ADDIS ABABA UNIVERSITY

COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCE
DEPARTMENT OF MATHEMATICS

FIXED POINT THEOREMS FOR GENERALIZED (α -ψ-φ-F)-RATIONAL CONTRACTION TYPE MAPPINGS IN α $-$COMPLETE B- METRIC SPACES AND ITS APPLICATIONS IN ORDINARY DIFFERENTIAL EQUATIONS.

By

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Approval Sheet

This is to certify that the thesis titled "**FIXED POINT THEOREMS FOR GENERALIZED (α - $\psi - \varphi$-F)-RATIONAL CONTRACTION TYPE MAPPINGS IN α −COMPLETE B- METRIC SPACES AND ITS APPLICATIONS IN ORDINARY DIFFERENTIAL EQUATIONS**" submitted in partial fulfillment of the requirement for the degree of Master of Science in Mathematics to the Department of Mathematics, Addis Ababa University, and is record of original research carried out by Deme Sani ID.No GSK/0778/09 under my supervision and no part of the thesis has been submitted for another degree or diploma. The assistance and the help received during the course of this investigation have been duly acknowledged.Therefore I recommended that it may be accepted as fulfilling the thesis requirement.

Dula Tolera (PhD)

Name of Advisor Signature Date

Examiner Signature

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Abstract

In this thesis we introduce Fixed point theorems for generalized (α-ψ-φ-F)- rational contraction type mappings in b-metric spaces and prove the existence and unique fixed point theorems for such mappings and also we give its application in ordinary differential equations. Our result Generalizes many fixed point theorems in the literature.

***Keywords**: b-metric space, fixed point, altering distance function, integral equation, admissible mapping. α--complete b-metric spaces, α--continuous mappings, triangular α- admissible mappings , C-class functions, generalized (α-ψ- φ -F)- rational-contraction type mapping.*

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1. Introduction

In the development of nonlinear analysis, fixed point theory occupies a prominent place in many aspects. It has been used in different branches of engineering and sciences. In particular, the famous Banach contraction principle, is very popular tool of mathematics in solving a great deal of problems in several branches of mathematics such as variational and linear inequalities, differentio-integral equation, and approximation theory. To overcome the problem of measure and the convergence induced by measurable functions, Bakhtin [5] or Czerwik [7] introduced an extension of metric space, which is called b-metric space or metric type space, and proved a more general Banach contraction principle in such space. Since then, many authors have been interested in investigating fixed point theorems for single-valued and set valued mappings in b-metric spaces (see [2, 6, 8, 9, 12–18, 20–24]). On the other hand, Khan et al. [11] introduced the concept of altering distance function, which is a control function that alters distance between two points in a metric space. Some mappings will become weak if they act with altering distance functions. Afterwards, many mathematicians obtained fixed point theorems associated with altering distance functions (see [14, 21–24]). Recently, Ansari [3] introduced C-class function in metric spaces and obtained some fixed point results. Subsequently, many scholars were interested in fixed point theorems for C-class function (see [4, 14, 19]). Throughout this paper, inspired and motivated by previous results in the existing literature, we give several fixed point results for C-class functions in b-metric spaces.Otherwise, we apply our results to prove the existence and uniqueness of a solution to ordinary differential equation with initial value conditions. In addition, we deal with the existence and uniqueness of a solution to a class of nonlinear integral equations. In the

sequel, we always denote by $\mathbb{N}, \mathbb{R}, \mathbb{R}^+$ the set of positive integers, real numbers, and nonnegative real numbers, respectively. The following definitions and results will be useful for proving our ma

2. preliminaries

***Definition* 2.1** ([**5**, **7**]) . Let X be a nonempty set and s > 1 be a given real number.

A function $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ is called a b-metric on X if, for $all\ x, y, z \in X$, the following conditions hold:

(b1) $d\ (x\ , y) = 0\ if\ and\ only\ if\ x = y$;

(b2) $d\ (x\ , y) = d\ (y\ , x)$;

(b3) $d\ (x\ , z) \leq s[d(x\ , y) + d(y, z)]$. In this case,

the pair (X , d) is called a b-metric space. It is evident that the class of b-metric space is larger than that of metric space since any metric space is a b-metric space with s = 1.

***Example* 2.1** ([**15**, **21**, **22**]) . Let (X , d) be a metric space and $\sigma_d : X \times X \to \mathbb{R}^+$ be defined by

and $\sigma_d(x , y) = [d(x, y)]^p$ for all x, y ∈ X, where p > 1 is a fixed real number. Then (X , σ_d) is a b-metric space with coefficient $s = 2^{p-1}$. In the following examples, we replace the coefficient $s = 2^{\frac{1}{p}}\ by\ s = 2^{\frac{1}{p}-1}$

***Example* 2.2** ([6,12,21 − 24]) . $Let\ 0 < p < 1\ and\ define\ L_p[a\ , b]\ by$

$$L_p[a\ , b] := \left\{ x(t) \middle| \int_a^b |x(t)|^p\, dt < \infty \right\},$$

where the mapping d : L_p[a , b]× L_p[a , b]→ $\mathbb{R}^+$ is defined

$d(x\,,y) \;=\; \left(\int_a^b |x(t)-y(t)|^p\,dt\right)^{\frac{1}{p}}$ $for\ each\ x \;=\; x(t)\,, y \;=\; y(t) \;\in$

$L_p[a\,,b]. Then\ (L_p[a\,,b], d)\ is\ a\ b$ −metric space with coefficient $s \;= 2^{\frac{1}{p}}$.

In fact, we only need to prove that condition (b_3) in Definition 2.1 holds. To this end,

let x = x(t), y = y(t), z = z(t), we show that

$$\left(\int_a^b |x(t)-y(t)|^p\,dt\right)^{\frac{1}{p}} \le 2^{\frac{1}{p}-1}\left[\left(\int_a^b |x(t)-y(t)|^p\,dt\right)^{\frac{1}{p}} + \left(\int_a^b |y(t)-z(t)|^p\,dt\right)^{\frac{1}{p}}\right]. \tag{1.1}$$

Denote u(t) = x(t) − y(t), v(t) = y(t) − z(t), then x(t) − z(t) = u(t) + v(t), so (1.1) becomes

$$\left(\int_a^b |u(t)+v(t)|^p\,dt\right)^{\frac{1}{p}} \le 2^{\frac{1}{p}-1}\left[\left(\int_a^b |u(t))|^p\,dt\right)^{\frac{1}{p}} + \left(\int_a^b |v(t)|^p\,dt\right)^{\frac{1}{p}}\right]. \tag{1.2}$$

Next we prove (1.2). Noticing the following inequalities,

$(a+b)^p \le a^p + b^p$ (a, b > 0, 0 < p ≤ 1),

$(a+b)^p \le 2^{p-1}(\,a^p + b^p)$ (a, b > 0, 0 < p ≥1) we have

$$\left(\int_a^b |u(t)+v(t)|^p\,dt\right)^{\frac{1}{p}} \;\le\; \left(\int_a^b (|u(t)|+|v(t)|)^p\,dt\right)^{\frac{1}{p}}$$

$$\le \left(\int_a^b (|u(t)|^p + |v(t)|^p)dt\right)^{\frac{1}{p}}$$

$$= \left(\int_a^b (|u(t)|^p\,dt + \int_a^b |v(t)|^p\,dt\right)^{\frac{1}{p}}$$

$$\le\; 2^{\frac{1}{p}-1}\left[\left(\int_a^b |x(t)-y(t)|^p\,dt\right)^{\frac{1}{p}} + \left(\int_a^b |y(t)-z(t)|^p\,dt\right)^{\frac{1}{p}}\right]$$

Example 2.3 ([**21**]) . $Let\ X \;=\; \{0,1,2\}$ and define the mapping $d:\; X \times X \;\to \mathbb{R}^+$

by d(0, 0) = d(1, 1) = d(2, 2) = 0, d(0, 1) = d(1, 0) = d(1, 2) = d(2, 1) = 1

and d(2, 0) = d(0, 2) = m,

where m > 2 is a real number. Then (X, d) is a b-metric space with coefficient s = $\frac{m}{2}$.

Definition 2. 2 ([**21**, **22**]). Let (X, d) be a b-metric space and $\{x_n\}$ a sequence in X. We say that

(1) $\{x_n\}$ b-converges $to\ x \in X\ if\ d(x_n, x) \rightarrow 0\ as\ n \rightarrow \infty$;

(2) $\{x_n\}$ is a b-Cauchy sequence if $d(x_m, x_n) \rightarrow 0\ as\ m, n \rightarrow \infty$;

(3) (X, d) is b-complete if every b-Cauchy sequence in X is b-convergent. Each b-convergent sequence in a b-metric space has a unique limit and it is also a b-Cauchy sequence.

Definition 2. 3 ([**2**]). $Let\ T: X \rightarrow X\ and\ \alpha: XxX \rightarrow [0, \infty). Then\ T\ is\ \alpha - admissibe\ if$

$\alpha\ (x, y) \geq 1 \ \Rightarrow\ \alpha\ (Tx, Ty) \geq 1.$

Definition 2. 4 [2]$Let\ \alpha: XxX \rightarrow [0, \infty). then\ amapping\ T: X \rightarrow X\ is\ atriangular\ \alpha - admissible\ if$

$a)\ T\ is\ \alpha - admissible$

$b)\ \alpha(x, z) \geq 1\ \&\ \alpha\ (z, y) \geq 1 \ \Rightarrow\ \alpha\ (x, y) \geq 1, x, y, z \epsilon X.$

Definition 2. 5 ([**2**]). $Let\ (X, d)\ be\ b - metric\ space\ and\ \alpha : XxX \rightarrow [0, \infty)$ a mapping on X is said to be α – complete if every b- Cauchy sequence $\{x_n\}$ in X with $\alpha\ (x_n, x_{n+1}) \geq 1$ b- converges in X.

Definition 2. 6 ([**14**]).A function ψ **:** [0,∞) → [0,∞) is called an altering distance function if the following properties are satisfied.

a) ψ is non – decreasing and continuous ;
b) $\psi\ (t) = 0 \Leftrightarrow t = 0.$

The family of all altering distance functions denoted by Ψ.

Definition 2. 7[**2**] $Let\ (X, d) be\ a\ metric\ space\ and\ \alpha: XxX \rightarrow [0, \infty)\ Amapping\ T: X \rightarrow X\ is\ said\ to\ be\ \alpha - continous\ mapping\ if\ each\ sequnece\ \{x_n\}\ in\ X\ with\ x_n \rightarrow x$ as

n→ ∞ and $\alpha(x_n, x_{n+1}) \geq 1$ for all nϵN∪{0] implies T$x_n \rightarrow Tx$ as n → ∞.

Definition 2. 8 ([**3**]). A mapping $F : [0, \infty)^2 \to \mathbb{R}$ is called a C-class function if

(i) $F(s, t) \leq s$ *for all* $s, t \geq 0$,

(ii) $F(s, t) = s \Rightarrow$ *either* $s = 0$ *or* $t = 0$,

(iii) $F(s, t)$ *is continuous in its variables*.

Example 2.4. Each of the following functions $F : [0, \infty)^2 \to \mathbb{R}$ is a C-class function:

(1) F(s, t) = s − t;

(2) F(s, t) = λs, where 0 < λ < 1;

(3) F(s, t) = $\frac{s}{(1+t)^r}$, r> 0

Definition 2.9 Volterra integral equation

a) of the **first** kind is expressed a

$$f(t) = \int_a^t k(t, s)\phi(s)ds$$

b) For the **second** it is usually represented as

$$\phi(t) = g(t) + \int_a^t k(t, s)\phi(s)ds$$

Where

- ϕ(t) is the unknown function we want to find
- k (t , s) is the kernel of the integral equation.
- g(t) is a given function.
- [a , t] is the interval in which the equation is defined.

Lemma 2. 1 [**8**] *Let* (X, d) be a b-metric space with coefficient s >1 and T : X → X be a mapping. Suppose that $\{x_n\}$ is sequence in X induced by $x_{n+1} = Tx_n$

such that $d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n)$.

for all $n \epsilon \mathbb{N} \cup \{0\}$ *where* $k \epsilon [0,1)$ *is constant. Then* $\{x_n\}$ *is b − cauchy sequence*.

3. MAIN RESULTS

Let (X, d) be a b-metric space with coefficients s >1 and $\alpha: XxX \rightarrow [0,\infty)$.T: X → X be a mapping. Assume that

$$M_1(x,y) = Max\{ d(x,y), \frac{d(x,Tx)d(y,TY)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}\} \quad (3.1)$$

$$M_2(x,y) = \max\{ d(x,y), \frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{1+s\{d(x,Tx)+d(y,Ty)}, \frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{1+s[d(x,Ty)+d(y,Tx)]}\} \quad (3.2)$$

$$M_3(x,y) = \max\{ d(x,y), \frac{d(x,Tx)d(y,Ty)}{1+s[d(x,y)+d(x,Ty)+d(y,Tx)]}, \frac{d(x,Ty)d(x,y)}{1+sd(x,Tx)+s^3[d(y,Tx)+d(y,Ty)]}\} \quad (3.3)$$

and $N(x,y) = min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$. (3.4)

Definition 3.1

Let (X, d) be a b-metric space with coefficients s >1 and $\alpha: XxX \rightarrow [0,\infty)$. A mapping T: X →X is said to be generalized (α- ψ -φ-F) rational contraction type mapping

$$\text{If } \alpha(x,y) \geq 1 \Rightarrow \psi(s^K d(Tx,Ty)) \leq F(\psi(M_i(X,Y)), \varphi(M_i(x,y))) + \beta N(x,y) \quad (3.5)$$

For all x , y ϵX , where $\alpha > 0$, $k > 0 and\ \beta > 0$ are constants , ψ, $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$

are altering distance functions F: $\mathbb{R}^{+} x\ \mathbb{R}^{+} \to \mathbb{R}$ is a C- class function, and M_i(x , y) (i=1,2,3) and $N(X , y)$ are defined by(3.1) − (3.4) .

Theorem 3.1

Let (X , d) be a b-metric space. Assume that α: X x X → $[0, \infty)$. And T: X→X are mappings. Suppose that the following conditions holds.

(i) (X , d) is an α – complete b- metric space;
(ii) T is a generalized ((α- ψ -φ-F) rational contraction type mapping;
(iii) T is triangular α – admissible mapping;
(iv) There exists x_0 ϵX , such that $\alpha(x_0 , Tx_0) \geq 1$;
(v) T is an α – continuous mapping

Then T has a fixed point x^* in X . more over for any $x_0 \epsilon X$, the iterative sequence $\{ T^n x_0\}$ be converges to the fixed point x^*.

Proof. Define a sequence $\{ x_n\}$ in X by $x_{n+1} = Tx_n$,(n ϵ ℕu {0}).

if $x_{n0} = x_{n0+1}$ for some $n_0 \epsilon$N, then $x_{n0} = x_{n0+1} = Tx_{n0+1}$.

In this case x_{n0} is a fixed point of T. Suppose that $x_n \neq x_{n+1}$ for all(n ϵ ℕu {0}).

Since T is $\alpha - admissible$ mapping and $\alpha(x_{0,} x_1) = \alpha(x_0 , Tx_0)$ we deduce that $\alpha(x_1 , x_2) = \alpha(Tx_0 , Tx_1) \geq 1$. On continuing this process we get that $\alpha\left(x_n , x_{n+1}\right) \geq 1$. for all (n ϵ ℕu {0}). Since T is a generalized (α- ψ -φ-F) $- rational\ contraction$ type mapping and $\alpha\left(x_n , x_{n+1}\right) \geq 1$ we $have$

$$\psi\ (s^k\ d\left(x_n , x_{n+1}\right) = \psi\ (s^k\ d(Tx_{n-1} , Tx_n))$$
$$\leq F(\psi(M_i(x_{n-1} , x_n)), \varphi(M_i(x_{n-1} , x_n))) + \beta N(x_{n-1} , x_n) \tag{3.6}$$
$$\leq \psi\ (M_i(x_{n-1} , x_n)) + \beta N(x_{n-1} , x_n) ,$$

$$where\ (M_1(x_n , x_{n-1})) = \max\{ d(x_n , x_{n-1}) , \frac{d(x_n , x_{n+1}) d(x_{n-1}, x_n)}{1+d(x_n , x_{n-1})}, \frac{d(x_n , x_{n+1}) d(x_{n-1} , x_n)}{1+d(x_{n+1}, x_n)} \}$$

$$\leq \max\{d(x_n , x_{n-1}), \frac{d(x_n , x_{n+1}) d(x_{n-1} , x_n)}{d(x_n , x_{n-1})}, \frac{d(x_n , x_{n+1}) d(x_{n-1}, x_n)}{d(x_{n+1} , x_n)} \} \tag{3.7}$$

$$= \max\{ d(x_n, x_{n-1}), d(x_n, x_{n+1}) \}$$

$$(M_2(x_n, x_{n-1}) = \max\{ d(x_n, x_{n-1}), \frac{d(x_n, x_{n+1})d(x_n, x_n) + d(x_{n-1}, x_n)d(x_{n-1}, x_{n+1})}{1+s[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)]},$$

$$\frac{d(x_n, x_{n+1})d(x_n, x_n) + d(x_{n-1}, x_n)d(x_{n-1}, x_{n+1})}{1+s[d(x_n, x_n) + d(x_{n-1}, x_{n+1})]} \}$$

$$= \max\{d(x_n, x_{n-1}), \frac{d(x_{n-1}, x_n)d(x_{n-1}, x_{n+1})}{1+s[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)]}, \frac{d(x_{n-1}, x_n)d(x_{n-1}, x_{n+1})}{1+sd(x_{n-1}, x_{n+1})]} \} \quad (3.8)$$

$$\leq \max\{d(x_n, x_{n-1}), \frac{sd(x_{n-1}, x_n)[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]}{1+s[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)]}, \frac{d(x_{n-1}, x_n)d(x_{n-1}, x_{n+1})}{1+sd(x_{n-1}, x_{n+1})]} \}$$

$$\leq \max\{d(x_n, x_{n-1}), \frac{sd(x_{n-1}, x_n)d(x_{n-1}, x_n) + d(x_n, x_{n+1})]}{s[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)]}, \frac{d(x_n, x_{n-1})}{s} \} \leq d(x_n, x_{n-1})$$

$$M_3(x_n, x_{n-1}) = \max\{ d(x_n, x_{n-1}), \frac{d(x_n, x_{n+1})d(x_{n-1}, x_n)}{1+s[d(x_n, x_{n-1}) + d(x_n, x_n) + d(x_{n-1}, x_{n+1})]},$$

$$\frac{d(x_n, x_n)d(x_n, x_{n-1})}{1+sd(x_n, x_{n+1}) + s^3[d(x_{n-1}, x_{n+1}) + d(x_{n-1}, x_n)]} \} \quad (3.9)$$

$$= \max\{ d(x_n, x_{n-1}), \frac{d(x_n, x_{n+1})d(x_{n-1}, x_n)}{1+s[d(x_n, x_{n-1}) + d(x_{n-1}, x_{n+1})]}, 0\}$$

$$\leq \max\{d(x_n, x_{n-1}), \frac{d(x_n, x_{n+1})d(x_{n-1}, x_n)}{d(x_n, x_{n-1})}\} = \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\}$$

and $N(x_n, x_{n-1}) = \min\{d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_n, x_n), d(x_{n-1}, x_{n+1})\}$

$$= \min\{d(x_n, x_{n+1}), d(x_{n-1}, x_n), 0, d(x_n, x_{n+1})\} = 0 \quad (3.10)$$

Next we prove $d(x_n, x_{n+1}) \leq \frac{1}{s^k} d(x_{n-1}, x_n)$, for all $n \epsilon \mathbb{N} \cup \{0\}$ (3.11)

If substitute $M_i = M_1$, then by (3.6), (3.7), and 3.10) it follows that

$$\psi(s^k d(x_n, x_{n+1}) \leq (\psi(M_1(x_{n-1}, x_n)) \leq \psi(\max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\} \quad (3.12)$$

since ψ is monotone, then it is easy to see from (3.12) that

$$s^k d(x_n, x_{n+1}) \leq \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\} \quad (3.13)$$

If $d(x_n, x_{n-1}) \leq d(x_n, x_{n+1})$, then by (3.13) it yields that

$d(x_n, x_{n+1}) \leq \frac{1}{s^k} \mathrm{d}(x_n, x_{n+1}) < d(x_n, x_{n+1})$, this is a contradiction. Hence,

$d(x_n, x_{n-1}) \geq d(x_n, x_{n+1})$. Again by (3.13), which implies that (3.11).

If $M_i = M_2$,then by (3.6) , (3.8) , and (3.10) it is obvious that

$\psi(s^k d(x_n, x_{n+1}) \leq \psi(M_2(x_{n-1}, x_n)) \leq \psi(d(x_n, x_{n-1}))$, again by monotonicity of ψ (3.11) holds.

If $M_i = M_3$,then by (3.6) , (3.9) , and (3.10) together with the case of $M_i = M_1$

Mentioned above we get (3.11) .As a consequence by (3.11) and lemma 2.1 we obtain $\{x_n\}$ is a b-cauchy sequence. Since (X, d) is α-b-complete there exists uϵX such that $\lim_{n\to\infty} x_n$ =u.

By condition (v) T is α-continuous we have

$$u = lim_{n\to\infty} T x_n = T lim_{n\to\infty} x_n = Tu. Therefore\ u = Tu. Hence\ T\ has\ a\ fixed\ point\ u.$$

Theorem 3.2 Let (X, d) be a b-metric space. Assume that $\alpha: XxX \to [0, \infty)$. $And\ T: X \to X$ are mappings. Suppose that the following conditions holds.

Let (X, d) be a b-metric space. Assume that $\alpha: XxX \to [0, \infty)$. $And\ T: X \to X$. Suppose that the following conditions are satisfied

I. (X , d) is an α – complete b- metric space;
II. T is a generalized **(**α-ψ-φ**-**F) rational contraction type mapping;
III. T is triangular α – admissible mapping;
IV. There exists x_0 ϵX , the such that $\alpha(x_0, Tx_0) \geq 1$;
V. If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n$\epsilon\mathbb{N} \cup \{0\}$ and $x_n \to x^* \epsilon X$ as n → ∞. Then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq 1$ for all k$\epsilon\mathbb{N}$.

Then T has a fixed point $x^* \epsilon X$ and$\{T^n x_0\}$ converges to x^*.

Proof. By analogous proof as in theorem 3.1 we can construct a convergent sequenc$\{x_n\}$ defined $by\ x_{n+1} = Tx_n\ for\ all\ (n\epsilon\mathbb{N} \cup \{0\})\ converging\ to\ x^* \epsilon X\ and \alpha(x_n, x_{n+1}) \geq 1\ for\ all$ n$\epsilon\mathbb{N} \cup \{0\}$.

$By\ condition\ (V)$ there exists a subsequence$\{ x_{n(K)} \}$ of $(x_n \}$ such that $\alpha(x_{n(k)}, x^*) \geq 1$ for all $k\epsilon\mathbb{N}$. Hence using generalized **(**α- ψ -φ-F) rational contraction type mappings we have

$$\psi\ (s^k\ d\big(x_{n(k)+1}, Tx^* \big) = \psi\big(Tx_{n(k)}, Tx^* \big)$$

$$\leq \mathrm{F}(\psi\ (M_i(x_{n(k)}, x^*)),\ \varphi M_i(x_{n(k)}, x^*))) + \beta \mathrm{N}(x_{n(k)}, x^*) \qquad (3.14)$$

$$\leq\ \psi\ (M_i(x_{n(k)}, x^*))\ ,$$

Where $M_1(x_{n(k)}, x^*) = \max\{ d(x_{n(k)}, x^*) , \frac{d(x_{n(k)}, Tx_{n(k)}) \mathrm{d}(x^*,\ Tx^*)}{1+d(x_{n(k)}, x^*)}, \frac{d(x_{n(k)}, Tx_{n(k)},) \mathrm{d}(x^*,\ Tx^*)}{1+d(Tx_{n(k)}, \mathrm{T}x^*)} \}$

$$\leq\ \max\{ d(x_{n(k)}, x^*) , \frac{d(x_{n(k)}, x_{n(k)+1}) \mathrm{d}(x^*, x^*)}{1+d(x_{n(k)}, x^*)},\ \frac{d(x_{n(k)}, Tx_{n(k)},) \mathrm{d}(x^*,\ Tx^*)}{1+d(Tx_{n(k)}, \mathrm{T}x^*)}\}$$

$$= \max\{d(x_{n(k)}, x^*) ,0,0\} = d(x_{n(k)}, x^*)\ , \qquad (3.14a)$$

$$M_2\ (x_{n(k)}, x^*) = \max\{ d(x_{n(k)}, x^*)\ ,\ \frac{d\big(x_{n(k)}, Tx^*\big)\big(d(x_{n(k)}, Tx^*\big) + \mathrm{d}(x^*, Tx^*) \mathrm{d}\big(x^*, T(x_{n(k)}\big)}{1+s\big[d\big(x_{n(k),} T(x_{n(k)} \big) + \mathrm{d}(x^*, Tx^*)\big]},$$

$$\frac{d(x_{n(k)}, Tx_{n(k)}) d(x_{n(k)}, Tx^*) + \mathrm{d}(x^*, Tx^*) \mathrm{d}(x^*, T(x_{n(k)})}{1+S[d\big(x_{n(k),} \mathrm{T}x^* \big) + \mathrm{d}\big(x^*, T(x_{n(k)}\big)]}\}$$

$$\leq \max\{ d(x_{n(k)}, x^*)\ ,\ \frac{d\big(x_{n(k)}, x^*\big)\big(d(x_{n(k)}, x^*\big) + \mathrm{d}(x^*, x^*) \mathrm{d}\big(x^*,\ x_{n(k)+1}\big)}{1+s\big[d\big(x_{n(k),} (x_{n(k)+1} \big) + \mathrm{d}(x^*, Tx^*)\big]},$$

$$\frac{d\big(x_{n(k)}, x_{n(k)+1} \big) d(x_{n(k)}, x^*) + \mathrm{d}(x^*, x^*) \mathrm{d}\big(x^*,\ x_{n(k)+1}\big)}{1+s\big[d\big(x_{n(k)}, x^* \big) + \mathrm{d}\big(x^*,\ x_{n(k)+1}\big)\big]} \quad \}$$

$$= \max\{d(x_{n(k)}, x^*)\ , d(x_{n(k)}, x^*) d(x_{n(k)}, x^*)\ ,0\ \}\ =\ d(x_{n(k)}, x^*)\ d(x_{n(k)}, x^*) \quad (3.14b)$$

$$M_3\ (x_{n(k)}, x^*) = \max\{ d(x_{n(k)}, x^*)\ ,\ \frac{d\big(x_{n(k)}, T(x_{n(k)}\big) d(x^*,\ Tx^*)}{1+s\big[d\big(x_{n(k)}, x^* \big) + \big(d(x_{n(k)}, Tx^*\big) + \mathrm{d}\big(x^*,\ T(x_{n(k)}\big)\big]},$$

$$\frac{d(x_{n(k)}\ , Tx^*)\big(\ d(x_{n(k)}\ , x^*\big)}{1+S[d\big(x_{n(k),}\ ,\ T(x_{n(k)} \big) + s^3[\mathrm{d}\big(x^*,\ T(x_{n(k)}\big) + d(x^*,\ Tx^*)]} \quad \}$$

$$\leq \max\{ d(x_{n(k)}, x^*)\ ,\ \frac{d\big(x_{n(k)}, (x_{n(k)+1} \big) d(x^*,\ x^*) \mathrm{d}(\ x^*,\ x^*)}{1+s\big[d\big(x_{n(k)}, x^*\big)\big) + \mathrm{d}\big(x_{n(k)}, x^*\big) + \mathrm{d}\big(x^*,\ (x_{n(k)+1}\big)\big]},$$

$$\frac{d\big(x_{n(k)}, x^*\big)\big)\big(d(x_{n(k)}, x^*\big)}{\big[\ (s(d\big(x_{n(k)}, (x_{n(k)+1} \big) + s^3[\mathrm{d}\big(x^*, (x_{n(k)+1}\big)\big]} \}$$

$$=\max\{ d(x_{n(k)},x^*),0,\frac{d(x_{n(k)}, x^*))(d(x_{n(k)}, x^*)}{1+s^3[d(x^*, (x_{n(k)+1})]}\}=d(x_{n(k)},x^*) \quad (3.14c)$$

And $N(x_{n(k)},x^*)=\min\{d(x_{n(k)}, Tx_{n(k)}), (d(x^*,Tx^*), (d(x_{n(k)}, Tx^*), d(x^*,T(x_{n(k)})\}$

$$=\min\{d(x_{n(k)},x_{n(k)+1}) , d(x^*,Tx^*), d(x_{n(k)},x^*), d(x^*, (x_{n(k)+1})\} \quad (3.14d)$$

Substituting equation (3.14a) - (3.14d) in equation (3.14) and letting k→ ∞ we obtain that

$\psi(d(x^*,Tx^*) \leq \psi(0) =0$.

Thus $\psi(d(x^*,Tx^*)=0$,this implies that $d(x^*,Tx^*)= 0$.hence $x^* = Tx^*$.

Theorem 3.3

In addition to the hypothesis of theorem 3.1 and theorem 3.2.

assume that for all x$\neq y \epsilon X$.There exists v ϵX such that $\alpha(x,v) \geq 1, \alpha(y,v) \geq 1$

and $\alpha(v,Tv) \geq 1$.Then T has unique fixed point.

Proof . suppose x^*& y^* are two fixed point of T such that $x^* \neq y^*$,then by assumption ,there exists v ϵX such that $\alpha(x^*,v) \geq 1, \alpha(y^*,v) \geq 1$and $\alpha(v,Tv) \geq 1$.Since T is triangular α-admissible ,we have $\alpha(x^*,T^n v) \geq 1$ and $\alpha(y^*,T^n v) \geq 1$ for all n$\epsilon\mathbb{N} \cup \{0\}$.

This implies that $\psi (d(x^*,T^{n+1}v)) = \psi (d(Tx^*,TT^n v))$

$$\leq F(\psi (M_i(x^*,T^n v)), \varphi M_i(x^*,T^n v))) + \beta N(x^*,T^n v)$$
$$\leq (\psi (M_i(x^*,T^n v))+ \beta N(x^*,T^n v),\text{for all } n\epsilon\mathbb{N} \cup \{0\}. \quad (3.15)$$

$$Where M_1(x^*,T^n v) =\max\{d(x^*,T^n v) ,\frac{d(x^*,Tx^*)d(T^n v , T^{n+1}v),}{1+d(x^*,T^n v)},\frac{d(x^*, Tx^*) d(T^n v ,T^{n+1}v)}{1+d(Tx^* ,T^{n+1}v)}\}$$

$$\leq \max\{ d(x^*,T^n v) ,0 , 0\} = d(x^*,T^n v),$$

$M_2(x^*,T^n v)$ =max{$d(x^*,T^n v)$,$\frac{d(x^*,\ Tx^*)(d(x^*,\ Tx^*)+d(T^n v\ ,\ T^{n+1}v)d(T^n v,\ Tx^*)}{1+sd(x^*,\ Tx^*)+d(T^n v\ ,T^{n+1}v\ )}$,

$$\frac{d(x^*,Tx^*)d(x^*,T^{n+1}v)+d(T^n v\ ,\ T^{n+1}v)d(T^n v\ ,Tx^*)}{1+s[d(x^*,T^n v\ )+d(T^n v\ ,Tx^*)]}\}$$

$$\leq \max\{\ d(x^*,T^n v)\ ,0\ ,0\} = d(T^n v\ ,Tx^*)$$

$M_3(x^*,T^n v)$ =max{$d(x^*,T^n v)$, $\frac{d(x^*,Tx^*)d(T^n v\ ,T^{n+1}v)}{1+s[d(x^*,\ T^n v)+d(x^*\ T^n v,\ )+d(T^n v\ ,Tx^*)\ ]}$,

, $\frac{d(x^*,T^{n+1}v\ )d(x^*\ ,T^n v)}{1+sd(x^*,Tx^*)+s^3\ [d(T^n v\ ,Tx^*\ )+d(T^n v,T^{n+1}\ )]}$

$$\leq \max\{d(x^*,T^n v\ ,\ \frac{d(x^*,T^{n+1}v)d(x^*,T^n v),}{1+s^3[d((T^n v\ ,\ Tx^*\ )}\ ,\}=d(T^n v\ ,Tx^*).$$

and $N(x^*,T^n v)$=min{$d(x^*,\ Tx^*)$, $d(T^n v\ ,T^{n+1}v)$, $d(x^*,T^{n+1}v)$, $d(T^n v\ ,Tx^*)$}

$$=\min\{\ 0\ ,0\ ,\ d(x^*,\ T^{n+1}v), d(T^n v\ ,Tx^*)\}= 0.$$

By theorem 3.1 ,we deduce that $\{T^n v\}$ converges to a fixed point z^* of T.This implies that limd($_{n\to\infty}$ $x^*,T^n v$) =d(x^*,z^*).

Taking n→ ∞ in (3.15) it follows that ψd($x^*,\ z^*$) = 0 or φd($x^*,\ z^*$)=0.

Therefore d($x^*,\ z^*$)= 0,hence $x^* = z^*$, similarly we can prove that $y^* = z^*$.

Hence $x^* = y^*$. Then T has a unique fixed point.

Corollary 2.4. Let (X , d) be a b-complete b-metric space with coefficient s > 1 and T: X → X be a mapping

such that $\alpha(x,y) = 1$ *implies* ψ(s^kd(Tx , Ty)≤ψ(M_i(x , y)) − ϕ(M_i (x , y)) + βN(x , y),

for all x , y ∈ X , where α > 0 ,k > 0, β > 0 are constants, ψ, ϕ : $\mathbb{R}^+ \to \mathbb{R}^+$ are altering distance functions, and M_i(x , y)(i = 1, 2, 3) and N(x , y) are defined by(3.1) - (3.4). Then for each i ∈ {1, 2, 3}, T has a unique fixed point in X. Moreover, for any x ∈ X , the iterative sequence {T^nx} (n∈ℕ ∪ {0}. b-converges to the fixed point.

Proof. Taking F (s , t) = s − t (s , t > 0) in Theorem 3.1, we obtain the desired result.

.**Corollary 2.5.** Let (X , d) be a b-complete b-metric space with coefficient s >1 and T : X → X be a mapping such that $\alpha(x, y) = 1$ *implies* s^kd(Tx , Ty) ≤ $\frac{M_i(x, y)}{1+M_i(x,y)}$

for all x, y ∈ X, where α > 0, k> 0 are aconstants, and M_i(x, y)(i = 1, 2, 3) are defined by (3.1)-(3.3). Then for each i ∈ {1, 2, 3}, T has a unique fixed point in X. Moreover, for any x ∈ X, the iterative sequence{T^nx} (n$\epsilon\mathbb{N} \cup \{0\}$) b-converges to the fixed point.

Proof. Take F (s , t) = $\frac{s}{1+s}$ (s > 0) and β = 0, the claim holds.

Corollary 2.6. Let (X , d) be a b-complete b-metric space with coefficient s > 1 and T : X → X be a mapping such that $\alpha(x, y) = 1$ *implies* s^kd(Tx , Ty) ≤ λM_i (x , y) ,

for all x , y ∈ X, where α > 0, k> 0 and 0 < λ < 1 are constants, andM_i (x , y)(i = 1, 2, 3) are defined by (3.1) - (3.3). Then for each i ∈ {1, 2, 3}, T has a unique fixed point in X. Moreover, for any x ∈ X, the iterative sequence{T^nx} (n$\epsilon\mathbb{N} \cup \{0\}$) b-converges to the fixed point.

Proof. *Take* $F\ (s\ , t)\ =\ \lambda s\ (s\ >\ 0)$ *and* $\beta\ =\ 0$, this completes the proof .

4. Applications

In this section we apply theorem 3.1 to solve the second order initial value problem of the form

$$\begin{cases} \frac{d^2y}{dx^2} + \phi_1(x)\frac{dy}{dx} + \ \phi_2(x)y = q(x), x\ \epsilon[0, x_0] \\ \qquad\qquad y(0) = c_0\ , y'(0) = c_1 \end{cases} \qquad (4.1)$$

where $\phi_1(x)\ , \phi_2\ (x)\ ,\ \ q(x)\epsilon\ C[0\ , x_0]$the set of all continuous real functions defined on$[0, x_0]$ are given and $c_{0,}\ c_1$ are constants.

Theorem 4 .1

Consider initial value problem (4.1), and set M = $\max_{0\le x,t\le x_0} |\phi_2\ (x)(t-x) - \phi_1(x)\ |$. If $2^k x_0^2 \mathrm{M}^2 < 1$ is satisfied or some $k > 0$,then(4.1) has a unique solution in $C([0, x_0])$. *Further* the solution is written as follows.

Y= $\sum_{n=0}^{\infty}\int_0^x$(x-t)$u_{n(t)}$dt +$c_1x+\ c_{0,}$

Where $u_{0(X)}$ =q(x)- $c_1\phi_{1(x)}$-$c_1x\phi_{2(x)}-c_0\phi_{2(x)}$,

$u_{n(x)=}\ \int_0^x[\phi_{2(x)}(t-x)-\phi_{1(x)}]\ u_{n-1}$(t) dt , n= 1, 2, ….

Proof .put u(x) = $\frac{d^2y}{dx^2}$, p(x)= $\frac{dy}{dx}$, then u(x) , p(x)$\epsilon\ C([0\ ,x_0])$,considering the initial conditions ,we obtain that

$$\frac{dy}{dx}=\int_0^x u(t)\,dt+c_{1,} \tag{4.2}$$

$$Y=\int_0^x p(s)ds+c_0\ =\ \int_0^x[\ \int_0^s u(t)dt+c_1\ ]ds+c_0 \tag{4.3}$$

$$=\int_0^x\int_0^s u(t)dtds+c_1x+c_0$$

$$=\int_0^x\ dt\int_t^x u(t)ds+c_1x+c_0=\int_0^x(x-t)u(t)dt+c_1x+c_0$$

Substituting (4. 2)and (4. 3) into (4 .1), we obtain that(4 .1) is equivalent to the following volterra type integral equation

$$U(x)=\int_0^x k(x\ ,t)u(t)dt+Q(x),$$

Where k(x , t)= $\phi_2(x)$(t- x) - $\phi_1(x)$, $Q(x)$=q(x) - $c_1\phi_1(x)$ - $c_1x\phi_2(x)$ - $c_0\phi_2(x)$.

Let X= $C([0\ ,x_0])$. put d : XxX → $\mathbb{R}^+$as d(u ,v)= $\max_{0\le x\le x_0}|u(x)-v(x)|^2$. It is clear that (X, d) is a b-complete b-metric space with coefficient s=2.

Define a mapping T : X → X by

$$Tu(x)=\int_0^x k(x\ ,t)u(t)dt+Q(x),$$

For any u , v$\epsilon\ C([0,x_0])$, we have

$$d(Tu\ ,Tv)=\max_{0\le x\le x_0}|\int_0^X k(x\ ,t)u(t)dt-\int_0^x k(x,t)v(t)\,dt|^2$$

$$=\max_{0\le x\le x_0}|\int_0^X k(x\ ,t)[u(t)-v(t)]dt|^2 \tag{4.4}$$

$\leq x_0^2 \mathrm{M}^2 \max\limits_{0 \leq x \leq x_0} |u(t) - V(t)|^2$

$= x_0^2 \mathrm{M}^2 \mathrm{d}(\mathrm{u}, \mathrm{v}).$

Let $\psi(\mathrm{t}) = \frac{1}{s^k}$ (t$\geq$ 0, $\mathrm{F}(\mathcal{E}, \eta) = s^{\mathrm{K}} x_0^2 \mathrm{M}^2 ((\mathcal{E}, \eta \geq 0)$, then ψ is an altering distance function and F is a c-class function because of $s^{\mathrm{k}} x_0^2 \mathrm{M}^2 = 2^k x_0^2 \mathrm{M}^2 < 1$. Otherwise by (4.4), we have

$\psi(s^{\mathrm{k}} \mathrm{d}(\mathrm{Tu}, \mathrm{Tv})) = \mathrm{d}(\mathrm{Tu}, \mathrm{Tv})$

$\leq x_0^2 \mathrm{M}^2 \mathrm{d}(\mathrm{u}, \mathrm{v}).$

$= s^{\mathrm{k}} x_0^2 \mathrm{M}^2 \psi\ \mathrm{d}(\mathrm{u}, \mathrm{v}).$

$\leq s^{\mathrm{k}} x_0^2 \mathrm{M}^2 \psi(M_i(\mathrm{u}, \mathrm{v})).$

$= \mathrm{F}(\psi(M_i(\mathrm{x}, \mathrm{y})), \varphi(M_i(\mathrm{x}, \mathrm{y}))) + \beta \mathrm{N}(\mathrm{x}, \mathrm{y}))$,where$\beta = 0, \varphi: \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}^+ \cup \{0\}$is any altering distance function, Owing to the above statement, all conditions of theorem 3.1 are satisfied ,then by **theorem 3.1** T has a unique fixed point in X .That is to say ,the initial value problem (4.1) has a unique solution in $C([0, x_0])$.

5. Conclusion

This thesis has established fixed point theorems for generalized (α−ψ− φ −F)-rational contraction mappings in α-complete b-metric spaces. We have demonstrated the application of these theorems to ordinary differential equations, showing how they can be used to prove existence and uniqueness of solution.

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