ADDIS ABABA UNIVERSITY

SCHOOL OF GRADUATE STUDIES

FACULTY OF COMPUTER AND MATHEMATICAL SCIENCES

DEPARTMENT OF MATHEMATICS

PROJECT ON

# A GENERALIZATION OF BOOLEAN RINGS

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**Submitted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics**

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# DECLARATION

I declare that this project has been composed by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship or any other similar title to me.

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# PERMISSION

This is to certify that this project is compiled by Abebaye Assefa in the Department of Mathematics, Addis Ababa University, under my supervision. I hereby also confirm that the project can be submitted for evaluation by examiners and eventual defense.

Dr. K.Venkateswarlu

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# ABSTRACT

In this paper we introduce the concept of Boolean ring and $R$ satisfy the identity $x^2 = x$ which, of course, implies the identity $x^2y - xy^2 = 0$. With this as motivation, we define a Boolean like ring and subBoolean ring $R$ to be a ring $R$ which satisfies the condition that $x^2y - xy^2$ is nilpotent for certain elements $x, y$ in $R$. A strongly Boolean ring is a ring which $x^2y = xy^2$ for some elements $x, y$ in $R$. The commutativity behavior of such rings is considered. Also, certain conditions which imply that these rings have a nil commutator ideal are established. We consider some conditions which imply that the subBoolean ring $R$ is commutative or has a nil commutator ideal. We also prove that a generalized Boolean ring with central idempotents must be nil or commutative. We further consider conditions which imply the commutativity of a generalized Boolean ring.

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# List of Mathematical Notations

| Notation | Meaning |
|---|---|
| $\mathbb{R}$ | The set of real numbers |
| $\mathbb{Q}$ | The set of rational numbers |
| $\mathbb{C}$ | The set of complex numbers |
| $\mathbb{Z}$ | The set of integers |
| $\cup$ | Union |
| $\cap$ | Intersection |
| $\subseteq$ | Subset of |
| $\nsubseteq$ | Not Subset of |
| $\subset$ | Proper subset of |
| $\cdots, \vdots$ | Continuous |
| $\forall$ | For all |
| $\in$ | An element of |
| $\notin$ | Not an element of |
| $\neq$ | Different from |
| $:$ | Such that |
| $N$ | The set of nilpotents |
| $C$ | The center of a ring $R$ |
| $J$ | The Jacobson radical of a ring $R$ |
| $E$ | The set of idempotents of $R$ |
| $R$ | A ring, not necessarily with identity |
| ■ | End of the proof |

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# INTRODUCTION

In this paper we proceed from the subdirect sum structure aspects of Boolean ring and by broadening the component rings of the structure. We generate an over class of rings which not alone includes as sub class that Boolean ring and generalized Boolean ring referred to. Boolean ring play an important role in application of electronics and computers. The concept has been generalized Boolean like, subBoolean and generalized Boolean rings. Boolean like rings of AL.Foster [4] arise naturally from general ring duality considerations and preserve many of the formal properties of Boolean ring. A Boolean like ring is a commutative ring with unity and is of characteristic two. It is clear that every Boolean ring is a Boolean like ring but not conversely. We also characterize all Boolean like ring and subBoolean ring.

The aim of this project is to present a brief discussion about Boolean ring, Boolean like ring, subBoolean ring and generalized Boolean ring.

In chapter one we will see four subtopics such that commutative ring, generalized periodic ring, subdirect product rings and subdirectly irreducible rings which are needed in writing the project are presented. In chapter two we will see three subtopics such that Boolean ring, Boolean like ring and subBoolean ring. Further we see some properties of Boolean like and subBoolean rings. Chapter three has one sub topic that is generalized Boolean ring and also some Theorems and some examples.

# CHAPTER ONE

# Preliminaries

In this chapter we introduce certain definitions and results concerning commutative ring, generalized periodic ring, subdirect product rings and subdirectly irreducible rings. Further we prove some motivating results to introduce the concept of Boolean ring, Boolean like ring, subBoolean ring, and generalized Boolean ring.

## 1.1. A commutative ring

In this subtopic we will see definitions of commutative rings.

**Definition 1.1.1:** A commutative ring $R$ is a set with two operations, addition and multiplication, such that:

1. $(R, +)$ is abelian;
2. $a(bc) = (ab)c$ for any $a, b, c \in R$ ( associative law );
3. $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for any $a, b, c \in R$ (distributive law);
4. $ab = ba$ for all $a, b \in R$ (commutative law );
5. There is an element $1 \in R$ with $1 \neq 0$ and with $1.a = a = a.1$ for any $a \in R$; 1 is identity element.

**Definition 1.1.2:** A sub set S of a commutative ring R is a sub ring of R if:

1) $1 \in S$;
2) If $a, b \in S$, then $a - b \in S$;
3) If $a, b \in$S, then $ab \in S$.

**Definition 1.1.3:** Let $R$ be a ring. A non empty subset $I$ of $R$ is called:

1) Left ideal of $R$ if:
    i. $I$ is a sub ring of $R$; and

ii. $a \in I, r \in R$ then $ra \in I$

2) Right ideal of $R$ if:
   i. $I$ is a subring of $R$; and
   ii. $a \in I,\ r \in R$ then $ar \in I$
3) A two sided ideal (or simply an ideal) of $R$ if both a left and right ideal of $R$.

**Definition 1.1.4:** An ideal $M$ of a ring $R$ is called maximal ideal if $M \neq R$ and for any ideal $I$ of $R$ if $M \subseteq I \subseteq R$ then $M = I$ or $I = R$.

**Definition 1.1.5:** An ideal $P$ of a commutative ring $R$ is called prime ideal if whenever $a, b \in R, ab \in P$ then $a \in P$ or $b \in P$.

**Definition 1.1.6:** The Jacobson radical of a ring R (or J) is the intersection of all maximal ideals.

**Definition 1.1.7:** The nil radical of a ring R is denoted by Nil R = $\{\ r \in R |\, r^n = 0$ for some $n > 0,\ r \in R\}$. This is an ideal in a ring $R$.

**Definition 1.1.8:** Let $(R, +, .)$ be a ring. If $x \in R$ and $x^2 = x$ then we say $x$ is idempotent.

**Remark:** 0 and 1 are idempotent in every ring R.

**Definition 1.1.9:** An element $r$ of a ring $R$ is called nilpotent element if there is a positive integer $n$ such that $r^n = 0$.

**Definition 1.1.10:** A ring R is called a reduced if Nil R$= 0$.

**Definition 1.1.11:** let $R$ be a ring. The center of R is the set $C(\mathrm{R}) = \{x \in \mathrm{R} \setminus\ ax = xa$ for all $a \in \mathrm{R}.\}$.

**Proposition1.1.12:** $C(\mathrm{R})$ is a subring of $R$.

**Definition 1.1.13:** let $R$ be a ring. Let $e^2 = e$ and $ex = xe$ for all $x \in R$ then $ex = xe$ is called central idempotent.

**Remark:** clearly 0, 1 are central idempotent elements in every ring $R$ with unity.

## 1.2 generalized periodic ring

In this subtopic we will see generalized periodic ring and we prove some theorems. Now we begin with the following definitions.

**Definition 1.2.1:** A ring $R$ is called periodic if for each $x \in R$ there exists distinct positive integers $m = m(x),\ n = n(x)$ such that $x^m = x^n$.

**Definition 1.2.2:** An element $x$ of a ring R is called potent if $x^k = x$ for some integer $k > 1$.

**Definition 1.2.3:** A ring $R$ is called subweakly periodic if $x \in R \backslash (J \cup C)$ can be written in the form $x = a + b$, where $a$ is nilpotent and $b$ is potent.

**Definition 1.2.4:** A ring $R$ is called generalized periodic if for every $x \in R$ such that $x \notin (N \cup C)$, We have $x^n - x^m \in (N \cap C)$, for some positive integers $m, n$ of opposite parity. Or equivalently, $x^n - x^{n+k} \in N \cap C$; $n, k \in Z^+$; $k$ odd; $(x \notin N \cup C)$.

**Theorem 1.2.5:** If $R$ is a generalized periodic ring, then $R$ is either commutative or periodic.

**Proof:** Let $N$ and $C$ denote the set of nilpotents and the center of $R$, respectively. Then to see this we need the following cases.

**Case 1:** For $N \subseteq C$, there exists $x \in R$ such that $x \notin C$ implies $x \notin (N \cup C)$, and hence there exists distinct positive integers $m, n$ such that $x^m - x^n \in N$, with $n > m$.

Suppose $(x^m - x^n)^k = 0$, then $(x - x^{n-m+1})^k x^{k(m-1)} = 0$,

This implies that $(x - x^{n-m+1})^{km} = (x - x^{n-m+1})^k x^{k(m-1)} g(x)$

$= 0$, where $g(\lambda) \in Z[\lambda]$

Thus, $x - x^{n-m+1} \in N, \forall x \notin C$, where $n - m + 1 > 1$ .

Recall that, in this case, we assumed that $N \subseteq C$, and $x - x^{n-m+1} \in C, \forall x \notin C$.

If $x \in C$, we see that $x - x^{n(x)} \in C$, for some $n(x) > 1$, where $x \in R$ (arbitrary).

Therefore, $R$ is commutative, by Theorem of Herstein [8].

**Case 2:** For $C \subseteq N$, then $x \notin N$ implies $x \notin (N \cup C)$, and hence there exist distinct positive integers $m, n$ such that $x^n - x^m \in N$, with $n > m$. Repeating the argument we see that:

$x - x^{n-m+1} \in N, \forall x \notin N, (n - m + 1 > 1)$. It is satisfied for all $x \in N$, we conclude that $x - x^{k(x)} \in N$, for some $k(x) > 1$, where $x \in R$ by Theorem of Chacron [10]. Hence $R$ is periodic.

**Case 3:** For $C \nsubseteq N$ $and$ $N \nsubseteq C$. Let $z \in C \backslash N, u \in N \backslash C$. this implies that: $z + u \notin C$ and $z + u \notin N$, and hence $(z+u)^n - (z+u)^m \in N$, for some integers $n > m \geq 1$. Since $z$ commutes with the nilpotent element $u$, implies that $z^n - z^m + u' \in N$, Where $u' \in N$, $u'$ commutes with $z$.

Hence $z^n - z^m \in N$, for $n > m \geq 1$. Now, the repetition of the argument used in the proof of $x - x^{n-m+1} \in N$ shows that $z - z^{n-m+1} \in N$, $\forall z \in C \backslash N$, where $(n - m + 1 > 1)$. Trivially $x - x^k \in N, \forall x \in N, \forall k \in Z^+$.

Finally, if $x \notin (N \cup C)$, then $x^n - x^m \in N$, for some integers $n > m \geq 1$. Again, we see that$x - x^{n-m+1} \in N, \forall x \notin (N \cup C)$, where $(n - m + 1 > 1)$. We conclude that $x - x^{k(x)} \in N$, for some $k(x) > 1$, where $x \in R$ (arbitrary). Thus, by Chacron's theorem [10], $R$ is periodic. This completes the proof. ■

**Lemma 1.2.6:** Let $R$ be a generalized periodic ring. If $e$ is any nonzero central idempotent in $R$ and $a \in N$, then $ea \in C$.

**Proof:** The proof is by contradiction. Suppose the lemma is false, and

Let $\lambda_0 \in N, e\lambda_0 \notin C$. (1)

Since $e \in C$ and $\lambda_0 \in N$, therefore $e\lambda_0$ is nilpotent.

Let $(e\lambda_0)^\alpha \in C, \forall \alpha \geq \alpha_0$, where $\alpha_0$ minimal.

Since $e\,\lambda_0 \notin C$ $(see$ (1)), therefore $\alpha_0 > 1$. let $\lambda = (e\lambda_0)^{\alpha_0 - 1}$.

Then, $\lambda = (e\lambda_0)^{\alpha_0 - 1} \in N,\ \lambda \notin C$ (by the minimality of $\alpha_0$),
$\lambda^k \in C,\ \forall\, k \geq 2,\ e \in C,\ e^2 = e \neq 0,\ e \notin N.$ (2)
Equation (2) implies that $e + \lambda \notin C$ and $e + \lambda \notin N$, and hence (see Definition 1.2, 4) $(e+\lambda)^{m'} - (e+\lambda)^{n'} \in C,$ (3)
Where $m', n'$ are of opposite parity. Combining (2) and (3), we see that (keep in mind that $e\lambda = \lambda$; see (2)) $(m' - n')e\lambda \in C,$ (4)
Where $m' - n'$ is an odd integer. Equation (2) also implies that $(-e + \lambda)$ is not in $(N \cup C)$, so$(-e+\lambda)^{m''} - (-e+\lambda)^{n''} \in N,$ (5)
Where $m'', n''$ are of opposite parity. Combining (2) and (5), we see that $(-e)^{m''} - (-e)^{n''} \in N$, and hence $2e \in N$, since $m''$ and $n''$ are of opposite parity. Therefore, $(2e)^{\gamma} = 0, \gamma \in Z^+$, and thus $2^{\gamma} e = 0$, which implies that $2^{\gamma} e\lambda \in C$; where $\gamma \in Z^+$. (6)
Now, combining (4) and (6), keeping in mind that $(2^{\gamma}, m' - n') = 1$, we see that $e\lambda \in C$, and hence, by (2), $\lambda = e\lambda \in C$, which contradicts (2). This contradiction proves the lemma. ■

Recall that for any $x, y \in R$ then $[x, y]$ denotes the commutator $xy - yx$.

**Theorem 1.2.7:** Suppose $R$ is a generalized periodic ring, and suppose that there exists an element $c \in C$, with $c \neq 0$, such that:

$c[x, y] = 0$ implies $[x, y] = 0, \forall\, x, y \in R$ (7), then $R$ is commutative.

**Proof:** We distinguish two cases.

**Case 1:** For $c \in N$. In this case, $c^k = 0$ for some positive integer $k$, and hence
$c^k[x, y] = 0, \forall\, x, y \in R.$ (8)

Combining (7) and (8), we see that:

$$c^k[x, y] = 0$$
$$\Rightarrow c[c^{k-1}x, y] = 0$$
$$\Rightarrow [c^{k-1}x, y] = 0$$
$$\Rightarrow c^{k-1}[x, y] = 0$$

$$\vdots$$

$$\Rightarrow c[x,y] = 0$$

$$\Rightarrow [x,y] = 0.$$

Thus, $c^k[x,y] = 0$ implies $[x,y] = 0$, and hence $R$ is commutative.

**Case 2:** For $c \notin N$. In view of Theorem 1.2.4, we may assume that $R$ is periodic. This implies, in particular, that $c^m$ is idempotent for some positive integer. Furthermore, $c^m = 0$ (Since $c \notin N$ in our present case).on two cases (since $c \in C$ also) $c^m = e$ is a nonzero central idempotent in $R$.

Let $a \in N$. By Lemma 1.2.6 we have $ea \in C$, and hence $[ea,x] = 0$ for all $x \in R$, which implies $[c^m a,x] = c^m[a,x] = 0, \ \forall x \in R$. (9)

The argument used in Case 1 of Theorem1.2.4 shows that $c^m[a,x] = 0$ implies $[a,x] = 0$, and hence (see (9)) $[a,x] = 0 \ \forall x \in R, \forall a \in N$.Thus, $R$ is a periodicring with the property that $N \subseteq C$. by Theorem of Herstein [10], it follows that $R$ is commutative, and the theorem is proved. ■

**Theorem 1.2.8:** Suppose $R$ is a generalized periodic ring. Suppose, further, that there exists a nonzero central element $c$ such that $ca = 0$ implies $a = 0$, $\forall \ a \in N$. Then $R$ is commutative.

**Proof:** If $R$ is a generalized periodic ring, then the nilpotents $N$ form an ideal and $R/N$ is commutative.

Let $x,y \in R$. For all $\bar{x},\bar{y}$ in $R/N$, $\bar{x}\,\bar{y} = \bar{y}\,\bar{x}$, and hence $[x,y] \in N$. Taking $a = [x,y] \in N$, then we see that:

$$c[x,y] = ca$$

$$c[x,y] = 0$$

$$[x,y] = 0, \forall \ x,y \in R.$$

Therefore $R$ is commutative. ■

## 1.3 subdirect product rings

In this subtopic we will see subdirect product ring and some examples. Now we begin with the following definitions.

**Definition 1.3.1:** A ring R is said to be sub direct product of the family of rings $\{R_i/i \in I\}$ if R is a sub ring of the direct product $\pi R_i$ such that $\pi_k(R) = R_k$ for all $k \in I$, where $i \in I$. $\pi_k : \pi R_i \rightarrow R_k$ is the canonical epimorphism, where $i \in I$.

**Example 1.3.2:** The direct product $\pi R_i$ is itself a sub direct product of the rings $R_i$. There could be other sub directs of the rings $R_i$.

If a ring $R$ is isomorphic to a subdirect product $T$ of rings $R_i$ for $i \in I$. T may be called a representation of $R$ as a sub direct product of the rings $R_i$.

**Theorem 1.3.3:** A ring $R$ has a representation as a subdirect product of rings $R_i$ for $i \in I$ if and only if for each $i \in I$, there exists an epimorphism $\phi_i \longrightarrow R_i$ such that if $r \neq 0$ in R, then $\phi_i(r) \neq 0$ for at least one $i \in I$.

**Example 1.3.4:** The ring $\mathbb{Z}$ is a sub direct of the field $\mathbb{Z}_p$ for all prime numbers $p$.

To prove this, let $\phi_p : \mathbb{Z} \longrightarrow \mathbb{Z}_p$ be the canonical epimorphism.

$\phi_p(n) = n(modp) = [n]_p$. If $r \neq 0$ in $\mathbb{Z}$, then $r$ can not be a multiple of all primes$p \in \mathbb{Z}$, and hence there is at least one prime number $p$ such that $\phi_p(r) \neq 0$.Then, $\mathbb{Z}$ is a fields $\mathbb{Z}_p$.

## 1.4 Subdirectly irreducible ring

In this subtopic we will see subdirectly irreducible ring and some examples. Now we begin with the following definitions.

**Definition 1.4.1:** A ring $R$ is sub directly irreducible if the intersection of all nonzero ideals of $R$ is not $\{0\}$.

Recall that a ring r is called simple if $R^2 \neq (0)$ and $R$ has no ideals other then (0) and $R$.

**Example 1.4.2:** A nonzero simple ring $R$ has no proper nonzero ideals, and hence the intersection of all nonzero ideals, is $R \neq \{0\}$. Thus, a nonzero simple ring is sub directly irreducible.

**Proposition 1.4.3:** Let $R$ be a subdirectly irreducible ring with identity and let $e$ be a central idempotent in $R$. Then $e = 0 \ or \ e = 1$.

**Proof:** Suppose $e$ is a central idempotent element with $e \neq 0$ and $e \neq 1$.

Let A$= eR$. $A$ is a two sided ideal since $e$ is central. Since $e^2 = e \neq 0$ and $e^2 \in A$, then $A \neq \{0\}$. Since $e \neq 1$ and $e$ is central, then there exists $r \in R$ such that $r - er \neq 0$.

Let $B = \{r - er | r \in R\}$.Then $B \neq \{0\}$. since $e$ is central and then $B$ is two sided ideal.

Now, we will show that $A \cap B = \{0\}$.

Let $x \in A \cap B$, then $x = er = r' - er'$ and $e^2r = er' - e^2r'$which implies that $er = er' - er' = 0$. Thus $x = 0$. Since $A \neq \{0\}$ and $B \neq \{0\}$, The $\cap$(all non zero ideals) $\subseteq A \cap B$, and hence

$\cap$(all non zero ideals) $\subseteq A \cap B = \{0\}$ which contradicts the hypothesis that $R$ is sub directly irreducible. Therefore, $e = 0$ or $e = 1$. ■

**Theorem 1.4.4:** (Birkhoff's theorem) every ring is isomorphic to a sub direct product of sub directly irreducible ring.

# CHAPTER TWO

# A generalization of Boolean rings

In this chapter we introduce the concept of Boolean ring, Boolean like ring and subBoolean ring. Further we see some properties and theorems of Boolean ring, Boolean like and subBoolean rings.

## 2.1 Booleans ring

In this subtopic we will see the structure of Boolean ring and some examples. Now we begin with the following definitions.

**Definition 2.1.1:** A Boolean ring is a ring in which every element is idempotent; that is $x^2 = x$ for all $x \in R$.

**Example 2.1.2: 1**) A ring which contain only 0 and 1 is the simplest Boolean ring.

2) The ring of integers modulo 2 is a Boolean ring denoted by ordering integer 2

i.e. (2,+, $*$ ) where 2= {0, 1} with + and $*$ are defined as:

$0 + 0 = 0 \quad 0 * 0 = 0$

$0 + 1 = 1 \quad 0 * 1 = 0$

$1 + 1 = 1 \quad 1 * 1 = 1$, Hence $x^2 = x$, for all $x \in 2$.

3) Let $P(x)$ be the power set of a set $x$. define addition and multiplication as follows: $A + B = (A \cap B^{'}) \cup (A^{'} \cap B)$ and $A.B = A \cap B$, then $(p(x), +, .)$ is Boolean ring.

**Proof:** 1. To show $(p(x), +)$ is an abelian group. Let $A, B, C \in P(x)$, clearly

a) $(A + B) + C = A + (B + C)$ i.e. addition is associative

b) $A + \emptyset = (A \cap \emptyset^{'}) \cup (A + \emptyset)$

$= (A \cap U) \cup \emptyset$

$= A \cup \emptyset$

$= A$ Similarly $\emptyset + A = A$

Hence, $\emptyset$ is the identity element for addition

c) $A + A = (A \cap A^{'}) \cup (A^{'} \cap A)$

$= \emptyset \cup \emptyset$

$= \emptyset$ for all $A \in P(x)$.

d) $A + B = (A \cap B^{'}) \cup (A^{'} \cap B)$

$= (B^{'} \cap A) \cup (B \cap A^{'})$

$= B + A$

Hence $(p(x), +)$ is an abelian group.

2). $A.(B.C) = A \cap (B \cap C)$

$= (A \cap B) \cap C$

$= (A.B).C$ i.e. multiplication is associative.

3) By the definition of addition and multiplication and application of $\cup$ and $\cap$

We also have:

$A.(B + C) = A.B + A.C$ and $(A + B).C = A.C + B.C$

Hence $(p(x), +, .)$ is a ring.

4). clearly $A.A = A \cap A = A$, for all $A \in P(x)$.

Therefore, $(p(x), +, .)$ is Boolean ring. ■

4). Anon trivial example of a Boolean ring is the set $2^x$ of all functions from an arbitrary non-empty set $x$ in to $2 = \{0,1\}$.

The elements of $2^x$ are called 2-valued function on $x$. This distinguished elements and operations in $2^x$ are defined point wise. This means that 0 and 1 in $2^x$are the functions defined for each $x \in X$ by $0(x) = 0$ and $1(x) = 1$. And if $f$ and $g$ are 2-valued functions on $x$ then the functions $f + g$ and $fg$ are defined by:

$((f + g)(x) = f(x) + g(x)$ And $(fg) = f(x)g(x)$

Hence $(ff)(x) = f^2(x) = f(x)f(x)$ for all $x \in 2^x$. Hence if $f(x) = 1$, then

$f^2(x) = f(x)f(x) = 1$

$= f(x)$ and $f(x) = 0$, then

$f^2(x) = f(x)f(x)$

$= 0$

$= f(x)$.

Hence $f^2 = f$ for all$f \in 2^x$.

5). $\mathbb{Z}_2^n$ as a Boolean ring for $n \geq 1$.

6). $\mathbb{Z}/_{2\mathbb{Z}}$ is a Boolean ring.

**Proposition 2.1.3:** A sub ring of a Boolean ring is Boolean. Furthermore, a homomorphic image of a Boolean ring is also Boolean.

**Proof:** Let $S$ be a sub ring of the Boolean ring $R$. Then, for every $x \in S$, $x$ is an element of $R$ and hence $x$ is idempotent.

Therefore $S$ is a Boolean.

Let $T$ be a homomorphic image of $R$ where $\pi: T \to R$ is a ring epimorphism.

Let $t \in T$, Then $t = \pi(r)$ for some $r \in R$.

Hence $t^2 = \pi(r)\pi(r)$

$= \pi(r^2)$

$= \pi(r)$ , since $r^2 = r$

$= t$, since $\pi(r) = t$.

Therefore every element of $T$ is idempotent. Therefore, $T$ is Boolean. ■

**Lemma 2.1.4**: Let $R$ be a Boolean ring. Then $\text{char}(R) = 2$.

**Proof:** Let $x, y \in R$. Then $x + x = (x + x)^2$

$= x^2 + 2x + x^2$

$= x + 2x + x$ (since $x^2 = x$ and $(x + x)^2 = x + x$)

Hence $2x = 0$. ■

**Lemma 2.1.5:** If a ring $R$ is Boolean, then $R$ is commutative.

**Proof:** let $x, y \in R$, we want to show that $xy = yx$.

Since $R$ is Boolean, we have $x^2 = x$ and $y^2 = y$, for all $x, y \in R$.

$$\begin{aligned} \text{Now } x + y &= (x+y)^2 \\ &= x^2 + xy + yx + y^2 \\ &= x + xy + yx + y \end{aligned}$$

By Lemma 2.1.4, $xy + yx = 0$ and $x = -x$

Hence $xy = yx$. ■

**Remark:** every Boolean ring $R$ is periodic ring.

**Proposition 2.1.6:** Let $R$ be a field. If $R$ is Boolean, then $R \cong \mathbb{Z}_2$.

**Proof:** Let $x \in R$ and $x \neq 0$.

Then $x^2 = x$ which implies that $x^2 - x = 0$ , hence $x(x-1) = 0.$ But $x^{-1}$ exists, thus $x = 1$.

Therefore $R = \{0,1\} \cong \mathbb{Z}_2$. ■

**Proposition 2.1.7:** Let $R$ be a Boolean ring with identity. Then every prime ideal is maximal in $R$.

**Proof:** Let P be a prime ideal of $R$. To show that P is maximal, we will show that ${}^{\mathrm{R}}/_{\mathrm{p}}$ is a field. Let $x + \mathrm{p} \in {}^{\mathrm{R}}/_{\mathrm{p}}$ where $x + \mathrm{p} \neq \mathrm{p}$ ; that is, $x \in P$ we have $(x + \mathrm{p})^2 = x^2 + \mathrm{p}$

$= \mathrm{x} + \mathrm{p}$ and hence $x^2 - x \in \mathrm{P}$ which gives that $x(x-1) \in$ p, since $x \notin P$ and $P$ is prime ideal , then $x - 1 \in P$.

Thus $x + \mathrm{P} = 1 + \mathrm{P}$. Therefore, ${}^{\mathrm{R}}/_{\mathrm{p}} = \{\mathrm{p}, 1 + \mathrm{p}\}$ and hence ${}^{\mathrm{R}}/_{\mathrm{p}}$ is a field.

Therefore every prime ideal is maximal in $R$. ■

**Corollary 2.1.8:** Let $R$ be a sub directly irreducible ring with identity $1 \neq 0$. If $R$ is Boolean, then $R \cong \mathbb{Z}_2$.

**Proof:** let $x \in R$, since $R$ is Boolean, then $x$ is a central idempotent ($R$ is commutative). But $R$ is sub directly irreducible with identity, then by proposition 2.1.6, $x = 0$ or $x = 1$.

Therefore$R \cong \mathbb{Z}_2$. ■

**Theorem 2.1.9:** If R is a finite Boolean ring, then $R$ has $2^k$ elements for some positive integer k.

**Proof:** Suppose $|R| = m$. We will show that $m = 2^k$ for some positive integer $k$. Suppose not, then $m$ has a prime factor $p$ other than 2. Since $R$ is an additive group, then by Cauchy's theorem $R$ has an element $x \neq 0$ of order $p$; that is, $p.x = 0$. Since $p$ is odd, then $p = 2n + 1$. Thus $(2n + 1).x = 0$, but $\text{char}(R) = 2$ by Lemma 2.1.4, and hence $x = 0$, a contradiction.

Therefore $R$ has $2^k$ elements for some positive integer $k$. ■

**Theorem 2.1.10:** let $R$ be a ring with identity $1 \neq 0$. then $R$ is Boolean iff $R$ is isomomorphic to a subdirect product of copies of the field $\mathbb{Z}_2$.

**Proof:** Let R be a Boolean ring. By Birkhoff's theorem, R is isomorphic to a subdirect product of nonzero subdirectly irreducible rings $R_i, i \in I$. for each $i \in I$, $R_i$ is the homomorphic image of $R$, and hence each $R_i$ is a Boolean ring with a nonzero identity. But, $R_i$ is a subdirectly irreducible ring, and then by corollary 2.1.8, $R \cong \mathbb{Z}_2$ for each $i \in I$.

Therefore, $R$ isomorphic subdirect product of copies of $\mathbb{Z}_2$.

**Conversely**: - Let $R$ be isomorphic to a subdirect product of copies of $\mathbb{Z}_2$.

Note that for $x \in \mathbb{Z}_2, x^2 = x$. Then every element $x \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2\times} \dots ..$, $x = (x_i)$ for $i \in I$, satisfies the identity $x^2 = x$ since $x^2 = (x_i)^2 = x$ for $i \in$I. thus, every element of the direct product $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2\times} \dots ..$, is idempotent. Since $R$ is isomorphic to a subdirect product of copies of $\mathbb{Z}_2$, then $R$ is isomorphic to a subring of

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2\times} \dots\dots$, This is Boolean.

Therefore, $R$ is Boolean. ■

## 2.2 Boolean-Like ring

In this subtopic we will see Boolean like ring. Further we see some examples. In Theorem 2.2.7 we characterize all commutative Boolean-Like and strongly Boolean-Like rings.

**Definition 2.2.1:** (1) A ring $R$ is called a Boolean-Like if $x^2\,y - xy^2 \in N$ for all $x,\ y,$ in $R \setminus (N \cup C)$.

(2) $R$ is called strongly Boolean-Like ring if $x^2\,y = xy^2$ for all $x,\ y,$ in $R \setminus (N \cup C)$. (1)

It is clear that all commutative rings are both Boolean-Like and strongly Boolean-Like; however the converse is not true.

**Remark:** every Boolean ring is Boolean like ring. But the converse is not true.

**Example**: $R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} : 0,1 \in GF(2) \right\}$ is both Boolean-Like and strongly Boolean-Like but it is not commutative.

The following two lemmas will be needed in the proofs of the main theorems.

**Lemma 2.2.2[8]:** Suppose $R$ is a ring such that $x - x^2 \in C$ for all $x$ in $R$. then $R$ is commutative.

**Lemma 2.2.3 [5]:** suppose $R$ is a ring in which each element $x$ is central, or potent in the sense that $x^k = x$ for some integer $k > 1$. then $R$ is commutative.

**Theorem 2.2.4:** A Boolean-Like ring $R$ with central nilpotents is commutative.

**Proof:** Since $N \subseteq C$, definition 2.2.1 implies that $x^2 y - xy^2 \in N$

for all $x, y \in R$.

We claim that $x - x^2 \in C$ for all $x \in R$.

Suppose not. Let $x \in R$ be such that $x - x^2 \notin C$. Then, $x \notin C$ and $x\,x^2 \notin C$, and $x^2(x - x^2) - x\left(x - x^2\right)^2 = x^4 - x^5 \in N$. Hence, for some $f(\lambda) \in Z[\lambda]$,

$$(x - x^2)^4 = (x - x^2)x^3 f(x)$$

$= (x^4 - x^5)\, f(x) \in N$, and thus $x - x^2 \in N \subseteq C$ .therefore, $x - x^2 \in C$, Contradicts to $x - x^2 \notin C$ .

This contradiction proves $x - x^2 \in C$

Therefore $R$ is commutative. ■

**Corollary 2.2.5:** A reduced Boolean-Like ring is commutative.

**Theorem 2.2.6:** Suppose $R$ is a Boolean-Like ring. Then,

(i) For all $x \in R \setminus C,\ x - x^2 \in N$.

(ii) For all $x \in R \setminus C,\ x^m = x^m e$ for some $m \geq 1$ and some $e \in xZ[x]$ for which $e^2 = e$.

(iii) Every sub ring and every homomorphic image of a Boolean-Like ring is Boolean-like.

**Proof:** (i) Suppose not. Let $x \notin C,\ x - x^2 \notin N$.

Then, $x \notin N, x^2 \notin N$.

**Case 1:** For $x^2 \notin C$. Then, $x \notin (NUC)$, $x^2 \notin (NUC)$, and hence by Definition 2.2.1, with $y = x^2$ we get $x^4 - x^5 \in N$.

Which, as we see in the proof of Theorem 2.2.4, implies that $x - x^2 \in N$, contradicting $x \notin C,\ x - x^2 \notin N$.

**Case 2:** For $x^2 \in C$. Then $x - x^2 \notin C$ (since $x \notin C$), and $x - x^2 \notin N$.

The net result is: $x - x^2 \notin (N \cup C)$ and $x \notin (N \cup C)$ either. So, by setting $y = x - x^2$ in Definition 2.2.1**,** we see that (2) holds, and thus, (as shown above), $x - x^2 \in N$, contradicting $x \notin C$, $x - x^2 \notin N$ again. This contradiction proves (i).

(ii) By part (i), if $x \notin C$, then $x - x^2 \in N$, and hence $\left( x - x^2\right)^m = 0$ for some $m \geq 1$. Thus, $x^m = x^m \left(x\, g(x)\right)$ for some $g(\lambda) \in Z[\lambda]$. Let $e = (x\, g(x))^m$. It is readily verified that $x^m = x^m e$ and $e^2 = e \in xZ[x]$ . This proves (ii).

(iii) This follows at once from the definition of a Boolean-Like ring. ■

**Theorem 2.2.7:** A Boolean-Like ring is commutative if and only if the idempotents of $R$ are central and $N \cap J$ is commutative.

**Proof:** Clearly a commutative Boolean-Like ring satisfies the above stated conditions on the idempotents and on $N \cap J$. To prove the converse, suppose that $R$ is a Boolean-Like ring for which $E \subseteq C, N \cap J$ is commutative, ( $E$ is the set of idempotents).

Claim that $N \subseteq J$

To prove this, let $a \in N$, $x \in R$. By Theorem 2.2.6 (i), $ax \in C$ or $ax - (a\,x)^2 \in N$. If $ax \in C$ then $ax \in N$. On the other hand, if $ax \notin C$, then by Theorem 2.2.6(ii), $(ax)^m = (ax)^m e$ for some idempotent $e \in axZ[ax]$ . Hence,

$$\begin{aligned} e &= ee \\ &= ear \\ &= aer, \text{ for some } r \in R \end{aligned}$$

Re-iterating, we see that $aer$ implies that:

$$\begin{aligned} e &= aer \\ &\vdots \\ &= a^k e r^k \text{ for all positive integers } k. \end{aligned}$$

Since $a \in N$, let $a^k = 0$. Then, by $a^k e r^k$ , $e = 0$, and hence

$$\begin{aligned} (ax)^m &= (ax)^m e \\ &= a^2 e r^2 \end{aligned}$$

$= 0$, Which implies that $ax \in N$. Hence, in any case, $ax \in N$, and thus $ax$ is right quasi regular for all $x \in R$. So $a \in J$, hence the claim. Since $N \subseteq J$, $N \cap J = N$ is commutative.

Claim $N$ is commutative. (1)

By Theorem 2.1.10, $R$ is isomorphic to a subdirect sum of subdirectly irreducible rings $R_i$ , $(i \in I)$. (1’)

Let $\beta: R \rightarrow R_i$ be the natural homomorphism of $R$ onto $R_i$ , and Let $\beta : x \rightarrow x_i$.

We prove that the set $N_i$ of nilpotents of $R_i \subseteq \beta(N) \cup C_i$, center of $R_i$. (2)

To prove this, let $d_i \in N_i, d_i \notin C_i$, and let $\beta(d) = d_i, d \in R$. Then $d \notin C$, and hence by Theorem 2.2.6. $d - d^2 \in N$. Suppose ${d_i}^k = 0$. Then, $d - d^{k+1} = (d - d^2) + d(d - d^2) + \ldots + d^{k-1}(d - d^2) \in N$ (Since $d - d^2 \in N$), this implies that $\beta(d - d^{k+1}) \in \beta(N)$.

Hence, $d_i - {d_i}^{k+1} \in \beta(N)$ ; that is, $d_i \in \beta(N)$, hence (2) holds.

Next to this we prove that:

Every element of $R_i$ is nilpotent or a unit or central. (3)

To prove this, let $x_i \in R_i \setminus C_i$, and let $\beta: x \rightarrow x_i$, $x \in R$, then $x \notin C$, and hence by Theorem 2.2.6: (iii) $x^m = x^m e$ for some $m \geq 1$, $e \in xZ[x]$, $e^2 = e$. Therefore, since $e \in C$ (by (1)), $e_i = \beta(e) \in C$ and, of course, ${e_i}^2 = e_i$. Thus ${x_i}^m = {x_i}^m e_i$, where $e_i$ is a central idempotent in $R_i$. Moreover, since $R_i$ is subdirectly irreducible, $e_i = 0$ or $e_i = 1$. If $e_i = 0$, then $x_i$ is nilpotent, while if $e_i = 1$ , then $x_i$ is a unit (Since $e_i \in x_i Z[x_i]$), which proves (3).

Next, we prove that $N_i \subseteq C_i$ . (4)

Suppose not. Then there exists $a_i \in N_i$ , $a_i \notin C_i$ , and thus for some $b_i \in R_i$ , $[a_i , b_i] \neq 0$. Hence $[a_i , b_i] \neq 0$, $a_i \in N_i$, $b_i \in R_i$. (5)

Now, by (2), $N_i \subseteq \beta(N) \cup C_i$ , and furthermore, $N$ is commutative, by (1). So $N_i$ is commutative. Combining this fact with (5), we see that $b_i \notin N_i$ and, of course, $b_i \notin C_i$. Hence, by (3), it follows that $b_i$ is a unit in $R_i$.

Moreover, by Theorem 2.2.6 $(i)$, $b_i - {b_i}^2 \in N_i$, and hence ${b_i}^{-1}(b_i - {b_i}^2) \in N_i$; that is, $1 - b_i \in N_i$, since $a_i \in N_i$ and $N_i$ is commutative, $[a_i , 1 - b_i ] = 0$ which implies that $[ a_i , b_i] = 0$, a contradiction (see (5)). Hence (5) holds.

Note that $R_i$, as a homomorphic image of $R$, satisfies (in particular) conclusion (i) Theorem 2.2.6.; that is,

$$x_i \in R_i \setminus C_i \text{ implies that } x_i - {x_i}^2 \in N_i. \qquad (6)$$

Combining (2), (6), and lemma 2.2.2 we conclude that $R_i$ is commutative, and hence the $R$ is commutative (see $(1')$). ■

Note that the above Theorem also holds for strongly Boolean-Like Rings.

**Corollary 2.2.8**: A Boolean-Like ring with central idempotents and commuting nilpotents is commutative.

Recall that an element $a \in R$ is said to be right quasiregulr if there is an element $a' \in R$ such that $a + a' + aa' = 0$. we call $a', a$ right quasiregulr. Ring $R$ is said to be a left primitive ring if and only if it has a faithful simple left $R$-module. Ring $R$ is said to be a division ring if it is a ring with unity and for each non zero $a \in R$, there exist an element $b \in R$ such that $ab = ba = 1$. If Jacobson radical $J(R) \neq 0$, then $R$ is called semisimple ring.

A concept related to commutatively is that of a ring having a nil commutator ideal.
In this connection, we have the following.

**Theorem 2.2.9:** The commutator ideal of a Boolean-Like ring $R$ is nil.

**Proof:** First, we show that the commentator ideal $C(R)$ is contained in J (7)
To prove this, recall that $R/J$ is isomorphic to a subdirect sum of primitive rings $R_i$ $(i \in I)$ by Theorem 2.2.6. (iii) $R_i$ is a Boolean-Like ring. If the primitive ring $R_i$ is a division ring, then for any $x_i$ which is not central, we have, by Definition 2.2.1, ${x_i}^2 ( x_i + 1 ) - x_i( x_i + 1)^2 = 0$, and hence $x_i = 0$ or $x_i = -1$, a contradiction. This contradiction proves that $R_i$ is

indeed commutative in this case. Next, suppose that $R_i$ is a primitive ring which is not a division ring. Since Definition 2.2.1 is inherited by all subrings and all homomorphic images of $R_i$, it follows, by Jacobson's density theorem [9], that there exists a division ring $D$ and an integer $k > 1$ such that the complete matrix ring $D_k$ satisfies Definition 2.2.1. This, however, is false, as can be seen by taking $x = E_{11}$, $y = E_{12} + E_{21}$, where $x, y \in D_k$. (To verify this, note that $(x^2y - x\,y^2)^3 = x^2y - x\,y^2 \notin N$).This contradiction proves that $R_i$ must be a division ring, and hence (as shown above), $R_i$ is commutative. So $R\,/\,J$ is commutative, hence (7) holds.

Our next goal is to show that $J \subseteq N \cup C$. (8)

Let $j \in J,\ j \notin C$. Then, by Theorem 2.2.6 (ii), $j^{\,m} = j^m e$ for some $m \geq 1$ and some $e \in jZ[j],\ e^2 = e,$ which implies that $e = 0$. Thus, $j^m = j^m e = 0$ and hence $j \in N,$ hence (8) holds.

Combining (7) and (8), we see that $C(R) \subseteq J \subseteq N \cup C$.

Next, we prove that $N \subseteq J$. (9)

Let $a \in N,\ x \in R,$ and suppose $a^n = 0$. Let $\bar{x} = \bar{x} + J \in R\,/J$. Since, by (7), $R\,/\,J$ is commutative, we have $(\bar{a}\bar{x})^n = (\bar{a})^{\,n}(\bar{x})^n = \bar{0}$, and hence $(a\,x)^n \in J \subseteq N \cup C$ (by (2)). Therefore, $(ax)^n \in N$ (which implies that $ax \in N$), or $(ax)^n \in C$. If $(ax)^n \in C$,then $((ax)^n)^n = (ax)^n\ (ax)^n\ \ldots\ (ax)^n = a^n r$, for some $r \in R$, and hence $((ax)^n)^n = 0$ (since $a^{\,n} = 0$ ). Again, $ax \in N$. It follows that $ax \in N$ for all $x$ in $R$, and thus $ax$ is right quasi regular for all $x$ in $R$. Hence, $a \in J$, hence (9) holds. Combining (9) and (8), we get $N \subseteq J \subseteq N \cup C$ (10)

Next, we prove that $N$ is an ideal. (11)

Let $a \in N,\ x \in R,$ by $(11)$, $a \in J,\ x \in R$, and hence $ax \in J \subseteq \cup\, C$ (by 11). So $ax \in N$ or $ax \in C,$ of course, if $ax \in C,$ then $ax \in N,$ and hence $ax \in N$ in any case. Similarly, $xa \in N$.

Finally, suppose $a, b \in N$. Then, by (11), $a, b \in J$ and hence $a - b \in J \subseteq N \cup C$.

So $a - b \in N$, or $a - b \in C$. If $a - b \in C$, then $a$ commutes with $b$ and hence again $a - b \in N$, hence (11) holds.

To complete the proof, since $N$ is an ideal (by (11)), $R/N$ is a Boolean-Like ring (by Theorem 2.2.6 (iii)), and hence by Theorem 2.2.6 (i) (with $R/N$ playing the role of $R$), every element of $R/N$ is central or potent. Therefore, by lemma 2.1.3 $R/N$ is commutative, which proves the theorem. ■

We now turn our attention to strongly Boolean-Like rings. In this connection, we have:

**Theorem 2.2.10:** Every strongly Boolean-Like ring $R$ with identity is commutative.

**Proof:** First, we prove that the set $N$ of nilpotents of $R$ is commutative.

To prove this, let $a, a' \in N$ and suppose $[a, a'] \neq 0$. Since $1 + a \in R \setminus (N \cup C)$ and $1 + a' \in R \setminus (N \cup C)$, It follows by definition 2.1.1 (2) that $(1 + a)^2 (1 + a') = (1 + a)(1 + a')^2$, and hence $1 + a = 1 + a'$, which contradicts the hypothesis that $[a, a'] \neq 0$, hence the set $N$ of nilpotents of $R$ is commutative. Next, we prove that all idempotents are central. To this end, suppose $e^2 = e$, $x \in R$, $a = xe - exe$. Suppose $a \neq 0$. Then $e \neq 0$, and hence $e \notin (N \cup C)$, since $e \notin C$. Also, $1 + a \notin C$ (since $1 + a \in C$ implies $a \in C$, and hence $ea = ae$, which yields the contradiction $0 = a$). Moreover, clearly $1 + a \notin N$, and thus $1 + a \notin (N \cup C)$. since $e \notin (N \cup C)$ and $1 + a \notin (N \cup C)$, it follows by Definition 2.2.1 (2) that $(1 + a)^2 e = (1 + a)e^2$, which implies that $(1 + a)e = e^2 = e$, and hence $ae = 0$; that is, $a = 0$, contradiction. This contradiction proves that $a = 0$, and thus $xe = exe$. A similar argument shows that $ex = exe$, and hence all idempotents are central

This proves the Theorem. ■

**Theorem 2.2.11:** Suppose $R$ is a strongly Boolean-Like ring with central idempotents.

Then, $R$ is isomorphic to a subdirect sum of rings $R_i$, where $R_i$ is either nil or Commutative.

**Proof:** Since $R$ is also a Boolean-Like ring, it follows, by Theorem 2.2.6 (ii), that for all $x \in R\backslash C,\ x^m = x^m e$ for some $m \geq 1,\ where\ e^2 = e \in xZ[x]$. (12) Write $R$ as a subdirect sum of subdirectly irreducible rings $R_i$. Since $R_i$ inherits (12) from the ring $R$ we see that for $all\, x_i \in R_i\ C_i\ , {x_i}^m = x^m{}_i e_i$ , where $m \geq 1,\ {e_i}^2 = e_i \in x_i Z[x_i]$ (13) Moreover, since (by hypothesis), $e$ is central in $R; e_i$ is central in the subdirectly irreducible ring $R_i$ , which implies that $e_i = 0$ or $e_i = 1$. If $R_i$ , does not have an identity. Then $e_i = 0$, and hence by (13),

$R_i = N_i\ U\ C_i,\ N_i$ is the set of nilpotents of $R_i, C_i$ is the center of $R_i$. (14)

It is easy to see that (14) implies that $R_i = N_i\ or\ R_i = C_i$ (if $R_i$ , does not have an identity). If, on the other hand, $1 \in R_i$, then $R_i$ is a strongly Boolean-Like ring with identity, and hence $R_i$ is commutative, by Theorem 2.2.10. This proves the Theorem. ■

We conclude with the following:

**Remark**: Theorem 2.2.7 and Theorem 2.2.10 are not true if we replace the exponent 2 in $(1)$ or $(2)$ by a prime $p > 2$. To see this, consider the ring

$$\boldsymbol{R} = \left\{ \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ \boldsymbol{0} & \boldsymbol{a^2} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{a} \end{bmatrix} : \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \boldsymbol{GF}(\boldsymbol{4}) \right\}$$

It is readily verified that:

$(i)$ $x^7 y = x y^7$ For all $x, y \in R\backslash(N \cup C)$;

$(ii)$ All idempotents of $R$ are central;

$(iii)$ $N$ is commutative;

$(iv)$ $1 \in R$.

But R is not commutative.

**Remark:** Theorem 2.2.10 is not true if $1 \notin R$, as can be seen by considering the four element ring of matrices in the introduction.

## 2.3 SubBoolean ring

In this subtopic we introduce the concept of SubBoolean ring and we prove some theorems. Further we see some examples. Theorem 2.3.11 below gives a characterization of subBoolean rings.

**Definition 2.3.1:** A ring R is called subBoolean if $x^2y - xy^2 \in N$ for all $x, y \in R\backslash(N \cup J \cup C)$.

Note that the class of subBoolean rings is quite large, all commutative rings, all nil rings, and all rings in which $J = R$. on the other hand, a sub Boolean ring need not be a Boolean or commutative.

**Example: 2.3.2:** $R = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} : 0,1 \in \ GF(2)\right\}$ is SubBoolean ring but neither Boolean nor commutative.

We begin with the following.

**Lemma 2.3.3:** If $R$ is a sub Boolean ring with central idempotents, then the set $N$ of nilpotents is contained in the Jacobson radical $J$ of $R$.

**Proof:** Suppose $a \in N,\ x \in R$. Suppose $ax \in (N \cup J \cup C)$.

If $ax \in N$, then $ax$ is right quasiregulr (r.q.r). Also, $ax \in J$ implies that $ax$ is r.q.r.

Now suppose $ax \in C$ (center of $R$). Then $(ax)^m = a^m x^m$ for all positive integer $m$, and hence $ax \in N$ (since $a \in N$), which again implies that $ax$ is r.q.r.

Next, consider the case $(ax)^2 \in (N \cup J \cup C)$. Again $(ax)^2 \in N$ implies that $ax$ is r.q.r., while $(ax)^2 \in C$ implies

$(ax)^{2k} = (ax)^2(ax)^2 \ldots (ax)^2 = a^k t$ for some $t \in R$, which implies that $ax \in N$ (since $a \in N$), and hence $ax$ is r.q.r.

Finally, if $(ax)^2 \in J$, then $(ax)^2$ is r.q.r., and hence is r.q.r. combining the above facts, we have:

If $ax \in (N \cup J \cup C)$ or $(ax)^2 \in (N \cup J \cup C)$, then $ax$ is r.q.r.

Now, suppose $ax \notin (N \cup J \cup C)$ and $(ax)^2 \notin (N \cup J \cup C)$. Then by Definition 2.3.1, $((ax)^{\ 2})^2(ax) - (ax)^2(ax)^2 \in N$. (1)

From (1), we see that $(ax)^n = (ax)^{n+1}g(ax)$ ; $g(\lambda) \in Z[\lambda]$ ; $n \geq 1$.

Let $e = [(ax)g(ax)]^n$. Then $e^2 = e,\ and\ (ax)^n = (ax)^n e$. Hence,

$(ax)^n = (ax)^n e;\ e = [(ax)g(ax)]^n\ ;\ e^2 = e;\ (a \in N)$. (2)

Suppose $a^m = 0$ (recall that $a \in N$). Since the idempotents are central, (2) readily implies that:

$$\begin{aligned} e &= ee \\ &= e\,[(ax)g(ax)]^n \\ &= eat \\ &= aet, \text{ for some } t \text{ in } R, \end{aligned}$$

And thus
$$\begin{aligned} e &= aet \\ &= a^2et^{\ 2} \\ &\ \vdots \\ &= a^m et^m \\ &= 0. \end{aligned}$$
Hence, by (3), $ax \in N$ and thus $ax$ is r.q.r.

Thus if $ax \notin (N \ \cup\ J\ \cup\ C)$ and $(ax)^2 \notin (N \cup\ J \cup R)$, then $ax$ is r.q.r.

Combining (1) and (2), we conclude that $ax$ is r.q.r. for all $x$ in $R$.

Hence $a \in J$, this proves the lemma. ■

**Theorem 2.3.4:** If R is a subBoolean ring with central idempotent, and then $R/_J$ is commutative.

**Proof:** By Lemma 2.3.3, $N \subseteq J$ and hence by Definition 2.3.1, $x^2y - xy^2 = 0$ for all noncentral elements $x, y \in R/_J$. Since the semisimple ring $R/_J$ is isomorphic to a subdirect sum of primitive rings $R_i$ $(i \in I)$. We have $x^2y - xy^2 = 0$ for all noncentral elements $x, y \in R_i$ $((i \in I)$.

**Case1:** If $R_i$ is a division ring. Suppose $R_i$ is not commutative. Let $x_i$ be a noncentral element of $R_i$. Then, $x_i{}^2(x_i + 1) - x_i(x_i + 1)^2 = 0$, and hence $x_i = 0$ or $x_i = 1$, a contradiction which proves that $R_i$ is commutative.

**Case 2:** If $R_i$ is a primitive ring which is not a division ring. In this case, by Jacobson's Density theorem [9], there exists a division ring $D$ and an integer $k > 1$ such that the complete matrix ring $D_k$ satisfies $x^2y - xy^2 = 0$ for all noncentral elements$x, y \in {}^{R}\!/_{J}$. This, however, is false, as can be seen by taking $x = E_{12},\ y = E_{12} + I_k,\ x, y \in D_k$. This contradiction shows that case 2 never occurs, which forces $R_i$ to be a division ring, and hence is $R_i$ commutative (see case 1). This proves the theorem. ■

**Theorem 2.3.5:** Suppose $R$ is a reduced ($N = \{0\}$) ring and $R$ is a subBoolean ring. Suppose, further, that $J$ is commutative. Then R is commutative.

**Proof:** Since $R$ is reduced, all idempotent are central, and hence by Theorem 2.3.4, ${}^{R}\!/_{J}$ is commutative**.** Therefore, since $J$ is commutative,

[[x, y], [z, t]] = 0 for all $x, y \in R$. (3)

Note that (3) is a polynomial identity which is satisfied by all elements of $R$. However, (3) is not satisfied by any $2 \times 2$ complete matrix ring over $GF(p)$ for any prime $p$, as can be seen by taking $[x, y] = [E_{11},\ E_{12}]$, $[z, w] = [E_{22}, E_{21}]$. Hence, the commutator ideal of $R$ is nil, and thus $R$ is commutative (since $N = \{0\}$). ■

**Remark:** The Jacobson radical of a Boolean ring is {0}.

**Corollary 2.3.6:** suppose $R$ is a ring with identity, and suppose $R$ is reduced and subBoolean. Then $R$ is commutative.

**Proof:** Let $j, j' \in J$ and suppose $[\,j, j'] \neq 0$, then, by Definition 2.3.1,

$(1 + j)^2(1 + j') - (1 + j)(1 + j')^2 \in N = \{0\}$, and hence

$(1+j)\{(1+j)-(1+j^{'})\}(1+j^{'})=0$, which implies that (since $1+j$ and $1+j^{'}$are units in $R$), $j=j'$ contradiction. This contradiction proves that $J$ is commutative, and the corollary follows from Theorem 2.3.5**.**

Hence $R$ is commutative. ■

**Theorem 2.3.7**: Suppose $R$ is a subBoolean ring with central idempotents, and suppose $J \subseteq C$. Then R is commutative.

**Proof:** By Lemma 2.3.3, $N \subseteq J$ and hence $N \subseteq J \subseteq C$, which, when combined with definition 2.3.1, yields $x^2y - xy^2 \in N$ for all $x, y \in {}^{R}/_{C}$.

Suppose $x \notin C$. Setting $y = -x$ in above equation, we get $2x^3 \in N$, and hence $2x \in N \subseteq C$ (See above).Thus, $2x \in C$ for all $x \in R$.

Next, we prove $x^2 \in C$ for all $x \in R$.

To see this, recall that by Theorem 2.3.4, $[x,y] \in J \subseteq C$ and hence $[x,y]$ is central for all $x, y \in R$. Using this fact and from above equation, we get

$$\begin{aligned}[x^2,y] &= x[x,y]+[x,y]\\ &= 2x[x,y]\\ &= x[2x,y]\\ &= 0.\ \text{Hence } x^2 \in C \text{ for all } x \in R.\end{aligned}$$

Now we prove this Theorem 2.3.7 by contradiction. Suppose $x \in C$ for some $x \in R$.Then, $x + x^2 \notin C$ and by above equation, we get $x(x+x^2) - x(x+x^2)^2 \in N$, and thus $x^3(x+x^2) \in N$.

Therefore, for some polynomial $g(\lambda) \in \mathbb{Z}[\lambda]$ ,we have that:

$$\begin{aligned}(x+x^2)^4 &= (x+x^2)^3(x+x^2)\\ &= (x^3g(x)(x+x^2)\\ &= x^3(x+x^2)g(x).\end{aligned}$$

This equation is a sum of pairwise commuting nilpotent elements $x + x^2 \in N \subseteq C$. (see above). Therefore, using above, $x^2 \in C$ for all $x \in R$ , we conclude that $x \in C$, for all $x \in R$, contradiction.

Hence $R$ is commutative. ■

**Theorem 2.3.8:** Suppose $R$ is a subBoolean ring with identity and with central idempotents. Suppose, further, that $J$ is commutative. Then R is commutative.

**Proof:** By Lemma $2.3.3$ $N \subseteq J$. We claim that $J \subseteq N \cup C$. (4)

Suppose not. Let $j \in J, j \notin N, j \notin C$. Since $N \subseteq J$, Definition 2.3.1 implies $x^2y - xy^2 \in N$ for all $x, y \in R \setminus (J \cup C)$. (5)

Note that $1 + j \notin J \cup C$, and $J^2 \subseteq C$ (since J is commutative). Therefore, $1 + j + j^2 \notin J \cup C$, and hence by equation (5), $(1 + j + j^2)^2(1 + j) - (1 + j + j^2)(1 + j)^2 \in N$ this implies $j^2(1 + j + j^2)(1 + j) \in N$ Since $(1 + j + j^2)^{-1}$ and $(1 + j)^{-1}$ are units in $R$, and since they both commute with $j$, it follows that $j^2 \in N$, and hence $j^2 \in N$, contradiction. This contradiction proves equation (4). In view of equation (4) and Definition 2.3.1, we have $x^2y - xy^2 \in N$ for all $x, y$ in $R \setminus (N \cup C)$. (6)

Now, suppose $x \notin N,\ x + 1 \notin N,\ x \notin C$ (and hence $x + 1 \notin C$). Then, by equation (6), we see that $x^2(x + 1) - x(x + 1)^2 \in N$, and thus $x(x + 1) \in N$. Since $x \in N$ or $x + 1 \in N$ implies that $x(x + 1) \in N$, we conclude that $x + x^2 = x(x + 1) \in N$ for all $x \in R\setminus C$ (7)

Since $x \in C$ implies $-x \in C$, we may repeat the above argument with $x$ replaced by $(-x)$ to get (see (7)) $x - x^2 \in N$ for all $x \in R\setminus C$ (7')

By Theorem 2.1.10, $R$ isomorphic to a subdirect sum of subdirectly irreducible rings $R_i (i \in I)$.

Let $\delta: R \rightarrow R_i$ be the natural homomorphism of $R$ onto $R_i$, and let $\delta : x \longrightarrow x_i$. We claim that:

The set $N_i$ of nilpotents of $R_i$ is contained in $\delta(N) \cup C_i$, (8)

Where $C_i$ denotes the center of $R_i$. To prove this, let $d_i \in N_i,\ d_i \notin C_i$, and let $\delta(d) = d_i, d \in R$.Then $d \notin C$, and hence by equation (7), $d - d^2 \in N$. Since $d_i$ is nilpotent, let ${d_i}^k = 0$, and observe that (since $d - d^2 \in N$), $d - d^{k+1} = (d - d^2)(1 + d + d^2 + \cdots + d^{k-1})^2 \in N$, this implies that $\delta(d - d^2) \in \delta(N)$. Thus $d_i - {d_i}^{k+1} \in \delta(N)$, and hence $d_i \in \delta(N)$, this proves equation (8). Our next goal is to prove that:

Every element of $R_i$ is nilpotent or a unit or central. (9)

To prove this, let $x_i \in R_i \backslash C_i$, and suppose $\delta(x) = x_i$, $x \in R$. Then $x \notin C$, and hence by equation $(7')$ , $x - x^2 \in N$, and thus $x^m = x^{m+1}g(x)$ for some $g(\lambda) \in Z[\lambda]$ and $m > 1$. the last equation implies that $x^m = x^m[xg(x)]^m$ and $[xg(x)]^m = e$ is idempotent. Therefore $x^m = x^m e$; $e = [xg(x)]^m$; $e^2 = e$. This reflects in $R_i$ as follows:

${x_i}^m = {x_i}^m e_i$; $e_i = [x_i g(x_i)]^m$; ${e_i}^2 = e_i$. (10)

Since, by hypothesis, the idempotents of $R$ are central, it follows that $e_i = \delta(e)$ is a central idempotent in the subdirectly irreducible ring $R_i$, and hence $e_i = 1$ or $e_i = 0$.

If $e_i = 0$, then by equation (10), $x_i$ is nilpotent. On the other hand, if $e_i = 1$, and then again by equation (10), $x_i$ is a unit in $R$, which proves (9). Next, we prove that:

Every unit $u_i$ in $R_i$ is central or $u_i = 1 + a_i$ for some nilpotent element $a_i \in N$. (11)

To prove this, suppose $u_i$ is a unit in $R_i$, which is not central, and suppose $\delta(d) = u_i, d \in R$. Then $d \in R \backslash C$, and hence by $(7')$, $d - d^2 \in N$, which implies that $u_i - {u^2}_i \in \delta(N)$. Therefore, ${u_i}^2 - u_i$ is nilpotent; say, $({u_i}^2 - u_i)^n = 0$ . Hence, $(u_i - 1)^n = 0$, And thus $u_i - 1 = a_i, a_i$ nilpotent; that is, $u_i = 1 + a_i, a_i \in N_i$ and hence (11) holds. Returning to (8), note that since $N \subseteq J$ (lemma 2.3.3) and $J$ is commutative (by hypothesis), $N$ itself is a commutative set, and hence by (8), the set $N_i$ of nilpotents of $R_i$ is commutative also. Moreover, by (9) and (11), the ring $R_i$ is generated by its nilpotent and central elements, and hence $R_i$ is commutative, which implies that the ground ring $R$ itself is commutative. ■

**Theorem 2.3.9:** A subBoolean ring with identity and with central nilpotents is necessarily commutative.

**Proof** First, we prove that the set $U$ of units of R is commutative.

Suppose not. Let $u, v$ be units in $R$ such that $[u, v] \neq 0$.Then, by Definition 2.3.1, $u^2v - uv^2 \in N$. Also, since $N \subseteq C, N$ is an ideal of $R$, and hence $u^{-1}(u^2v - uv^2)v^{-1} \in N$, which implies that $u - v \in N \subseteq C$. Thus,$[u, v] = 0$ contradiction. This contradiction proves the set $U$ of units of R is commutative. Let $j, j' \in J$. Then The set $U$ of units of R is commutative,$[1 + j, 1 + j'] = 0$, and hence $[j, j'] = 0$; that is, $J$ is commutative. Furthermore, since all nilpotents are central, the idempotents of $R$ are all Central.

Therefore, by Theorem 2.3.8, $R$ is commutative. ■

**Theorem 2.3.10:** Suppose $R$ is a subBoolean ring. Suppose, further, that the idempotents of $R$ are central and $J$ is commutative. If, in addition, $R$ is subweakly periodic, then $R$ is Commutative (and conversely).

**Proof:** To begin with, if zero is the only potent element of $R$, then (by definition of a subweakly periodic ring), $R = N \cup J \cup C = J \cup C$ (since $N \subseteq J$, by Lemma 2.3.3), and hence $R$ is commutative, since $J$ is commutative. Thus, we may assume that $R$ has a nonzero potent element. Let $a$ be any nonzero potent element of $R$, and let $a^K = a$ with $k > 1$. Let $e = a^{K-1}$. Then e is a nonzero idempotent which, by hypothesis, is central. Hence, $eR$ is a ring with identity. Moreover, $eR$ is a subBoolean ring (keep in mind that the Jacobson radical of $eR$ is $eJ$, where $J$ is the Jacobson radical of $R$). Also, the idempotents of $eR$ are central, and the Jacobson radical of $eR$ (namely, $eJ$) is commutative. Hence, by Theorem 2.3.7 $eR$ is commutative. Let $y \in R$. Then $e[a, y] = [ea, ey] = 0$. Recalling that $e = a^{K-1} \in C$ and $a^K = a$, it follows that:

$$
\begin{aligned}
0 &= e[a,y] \\
&= a^{K-1}[a,y] \\
&= a^K y - a^{K-1} y a \\
&= a^K y - y a^K \\
&= ay - ya, \text{ for all } y \text{ in } R, \text{ and hence}
\end{aligned}
$$

All potent elements of $R$ are central. (12)

To complete the proof, let $x, y \in R \setminus (J \cup C)$. Then

$x = a + b,\ y = a' + b';\ a,\ a' \in N;\ b,\ b'$ Potent. (13)

By Lemma 2.3.3, $N \subseteq J$ and hence by (13),

$x = a + b,\ y = a' + b';\ a,\ a' \in J;\ b,\ b'$ potent. (13')

Therefore, by (12) and the hypothesis that $J$ is commutative,

$$
\begin{aligned}
[x,y] &= [a+b, a'+b'] \\
&= [a,a'] \\
&= 0 \text{ (see } (13'))\text{.}
\end{aligned}
$$

By a similar argument, $[x,y] = 0$ also if $x \in J \cup C$ or $y \in J \cup C$. This completes the proof. ■

A concept related to commutativity is the notion that the commutator ideal is nil. In this connection, we have the following theorem.

**Theorem 2.3.11:** Suppose $R$ is a subBoolean ring with identity and with central idempotents. Then the commutator ideal of $R$ is nil.

**Proof:** First we prove that $J \subseteq N \cup C$

Suppose not. Let $j \in J, j \notin N, j \notin C$. Then $1 + j \notin J, 1 + j \notin N, 1 + j \notin C$. We now distinguish two cases.

**Case 1:** If $j^2 \notin C$. In this case, $1 + j^2 \notin J,\ 1 + j^2 \notin N,\ 1 + j^2 \notin C$. Hence, by Definition 2.3.1, $(1 + j)^2(1 + j^2) - (1 + j)(1 + j^2)^2 \in N$

And thus $j(1 - j^4) \in N$.Since $(1 - j^4)^{-1}$ is a unit in $R$ which commutes with $j$, it follows that $j \in N$, contradiction.

**Case 2:** $If\ \ j^2 \in C$. In this case, a similar argument shows that, since

$1 + j + j^2 \notin (N \cup J \cup C$ and $1 + j \notin (N \cup J \cup C)$,

$(1 + j)^2(1 + j + j^2) - (1 + j)(1\ + j + j^2)^2 \in\ N$, This implies $j^2(1 +\ j)(1 + j + j^2) \in N$. Since $[(1 + j)(1 + j + j^2)]^{-1}$ is a unit in $R$ which commutes with $j^2$, it follows that $j^2 \in N$, and hence $j^2 \in N$, contradiction. This contradiction (in both cases) $J \subseteq\ N \cup C$.

Next, we prove that $N$ is an ideal of $R$.

By Lemma 2.3.3, $N \subseteq\ J,$ which when combined with $J \subseteq\ N \cup C$ yields $N \subseteq J \subseteq N \cup C$.

Now, suppose $a, b \in N$. Then, $a, b \in J$ and hence $a - b \in J \subseteq (N \cup C)$ which implies $a - b \in N$ or $a - b\ \in C$, and thus $a - b \in N$ (in either case). Next, suppose $a \in N,\ x \in R$. Then, $a \in\ J$ and hence $ax \in J \subseteq (N \cup C)$, which implies $ax \in N$ or $ax \in C$. If $ax \in C$, then $(ax)^k = a^k x^k$ for all $k \geq 1$, and hence $ax \in N$ (since $a \in N$). So in either case, $ax \in N$. Similarly $ax \in N$, hence $N \subseteq J \subseteq N \cup C$ holds.

Returning to $J \subseteq\ N \cup C$, we see that $N \cup J \cup C = N \cup C$, which when combined with Definition 2.3.1 shows that $x^2y\ - xy^2\ \in N$ for all $x, y \in R \setminus (N \cup C)$. $x^2y - xy^{\ 2} = 0$ for all noncentral elements $x, y \in R/N$. Suppose $x \in R/N$ is noncentral. Then $x + 1 \in R/N$ is non central also and hence by $x^2y - xy^{\ 2} = 0$ for all noncentral elements $x, y \in R/N,\ x^2(x + 1)\ -\ x(x + 1)^2\ = 0$. Therefore, $x(1 + x) = 0$ this implies that $x(1 + x)(1 - x) = 0$, that is, $x^3 = x$ (if $x$ is noncentral). Hence every element of $R/N$ is central or potent (satisfying $x^3 = x$).

It follows, by Lemma 2.2.3 that R/N is commutative, and hence the commutator ideal of $R$ is nil. ■

We conclude with the following:

**Remark:** If in the definition of a subBoolean ring (see Definition 2.3.1**),** we replace the exponent 2 by $n$, where $n$ is a fixed positive integer other than 2, then neither Theorem2.3.8 nor Theorem 2.3.10 is necessarily true.

To see this, let $\boldsymbol{R} = \left\{\begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ \boldsymbol{0} & \boldsymbol{a^2} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{a} \end{bmatrix} : \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \boldsymbol{GF}(\boldsymbol{4})\right\}$

It can be verified that $R$ satisfies the condition;

$x^7y - xy^7 \in N$ for all $x, y$ in $R$.

Furthermore, $R$ satisfies all the hypotheses of both Theorem 2.3.8 and Theorem 2.3.10 (except, of course, the exponent 2 is now replaced by 7). But $R$ is not commutative.

# CHAPTER THREE

# Generalized Boolean ring

In this chapter we see the concept of generalized Boolean ring and we prove some Theorems. Further we see some examples. Now we begin with the following.

**Definition 3.1** A ring $R$ is called generalized Boolean if for every $x \in R$ such that $x \notin (N \cup C)$, there exists an even positive integer $n$ such that $x - x^n \in (N \cap C)$.

A generalized Boolean ring is also a generalized periodic ring (see definitions 1.2.4 and 3.1**).** Therefore we have the following corollary and remarks which is proved by the authors in [6].

**Corollary 3.2:** If R is a generalized Boolean ring, then $R$ is either commutative or Periodic.

**Remark 1:** Suppose that R is a generalized periodic ring with identity 1. Then, R is commutative.

Note that remark 1 follows at once by taking $c = 1$ in theorem 1.2.7.

**Remark 2:** A generalized Boolean ring with identity 1 is necessarily commutative.

**Theorem 3.3:** A generalized Boolean ring $R$ with central idempotents is necessarily nil $(R = N)$ or commutative $(R = C)$.

**Proof:** Since R is also a generalized periodic ring, therefore by Theorem 1.2.5 $R$ is commutative or periodic. If $R$ is commutative, there is nothing to prove. So we may assume that $R$ is periodic. We now distinguish two cases.

**Case 1:** $(C \subseteq N)$. Recall that, by hypothesis, the set E of idempotents is central, and hence $E \subseteq C \subseteq N$ (in the present case). Thus, $E \subseteq N,$ and hence $E = \{0\}$. Therefore, zero is the only idempotent of $R$. (1)

Let $x \in R$. Since $R$ is periodic, therefore $x^k$ is idempotent for some positive integer $k$, and hence by (1), $x^k = 0$, hence $R$ is nil.

**Case 2:** $(C \subseteq N)$. Then, for some $c \in R$, we have:

$c \in C$, where $c \notin N$. (2)

Again, since $R$ is periodic, $c^m$ is idempotent for some positive integer $m$. Moreover, $c^m \neq 0$ (since $c \notin N$ ). The net result is (see (2))

$e = c^m$ is a nonzero central idempotent of $R$ (3)

Now, suppose $a \in N$. Since $0 \neq e \in C$ and $a \in N$, therefore $e + a \notin N$. Suppose $a \notin C$. Then $e + a \notin C$ (since $e \in C$), and hence $e + a \notin (N \cup C)$. Therefore, by Definition 3.1, $(e + a) - (e + a)^n \in (N \cap C)$, for some even integer $n \geq 2$. (4)

Since $R$ is also a generalized periodic ring, therefore by lemma 1.2.6 (see (3)) $ea^i \in C, \forall i \in \{1, \dots, n-1\}$, where $(0 \neq e = e^2, e \in C, a \in N)$(5) combining (4) and (5), we see that:

$a - a^n \in C, \forall a \in N \backslash C$. (6)

Since (6) is trivially satisfied for $a \in (N \cap C)$, therefore

$a - a^n \in C, \forall a \in N$, where $n \geq 2$. (7)

We claim that $N \subseteq C$. (8)

The proof is by contradiction. Suppose (8) is false. Then, for some $a \in R$, we have $a \in N$, where $a \notin C$. (9)

Since $a \in N$, there exists a positive integer $\delta_0$ such that:

$a^\delta \in C, \forall \delta \geq \delta_0$, where $\delta_0$ minimal. (10)

Moreover, since $a \notin C$ (see (9)), therefore $\delta_0 > 1$. Now, applying (7) to the nilpotent element $(a^{\delta_0 - 1})^n$, we see that:

$a^{\delta_0 - 1} - (a^{\delta_0 - 1})^n \in C$, for some $n = n\left(a^{\delta_0 - 1}\right) \geq 2$. (11)

Furthermore, since $(\delta_0 - 1)n \geq (\delta_0 - 1)2 \geq \delta_0$ (since $\delta_0 \geq 2$), (10) implies that $(a^{\delta_0 - 1})^n = a^{(\delta_0 - 1)n} \in C$. (12)

Combining (11) and (12), we conclude that $a^{\delta_0-1} \in C$, which contradicts the minimality of $\delta_0$ in (10). This contradiction proves (8). Since $R$ is a periodic ring satisfying (8), therefore, by theorem of Herstein [10], $R$ is commutative. ■

**Corollary 3.4:** A generalized Boolean ring with central idempotents and commuting nilpotents is commutative.

**Corollary 3.5:** If $R$ is a generalized Boolean ring, and if $R$ is 2-torsion-free, then $R$is nil or commutative.

**Proof:** We claim that all idempotents of $R$ are central. Suppose not, and suppose $e$ is a non central idempotent in $R$. Then $-e \notin (N \cup C)$, and hence (see Definition 3.1) $(-e)-(-e)^n \in C$, where $n$ even. Thus, $2e \in C$, and hence $[2e, x] = 0$ for all $x$ in $R$. Since $R$ is 2-torsion-free, $2[e, x] = 0$ implies $[e, x] = 0$, and thus $e \in C$, a contradiction. This contradiction proves that all idempotents of $R$ are central, and hence $R$ is nil or commutative, by Theorem 3.3. ■

**Theorem 3.6:** Let $R$ be a generalized Boolean ring in which every finite subring is either commutative or nil. Then $R$ is either commutative or nil.

**Proof:** By contradiction. Thus, suppose $R$ is a generalized Boolean ring such that every finite sub ring of $R$ is either commutative or nil. Suppose, further, that $R$ is not commutative and not nil either. By Theorem 3.3, there must exist a non central idempotent element $e \in R$, and hence $e \notin (C \cup N)$. Thus (see Definition 3.1**)**, since $-e \notin (C \cup N)$, $(-e)-(-e)^n \in (N \cap C)$, where $n$ even.This implies that $2e \in (N \cap C)$, and hence $(2e)^k = 2^k e = 0$, for some $k \in \mathbb{Z}$. Since $e \in C$, we must have the following:

Either $ex - exe \neq 0$ for some $x \in R$, or $x'e - ex'e \neq 0$ for some $x' \in R$. Suppose $u = ex - exe = 0$. Then, $eu = u = 0 \neq ue = u^2$, where ($u = ex - exe \neq 0$). Moreover, $2u = [2e, ex]$
$= 0$ (since $2e \in C$).

Furthermore, the sub ring generated by $e$ and $u$ is $\langle e, u \rangle = \{re + su \mid r, s \in Z\}$.

Since $2^k e = 0$ and $2u = 0$, the subring $\langle e, u \rangle$ is finite. Indeed, $\langle e, u \rangle = \{re + su \mid 1 \leq r \leq 2^k$, where $1 \leq s \leq 2\}$. On the other hand, if $x'e - ex'e = 0$ for some $x' \in R$ (the only other possibility), then the sub ring, $\langle e, v \rangle$ generated by $e$ and $v = x'e - ex'e$ is (as is readily verified) hypothesis. This contradiction proves the theorem. ■

**Remark:** A careful examination of the proof of Theorem 3.6 shows that we only need to assume that "every subbing $S$, with $|S| = 2^m$ for some positive integer $m$, is Commutative or nil" in order for the ground generalized Boolean ring $R$ to be commutative or nil. Indeed, $|\langle e, u \rangle| = 2^k . 2 = 2^{k+1}$, since the representation of any $x$ in this subbing in the form $x = re + su$; $r, s \in \mathbb{Z}$, is unique. For, suppose $x = re + su$
And $x = r'e + s'u$. Then, $(r - r')e = (s' - s)u$. Recall that $2u = 0$, and $ue = 0$. Thus, if $s' - s$ is even, then (r − $r'$ ) e = 0, and hence $re = r'e, su = s'u$. on the other hand, if $s' -$ s is odd, then $(r - r')e = u$, and hence $(r - r')ee = ue = 0$. Again, we obtain $re = r'e, su = s'u$.

We conclude with the following examples:

**Example 3.7:** Let $R = \left\{ \begin{bmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{bmatrix} : a, b, c \in GF(4) \right\}$

It is readily verified that the idempotents of $R$ are central and

$x - x^7 = 0, \forall x \in R\backslash(N \cup C)$, but $R$ is neither nil commutative. Hence, Theorem 3.3 is not true if we drop the hypothesis that "n is even" in the Definition of a generalized Boolean ring.

**Example 3.8:** Let $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in GF(3) \right\}$

This example shows that we cannot drop the hypothesis that "$N$ is commutative" in Corollary 3.4 (note that $R$ is not commutative.)

**Example 3.9:** Let $R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} : 0,1 \in GF(2) \right\}$

This example shows that we cannot drop the hypothesis that "the idempotents are central" in Corollary 3.4. (Note that $R$ not commutative.) This example also shows that we cannot drop the hypothesis that "$R$ is 2-torsion-free" in Corollary 3.5 Note that, in this ring $R, x - x^2 = 0$ for all $x \in R\backslash(N \cup C)$, even more is true. This ring $R$ also shows that we cannot drop the hypothesis that "$1 \in R$" in Remark1, nor the hypothesis that "$1 \in R$" in Remark 2.

Returning to the ring $R$ in Example 3.7, we see that this ring further shows that we cannot drop the hypothesis that "$m$ and $n$ are of opposite parity" in the Definition of a generalized periodic ring in connection with remark1, or the hypothesis that "$n$ is even" in the Definition of a generalized Boolean ring as far as remark 2 is concerned.

(Recall that $x - x^7 = 0$ for all $x \in R\backslash(N \cup C)$).

**Example 3.10:** Let $S$ be any noncom mutative ring such that $S^3 = (0)$. (For example, we may take $S$ to be the ring of all $3 \times 3$ strictly upper triangular matrices over a field F). Let $R = GF(4) \oplus S$. It is readily verified that $x^3 = x^6$ for all $x \in R$, and hence $R$ is indeed a generalized periodic ring. Moreover, the only idempotents of $R$ is $(0,0)$ and $(1,0)$, and thus the idempotents of $R$ are certainly central. Had $R$ been a generalized Boolean ring, then, by

Corollary 3.3, $R$ would have to be either nil or commutative, which is certainly false here (recall that $S$ is not commutative). This example shows that the set of generalized periodic rings is a wider class than that of generalized Boolean rings, and thus Corollary 3.3 does not hold for generalized periodic rings.

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