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Addis Ababa University

School of Graduate Studies

Faculty of Computer and Mathematical Sciences

Department of Mathematics

Project

On

Solutions to Dirichlet Problem and Neumann Problem

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A Project submitted to the School of Graduate Studies
of Addis Ababa University in Partial fulfillment of the requirements for the Degree of Master of Science in Mathematics

January, 2011

Addis Ababa, Ethiopia

Addis Ababa University
school of Graduate Studies

Faculty of Computer and Mathematical Sciences
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Declaration

"I declare that this project has been composed by me and that no part of the project has formed the basis for the award of any degree, diploma, associateship, fellowship or any other similar title to me.

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Advisor's Signature

ACKNOWLEDGEMENT

In preparing this project I would like to express my appreciation to my advisor Dr. Tsegaye Gedif for his patience in repeatedly reading the draft manuscript of this project and for making constructive comments and suggestions from which I have immeasurably benefited in sharpening my understanding, predominantly, on the area I study.

Second, I would like to extend grateful acknowledgement to my family in general my father, my mother and brothers.

Last but not least, I am also indebted for many friends and to all those who helped me one way or another for the accomplishment of my study.

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2011

ABSTRACT

The complete set of this project focuses on solvability of the Dirichlet problem of the type

$$\begin{cases} \Delta u = f & in\ \Omega \\ u = \varphi & on\ \partial\Omega \end{cases}$$

and the Neumann problem of the type

$$\begin{cases} \Delta u = f\ in\ \Omega \\ \frac{\partial u}{\partial \upsilon} = h\ on\ \partial\Omega \end{cases}$$

The explicit solutions of the problem in particular when $f = 0$, in elliptic linear operator specifically in Laplace's operator and also discuss the integral representation for the solution of the Dirichlet problem for harmonic functions in upper half space and in a ball in the whole of $\mathbb{R}^n$.

Keywords Ball, Dirichlet problem, half space, harmonic functions, integral representation, Neumann problem

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Introduction

Laplace's equation is one of the most important partial differential equation in applied mathematics. Because it occurs in gravity, electrostatic, steady state heat conduction, compressible fluid flow and so on…

In n-dimensional, u depends on n-coordinates $x_1, x_2, \ldots, x_n$ .The Laplace equation in n-dimensional case given as

$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial {x_i}^2}$, where i=1, 2, 3… n, whose solutions are harmonic functions.

We shall limit our discussion on Dirichlet problem, integral representation of harmonic function in upper half space and in a ball and Neumann problem. This project consists of two chapters. The first chapter is about preliminaries which simply to make the reader familiar with the concept of elliptic Partial differential equations of second order specifically about Laplace's equation. In this chapter definitions and notations, result of advanced calculus, change of variable formula, basic property of harmonic function, distribution, and related result are discussed. The second chapter deals mainly on integral representation of harmonic functions and it discusses fundamental solution, Newtonian potential, Dirichlet problem, Integral representation of harmonic function a in upper half-Space, Integral representation of harmonic function in a Ball , Neumann problem and related result.

CHAPTER ONE

PRELIMINARIES

The purpose of this chapter is to fix some terminology which will be used through the project, to present a few analytical tools which will be used in later chapter.

1.1 Notations and Definitions

Points and Sets in Euclidean Space

$\mathbb{R}$ will denote the set of real numbers.

$\mathbb{R}^n$ denotes the euclidean space of n-dimensional for $n > 1$. We will be working in $\mathbb{R}^n$.

$:=$ denotes "equals by definition"; it is used to stress that an equation is defining someth-

ing

■ denote the end of a proof.

Ω always is an open subset of $\mathbb{R}^n$.

$\partial\Omega$ denote the boundary Ω.

$\bar{\Omega}$ will denote the closure Ω (i.e. $\bar{\Omega} = \Omega \cup \partial\Omega$).

If x and y are points in $\mathbb{R}^n$, we set

$$x.y = \sum_{i=1}^{n} x_i y_i \text{ and } |x| = (x.x)^{\frac{1}{2}}$$

We use the following notations for sphere and balls; if $x \in \mathbb{R}^n$ and $r > 0$

- $B(x,r) := \{y \in \mathbb{R}^n \,;\, |x - y| < r\}$ (open ball)
- $\bar{B}(x,r) := \{y \in \mathbb{R}^n \,;\, |x - y| \le r\}$ (closed ball)
- $S(x,r) = \partial\Omega := \{y \in \mathbb{R}^n \,;\, |x - y| = r\}$ is called a sphere (circle if n=2) center at x and radius r.

Definition A partial differential equation (PDE) is an equation involving derivatives of an unknown function $u:\Omega \to \mathbb{R}$, where Ω is an open subset of $\mathbb{R}^n$, $n \geq 2$.

Let $u(x) = u(x_1, \dots, x_n)$ be a differentiable function. We use any of the following notations to denote the partial derivative of u with respect to the i^{th} variable x_i.

$$u_{x_i}, \partial_{x_i}u, D_i u, \frac{\partial u}{\partial x_i}.$$

The second partial derivative of a twice continuously differentiable functions u with respect to x_i and x_j will be denoted by any of the following notations.

$$u_{x_i x_j}, \partial^2{}_{x_i x_j}u, D_{ij}u, D_{x_i x_j}u, \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $s^{n\times n}$ be the set of $n \times n$ real symmetric matrices.

A second order partial differential equation on Ω in an unknown $u(x) = u(x_1, \dots, x_n)$ is an equation of the form

$$F(x, u, Du, D^2u) = 0 \tag{1.1}$$

Where $F:\Omega \times \mathbb{R} \times \mathbb{R}^n \times S^{n\times n} \longrightarrow \mathbb{R}$ .A typical point γ of $\Gamma = \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^{n\times n}$ is given by $\gamma = (x, z, \eta, r)$ where $x \in \Omega, z \in \mathbb{R}, \eta \in \mathbb{R}^n, r \in S^{n\times n}$.

Definition The second order partial differential equation (1.1) is called linear if it is of the form

$$\sum_{i,j=1}^{n} a_{ij}(x)D_{ij}u + \sum_{i=1}^{n} b_i(x)D_i u + c(x)u + d(x) = 0$$

Example The Laplace equation $\Delta u = 0$, where $\Delta u = \sum_{i=1}^{n} D_{ii}u$, here

$$F(x, z, \eta, r) = tr(r)$$

A partial differential equation (1.1), and assume that F is differentiable in the variable r.we extend F to the whole space of $n \times n$ matrices by say $F(x, z, \eta, r) = F\left(x, z, \eta, \frac{1}{2}(A + A^T)\right)$, where $(x, z, \eta, A) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^{n\times n}$. the $n \times n$ matrix $[F_{ij}(\gamma)]_{n\times n}$ is symmetric where $F_{ij} := \frac{\partial F}{\partial r_{ij}}$

Definition The equation (1.1) is said to be elliptic at a point $\gamma = (x, z, \eta, r) \in \Gamma = \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^{n\times n}$ if and only if the matrix $[F_{ij}(\gamma)]_{n\times n}$ is positive definite, that is $\sum_{i,j=1}^{n} \frac{\partial F}{\partial r_{ij}} \xi_i \xi_j > 0$ for all $\xi \in \mathbb{R}^n \backslash \{0\}$

Equivalently, the partial differential equation (i) is elliptic at γ if and only if the all Eigen values (they depend on γ) of $[F_{ij}(\gamma)]_{n\times n}$ are positive.

Among the most important of all partial differential equations are undoubtedly Laplace's equations.

$$\Delta u = 0 \qquad (1.2)$$

Where x$\in \Omega$ and the unknown is $u : \bar{\Omega} \to \mathbb{R}$, $u = u(x)$,where $\Omega \subset \mathbb{R}^n$ is a given open set.

Definition A C^2 function u satisfying (1.2) is called a harmonic function.

EXAMPLES

1) Show that $u(x) = a.x$ is harmonic for $a \in \mathbb{R}^n$

Proof

$u(x) = a.x$

$\quad = a_1x_1 + a_2x_2+, \dots, a_nx_n$

$D_i u(x) = a_i$

$D_{ii}u(x) = 0$

$\Delta u(x) = \sum_{i=1}^{n} D_{ii}\, u(x) = 0$

Hence $u(x) = a.x\ is\ harmonic\ functon$

2) Show that $u(x) = e^{x_1+,\dots,+x_{n-1}} \sin(\sqrt{n-1}\, x_n)$ is harmonic on $\mathbb{R}^n$

Proof

$u_{x_i} = e^{x_1+,\dots,+x_{n-1}} \sin(\sqrt{n-1}\, x_n)$, For i=1,…, n-1

$u_{x_ix_i} = e^{x_1+,\dots,+x_{n-1}} \sin(\sqrt{n-1}\, x_n)$, For i=1,…, n-1

$u_{x_n} = \sqrt{n-1}\, e^{x_1+,\dots,+x_{n-1}} \cos(\sqrt{n-1}\, x_n)$ $, i = n$

$u_{x_nx_n} = -(n-1)e^{x_1+,\dots,+x_{n-1}} \sin(\sqrt{n-1}\, x_n)$, i=n

$\Delta u(x) = \sum_{i=1}^{n-1} u_{x_ix_i} + u_{x_nx_n}$

$\Delta u(x) = 0$

Hence $u(x) = e^{x_1,\dots,x_{n-1}} \sin(\sqrt{n-1}\, x_n)$ is a harmonic function

Function Space

Let $\Omega \subseteq \mathbb{R}^n$ be a domain .The following function space will be used.

- $C(\Omega)$ or $C^0(\Omega)$ will stand for the space of continuous functions on Ω
- Let $k \geq 1$ be an integer.
    - $C^k(\Omega) := \{u: \Omega \to \mathbb{R}: D^\alpha U\ is\ continuous\ in\ \Omega\ for\ all\ |\alpha| \leq k\}$.

- $C^k(\overline{\Omega}) := \{u \in C^k(\Omega): D^\alpha U \text{ is uniformly continuous in } \Omega \text{ for all } |\alpha| \leq k\}$
- $C_c{}^k(\Omega) := \{u \in C^k(\Omega): supp(u) \text{ is a compac subset of } \Omega\}$, where $supp(u) = \overline{\{x \in \Omega: u(x) \neq 0\}}$.
- $C^\infty(\Omega) := \{u: \Omega \to \mathbb{R}: u \text{ is infinitely differentiable in } \Omega\}$
- $C_c{}^\infty(\Omega)$ will be the space of infinitely differentiable function in Ω with compact support.

* $L_\infty(\Omega)$ is the set of all bounded measurable functions in Ω; the norm is defined by

$$\|u\|_{\infty,\Omega} = ess \sup_{x\in\Omega} |u(x)|$$

The support of a function u, denoted by *supp u*, is the complement of the largest open set on which $u = 0$, if $\Omega \subseteq \mathbb{R}^n$, we denote by $C_0{}^\infty(\Omega)$ the space of all C^∞ functions on $\mathbb{R}^n$ whose support is compact and lies in Ω.(In particular, if Ω is open such functions vanish near $\partial\Omega$.)

1.2 Result from Advanced Calculus

If u is a differentiable function defined near $\partial\Omega$.We can then define the normal derivative of u on $\partial\Omega$ by

$$\frac{\partial u}{\partial v} = v.\, grad\ u = v.\, Du$$

The Divergence Theorem Let $v(x) = (v_1(x), \ldots, v_n(x))$ be the unit out ward normal to $\partial\Omega$ at $x \in \partial\Omega$. Ω be a c^1 boundary and $u \in c^1(\overline{\Omega})$

$\int_\Omega u_{xi}\, dx = \int_{\partial\Omega} u\, v_i d\sigma$, More generally we have

$\int_\Omega divw dx = \int_{\partial\Omega} w\, v d\sigma$ *[Diveregence theorem]*

Where w is a c^1 vector field on $\overline{\Omega}$ and the dot (.) denotes the euclidean product of vectors in $\mathbb{R}^n$, and $d\sigma$ is the volume element of $\partial\Omega$.

If $x \in \mathbb{R}^n$, $x = |x|\frac{x}{|x|} = rw$, where $r \in (0,\infty)$ and $w = \frac{x}{|x|} \in s^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ which is a unit sphere. The formula $x = rw$ is called the polar coordinate representation of x. Lebesgue measure is given in polar coordinate by

$$dx = r^{n-1}\, dr\, d\sigma(w)$$

Where $d\sigma$ is surface measure on s^{n-1}.

If f is lebesgue measurable function in $\mathbb{R}^n$ such that either $f \geq 0$ in $\mathbb{R}^n$ or $f \in L^1(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} f(x)\, dx = \int_0^{\infty}\int_{s^{n-1}} f(rw)\, r^{n-1} d\sigma(w) dr$$

And, given $f \in L^1(B)$ where $B = B(x^0, r)$

$$\int_{B(x^0,r)} f(x)dx = \int_0^{r}\int_{s^{n-1}} f(x^0 + rw)\, r^{n-1} d\sigma(w) dr$$

1.3 Change of Variable Formula

Let Ω be an open set in $\mathbb{R}^n$, and $\Psi: \Omega \to \mathbb{R}^n$ be a one-to-one c^1 function such that $\Psi^{-1}: \Psi(\Omega) \to \Omega$ is also c^1.suppose that f is lebesgue measurable on $\Psi(\Omega)$.Then $fo\Psi$ is lebesgue measurable on Ω, and

$\int_{\Psi(\Omega)} f(x)dx = \int_{\Omega} f(\Psi(x))\, |det J\Psi(x)| dx$. Where $J\Psi(x)$ is the jacobian matrix of Ψ at $x \in \Omega$.

We also need to fix some notations

- ω_n denote the volume of the unit ball B (0, 1) in $\mathbb{R}^n$ (i.e. $\omega_n := |B(0,1)|$).It follows that $|\partial B(0,1)| := n\omega_n$.

Let $\Psi(y) = x + ry$.Then $J\Psi(y) = rI$.

Note that

$|B(x,r)| = \int_{B(x,r)} dy = \int_{\Psi(B(0,1))} dy$

$= \int_{B(0,1)} |\det(J\Psi(x))| dx$

Since $\Psi(y) = (x_1 + ry_1, x_2 + ry_2, \dots, x_n + ry_n)$

$J\Psi(x) = \begin{bmatrix} r & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r \end{bmatrix}$

$|\det J\Psi(x)| = r^n$

Thus $|B(x,r)| = \int_{B(0,1)} r^n dy = r^n \int_{B(0,1)} dy = r^n |B(0,1)| = r^n \omega_n$

Therefore,

$|B(x,r)| = r^n \omega_n$

Similarly, $|\partial B(x,r)| = r^{n-1} n\omega_n$

❖ For $f \in L^1(\mu), and$ a measurable set E, which $\mu(E) \neq 0$.we use the notation

$$⨍_E f d\mu := \frac{1}{\mu(E)} \int_E f d\mu$$

(The integral average of f *over* E

1.4 Basic property of Harmonic Function

Green's Theorem

Let $u, w \in c^2(\overline{\Omega})$.Then we have

i) $\int_\Omega u \Delta w dx = -\int_\Omega Du.Dw dx + \int_{\partial\Omega} u \frac{\partial w}{\partial v} d\sigma$ (Green's First Identity)

(Here Du is the gradient of u)

ii) $\int_\Omega (u\Delta w - w\Delta u) dx = \int_{\partial\Omega} (u\frac{\partial w}{\partial v} - w\frac{\partial u}{\partial v}) d\sigma$ (Green's Second Identity)

Proof i) $div(uDw) = Du.Dw + u\Delta w$

$$\begin{aligned} div(uDw) &= div((uw_{x1}, \dots, uw_{xn})) \\ &= \sum_{i=1}^{n} \frac{\partial}{\partial xi}\left(u\frac{\partial w}{\partial xi}\right) \\ &= \sum_{i=1}^{n} \frac{\partial u}{\partial xi}.\frac{\partial w}{\partial xi} + \sum_{i=1}^{n} u\frac{\partial^2 w}{\partial xi^2} \\ &= Du.Dw + u\Delta w \end{aligned}$$

Integrating with respect to dx on Ω

$$\begin{aligned} \int_\Omega Du\, dwdx + \int_\Omega u\, \Delta wdx &= \int_\Omega div(uDw)\, dx \\ &= \int_{\partial\Omega} uDw.\upsilon\, d\sigma \quad \text{by (Divergence Theorem)} \\ &= \int_{\partial\Omega} u\frac{\partial w}{\partial \upsilon}\, d\sigma \end{aligned}$$

Therefore,

$$\int_\Omega u\Delta wdx = -\int_\Omega Du.Dwdx + \int_{\partial\Omega} u\frac{\partial w}{\partial \upsilon}\, d\sigma \tag{1.3}$$

ii) First consider the following equation

$$\int_\Omega w\Delta udx = -\int_\Omega Du.Dwdx + \int_{\partial\Omega} w\frac{\partial u}{\partial \upsilon}\, d\sigma \tag{1.4}$$

Subtract equation (1.4) from equation (1.3); we get Green's second identity

$$\int_\Omega (u\Delta w - w\Delta u)dx = \int_{\partial\Omega}\left(u\frac{\partial w}{\partial \upsilon} - w\frac{\partial u}{\partial \upsilon}\right)d\sigma \quad \blacksquare$$

The Mean Value Theorem If $B(x, r)$ is a ball with center x and radius r which is completely contained in the open set $\Omega \subset \mathbb{R}^n$, then the value $u(x)$ of a harmonic function $u: \Omega \to \mathbb{R}$ at the center of the ball is given by the average value of u on the surface of the ball; this average value is also equal to the average value of u in the interior of the ball. In other words

$$u(x) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} ud\sigma = \frac{1}{n\omega_n r^n} \int_{B(x,r)} udy$$

Where, ω_n is the volume of the unit ball in n dimensions and σ is the $n-1$ dimensional surface measure.

Theorem (Maximum principle for harmonic functions)

If u is harmonic ($u \in c^2(\Omega) \cap c(\overline{\Omega}), \Delta u = 0$), in a bounded Ω. Then

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$$

Moreover, if Ω is connected and if the maximum of u is obtained in Ω, then u is constant.

Remark

- The second statement is also referred to as the strong maximum principle
- The assertion holds also the minimum of u, replace u by $-u$.

Proof we prove the strong maximum principle. Suppose that $M = \max_{\overline{\Omega}} u$ is attained at $x_0 \in \Omega$, then there exists a ball $B(x_0, r) \subset \Omega,\ r > 0$. Mean value property

$$M = u(x_0) = ⨍_{B(x_0,r)} u(y)dy$$

$$\leq ⨍_{B(x_0,r)} M dy = M$$

We have equality

$\Rightarrow u = M$ on $B(x_0, r)$

If $u(z) < M$ for a point $z \in B(x_0, r)$, then we get a strict inequality

$\mathcal{M} = \{x \in \Omega, u(x) = M\}$

$\mathcal{M}$ is a closed in Ω since u is constant; $\mathcal{M}$ is also open in Ω since it can be viewed as the union of open balls

$\Rightarrow \mathcal{M} = \Omega,\ u$ is constant in Ω. ∎

1.5 Distributions

Distributions (or generalized functions) are objects that generalize functions. Distributions make it possible to differentiate functions whose derivatives do not exist in the classical sense. In particular, any locally integrable function has a distributional derivative.

Definition A distributions $f \in D'$ on a non empty set $\Omega \subset \mathbb{R}^n$ is any continuous linear functional defined on the space of basic functions from D.

i.e.

☞ We write the value of the functional f on basic function φ as $< f, \varphi >= f(\varphi)$ which is a (complex) real numbers.

☞ The distribution $f \in D'$ is a linear functional on the space of basic functions D. That is if $\varphi, \psi \in D$ and $\lambda, \mu \in \mathbb{C}$, then

$$< f, \lambda\varphi + \mu\psi > = f(\lambda\varphi + \mu\psi)$$

$$= \lambda < f, \varphi > +\mu < f, \psi >$$

☞ $f \in D'$ is a continuous functional on D

i.e. if $\varphi_n \to \varphi$ in D as $n \to \infty$, then $< f, \varphi_n > \to < f, \varphi >$

A simple example of a distribution is the Dirac delta δ, defined by

$$\delta(\varphi) =< \delta, \varphi >= \varphi(0)$$

Convolution

Let f and g be locally integrable functions on $\mathbb{R}^n$. The convolution $f * g$ of f and g is defined formally by

$f * g(x) = \int f(x-y)g(y)dy = g * f(x)$

(The two integrals are equal by change of variables)

The convolution of any distribution u with the Dirac delta function δ exists and is equal to u (i.e. $u * \delta = u$).The meaning of this formula is that any distribute on u may be expand- ing in terms of delta functions $(u(x) = \int u(y)\delta(x-y))$

CHAPTER TWO

INTEGRAL REPRESENTATION OF HARMONIC FUNCTIONS

In this chapter we introduce the Green's function, the Poisson's kernel and we develop the Poisson's integral formula which is integral representation of harmonic function in upper half space and in a ball and lastly we shall see Neumann problem.

One good strategy for investigating any partial differential equation is first to identify some explicit solutions and then, provided the partial differential equation(PDE) is linear, to assemble more complicated solutions out of the specific ones, Furthermore ,in looking for explicit solutions is often wise to restrict attention to classes of functions with certain symmetry property. Since Laplace's equation is invariant under rotations. Now, a typical rotation invariant quantity is the distance function from a point, for instance from the origin, that is $r = |x|$. Thus, let us look for radially symmetric harmonic function $u = \gamma(|x|)$.

Definition 2.1 A radial function is a function of the form $u(x) = \gamma(|x|)$ for some functions $\gamma: [0, \infty) \to \mathbb{R}$.

Let us therefore attempt to find a solution u of Laplace's equation

$$\Delta u = 0 \qquad in\, \Omega \tag{2.1}$$

having the form $u(x) = \gamma(r)$, where $r = |x| = \sqrt{{x_1}^2+, \dots, +{x_n}^2}$ and γ to be selected so that $\Delta u = 0$ holds.

First note for $i = 1, \dots, n$ that

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}({x_1}^2+, \dots, +{x_n}^2)^{\frac{-1}{2}}.2x_i = \frac{x_i}{r}, (x \neq 0)$$

We thus have

$$\frac{\partial u}{\partial x_i} = \gamma_r(r)\frac{\partial r}{\partial x_i} = \gamma_r(r)\frac{x_i}{r}$$

$$\frac{\partial^2 u}{\partial {x_i}^2} = \frac{\partial}{x_i}\left(\gamma_r(r)\frac{x_i}{r}\right)$$

$$= \gamma_{rr}\frac{{x_i}^2}{r^2} + \gamma_r\left(\frac{1}{r} + x_i \partial x_i \frac{1}{r}\right)$$

Where

$$\partial x_i \frac{1}{r} = \partial x_i ({x_1}^2+,\ldots,+{x_n}^2)^{\frac{-1}{2}} = -\frac{1}{2}({x_1}^2+,\ldots,+{x_n}^2)^{\frac{-3}{2}} 2x_i = -\frac{x_i}{r^3}$$

Then,

$$\frac{\partial^2 u}{\partial {x_i}^2} = \gamma_{rr}\frac{{x_i}^2}{r^2} + \gamma_r\left(\frac{1}{r} - \frac{{x_i}^2}{r^3}\right)$$

Thus

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial {x_i}^2}$$

$$= \sum_{i=1}^{n} \gamma_{rr}\frac{{x_i}^2}{r^2} + \gamma_r\left(\frac{1}{r} - \frac{{x_i}^2}{r^3}\right)$$

$$= \gamma_{rr} + \gamma_r\left(\frac{n}{r} - \frac{1}{r}\right)$$

Result: If $u(x) = \gamma(r)$ is a radially symmetric solution of Laplace's equation then

$$\gamma_{rr} + \gamma_r\left(\frac{n-1}{r}\right) = 0 \tag{2.2}$$

2.1 The Fundamental solution of the Laplacian and the Newtonian Potential

2.1.1 The Fundamental Solution of the Laplacian

Special solution radially for solving $\Delta u = 0$ *of the* previous equation (2.2)

Assuming $\gamma(r) \neq 0$

$$\frac{\gamma_{rr}}{\gamma_r} = \frac{1-n}{r} \qquad \text{Integrate in } r$$

$$\Rightarrow \ln\gamma_r = (1-n)\ln r + \ln c_1$$

$$\Rightarrow \gamma_r = c_1 r^{1-n} \qquad where\ \ln c_1, is\ a\ constant$$

Integrate again

$$n = 2: \gamma_r = \frac{c_1}{r} \Rightarrow \gamma = c_1 \ln r + c_2$$

$$n \geq 3: \gamma_r = \frac{c_1}{r^{n-1}} \Rightarrow \gamma = -\frac{1}{n-2}\frac{c_1}{r^{n-2}} + c_2$$

Consequently if $r > 0$, we have

$$\gamma(r) = \begin{cases} \frac{1}{2-n}\frac{c_1}{r^{n-2}} + c_2 & if\ n > 2 \\ c_1 lnr + c_2 & if\ n = 2 \end{cases} \tag{2.3}$$

Where $c_1 and\ c_2$ are constants. That is $u: x \longmapsto \gamma(|x|)$ is a solution of $\Delta u = 0$ in $\mathbb{R}^n\backslash\{0\}$.

Note: $\Delta\gamma = 0$ for $x \neq 0$ but not defined if $x = 0$.

$\Delta\gamma$ can be defined everywhere in the sense of distributions.

$\Delta\gamma = \delta_0 ='' \ Dirac\ Mass''$

$$\delta_0(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$$

$$\Delta\gamma(x-y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$$

Formally, this suggest a solution formula for Poisson's equation

$$\Delta u = f \quad in\ \mathbb{R}^n$$

Namely

$$u(x) = \int_{\mathbb{R}^n} \gamma(x-y) f(y) dy$$

Therefore we set that by the translation invariance of the Laplace operator the function

$$u: x \longmapsto \gamma(|x-y|)$$

is a solution of $\Delta u = 0\ \ on\ \mathbb{R}^n\backslash\{y\}$ .we take $c_2 = 0$ but would like to choose the constant c_1 such that

$$\Delta_x \gamma(|x-y|) = \delta_y(x) \qquad (x \in \mathbb{R}^n\backslash\{y\})$$

in the sense that

$$\int_{\mathbb{R}^n} \gamma(|x-y|)\, \Delta\varphi(y) dy = \varphi(x) \qquad For\ all\ \varphi \in c_c^{\infty}(\mathbb{R}^n)$$

For this, suppose $\varphi \in c^{\infty}(\mathbb{R}^n)$ say φ is supported in the open set Ω, and let $\epsilon > 0$ small enough such that $B_\epsilon(x) \subseteq \Omega$. let $\Omega_\epsilon = \Omega\backslash\overline{B_\epsilon}(x)$.

Now we will assume for the case $n \geq 3$

Let $\upsilon(y)$ denote the outer normal to $\partial\Omega_\epsilon$ at y. Note that at $y \in \partial B_\epsilon(x)$

$$\upsilon(y) = \frac{x-y}{|x-y|}$$

By Green's second identity we see that

$$\int_{\Omega_\epsilon} (\gamma(|x-y|)\Delta\varphi(y) - \Delta\gamma(|x-y|)\varphi(y))dy$$

$$= \int_{\partial\Omega_\epsilon} \left(\gamma(|x-y|)\frac{\partial\varphi}{\partial\upsilon} - \varphi(y)\frac{\partial\gamma(|x-y|)}{\partial\upsilon}\right) d\sigma(y)$$

Since $\Delta\gamma(|x-y|) = 0$ on Ω_ϵ, and recalling that $\varphi = 0$ near $\partial\Omega$ we see that

$$\int_{\Omega_\epsilon} \gamma(|x-y|)\Delta\varphi(y)dy = \int_{\partial\Omega_\epsilon} \left(\gamma(|x-y|)\frac{\partial\varphi}{\partial\upsilon} - \varphi(y)\frac{\partial\gamma(|x-y|)}{\partial\upsilon}\right) d\sigma(y)$$

$$= \int_{|x-y|=\epsilon} \left(\gamma(|x-y|)\frac{\partial\varphi}{\partial\upsilon} - \varphi(y)\frac{\partial\gamma(|x-y|)}{\partial\upsilon}\right) d\sigma(y)$$

$$= \frac{c_1}{(2-n)\,\epsilon^{n-2}}\int_{|x-y|=\epsilon} \frac{\partial\varphi}{\partial\upsilon} d\sigma(y) - \int_{|x-y|=\epsilon} \varphi(y)\frac{\partial\gamma(|x-y|)}{\partial\upsilon} d\sigma(y)$$

For all $\epsilon > 0$, and $\gamma(|x-y|)\Delta\varphi(y)$ is integrable on $\mathbb{R}^n$, by the lebesgue dominating convergence theorem we see that

$$\lim_{\epsilon\to 0}\int_{\Omega_\epsilon} \gamma(|x-y|)\Delta\varphi(y)dy = \int_{\Omega} \gamma(|x-y|)\Delta\varphi(y)dy$$

Since $\varphi \in c^{\infty}(\Omega)$, and hence $D\varphi\,(y)$ is bounded it is clear that

$$\lim_{\epsilon\to 0}\frac{c_1}{(2-n)\,\epsilon^{n-2}}\int_{|x-y|=\epsilon} \frac{\partial\varphi}{\partial\upsilon}\, d\sigma(y) = 0$$

Note that this limit is still true provided that $D\varphi$ is locally bounded on Ω.
Also, on $\partial B_\epsilon(x)$

$$\frac{\partial\gamma(|x-y|)}{\partial\upsilon} = D_y\,\gamma(|x-y|).\upsilon$$

$$= D_y\,\gamma(|x-y|)\,\tfrac{x-y}{|x-y|}$$

$$= \tfrac{c_1}{(2-n)}(n-2)\,\tfrac{y-x}{|x-y|}|x-y|^{1-n}.\tfrac{x-y}{|x-y|}$$

$$= -c_1|x-y|^{1-n}$$

Therefore

$$-\int_{|x-y|=\epsilon} \varphi(y)\frac{\partial\gamma(|x-y|)}{\partial\upsilon} d\sigma(y) = \frac{c_1}{\epsilon^{n-1}}\int_{|x-y|=\epsilon} \varphi(y)d\sigma(y)$$

$$= c_1 n\omega_n \fint_{|x-y|=\epsilon} \varphi(y)d\sigma(y)$$

$$\to c_1 n\omega_n \varphi(x) \ as\ \varepsilon \to 0$$

Therefore we see that

$$\int_{\Omega} \gamma(|x-y|)\Delta\varphi(y)dy = c_1 n\omega_n \varphi(x)$$

Hence we choose $c_1 = \frac{1}{n\omega_n}$

similarly For case n=2, we have

$$\int_{\Omega} \gamma(|x-y|)\Delta\varphi(y)dy = c_1 2\pi\varphi(x), since\ |\partial B(0,1)| = 2\pi, when\ n = 2$$

Hence again we choose $c_1 = \frac{1}{2\pi}$

Therefore for $x \neq 0$, we define

$$\Gamma(x) = \begin{cases} \frac{1}{2\pi} ln|x| & if\ n = 2 \\ \frac{1}{n(2-n)\omega_n |x|^{n-2}} & if\ n > 2 \end{cases}$$

Definition 2.1.1.1:-The function

$$\Gamma(|x-y|) = \begin{cases} \frac{1}{2\pi} ln|x-y| & if\ n = 2 \\ \frac{1}{n(2-n)\omega_n |x-y|^{n-2}} & if\ n > 2 \end{cases} \tag{2.4}$$

defined for $x \in \mathbb{R}^n \backslash \{y\}, x \neq 0$, is *the fundamental solution* of Laplace's equation.

We will sometimes slightly abuse notation and write $\Gamma(x-y) = \Gamma(|x-y|)$ to emphasize that the fundamental solution is radial.

The partial differential equation (2.1) is unchanged, and so $x \longmapsto \Gamma(x-y)$ is also harmonic as a function of $x, x \neq y$.let us now take $f: \mathbb{R}^n \to \mathbb{R}$ and note that the mapping $x \longmapsto \Gamma(x-y)f(y)$ $(x \neq y)$ is harmonic for each point $y \in \mathbb{R}^n$, and thus so is the sum of finitely many such expressions built for different points y.

This reasoning might suggest that the convolution

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x-y)f(y)dy \tag{2.5}$$

$$u(x) = \begin{cases} \frac{1}{2\pi}\int_{\mathbb{R}^2} ln|x-y|f(y)dy & if\ n=2 \\ \frac{1}{n(2-n)\omega_n}\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-2}} & if\ n>2 \end{cases}$$

is a solution of $\Delta u = f$ at least when $f \in c_0^{\infty}(\mathbb{R}^n)$. In the next theorem we show that this is indeed the case.

Theorem 2.1.1.1: Let $f \in c_c^{\ 2}(\mathbb{R}^n)$.Then the function defined by (2.5) is in $c^2(\mathbb{R}^n)$ and satisfies $\Delta u = f$ on $\mathbb{R}^n$.

Proof

- we first rewrite the integral, by change of variable in (2.5) as

$$u(x) = \int_{\mathbb{R}^n}\Gamma(y)f(x-y)dy$$

Since $\Gamma(y)\Delta_x f(x-y)\ in\ \mathbb{R}^n$ uniformly in x, we can differentiate under the integral sign.

Hence

$$\frac{u(x+he_i)-u(x)}{h} = \int_{\mathbb{R}^n}\Gamma(y)[\frac{f(x+he_i-y)-f(x-y)}{h}]dy$$

Where $h \neq 0\ and\ e_i = (0,\dots,1,\dots,0)$, the 1 in the i^{th} slot .but

$$\frac{f(x+he_i-y)-f(x-y)}{h} \to \frac{\partial f}{\partial xi}(x-y)$$

Uniformly on $\mathbb{R}^n$ as $h \to 0$, and thus

$$\frac{\partial u}{\partial x_i}(x) = \int_{\mathbb{R}^n}\Gamma(y)\frac{\partial f}{\partial x_i}(x-y)dy \qquad (i=(1,\dots,n))$$

Similarly

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = \int_{\mathbb{R}^n}\Gamma(y)\frac{\partial^2 f}{\partial x_i \partial x_j}(x-y)dy \qquad (\mathrm{i,j}=1,\dots,\mathrm{n}) \qquad (2.6)$$

And

$$\Delta u = \int_{\mathbb{R}^n}\Gamma(y)\Delta_x f(x-y)dy$$

As the expression on the right hand side of (2.6) is continuous in the variable x, we see $u \in c^2(\ \mathbb{R}^n)$.

- Since Γ blows up to 0, we will need for subsequent calculations to isolate this singularity inside a small ball. So for $\epsilon > 0$ we write

$$\Delta u(x) = \int_{B(0,\epsilon)} \Gamma(y)\Delta_x f(x-y)dy + \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \Gamma(y)\Delta_x f(x-y)dy$$

$$= I_\epsilon(x) + J_\epsilon(x)$$

We wish to show below that

$\lim_{\epsilon\to 0} I_\epsilon(x) = 0$, and $\lim_{\epsilon\to 0} J_\epsilon(x) = f(x)$

$|I_\epsilon| \le |D^2 f|_{L^\infty(\mathbb{R}^n)} \int_{B(0,\epsilon)} |\Gamma(y)|dy$

$$\le \begin{cases} c\epsilon\,|ln\epsilon| - \epsilon & (n = 2) \\ c\epsilon^2 & (n \ge 3) \end{cases}$$

In particular: $I_\epsilon \to 0$ as $\epsilon \to 0$.Now we consider $J_\epsilon(x), first\ we\ note\ that$ $\Delta_x f(x-y) = \Delta_y f(x-y)$, since $f \in c_c^{\infty}(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n \setminus B(0,\epsilon)} \Gamma(y)\Delta_x f(x-y)dy = \int_{\Omega \setminus B(0,\epsilon)} \Gamma(y)\Delta_x f(x-y)dy$$

$$= \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \Gamma(y)\Delta_y f(x-y)dy$$

By Green's second identity, and noting that $\Gamma(x-y)$ is harmonic on $\Omega \setminus B\,(0,\epsilon)$ we find that

$$J_\epsilon(x) = \int_{\Omega \setminus B(0,\epsilon)} \Gamma(y)\Delta_y f(x-y)dy$$

$$= \int_{\partial(\Omega \setminus B(0,\epsilon))} \left(\Gamma(y)\frac{\partial f}{\partial \upsilon}(x-y) - f(x-y)\frac{\partial \Gamma(y)}{\partial \upsilon}\right) d\sigma(y)$$

$$= \int_{|y|=\epsilon} \left(\Gamma(y)\frac{\partial f}{\partial \upsilon}(x-y) - f(x-y)\frac{\partial \Gamma(y)}{\partial \upsilon}\right) d\sigma(y)$$

It is easy to see that

$$\lim_{\epsilon\to 0} \int_{|y|=\epsilon} \Gamma(y)\frac{\partial f}{\partial \upsilon}(x-y)d\sigma(y) = 0$$

As before we see that

$$\frac{\partial \Gamma}{\partial \upsilon}(y) = -\frac{1}{n\omega_n \epsilon^{n-1}} \quad on\ \partial B(0,\epsilon)$$

Therefore

$$-\lim_{\epsilon\to 0}\int_{|y|=\epsilon} f(x-y)\frac{\partial\Gamma(y)}{\partial \upsilon}d\sigma(y) = \lim_{\epsilon\to 0}\frac{1}{n\omega_n\epsilon^{n-1}}\int_{|y|=\epsilon} f(x-y)d\sigma(y)$$

$$= \lim_{\epsilon\to 0} ⨍_{|y|=\epsilon} f(x-y)d\sigma(y)$$

$$= f(x)$$

Hence

$$\Delta u = f \text{ on } \mathbb{R}^n \quad ■$$

2.1.2 The Newtonian Potential

The Fundamental solution of Laplacian is

$$\Gamma(x) = \begin{cases} \frac{1}{2\pi} ln|x| & if\ n=2 \\ \frac{1}{n(2-n)\omega_n |x|^{n-2}} & if\ n \geq 3 \end{cases}$$

By choosing $c_2 = 0$ and $c_1 = \frac{1}{2\pi}$ if n=2, $c_1 = \frac{1}{n\omega_n}$ if $n \geq 3$, But if we choose $c_2 = 0$ and $c_1 = \frac{1}{2\pi}$ if n=2, $c_1 = \frac{1}{4\pi}$ if n=3, we have

$$\Gamma(x) = \begin{cases} \frac{1}{2\pi} ln|x| & if\ n=2 \\ -\frac{1}{4\pi|x|} & if\ n=3 \end{cases}$$

is also called a fundamental solution for the Laplace operator Δ, the above choose of the constant c_1 is made in order to have $\Delta\Gamma(x) = \delta(x)$. Where $\delta(x)$ denotes the Dirac measure at $x = 0$.

The physical meaning of Γ is remarkable: if $n = 3$, in standard units, $-4\pi\Gamma$ represents the electrostatic potential due to a unitary charge located at the origin and vanishing at infinity.

Clearly, if the origin is replaced by a point y, the corresponding potential is $\Gamma(x-y)$ and

$$\Delta_x\Gamma(x-y) = \delta(x-y)$$

By symmetry, we also have $\Delta_y\Gamma(x-y) = \delta(x-y)$

Suppose that $f(x)$ is the density of a charge located inside a compact set in $\mathbb{R}^3$. Then $\Gamma(x-y)f(y)dy$ represents the potential at x due to charge $f(y)dy$ inside a small region of volume dy around y. The full potential given by the sum of all the contributions we get

$$u(x) = \int_{\mathbb{R}^3} \Gamma(x-y)f(y)dy = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{f(y)}{|x-y|}dy$$

Which is the convolution between f and Γ and it is called the **Newtonian potential** of f. Formally, we have

$$\Delta u(x) = \int_{\mathbb{R}^3} \Delta_x \Gamma(x-y)f(y)dy = \int_{\mathbb{R}^3} \delta(x-y)f(y)d = f(x)$$

under suitable hypothesis on f.

2.2 The Dirichlet Problem

The Dirichlet problem is the problem of finding a function which solves a specified PDE in the interior of a given region takes prescribed values on the boundary of the region.

In this section we show the existence and uniqueness of a solution to the dirichlet problem

$$\begin{cases} \Delta u = 0 & in\ \Omega \\ u = \varphi & on\ \partial\Omega \end{cases} \tag{2.7}$$

where $\varphi \in c(\partial\Omega)$ and for some $x \in \mathbb{R}^n$.

Theorem 2.2.1 :(Green's Representation Formula)

Let Ω be a domain with C^1 boundary. If $u \in C^2(\overline{\Omega})$ .Then for any $x \in \Omega$.

$$u(x) = \int_{\Omega} \Gamma(x-y)\Delta u(y)dy - \int_{\partial\Omega}(\Gamma(x-y)\frac{\partial u(y)}{\partial v} - u(y)\frac{\partial \Gamma}{\partial v}(x-y))d\sigma(y) \tag{2.8}$$

Proof

$\Gamma(x-y)$ is harmonic in y in $\mathbb{R}^n \backslash \{x\}$ for $n \geq 2$. let $x \in \Omega$. then $B(x,\epsilon) \subset\subset \Omega$ for some $\epsilon > 0$. Let $\Omega_\epsilon = \Omega \backslash B(x,\epsilon)$

Now $u(y)$ *and* $\Gamma(x-y)$ are in $c^2(\overline{\Omega_\epsilon})$ .We apply Green's second identity

$$\int_{\Omega_\epsilon} (\Gamma(x-y)\Delta u(y) - u(y)\Delta\Gamma(x-y))dy$$

$$= \int_{\partial\Omega_\epsilon} (\Gamma(x-y)\frac{\partial u}{\partial \upsilon}(y) - u(y)\frac{\partial\Gamma(x-y)}{\partial \upsilon})d\sigma(y) \qquad (2.9)$$

$$\int_{\Omega_\epsilon} \Gamma(x-y)\Delta u(y)dy = \int_{\partial\Omega_\epsilon} \Gamma(x-y)\frac{\partial u}{\partial \upsilon}(y)d\sigma(y) - \int_{\partial\Omega_\epsilon} u(y)\frac{\partial\Gamma(x-y)}{\partial \upsilon}d\sigma(y)$$

Now we note that

$$\lim_{\epsilon\to 0}\int_{\Omega_\epsilon} \Gamma(x-y)\Delta u(y)dy = \int_{\Omega} \Gamma(x-y)\Delta u(y)dy$$

We make the following claims about the limits of the other two terms as $\epsilon \to 0$

Claim1:

$$\lim_{\epsilon\to 0}\left[-\int_{\partial\Omega_\epsilon} u(y)\frac{\partial\Gamma}{\partial \upsilon}(x-y)d\sigma(y)\right] = -\int_{\partial\Omega} u(y)\frac{\partial\Gamma}{\partial \upsilon}(x-y)d\sigma(y) + u(x)$$

Claim2:

$$\lim_{\epsilon\to 0}\left[\int_{\partial\Omega_\epsilon} \Gamma(x-y)\frac{\partial u}{\partial \upsilon}(y)d\sigma(y)\right] = \int_{\partial\Omega} \Gamma(x-y)\frac{\partial u}{\partial \upsilon}(y)d\sigma(y)$$

Proof of claim 1

$$-\int_{\partial\Omega_\epsilon} u(y)\frac{\partial\Gamma}{\partial \upsilon}(x-y)d\sigma(y)$$

$$= -\int_{\partial\Omega} u(y)\frac{\partial\Gamma}{\partial \upsilon}(x-y)d\sigma(y) + \int_{\partial B(x,\epsilon)} u(y)\frac{\partial\Gamma}{\partial \upsilon}(x-y)d\sigma(y)$$

Now,

$$\Gamma(x-y) = \frac{1}{n(2-n)\omega_n}|x-y|^{2-n}$$

$$\nabla_y\Gamma(x-y) = \frac{1}{n\omega_n}|x-y|^{1-n}\frac{y-x}{|y-x|}$$

$$\frac{\partial\Gamma}{\partial \upsilon}(x-y) = \nabla_y\Gamma(x-y).\upsilon(y)$$

$$= \nabla_y\Gamma(x-y).\frac{x-y}{|x-y|}$$

And the outward normal derivative on $B(x,\epsilon)$ is

$$\upsilon(y) = \frac{x-y}{|x-y|} \qquad for\ y \in \partial B(x,\epsilon)$$

On the $\partial B(x,\epsilon)$, we have $\Gamma(x-y) = \Gamma(\epsilon)$

Therefore,

$\frac{\partial \Gamma}{\partial \upsilon}(\epsilon) = \frac{1}{n\omega_n}\epsilon^{1-n}$

Therefore,

$$\int_{\partial B(x,\epsilon)} u(y)\frac{\partial \Gamma}{\partial \upsilon}(x-y)d\sigma(y) = \frac{1}{n\omega_n}\int_{\partial B(x,\epsilon)}\frac{u(y)}{|x-y|^{n-1}}d\sigma(y)$$

$$= \frac{\partial \Gamma(\epsilon)}{\partial \epsilon}\int_{\partial B(x,\epsilon)} u(y)d\sigma(y)$$

$$= \frac{1}{n\omega_n \epsilon^{n-1}}\int_{\partial B(x,\epsilon)} u(y)d\sigma(y)$$

$$= \frac{1}{|\partial B(x,\epsilon)|}\int_{\partial B(x,\epsilon)} u(y)d\sigma(y)$$

Thus as $\epsilon \to 0$

$$\frac{1}{|\partial B(x,\epsilon)|}\int_{\partial B(x,\epsilon)} u(y)d\sigma(y) \to u(x)$$

Therefore, we have

$$\lim_{\epsilon\to 0}\int_{\partial B(x,\epsilon)} u(y)\frac{\partial \Gamma}{\partial \upsilon}(x-y)d\sigma(y) = u(x)$$

Proof of claim 2

Now we know

$$\int_{\partial\Omega_\epsilon} \Gamma(x-y)\frac{\partial u}{\partial \upsilon}(y)d\sigma(y)$$

$$= \int_{\partial\Omega} \Gamma(x-y)\frac{\partial u}{\partial \upsilon}(y)d\sigma(y) - \int_{\partial B(x,\epsilon)} \Gamma(x-y)\frac{\partial u}{\partial \upsilon}(y)d\sigma(y)$$

We just need to show that

$$\int_{\partial B(x,\epsilon)} \Gamma(x-y)\frac{\partial u}{\partial \upsilon}(y)d\sigma(y) \to 0 \text{ As } \epsilon \to 0$$

Substituting in the explicit formula for Γ for $n \geq 3$

We see that

$$\left|\int_{\partial B(x,\epsilon)} \Gamma(x-y)\frac{\partial u}{\partial \upsilon}(y)d\sigma(y)\right| \leq \frac{1}{n(2-n)\omega_n}\int_{\partial B(x,\epsilon)}\frac{1}{|x-y|^{n-2}}\left|\frac{\partial u}{\partial \upsilon}(y)\right|d\sigma(y)$$

$$\leq \left|\frac{\partial u}{\partial \upsilon}(y)\right|_{L^\infty(B(x,\epsilon))}\frac{1}{n(2-n)\omega_n\epsilon^{n-2}}\int_{\partial B(x,\epsilon)} d\sigma(y)$$

$$\leq c\epsilon\frac{1}{|\partial B(x,\epsilon)|}\int_{\partial B(x,\epsilon)} d\sigma(y)$$

$$= c\epsilon$$

Therefore, as $\epsilon \to 0$

$\int_{\partial B(x,\epsilon)} \Gamma(x-y)\frac{\partial u}{\partial \upsilon}(y)d\sigma(y) \to 0$

After taking limit as $\epsilon \to 0$ (2.9) becomes

$\int_{\Omega} \Gamma(x-y)\Delta u(y)dy = \int_{\partial\Omega} \Gamma(x-y)\frac{\partial u}{\partial \upsilon}(y)d\sigma(y) - \int_{\partial\Omega} \mathrm{u}(y)\frac{\partial \Gamma}{\partial \upsilon}(x-y)d\sigma(y) + u(x)$

Hence

$$u(x) = -\int_{\partial\Omega}\left[\Gamma(x-y)\frac{\partial u}{\partial \upsilon}(y)d\sigma(y) - \mathrm{u}(y)\frac{\partial \Gamma}{\partial \upsilon}(x-y)\right]d\sigma(y) + \int_{\Omega}\Gamma(x-y)\Delta u(y)dy \qquad \blacksquare$$

***Lemma* 2.2.2:** If $u \in C^2(\overline{\Omega})$ *is harmonic in* Ω, *then*

$$u(x) = -\int_{\partial\Omega}\left(\Gamma(x-y)\frac{\partial u}{\partial \upsilon}(y) - \mathrm{u}(y)\frac{\partial \Gamma}{\partial \upsilon}(x-y)\right)d\sigma(y)$$

Proof

The proof follows from the above theorem

$u(x) = \underbrace{\int_{\Omega}\Gamma(x-y)\Delta u(y)dy}_{=0} - \int_{\partial\Omega}(\Gamma(x-y)\frac{\partial u}{\partial \upsilon}(y) - \mathrm{u}(y)\frac{\partial \Gamma(x-y)}{\partial \upsilon})d\sigma(y)$, since u is harmonic.

Therefore

$u(x) = -\int_{\partial\Omega}\left(\Gamma(x-y)\frac{\partial u}{\partial \upsilon}(y) - \mathrm{u}(y)\frac{\partial \Gamma}{\partial \upsilon}(x-y)\right)d\sigma(y) \quad \blacksquare$

We would like to use the Green's representation formula (2.8) to solve $\begin{cases}\Delta u = f \ in\ \Omega \\ u = \varphi \ on\ \partial\Omega\end{cases}$. If we knew Δu on Ω and u on $\partial\Omega$ and $\frac{\partial u}{\partial \upsilon}$ on $\partial\Omega$, then we could solve for u. But we don't know all this information .We know Δu on Ω and u on $\partial\Omega$. We want to eliminate one unknown term by adjusting Γ.

We proceed as follows. For each $x \in \Omega$, we introduce a **corrector function** $\phi^x(y)$ which satisfies the following boundary value problem.

$$\begin{cases}\Delta\phi^x(y) = 0 & in\ \Omega \\ \phi^x(y) = -\Gamma(x-y) & on\ \partial\Omega\end{cases} \qquad (2.10)$$

We use Green's second identity on Ω with $u(y)$ and $\phi^x(y)$ we get

$$\int_{\Omega}\phi^x(y)\Delta u(y)dy = \int_{\partial\Omega}\left(\phi^x(y)\frac{\partial u}{\partial \upsilon}(y) - \mathrm{u}(y)\frac{\partial \phi^x}{\partial \upsilon}(y)\right)d\sigma(y) \qquad (2.11)$$

Subtracting (2.11) to (2.8) we have

$$u(x) = \int_{\Omega}\left(\Gamma(x-y)+\phi^{x}(y)\right)\Delta u(y)dy - \int_{\partial\Omega}\left(\left(\Gamma(x-y)+\phi^{x}(y)\right)\frac{\partial u}{\partial \upsilon}(y) - u(y)\frac{\partial(\Gamma(x-y)+\phi^{x}(y))}{\partial \upsilon}\right)d\sigma(y)$$

$$u(x) = \int_{\Omega}\left(\Gamma(x-y)+\phi^{x}(y)\right)\Delta u(y)dy + \int_{\partial\Omega} u(y)\frac{\partial(\Gamma(x-y)+\phi^{x}(y))}{\partial \upsilon}d\sigma(y) - \int_{\partial\Omega}\left(\Gamma(x-y)+\phi^{x}(y)\right)\frac{\partial u}{\partial \upsilon}(y)d\sigma(y) \quad (2.12)$$

Definition 2.2.1 $G(x,y) := \Gamma(x-y)+\phi^{x}(y). \forall x,y \in \Omega, x \neq y$ is called the Green's function for the laplacian on Ω.

Theorem 2.2.3 (Representation formula)

If $u \in c^{2}(\overline{\Omega})$ solves the Dirichlet problem

$$\begin{cases}\Delta u = f \; in \; \Omega \\ u = \varphi \; on \; \partial\Omega\end{cases}$$

Then

$$u(x) = \int_{\Omega} G(x,y)f(y)dy + \int_{\partial\Omega} \varphi(y)\frac{\partial G}{\partial \upsilon}(x,y)d\sigma(y)$$

Proof

From equation (2.12) we have

$$u(x) = \int_{\Omega}\left(\Gamma(x-y)+\phi^{x}(y)\right)\Delta u(y)dy + \int_{\partial\Omega} u(y)\frac{\partial(\Gamma(x-y)+\phi^{x}(y))}{\partial \upsilon}d\sigma(y) - \int_{\partial\Omega}\left(\Gamma(x-y)+\phi^{x}(y)\right)\frac{\partial u}{\partial \upsilon}(y)d\sigma(y)$$

$$u(x) = \int_{\Omega} G(x,y)\Delta u(y)dy + \int_{\partial\Omega} u(y)\frac{\partial G}{\partial \upsilon}(x,y)d\sigma(y) - \int_{\partial\Omega} G(x,y)\frac{\partial u}{\partial \upsilon}(y)d\sigma(y)$$

Since $G(x,y) = 0 \; \forall y \in \partial\Omega \; and \; \forall x \in \Omega$ now becomes

$$u(x) = \int_{\Omega} G(x,y)\Delta u(y)dy + \int_{\partial\Omega} u(y)\frac{\partial G}{\partial \upsilon}(x,y)d\sigma(y)$$

$$u(x) = \int_{\Omega} G(x,y)f(y)dy + \int_{\partial\Omega} \varphi(y)\frac{\partial G}{\partial \upsilon}(x,y)d\sigma(y) \quad \blacksquare$$

Green's function= fundamental solution with zero boundary data

$$\begin{cases}\Delta_{y}G(x,y) = \delta_{y}(x) & y \in \Omega \\ G(x,y) = 0 & y \in \partial\Omega\end{cases}$$

Corollary 2.2.4

If u is a harmonic in Ω, i.e $f \equiv 0$ in Ω, then

$$u(x) = \int_{\partial\Omega} \varphi(y) \frac{\partial G}{\partial \upsilon}(x,y)d\sigma(y) \qquad (x \in \Omega)$$

Theorem 2.2.5: (symmetry of Green's Functions)

Green's function $G(x,y)$ is symmetric in $\Omega \times \Omega$, i.e. $G(x,y) = G(y,x)$ for $x \neq y \in \Omega$.

Proof

Pick $x_1, x_2 \in \Omega \; with \; x_1 \neq x_2$.choose $r > 0$ small such that $B(x_1,r) \cap B(x_2,r) = \emptyset$ .Set $G_1(y) = G(x_1, y)$ and $G_2(y) = G(x_2,y)$ .We apply Green's formula in $\Omega \backslash B(x_1,r) \cup B(x_2,r)$ and get

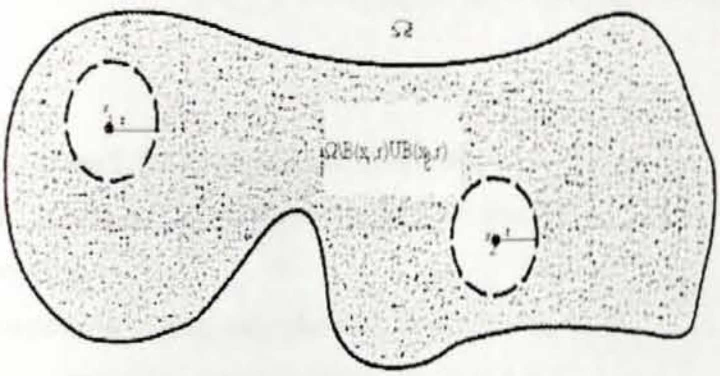

Figure 1

$$\int_{\Omega\backslash B(x_1,r)\cup B(x_2,r)} (G_1 \Delta G_2 - G_2 \Delta G_1)\, dy$$
$$= \int_{\partial\Omega} (G_1 \frac{\partial G_2}{\partial \upsilon} - G_2 \frac{\partial G_1}{\partial \upsilon}) d\sigma - \int_{\partial B(x_1,r)} (G_1 \frac{\partial G_2}{\partial \upsilon} - G_2 \frac{\partial G_1}{\partial \upsilon}) d\sigma$$
$$- \int_{\partial B(x_2,r)} (G_1 \frac{\partial G_2}{\partial \upsilon} - G_2 \frac{\partial G_1}{\partial \upsilon}) d\sigma$$

Since G_i is harmonic for $y \neq x_i, i = 1,2$,and vanishes on $\partial\Omega$ we have

$$\int_{\partial B(x_1,r)} (G_1 \frac{\partial G_2}{\partial \upsilon} - G_2 \frac{\partial G_1}{\partial \upsilon}) d\sigma + \int_{\partial B(x_2,r)} (G_1 \frac{\partial G_2}{\partial \upsilon} - G_2 \frac{\partial G_1}{\partial \upsilon}) d\sigma = 0$$

Note that the left side has the same limit as the left side in the following as $r \to 0$

$$\int_{\partial B(x_1,r)} (\Gamma \frac{\partial G_2}{\partial \nu} - G_2 \frac{\partial \Gamma}{\partial \nu}) d\sigma + \int_{\partial B(x_2,r)} (G_1 \frac{\partial \Gamma}{\partial \nu} - \Gamma \frac{\partial G_1}{\partial \nu}) d\sigma = 0$$

While we have

$$\int_{\partial B(x_1,r)} \Gamma \frac{\partial G_2}{\partial \nu} d\sigma \to 0, \int_{\partial B(x_2,r)} \Gamma \frac{\partial G_1}{\partial \nu} d\sigma \to 0 \; as \; r \to 0$$

And

$$\int_{\partial B(x_1,r)} G_2 \frac{\partial \Gamma}{\partial \nu} d\sigma \to G_2(x_1), \int_{\partial B(x_2,r)} G_1 \frac{\partial \Gamma}{\partial \nu} d\sigma \to G_1(x_2) \; as \; r \to 0$$

This implies

$$-G_2(x_1) + G_1(x_2) = 0, or \; G(x_2, x_1) = G(x_1, x_2) \quad \blacksquare$$

Theorem 2.2.0 (Uniqueness Theorem)

The solution of the Dirichlet problem is unique.

Proof

Suppose u_1 and u_2 are solutions, take $u_1 = u_2$ on $\partial\Omega$

$\max_{\bar{\Omega}}(u_1 - u_2) = \max_{\partial\Omega}(u_1 - u_2) = 0$

$\Rightarrow u_1 - u_2 = 0$ in Ω

$\Rightarrow u_1 = u_2$ in Ω

Hence there exist at most one solution

2.3 Integral Representation of Harmonic Function in Upper Half-space

2.3.1 Green's Function for Upper Half –Space

In this subsection we will build Green's functions for the half-space $\mathbb{R}_+^n$. Everything depends up on our explicitly solving the corrector problem (2.10) in these regions and this in turn depends up on geometric reflection tricks.

$\mathbb{R}_+^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n / x_n > 0\}$.

Definition 2.3.1.1 If $x = (x_1, \dots, x_{n-1}, x_n) \in \mathbb{R}_+^n$, its reflection in the plane $\partial\, \mathbb{R}_+^n$ is the points $x^* = (x_1, \dots, x_{n-1}, -x_n)$

We will solve problem (2.10) for the half-space by setting

$\phi^x(y) = -\Gamma(x^* - y) = -\Gamma(x_1 - y_1, x_2 - y_2, \dots, x_{n-1} - y_{n-1}, -x_n - y_n)\ \ (x, y \in \mathbb{R}_+^n)$

The idea is that the corrector ϕ^x is built from$-\Gamma$ by "reflecting the singularity" from $x \in \mathbb{R}_+{}^n$ to $x^* \notin \mathbb{R}_+^n$, we note

$\phi^x(y) = -\Gamma(x - y)$ If $y \in \partial\, \mathbb{R}_+^n$ and thus

$\begin{cases} \Delta\phi^x = 0 & in\ \mathbb{R}_+^n \\ \phi^x = -\Gamma(x - y) & \text{on } \partial\mathbb{R}_+^n \end{cases}$ are required.

Definition 2.3.1.2 Green's function for the half space $\mathbb{R}_+^n$ is

$$G(x, y) = \Gamma(x - y) - \Gamma(x^* - y)\, , (x, y \in \mathbb{R}_+^n, x \neq y)$$

2.3.2 Using Green's function to solve Laplace's equation in Upper Half-space

Suppose now u solves the boundary value problem

$$\begin{cases} \Delta u = 0 & in\ \mathbb{R}_+^n \\ u = \varphi & on\ \ \partial\, \mathbb{R}_+^n \end{cases} \tag{2.13}$$

Now

For $n \geq 3$:

Then

$$\frac{\partial G}{\partial y_n}(x,y) = \frac{\partial \Gamma(x-y)}{\partial y_n} - \frac{\partial \Gamma(x^{*}-y)}{\partial y_n}$$

$$= \frac{x_n - y_n}{n\omega_n |x-y|^n} + \frac{x_n + y_n}{n\omega_n |x^{*}-y|^n}$$

$$= \frac{1}{n\omega_n}\left[\frac{x_n - y_n}{|x-y|^n} + \frac{x_n + y_n}{|x^{*}-y|^n}\right]$$

$$= \frac{2x_n}{n\omega_n |x-y|^n}$$

Consequently if $y \in \partial\ \mathbb{R}^n_+, y_n = 0, |x-y| = |x^{*}-y|$

$$\frac{\partial G}{\partial \upsilon}(x,y) = \frac{\partial G}{\partial y_n}(x,y) = \frac{2x_n}{n\omega_n |x-y|^n}$$

Then from representation formula we have

$$u(x) = \frac{2x_n}{n\omega_n}\int_{\partial \mathbb{R}_+{}^n} \frac{\varphi(y)}{|x-y|^n} d\sigma(y) \quad (x \in \mathbb{R}^n_+) \tag{2.14}$$

which is called a **Poisson's integral formula (integral representation of harmonic functions in upper half space)**. And the function $k(x,y) = \frac{2x_n}{n\omega_n |x-y|^n}$ $(x \in \mathbb{R}^n_+, y \in \partial\ \mathbb{R}^n_+)$ is a **Poisson's kernel.**

For $\boldsymbol{n} = \mathbf{2}$. let $\mathbb{R}^2_+$ be the upper half-plane in $\mathbb{R}^2$ that is

$$\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$$

We will look for the Green's function for $\mathbb{R}_+{}^2$.In particular we need to find a corrector function ϕ^x for each $x \in \mathbb{R}^2_+$, such that

$$\begin{cases} \Delta\phi^x = 0 & in\ \ \mathbb{R}^2_+ \\ \phi^x = -\Gamma(x-y) & \text{on } \partial\,\mathbb{R}^2_+ \end{cases}$$

Fix $x \in \mathbb{R}^2_+$.we know $\Delta_y \Gamma(x-y) = 0$ for $x \neq y$ .therefore, if we choose $z \notin \Omega$, then $\Delta_y \Gamma(z-y) = 0$ for all $y \in \Omega$ .Now, if we choose $z = z(x)$ appropriately, $z \notin \Omega$, such that $\Gamma(z-y) = \Gamma(x-y)$ for $y \in \Omega$, then letting $\phi^x(y) = -\Gamma(x-y)$ we will have found a corrector function ,Recall for $n = 2$

$$\Gamma(z-y) = \frac{1}{2\pi} ln|z-y|$$

Therefore, $\Gamma(z-y)$ is a function of $|z-y|$, for $x = (x_1, x_2) \in \mathbb{R}^2_+$, we see that for all $y \in \partial\, \mathbb{R}^2_+$

$$|x-y| = |(x_1, x_2) - (y_1, 0)| = |(x_1, -x_2) - (y_1, 0)| = |x^* - y|$$

Where $x^* = (x_1, -x_2)$ is the reflection of x in the plane

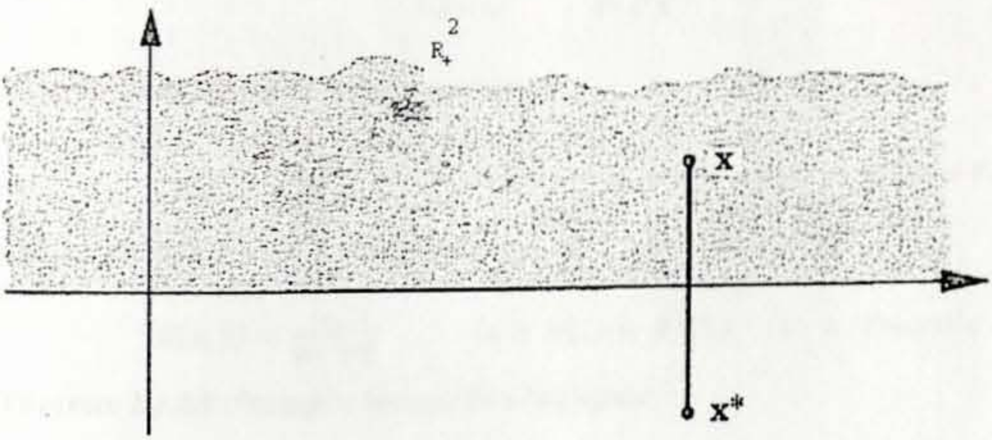

Figure 2

Therefore, a Green's function for the upper half-plane is given by

$$G(x,y) = -\Gamma(x^* - y) + \Gamma(x-y)$$
$$= -\frac{1}{2\pi} ln|x^* - y| + \frac{1}{2\pi} ln|x-y|$$

Then

$$\frac{\partial G}{\partial y_2}(x,y) = -\frac{\partial\Gamma(x^* - y)}{\partial y_2} + \frac{\partial\Gamma(x-y)}{\partial y_2}$$
$$= \frac{1}{2\pi}\frac{x_2 - y_2}{|x-y|^2} + \frac{1}{2\pi}\frac{x_2 + y_2}{|x^* - y|^2}$$
$$= \frac{1}{2\pi}\left[\frac{x_2 - y_2}{|x-y|^2} + \frac{x_2 + y_2}{|x^* - y|^2}\right]$$
$$= \frac{x_2}{\pi|x-y|^2}$$

Consequently if $y \in \partial\,\mathbb{R}^2_+$,$y_2 = 0, |x-y| = |x^* - y|$

$$\frac{\partial G}{\partial \upsilon}(x,y) = \frac{\partial G}{\partial y_2} - (x,y) = \frac{x_2}{\pi|x-y|^2}$$

Suppose now u solves the boundary value problem

$$\begin{cases} \Delta u = 0 & in\ \mathbb{R}^2_+ \\ u = \varphi & on\ \partial\,\mathbb{R}^2_+ \end{cases}$$

Then from representation formula we have

$u(x) = \frac{x_2}{\pi}\int_{\partial\mathbb{R}_+{}^2} \frac{\varphi(y)}{|x-y|^2}d\sigma(y)$ $(x \in \mathbb{R}^2_+)$ which is called a Poisson's integral formula. And the function

$k(x,y) = \frac{x_2}{\pi|x-y|^2}$ $(x \in \mathbb{R}^2_+, y \in \partial\,\mathbb{R}^2_+)$ is a Poisson's kernel.

Theorem 2.3.2.1 (Poisson's formula for a half-space)

Assume $\varphi \in c(\mathbb{R}^{n-1}) \cap l^\infty(\mathbb{R}^{n-1})$,and $u(x) = \frac{2x_n}{n\omega_n}\int_{\partial\mathbb{R}_+{}^n} \frac{\varphi(y)}{|x-y|^n}d\sigma(y)$ .Then

(i) $u \in c^\infty(\mathbb{R}_+{}^n) \cap l^\infty(\mathbb{R}_+{}^n)$

(ii) $\Delta u = 0$ in $\mathbb{R}_+{}^n$

(iii) $\lim_{x\to x^0} u(x) = \varphi(x^0)$ for each point $x^0 \in \partial\mathbb{R}_+{}^n, x \in \mathbb{R}_+{}^n$

Proof

i) For each fixed x,the mapping $y \mapsto G(x,y)$ is harmonic except for $y = x$.As $G(x,y) = G(y,x)$ according to theorem 2.2.5, $x \mapsto G(x,y)$ is harmonic except for $x = y$.Thus $x \mapsto \frac{\partial G}{\partial y_n}(x,y) = k(x,y)$ is harmonic for $x \in \mathbb{R}_+{}^n$, $y \in \partial\mathbb{R}_+{}^n$

ii) A direct calculation, the detail of which we omit

$$1 = \int_{\partial\mathbb{R}_+{}^n} k(x,y)\, d\sigma(y) \tag{2.15}$$

For each $x \in \mathbb{R}_+{}^n$, as φ is bounded, u defined by (2.14) is likewise bounded. Since $x \mapsto k(x,y)$ is symmetric for $x \neq y$, we easily verify as well $u \in c^{\infty}(\mathbb{R}_+{}^n)$ with

$$\Delta u(x) = \int_{\partial\mathbb{R}_+{}^n} \Delta_x k(x,y)\varphi(y)\, d\sigma(y) = 0$$

$$\Rightarrow \Delta u = 0$$

iii) Now fix $x^0 \in \partial\mathbb{R}_+{}^n$,for $\epsilon > 0$,choose $\delta > 0$ so small that

$|\varphi(y) - \varphi(x^0)| < \epsilon$ If $|y - x^0| < \delta$, $y \in \partial\mathbb{R}_+{}^n$.Then if $|x - x^0| < \frac{\delta}{2}$, $x \in \mathbb{R}_+{}^n$

$$|u(x) - \varphi(x^0)| \leq \left|\int_{\partial\mathbb{R}_+{}^n} k(x,y)[\varphi(y) - \varphi(x^0)]\, d\sigma(y)\right| \tag{2.16}$$

$$|u(x) - \varphi(x^0)| \leq \int_{\partial\mathbb{R}_+{}^n \cap B(x^0,\delta)} k(x,y)|\varphi(y) - \varphi(x^0)|\, d\sigma(y)$$
$$+ \int_{\partial\mathbb{R}_+{}^n \setminus B(x^0,\delta)} k(x,y)|\varphi(y) - \varphi(x^0|\, d(y) \tag{2.17}$$

$$=: I + J$$

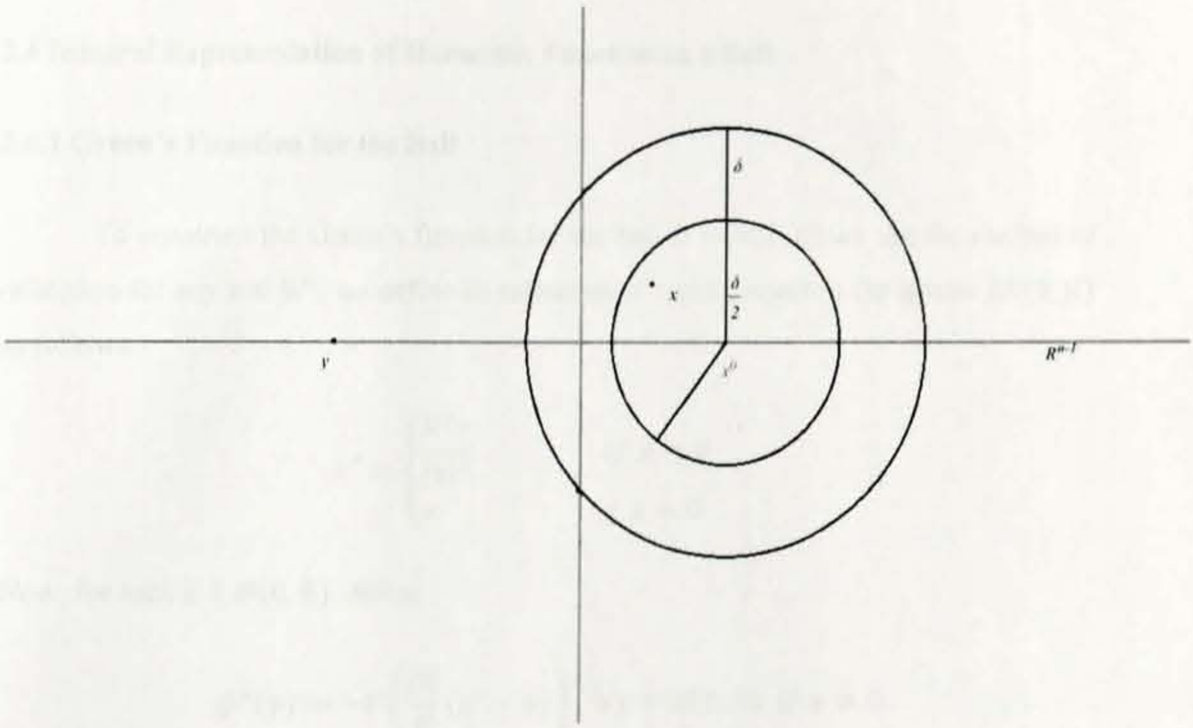

Figure 3

Now (2.15) and (2.16) implies

$I \le \epsilon \int_{\partial \mathbb{R}_+{}^n} k(x,y)\, dy = \epsilon$

Furthermore if $|x - x^0| < \frac{\delta}{2}$ *and* $|y - x^0| \ge \delta$, *we have*

$|y - x^0| \le |y - x| + \frac{\delta}{2} \le |y - x| + \frac{1}{2}|y - x^0|$; *and so*

$|y - x| \ge \frac{1}{2}|y - x^0|$ Thus

$J \le 2\|\varphi\|_{L^\infty} \int_{\partial \mathbb{R}_+{}^n \setminus B(x^0,\delta)} k(x,y) d\sigma(y)$

$\le 2^{n+2} \frac{\|\varphi\|_{L^\infty} x_n}{n\omega_n} \int_{\partial \mathbb{R}_+{}^n \setminus B(x^0,\delta)} |y - x^0|^{-n} d\sigma(y) \to 0$ *as* $x_n \to 0$

Combining this calculate with estimate (2.17), we deduce $|u(x) - \varphi(x^0| \le 2\epsilon$ provided $|x - x^0|$ is sufficiently small. ■

2.4 Integral Representation of Harmonic Function in a Ball

2.4.1 Green's Function for the Ball

To construct the Green's function for the ball $B = B(0, R)$.we use the method of reflection for any $x \in \mathbb{R}^n$, we define its reflection x^* with respect to the sphere $\partial B(0, R)$ as follows

$$x^* = \begin{cases} \frac{R^2 x}{|x|^2} & if\ x \neq 0 \\ \infty & if\ x = 0 \end{cases}$$

Now, for each $x \in B(0, R)$, define

$$\phi^x(y) := -\Gamma\left(\frac{|x|}{R}(x^* - y)\right) \quad \forall y \in B(0, R) \ if\ x \neq 0$$

$$\phi^x(y) = -\frac{1}{n(2-n)\omega_n(\frac{|x|}{R})^{n-2}|x^* - y|^{n-2}}$$

And

$$\phi^0(y) = -\Gamma(R)$$

Note that $\phi^x$is harmonic in $B(0, R)$ for any $x \in B(0, R)$

Lemma 2.4.1.1 For $x, y \in B(0, R)$.Then

$$\left|R\frac{x}{|x|} - |x|y\right| = \left|R\frac{y}{|y|} - |y|x\right|$$

Proof

This follows by squaring each side and showing that these reduce to the same quantity

$$\left|R\frac{x}{|x|} - |x|y\right|^2 = \left(R\frac{x}{|x|} - |x|y\right)(R\frac{x}{|x|} - |x|y)$$

$$= R^2 \frac{|x|^2}{|x|^2} - Rxy - Rxy + |x|^2|y|^2$$

$$= R^2 - 2Rxy + |x|^2|y|^2$$

$$\left|R\frac{y}{|y|} - |y|x\right|^2 = \left(\left|R\frac{y}{|y|} - |y|x\right|\right)\left(\left|R\frac{y}{|y|} - |y|x\right|\right)$$

$$= R^2 \frac{|y|^2}{|y|^2} - Rxy - Rxy + |x|^2|y|^2$$

$$= R^2 - 2Rxy + |x|^2|y|^2$$

Thus, $\left|R\frac{x}{|x|} - |x|y\right|^2 = \left|R\frac{y}{|y|} - |y|x\right|^2$

$\Rightarrow \left|R\frac{x}{|x|} - |x|y\right| = \left|R\frac{y}{|y|} - |y|x\right|$

Hence

$$\left|R\frac{x}{|x|} - |x|y\right| = \left|R\frac{y}{|y|} - |y|x\right| \qquad \blacksquare$$

To show that $\phi^x(y) = -\Gamma(x - y)$ for $(x, y) \in B \times \partial B$

Now

$$\frac{|x|}{R}|x^* - y| = \frac{|x|}{R}\left|\frac{R^2 x}{|x|^2} - y\right|$$

$$= \left|\frac{Rx}{|x|} - |x|(\frac{y}{R})\right|$$

$$= \left| \frac{R\frac{y}{R}}{\left|\frac{y}{R}\right|} - \left|\frac{y}{R}\right| x \right| \quad \text{by the lemma 2.4.1}$$

$$= |x - y| \quad if\ y \in \partial B(0, R)$$

$\phi^x(y) = -\Gamma\left(\frac{|x|}{R}(x^* - y)\right) = -\Gamma(x - y)$, if $x \in B(0,R)$ and $y \in \partial B(0,R)$

Therefore, the Green's function of the ball $B(0,R)$ is

$$G(x,y) = \phi^x(y) + \Gamma(x-y) \qquad (2.18)$$

$$G(x,y) = \begin{cases} -\Gamma\left(\frac{|x|}{R}(x^* - y)\right) + \Gamma(x-y) & \text{if } x \neq 0 \\ -\Gamma(R) + \Gamma(y) & \text{if } x = 0 \end{cases}$$

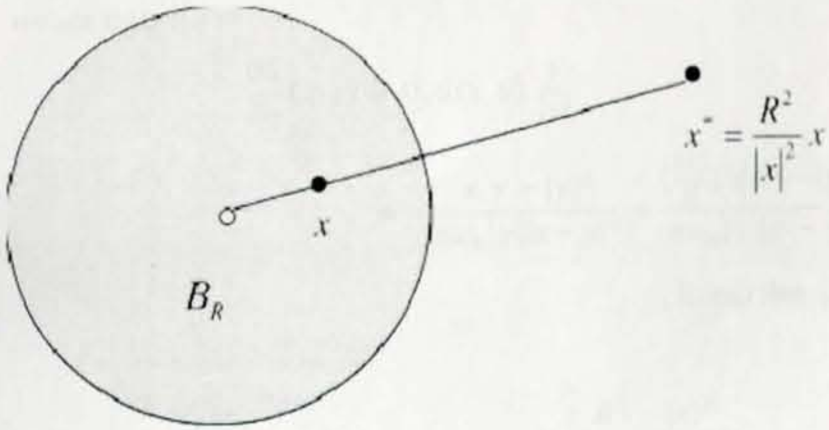

Fig 3. The image x^* of x in the construction of the Green's function for the sphere

2.4.2 Using Green's function to solve Laplace's equation in a Ball

Assume now u solves the boundary value problem

$$\begin{cases} \Delta u = 0 & in\ B(0,R) \\ u = \varphi & On\ \partial B(0,R) \end{cases}$$

Then using the representation formula in theorem 2.2.3 we see

$$u(x) = \int_{\partial B(0,R} \varphi(y) \frac{\partial G}{\partial \upsilon}(x-y) d\sigma(y) \tag{2.19}$$

According to formula (2.18)

$$D_y G(x,y) = \Gamma'(|x-y|) D_y(|x-y|) - \frac{1}{(\frac{|x|}{R})^{n-2}} \Gamma'(|x^* - y|) D_y(|x^* - y|)$$

$$= \frac{1}{n\omega_n |x-y|^{n-1}} \cdot \left(-\frac{x-y}{|x-y|}\right) - \frac{1}{(\frac{|x|}{R})^{n-2}} \frac{1}{n\omega_n |x^* - y|^{n-1}} \left(-\frac{x^* - y}{|x^* - y|}\right)$$

$$= -\frac{x-y}{n\omega_n |x-y|^n} + \frac{(\frac{|x|}{R})^2 (x^* - y)}{n\omega_n ((\frac{|x|}{R})|x^* - y|)^n}$$

Now if $\upsilon = \frac{y}{|y|}$ is the outer unit normal vector field on $\partial B(0,R)$, then for $y \in \partial B(0,R)$ we see that

$$\frac{\partial G}{\partial \upsilon}(x,y) = D_y G(x,y) . \frac{y}{|y|}$$

$$= -\frac{x.y - |y|^2}{n\omega_n |y| |x-y|^n} + \frac{(\frac{|x|}{R})^2 (x^* y - |y|^2)}{n\omega_n |y| |x-y|^n}$$

, Recall that $\left|\frac{|x|}{R}(x^* - y)\right| = |x-y|$

$$= \frac{R^2 - |x|^2}{n\omega_n R |x-y|^n}$$

Hence formula (2.19) yield the representation formula

$u(x) = \frac{R^2 - |x|^2}{n\omega_n R} \int_{\partial B(0,R)} \frac{\varphi(y)}{|x-y|^n} d\sigma(y)$ This is called a ***Poisson's integral formula(integral representation of harmonic functions in a ball).***

Where the function $k(x,y) = \frac{R^2 - |x|^2}{n\omega_n R |x-y|^n}$, $(x \in B(0,R), y \in \partial B(0,R))$ is a **Poisson's kernel** for the ball $B(0,R)$.

Theorem 2.4.1.1 (Poisson representation formula; solution of the Dirichlet problem on Ball)

Let $\varphi \in c(\partial B)$, let $u(x) = \begin{cases} \frac{R^2-|x|^2}{n\omega_n R}\int_{\partial B(0,R)} \frac{\varphi(y)}{|x-y|^n} d\sigma(y) & |x| < R \\ \varphi(x) & |x| = R \end{cases}$

Then $u \in c^2(B) \cap c(\overline{B})$ and solves the dirichlet problem $\begin{cases} \Delta u = 0 & in\ B \\ u = \varphi & on\ \partial B \end{cases}$

Proof

1) It is clear that u is infinitely differentiable .To compute the laplacian Δu we will use the following simple formula .If $w, v \in c^2(\Omega)$,then $\Delta(vw) = v\Delta w + 2Dw.Dv + w\Delta v$

First note

$$D\left(\frac{R^2 - |x|^2}{n\omega_n R}\right) = \frac{1}{n\omega_n R} D(|x|^2) = \frac{-2x}{n\omega_n R}$$

, and

$$\Delta\left(\frac{R^2 - |x|^2}{n\omega_n R}\right) = -\frac{1}{n\omega_n R}\Delta(|x|^2) = \frac{-2}{\omega_n R}$$

Next we note the boundary integral $\int_{\partial B(0,R)} \frac{\varphi(y)}{|x-y|^n} d\sigma(y)$is infinitely differentiable and that one can differentiate under the integral .Let us note that

$$D(|x-y|^{-n}) = -n|x-y|^{-n-2}(x-y), \text{and } \Delta(|x-y|^{-n}) = 2n|x-y|^{-n-2}$$

Therefore,

$$\Delta u(x) = \Delta\left(\frac{R^2 - |x|^2}{n\omega_n R}\right)\int_{\partial B(0,R)} \frac{\varphi(y)}{|x-y|^n} d\sigma(y)$$
$$+ 2D\left(\frac{R^2 - |x|^2}{n\omega_n R}\right).D\left(\int_{\partial B(0,R)} \frac{\varphi(y)}{|x-y|^n} d\sigma(y)\right)$$
$$+ \frac{R^2 - |x|^2}{n\omega_n R}\Delta(\int_{\partial B(0,R)} \frac{\varphi(y)}{|x-y|^n} d\sigma(y))$$

$$= \frac{-2}{\omega_n R}\int_{\partial B(0,R)} \frac{\varphi(y)}{|x-y|^n} d\sigma(y) + 2\left(-\frac{2x}{n\omega_n R}\right)\int_{\partial B(0,R)} -n(x-y)\frac{\varphi(y)}{|x-y|^{n+2}} d\sigma(y)$$
$$+ \frac{R^2-|x|^2}{n\omega_n R}\int_{\partial B(0,R)} \frac{2n\varphi(y)}{|x-y|^{n+2}} d\sigma(y)$$

$$= \frac{-2}{\omega_n R}\int_{\partial B(0,R)} \frac{\varphi(y)|x-y|^2}{|x-y|^{n+2}} d\sigma(y) + \left(\frac{1}{\omega_n R}\right)\int_{\partial B(0,R)} 4(|x|^2 - x.y)\frac{\varphi(y)}{|x-y|^{n+2}} d\sigma(y)$$
$$+ \frac{R^2-|x|^2}{\omega_n R}\int_{\partial B(0,R)} \frac{2\varphi(y)}{|x-y|^{n+2}} d\sigma(y)$$
$$= -\frac{1}{\omega_n R}\int_{\partial B(0,R)} \frac{\varphi(y)}{|x-y|^{n+2}}(2|x-y|^2 - 4(|x|^2 - x.y) - 2R^2 + 2|x|^2) d\sigma(y)$$

$$= 0$$

A much simpler argument would be to notice that, since $G(x,y)$is harmonic in $x \in B(0,R)$for each $y \in \partial B(0,R)$

$$\frac{\partial G}{\partial v}(x,y) = D_y G(x,y).v(y) \text{ is also harmonic in } x \text{ for each } y \in \partial B(0,R)$$

Therefore u is harmonic.

2) Let us now show that u is continuous on $\partial B(0,R)$ with $u = \varphi$ on $\partial B(0,R)$.Let $x^0 \in \partial B(0,R)$

$$u(x) - u(x^0) = u(x) - \varphi(x^0)$$
$$= \int_{\partial B(0,R)} k(x,y)\varphi(y) d\sigma(y) - \int_{\partial B(0,R)} k(x,y)\varphi(x^0) d\sigma(y)$$

Let $\epsilon > 0$ be given, since φ is continuous at x^0 there is $\delta > 0$ such that

$|\varphi(y) - \varphi(x^0)| < \frac{\epsilon}{2}\ \forall\, y \in \partial B$ with $|y - x^0| < \delta$

Now for $x \in B$, but $K(x,y) \geq 0$

$|\varphi(x) - \varphi(x^0)| \leq \int_{\partial B(0,R)} k(x,y)|\varphi(y) - \varphi(x^0)| d\sigma(y)$

$$= \int_{|y-x^0|<\delta} k(x,y)|\varphi(y) - \varphi(x^0)| d\sigma(y)$$

$$+ \int_{|y-x^0|\geq\delta} k(x,y)|\varphi(y) - \varphi(x^0)| d\sigma(y)$$

$$, y \in \partial B$$

But $|\varphi(y) - \varphi(x^0)| \leq |\varphi(y)| + |\varphi(x^0)| \leq 2\max_{\partial B}|\varphi|$

$|\varphi(x) - \varphi(x^0)| \leq \frac{\epsilon}{2}\int_{|y-x^0|<\delta} k(x,y)d\sigma(y) +$

$$2\max_{\partial B}|\varphi| \frac{R^2-|x|^2}{n\omega_n R}\int_{|y-x^0|\geq\delta}\frac{1}{|x-y|^n}d\sigma(y)$$

$$= \frac{\epsilon}{2} + 2\max_{\partial B}|\varphi|\frac{R^2 - |x|^2}{n\omega_n R}\int_{|y-x^0|>\delta}\frac{1}{|x-y|^n}d\sigma(y)$$

Now consider $x \in B$ with $|x - x^0| < \frac{\delta}{2}$ and $|x-y| \geq |y - x^0| - |x^0 - x| \geq \delta - \frac{\delta}{2} = \frac{1}{2}\delta$

Therefore $\int_{|y-x^0|>\delta}\frac{1}{|x-y|^n}d\sigma(y) \leq \int_{|y-x^0|>\delta}\frac{1}{(\frac{\delta}{2})^n}d\sigma(y)$. provided that $|x - x^0| < \frac{\delta}{2}$

Thus $$|u(x) - u(x^0)| \leq \frac{\epsilon}{2} + 2\max_{\partial B}|\varphi|\frac{R^2-|x|^2}{n\omega_n R}\underbrace{(\tfrac{2}{\delta})^n\int_{\partial B}d\sigma(y)}_{(\frac{2}{\delta})^n n\omega_n R^{n-1}}$$

$$\leq \frac{\epsilon}{2} + 2\max_{\partial B}|\varphi|\,(\tfrac{2}{\delta})^n R^{n-2}(R^2 - |x|^2)$$

Choose $0 < \delta^{'} < \delta$ so that $|x|^2$ is very close to R^2 such that

$$2\max_{\partial B}|\varphi|\,(\frac{2}{\delta})^n R^{n-2}(R^2 - |x|^2) < \frac{\epsilon}{2}$$

Therefore

$$|u(x) - u(x^0)| = |u(x) - \varphi(x^0)| < \epsilon \; for \; |x - x^0| < \frac{\delta^{'}}{2} \qquad \blacksquare$$

2.4.3 Harnack's Inequality and Liouville's theorem

From the mean value and Poisson's formulas we deduce the inequality, known as harnack's inequality:

Theorem 2.4.1.2 (Harnack's Inequality)

Let u be harmonic and nonnegative in the ball $B = B(0,R) \subset \mathbb{R}^n$.Then for any $x \in B(0,R)$

$$\frac{(R)^{n-2}(R-|x|)}{(R+|x|)^{n-1}}u(0) \le u(x) \le \frac{(R)^{n-2}(R+|x|)}{(R-|x|)^{n-1}}u(0)$$

Proof

Let $R > 0$ be arbitrary and let $x \in B(0,R)$ .Then by the Poisson integral formula

$$u(x) = \frac{R^2-|x|^2}{n\omega_n R}\int_{\partial B(0,R)}\frac{u(y)}{|x-y|^n}d\sigma(y)$$, for $x \in B(0,R)$ and $y \in \partial B(0,R)$

$|x-y| \le |x| + |y| = |x| + R$, also $|x-y| \ge |y| - |x| = R - |x|$

Therefore $R - |x| \le |x-y| \le R + |x|$

$$\frac{1}{(R+|x|)^n} \le \frac{1}{(|x-y|)^n} \le \frac{1}{(R-|x|)^n}$$

Multiply through by $u(y)$ and integrating on $\partial B(0,R)$ we get

$$\frac{1}{(R+|x|)^n}\int_{\partial B(0.R)}u(y)\,d\sigma(y) \le \int_{\partial B(0.R)}\frac{u(y)}{(|x-y|)^n}d\sigma(y) \le \frac{1}{(R-|x|)^n}\int_{\partial B(0.R)}u(y)\,d\sigma(y)$$

Next multiply the inequality by $\frac{R^2-|x|^2}{n\omega_n R}$

We find

$$\frac{R^2-|x|^2}{n\omega_n R}\frac{1}{(R+|x|)^n}\int_{\partial B(0.R)}u(y)\,d\sigma(y) \le \frac{R^2-|x|^2}{n\omega_n R}\int_{\partial B(0.R)}\frac{u(y)}{(|x-y|)^n}d\sigma(y)$$

$$\le \frac{R^2-|x|^2}{n\omega_n R}\frac{1}{(R-|x|)^n}\int_{\partial B(0.R)}u(y)\,d\sigma(y$$

$i.e$

$$\frac{R-|x|}{n\omega_n R}\frac{1}{(R+|x|)^{n-1}}\int_{\partial B(0.R)} u(y)\,d\sigma(y) \leq \frac{R^2-|x|^2}{n\omega_n R}\int_{\partial B(0.R)} \frac{u(y)}{(|x-y|)^n}\,d\sigma(y)$$

$$\leq \frac{R+|x|}{n\omega_n R}\frac{1}{(R-|x|)^{n-1}}\int_{\partial B(0.R)} u(y)\,d\sigma(y$$

Recall that $|\partial B(0,R)| = n\omega_n R^{n-1}$

Therefore using the mean value property and Poisson formula we find that
$$\frac{(R)^{n-1}(R-|x|)}{R(R+|x|)^{n-1}}\fint_{\partial B(0.R)} u(y)\,d\sigma(y) \leq u(x) \leq \frac{(R)^{n-1}(R+|x|)}{R(R-|x|)^{n-1}}\fint_{\partial B(0.R)} u(y)\,d\sigma(y)$$

$again\ recall$: $u(0) = \fint_{\partial B(0.R)} u(y)\,d\sigma(y)$

$$\frac{(R)^{n-2}(R-|x|)}{(R+|x|)^{n-1}}u(0) \leq u(x) \leq \frac{(R)^{n-2}(R+|x|)}{(R-|x|)^{n-1}}u(0)$$

Corollary 2.4.1.4 (Liouville's Theorem)

If $u \geq 0$ is harmonic in $\mathbb{R}^n$, then it is a constant in $\mathbb{R}^n$.

Proof

The function u is harmonic and nonnegative

Fix $x \in \mathbb{R}^n$ and choose $R > |x|$; Harnack's inequality gives

$$\frac{(R)^{n-2}(R-|x|)}{(R+|x|)^{n-1}}u(0) \leq u(x) \leq \frac{(R)^{n-2}(R+|x|)}{(R-|x|)^{n-1}}u(0)$$

Letting $R \to \infty$, and observing that

$$\lim_{R\to\infty}\frac{(R)^{n-2}(R-|x|)}{(R+|x|)^{n-1}} = \lim_{R\to\infty}\frac{(R)^{n-2}(R+|x|)}{(R-|x|)^{n-1}} = 1$$

Thus we find

$$u(0) \leq u(x \leq u(0)$$

$$\Rightarrow u(x) = u(0)$$

Since x is arbitrary, we conclude

$$u = u(0)$$

Hence u *is constant* ■

2.5 The Neumann Problem

We can find a representation formula for the solution of a Neumann problem as well.

Theorem 2.5.1

If u be a solution of the Neumann problem

$$\begin{cases} \Delta u = f \ \ in \ \Omega \\ \frac{\partial u}{\partial v} = h \ \ on \ \partial\Omega \end{cases} \tag{2.20}$$

where f and h have to satisfy the solvability condition(compatibility condition) $\int_{\partial\Omega} h(y)d\sigma(y) = \int_{\Omega} f(y)dy$.Then

$$u(x) - \frac{1}{|\partial\Omega|}\int_{\partial\Omega} u(y)d\sigma(y) = \int_{\Omega} N(x,y)f(y) - \int_{\partial\Omega} N(x,y)h(y)d\sigma(y)$$

Proof

Keeping in mind that u is uniquely determined up to additive constant. From theorem 2.2.1 we can write

$$u(x) = \int_{\Omega} \Gamma(x-y)f(y) - \int_{\partial\Omega} \Gamma(x-y)h(y)d\sigma(y) + \int_{\partial\Omega} u(y)\frac{\partial\Gamma}{\partial v}(x-y)]d\sigma(y) \tag{2.21}$$

and this time we get the integral, containing the unknown data u on $\partial\Omega$. Mimicking what we have done for the Dirichlet problem. We try to find an analog of the Green's function, that is a function N=N(x, y) given by

$$N(x,y) = \Gamma(x,y) + \phi^x(y)$$

Where for x fixed, ϕ^x is a solution

$$\begin{cases} \Delta_y \phi^x = 0 & in\ \Omega \\ \frac{\partial \phi^x}{\partial \upsilon} = -\frac{\partial \Gamma}{\partial \upsilon}(x-y) & on\ \partial\Omega \end{cases}$$

in order to have $\frac{\partial N}{\partial \upsilon}(x,y) = 0$ on $\partial\Omega$.But this Neumann problem has no solution because the compatibility condition

$$-\int_{\partial\Omega} \frac{\partial \Gamma}{\partial \upsilon}(x-y)\, d\sigma(y) = 0$$

is not satisfied. In fact letting $u = -1$ in (2.8) we get

$$-\int_{\partial\Omega} \frac{\partial \Gamma}{\partial \upsilon}(x-y)\, d\sigma(y) = -1 \tag{2.22}$$

Thus, taking in the account (2.22), we require ϕ^x to satisfy

$$\begin{cases} \Delta_y \phi^x = 0 & in\ \Omega \\ \frac{\partial \phi^x}{\partial \upsilon} = -\frac{\partial \Gamma}{\partial \upsilon}(x-y) + \frac{1}{|\partial\Omega|} & on\ \partial\Omega \end{cases} \tag{2.23}$$

In this way $\int_{\partial\Omega}(-\frac{\partial \Gamma}{\partial \upsilon}(x-y) + \frac{1}{|\partial\Omega|})d\sigma(y) = 0$ and (2.23) is solvable. Note that, with the choose of ϕ^x, we have

$$\frac{\partial N}{\partial \upsilon}(x-y) = \frac{1}{|\partial\Omega|}\ on\ \partial\Omega \tag{2.24}$$

Apply Green's identity to u and ϕ^x we find

$$0 = \int_\Omega \phi^x(y)f(y) - \int_{\partial\Omega} \phi^x(y)h(y)d\sigma(y) + \int_{\partial\Omega} u(y)\frac{\partial \phi^x}{\partial \upsilon}(y)d\sigma(y) \tag{2.25}$$

Adding (2.25) to (2.21) and using (2.24) we obtain

$$u(x) - \frac{1}{|\partial\Omega|}\int_{\partial\Omega} u(y)d\sigma(y) = \int_\Omega N(x,y)f(y) - \int_{\partial\Omega} N(x,y)h(y)d\sigma(y)$$

Thus the solution of the Neumann problem can also written as the sum of two potentials, up to the additive constant $C = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} \mathrm{u}(y)d\sigma(y)$, the mean value of u .

when $f = 0$, i.e. $\begin{cases} \Delta u = 0 \ \ in\ \Omega \\ \frac{\partial u}{\partial v} = h \ \ on\ \partial\Omega \end{cases}$, then we have

$$u(x) - \frac{1}{|\partial\Omega|}\int_{\partial\Omega} \mathrm{u}(y)d\sigma(y) = -\int_{\partial\Omega} N(x,y)h(y)d\sigma(y)$$

Thus the function $N(x,y)$ is called Neumann function (also Green's function for the Neumann problem) ■

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