ADDIS ABABA UNIVERSITY

COLLEGE OF NATURAL SCIENCE

DEPARTMENT OF MATHEMATICS

Fuzzy Goal Programming Approach For Tri-level Nonlinear Programming

By

Teklay Hailay Tsegay

Advisor: Semu Mitiku Kassa(PhD)

A Thesis Submitted to the Department of Mathematics of Addis Ababa University in Partial Fulfillment of the Requirements for the Master of Science(Msc) Degree in Mathematics(Optimization)

December, 2014
Addis Ababa, Ethiopia

Declaration

This thesis is my original work and has not been presented for a degree in any other University and all the sources of information used for the thesis have been fully acknowledged.

Teklay Hailay Tsegay Signature:_____________

This thesis is submitted for examination with my approval as a university advisor:

Dr. Semu Mitiku Kassa Signature:_____________

Addis Ababa University
College Of Computational And Natural Sciences
School of Graduate Studies
Department of Mathematics

The undersigned hereby certify that they have read and recommend to the department of Mathematics for acceptance of this thesis entitled "**Fuzzy Goal Programming Approach For Tri-level Nonlinear Programming**" by Teklay Hailay in partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

This thesis is approved by board of examiners:

Chairman of the department or Graduate committee: ____________________

Sign.____________________ Date____________________

Examiner 1: ____________________

Sign.____________________ Date____________________

Examiner 2: ____________________

Sign.____________________ Date____________________

Contents

Abstract

Multi-level programming problem (MLPP) is an optimization problem which has other optimization problems in its constraint set and has a decision maker for each objective function controlling part of the variables. In this thesis we try to investigate some special multilevel programming problem which is a tri-level programming with the quadratic fractional objective functions and polyhedral constraints. Fuzzy goal programming approach is one of the methods used for solving multilevel programming problems through fuzzy goal programming (FGP) model formulation, corresponding objectives of equivalent multi-level programming problem are transformed into fuzzy goals (membership functions) by means of assigning an aspiration level to each of them and suitable membership function is defined for each objectives. Then achievement of the highest membership value of each of the fuzzy goals is formulated by minimizing the sum of negative deviational variables. However, due to the conflicting nature of each DM's objective function and hierarchical nature of the problem, solving the formulated problem for one level may not produce a satisfactory solution for the system. So this nature of the problem can be controlled by defining a new membership function of first level decision maker (FLDM) and second level decision maker (SLDM) based on the solution we obtained in the first formulation. Therefore, after some iterations FLDM and SLDM arrive at a satisfactory level and this intern produce a satisfactory solution for the system.

Keywords: Tri-level programming, Quadratic fractional programming, Fuzzy goal programming, membership functions, deviational variables.

Acknowledgement

First and foremost, I would like to express my sincere gratitude to my advisor, Dr. Semu Mitiku for his valuable discussions, comments and suggestions throughout my study.

I would like to thank Wollo University (WU), for providing this opportunity and Mathematics Department of Addis Ababa University (AAU), for accepting my application and giving different support throughout my study.

My thanks also goes to staff members of Mathematics department in Addis Ababa University (AAU) for their valuable cooperation in all rounds.

I continue to be grateful to the financial support from International Science Program(ISP), Sweden through the Department of Mathematics, AAU.

Finally, unique thanks go to my family and friends for their love and encouragement over the years of my study.

List of Figures

List of Abbreviations

MLPP	Multi-level programming problem
FLDM	First level decision maker
SLDM	Second level decision maker
TLDM	Third level decision maker
BLPP	Bi-level programming problem
TLPP	Tri-level programming problem
QFTLPP	Quadratic fractional programming problem
NLMLPP	Nonlinear multi level programming problem
DM	Decision Maker
FGP	Fuzzy Goal Programming
NP	Non-deterministic Polynomial-time
FBLPP	Fractional Bilevel Programming Problem
FPP	Fractional Programming Problem
QFMLPP	Quadratic Fractional Multilevel Programming Problem
LFBLPP	Linear Fractional Bilevel Programming Problem
NLFBLPP	Nonlinear Fractional Bilevel Programming Problem
LFTLPP	Linear Fractional Trilevel Programming Problem
NLFTLPP	Nonlinear Fractional Trilevel Programming Problem
LGP	Linear Goal Programming
FP	Fuzzy Programming

Chapter 1

Introduction

1.1 Background

Multi-level programming problem (MLPP) concerns with decentralized programming problems with multiple decision makers (DMs) in multi-level or hierarchical organizations, in which one decision maker (DM) is located at each of the hierarchical decision making levels and controls separately a decision vector for optimizing his/her own objective. Multilevel programming problems also known as hierarchical optimization. In hierarchical optimization, the constraint domain is implicitly determined by a series of optimization problems which must be solved in a predetermined sequence. Hierarchical optimization is a generalization of mathematical programming. In the real world, we often encounter situations where there are two or more decision makers in an organization with a hierarchical structure, and they make decisions in turn or at the same time so as to optimize their objective functions. In most of hierarchal organizations decision makers can make their decision cooperatively or non cooperatively. Under the assumption that the DMs do not have motivation to cooperate mutually, a Stackelberg solution is adopted as a reasonable solution for the situation. It is assumed that the DM at the upper level (leader) and the DM at the lower level (follower) completely know their objective functions and the constraints of the problem and they do not have any motivation to cooperate with each other, and the leader first makes a decision and then the follower specifies a decision so as to optimize the objective function of itself with full knowledge of the decision of the leader. Under this assumption, the leader also makes a decision such that the objective function of the leader is optimized. Then, a solution defined as the above mentioned procedure is called a Stackelberg (equilibrium) solutions, which has been employed as a solution concept for two-level mathematical programming problems. When the Stackelberg solution is employed, it is assumed that there is no communication between the two DMs, or they do not make any binding agreement even if there exists such communication [4, 11, 41].

On the other hand, under the assumption that the DMs have motivation to cooperate mutually, a satisfactory solution for all the DMs is adopted as a reasonable solution for the situation [4, 41]. Because, some times the decision maker's objective functions may conflict each other and conflict provides the decision dissatisfaction with the status. In such situation, although the execution of decision is sequential from an upper-level (leader) to a lower level(follower), the decision for optimizing the objective of an upper-level DM is often affected by the reaction of a lower-level DM due to his/her dissatisfaction with the decision, because the objectives at different levels often conflict each other. In such cases, the problem for proper distribution of decision powers to the DMs is often encountered in most of the hierarchical decision situations [31, 42].

In general, multi-level programming problem has the following common characteristics [44]:

- Interactive decision making units exist within a predominantly hierarchical structure;
- The execution of decisions is sequential from higher level(leader) to lower level(follower);
- Each decision making unit independently controls a set of decision variables and is interested in maximizing or minimizing its own objective but is affected by the reaction of lower level decision makers (DMs) due to their dissatisfaction with the decision of the higher level DMs. Due to their dissatisfaction with the decision of the higher level DMs, decision deadlock may arise frequently in the decision-making situation.
- The external effect on a decision maker's problem can be reflected in both objective function and the set of feasible decisions.

Generally, the basic concept of multi-level decision making is that an upper-level decision-maker sets his or her goal and/or decision and then asks each subordinate level of the organization for their optima. The decisions of the lower levels are then submitted and modified by the upper-level with consideration of the overall benefits of the organization. This mutually interactive process is continued until it reaches a solution, which is satisfactory to all the decision-makers. Obviously, the degree of interaction and the degree of satisfaction depend on the management style of the upper level. This decision-making process is extremely useful to the hierarchy decentralized organizations such as the various manufacturing and service companies [31].

Although there exist many optimization tools such as the decomposition principle, goal programming, multi-objective programming, and game theory, almost all of these traditional approaches cannot meet the common features of the decision process of a multi-level decentralized organization. Duo-ploy or multiploy game-like decision making process, also known as bi-1evel or multi-level programming, is developed for hierarchical systems. However, these approaches generally assume that the lower level decision is fina1 and no further interaction is required. Although we could modify the procedure to allow continuous interactions, the computational requirements would be tremendous if continuous interaction was carried out. In order to increase the computational efficiency of the basic multi-level programming algorithms, a completely different philosophy of exploring the typical fuzziness, vagueness, or the not-well-defined nature of a large decentralized hierarchical organization using fuzzy set theory was proposed in different researches [2, 4, 5, 13, 33]. Although much more research is needed, the resulting fuzzy interactive sequential approach appears to be a useful and powerful one. The advantages are that not only the computational requirements are reduced tremendously, the representation of the system is also more realistic. In other words, the traditional approach is trying to solve a non-existing problem by requiring an unrealistically accurate model and by ignoring the inherent fuzziness of large organizations [31].

Depending on the number of decision makers, multilevel programming problem can be divided in to different formats like; bilevel programming, trilevel programming, four level programming, etc which has two, three and four decision makers, respectively. Due to the interaction between each decision maker, as the number of level increases the level of complexity increases. Therefore, bilevel programming problem is the simplest one when we compare with the other multilevel programming problems,i.e, bilevel programming problem is relatively easy to understand as compared to the others. Because of this, different previous results concentrate on bilevel programming problems. However, due to the interaction between the leader and follower it is not that much easy to understand the degree of interaction and to find the satisfactory(best) solution for the system(organization) [31].

Depending on the functions involved in the objective and constrains of the problem, multilevel programming can be divided in to linear and nonlinear. If all the functions involved on the problem are linear, then the problem is called linear multilevel programming problem. If at least one function in the objectives or constraints of the problem is nonlinear, then he problem is called nonlinear multilevel programming problem. Unfortunately, multilevel programming problems are not easy to solve because of its implicit nature or the constraint is also contains an optimization problem. But, comparatively linear multilevel programming problem is more simple than nonlinear multilevel programming problem in order to find the best(satisfactory) solution. Therefore, there are different methods like; vertex enumeration,fuzzy approach and fuzzy goal programming which were introduced by different researchers in different situations in order to solve linear multilevel programming problems.

Most of the computational algorithms are developed for the simplest linear BLPPs. But, unfortunately, even for this simplest problem, there exist no simple solutions. In fact, it has been proved that for the simplest two levels linear case the problem is NP-hard [12]. Furthermore, the geometric properties of the bi-Level linear problem are much more complicated than the usual mathematical programming problem and thus the set of feasible solution is nonconvex and non-unique. Another difference is that no general hypothesis on cost function, which will guarantee Pareto optimal, can be obtained even for the linear BLPPs unless both the upper and lower objectives coincide, in which case, both DMs completely cooperate and lead to the optimal solution of BLPP which is Pareto optimal. Thus, the solutions of BLPP or MLPP are, in general, not Pareto optimum. The difficulties in obtaining an effective solution for the multi-level problems are two folds: the complexity caused by the interactions of the various DMs in the various levels and the implicit nature caused by these interactions. Due to the imbedding of the lower level problems into the upper levels, most optimization or decision making techniques are too restrictive and cannot meet the flexibility demands. Furthermore, because of the limited freedom of the lower decision makers, there exists no solution of the overall problem [4].

1.1.1 Fractional Multilevel Programming

In many real world decision making situation, decision makers (DMs) have to optimize the objective functions which are ratio of two functions of decision variables. This type of optimization problem is called fractional programming problem (FPP) [18]. The objective function of FPP may be represented by the ratio of purchasing cost and selling cost, ratio of the productions of two major crops, ratio of death and birth of people of a certain region, ratio of the full time workers and part time workers, ratio of salary and bonus etc. When both the numerator and denominator are linear functions and all the constraints are linear, then it is called linear FPP and if any one of the numerator or denominator is nonlinear, then it is called nonlinear FPP. In addition to this, nonlinear multilevel programming can be linear fractional form, quadratic form and quadratic fractional forms. Linear fractional multilevel programming involves objective function in fractional form i.e.,$f(x) = \frac{f_1(x)}{f_2(x)}$ with $f_1(x)$ and $f_2(x)$ are linear functions and all the constraints are linear. Quadratic fractional multilevel programming problems (QFMLPPs) involve objective functions in quadratic fractional form, i.e., $f(x) = \frac{f_1(x)}{f_2(x)}$ at each level with the assumption that $f_1(x)$ and $f_2(x)$ are quadratic functions and all the constraints are linear [27].

In a bi-level programming problem (BLPP), if the objective functions are linear fractional forms and the constraints are linear, then the problems are termed as linear fractional bi-level programming problems (LFBLPPs) and if they are of nonlinear fractional forms, they are termed as nonlinear fractional bi-level programming problems (NLFBLPPs). Quadratic FBLPP (QFBLPP) is one type of NLFBLPP [13].

Similarly, in tri-level programming problems (TLPP), if the objective functions are linear fractional form and the constraints are linear, then the problem is called linear fractional tri-level programming problems (LFTLPP) and if they are of nonlinear fractional forms, they are termed as nonlinear fractional tri-level programming problems (NLFTLPPs). In addition to this, if the objective functions are quadratic fractional forms and the constraints are linear, which is one type of NLFTLPPs, then the problem is known as quadratic fractional tri-level programming problems (QFTLPPs) [22].

Three-level programming problem is a class of multilevel programming problem in which there are three independent decision-makers (DMs). Each DM attempts to optimize its objective function and is affected by the actions of the others DMs. Generally, the characteristics of TLPP problem are summarized as follows [34]:

- The system has interacting decision-making units with a hierarchal structure.
- The third level decision-maker (TLDM) executes its policies after, and

in view of the decision of the second level decision maker (SLDM); who in turn do the same action with the first level decision maker (FLDM).

- Every level optimizes its own objective independently of other levels, but is affected by the actions and reactions of other levels.
- The effect of the FLDM on the lower level problem could be reflected in both its objective function and the set of feasible decision.

In general, multilevel programming consists of decision makers in the hierarchy who make decisions in a structured, leader-follower ordering. By convention, if the hierarchy has n decision makers, decision maker one (the decision maker at the top of the hierarchy), goes first. He/she makes his/her decisions based on his/her objective function and the rational reactions of the decision makers who make their decisions after him/her. After the first decision maker has made his/her decisions, decision maker two makes his/her decisions based on his/her objective function, the decisions made by the first decision maker and the rational reactions of the decision makers who make their decisions after him/her. This process continues down the hierarchy until the final decision maker (decision maker n) makes his/her decisions based on his/her objective function and decisions made by the $n-1$ decision makers above him/her [22].

Multi-level programming is computationally more complex and expensive than conventional mathematical programming. This approach can be a powerful analytical tool once solutions difficulties are overcome. However, there are different methods that we would use to solve multilevel programming problems. Fuzzy goal programming approach is among these methods that we use to solve MLPP. To formulate the fuzzy goal programming (FGP) model of the problem, the fuzzy goals of the objectives are determined by determining individual optimal solution. The fuzzy goals are then characterized by the associated membership functions which are transformed into fuzzy flexible membership goals by means of introducing over- and under-deviational variables and assigning highest membership value (unity)as aspiration level to each of them. The method of variable change on the under- and over-deviational variables of the membership goals associated with the fuzzy goals of the model is introduced to solve the problem efficiently by using linear goal programming (LGP) methodology [2].

Several solution approaches for BLPP as well as QBLPP and QFBLPP problems as a special case have been deeply investigated [10, 13, 15, 35] by the pioneer researchers in the field from the view point of their potential use to different real-life hierarchical decision systems. Some of these approaches are classical (Extreme point search, Transformation approach and Descent and heuristic), fuzzy programming and fuzzy goal programming. But the use of a classical approach often leads to the paradox that the decision power of a lower-level DM dominates that of a higher-level DM.

In order to overcome the shortcomings of the classical approaches, a fuzzy programming (FP) approach to hierarchical decision problems has been introduced by Lai [30] in 1996. Lai's solution concept has been further extended by Shih et al. [44] and Shih and Lee [46] to make a reasonable balance of decision powers to the DMs. The main difficulty with a conventional FP approach is that re-evaluation of the problem again and again by re-defining the elicited membership values of the objectives is involved to reach a satisfactory decision. To avoid such a computational difficulty, goal satisficing method in GP [23] for minimizing the regrets of the DMs in case of a BLPP has been studied by Moitra and Pal [33] in the recent past (2002).

Several three level programming problems and their solution methods have been presented, such as, a hybrid extreme-point search algorithm, mixed-integer problem with complementary slackness, and the penalty function approach. These approaches have been used widely in searching for the optimal solutions. Apart from these approaches, fuzzy sets have been employed to formulate and solve three-level non-linear multi-objective decision-making [34].

In the recent past, linear fractional bilevel programming and linear fractional multilevel programming were studied by P. Dey and S. Pramanik[20], K. Lachhwani and P. Poonia[27], respectively. P. Dey and S. Pramanik has been transformed by formulating a membership function for each objective function, relaxing the decision variable of all the decision makers and finally use a first order Taylor series about the individual solution for the nonlinear part. But, K. Lachhwani and P. Poonia were transformed the problem defining a membership function of the numerator and denominator of each DM and decision variables of the problem. Finally, the problem becomes a linear programming. B. Pal and N. Moitra[35] used fuzzy goal programming approach for solving quadratic BLP in 2003. Also, a solution approach for quadratic fractional BLP was studied by A. Biswas and K. Bose[13]. Generally, most of the previous results are concentrated on linear BLPP and linear fractional, quadratic, and quadratic fractional BLPPs which are nonlinear BLPP formats. But, fuzzy goal programming approaches (FGP) for quadratic fractional three level (trilevel) programming and in general for nonlinear three level programming did not yet appear in literature.

In this study, we apply fuzzy goal programming approach for solving quadratic fractional trilevel programming problems. In this approach, first we formulate the model by transforming the membership functions defined for the fuzzy goals of the problem in to flexible goals and then through solving it we obtained a better solution. In the model formulation process, we transform the membership functions defined for the fuzzy goals of the problem into flexible goals by assigning the highest degree (unity) of the membership functions as their aspiration level.

1.2 Mathematical Formulation of Multilevel Programming

Mathematically, multilevel programming problems(MLPPs) can be formulated as[50]:

$$\begin{aligned}
\max/\min_{x_1} f_1(x) \;\; & where \;\; x_2, x_3, \ldots, x_n \;\; solve, \\
& s.t \;\; x \in S_1 \\
\max/\min_{x_2} f_2(x) \;\; & where \;\; x_3, x_4, \ldots, x_n \;\; solve, \\
& s.t \;\; x \in S_2 \\
& . \\
& . \\
& . \\
\max/\min_{x_{n-1}} f_{n-1}(x) \;\; & where \;\; x_n \;\; solves \\
& s.t \;\; x \in S_{n-1} \\
& \max/\min_{x_n} f_n(x) \\
& x \in S_n
\end{aligned} \tag{1.1}$$

where $f_i : R^m \to R$, $S_i \subseteq R^m$, and $x_i \in R^{m_i}$, for $i = 1, 2, 3, \ldots, n$.

In the above mathematical formulation of a multilevel programming problem(MLPP), consider the problem in (1.1) composed of n levels each characterized by individual objective functions f_i for $i = 1, 2, 3, ..., n$ which are to be minimized or maximized by the respective decision makers. Assume

that the decision variable space R^m is partitioned among n levels, such that $(x_1, x_2, ..., x_n) \in X_1 \times X_2 \times \cdots \times X_n = S \subset R^m$, where $X_i \subset R^{m_i}$, $\sum_{i=1}^{m} m_i = m$ and S be a nonempty set. Suppose that the decisions are made sequentially beginning with DM_1 (called the leader DM)who has control over a vector $x_1 \in X_1$, followed by DM_2 who has control over a vector $x_2 \in X_2$ down through DM_n who has control over a vector $x_n \in X_n$.
If $n = 1$, then problem (1.1) becomes a programming problem with single objective function. Thus, multilevel programming is a generalization of the classical mathematical programming problems. If $n = 2$, problem (1.1) called BLPP and also problem (1.1) is called TLPP if $n = 3$.

1.2.1 Bilevel Programming

Bilevel programming problem (BLPP) is a special case of a multilevel programming problem (MLPP) of a large hierarchical decision system. In a BLPP, two decision makers (DMs) are located at two different hierarchical levels, each independently controlling one set of decision variables and with different and perhaps conflicting objectives[33].
Let $(x, y) \in X \times Y = S \subset R^{n_1} \times R^{n_2}$, and $f, F : R^{n_1} \times R^{n_2} \to R$,
$g = [g_1, g_2, .., g_r] : R^{n_1} \times R^{n_2} \to R^r$, $h = [h_1, h_2, ..., h_{r'}] : R^{n_1} \times R^{n_2} \to R^{r'}$
$G = [G_1, G_2, .., G_p] : R^{n_1} \times R^{n_2} \to R^p$, $H = [H_1, H_2, ..., H_{p'}] : R^{n_1} \times R^{n_2} \to R^{p'}$.
The general BLPP can then be defined as:

$$\min_{x \in X} F(x, y)$$
$$s.t : G(x, y) \leq 0$$
$$H(x, y) = 0$$
$$\min_{y \in Y} f(x, y)$$
$$s.t : g(x, y) \leq 0$$
$$h(x, y) = 0, \ \ where \ \ y \ \ solves \tag{1.2}$$

We call the top DM who first makes a decision the leader and the other DM the follower. In non-cooperative situations, for each decision made by the leader, the follower with a full knowledge of the decision of the leader, reacts with an optimal solution of its problem. According to this rule the leader also makes a decision which optimizes its objective function. The optimal decision of the leader with an optimal reaction of the follower after the leader's decision is called a stackelberg equilibrium in the field of game theory and economics and we refer to it as a stackelberg solution, and we refer to it as a Stackelberg solution[50].

Furthermore the relaxed problem associated with BLPP can be stated as:

$$\min_{x,y} F(x,y) \tag{1.3}$$

$$subject\ to\ G(x,y) \leq 0, H(x,y) = 0, g(x,y) \leq 0, h(x,y) = 0$$

and its optimal value is a lower bound for the optimal value of the BLPP.

Some of the defining components of the BLPP in (1.2) can be defined as [50]:

- Constraint region of the BLPP

  $$\Omega = \{(x,y) : x \in X, y \in Y, G(x,y) \leq 0, H(x,y) = 0, g(x,y) \leq 0, h(x,y) = 0\}$$

- For each given $x^* \in X$, the follower's feasible region is :

  $$\Omega(x^*) = \{y : y \in Y, g(x^*, y) \leq 0, h(x^*, y) = 0\}$$

- Projection of Ω onto the leader's decision space gives :

  $$\Omega(X) = \{x : \exists y, (x,y) \in \Omega\}$$

- For each given $x^* \in \Omega(X)$, the follower's reaction set is :

  $$M(x^*) = \{y : y \in argmin\{f(x^*, y), y \in \Omega(x^*)\}\}$$

- The Induced Region at the upper level is :

  $$R = \{(x,y) : x \in X, G(x,y) \leq 0, H(x,y) = 0, y \in M(x)\}$$

The induced region R is the feasible set of the BLPP. It is usually nonconvex and, in the presence of upper level constraints, can be disconnected or even empty.

If $f(x,y)$ and $h(x,y)$ are convex functions and h is a linear function in y for all values of x, then the lower level problem is convex. As a result, for any fixed x the lower level problem has a unique solution. The advantage of dealing with the convex lower level problem of the BLPP is that under an appropriate constraint qualification, the lower level problem can be replaced by its Karush-Kuhn-Tucker (KKT) conditions to obtain an equivalent (one-level) mathematical program. However, despite their designation, BLPP with convex lower level problem have nonconvex induced regions that can be disconnected or even empty in the presence of upper level constraints[49]. Moreover, a BLPP need not have a solution,i.e., restricting the functions F, G, H, f, h, g to be continuous and bounded DOES NOT guarantee the existence of a solution and this can be illustrate by the following example (1.1).

Now let us describe the stackelberg BLPP by looking at the follower's reaction to the leader's action. Given that a strategy $x^* \in X$ is chosen by the leader, the follower reacts with its optimal decisions in $M(x^*) \subset Y$ by solving the optimization problem given by:

$$\begin{aligned} &\min f(x^*, y) \\ &\textit{subject to } g(x^*, y) \leq 0 \\ &h(x^*, y) = 0 \\ &y \in Y \end{aligned} \tag{1.4}$$

If for each $x^* \in X$, (1.4) has a unique optimal solution ,$M(x^*)$, then we can reformulate the BLPP at the upper level as:

$$\begin{aligned} &\min F(x, M(x)) \\ &\textit{subject to } G(x, M(x)) \leq 0 \\ &H(x, M(x)) = 0 \\ &x \in X \end{aligned} \tag{1.5}$$

In addition, if F is continuous on $(x, M(x)); \forall x \in X$ and X is nonempty compact set then the problem in (1.5) attains its optimal solution (Weirstrass Theorem).

Example 1.1.

$$\begin{aligned} &\min_{x_1,x_2} F(x_1, x_2, y_1, y_2) = (x_1 \;\; x_2) \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ &\textit{subject to } x_1 + x_2 = 1 \\ &x_1, x_2 \geq 0 \\ &\min_{y_1,y_2} f(x_1, x_2, y_1, y_2) = (x_1 \;\; x_2) \begin{pmatrix} -1 & -4 \\ -3 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ &\textit{subject to } y_1 + y_2 = 1 \\ &y_1, y_2 \geq 0 \end{aligned}$$

The solution $\hat{y}$ to the followers problem as a function of x is:

$$\hat{y}(x) = \begin{cases} (1,0), & for\ \ x_1 + 3x_2 > 4x_1 + 2x_2, i.e., x_1 < \frac{1}{4} \\ y_1 + y_2 = 1, & for\ \ x_1 = \frac{1}{4} \\ (0,1), & for\ \ x_1 > \frac{1}{4} \end{cases}$$

where $(x_1, x_2) = x$ and $y = (y_1, y_2)^T$
Substituting these values into the leader's problem gives:

$$\min_x F = \begin{cases} 2x_1 + 4x_2, & ; x_1 < \frac{1}{4} \\ 2y_1 + \frac{3}{2}\ (0 \leq y_1 \leq 1), & ; x_1 = \frac{1}{4} \\ 3x_1 + x_2, & ; x_1 > \frac{1}{4} \end{cases}$$

$$subject\ \ to\ \ : x_1 + x_2 = 1$$

$$x_1, x_2 \geq 0$$

For $x_1 < \frac{1}{4}$, the leader's problem becomes:

$$\min_x F = 2x_1 + 4x_2$$

$$subject\ \ to\ \ : x_1 + x_2 = 1$$

$$x_1, x_2 \geq 0$$

This implies that

$$\min_{x_1} F = -2x_1 + 4$$

$$subject\ \ to\ \ : 0 \leq x_1 < \frac{1}{4}$$

Since the constraint set is not compact and the objective function has minimum value at $x = \frac{1}{4}$ which is not in the constraint set, the leader's problem has no solution for $x_1 < \frac{1}{4}$.
For $x_1 > \frac{1}{4}$, the leader's problem becomes:

$$\min_x F = 3x_1 + x_2$$

$$subject\ \ to\ \ : x_1 + x_2 = 1$$

$$x_1, x_2 \geq 0$$

This implies that

$$\min_{x_1} F = 2x_1 + 1$$

$$subject\ \ to\ \ : \frac{1}{4} < x_1 \leq 1$$

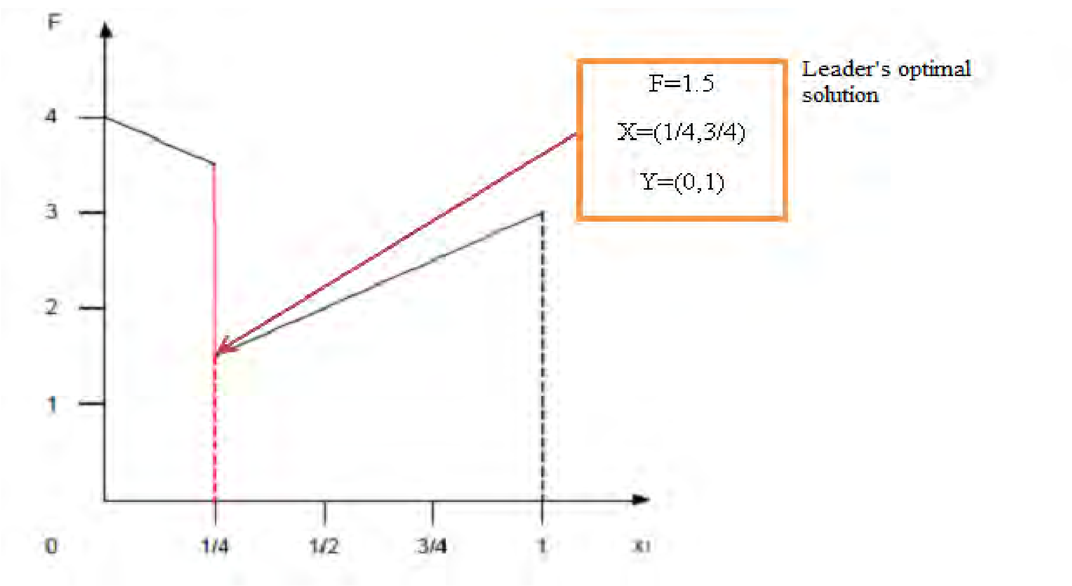

Figure 1.1: Solution space and optimal solution for leader's problem

Since the constraint set is not closed, the leader's problem has no solution for $\frac{1}{4} < x_1$.
Finally, for $x_1 = \frac{1}{4}$, $x_2 = \frac{3}{4}$ and the leader's problem is:

$$\min_x F = 2y_1 + \frac{3}{2} \ (0 \leq y_1 \leq 1)$$

$$subject\ to \ : x_1 + x_2 = 1$$

$$x_1, x_2 \geq 0$$

which is independent of the decision variable x.

Therefore, at $x = (\frac{1}{4}, \frac{3}{4})$ follower's optimal value is $f = -\frac{5}{2}$ at any point on the line $y_1 + y_2 = 1$ and the corresponding solution for the leader is $F = 2y_1 + \frac{3}{2}$ which implies $F \in [1.5, 3.5]$. Consequently, there is no way for the leader to guarantee the achievement of its minimum payoff,i.e., no solution. In addition to this, restricting the functions $F, f, h = x_1 + x_2 - 1$, and $g = y_1 + y_2 - 1$ to be continuous and bounded does not guarantee the existence of a solution.

However, the leader's objective function $F = 2x_1 + 1(0 \leq y_1 \leq 1)$ has minimum value at $y_1 = 0$ and $y_2 = 1$. So, $(x_1, x_2, y_1, y_2) = (\frac{1}{4}, \frac{3}{4}, 0, 1)$ is leader's optimal solution. The solution space and optimal solution for leaders problem of example (1.1) can be seen graphically as figure(1.1).

Example 1.2.

$$\min_x F(x,y) = x - 4y$$
$$\min_y f(x,y) = y$$
$$-x - y \leq -3$$
$$-2x + y \leq 0$$
$$2x + y \leq 12$$
$$-3x + 2y \geq -4$$
$$x, y \geq 0$$

The constraint region of the BLPP is

$$\Omega = \{(x,y) : -x-y \leq -3, -2x+y \leq 0, 2x+y \leq 12, -3x+2y \geq -4, x \geq 0, y \geq 0\}$$

.

$$M(x) = \{y : y = 3 - x,\ \ for\ \ 1 \leq x \leq 2,\ \ and\ \ y = 1.5x - 2,\ \ for\ \ 2 \leq x \leq 4\}$$

The inducible region of BLPP is

$$R = \{(x,y) : x \in [1,4], y \in M(x)\}$$

From the figure (1.3) we can observe that the feasible region S is convex, but the inducible region R is nonconvex and nondifferentiable. The optimal solution of the problem in R is $(x^*, y^*) = (4,4)$ with optimal values $F^* = -12$ and $f^* = 4$.

However, the optimal solution $(x^*, y^*) = (3,4)$ dominates the optimal solution $(x^*, y^*) = (4,4)$, because the optimal value in the second solution $F^* = -13$ is better for the first level DM and there is no any change in the second level DM optimal value.

Moreover, the solution $(3,6)$ is best for all the DMs with $F = -21, f = 6$ which is not in the inducible region. If we project $(3,6)$ to the inducible region, the solution is $(3,2.5) \in R^2$ with $F = -7, f = 2.5$ which is not satisfactory solution. Graphically we can observe this situation from figure (1.3).
Therefore, there are solutions in the feasible region but not in the inducible region which dominates the solution obtained in the inducible region.

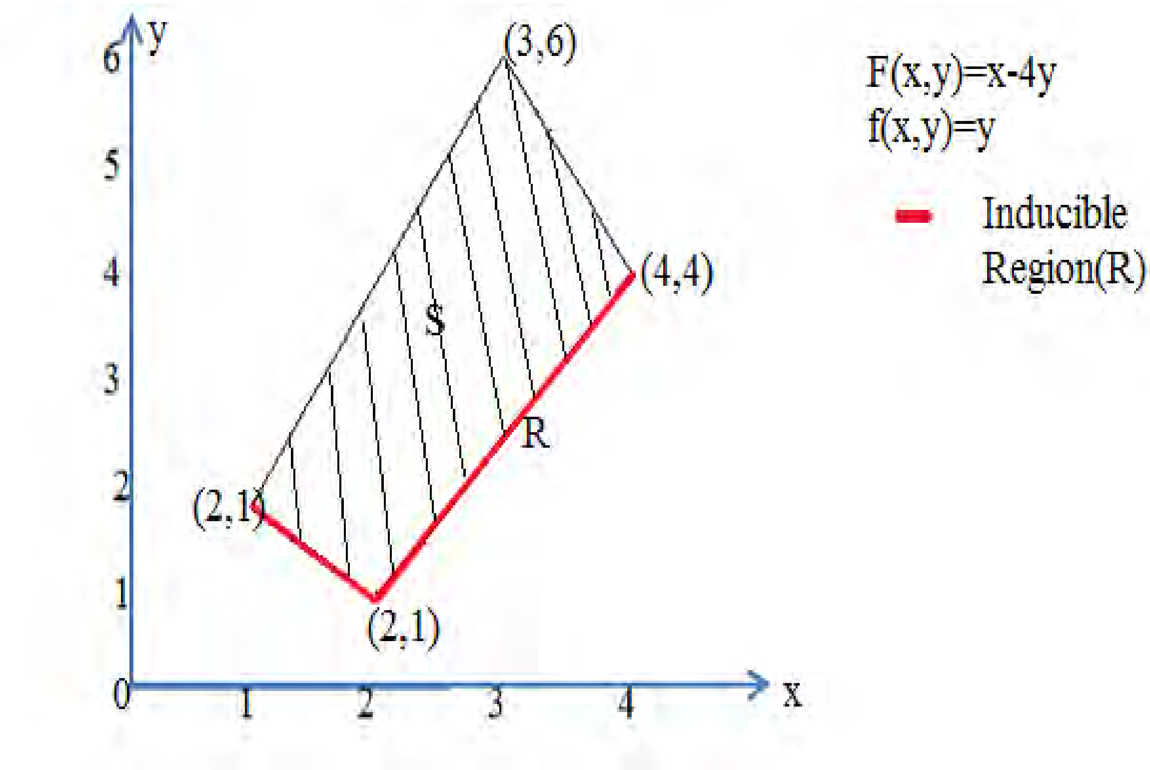

Figure 1.2: Inducible and feasible region of the problem

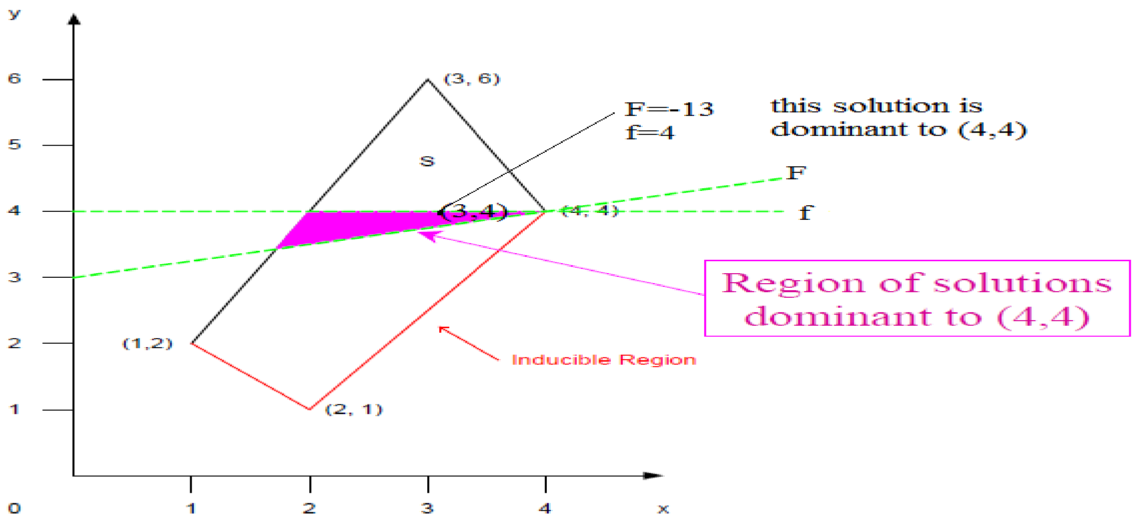

Figure 1.3: Inducible region and region of dominance solutions

1.2.2 Tri-level Programming

A tri-level programming problem is formulated for a problem in which three decision makers make decision successively and each decision maker controls part of the decision variables in order to optimize their single objective functions. In other words, tri-level programming is known as three level programming problem(TLPP). In most of the real practice, it is natural that the decision makers behave cooperatively rather than noncooperatively. The interactive decision making is generally applicable in large hierarchy organizations, which are characterized by the mutual interactions in a top to bottom sequence and by the vague and not-well-defined nature of a large hierarchy organization. The decision of a typical tri-level process is carried out in the following manner. The upper-level DM, the leader, makes his or her decision first with full information about all levels, and then the second level DM, makes his or her decision with full information about the third level and based on decision of the first level;and finally the third level decision maker makes the decision based on the full knowledge of the first and second decision. In most practical cases, the third-level decision is the final decision and no time or energy for the upper levels(first and second levels) to make another decision afterward and the second level makes decision based on the first level decision and full information about the third level and no time or energy for the first level to make another decision afterward. Notice that although the third level cannot control the decision variables of the first and second levels, the final decision of the lower does eventually influence the first and second levels and the overall results, because every objective function depends on all the decision variables, i.e., to determine the optimal value we have to know all decision variable values. Besides this the second level cannot control the decision of the first level, but the final decision of the lower eventually influence the overall results.

However, the cooperation between each decision maker may be controlled by the interest of the decision makers. The degree of interaction between each consecutive decision makers can be expressed by adding particular constraints.

Mathematically, tri-level programming problem(TLPP) can be defined as follows:

$$\min_{x_1} f_1(x)$$
$$s.t \;\; x \in S_1, \;\; where \;\; x_2, x_3 \;\; solve$$
$$\min_{x_2} f_2(x)$$

$$s.t \;\; x \in S_2, \;\; where \;\; x_3 \;\; solves$$
$$\min_{x_3} f_3(x)$$
$$s.t \;\; x \in S_3 \tag{1.6}$$

where

$$f_i : R^n \to R, \;\; for \;\; i = 1, 2, 3$$

and $S_i \subseteq R^n \;\; for \;\; i = 1, 2, 3$, $n = n_1 + n_2 + n_3$ and $x_i \in R^{n_i}$ and

$$x = (x_1, x_2, x_3) \in R^n$$

A trilevel programming problem, in equation (1.6), comprises of three sub-problems, one at each optimization level with the following basic definitions of sets[3]:

- The set
$$\Omega_3(x_1, x_2) = \{x_3 \in R^{n_3} : (x_1, x_2, x_3) \in S_3\}$$
is called a feasible set for the third level problem.

- The set
$$\Psi_3(x_1, x_2) = \{x_3 \in R^{n_3} : x_3 \in \arg\min \{f_3(x_1, x_2, x_3) : x_3 \in \Omega_3(x_1, x_2)\}\}$$
is called the rational reaction set for the third level problem.

- The set
$$\Omega_2(x_1) = \{(x_2, x_3) \in R^{n_2} \times R^{n_3} : (x_1, x_2, x_3) \in S_2 \cap S_3, x_3 \in \Psi_3(x_1, x_2)\}$$
is called a feasible set for the second level problem.

- The set
$$\Psi_2(x_1) = \{(x_2, x_3) \in R^{n_2} \times R^{n_3} : x_2 \in \arg\min \{f_2(x_1, x_2, x_3) : (x_2, x_3) \in \Omega_2(x_1)\}\}$$
is called the rational reaction set for the second level problem.

The above definition show us the nature of the rational reaction sets, which describe the dependence of the decisions taken at the upper levels on the decisions taken at lower levels. Because of this sequential nature of the rational reaction sets, multilevel programming problem(MLPP) are generally very complex and are difficult to solve.

The individual constraints S_i shows the degree of interaction between each decision maker and also a solution of the system may not in the intersection of $S_i(\cap_{i=1}^{3} S_i)$. That means the solution is in the union of S_i and each DM has it's own criteria. However, in different real life situations, the constraints of multilevel decision making problems is common and in this common constraint region each DM has an interest to find a best(satisfactory) solution of the system through controlling part of the decision variables. Hence, the solution we obtained is in the intersection set and the problem(1.6) reduces to

$$\begin{aligned} &\min_{x_1} f_1(x) \;\; where \;\; x_2, x_3 \;\; solves \\ &\min_{x_2} f_2(x) \;\; where \;\; x_3 \;\; solves \\ &\min_{x_3} f_3(x) \\ &s.t \;\; x \in S = \cap_{i=1}^{3} S_i \end{aligned} \tag{1.7}$$

In the above problem (1.7) the degree of interaction between each decision maker is the same and all DMs are interested to optimize their own objective function within the same feasible region.

Depending on the functions involved in the problem(1.6), the problem can be named in different ways:

- If all the functions involved in the problem are linear, then (1.6) is called linear tri-level programming problems.
- If at least one of the objective functions or constraints is nonlinear, then the problem is called tri-level nonlinear programming problem(TLNLPP).

The linear three level programming problem (TLPP) can be defined as a three person, non-zero sum game with perfect information in which players move sequentially from top to bottom level with all the objective functions and all the constraints are linear. Mathematically, linear tri-level programming problem can be expressed as:

$$\begin{aligned} &\min_{x_1} f_1(x) = C_{11}x_1 + C_{12} + C_3 x_{13} \;\; where \;\; x_2, x_3 \;\; solves \\ &\min_{x_2} f_2(x) = C_{21}x_1 + C_{22} + C_3 x_{23} \;\; where \;\; x_3 \;\; solves \\ &\min_{x_3} f_3(x) = C_{31}x_1 + C_{32} + C_{33}x_3 \\ &s.t \;\; x = (x_1, x_2, x_3) \in S \end{aligned} \tag{1.8}$$

where

$$S = \{(x_1, x_2, x_3) \in R^n | A_1x_1 + A_2x_2 + A_3x_3 \leq b, \ \ and \ \ x_1, x_2, x_3 \geq 0\}$$

and x_1, x_2, x_3 are the control or decisions variables for the first, second and third levels, respectively.

Equation (1.8) represents a three-level hierarchical process, whose objective is to minimize $f = (f_1, f_2, f_3)$ by choosing the decisions $x = (x_1, x_2, x_3)$ in a top to bottom manner.

Tri-level nonlinear programming problem is a TLPP in which either the objectives of the decision maker or constraints of the decision maker are described in nonlinear functions. Depending on the functions involve in the problem, the TLNLPP can be named in different ways as follows:

- If the objective functions of the DMs are linear fractional form and the constraints are expressed in affine functions, then the three level programming problem is called linear fractional three level(tri-level) programming problems (LFTLPP).
- If the objective functions of the DMs are quadratic and the constraints are expressed in affine functions, then the three level programming problem is called quadratic three level(tri-level) programming problems (QTLPP).
- If the objective functions of the DMs are quadratic fractional and the constraints are expressed in affine functions, then the three level programming problem is called quadratic fractional three level(tri-level) programming problems (QFTLPP).
- If the objective functions of the DMs are convex-concave fractional and the constraints are expressed in affine functions, then the three level programming problem is called convex-concave fractional three level(tri-level) programming problems (CCFTLPP).

Chapter 2

Literature Review and Scientific Motivation

2.1 Literature Review

Candler and Townsley [15] first introduced the concept of Bilevel Programming Problem (BLPP) as a mathematical program that contains an optimization problem in the constraints. To handle such problem different techniques were developed such as vertex enumeration method [27] and transformation approaches[10, 11] which are effective only for very simple types of problems. In these methods the decision makers (DMs) have no cooperating attitude with each other. So, decision deadlock arises frequently due to followers dissatisfaction with the solution.

However, several algorithmic approaches have been developed that can solve convex bilevel and trilevel programming problems. For linear multilevel programming problems (MLPPs) vertex enumeration methods have been used and for nonlinear multilevel programming problems (MLPPs) many of the methods try to transform the lower level problems by using the Karush-Kuhn-Tucker (KKT) conditions or penalty functions. However, the relaxed KKT conditions may result in non optimal reactions from the lower levels or in ambiguity to choose one solution among the optimal reaction set. Moreover, it seems too difficult to extend such an approach beyond two levels, because of the non-convex constraints introduced due to complementarity conditions. If the lower levels are convex, their parametric solutions are unique in each region, where the solution is stable. But many practical problems that are modeled using multilevel programming problem (MLPP) may contain non-convex terms in their lower level problems. To overcome this, the process of convexification of the lower level problems to underestimate them by convex functions (if they are nonconvex) at each iteration and use multi-parametric programming (MPP) approach to propose a branch-and-bound algorithm to find a global approximate solution for multilevel problems with non-convexity at their inner levels[26].

Zimmermann [54] first applied fuzzy set theory in decision making problems with several conflicting objectives in 1978. In Zimmermann's solution concept, the theory of fuzzy sets has been employed to formulate and solve fuzzy linear multi-objective programming problems through construction of membership function of the corresponding objective functions. In 1996 Lai and Hwang [29] introduced the concept of membership functions of fuzzy sets in BLPPs. Lai's solution concept was then extended to linear multilevel programming problems by Shih et al. [45] and a supervised search procedure with the use of max-min operator of Bellman and Zadeh [8] was proposed.

Fuzzy programming (FP) approach was proposed by Lai [30] and Shih et.al [44] to solve MLPPs using the concept of tolerance membership functions and multiple objective decision making.The idea is to use the basic fuzziness and vagueness nature of such large hierarchical systems to make the complexity tractable. The upper level defines his or her tolerances by the use of membership function which constrains the lower level DM's feasible space. However, Lai's and Shih's solution concept is not the usual practice for solving real-world decision making problems, managerial decisions almost always allow some compensation between different achievements so that a balance of the objectives can be obtained. To overcome such problems Shih and Lee [46] further extended Lai's [30] concept by introducing the compensatory fuzzy operator for solving MLPPs. Thus, in this approach, multiple decision units in the same level can have unequal objectives. But, to find the satisfactory solution for the system there is a possibility of rejecting the elicit membership function again and again. In 1997, fuzzy goal programming technique (FGP) introduced by Mohamed [32] for proper distribution of decision powers to the DMs to arrive at a satisficing decision for the overall benefits of the organization was developed to overcome the possibility of rejecting the solution again and again. The main advantage of the FGP algorithm is that the possibility of rejecting the solution again and again by the upper level DMs and re-evaluation of the problem repeatedly, by redefining the elicited membership functions, needed to reach the satisfactory decision does not arise. The FGP of Mohamed [32] was extended to solve multi objective linear fractional programming problems in [36], bi-level programming problems in [33], bi-level quadratic programming problems in [35], and in 2007 a fuzzy goal programming (FGP) approach to linear multi-level programming problems with single objective function in each level, is proposed [37].

Lachhwani and Poonia [27] proposed FGP approach for multi-level linear fractional programming problem by constructing the tolerance membership functions for the fuzzily described numerator and denominator part of the objective functions of all levels as well as the control vectors of the higher level decision makers are respectively defined by determining individual optimal solutions of each of the level decision makers. Baky [6] suggested two new techniques with FGP approach based on solution preferences by the decision maker at each level to solve new type of problem multi-level multi-objective linear programming (ML-MOLP) problems through the fuzzy goal programming (FGP) approach. Sinha and Baky [3] presented interactive balance space approach for solving multi-level multi objective programming problems. Baky [5] proposed FGP algorithm for solving decentralized bi-level multi-objective programming (DBL-MOP) problems with a single decision maker at the upper level and multiple decision makers at the lower level. However, in 2013 Lachhwani [28] modified FGP approach for the problem in which solution preferences by decision maker at each level and sequential order of decision making process are not taken into account by the proposed technique.

Non-linear BLPP, such as QBLPP, was studied by Pal and Moitra [35]. In their approach, they formulate fuzzy quadratic programming model to minimize the deviation variables of each level by using Hamming distance [53],i.e, first formulate a membership function of the objective function for each decision maker and bound the corresponding decision variables, then finally formulate the new problem using weighted sum concept. Then they transform the quadratic model into an equivalent non-linear FGP model to achieve the highest degree (unity) of satisfaction to the extent possible for the leader and follower. In the decision making process, linear approximation technique suitable for non-linear goal programming studied by Ignizio [23] is applied to obtain satisfactory solution. Finally, they formulate priority based FGP model taking decision variables at first priority level and objective goals at second priority level without considering the system constraints.

Osman et al. [34] extended the fuzzy approach of Abo-Sinna [1] for solving non-linear bi-level and tri-level multi-objective decision making under fuzziness. Their method based on the concept that the lower level decision maker maximizes membership goals taking a goal or preference of the upper level decision maker (ULDM) into consideration. The level DMs elicit non-linear membership functions of fuzzy goals for their non-linear objective functions and especially the upper level decision maker (ULDM) specifies linear fuzzy goals for the decision variables. Lower level decision maker (LLDM) solves a fuzzy programming with a constraint on a satisfactory degree of ULDM. However, there is a possibility that their fuzzy approach offers undesirable solution because of inconsistency among the fuzzy goals of the non-linear objective functions and linear fuzzy goals of the decision variables [21].

Multilevel programming is either linear or nonlinear programming problems depending on the functions involved in the problem. For linear multilevel programming problems different methods such as vertex enumeration, transformation approach, fuzzy approach, and fuzzy goal approaches has been proposed. However, vertex enumeration approach is applicable for linear multilevel programming problems, and if there is large number of vertices, the vertex enumeration approach takes more time to obtain the solution, i.e., the problem becomes NP-hard. In transformation approach, transformation of the multilevel programming to single level programming using penalty function or KKT conditions has been used. But, this approach seems difficult to extend more than bilevel programming problems, because of the non-convex constraints introduced due to complementarity conditions. Then, the problem becomes more complex to control. Although a transformation approach seems appropriate for bilevel programming problem, it is only possible for problems with convex lower level problem. Moreover, if the lower level DM's solution is not unique for particular value of upper level DM's decision variable, then it is difficult to find the upper level solution based on the lower level solution. To overcome all those problems, fuzzy goal programming approach was developed for linear multilevel, liner fractional multilevel, quadratic BLPP and quadratic fractional BLPPs. For the linear part, simply transform the problem in to new single level linear programming using membership function of each Decision maker and control variables and then solve the transformed problem using techniques linear programming. However, for the nonlinear part, the new formulated problem using fuzzy goal programming is also nonlinear and so it is not as simple as linear programming. To handle such problems, different researchers have been used a Taylors series approximation for the nonlinear part. But, this approximation may not produce satisfactory solution if the degree of nonlinearity is high. That is why different results has been concentrate on quadratic and quadratic fractional bilevel programming problems, because such functions are not that much difficult to study its behaviour and approximation is better.

In addition, if the number of level increases the complexity of the problem increases. Thus, trilevel programming problem is more complex than bilevel programming problem. In fuzzy goal programming approach, for bilevel programming, first define a membership functions of each objective function and control variable of the first level decision maker and finally formulate a single level programming which is simpler to solve than the original problem. But, the membership function of the control variable is linear and the membership function of objective function of each decision maker is nonlinear due to the nonlinearity of objective functions. So, small change of the decision variable may produce large change in the corresponding objective function and this may give us unstable solution. This implies that there may be inconsistency in the constraint set of the formulated problem. Moreover, if the problem has more than two decision makers with more than two part of the decision variables, it seems difficult to control every interaction between each decision makers. The interaction between the first level decision maker and the last decision maker is indirect and it could be difficult to define this interaction explicitly, because the objective function of each decision makers are nonconvex,i.e., the lower level problem is nonconvex. In addition to this, different results were used a weighted sum to solve BLPP after formulating a membership function of each DM's objective function and control variables. Some of the investigators were used the same weight(unity weight) and some of them were used a different weight which is the reciprocal of the distance between the individual solution(maximum and minimum). But, the weight may not be the same, because the importance of each decision maker is not always the same and also the reciprocal weight may produce non satisfactory solution, because a large distance between the two maximum and minimum objective value may give us a large value of the negative deviation. Therefore, for all the cases there is a possibility of producing non satisfactory solution of the system.

In this thesis we apply a fuzzy goal programming approach with the weighted sum concept to control the interaction between each decision maker of the tri-level nonlinear programming and then we use matlab solver to find the best solution of the new formulated nonlinear problem. Through iterating the weight we could found a better solution and after some iteration we obtained a satisfactory solution. Moreover, the weights we use in this thesis are distances between a maximum and minimum values of the objective functions of each decision maker as it is and due to the nature of the decision makers they are different.

2.2 Scientific Motivation

As we have seen in the literature review, there is no way to solve a general MLPPs(linear and nonlinear). Most existing approaches work with BLPPs specially with the linear BLPPs and BLPPs having nice properties of the involved functions, such as continuity, differentiability, convexity,. . . etc. Moreover, there are different appropriate methods for linear multilevel programming(vertex enumeration and fuzzy goal programming methods). But for general multilevel nonlinear programming, no existing appropriate approach have been proposed to solve the problem. However, for BLPP there are different techniques to find the satisfactory solution. Starts from that point of view our interest is to apply the concept of fuzzy set theory for a TLNPPs. We all need to get satisfactory solution within acceptable time such that all the DMs agreed, instead of finding the best solution of all levels. Because, if there is a conflict between each DMs (loosing of one DM gains the other DM), it is not possible to find a best solution which is also a best solution for each DM. In different real world decision problems, conflicting of the corresponding objective functions is happened. The most significant advantage of FGP approach in MLPPs over the other traditional approaches(vertex enumeration, transformation approach) is that solving of complex real world problems through transforming the problem in to simple and understandable problem and after that we can find the satisfactory solution simply. Some researchers have tried to apply FGP approach partially to solve linear fractional, quadratic, and quadratic fractional BLPPs through linearizing the nonlinear part using Taylor series, so that they only receive partial benefit from it[13, 20, 27, 35, 43]. Moreover, the increase of only one level, the solution of the problem is much more complicated compared to the BLPP. Still now, no existing algorithms have proposed the whole concept of FGP to general nonlinear MLPs. Therefore, the huge hierarchical structures of the real world problems which mostly contains nonlinear objective functions and constraint functions, the lack of efficient technique for general MLPPs and the performance of fuzzy goal programming (FGP) approach in linear multilevel programming problems, motivated us to apply the concept of FGP approach by introducing some modifications of the existing approaches in QFBLPPs, for efficiently implementing it to general some nonlinear tri-level programming problems.

Chapter 3

Fuzzy Goal Programming Approach For Tri-level Nonlinear Programming

In the real world, we often encounter decision making situations involving multiple decision makers (DMs). Especially in industrial or governmental decision making situations, DMs have different interest and decision priority. A three-level programming problem is one of the mathematical optimization models which has three decision makers as leader(decision maker one),decision maker two and decision maker three sequentially and each decision maker controls part of the variables. In three level programming problems each decision maker has only one objective function and controls part of the decision variables.

The three level programming problem (TLPP), whether from the standpoint of the three-ploy Stackelberg behavior or from the interactive organizational behavior, is a very practical problem and encountered frequently in actual practice. For example, Cassidy et al. [16] considered the distribution of federal budget. Federal government, the first level, allocates the money to the various states; the states, as the second level of managers, allocate money to the various cities and the city can further allocate to different projects. The behavior of the lower hierarchies are not within the control of the higher hierarchies except through legislation and existing laws, and thus the allocation of budget of the federal government has the same characteristics as that of the Stackelburg behavior [31].

An important feature of multilevel optimization problems is that a planner at one level of the hierarchy may have his objective function determined, in part, by variables controlled at other levels. However, his control instruments may allow him to influence the policies at other levels, and thereby improve his own objective function. Such policies may include the control of the allocation and use of resources at lower levels, and the control of the benefits conferred upon subordinate levels [9].

Tri-level programming is known to be extremely hard to solve. But, for linear and convex multilevel programming problems different approaches are developed such as fuzzy approach and transformation approach etc. However, in this study we concentrate only on tri-level nonlinear programming problems. One way to handle such hierarchies is to focus on one level and include other levels' behaviors as assumptions. The main problem in trilevel nonlinear programming problem(TLNLPP) is the way of making decision, because each decision maker wants to maximize their profit or minimize their cost within the same or intersecting feasible region. But, the objective functions of the decision makers may conflict each other. That means there is a cooperation between each decision makers and the decision of one decision maker affects directly or indirectly the decisions of the other decision makers. So, to make fair distribution of power to each decision makers, we can use different techniques like fuzzy programming and transformation approach. However, in this study we are considering only the fuzzy programming approach to handle the problem using fuzzy set theoretic concepts.

Moreover, to distribute the power to each of the decision makers we can use different techniques such as fuzzy programming approach and fuzzy goal programming approach. In this thesis we try to apply the fuzzy goal programming approach in order to find the satisfactory solution of the tri-level nonlinear programming problem.
Now, in order to apply the method first we have to discuss some concepts about goal programming and fuzzy set theory.

3.1 Fuzzy Goal Programming

3.1.1 Goal Programming

Goal programming is an optimization technique designed to handle decision-making situations where a number of conditions characterized as goals are to be met as closely as possible. In goal programming, although one can expect that there is a set of feasible solutions satisfying the constraints, none of them may simultaneously satisfy all the conflicting goals of the organization.

The objective of goal programming is to find a solution that satisfies the true constraints and comes closest to meeting the stated goals.

Goal programming approaches analyze how much a proposed solution deviates from each stated goals. Accordingly, for each goal a pair of deviation variables are defined (one equaling the amount by which the solution overachieves the goal; the other equaling the amount by which it fails to meet the goal).
Before formulating the goal programming we have to set some basic definitions as follows.

Definition 3.1.1.1. *[25]* ***[Goal]*** A goal refers to a criterion and a numerical level, known as a target level, which the decision maker(s) desire to achieve on that criterion.

There are three principal types of goals that can occur in a goal programming model;

- Achieve at most the target level(Goal type 1)
- Achieve at least the target level(Goal type 2)
- Achieve the target level exactly(Goal type 3)

Definition 3.1.1.2. *[25]* ***[Deviational Variable]*** A deviational variable measures the difference between the target level on a criterion and the value that is able to be achieved in a given solution.

- If the achieved value is above the target level then the difference is given by the value of the positive deviational variable.
- If the achieved value is below the target level then the difference is given by the value of the negative deviational variable.

The essence of goal programming is the minimization of unwanted deviational variables. For goal type 1 ('less is better'), the positive deviational variable is said to be the unwanted deviational variable . For goal type 2('more is better), the negative deviational variable is said to be the unwanted deviational variable. For goal type 3, both positive and negative deviational variables are said to be unwanted deviational variables.

In general, goal programming is a powerful and effective methodology for the modeling, solution, and analysis of problems having multiple and conflicting goals and objectives, it has often been cited as being the "workhorse" of multiple objective optimization (i.e.,the solution to problems having multiple, conflicting goals and objectives) as based on its extensive list of successful applications in actual practice [42].

Algebraically speaking, we allow our generic goal programme to have Q goals, which we give the index $q = 1, ..., Q$. We also define n decision variables which we shall term $x = (x_1, x_2, ..., x_n)$. These are the factors over which the decision maker(s) have control and define the decision to be made. Each goal has an achieved value $f_q(x)$, on its underlying criterion which is a function of the decision variables. Note that in this generic form no assumptions have yet been made about the nature of the decision variables of goals. The decision maker(s) sets a numeric target level for each goal which is denoted by b_q. This leads to the basic formulation of the q^{th} goal:

$$f_q(x) + d_q^- - d_q^+ = b_q$$

where d_q^- is the negative deviational variable of the q^{th} goal that represents the level by which the target level is under-achieved and d_q^+ is the positive deviational variable of the q^{th} goal that represents the level by which the target level is over-achieved. The two deviational variables are constrained to take non-negative values and both cannot take a non-zero value simultaneously.

The decision maker must then decide which deviational variables are unwanted and these are penalized in an achievement function. There are three basic types of penalization:

- A typical d_q^+ goal would involve cost, where any positive deviation above the goal level would be penalized.
- A typical d_q^- goal would involve profit, where any negative deviation below the goal level would be penalised.
- A typical $d_q^+ + d_q^-$ goal would involve a workforce-level target, where any negative or positive deviation from the target level would be penalised.

We note that the set of goals are sometimes termed as soft constraints. That is, the decision maker desires to meet each goal but if the goal is not achieved, then this does not imply that the solution is infeasible. Goal programming also allows for an addition of a set of linear programming style hard constraints whose violation will make the solution infeasible. These are modeled by adding the condition $x \in S$ where S is the feasible region made up of points in decision space that satisfy all of the constraints and sign restrictions. Finally, the unwanted deviational variables need to be brought together in the form of an achievement function whose purpose is to minimize them and thus ensure that a solution that is "as close as possible" to the set of desired goals is found. The exact nature of the achievement is dependent on the goal programming variant used, so in our generic form it is simply represented by a generic function of the deviational variables:

$$Min \;\; \lambda = h(d^-, d^+)$$

where d^- is the vector of q negative deviational variables and d^+ is the vector of q positive deviational variables. Although this function is termed as an achievement function, in reality it measures the "lack" of achievement of the goals. That is the distance from the target to the achieved level of the goals.

The above considerations lead to the generic algebraic form of the goal programming:

$$Min \;\; \lambda = h(d^-, d^+) \tag{3.1}$$

subject to:

$$f_q(x) + d_q^- - d_q^+ = b_q, q = 1, ..., Q$$
$$x \in S$$
$$d_q^-, d_q^+ \geq 0, q = 1, ..., Q$$

Note that the condition $d^-.d^+ = 0$ is also required to hold, but this will automatically be satisfied due to the nature of the goal programming minimisation.

Depending on the objective functions of the problem, goal programming can be fractional goal programming and nonlinear goal programming. A fractional goal programming has one or more goals of the form

$$\frac{f_q(x)}{g_q(x)} + d_q^- - d_q^+ = b_q$$

where $\frac{f_q(x)}{g_q(x)}$ is a generic function of the decision variables. This type of goal programming is mentioned by Romero [38] as arising in the fields of financial planning, production planning, and engineering. This variant also occurs in some goal programming based methods for deriving weighting vectors from pairwise comparison matrices [19]. Romero also warns on the perils of simple liberalization to the following form:

$$f_q(x) + d_q^- - d_q^+ = b_q.g_q(x)$$

This is not a valid mathematical transformation as the deviational variables should have been multiplied by the function $g_q(x)$ as well, giving

$$f_q(x) + d_q^-.g_q(x) - d_q^+ g_q(x) = b_q.g_q(x)$$

Hence the new terms $d_q^- g_q(x)$ and $d_q^+ g_q(x)$ mean that linearisation has not been achieved. The advent of more powerful non-linear programming technology and computing power; heuristics such as multi-objective evolutionary methods and advances in exact methods for solving fractional goal programmes have made the need for linearisation less urgent. In many cases the model can now be solved without linearisation. Fractional goal programmes also require their own adapted form of Pareto efficiency detection and restoration procedure in multiobjective programming problems.

The extra level of difficulty of solving a non-linear goal programme over a linear goal programme can be thought of as similar to the extra level of difficulty in solving a non-linear programme over a linear programme. Non-linearity can occur in the achievement function, goals, and/or constraints. Thus linearisation techniques are one possibility to obtain an approximate solution if the degree of non-linearity is not too large. But, if the degree of non-linearity is big the linearization technique have the possibility to give undesired solutions. A further possibility is to use one of the heuristic or meta-heuristic techniques, although this will require greater investment of time in setting up a bespoke model and fine-tuning the model parameters.

3.1.2 Fuzzy Set

Informally, a fuzzy set is a class of objects in which there is no sharp boundary between those objects that belong to the class and those that do not. A more precise definition is stated as follows.

Definition 3.1.2.1. *[3](Fuzzy Set)* Let X denote a universal set. Then, a fuzzy set $\tilde{A}$ is defined by its membership function

$$\mu_{\tilde{A}} : X \to [0, 1]$$

which assigns to each element $x \in X$ a real number $\mu_{\tilde{A}}(x)$ in the interval [0,1], where the value of $\mu_{\tilde{A}}$ at x represents the grade of membership of x in $\tilde{A}$. Thus, the nearer the value of $\mu_{\tilde{A}}(x)$ is unity, the higher the grade of membership of x in $\tilde{A}$.

A fuzzy subset $\tilde{A}$ can be characterized as a set of ordered pairs of element x and grade $\mu_{\tilde{A}}(x)$ and is often written

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\}$$

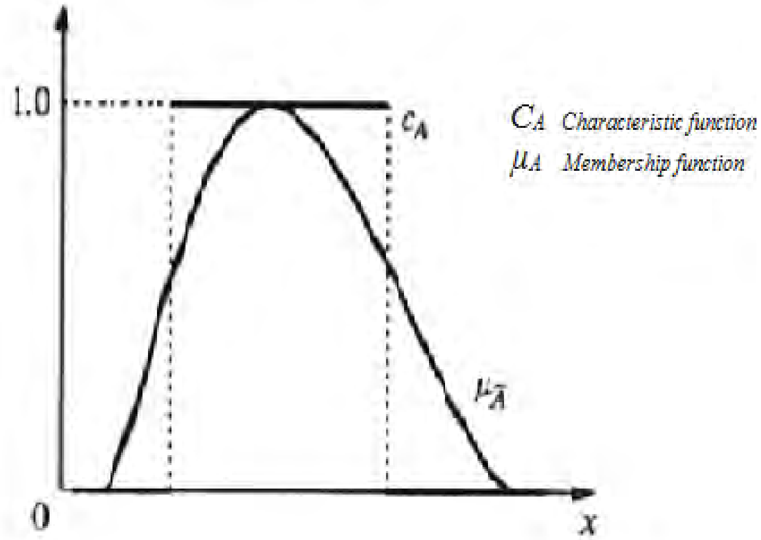

Figure 3.1: Fuzzy membership function and characteristic function

When the membership function $\mu_{\tilde{A}}(x)$ contains only the two points 0 and 1, then $\mu_{\tilde{A}}(x)$ is identical to the characteristic function

$$C_{\tilde{A}} : X \to \{0, 1\}$$

and hence, $\tilde{A}$ is no longer a fuzzy subset, but an ordinary set A.
From these definitions, one finds that a fuzzy set $\tilde{A}$ is a natural extension of an ordinal set A.

Figure (3.1) illustrates the membership function $\mu_{\tilde{A}}(x)$ of a fuzzy subset $\tilde{A}$ together with the characteristic function $C_A(X)$ of an ordinary set A.

Definition 3.1.2.2. *[3]* The membership function $\mu_{\tilde{C}}$ of the intersection of two fuzzy sets $\tilde{C} = \tilde{A} \bigcap \tilde{B}$ is pointwise defined for all $x \in X$ by

$$\mu_{\tilde{C}}(x) = min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$$

Definition 3.1.2.3. *[3]* The membership function $\mu_{\tilde{D}}$ of the union of fuzzy sets $\tilde{D} = \tilde{A} \bigcup \tilde{B}$ is pointwise defined for all $x \in X$ by

$$\mu_{\tilde{D}}(x) = max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$$

Usually, the maximum (or minimum) of a function f over a given domain D is attained at a precise point x_0. However, we may be interested in the behavior of the function in a neighborhood of x_0; the concept of a maximizing set (minimizing set) provides a tool for modeling this situation. The notion of an extremum also must be generalized to deal with problems such as an extremum of a function over a fuzzy domain or an extremum of a fuzzy function over a domain.

Definition 3.1.2.4. *[55]* Let f be a real-valued function whose domain is a set X and f is assumed to be bounded from below by $inf(f)$ and from above by $sup(f)$. The maximizing set is a fuzzy set M in X such that:

$$\forall x \in X; \mu_M(x) = \frac{f(x) - inf(f)}{sup(f) - inf(f)}$$

We always have

$$\mu_M(x_0) = 1$$

for all x_0 such that $f(x_0) = sup(f)$ and

$$\mu_M(x_0) = 0$$

for all x_0 such that $f(x_0) = inf(f)$.
Clearly, the maximizing set provides essential information about the effect on the value of the objective function f of choosing values of x other than x_0.

Definition 3.1.2.5. *[55]* Let f be a real valued function and $\tilde{C}$ a fuzzy constraint (solution space). If $f(x)$ is bounded on $\tilde{C}$, then the maximizing set over a fuzzy constraint $\tilde{MC}(f)$, is defined by its membership function

$$\mu_{M\tilde{C}(f)}(x) = \begin{cases} 0, if \;\; f(x) \leq inf_{\tilde{C}}(f) \\ \frac{f(x) - inf_{\tilde{C}}(f)}{sup_{\tilde{C}}(f) - inf_{\tilde{C}}}, if \quad inf_{\tilde{C}}(f) < f(x) < sup_{\tilde{C}}(f) \\ 1, if \;\; f(x) \geq sup_{\tilde{C}}(f) \end{cases} \tag{3.2}$$

The above two definitions (3.1.2.4) and (3.1.2.5) are defined for maximization of a function over a fuzzy(crisp) set only. But, for minimization of a function over a fuzzy set(crisp set) the following definitions could be formulated.

Definition 3.1.2.6. Let f be a real-valued function whose domain is a set X and f is assumed to be bounded from below by $inf(f)$ and from above by $sup(f)$. The minimizing set is a fuzzy set M in X such that:

$$\forall x \in X; \mu_M(x) = \frac{sup(f) - f(x)}{sup(f) - inf(f)}$$

From this definition we always have

$$\mu_M(x_0) = 1$$

for all x_0 such that $f(x_0) = inf(f)$ and

$$\mu_M(x_0) = 0$$

for all x_0 such that $f(x_0) = sup(f)$.
Clearly, the minimizing set provides essential information about the effect on the value of the objective function f of choosing values of x other than x_0. For instance for any $\bar{x} \in M$ other than x_0 , $\mu_M(x_0) \geq \mu_M(\bar{x})$ if $f(x_0) = inf(f)$ and $\mu_M(x_0) \leq \mu_M(\bar{x})$ if $f(x_0) = sup(f)$.

Definition 3.1.2.7. Let f be a real valued function and $\tilde{C}(x)$ a fuzzy constraint (solution space). If $f(x)$ is bounded on $\tilde{C}$, then the minimizing set over a fuzzy constraint $\tilde{MC}(f)$, is defined by its membership function

$$\mu_{M\tilde{C}(f)}(x) = \begin{cases} 0, if \;\; f(x) \geq sup_{\tilde{C}}(f) \\ \frac{sup_{\tilde{C}}(f)-f(x)}{sup_{\tilde{C}}(f)-inf_{\tilde{C}}}, if \quad inf_{\tilde{C}}(f) < f(x) < sup_{\tilde{C}}(f) \\ 1, if \;\; f(x) \leq inf_{\tilde{C}}(f) \end{cases} \tag{3.3}$$

Two important elements of decision making are the goals of the decision that are represented by the maximized objective function and the imposed constraints that confine the search space. Fuzzy decision making essentially replaces the crisp goals and the constraints with their fuzzy equivalents. The following definitions are due to Bellman and Zadeh[47].

Definition 3.1.2.8. *[47]* Let S be a given set of possible alternatives which contains a solution to a decision making problem under consideration. A fuzzy goal G is a fuzzy set on S characterized by its membership function

$$\mu_S : S \to [0, 1],$$

which represents the degree to which the alternatives satisfy the specified decision goal.

In general, a fuzzy goal indicates that a target should be obtained, but it also quantifies the degree to which the target is fulfilled.

Definition 3.1.2.9. *[47]* Let S be a given set of possible alternatives which contains a solution to a decision making problem under consideration. A fuzzy constraint C is a fuzzy set on S characterized by its membership function

$$\mu_C : C \to [0, 1],$$

which constrains the solution to a fuzzy region within the set of possible solutions.

A fuzzy constraint is a generalization of a crisp constraint. In fact, the support of a fuzzy constraint determines the set of alternatives in which the solution to the decision making problem lies. The support defines a crisp set and thus it defines the crisp constraints to the problem. The solution set, however, is not the crisp constraint set as is the case in conventional optimization. Instead, certain alternatives in the solution set satisfy the constraints completely, while others violate them to some degree.

Realizing that a decision should satisfy the decision goals as well as the decision constraints, Bellman and Zadeh suggested the following model for the fuzzy decision.

Definition 3.1.2.10. *[47]* Let S be a given set of possible alternatives which contains a solution to a decision making problem under consideration. Let G be the set of fuzzy goals for the decision, represented by the membership function $\mu_G(x)$, $x \in S$, and let C be the set of fuzzy constraints represented by the membership function $\mu_C(x)$, $x \in S$. Then the fuzzy decision F results from the intersection of the fuzzy decision goals and the fuzzy constraints, i.e.,

$$F = G \bigcap C.$$

The fuzzy decision is characterized by its membership function

$$\mu_F(x) = \mu_G(x) \bigwedge \mu_C(x)$$

where $\bigwedge$ denotes the minimum operation.

Definition 3.1.2.11. *[47]* The optimal decision x^* in fuzzy decision making is the decision with the largest membership value, also called the maximizing decision, which is defined by

$$x^* = \arg\max_{x \in C}(\mu_G(x), \mu_C(x))$$

It is important to realize that the distinction between the goals and the constraints disappear in this model. Essentially, both the goals and the constraints are represented by membership functions defined on the set of possible alternatives. The decision function (the conjunction in the model) makes an appropriate combination of the goals and the constraints. Because of the symmetry between the goals and the constraints, this model is sometimes called a symmetric model. Since both the goals and the constraints are represented by fuzzy sets, it is quite possible that the goals are achieved only partially while the constraints are violated slightly.

3.1.3 Fuzzy Goal Programming

The standard Goal Programming (GP) model considers the aspiration levels (goals) as precise and deterministic. However, in practice, there are many decision-making situations where the decision-maker is not able to establish the goal values precisely. The goals fuzziness is more related to the nature of the objectives involved in the decision making situation. The Fuzzy Goal Programming (FGP) Model has been developed in the earliest of the 80s to deal with such situations. The concept of membership functions, based on fuzzy set theory, has been used for modelling the fuzziness of the goals in the GP [7].

Fuzzy goal programming utilises fuzzy set theory (Zadeh, 1965) to deal with a level of imprecision in the goal programming model. This imprecision is normally related to the goal target values but could also be related to other aspects of the goal programme such as the priority structure. There are various possibilities for measuring the fuzziness around the target goals, each of which leads to a different fuzzy membership function. These functions model the drop in dissatisfaction from a state of total satisfaction (where the membership function takes the value 1) to a state of total dissatisfaction (where the membership function takes the value 0). There are many possible fuzzy membership functions, the algebraic structure of the most common linear fuzzy membership functions are outlined below[25]:

- Algebraically, positive deviations (Right sided) penalised by the linear function:

$$\mu[f_q(x)] = \begin{cases} 1, f_q(x) \leq b_q \\ 1 - \frac{f_q(x) - b_q}{p_{max}}, b_q \leq f_q(x) \leq b_q + p_{max} \\ 0, f_q(x) \geq b_q + p_{max} \end{cases} \tag{3.4}$$

- Algebraically, negative deviations (Left sided) penalised by the linear function:

$$\mu[f_q(x)] = \begin{cases} 1, f_q(x) \geq b_q \\ 1 - \frac{b_q - f_q(x)}{p_{max}}, b_q - p_{max} \leq f_q(x) \leq b_q \\ 0, f_q(x) \leq b_q - p_{max} \end{cases} \tag{3.5}$$

  where p_{max} is the maximum difference between the target level b_q and the criterion in which the decision maker want to achieve,i.e.,

$$0 \leq |b_q - f_q(x)| \leq p_{max}.$$

In negative deviations,$0 \leq b_q - f_q(x) \leq p_{max}$ as

$$|b_q - f_q(x)| = b_q - f_q(x).$$

and
In positive deviations, $0 \leq f_q(x) - b_q \leq p_{max}$, as

$$|b_q - f_q(x)| = f_q(x) - b_q.$$

Graphically, positive deviation (Right sided) and negative deviation (Left sided) can be expressed as in figures 3.2 and 3.3, respectively:

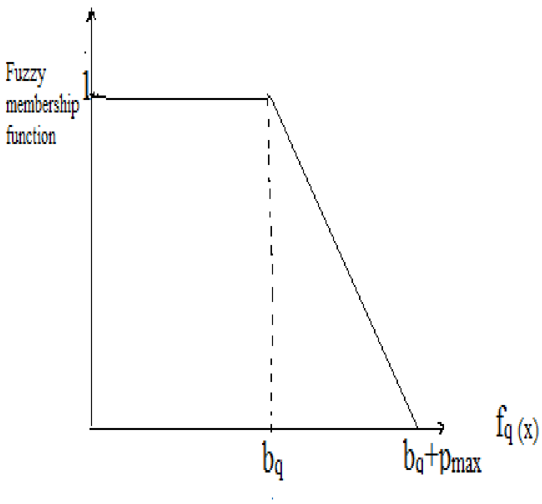

Figure 3.2: Right sided fuzzy membership function

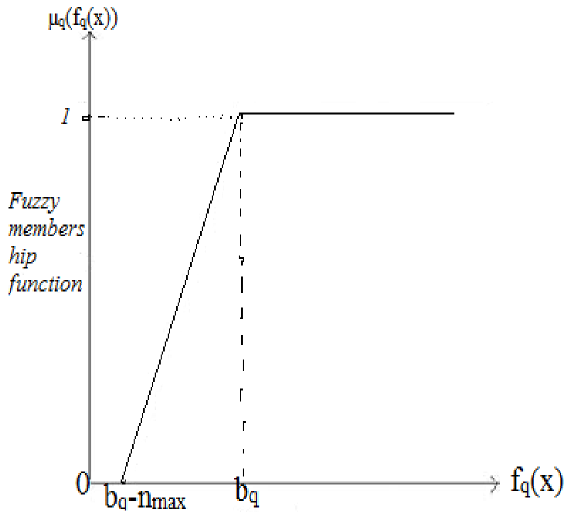

Figure 3.3: Left sided fuzzy membership function

3.2 Problem Formulation and Solution Concept of Tri-level Nonlinear programming

In a hierarchical decision system, where all the DMs are motivated to cooperate with each other and each DM tries to optimize his/her own benefit, they pay series attention to the interest of the other. But, making fair distribution of power to the decision makers is not that much easy and explicit. So, to solve such kind of problems, different researchers have been tried to introduce fuzzy set theory and using this concept different methods developed such as fuzzy programming and fuzzy goal programming[35]. Since the DMs may interested to make decision cooperatively, the objectives of the DMs may conflict in nature. Now, in this thesis, we try to apply fuzzy goal programming(FGP) method to solve the TLPP by transforming the TLPP in to conventional single level programming problem and iterate to arrive at the satisfactory solution.

Consider a Quadratic fractional TLPP containing minimization type objective functions at each level. Suppose that DM$_i$ denotes the DM at the i^{th} -level($i = 1, 2, 3$) and DM_1,DM_2 and DM_3 controls the decision vector $x \in X \subseteq R^{n_1}$,$y \in Y \subseteq R^{n_2}$,and $z \in Z \subseteq R^{n_3}$, respectively, where $n = n_1 + n_2 + n_3, x = (x_1, x_2, x_3, ..., x_{n_1}), y = (y_1, y_2, y_3, ..., y_{n_2}), z = (z_1, z_2, z_3, ..., z_{n_3}) and (x, y, z) \in R^n$. Let F_1, F_2 and F_3 are the nonlinear objective functions of the first DM, second DM and third DM with respective controlling vector of decision variables x, y and z.
Mathematically, the QFTLPP of minimization type can be formulated as:
Find $X(x_1, x_2, x_3)$ so as to

$$\begin{aligned} &\min_x F_1(x, y, z) \ \ where \ \ y, z \ \ solves, \\ &\min_y F_2(x, y, z) \ \ where \ \ z \ \ solves, \\ &\min_z F_3(x, y, z) \\ &subject \ \ to : (x, y, z) \in S \subseteq \mathbb{R}^n \end{aligned} \tag{3.6}$$

Where

- $S = \{(x, y, z) \in \mathbb{R}^n | A_1x + A_2y + A_3z\{\leq, \geq, =\}b : x, y, z \geq 0\}$ and $\bar{X} = (x, y, z) \in \mathbb{R}^n$,
- Feasible region S is a nonempty bounded set.
- C_{ij} and b $(i = 1, 2, 3; j = 1, 2)$ are constant vectors.
- $D_{ij}(i = 1, 2, 3; j = 1, 2)$ are constant symmetric matrices.
- $C_{ij}x + \frac{1}{2}\bar{X}^T D_{ij}\bar{X} + \alpha_{ij}(i = 1, 2, 3; j = 1, 2)$ is positive for all $\bar{X} \in S$
- $F_i(\bar{X}) = \frac{\alpha_{i1}+C_{i1}\bar{X}+\frac{1}{2}\bar{X}^T D_{i1}\bar{X}}{\alpha_{i2}+C_{i2}\bar{X}+\frac{1}{2}\bar{X}^T D_{i2}\bar{X}}$, for all $i = 1, 2, 3$

Now we need to solve problem(3.6) using fuzzy goal programming approach and find the satisfactory solution for all the decision makers who work co-operatively. In real life, different decision makers may or may not have a conflict in nature, and so all the decision makers may or may not cooperate each other. Therefore, if the objective functions of the DMs are conflict each other then they motivate to cooperate each other and if there is no conflict between each objective function of the DMs then it not necessary to cooperate each other.

In order to apply the fuzzy goal programming approach, we have to know the imprecise aspiration level of all the objective functions and all the objective functions must be bounded on the constraint set. However, the guarantee for boundedness of the objective functions is obtained from the compactness of the constraint set and continuity of the objective functions,

$$F_1(\bar{X}) = \frac{f_{11}(\bar{X})}{f_{12}(\bar{X})},\ \ F_2(\bar{X}) = \frac{f_{21}(\bar{X})}{f_{22}(\bar{X})}$$

and $F_3(\bar{X}) = \frac{f_{31}(\bar{X})}{f_{32}(\bar{X})}$ where

$$f_{ij}(\bar{X}) = \alpha_{ij} + c_{ij}\bar{X} + \frac{1}{2}\bar{X}^T D_{ij}\bar{X}, i = 1:3$$

and $j = 1:2$ and $\bar{X}^T$ denotes transpose of decision vector.

The QFTLPP could be solved using the fuzzy goal programming approach by constructing a membership function and transforming the QFTLPP in to single level fuzzy goal programming problem. In the following section, the construction of membership function and transformation of the QFTLPP in to single level programming problem will be discussed step by step.

3.2.1 Construction of membership functions

We know that the DMs are interested in minimizing their own objective values over the same feasible region S. Then, the optimal solution of each decision maker when determined in isolation is acceptable and also can be considered as an acceptable aspiration level of the corresponding objective function. Although the solutions of the DMs are different, each DM needs to make decision cooperatively with small violation from the individual optimal solution as much as possible. Besides to this, all the decision makers are interested to find a satisfactory solution in the neighborhood of the individual solutions with small violation from the individual solution in such a way that all the DMs are agree as much as possible. So, the fuzziness or vagueness of the decision maker's judgment is occurred. The fuzziness is faced in this because we do not have full information about the violation of the individual solution, i.e., "how much the violation is" from the individual solution. That means, each decision maker face a difficulty to make decision because of the conflict between each objective functions of the DMs. Therefore, introducing a fuzzy set theory in MLPPs is better to find a satisfactory solution for the organization or hierarchical system.

Fuzzy set theory provides a strict mathematical framework (there is nothing fuzzy about fuzzy set theory!) in which vague conceptual phenomena can be precisely and rigorously studied. It can also be considered as a modelling language well suited for situations in which fuzzy relations, criteria, and phenomena exist. The fuzzy objective function is characterized by its membership function and so are the constraints. Since we want to satisfy (optimize) the objective function as well as the constraints, a decision in a fuzzy environment is defined by analogy to nonfuzzy environments as the selection of activities which simultaneously satisfy objective function(s) and constraints. The decision in a fuzzy environment can therefore be viewed as the intersection of fuzzy constraints and fuzzy objective function(s). The relationship between constraints and objective functions in a fuzzy environment are therefore fully symmetric, that is, there is no longer a difference between the former and the latter.

Mathematically, the concept of fuzzy set theory could be introduced as follows:
Let

$$F_i^b = \min_{(x,y,z)\in S} F_i(x,y,z) = F_i(x^{b_i}, y^{b_i}, z^{b_i}), i = 1:3$$

Then $(x^{b_1}, y^{b_1}, z^{b_1}), (x^{b_2}, y^{b_2}, z^{b_2})$, and $(x^{b_3}, y^{b_3}, z^{b_3})$ are absolutely acceptable optimal solutions to the respective DMs, but the individual solution of one DM may not be satisfactory for the other DM.
Now, $F_i(\bar{X}) \geq F_i^b$ for all $i = 1,2,3$ and for all $\bar{X} = (x,y,z) \in S$ as F_i^b, $i = 1,2,3$ is the individual optimal value. But, our interest is to find an optimal value which is greater than the individual optimal value in such a way that all the DMs agree. Since F_i^b is the individual optimal value, there is no any $\bar{X}^* \in S$ such that $F_i(\bar{X}^*) < F_i^b$. So the vagueness is in one side which is greater than or equal to the individual optimal value. Mathematically, the fuzzy theoretic concept can be expressed as:

$$F_i(\bar{X}) \preceq F_i^b, \;\; i = 1,2,3$$

where the operation $\preceq$ reads as "essentially less than or equal to", i.e., small violation of the usual operation "less than or equals to" ($F_i(\bar{X}) \leq F_i^b$) is possible.
Since $F_i^b(i = 1, 2, 3)$ is the individual optimal value, there is no any $\bar{X}(x, y, z) \in S$ such that $F_i(x, y, z) < F_i^b(i = 1, 2, 3)$. Although the violation is always above the individual optimal value, there is no any restriction on the violation,i.e., "to what extent is the violation?". To avoid such problem we have to identify the above limitation of the violation and then up to the above limitation the violation is possible otherwise not. If the above limitation is $b_i(i = 1, 2, 3)$, then the fuzzy objectives of the decision makers can be expressed using membership function.

The individual solution of each DM could be found using parametric methods,.i.e., Dinkelbach's algorithm which is appropriate for nonlinear fractional programming problems[48]. using this algorithm , $(x^{b_1}, y^{b_1}, z^{b_1})$, $(x^{b_2}, y^{b_2}, z^{b_2})$ and $(x^{b_3}, y^{b_3}, z^{b_3})$ are the individual solutions of the first, second , and third decision levels, respectively. Since the first decision maker controls only the decision variable x, x^{b_1} is the best solution for the first decision maker. However, due to the cooperation between each decision maker, the lower decision makers may be dissatisfied with this solution. Thus, in order to satisfies the lower level decision makers the decision variable controlled by the first level must be varied. Similarly, y^{b_2} is best solution for the second decision maker, but the value of x is varied. Therefore, the variation of the variables can be restricted based on the solutions we obtained individually, because each decision maker is interested on the solution approximately closest to the individual solution. Consequently, there is a variation of the corresponding objective value.

Naturally, tri-level programming problem has the property of DM1 specifies his/her control variable value x in such a way that the objective value is minimum and take the other control variables as parameters and then DM2 specifies his/her control variable value y based on the decision of DM1 such that his objective value is minimum and control variable of DM3 is considered as parameter. Finally, DM3 decides his/her control variable value based on the previously specified control variable values x and y. Therefore, solution of the tri-level programming problem is (x, y, z). However, this nature of tri-level programming problem may not be easily controllable. To handle such difficulty, fuzzy set theory concept is introduced by different researchers and they transformed the model to another conventional model through expressing the control variables of DM1 and DM2 and also objective function of all the DMs by membership function. However, if the objective functions are nonlinear, inconsistency between the membership function of control variables and objective functions may be occurred. To overcome such problem, make free the control variables and then find the satisfactory solution through updating the membership function of objective function goals.

Remark: If $(x^{b_1}, y^{b_1}, z^{b_1}) = (x^{b_2}, y^{b_2}, z^{b_2}) = (x^{b_3}, y^{b_3}, z^{b_3})$, then there is no any conflict between each objectives of the DM and the individual solution of each DM also a best solution for the QFTLPP.

However, due to conflicting nature of the objectives of the DMs, the individual solutions of the decision makers are might be unequal. So the DMs could be forced to find a satisfactory solution around the individual optimal solution. Therefore, the respective fuzzy tolerance values of the objectives of leader and followers appear as:

$$F_1(x, y, z) \preceq F_1^b, F_2(x, y, z) \preceq F_2^b, \quad and \quad F_3(x, y, z) \preceq F_3^b \tag{3.7}$$

The operation $\preceq$ reads as "essentially less than or equal to". The $\preceq$ indicates that small violation from the usual operation $\leq$ is possible and equation (3.7) is called the fuzzy goals of the decision makers.

Moreover,

$$F_1^b \leq F_1(x^{b_2}, y^{b_2}, z^{b_2}), F_1(x^{b_3}, y^{b_3}, z^{b_3});$$
$$F_2^b \leq F_2(x^{b_1}, y^{b_1}, z^{b_1}), F_2(x^{b_3}, y^{b_3}, z^{b_3})$$

and

$$F_3^b \leq F_3(x^{b_1}, y^{b_1}, z^{b_1}), F_3(x^{b_2}, y^{b_2}, z^{b_2})$$

Therefore,

$$F_i^b \leq F_i(x, y, z) \leq F_i^1 = \max_{(x,y,z)\in S} F_i(x, y, z), for\ all\ (x, y, z) \in S\ and\ i = 1 : 3$$

and the fuzzy goal $F_i(x) \preceq F_i^b$ can be restricted on the interval $[F_i^b, F_i^1]$. That means, if the violation is above F_i^1 it is impossible and if the value of $F_i(x)$ is strictly less than the individual optimal value then it is a best solution but it may not happen. Hence, it is better to search the optimal value of $F_i(x)$ on the interval $[F_i^b, F_i^1]$ such that all the DMs are satisfied. In order to find such satisfactory optimal value first we have to construct the membership function that describes the degree of closeness to the individual optimal value.

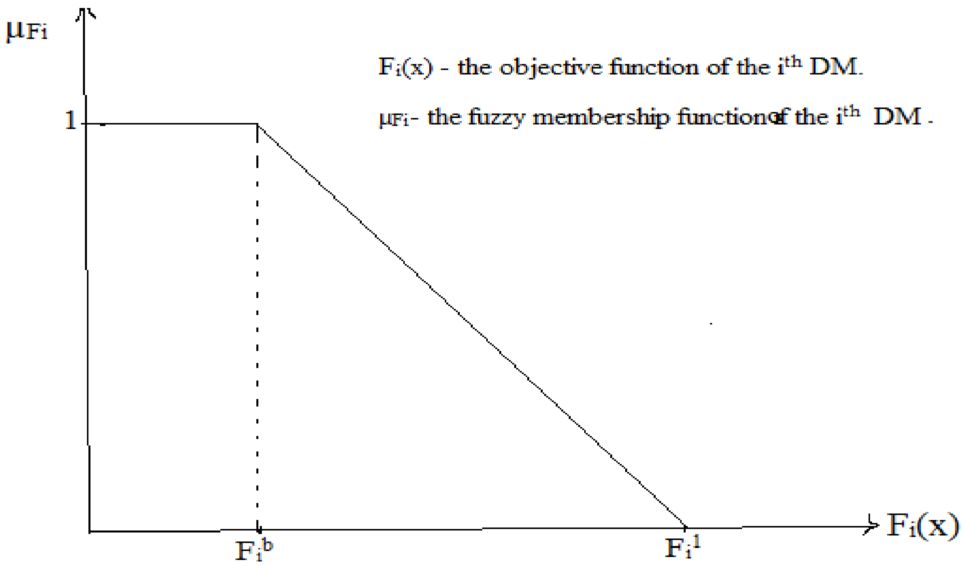

Figure 3.4: Fuzzy membership function of the i^{th} DM

Now the membership function of the i^{th} DM can algebraically be formulated as:

$$\mu_i(F_i(\bar{X})) = \begin{cases} 1, & if \;\; F_i(\bar{X}) < F_i^b \\ \frac{F_i^1 - F_i(\bar{X})}{F_i^1 - F_i^b}, & if \;\; F_i^b \leq F_i(\bar{X}) \leq F_i^1 \\ 0, & if \;\; F_i(\bar{X}) > F_i^1 \end{cases} \quad , for \;\; i = 1, 2, 3 \qquad (3.8)$$

where $F_i^1 = max_{(x,y,z)} F_i(x, y, z), \;\; for \;\; i = 1 : 3$

Some properties of the membership functions are:
Property 1: $\mu_i(F_i(\bar{X}^*)) \leq \mu_i(F_i(\bar{X}))$ if and only if $F_i(\bar{X}) \leq F_i(\bar{X}^*)$ for $\bar{X}^*, \bar{X} \in S$ and for all $i = 1, 2, 3$.

Property 2: Maximization of the membership function $\mu_{F_i}(x)$ in feasible set S implies finding of the satisfactory solution for the i^{th} DM.

Property 3: As $\mu_i(F_i(\bar{X})) \to 1$, $F_i(x) \to F_i^b$. That means, as $\mu_i(F_i(\bar{X}))$ close to unity the solution becomes better.
Graphically, the membership function of the i^{th} DM can be describe as in figure (3.4):
From the above properties, we can conclude that finding the best objective value for each DM is equivalent with maximization of the corresponding

membership function. Therefore, the original TLPP is equivalent with the following problem (3.9).

$$
\begin{aligned}
DM_1 &\quad \max \mu_1(F_1(\bar{X}))\\
DM_2 &\quad \max \mu_2(F_2(\bar{X}))\\
DM_3 &\quad \max \mu_3(F_3(\bar{X}))\\
&\textit{subject to } \bar{X} \in S
\end{aligned} \tag{3.9}
$$

The fuzzy goal programming problem can be formulated as in the following section.

3.2.2 Formulation of the FGPP for QFTLPP

In the formulation of fuzzy goal programming problem (FGPP), we use the concepts of membership functions at each level, and develop a new conventional programming problem for solving the trilevel nonlinear programming problem (TLNLPP). To transform the TLNLPP, we propose that the first level decision maker (FLDM) defines his/her objective functions with possible tolerances, which are described by linear membership functions of fuzzy set theory and fuzzy decision. This information is delivered to the second level decision maker (SLDM) who defines his/her objective functions with possible tolerances in view of the first level decision maker (FLDM). Finally, the TLDM solves his/her problem under restrictions of the FLDM and SLDM requirements. Then, the TLDM presents his/her solution to the FLDM. If the FLDMs reject this proposal, the FLDM must update and change former goals as well as their corresponding tolerances also the SLDM must do the same until a satisfactory solution is reached. After some iterations we arrive at compromise(satisfactory) solution of the TLNLP problem.

In decision making situation, the aim of each decision maker is to achieve highest membership value(unity) of the associated fuzzy goal in order to obtain the absolute satisfactory solution. However, in real practice, achievement of all membership values to the highest degree(unity) is not possible, because of conflicting objectives. Therefore, decision policy for minimizing the regrets of the DMs for all levels should be taken into consideration. Hence, each DM should try to maximize his/her membership function as close as possible to unity through minimizing its negative-deviational variables. Therefore, in effect we are simultaneously optimizing all the objective functions.

In fuzzy programming (FP) approaches, the highest degree of membership function is one. Therefore, the aspiration level of each membership function defined in (3.8) is unity. Since the objective functions of the DM may conflict in nature, it is not possible to find $\bar{X} \in S$ such that the value of all the membership functions is unity at that point. But, we can find $\bar{X} \in S$ in which the value of the membership function is close to unity as much as possible at $\bar{X}$. Therefore, there is a deviation between the possible maximum value of the membership function and unity. Consequently, every DM needs to minimize those unwanted deviations as much as possible.
Mathematically, the unwanted deviation can be defined as:

$$|\mu_i^1(F_i(x,y,z)) - 1| = \begin{cases} d_i^-, & if \ \mu_i(F_i(x,y,z)) \leq 1, \quad for \ i = 1,2,3 \\ d_i^+, & if \ \mu_i(F_i(x,y,z)) \geq 1 \end{cases}$$

and d_i^- is called under-deviation, d_i^+ is called over-deviation. But, since $\mu_i(F_i(x,y,z)) \leq 1$ for all $(x,y,z) \in S$, $d_i^+ = 0$, i.e., there is no over-deviation. So, as in Mohamed [32], for the defined membership functions in(3.8), the flexible membership goals for all the levels can be presented as:

$$\mu_i^1(F_i(x,y,z)) + d_i^- - d_i^+ = 1, i = 1,2,3 \tag{3.10}$$

Or equivalently

$$\frac{F_i^1 - F_i(x,y,z)}{F_i^1 - F_i^b} + d_i^- - d_i^+ = 1, i = 1,2,3 \tag{3.11}$$

where $d_i^-, d_i^+ \geq 0$ with $d_i^-.d_i^+ = 0$ represents the under -and over-deviations, respectively, from the aspired levels.

Note that d_i^- and d_i^+ are the unwanted deviations from the aspiration level of the membership function. Since the essence of fuzzy goal programming is to minimize the unwanted deviations, we have to minimize d_i^- and d_i^+ as much as possible. But, $0 \leq \mu_{F_i}(F_i(x)) \leq 1$ implies that there is no any over-deviation,i.e.,$d_i^+ = 0$. Hence, only d_i^- is unwanted deviation and equation (3.11) becomes:

$$\frac{F_i^1 - F_i(x,y,z)}{F_i^1 - F_i^b} + d_i^- = 1, i = 1,2,3 \tag{3.12}$$

From equation (3.12), any $(x^*, y^*, z^*) \in S$ such that

$$|\frac{F_i^1 - F_i(x^*,y^*,z^*)}{F_i^1 - F_i^b} - 1| = d_i^-$$

minimum is a satisfactory solution for the i^{th} DM. For instance, if

$$|\frac{F_i^1 - F_i(x^*, y^*, z^*)}{F_i^1 - F_i^b} - 1| = d_i^- = 0$$

then $F_i(x^*, y^*, z^*) = F_i^b$ which is the individual optimal value. Now, we need to minimize the unwanted deviation for each membership function simultaneously. Then, we can transform the problem in to single objective optimization by weighted sum method. Therefore, considering the goal achievement problem of the goals at the same priority level, the equivalent fuzzy quadratic fractional TLPP goal programming model of the problem can be presented as:

$$\begin{aligned} \min \lambda &= \sum_{i=1}^{3} w_i d_i^- \\ subject\ to\ &: \mu_i^1(F_i(x, y, z)) + d_i^- = 1, i = 1, 2, 3 \\ &(x, y, z) \in S \\ &d_i^- \geq 0, i = 1, 2, 3 \end{aligned} \tag{3.13}$$

Where the numerical weights w_i represent the relative importance of achieving the aspired levels of the respective fuzzy goals subject to the constraint set in the decision situation. In different previous results of QFBLPP [35, 43], the values of w_i are determined as

$$w_i = \frac{1}{F_i^1 - F_i^b}, i = 1, 2, 3$$

.
However, this numerical value of the weights may be give us non satisfactory solution of the system. Because, the product $w_i d_i^-$ indicates if w_i is "some what large", then $F_i^1 - F_i^b$ is "some what small" and d_i^- must be "some what small". Thus, $(F_i^1 - F_i^b)d_i^-$ is small and this is our interest. But, if w_i is "some what small", then $F_i^1 - F_i^b$ is "some what large" and d_i^- can be possible to take "some what large value". Therefore, there is a possibility to take large value of $(F_i^1 - F_i^b)d_i^-$ in the problem. Consequently, there is a possibility that the optimal value at the satisfactory solution so far from the individual optimal value. Hence, it is better to take the value

$$w_i = F_i^1 - F_i^b \tag{3.14}$$

Problem (3.13) show us the FLDM and SLDM identifies his/her possible values of objective functions in an interval forms and then give to the lower level DM, finally the TLDM search a satisfactory solution based on the FLDM and

SLDM restrictions. However, the restriction of the first and second level DMs express in fuzzy sense to apply fuzzy programming approach. Then, the satisfactory solution can be obtained through finding the compromise solution of the problem expressed using membership function of the DMs.

However, the FLDM may not be satisfied with the solution we obtained by solving the problem formulated in (3.13). In this case, the first level DM must be update his/her membership function(restriction of his/her objective value) and then give to the SLDM, SLDM also update his/her membership function based on the FLDM restriction obtained. Thus, based on those new restriction of the FLDM and SLDM the TLDM seek a new compromise solution to all the DMs. Until the FLDM and SLDM are satisfied with the compromise solution this procedure continue and finally a satisfactory solution obtained.

From equation (3.12) we have $F_i(x,y,z) - (F_i^1 - F_i^b)d_i^- = F_i^b, i = 1,2,3.$ Since the objective function of each DM are fractional, $F_i(x,y,z) = \frac{f_{1i}(x,y,z)}{f_{2i}(x,y,z)}$ where the denominator and nominator are quadratic forms. Now, let

$$\frac{1}{f_{2i}(x,y,z)} = t_i > 0$$

as $f_{2i}(x,y,z) > 0$ for all $i = 1,2,3$. Therefore, equation (3.12) becomes $f_{1i}(x,y,z)t_i - (F_i^w - F_i^b)d_i^- = F_i^b, i = 1,2,3$ and consequently, problem (3.13) can be modified as:

$$\begin{aligned} \min \;\; \lambda &= \sum_{i=1}^{3} w_i d_i^- \\ \textit{subject to } &: f_{1i}(x,y,z)t_i - (F_i^w - F_i^b)d_i^- = F_i^b, i = 1,2,3 \\ &(x,y,z) \in S \\ &d_i^- \geq 0 \textit{ and } t_i > 0, i = 1,2,3 \\ &f_{2i}(x,y,z)t_i = 1, i = 1,2,3 \end{aligned} \tag{3.15}$$

Now we prove the equivalence between conventional problem (3.13) and conventional non-linear programming problem (3.15) in the following theorem.

Theorem 3.1. If (3.13) reaches at a optimal solution $\bar{X} = \bar{X}^*$, then (3.15) also reaches at the optimal solution $\bar{x}^*, t_i^*$ and the values of achievement functions at these points are equal.

Proof. Let $\bar{X} = \bar{X}^*$ be an optimal solution of problem (3.13). Then It follows that corresponding values of

$$\mu_i^1(F_i(\bar{X}^*)) + d_i^{-*} = 1, i = 1, 2, 3$$

$$\frac{f_{i1}(\bar{X}^*)}{f_{i2}(\bar{X}^*)} + (F_i^1 - F_i^b)d_i^{-*} = F_i^b$$

It follows that corresponding values of

$$f_{i1}(\bar{X}^*)t_i^* = \frac{f_{i1}(\bar{X}^*)}{f_{i2}(\bar{X}^*)}$$

Thus, $t_i^* = \frac{1}{f_{i2}(\bar{X}^*)}$ or $t_i^* f_{i2}(\bar{X}^*) = 1$, $i = 1, 2, 3$
And also since the weight of problem (3.13) and (3.15) are the same and

$$d_i^{-*} = 1 - \mu_i^1(F_i(\bar{X}^*)) = \frac{F_i^b - f_{i1}(\bar{X}^*)t_i^*}{F_i^1 - F_i^b}, i = 1, 2, 3$$

the objective function of problem (3.13) is the same with the objective function of problem(3.15). □

Now the problem we formulated in (3.13) may not be equivalent with the original trilevel programming problem. However, the optimal solution we obtained from problem (3.13) is a candidate solution of the original trilevel programming problem. Through updating the weight of deviational variables we could also obtained new candidate solution and this candidate solution is as good as or better than the previous one. So, iteratively we obtained a set of candidate solutions and from those candidate solutions we can choose a satisfactory solution using the criteria developed. Therefore, problem (3.13) is not equivalent with the original trilevel programming problem, but we can find the solution of the trilevel programming problem by solving the new formulated problem iteratively.

Although we obtained a solution by solving problem (3.13) it may not be satisfactory for the first level DM. Therefore, we have to seek another better solution for the first level DM through minimizing the possible relaxation of first level and second level DM objective functions. So the new solution we obtained is better than the first one. The way of finding better solution can be followed as in the following manner.

Assume the solution of problem (3.13) is (x_1, y_1, z_1). Then, if FLDM and SLDM are satisfied with this solution, then this solution is satisfactory for all

the DMs. Otherwise, update the membership function of the first and second level DMs. But,"**how can we update the corresponding membership function?"**.
Assume the FLDM is not satisfied with the solution (x_1, y_1, z_1). Since $F_1^b \leq F_1(x_1, y_1, z_1) \leq F_1^1$ and $F_1(x_1, y_1, z_1)$ is not optimal value, restrict the optimal value of FLDM on the interval $[F_1^b, F_1^2]$ where $F_1^2 = F_1(x_1, y_1, z_1)$. Then define a new membership function on this interval. Moreover, the SLDM also updates his membership function, because this DM depends on the choice of FLDM. Consequently, the following new problem is formulated in the second iteration;

$$\min \ \lambda = \sum_{i=1}^{3} w_i d_i^-$$

$$subject\ to \ : \mu_i^2(F_i(x, y, z)) + d_i^- = 1, i = 1, 2$$

$$\mu_3(F_3(x, y, z)) + d_3^- = 1$$

$$(x, y, z) \in S$$

$$d_i^-, d_i^+ \geq 0, i = 1, 2, 3 \tag{3.16}$$

where $w_i = F_i^2 - F_i^b, i = 1, 2$, $w_3 = F_3^1 - F_3^b$,

$$\mu_i^2(F_i(x, y, z)) = \frac{F_i^2 - (F_i(x, y, z)}{F_i^2 - F_i^b}, i = 1, 2$$

and

$$\mu_3^2(F_3(x, y, z)) = \frac{F_3^2 - (F_3(x, y, z)}{F_3^2 - F_3^b}$$

By solving this equation we obtained a new compromise solution (x_2, y_2, z_2) and then check whether the first level and second level DMs are satisfied or not with this solution. If not, continue to the next iteration. Continuing in this manner, at the k^{th} iteration, the problem becomes:

$$\min \ \lambda = \sum_{i=1}^{3} w_i d_i^-$$

$$subject\ to \ : \mu_i^k(F_i(x, y, z)) + d_i^- = 1, i = 1, 2$$

$$\mu_3(F_3(x, y, z)) + d_3^- = 1$$

$$(x, y, z) \in S$$

$$d_i^- \geq 0, i = 1, 2, 3 \tag{3.17}$$

where $w_i = F_i^k - F_i^b, i = 1,2$, $w_3 = F_3^1 - F_3^b$ and

$$\mu_i^k(F_i(x,y,z)) = \frac{F_i^k - F_i(x,y,z)}{F_i^k - F_i^b}, i = 1,2$$

and $F_i^k = F_i(x_{k-1}, y_{k-1}, z_{k-1}), k = 2,3,4,\ldots$
From the above formulation of the problems in different iterations, we can observe that as the number of iteration increases, the weight w_i decreases proportionally. Consequently, the possible value of deviational variables that we take in each iteration controls based on the corresponding weight.

From all those constructions, the following results can be formulated.

Proposition 3.1. If $[F_i^b, F_i^{k+1}] \subseteq [F_i^b, F_i^k]$, then $\mu_i^{k+1}(F_i(x,y,z)) \leq \mu_i^k(F_i(x,y,z))$,for any $(x,y,z) \in S$ and $k = 1,2,3,...,$ $i = 1,2$.

Proof. Assume $\mu_i^k(F_i(x,y,z)) = \frac{F_i^k - F_i(x,y,z)}{F_i^k - F_i^b}$ and $[F_i^b, F_i^{k+1}] \subset [F_i^b, F_i^k]$. Then, $F_i^{k+1}] \leq F_i^k]$.
Now

$$\mu_i^k(F_i(x,y,z)) - \mu_i^{k+1}(F_i(x,y,z)) = \frac{F_i^k - F_i(x,y,z)}{F_i^k - F_i^b} - \frac{F_i^{k+1} - F_i(x,y,z)}{F_i^{k+1} - F_i^b}$$

$$\mu_i^k(F_i(x,y,z)) - \mu_i^{k+1}(F_i(x,y,z)) = \frac{(F_i^k - F_i(x,y,z))(F_i^{k+1} - F_i^b) - (F_i^{k+1} - F_i(x,y,z))(F_i^k - F_i^b)}{(F_i^{k+1} - F_i^b)(F_i^k - F_i^b)}$$

Let $w_i^{k+1} = F_i^{k+1} - F_i^b$ and $w_i^k = F_i^k - F_i^b$. Then

$$\begin{aligned}
\mu_i^k(F_i(x,y,z)) - \mu_i^{k+1}(F_i(x,y,z)) &= \frac{(F_i^k - F_i(x,y,z))(w_i^{k+1}) - (F_i^{k+1} - F_i(x,y,z))(w_i^k)}{w^k w_i^{k+1}} \\
&= \frac{F_i^k w_i^{k+1} - F_i(x,y,z) w_i^{k+1} - F_i^{k+1} w_i^k + F_i(x,y,z) w^k}{w_i^k w_i^{k+1}} \\
&= \frac{F_i^k w_i^{k+1} - F_i^{k+1} w_i^k + F_i(x,y,z)(w_i^k - w_i^{k+1})}{w_i^k w_i^{k+1}} \\
&\geq \frac{F_i^k w_i^{k+1} - F_i^{k+1} w_i^k + F_i^b(w_i^k - w_i^{k+1})}{w_i^k w_i^{k+1}}, \ as \ F_i^b \leq F_i(x,y,z) \\
&= \frac{w^{k+1}(F_i^k - F_i^b) + w_i^k(F_i^b - F_i^{k+1})}{w_i^k w_i^{k+1}} \\
&= \frac{w^{k+1} w^k - w_i^k w_i^{k+}}{w_i^k w_i^{k+1}} = 0, \ since \ w^k = F_i^k - F_i^b
\end{aligned}$$

This implies that $\mu_i^k(F_i(x,y,z)) - \mu_i^{k+1}(F_i(x,y,z)) \geq 0$.
Therefore,$\mu_i^k(F_i(x,y,z)) \geq \mu_i^{k+1}(F_i(x,y,z))$ □

From this result, we can conclude that there is another $(\bar{x}, \bar{y}, \bar{z}) \in S$ in which $F_i(\bar{x}, \bar{y}, \bar{z}) \in [F_i^b, F_i^k], i = 1, 2$ such that

$$\mu_i^k(F_i(x, y, z)) = \mu_i^{k+1}(F_i(\bar{x}, \bar{y}, \bar{z})) \geq \mu_i^{k+1}(F_i(x, y, z)).$$

Hence, $(\bar{x}, \bar{y}, \bar{z})$ is a better solution than (x, y, z).

Proposition 3.2. $\mu_i^k(F_i(x_k, y_k, z_k)) \leq \mu_i^k(F_i(x_{k+1}, y_{k+1}, z_{k+1}))$ if and only if $F_i(x_k, y_k, z_k) \geq F_i(x_{k+1}, y_{k+1}, z_{k+1})$, for $i = 1, 2$ and $k = 1, 2, 3, ...$

Proof. From the above definition of membership functions

$$\mu_i^k(F_i(x, y, z)) = \frac{F_i^k - F_i(x, y, z)}{F_i^k - F_i^b}, i = 1, 2$$

where $F_i^k = F_i(x_{k-1}, y_{k-1}, z_{k-1}), i = 1, 2.$
For $k = 1$, $(x_0, y_0, z_0) = \arg\max_{(x,y,z)\in S} F_i(x, y, z)$.

Assume $\mu_i^k(F_i(x_k, y_k, z_k)) \leq \mu_i^k(F_i(x_{k+1}, y_{k+1}, z_{k+1}))$. Then,

$$\frac{F_i^k - (F_i(x_k, y_k, z_k)}{F_i^k - F_i^b} \leq \frac{F_i^k - (F_i(x_{k+1}, y_{k+1}, z_{k+1})}{F_i^k - F_i^b}.$$

After simple simplification, we obtained $F_i(x_k, y_k, z_k) \geq F_i(x_{k+1}, y_{k+1}, z_{k+1})$

Suppose $F_i(x_k, y_k, z_k) \geq F_i(x_{k+1}, y_{k+1}, z_{k+1})$. Then,

$$-F_i(x_k, y_k, z_k) \leq -F_i(x_{k+1}, y_{k+1}, z_{k+1})$$

$$F_i^k - F_i(x_k, y_k, z_k) \leq F_i^k - F_i(x_{k+1}, y_{k+1}, z_{k+1})$$

Since $F_i^k > F_i^b$, for all $k = 1, 2, 3, \ldots, i = 1, 2, 3,$

$$\frac{F_i^k - F_i(x_k, y_k, z_k)}{F_i^k - F_i^b} \leq \frac{F_i^k - F_i(x_{k+1}, y_{k+1}, z_{k+1})}{F_i^k - F_i^b}$$

Therefore, $\mu_i^k(F_i(x_k, y_k, z_k)) \leq \mu_i^k(F_i(x_{k+1}, y_{k+1}, z_{k+1}))$.

□

From the above result we can conclude that a solution that have more membership function value is better than a solution that have less membership function value.

Proposition 3.3. For the FLDM and SLDM the solution obtained in the $(k+1)^{th}$ iteration is as good as or better than the solution obtained in the k^{th} iteration.

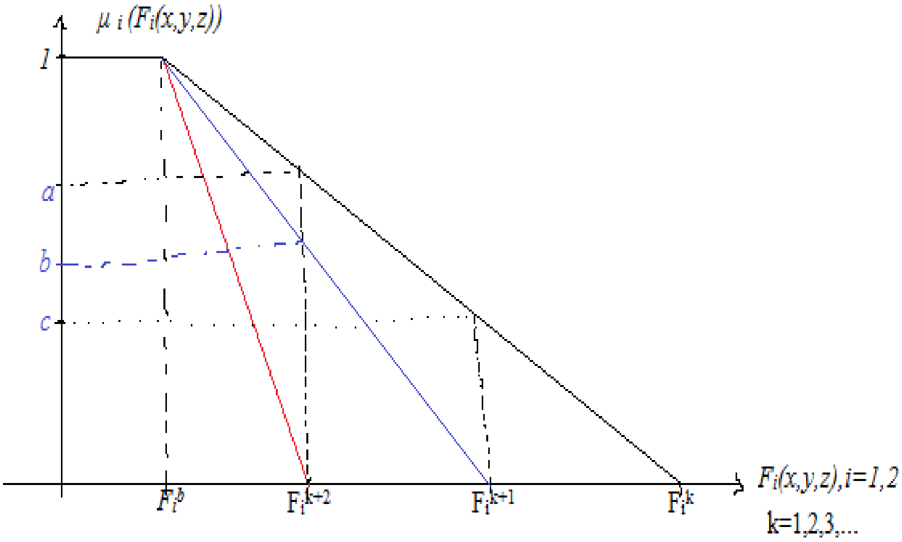

Figure 3.5: The relationship between each membership function for any consecutive iterations

Proof. Assume $\mu_i^k(F_i(x,y,z)) = \frac{F_i^k - F_i(x,y,z)}{F_i^k - F_i^b}$,for $k = 1, 2, ...$ and i=1,2. Then, for $k = 1$ $F_i^k = F_i^w = \max_{(x,y,z)\in S} F_i(x,y,z)$ for all $i = 1, 2, 3$. But, for $k = 2$, $F_i^k = F_i(x_k, y_k, z_k)$, where (x_k, y_k, z_k) is the solution obtained by solving problem (3.13).
In the first iteration,i.e., $k = 1$, $F_i^b \leq F_i(x,y,z) \leq F_i^w$ for all $(x,y,z) \in S$. Then, $F_i^b \leq F_i(x_k, y_k, z_k) \leq F_i^w$.
In the second iteration,i.e., $k = 2$,update the value of the maximum tolerance by the optimal value we obtained in the first iteration,i.e., $F_i^k = F_i(x_k, y_k, z_k)$. Then,based on this maximum tolerance, define the new membership function $\mu_i^k(F_i(x,y,z)) = \frac{F_i^k - F_i(x,y,z)}{F_i^k - F_i^b}$, for $i = 1, 2$ only and then find a new solution by solving the problem in (3.17) called $(x_k, y_k, z_k), k = 2$. Since the membership function is always between 0 and 1,

$$F_i^b \leq F_i(x_k, y_k, z_k) \leq F_i^k \leq F_i^w, \quad for \;\; k = 2, i = 1, 2.$$

This implies that,$F_i(x_2, y_2, z_2) \leq F_i(x_1, y_1, z_1)$(because,$F_i^2 = F_i(x_1, y_1, z_1), i = 1, 2)$. Therefore,for the FLDM and SLDM the solution obtained in the second iteration is better than the solution obtained in the first iteration. □

From the above result, we can generalize that for any two consecutive iterations $k, k+1$, the solution obtained in the $(k+1)^{th}$ iteration is as good as or better than the solution obtained in the k^{th} iteration, for both DMs the

FLDM and SLDM. But, the third level DM solution in the $(k+1)^{th}$ iteration is as good as or better than the $(k)^{th}$ iteration's solution. This implies that, through small change in the solution of the TLDM, we have to find a new better compromise solution of the system.

Therefore, this iteration continues until the first level DM and second level DM are satisfied with the solution we obtained. But, the question "when is the solution satisfactory?" can be raised. So, still now no guarantee for stopping criteria of the iteration. For our case the stopping criteria is when

$$|F_i(x_{k+1}, y_{k+1}, z_{k+1}) - F_i(x_k, y_k, z_k)|, i = 1, 2, 3$$

is sufficiently close to Zero . This implies that there is no more improvement in the objective value and so we arrive at the satisfactory solution. If $|F_i(x_{k+1}, y_{k+1}, z_{k+1}) - F_i(x_k, y_k, z_k)| = 0$, for all $i = 1, 2, 3$, then the solution is best(satisfactory). However, the iteration may take too long time without any more improvement or may not arrive at finite time at all. To solve such kind of problem, we set a tolerance ϵ which is sufficiently close to zero. Therefore, if $|F_i(x_{k+1}, y_{k+1}, z_{k+1}) - F_i(x_k, y_k, z_k)| < \epsilon$, for all $i = 1, 2, 3$, stop. Otherwise, seek a new solution through updating the corresponding membership functions. However, to check the stopping criteria for each DM, it seems difficult and lengthy. To overcome such problems, suppose $|F_i(x_{k+1}, y_{k+1}, z_{k+1}) - F_i(x_k, y_k, z_k)| = \epsilon_i$ and then, replace the stopping criteria with $\sum_{i=1}^{3} \epsilon_i < \epsilon$. When $\sum_{i=1}^{3} \epsilon_i < \epsilon$, $\epsilon_i < \epsilon$ for each $i = 1, 2, 3$. Otherwise, search a new optimal solution through updating the membership functions of the SLDM and FLDM.

Generally, the proposed technique seems; the FLDM and SLDM specify the preferred values of their respective goals with certain amount of tolerance. This information is represented implicitly by the use of membership functions and passed to the lower-level DM(TLDM). The lower-level DM obtains his or her optimum based on goals and preferences of the upper level and then presents the results to the upper level. If the upper level agrees with the proposed solutions, a final decision is reached and, for the convenience of description, this decision or solution will be referred to as a satisfactory solution. If he or she rejects this proposal, the DMs in both levels will need to re-evaluate and change the goals and decisions as well as their corresponding tolerances. This mutually interactive process is continued until a satisfactory solution is reached. This strategy is very flexible. Since the DMs in both levels first seek their optimal solutions in isolation, it does not violate the noncooperate idea. However, the strategy does require a certain degree of coordination between the different levels.

3.2.3 General description of the algorithm

The proposed algorithm is designed for efficiently searching a satisfactory solution of TLPP having a common constraint set S, which is closed and bounded(compact) set and quadratic fractional objective function.

1. Calculate the individual minimum and maximum of each objective function in the three levels under the given constraints,i.e.,

$$F_i^b = \min_{(x,y,z)\in S} F_i(x,y,z) \ and \ F_i^1 = \max_{(x,y,z)\in S} F_i(x,y,z), i = 1,2,3$$

2. If the individual solutions of all DMs are equal, then stop with the optimal solution we obtained. Otherwise, go to step 3.

3. Set the fuzzy goals $F_i^b \preceq F_i^b$ and the upper tolerance limits F_i^1 for all the objective functions of the DMS. That is $F_i^b \leq F_i(x,y,z) \leq F_i^1, i = 1,2,3$.

4. Elicit the quadratic fractional membership functions for each of the objective functions in the three levels as:

$$\mu_i(F_i(x,y,z)) = \frac{F_i^1 - F_i(x,y,z)}{F_i^1 - F_i^b}, i = 1,2,3$$

5. Formulate the Model (3.13) for the QFTLP problem.

6. Solve Model (3.13)using matlab software to get a candidate solution (x_1, y_1, z_1) for the QFTLP problem.

7. If the stopping criteria is satisfied with the candidate solution in Step 6, go to Step 8, else go to Step 9. That is, if $k = 1$ and $\sum_{i=1}^{3} |F_i^{k+1} - F_i^k| = \sum_{i=1}^{3} \epsilon_i < \epsilon$, where $F_i^{k+1} = F_i(x_1, y_1, z_1)$ and $F_i^k = F_i^b$ stop, Otherwise go to step 8.

8. Stop with a satisfactory solution to the QFTLP problem,(x_1, y_1, z_1).

9. For $k = k+1$ modify the upper tolerance limit of FLDM,F_1^k and SLDM, F_2^k objective functions and then go to step 4. That is, replace F_i^k by $F_i^{k+1} = F_i(x_k, y_k, z_k)$ and then go to step 3.

The flow chart of the algorithm can describe as in figure (3.6).

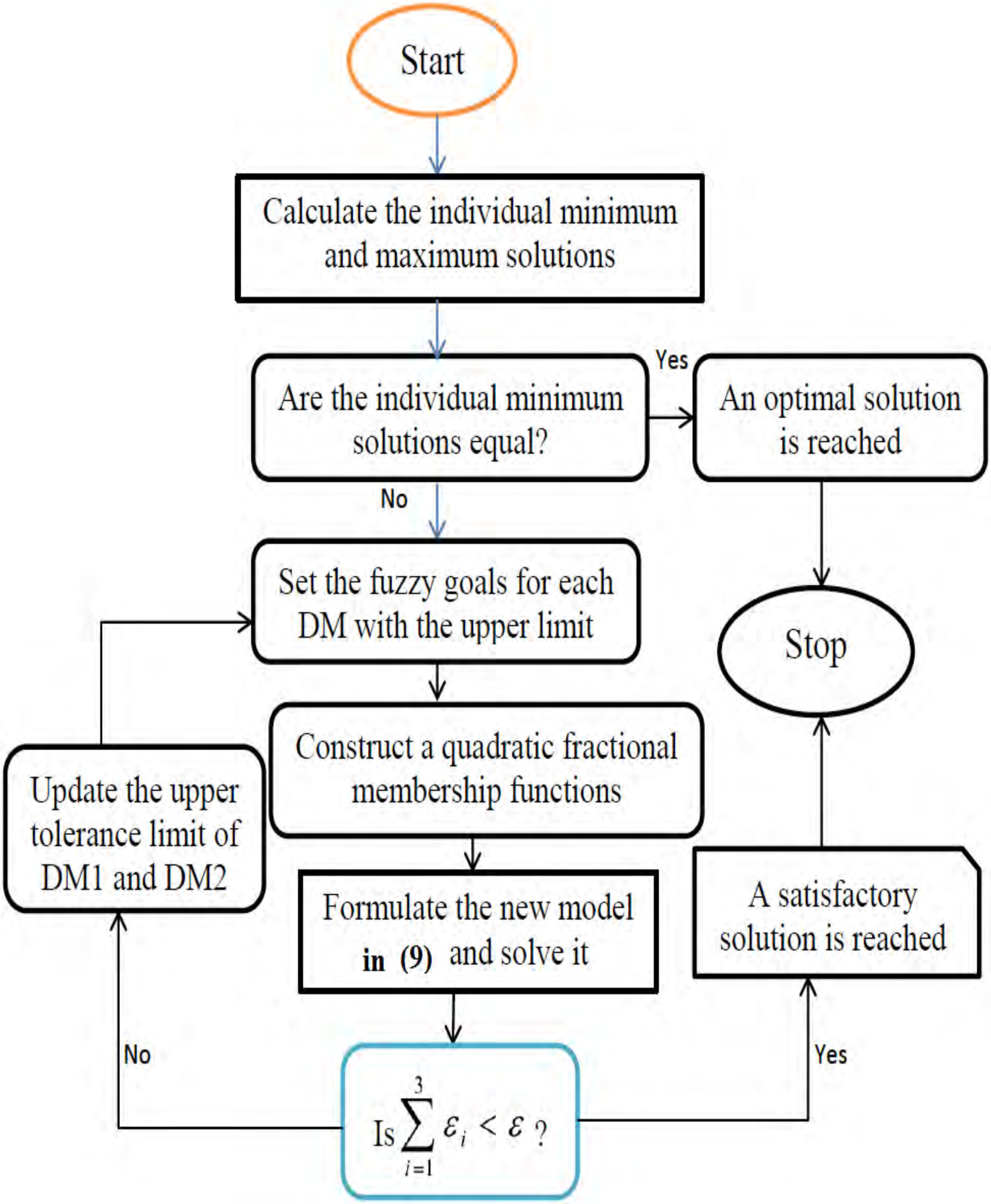

Figure 3.6: Flowchart of the FGP algorithm

3.3 Numerical Approximation

The optimization problem formulated in (3.15) seems difficult to find the best solution by hand. To overcome such difficultly we can use optimization toolbox in MATLAB. Optimization Toolbox provides functions for finding parameters that minimize or maximize objectives while satisfying constraints. The toolbox includes solvers for linear programming, mixed-integer linear programming, quadratic programming, nonlinear optimization, and nonlinear least squares. You can use these solvers to find optimal solutions to continuous and discrete problems, perform tradeoff analyses, and incorporate optimization methods into algorithms and applications. Moreover, optimization Toolbox provides widely used optimization algorithms for solving nonlinear programming problems in MATLAB. The toolbox includes solvers for unconstrained and constrained nonlinear optimization and solvers for least-squares optimization. However, the problem we formulated in this work is constrained nonlinear programming and then we use only the toolbox of such problem. Optimization Toolbox uses four algorithms to solve constrained nonlinear programming problems[17]:

- The interior point algorithm is used for general nonlinear programming and also useful for large-scale problems that have sparsity or structure, and tolerates user-defined objective and constraint function evaluation failures. It is based on a barrier function, and optionally keeps all iterations strictly feasible with respect to bounds during the optimization run.

- The SQP algorithm is used for general nonlinear programming and it honors bounds at all iterations and tolerates user-defined objective and constraint function evaluation failures.

- The active-set algorithm is appropriate for general nonlinear programming problems.

- The trust-region reflective algorithm is used for bound constrained problems or linear equalities only and it is especially useful for large-scale problems.

Optimization toolbox includes different solvers, such as "ga" and "fmincon" which are appropriate for general constrained nonlinear programming and smooth constrained nonlinear programming, respectively. Therefore,the solver fmincon applies to most smooth objective functions with smooth constraints and since the problem we use in this work contains only smooth objective function and smooth constraints this solver is efficient. It is not listed as a preferred solver for least squares or linear or quadratic programming because the listed solvers are usually more efficient[17].

In the above QFTLPP(3.15) the individual solutions must be global, because if the solution is local, it is not simple to define the membership function of each DM. Thus, to fulfil the condition, any local minimum is a global minimum, the functions in the numerator and denominator are convex and concave, respectively. Then, the fractional objective function of each DM is quasiconvex, but not convex. Moreover,if D_{i1} is positive semidefinite and D_{i2} is negative semidefinite then each objective functions of the DMs are quasiconvex. Therefore, the individual solution of each DM can be find using matlab software('fmincon' command) or Dinkelbach's algorithm.

Naturally, multilevel programming problem is not easy to compare two approximate results by just looking at the values of the objective functions and the constraint region. Thus, the satisfactory solution is identified by checking the FLDM is satisfied or not, i.e., if the FLDM is satisfied with the solution we obtained, stop. Otherwise search another solution through updating the membership function of FLDM and SLDM. However, the new formulated problem may not seems easy to solve. To avoid such difficulty, approximate the solution using matlab software('fmincon' solver). Therefore, in each iteration we use the "fmincon" solver of the matlab software.

3.3.1 Numerical examples

To demonstrate the proposed FGP procedure, consider the following quadratic fractional trilevel programming problem:

Example 3.1.

$$\min_{x} \ F_1(x,y,z) = \frac{(x-3)^2 + y^2 + z^2}{(x-2)^2 + y^2 + z^2 + 1}$$

$$\min_{y} \ F_2(x,y,z) = \frac{(x-2)^2 + (y+1)^2 + (z-1)^2}{(x-1)^2 + (y+2)^2 + z^2}$$

$$\min_{z} \ F_3(x,y,z) = \frac{(x-3)^2 + (y-1)^2 + (z+1)^2}{(x-2)^2 + (y+2)^2 + (z+1)^2}$$

subject to:

$$2x + y + z \leq 8, x + 2y + z \leq 6, \ \textit{and} \ x, y, z \geq 0$$

First, each DM solves his/her problem individually as follows using different techniques and software packages:
For the FLDM,

$$\min_{x} \; F_1(x,y,z) = \frac{(x-3)^2 + y^2 + z^2}{(x-2)^2 + y^2 + z^2 + 1}$$

subject to:

$$2x + y + z \le 8, x + 2y + z \le 6, \; and \; x, y, z \ge 0$$

For the SLDM,

$$\min_{x} \; F_1(x,y,z) = \frac{(x-3)^2 + y^2 + z^2}{(x-2)^2 + y^2 + z^2 + 1}$$

subject to:

$$2x + y + z \le 8, x + 2y + z \le 6, \; and \; x, y, z \ge 0$$

For the TLDM,

$$\min_{z} \; F_3(x,y,z) = \frac{(x-3)^2 + (y-1)^2 + (z+1)^2}{(x-2)^2 + (y+2)^2 + (z+1)^2}$$

subject to:

$$2x + y + z \le 8, x + 2y + z \le 6, \; and \; x, y, z \ge 0$$

Using matlab solvers the individual optimal values and solutions of the leader(FLDM), SLDM, and TLDM are $F_1^b = 0$ with $(x, y, z) = (3, 0.001, 0.001)$, $F_2^b = 0.157$ with $(x, y, z) = (2.186, 0, 1.186)$, and $F_3^b = 0.084$ with $(x, y, z) = (3.092, 1.275, 0)$, respectively. Then, the fuzzy objective goals appear as:

$$F_1(x,y,z) \preceq 0, \; F_2(x,y,z) \preceq 0.157, \; and \; F_3(x,y,z) \preceq 0.084$$

.
In addition to this the maximum value of each objective function of the DMs can find through replacing the objective function F_i of each DM by $-F_i$ and then solving the problem using different matlab commands.
For the FLDM,

$$\min_{x} \ F_1(x,y,z) = -\frac{(x-3)^2+y^2+z^2}{(x-2)^2+y^2+z^2+1}$$

subject to:

$$2x+y+z \leq 8, x+2y+z \leq 6, \ \ and \ \ x,y,z \geq 0$$

The optimal solution is $(1, 0.001, 0.001)$ with optimal value 2.
For the SLDM,

$$\min_{x} \ F_1(x,y,z) = -\frac{(x-3)^2+y^2+z^2}{(x-2)^2+y^2+z^2+1}$$

subject to:

$$2x+y+z \leq 8, x+2y+z \leq 6, \ \ and \ \ x,y,z \geq 0$$

For the TLDM,

$$\min_{z} \ F_3(x,y,z) = -\frac{(x-3)^2+(y-1)^2+(z+1)^2}{(x-2)^2+(y+2)^2+(z+1)^2}$$

subject to:

$$2x+y+z \leq 8, x+2y+z \leq 6, \ \ and \ \ x,y,z \geq 0$$

Therefore, the maximum values of FLDM, SLDM, and TLDM are $F_1^b = 2.000$ with $(x,y,z) = (1, 0.001, 0.001)$, $F_2^b = 1.200$ with $(x,y,z) = (0,0,0)$, and $F_3^b = 1.222$ with $(x,y,z) = (0,0,0)$, respectively.

Hence, the aspiration levels and upper tolerance limits of all objective functions for the QFTLP problem can be summarized in table (3.1).

	F_1	F_2	F_3
F_i^b	0	0.157	0.084
F_i^1	2.000	1.200	1.222

Table 3.1: Aspiration level and upper tolerance limits of objective function

Now, by using the above tolerance ranges the quadratic fractional membership function of each DM can be defined as:

$$\frac{2 - \frac{(x-3)^2+y^2+z^2}{(x-2)^2+y^2+z^2+1}}{2-0} + d_1^- = 1,$$

$$\frac{1.200 - \frac{(x-3)^2+y^2+z^2}{(x-2)^2+y^2+z^2+1}}{1.200 - 0.157} + d_2^- = 1$$

and

$$\frac{1.222 - \frac{(x-3)^2+(y-1)^2+(z+1)^2}{(x-2)^2+(y+2)^2+(z+1)^2}}{1.222 - 0.084} + d_2^- = 1$$

$$d_1^-, d_2^-, d_3^- \geq 0$$

Therefore, the proposed fuzzy goal programming (FGP) model for solving QFTLPP is:

$$\min \ \lambda = (2.00 - 0) \times d_1^- + (1.200 - 0.157) \times d_2^- + (1.222 - 0.084) \times d_3^-$$

subject to

$$\frac{2 - \frac{(x-3)^2+y^2+z^2}{(x-2)^2+y^2+z^2+1}}{2 - 0} + d_1^- = 1$$

$$\frac{1.200 - \frac{(x-3)^2+y^2+z^2}{(x-2)^2+y^2+z^2+1}}{1.200 - 0.157} + d_2^- = 1$$

$$\frac{1.222 - \frac{(x-3)^2+(y-1)^2+(z+1)^2}{(x-2)^2+(y+2)^2+(z+1)^2}}{1.222 - 0.084} + d_2^- = 1$$

$$2x + y + z \leq 8, x + 2y + z \leq 6, \ \ and \ x, y, z \geq 0 \tag{3.18}$$

$$d_1^-, d_2^-, d_3^- \geq 0$$

Equivalently problem(3.18) can be written as:

$$\min \ \lambda = 2.00 \times d_1^- + 1.043 \times d_2^- + 1.138 \times d_3^-$$

subject to

$$[(x-3)^2 + y^2 + z^2]t_1 - 2d_1^- = 0$$

$$[(x-2)^2 + y^2 + z^2 + 1]t_1 = 1$$

$$[(x-3)^2 + y^2 + z^2]t_2 - 1.043d_2^- = 0.157$$

$$[(x-2)^2 + y^2 + z^2 + 1]t_2 = 1$$

$$[(x-3)^2 + (y-1)^2 + (z+1)^2]t_3 - 1.138d_2^- = 0.084$$

$$[(x-2)^2 + (y+2)^2 + (z+1)^2]t_3 = 1$$

$$2x + y + z \leq 8, x + 2y + z \leq 6, \tag{3.19}$$

$$x, y, z, d_1^-, d_2^-, d_3^- \geq 0, \ \ and \ \ t_1, t_2, t_3 > 0$$

Therefore, problem (3.19) is nonlinear programming with all the functions involved in the objective and constraint are continuous and differentiable and also we can approximate the solution using the matlab solver "fmincon". However, the solution we obtained from problem (3.19) may not a solution for the original QFTLPP. To obtained the solution of original problem, solve problem (3.19) iteratively by updating the upper boundary of the objective value. The optimal solution of the problem can be observe as follows:

Solving the new problem using the proposed approach, we obtained an optimal solution of the problem

$$(x^*, y^*, z^*) = (3.03849843458209, 0.279156133777985, 2.2648097564041e{-}005)$$

with the optimal values

$$F_1(x^*, y^*, z^*) = 0.0606712057718949$$

$$F_2(x^*, y^*, z^*) = 0.397229301543659$$

$$F_3(x^*, y^*, z^*) = 0.210334877156276$$

of FLDM, SLDM and TLDM, respectively.

Example 3.2. *[3]*

$$\min_x \ F_1(x, y, z) = -6x + 2y$$

$$\min_y \ F_2(x, y, z) = 2y + z_2$$

$$\min_z \ F_3(x, y, z) = -z_1^2 + y + z_1$$

subject to

$$x + 2y \leq 1$$

$$-2x + y + z_1 \leq 0$$

$$-2z_1 + z_2 + x \leq 0$$

$$-2z_1 - 6z_2 + y \leq 0$$

$$0 \leq x, y, z_1 \leq 1, \ \ 0 \leq z_2 \leq \frac{1}{2}$$

First find the individual optimal values and solutions of each DM using matlab solver as follows:
For the first level DM

$$\min_{(x,y,z)} \; F_1(x,y,z) = -6x + 2y$$

subject to

$$x + 2y \leq 1$$
$$-2x + y + z_1 \leq 0$$
$$-2z_1 + z_2 + x \leq 0$$
$$-2z_1 - 6z_2 + y \leq 0$$
$$0 \leq x, y, z_1 \leq 1, \;\; 0 \leq z_2 \leq \frac{1}{2}$$

Then $(0.999999, 0, 0.859440, 0.231309)$ is an optimal solution of DM one with optimal value -6.
For the second level DM

$$\min_{(x,y,z)} \; F_2(x,y,z) = 2y + z_2$$

subject to

$$x + 2y \leq 1$$
$$-2x + y + z_1 \leq 0$$
$$-2z_1 + z_2 + x \leq 0$$
$$-2z_1 - 6z_2 + y \leq 0$$
$$0 \leq x, y, z_1 \leq 1, \;\; 0 \leq z_2 \leq \frac{1}{2}$$

Then $(0.583546, 0.000000, 0.785366, 0.000000)$ is an optimal solution of DM two with optimal value 0.000000.
and for the third level DM

$$\min_{(x,y,z)} \; F_3(x,y,z) = -z_1^2 + y + z_1$$

subject to

$$x + 2y \leq 1$$
$$-2x + y + z_1 \leq 0$$
$$-2z_1 + z_2 + x \leq 0$$

$$-2z_1 - 6z_2 + y \leq 0$$

$$0 \leq x, y, z_1 \leq 1, \;\; 0 \leq z_2 \leq \frac{1}{2}$$

Then $(0.000000, 0.000000, 0.000000, 0.000000)$ is an optimal solution of DM three with optimal value 0.000000.

Therefore, the fuzzy goal of each DM could be defined as:

$$F_1(x, y, z) \preceq -6.000000, \;\; F_2(x, y, z) \preceq 0.0000 \;\; and \;\; F_3(x, y, z) \preceq 0.0000$$

To define the membership function of each DM we have to know the maximum value of FLDM, SLDM, and TLDM. So the maximum values of FLDM, SLDM, and TLDM is 0.0000, 1.125 and 0.56, respectively. Therefore, the membership function of each DM can be defined as follows:

$$\frac{6x - 2y}{6.000000} + d_1^- = 1$$

$$\frac{1.125 - 2y - z_2}{1.125} + d_2^- = 1$$

and

$$\frac{0.56 + z_1^2 - y - z_1}{0.56} + d_3^- = 1$$

$d_1^-, d_2^-, d_3^- \geq 0$

Therefore, the proposed fuzzy goal programming (FGP) model is:

$$\min \lambda = 6 \times d_1^- + 1.125 \times d_2^- + 0.56 \times d_3^-$$

subject to

$$6x - 2y + 6.000000 \times d_1^- = 6.000000$$

$$2y + z_2 - 1.125 \times d_2^- = 0$$

$$z_1^2 - y - z_1 + 0.56 \times d_3^- = 0$$

$$x + 2y \leq 1$$

$$-2x + y + z_1 \leq 0$$

$$-2z_1 + z_2 + x \leq 0$$

$$-2z_1 - 6z_2 + y \leq 0$$

$$0 \leq x, y, z_1 \leq 1, \;\; 0 \leq z_2 \leq \frac{1}{2}$$

$$d_1^-, d_2^-, d_3^- \geq 0 \tag{3.20}$$

Problem (3.20) contains continuous and differentiable functions in the objective as well as in the constraints. So using the matlab solvers and the proposed method we obtained a satisfactory solution

$$(x, y, z_1, z_2) = (1.00000000000000, 0.00000000000000, 1.00000000000000, 0.00000000000000)$$

with the corresponding optimal values

$$(F_1^*, F_2^*, F_3^*) = (-6.00000000000000, 0.00000000000000, 0.00000000000000)$$

.

However, using "A multi-parametric programming algorithm" [3], the solution of this trilevel programming problem is $(0.6, 0.2, 0.271, 0.047)$ with the optimal value $(F1, F2, F3) = (-3.2, 0.447, 0.16853)$.

When we compare those two solutions of the trilevel programming problems, they are different and the objective values of all DMs we obtained using fuzzy goal programming approach is less than the objective values of each DM that obtained using "A multi-parametric programming algorithm"[3]. Hence, the solution we obtained in this thesis (using fuzzy goal programming approach) is better.

Example 3.3.

$$\min_x f_1(X) = \frac{(x-3)^2 + (y-2)^2 + (z-1)^2}{x + 3y + (z+1)^2}$$

$$\min_y f_2(X) = \frac{(x-1)^2 + (y-1)^2 + (z-1)^2}{(x+2)^2 + (y-2)^2 + (z-3)^2}$$

$$\min_z f_3(X) = \frac{5x^2 + y^2 + z}{(x+2)^2 + 4y}$$

subject to

$$y + 2z \leq 10, 0 \leq x \leq 5, 0 \leq z \leq 4, 0 \leq y$$

To solve this problem using the proposed method first we have to find the individual maximal and minimal solutions of each DM using the matlab solver and then the solution of DM_1, DM_2 and DM_3 obtained as in table (3.2).

Max. and Min. value	DM_1	DM_2	DM_3
f_i^b	0.000000	0.000000	0.000000
f_i^1	14.000000	0.176471	1.000000
Individual best soln (X^*)	(3,2,1)	(1,1,1)	(0,0,0)

Table 3.2: Individual solution of each DM with optimal values

Therefore, the proposed fuzzy goal programming (FGP) model becomes:

$$\min \lambda = 14.000000 \times d_1^- + 0.176471 \times d_2^- + 1.000000 \times d_3^-$$

subject to

$$((x-3)^2 + (y-2)^2 + (z-1)^2)t_1 - 14.00000d_1^- = 0$$
$$((x-1)^2 + (y-1)^2 + (z-1)^2)t_2 - 0.176471d_2^- = 0$$
$$(5x^2 + y^2 + z)t_3 - 1.000000d_3^- = 0$$
$$(x + 3y + (z+1)^2)t_1 - 1 = 0$$
$$((x+2)^2 + (y-2)^2 + (z-3)^2)t_2 - 1 = 0$$
$$((x+2)^2 + 4y)t_3 - 1 = 0$$
$$y + 2z \leq 10, x \leq 5, z \leq 4$$
$$x, y, z, d_1^-, d_2^-, d_3^- \geq 0 \tag{3.21}$$

Solving problem (3.21) using the proposed approach , one can get a satisfactory solution $(x, y, z) = (1.407947, 1.946424, 0.765045)$ with the corresponding optimal value $(f_1, f_2, f_3) = (0.250198, 0.067261, 0.745636)$.

Example 3.4. *[27]*

$$\min_{x_1,x_2} f_1(x,y,z) = \frac{-7x_1 - 3x_2 + 4y - 2z}{x_1 + x_2 + y + 1}$$
$$\min_{y} f_2(x,y,z) = \frac{-x_2 - 3y + 4z}{x_1 + x_2 + y + 2}$$
$$\min_{z} f_3(x,y,z) = \frac{-2x_1 - x_2 - y - z}{x_1 + x_2 + y + 3}$$

subject to

$$x_1 + x_2 + y + z \leq 5$$
$$x_1 + x_2 - y - z \leq 2$$
$$x_1 + x_2 + y \geq 1$$
$$x_1 - x_2 + y + 2z \leq 4$$
$$x_1 + 2y + 2z \leq 3$$
$$z \leq 2$$
$$x_1, x_2, y, z \geq 0$$

To find a satisfactory solution of the trilevel programming, first we have to find the individual maximum and minimum values of each DMs using matlab solver. Now the individual optimal solution of DM_1, DM_2 and DM_3 are $(x_1, x_2, y, z) = (2.333333, 0, 0, 0.333333)$, $(x_1, x_2, y, z) = (0, 0, 1.5, 0)$ and $(x_1, x_2, y, z) = (2.333333, 0, 0, 0.333333)$ with the optimal values $(f_1^b, f_2^b, f_3^b) = (-5.1, -1.285714, -0.9375)$, respectively. In addition to this, the maximum values of DM_1, DM_2 and DM_3 are $f_1^1 = 2.4, f_2^1 = 1.666667$ and $f_3^1 = -0.25$, respectively.

Using all the above information, the new fuzzy goal programming model becomes:

$$\min \lambda = 4.85d_1^- + 2.952381d_2^- + 3.3375d_3^-$$

subject to

$$\begin{aligned}
&\frac{-7x_1 - 3x_2 + 4y - 2z}{x_1 + x_2 + y + 1} - 4.85d_1^- + 5.1 = 0\\
&\frac{-x_2 - 3y + 4z}{x_1 + x_2 + y + 2} - 2.952381d_2^- + 1.285714 = 0\\
&\frac{-2x_1 - x_2 - y - z}{x_1 + x_2 + y + 3} - 3.3375d_3^- + 0.9375 = 0\\
&x_1 + x_2 + y + z \leq 5\\
&x_1 + x_2 - y - z \leq 2\\
&x_1 + x_2 + y \geq 1\\
&x_1 - x_2 + y + 2z \leq 4\\
&x_1 + 2y + 2z \leq 3\\
&z \leq 2\\
&x_1, x_2, y, z, d_1^-, d_2^-, d_3^- \geq 0
\end{aligned} \tag{3.22}$$

Equivalently, problem (3.22) can be written as:

$$\min \lambda = 4.85d_1^- + 2.952381d_2^- + 3.3375d_3^-$$

subject to

$$\begin{aligned}
&(-7x_1 - 3x_2 + 4y - 2z)t_1 - 4.85d_1^- + 5.1 = 0\\
&(-x_2 - 3y + 4z)t_2 - 2.952381d_2^- + 1.285714 = 0\\
&(-2x_1 - x_2 - y - z)t_3 - 3.3375d_3^- + 0.9375 = 0\\
&(x_1 + x_2 + y + 1)t_1 - 1 = 0
\end{aligned} \tag{3.23}$$

$$(x_1 + x_2 + y + 2)t_2 - 1 = 0$$
$$(x_1 + x_2 + y + 3)t_3 - 1 = 0$$
$$x_1 + x_2 + y + z \leq 5$$
$$x_1 + x_2 - y - z \leq 2$$
$$x_1 + x_2 + y \geq 1$$
$$x_1 - x_2 + y + 2z \leq 4$$
$$x_1 + 2y + 2z \leq 3$$
$$z \leq 2$$
$$x_1, x_2, y, z, d_1^-, d_2^-, d_3^- \geq 0 \text{ and } t_1, t_2, t_3 > 0$$

After implementing the proposed approach on problem (3.23) we obtained a satisfactory solution $X^* = (x_1, x_2, y, z) = (2.333333, 0.000000, 0.000000, 0.333333)$ with an optimal value $(f_1, f_2, f_3) = (-5.099999, 0.307692, -0.9374999)$. Therefore, for DM$_1$ we have $\mu_1(X^*) = 0.999999$, for DM$_2$ $\mu_2(X^*) = 0.460298$ and for DM$_2$ $\mu_3(X^*) = 0.999999$.

However, the result reported in [27] are $(f_1, f_2, f_3) = (-3.42738, -1.642437, -0.751564)$ with satisfactory solution $X^* = (1.0000, 0.0000, 0.0000, 1.0000)$. So it is difficult to compare those two solution, because the objective value of some DMs (DM$_1$ and DM$_3$) improves but some of them (DM$_2$) are not.

Conclusion

The complexity of the solution approach for MLPP is confirmed by looking its simplest version, BLPP. Especially if nonconvexity appears in the inner level problem, most of the existing algorithms fail to work. However, in this thesis we have described a finding of satisfactory solution strategy for the tri-level programming problem with fractional objective functions (nonconvex property in the problems) and based on the combination of fuzzy goal programming approach and weighted sum technique. The proposed approach is suitable for tri-level programming problems involving fractional objective functions (special case for the nonconvex problem) and linear terms in the constraint sets. The proposed methodology yields a satisfactory solution of TLP problem with a higher degree of satisfaction by updating the weight of deviational variables. The main advantage of the proposed technique is that it is simple, and requires less computational work as it finally converts nonlinear TLPP problem into nonlinear programming problem which can be easily solved using nonlinear techniques or software packages like matlab, LINGO etc.

However, the proposed methodology can be further extended to solve general multi-level nonlinear programming problem, multi-level integer programming problem and multilevel multiobjective programming problem as future research work.

Bibliography

[1] Abo-Sinna, M. A. *"A bilevel non-linear multi-objective decision making under fuzziness"*, Journal of Operational Research Society of India, Vol.38, No.5, pp.484-495, 2001.

[2] Abo-Sinna, M. A. and Baky, I. A.,*"Interactive balance space approach for solving bilevel multi-objective programming problems"*, Advances in Modelling and Analysis B, Vol.49, No.3-4, pp.43-62, 2006.

[3] Abo-Sinnaa, M. A and Baky, I. A., *"Interactive balance space approach for solving multi-level multiobjective programming problems"*, Information Sciences, Vol.177, No.16, pp.3397-3410, 2007.

[4] Azad, A. K., Sakawa, M., Kato, K. and Katagiri, H. *"Interactive fuzzy programming for multi-level nonlinear integer programming problems through genetic algorithms"*, Asia Pacific Management Review: An International Journal, Vol.10, No.1, pp.223-236, 2006.

[5] Baky, I. A., *"Fuzzy goal programming algorithm for solving decentralized bilevel multiobjective programming problems"*, Fuzzy Sets and Systems, Vol.160, No.18, pp.2701-2713, 2009.

[6] Baky, I. A., *"Solving multi-level multi-objective linear programming problems through fuzzy goal programming approach*, Applied Mathematical Modelling, Vol.34, No.9, pp.2377-2387, 2010.

[7] Belaid, A.,Jean-Marc, M. and Amal, H., *"Fuzzy Goal Programming Model: an overview of the current state of the art"*, Journal of multicriteria decision analysis, Vol.16, Issue 5-6, pp.149-161, 2009.

[8] Bellman, R. E. and Zadeh, L. A, *"Decision-making in a fuzzy environment"*, Management Science, Vol. 17, No.4, pp.141-164, 1970.

[9] Bialas, W. F and Karwan, M. H., *"Mathematical Methods for Multilevel Planning"*, Research Report No. 79-2, 1979.

[10] Bialas, W. F and Karwan, M. H.,*"On Two-Level Optimization"*, IEEE Transactions on Automatic Control, Vol.27, No.1, pp.211-214, 1982.

[11] Bialas, F. and Karwan, H., *"Two level linear programming"*, Management Science, Vol.30, No.8, pp.1004-1020, 1984.

[12] Blair, C., *"The computational complexity of mulilevel linear programs"*, Annals of Operations Research, Vol.34, Issue 1, pp.13-19, 1992.

[13] Biswas, A. and Bose, K., *"Fuzzy Goal Programming Approach for Quadratic Fractional Bilevel Programming"*, In: Proceeding of the 2011 International Conference on scientiic computing(CSC 2011), pp.143-149, CSREA Press, USA.

[14] Caballero, R. and Hernández, M., *"Restoration of efficiency in a goal programming problem with linear fractional criteria"*, European Journal of Operational Research, Vol.172, Issue 1, pp.31-39, 2006.

[15] Candler, W. and Townsley, R., *"A Linear Two-Level Programming Problem"*, Computers and Operations Research, Vol. 9, No.1, pp.59-76, 1982.

[16] Cassidy, R., Kirby, M. and Raik, W., *"Efficient distribution of resources through three levels of government"*, Management science, Vol.17, No.8, pp.462-473, 1971.

[17] Coleman, T., Branch, M and Grace, A., *"Optimization Toolbox Users Guide:Computation, Visualization and Programming"*, The Math works Inc., http://www.mathworks.com, 1999.

[18] Craven, D., *"Fractional Programming"*, Heldermann Verlag 1988, Berlin, 2006.

[19] Despotis, D., *"Fractional minmax goal programming: A unified approach to priority estimation and preference analysis in MCDM"*, The Journal of the Operational Research Society, Vol.47, No.8, PP.989-999, 1996.

[20] Dey, P. P. and Pramanik, S., *"Goal Programming Approach to Linear Fractional Bilevel Programming Problem Based on Taylor Series Approximation"*, International Journal of Pure and Applied Sciences and Technology, Vol.6, No.2, pp.115-123, 2011.

[21] Ehrgott, M. and Gandibleux, X., *"Multiple criteria optimization: state of art annotated bibliographic surveys"*, International Series in Operations Research and Management Science, Vol.52, Springer, 2002.

[22] Esubalew, L., *"Linear three-level programming problem with the application to hierarchical organization"*, MSc Thesis, Department of Mathematics, Addis Ababa University, 2007.

[23] Ignizio, P., *"Goal programming and extensions"*, Lexington, MA: Lexington DC., 1976.

[24] Inuiguchi, M., Ichihashi, H. and Kume, Y., *"A solution algorithm for fuzzy linear programming with piecewise linear membership functions"*, Fuzzy Sets and Systems, Vol.34, Issue 1, pp.15-21, 1990.

[25] Jones, D. and Tamiz, M., *"Practical Goal Programming"*, Springer New York Dordrecht Heidelberg London, 2010.

[26] Kassa, A. M. and Kassa, S. M., *"A multi-parametric programming algorithm for special classes of non-convex multilevel optimization problems"*, An International Journal of Optimization and Control: Theories and Applications, Vol.3, No.2, pp. 133-144, 2013.

[27] Lachhwani, L. and Poonia, M. P., *"Mathematical solution of multilevel fractional programming problem with fuzzy goal programming approach"*, Journal of Industrial Engineering International, Vol.8, No.12, doi:10.1186/2251-712x-8-16, 2012.

[28] Lachhwani, L., *"On solving multi-level multi objective linear programming problems through fuzzy goal programming approach"*, Vol.51, No.4, pp.624-637, 2014.

[29] Lai, Y. J, and Hwang, C. L., *"Fuzzy Mathematical Programming; Methods and Applications"*, Springer, Berlin, 1996.

[30] Lai, Y. J, *"Hierarchical optimization: a satisfactory solution"*, Fuzzy Sets and Systems, Vol.77, No.3, pp.321-335, 1996.

[31] Lee, E. S. and Shih, H. -S., *"Fuzzy and Multi-level Decision Making: An Interactive Computational Approach"*, Springer, London, UK, 2001.

[32] Mohamed, R. H., *"The relationship between goal programming and fuzzy programming problems"*, Fuzzy Sets and Systems, Vol.89, No.2, pp.215-222, 1997.

[33] Moitra, B. N. and Pal, B. B, *"A fuzzy goal programming approach for solving bilevel programming problems"*, in Advances in Soft ComputingAFSS, N. R. Pal and M. Sugeno, Eds., vol.2275 of Lecture Notes in Computer Science, pp.9198, Springer, Berlin, Germany, 2002.

[34] Osman, M. S, Abo-Sinna, M. A, Amer, A. H and Emam, O. E, *"A multi-level non-linear multi-objective decision-making under fuzziness"*, Applied Mathematics and Computation, Vol.153, No.1, pp.239-252, 2004.

[35] Pal, B. B. and Moitra,B. N., *"A fuzzy goal programming procedure for solving quadratic bilevel programming problems"*, International Journal of Intelligent Systems, Vol.18, No.5, pp.529-540, 2003.

[36] Pal, B. B., Moitra, B. N. and Maulik, U., *"A goal programming procedure for fuzzy multiobjective linear fractional programming problems"*, Fuzzy Sets and Systems, Vol.139, No.2, pp.395-405, 2003.

[37] Pramanik, S. and Roy, T. K., *"Fuzzy goal programming approach to multi-level programming problem"*, European Journal of Operational Research, Vol.176, No.2, pp.1151-1166, 2007.

[38] Romero, C., *"Handbook of Critical Issues in Goal Programming; Mathematical Social Sciences 22*, Oxford:Pergamon Press, 1991.

[39] Saber, M and Ravindran, A., *"Nonlinear goal programming theory and practice: A survey"*, Computers and Operations Research, Vol.20, No.3, pp.275-291, 1993.

[40] Sakawa, M., *"Large Scale Interactive Fuzzy Multiobjeetive Programming"*, New York: Physica-Verl., 2000.

[41] Sakawa, M., Katagiri, H. and Matsui, T., *"Stackelberg Solutions to Noncooperative Two-Level Nonlinear Programming Problems through Evolutionary Multi-Agent Systems"*, Faculty of Engineering, Hiroshima University, Japan, Multi-Agent Systems - Modeling, Control, Programming, Simulations and Applications, Dr. Faisal Alkhateeb (Ed.), ISBN: 978-953-307-174-9, InTech, DOI: 10.5772/15628, 2011.

[42] Sakawa, M. and Nishizaki, I., *"Cooperative and Noncooperative Multi-Level Programming"*, Springer Dordrecht Heidelberg London New York, 2009.

[43] Saraj, M. and Sadeghi, S., *"Using fuzzy goal programming in solving quadratic bi-level fractional multi-objective programming problems"*, Annals of Fuzzy Mathematics and Informatics, Vol.7, No.1, pp.1-13, 2014.

[44] S. Shih, J. Lai, and S. Lee, *"Fuzzy approach for multi-level programming problems"*, Computers and operation research, vol.23, Issue 1, pp.73-91, 1983.

[45] Shih, H. S., Lai, Y. J. and Lee, E. S, *"Fuzzy approach for multi-level programming problems"*, Computers and Operations Research, Vol. 23, No.1, pp.73-91, 1996.

[46] Shih, H. S. and Lee, E. S,, *"Compensatory fuzzy multiple decision making"*, Fuzzy Set Systems, Vol.114, Issue 1, pp.71-87, 2000.

[47] Joaõ, M. C. and Uzay, K., *"Fuzzy Decision Making in Modeling and Control"*, world scientific series in robotics and intelligent systems, vol. 27, CJ Harris(University of Southampton), 2002.

[48] Stancu-Minasian, I. M., *"Fractional Programming:Theory, Methods, and applications"*, Kluwer Academic Publishers, The Netherlands, 1997.

[49] Vicente, L. N. and Calamai, P. H., *"Bilevel and Multilevel Programming:A Bibliography Review"*, Journal of Global Optimization, Vol.5, No.3, pp.291-306, 1994.

[50] Weldemariam, A. T., *"Systematic evolutionary algorithm for a general multilevel stackelberge problems with bounded decision variables(SEAMSP)"*, Msc Thesis, Department of Mathematics, Addis Ababa University, 2013.

[51] Wen, U. and Biala, F. W., *"The hybrid algorithm for solving the three-level linear programming problem"*, Computers and Operations Research, Vol.13, No.4, pp.367-377, 1986.

[52] Yang, T., Ignigio, J. P. and Kim, H. J, *"Fuzzy programming with nonlinear membership functions: Piecewise linear approximation"*, Fuzzy Sets and Systems, Vol.41, Issue 1, pp.39-53, 1991.

[53] Zeleny, M., *"Multiple criteria decision making"*, McGraw-Hill, New York, 1982.

[54] Zimmermann, H. J., *"Fuzzy programming and linear programming with several objective functions"*, Fuzzy Sets and Systems, Vol.1, No.1, pp.45-55, 1978.

[55] Zimmermann, H. J, *"Fuzzy Set Theory and Its Applications"*, operation research, New York, 2001.