzero ideal J *of* S (1)

Applying theorem 2.4 again, there is a nonzero ideal K *of* R *s.t* $KB \subseteq BJ$ (2)

By proof of part (a), $K \nsubseteq I$ and hence $K + I = R$ *and so* $KB + IB = B$ (3)

Combining (1),(2) and (3),we get $C \subseteq B$

Hence $C = B$ $\rightarrow\leftarrow$

Therefore B/C is faithful simple right S-module

Therefore S is right primitive. ■

Corollary 2.19: let R be noetherian ring which is a subring of a ring S such that $_RS$ *and* S_R are finitely generated, and assume that S is prime ring . Then S is right (left) primitive if and only if at least one minimal prime ideal of R is right (left) primitive, if and only if every minimal prime ideal of R is right (left) primitive.

Proof: we need to show that TFAE

a) S is right (left) primitive
b) at least one minimal prime ideal of R is right (left) primitive
c) every minimal prime ideal of R is right (left) primitive

(a⟹b) since R is noetherian, there exists finitely many minimal prime ideals

Let Q be a minimal prime ideal of R. Then since S is prime, 0 is prime ideal of S, it follows 0 is prime ideal of R

$\Rightarrow 0 = 0 \cap R$ is a prime ideal of R

Hence Q is minimal prime ideal of R over $P \cap R$ where P is prime ideal of S

Thus by lemma 2.17, there are nonzero bimodules $,_{R/Q}A_S$ *and* $,_S B_{R/Q}$ which are finitely generated and torsion free on each side (1)

Since S is right (left) primitive by theorem 2.18, (b) R/Q is right (left) primitive

$\Rightarrow$ zeros of $R/Q = Q$ is right (left) primitive ideal of R

$(b \Rightarrow c)$ Since Q arbitrary, every minimal prime ideal of R is right (left) primitive

$(c \Rightarrow a)$ Q is right (left) primitive ideal $\Rightarrow R/Q$ is right (left) primitive ring

As (1), taking $,_{R/Q}A_S$, we see by theorem 2.18, (b), S is is right (left) primitive ring. ∎

Definition: A Jacobson ring (sometimes called a Hilbert ring) is a ring in which every prime ideal is semi primitive, i.e every prime factor ring has zero Jacobson radical.

Corollary 2.20: (Cortzen-Small) let R be a noetherian ring which is a subring of a ring S such that $_RS$ *and* S_R are finitely generated .if R is Jacobson ring, then so is S.

Proof: Assume $S \neq 0$ and let P be a prime ideal of S. Then we need to show P is semi primitive.

Since R is noetherian, there is a minimal prime ideal Q of R

$\Rightarrow$ Q is semi primitive ideal since R is Jacobson

$\Rightarrow R/Q$ is semi primitive ring (1)

On the other hand by lemma 2.17, we have a nonzero bimodule $,_{R/Q}A_{S/P}$ which is finitely generated and torsion free on each side

As (1) we see by theorem 2.18,(a), S/P is semi primitive ring

Hence P is semi primitive ideal of S

Therefore S is Jacobson ring. ∎

2.6 Bimodule composition series

Most terminology relating to submodules of a module can be directly carried over to sub-bimodules of a bimodule.For instance, a bimodule C_R is simple provided C is nonzero and its only sub-bimodules are 0 and C.

Definition : A bimodule composition series for C is a chain

$C_0 = 0 < C_1 < C_2 < \cdots < C_m = C$ of sub-module of C such that C_i/C_{i-1} for i=1, . . . ,m is a simple bimodule.

Theorem 2.21 : le ${}_TC_R$ be a bimodule such that ${}_TC$ has finite length and let

$C_0 = 0 < C_1 < C_2 < \cdots < C_m = C$ be a bimodule composition series for C.

Let $Q_i = r.ann_R(C_i/C_{i-1})$ for i=1, . . . ,m

a) Each Q_i is a prime ideal of R, and if R/Q_i is right Goldie, then C_i/C_{i-1} is torsion free as a right (R/Q_i)-module
b) If $D_0 = 0 < D_1 < D_2 < \cdots < D_m = C$ is another bimodule composition series for C, then $n = m$, and there exists a permutation π *of* $\{1,2,\cdots,m\}$ such that $r.ann_R(D_i/D_{i-1}) = Q_{\pi(i)}$ for i=1, . . . ,m

<u>Proof:</u> To show Q_i is aprime ideal of R for each $i \in \{1,2,\cdots,m\}$

Let I, J be any ideals of R. Then we need to show $IJ \subseteq Q_I \Rightarrow I \subseteq Q_i \; or \; J \subseteq Q_i$

Or by contra positive, we need to show $I \nsubseteq Q_i \; and \; J \nsubseteq Q_i \Rightarrow IJ \nsubseteq Q_i$

Suppose $I \nsubseteq Q_i \; and \; J \nsubseteq Q_i$. Then we need to show $IJ \nsubseteq Q_i$

Now $I \nsubseteq Q_i \Rightarrow \exists\, a \in I \; s.t \; a \notin Q_i = r.ann_R(C_i/C_{i-1})$

$\Rightarrow \exists\, y \in C_i \; s.t \; (y + C_{i-1}) \neq C_{i-1}$

$\Rightarrow ya + C_{i-1} \neq C_{i-1} \Rightarrow ya \notin C_{i-1}$ (1)

On the other hand, $x = y + C_{i-1} \in C_i/C_{i-1}$ and $a \in I \Rightarrow xa \in (C_i/C_{i-1})I$

But $xa = (y + C_{i-1})a = ya + C_{i-1} \neq C_{i-1}$ by (1)

Hence $xa \notin \{C_{i-1}\}$. Thus $(C_i/C_{i-1})I \nsubseteq \{C_{i-1}\}$

Therefore $(C_i/C_{i-1})I \neq \{C_{i-1}\}$

Similarly $J \nsubseteq Q_i$ implies $(C_i/C_{i-1})J \neq \{C_{i-1}\}$

Since C_i/C_{i-1} is simple bimodul and $(C_i/C_{i-1})I$, $(C_i/C_{i-1})J$ are sub-bimodule of C_i/C_{i-1}, we have $(C_i/C_{i-1})I = C_i/C_{i-1} = (C_i/C_{i-1})J$

Thus, $(C_i/C_{i-1})(IJ) = [(C_i/C_{i-1})I]J = (C_i/C_{i-1})J \neq C_{i-1}$

$\Rightarrow IJ \nsubseteq Q_i$

Therefore Q_i is prime ideal of R .

To show C_i/C_{i-1} is torsion free as a right (R/Q_i)-module

Assume that R/Q_i is right Goldie. Then R/Q_i is semi prime right Goldie ring

$\Rightarrow$ we have a theorem in chapter-1 namely, if A is any right (R/Q_i)-module, then t(A) is a torsion submodule of A as a right (R/Q_i)-module

Since C_i/C_{i-1} is an (R/Q_i)-module (we will see latter), there is a torsion submodule of C_i/C_{i-1} as a right (R/Q_i)-module

Let D be the torsion submodule of C_i/C_{i-1} as a right (R/Q_i)-module. Then since C_i/C_{i-1} is a right R-module, D is a right R-module

Since TD is both left T-module and right R-module and $,_T (C_i/C_{i-1})$ is noetherian , $,_T (TD)$ is finitely generated.it follows $,_T D$ is finitely generated and hence D is left T-module. Thus D is a sub-bimodule of C_i/C_{i-1}

Since C_i/C_{i-1} is simple bimodule, either $D = 0$ *or* $D = C_i/C_{i-1}$

Suppose $D = C_i/C_{i-1}$ .Then C_i/C_{i-1} is torsion as a right (R/Q_i)-module

Since ${}_T(C_i/C_{i-1})$ is finitely generated (say,$\{a_1 + C_{i-1},\cdots, a_n + C_{i-1}\}$ be generating set)
,$x \in C_i/C_{i-1} \Rightarrow x = t_1(a_1 + C_{i-1}) +\cdots +t_n(a_n + C_{i-1})$ where

$x \in C_i/C_{i-1}$,$i = 1,\cdots, n$

As C_i/C_{i-1} is torsion, $x \in C_i/C_{i-1}$ implies $\exists$ *a regular element* $c \in R/Q_i$ $s.t$ $xC = C_{i-1}$
$\Rightarrow [t_1(a_1 + C_{i-1}) +\cdots +t_n(a_n + C_{i-1})]c = C_{i-1}$

$$\Rightarrow (t_1a_1)c + C_{i-1} +\cdots +(t_na_n)c + C_{i-1} = C_{i-1}$$

$\Rightarrow t_1a_1c +\cdots +t_na_nc + C_{i-1} = C_{i-1}$ Since C_i is an (T, R)-bimodule

$\Rightarrow a + C_{i-1} = C_{i-1}$ Where $a = t_1a_1c +\cdots +t_na_nc \in C_i$

$\Rightarrow a \in C_{i-1} \Rightarrow C_i \subseteq C_{i-1}$

Thus $C_i = C_{i-1}$ (2)

On the other hand C_i/C_{i-1} is faithful right (R/Q_i)-module *i.e* $ann_{R/Q_i}(C_i/C_{i-1}) = \{Q_i\}$

Let $r + Q_i \in ann_{R/Q_i}(C_i/C_{i-1})$ Then $r \in Q_i$ for all $r \in R$

$\Rightarrow (a + C_{i-1})r = C_{i-1}$ $a \in C_i$

$\Rightarrow C_{i-1}r = C_{i-1}$ for each r by (2) $\rightarrow\leftarrow$

Hence $D = 0$

Therefore C_i/C_{i-1} is torsion free overR/Q_i.

b) This is immediate from the Jordan-Holder theorem(see chapter-1) which says that n=m and there is a permutation π *of* $\{1,2, \cdots, m\}$ that $D_i/D_{i-1} \cong C_{\pi(i)}/C_{\pi(i)-1}$ for $i = 1,2, \cdots, m$ ■

lemma 2.22: Let ${}_TC_R$ be such that ${}_TC$ has finite length and R is right noetherian. let

$C_0 = 0 < C_1 < C_2 < \cdots < C_m = C$ be a bimodule composition series for C and

$B_0 = 0 < CB_1 < B_2 < \cdots < B_m = C$ be a right affiliated series for C. Let $Q_i = r.ann_R(C_i/C_{i-1})$ for i=1, . . . , m and $P_j = r.ann_R(B_J/B_{J-1})$ for J=1, . . . ,n . Then $\{Q_1, \cdots, Q_m\} = \{P_1, \cdots, P_N\}$

Proof: we may assume that$C \neq 0$. Since the set $\{Q_1, \cdots, Q_m\}$ is independ

ent of the choice of a bimodule composition series for C (by theorem 2.21), we may assume that our bimodule composition series is a refinement of the given right affiliated series.

Thus there exist integers $i(0) = 0 < i(1) < \cdots < i(n) = n$ $s.t$ $C_{i(j)} = B_j$ for j=o . . . n

To prove the lemma it suffices to show that $Q_i = P_j$ for j=1 . . . n and i=i(j-1)+1, . . . i(j)

That is we need only prove the lemma for each of the bimodules B_J/B_{J-1}.

Hence we may assume that the right affiliated series for C is just $B_0 = 0 < B_1 = C$ and we must prove that all $Q_i = P_1$

Since $P_1 = r.ann_R(B_1) = r.ann_R(C) \subseteq Q_i$ for i=1 . . . m

We may replace R by R/P_1.Thus we may assume that R is prime and C_R is faithful, and it remain to show that all $Q_i = 0$

Suppose $Q_i \neq 0$ for some i. Then since R is a prime right Goldie ring, Q_i contains a regular element (see chapter-1)

On the other hand, by theorem 2.7 C is torsion free a right R-module.

As C_i is a bimodule which has finite length on the left, lemma 2.10 say that C_i is divisible as a right R-module, $i.e$ $C_i x = C_i \not\subset C_{i-1}$ for all regular element $x \in R$

Whence the ideal $Q_i = r.ann_R(C_i/C_{i-1})$ contains no regular element →←

Thus $Q_i = 0$

Therefore $Q_i = P_1$ ∎

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