THE CONTRIBUTION OF COULOMBIC PRESSURE TO THE STABILITY OF THIN KEPLERIAN ACCRETION DISCS AROUND A NEUTRON STAR WITH AXISYMMETRIC MAGNETIC DIPOLE

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Table of Contents

List of Figures

Abstract

We determine the stability of an axisymmetric geometrically thin and optically thick accretion disc around a magnetized neutron star in the region where gas pressure is dominant. Analysis is made with the inclusion of pressure from coulombic sources. The opacity in the middle region is mainly due to electron scattering whereas that in the outer region is mainly due to free-free emission. Starting from the vertically integrated non-relativistic hydrodynamics equations we set up the basic equations which govern the structure of the disc and for the stability analysis of the disc model we have kept the time dependencies in the equations.

Although we include the effect of coulombic pressure it is thermally stable as if there were gas pressure only and our graphical solution given at the end of this manuscript shows that the disc is viscously unstable.

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Introduction

Over the last three decades, fluid dynamical studies of accretion disks around a magnetized neutron star have been extensively performed. The accretion disk is likely to be formed when the compact star is a member of a close binary system and matter transferred from a giant type star onto its compact companion has high angular momentum. Shakura and sunyaev (1973) initiated this discussion considering a very simplistic but effective standard model of a geometrically thin, optically thick accretion disk. They were able to obtain an analytical solution of height integrated hydrodynamical equations by using Newtonian gravitational potential.

The study of stability of the accretion disc is one of the important criteria in this context. The stability of geometrically thin accretion discs has been studied extensively after the construction of standard α-discs. According to the standard theory of accretion discs (Shakura and Sunyaev 1973), the middle and outer parts of the disc are dominated by the gas pressure. Those regions have been found to be stable to the thermal and viscous modes but pulsationary unstable to the acoustic modes (Blumenthal,Yang, and Lin 1984). Some recent research work about the isothermal accretion disc also obtained similar results (Wallinder 1990; Wu et al.1995b; Wu, Yang, and Yang 1994).

Some early analyses about the stability of gas pressure dominated discs have incorporated azimuthal perturbations (Livio and Shaviv 1977,1981; Van Hon, Wesemael, and Winger 1980.). However, the radial perturbations were neglected in all the studies . McKee (1991) has investigated the contribution of gas pressure to the stability of a standard alpha-disc. He found that the disc is stable when $\beta < 0.6$ (β is the ratio of gas pressure to the total pressure). This implies that a gas pressure dominated disc is more stable.

It has been also found that the disc is thermally and viscously unstable if it is optically thick and radiation pressure dominated (Pringle,Rees and Pocholczyk, 1973; Lightman and Eardely, 1976; Shahura and Sunyaev, 1976). There is also a possible mode of pulsational overstability. In this case, one looks for instabilities in which oscillations on the orbital timescale grow in amplitude because of the effects of viscosity (Lin and Paploizou, 1996). Kato (1978) considered the evolution of small perturbations of all three components of velocity as well as T and S. He found that the disc experience pulsational instability besides the viscous instabilities and thermal instabilities. If a geometrically thin disc is optically thin, it has been found also that it is viscously stable but thermally unstable (Piran, 1978). Those instabilities are believed to be relevant to some light variation observed in many systems such as X-ray binaries. In the standard α model, the viscous heating is balanced by radiative cooling. However, if the radiative cooling is not efficient, the advection will be non negligible. Particularly in an optically thin disc, the radiative cooling rate is so slow that most of the viscous generated energy is advected radially. Recently, the accretion disc models with gas pressure dominated with either electron scattering or free-free opacity have been studied(Abramowicz etal., 1988; Kato, Honma and Matsumaoto, 1988;Narayan and Popham, 1993; Narayan and Yi, 1994, 1995a , 1996b; Abramowicz et al., 1995; Chen et al., 1995; Chen, 1995). The gas dominated disk model with electron scatering opacity has also been adopted successfully to explain the observations of low and high luminosity systems (Nayan, Yi and Mahadeven, 1995; Narayan, Mcclintock and YI, 1996).

The possibility of steady and stable disc formation by incoming matter toward a neutron star is allowed only for a certain sets of initial parameters. Abramowicz Zurek (1981) studied the effective of angular momentum on the accretion and the corresponding stability of the transonic nature of the in falling matter on to the star. The gas elements in the disc lose angular momentum, due to the interaction or friction between adjacent layers and spiral inwards. Part of the released gravitational energy increases the kinetic energy of the rotation and the other part is converted in to thermal energy which is radiated from the disk surface. Thus, viscosity converts gravitational potential energy in efficient manner in to radiation Accretion disk around a compact star has been thought to play an important role in various X-ray sources. The interaction between a magnetized star and a surrounding accretion disc is one of the most poorly understood aspects of accretion.The magnetic field of the star penetrates the surrounding accretion disc and couples the two. According to the Ghosh and Lamb (1979) model, the part of the accretion disc that is located inside the co rotation radius provides a spin up torque on the star, since it is rotating faster than the star, while the more slowly rotating outer part of the accretion disc brakes the star. The net torque is determined by the location of the inner edge of the disc, which moves inwards as the accretion rate increases, thereby increasing the spin up-torque on the star.

In this respect Shapiro et al. (1976) gave a detailed two-temperature disk model which might be promising if radiation pressure dominated inner region of the disk was secularly unstable. On the other hand, Bisnovatyi-Kogan and Blinnikov (1977) proposed a corona disk model where the accretion process would not be steady, by investigation of particle motions taking account of the radiation from the disk and gravitational field of a black hole.

In all the studies of stability mentioned above, we see that although a gas pressure dominated region with either an electron scattering dominated or free-free absorbation opacity is studied, consideration of the effect of coulombic pressure is not made. The aim of this work is then to expound on the effect of coulombic pressure on the stability of thin Keplerian accretion discs.

Chapter 1

COULOMBIC PRESSURE

Since we are concerned with real gases which have direct applications in astrophysics as compared to ideal gases we have to include the effect of interactions between the various charged components of the system. We now include coulombic potential on top of gravitational potential,V_0.

Recall that from the kinetic theory of gases, gas pressure is given by

$$P = \frac{2}{3}\frac{K}{v} \tag{1.0.1}$$

where K is average kinetic energy of the gas. From Virtial theorem we have $K = \frac{-V_0}{2}$ thus,

$$P = \frac{-1}{3}\frac{V_0}{v} \tag{1.0.2}$$

where $V_0 < 0$.

In the presence of coulombic potential U_C, we expect the total pressure to carry additional term P_{coul}. Since $U_c > 0$, We expect

$$P_{coul} = \frac{1}{3}\frac{U_c}{v} \tag{1.0.3}$$

Hence, the total pressure in the absence of degeneracy becomes

$$P = \frac{2}{3}\frac{K}{v} + \frac{1}{3}\frac{U_c}{v} \tag{1.0.4}$$

So to find P_{coul} one has to calculate first U_c. Since our system is a plasma that is, polarizable, the potential of every ion is shielded and its long range effect is cut short. When ever we want to consider the behavior of a gas on a length scales comparable to the mean free path between collisions, we must use the idea of plasma physics that will be important to our study of accretion. A plasma differs from an atomic gas or molecular gas in that it consists a mixture of two gases of electrically charged particles, an electron gas and an ion gas, with very different particle masses m_e and m_i

The electrons and ions interact with each other through their electrostatic coulombic attractions and repulsions. These coulomb forces decrease only slowly ($\propto r^{-2}$) with distance and do not have a characteristic length scale. Thus, a plasma particle interacts with many others at any one instant, and this makes the description of the collisions more complicated than in atomic or molecular gases, where the inter particle forces are very short range.

A further complication arises from the great differences in particle masses m_e and m_i. Since collisions between particles of very different masses can transfer only a small fraction of the kinetic energy of order $m_e/m_i << 1$, it is possible for electrons and ions to have significantly different temperatures over appreciable time scales. These two properties- the long range nature of the coulomb force and the disparity in electron and ion masses give the physics of plasmas its particular character. A further series of complex phenomena occurs when the plasma is permeated by a large scale magnetic field, this is particularly relevant for the study of gas accreting on highly magnetized neutron stars.

Formally the shielded potential is given by

$$V \propto \frac{e^{-(const)r}}{r} \tag{1.0.5}$$

where r is the distance between the two interacting particles.
It is calculated from the requirement that,

,

$$\rho_q = \rho_0 exp(\frac{-zeV}{k_\beta T})$$

where $\rho_q = e\sum_z(zn_z)$ is the charge density and $n_z = n_0 exp(\frac{-zeV}{k_\beta T})$ is the number of ions per unit volume, for our system to be nearly perfect we expect, $n_z << 1$. Since ρ_q satisfies Poisson's equation,

$$\nabla^2 V = -4\pi\rho_q = -4\pi e\sum_z(zn_z)$$

Or

$$\nabla^2 V = -4\pi e\sum_z(zn_0)exp(\frac{-zeV}{k_\beta T}).$$

For slightly real system $zeV << k_\beta T$, thus

$$\nabla^2 V \approx -4\pi e(1 - \frac{-zeV}{k_\beta T})\sum_z(zn_0)$$

using the fact that the number densities of ions and electrons at any point must be approximately equal, and therefore a plasma must always be close to charge neutrality: even a small charge imbalance would result in very large electric fields which would act to move the plasma particles so as to restore neutrality very quickly.
Hence $e\sum_z(zn_0) = 0$, We have

$$\nabla^2 V \approx \frac{4\pi e^2}{k_\beta T}\sum_z(z^2 n_0 V)$$

Or

$$\nabla^2 V \approx K_D^2 V,$$

where

$$K_D^2 = \frac{4\pi e^2}{k_\beta T}\sum_z(z^2 n_0)$$

From the fact that $\frac{(ze)^2}{K_\beta T}$ has the dimension of length and n_0 has the dimension of inverse length cube we notice that K_D is an inverse length.The solution to Poisson's equation is easily determined by inspection based on the facts

i) $\rho_q = \rho_0 exp(\frac{-zeV}{k_\beta T})$

ii) $V \propto \frac{1}{r}$.

We therefore expect the solution to be of the form

$$V = const.\frac{e^{-K_D r}}{r}$$

This can be checked by direct substitution in to $\nabla^2 V \approx K_D^2 V$, and to find the constant we use the boundary condition that $\lim_{r\to\infty} V = \frac{ez}{r}$, implying $const = \frac{ez}{r}$,

Hence

$$V = ez\frac{e^{-K_D r}}{r}.$$

The fact that we have weak interaction is telling us, $K_D r << 1$.

In this case $e^{-K_D r}$ may be written as

$$e^{-K_D r} \approx 1 - K_D$$

And it gives, $V(r) = \frac{ez}{r} + \Phi_z$, where $\Phi_z = -ezK_D$, is the potential due to other charges at the ion.

The coulomb energy per unit volume is commonly given as

$$\frac{U_c}{v} = \frac{1}{2}\sum_z (zen_0\Phi_z), where$$

$$\Phi_z = -ezK_D, K_D = (\frac{4\pi f e^2 \rho}{k_\beta m_p T})^{\frac{1}{2}}$$

Here we use the parameter $f = \sum(z^2 + z)\frac{x_z}{A_z}$, where x_z is the mass abundance of the gas and A_z is atomic weight. Since we are considering the disk system (whirling gas) as a hydrogen gas for which the atomic number, $z = 1$ and its abundance $x_z = 1$ and its atomic weight $A_z = 1.0079$, thus for hydrogen $f = 2.0079$ and the mean molecular weight is $\mu = 0.6$, $\frac{U_c}{v} = -e^3(\frac{\pi}{k_\beta T})^{\frac{1}{2}}(f\rho N_o)^{\frac{3}{2}}$. Thus the coulomb pressure, $P_{coul} = \frac{1}{3}\frac{U_c}{v}$, followed from this will be

$$P_{coul} = -\frac{1}{3}e^3(\frac{\pi}{k_\beta T})^{\frac{1}{2}}(f\rho N_o)^{\frac{3}{2}}.$$

And hence the total pressure, $P = P_g + P_{coul}$ would be

$$P = \frac{N_o k_\beta T \rho}{\mu} - \frac{1}{3}e^3(\frac{\pi}{k_\beta T})^{\frac{1}{2}}(f\rho N_o)^{\frac{3}{2}}. \quad (1.0.6)$$

Chapter 2

STRUCTURE EQUATION

In this section we set up the basic equations which govern the structure of the thin accretion disc around a neutron star with a magnetic dipole field. These equations are basically derivable from the equations of non relativistic magnetohydrodynamics. The disc structure we consider was first investigated by Shakure and Sunyaev but is modified. It is modified since we include the effect of coulombic pressure on the top of gas and radiation pressure. To specify a disc model we need give a viscosity prescription and a relation for the opacity equation. In addition, since we are interested in the portion of the disc where gas pressure is dominant.We shall drop the radiation pressure term $\frac{4\sigma}{3c}T_c^4$ from the equation of state.

2.1 Assumptions

We consider an axisymmetric disk structure of accreting matter and employ a cylindrical coordinate system (r, φ, z) with the z-axis chosen as the axis of rotation. The basic structure of a disc is the same in all systems. A disc rotates around a central object, usually the central object provides all of the gravitational force on the disc, but in some cases there are additional sources of gravity, such as the self-gravity of the disc it self, the orbit of the material is close to circular. The rotation of the disc is generally differential, so that the rotation velocity and the rotation period change with distance from the center. If all of the gravitational force is provided by the central object, the disc is Keplerian disc,

and the rotation period depends on radius according to the Keplerian laws of orbital motion: $T \propto r^3$, where T is the orbital period and r is the distance from the center of the disc. Therefore, throughout the analysis of this paper the following assumptions will made:

1. Relativistic effects and self-gravity effects, and hence gravitational instabilities are therefore neglected. This assumption will be valid for disks around magnetized neutron stars.

2.The disc is assumed to consist of fully ionized hydrogen gas.

3.The viscosity may be adequately represented by an α- disc model such that

$-f_{r\varphi} = \alpha P$, where $f_{r\varphi}$ is the only appreciable component of the stress tensor and P is the total pressure. the viscosity parameter, α, will be assumed constant within the interval $10^{-3} < \alpha \leq 1.0$

4.The disc is thin (disc half thickness= $H << r$ =disc radius) and all equations will be averaged over the vertical structure.

5.Vertical hydrostatic equilibrium is maintained as the instabilities develop. This is certainly true in the case of the viscous instability. However, the vertical time scale t_z, becomes comparable to the thermal time scale, t_{th}, for alpha near unity $t_{th} \approx \alpha^{-1} t_z$. As pointed out by Pringle(1976), the growth rate for the thermal instability may be an underestimate due to this assumption.

2.2 Mass Conservation

The continuity equation that describes the mass conservation for a fluid flow of density ρ flowing at velocity $v = (v_r, v_\varphi, v_z)$ in unit time is given by

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho v) = 0 \tag{2.2.1}$$

which is equivalent with $\frac{\partial\rho}{\partial t} + \rho \times \nabla.v + v.\nabla\rho = 0$

In steady state and v non zero the continuity equation takes the form

$$\nabla.(\rho v) = 0 \tag{2.2.2}$$

For thin cylindrically symmetric disc Eq.(2.2.1) can be expressed as

$$\partial_t\rho = \frac{1}{r}\partial_r(r\rho v_r) + \partial_\varphi(\rho v_\varphi) + \partial_z(\rho v_z)$$

But from the cylindrical symmetry the fluid variables are independent of φ (dropping the φ derivatives), thus we get

$$\partial_t\rho + \frac{1}{r}\partial_r(r\rho v_r) + \partial_z(\rho v_z) = 0$$

So for a thin axisymmetric disc after neglecting the vertical out flow from the disc, the gas will have a radial component velocity only:

$$\partial_t\rho + \frac{1}{r}\partial_r(r\rho v_r) = 0 \tag{2.2.3}$$

Let us integrate this equation over the $z-direction$

$$\partial_t \int \rho dz + \frac{1}{r}\partial_r \int r\rho v_r dz = 0 \tag{2.2.4}$$

Introducing a useful variable called surface density

$$S(r,t) = \int_{-H}^{H} \rho dz$$

where ρ is the vertically average density and H is half thickness. For thin discs (height much smaller than the radial variable), we can neglect the vertical variation of v_r in the second integral of Eq.(2.2.4), and thus obtain

$$\partial_t S + \frac{1}{r}\partial_r(rSv_r) = 0$$

Introducing also a quantity called accretion rate,

$$\dot{M}(r) = -2\pi r \int_{-H}^{H} \rho v_r dz = -2\pi r v_r S \tag{2.2.5}$$

Thus we also have

$$\partial_r \dot{M} = 2\pi r \partial_t S$$

For steady state,

$$\partial_r \dot{M} = 0$$

which means $\dot{M}$ = constant (independent of radius).

2.3 The Viscosity

In real fluid the transport of momentum occurs in part by the transport fluid volumes having different velocities. But additional transfer is caused by the internal friction between particles moving with adjacent layers of the fluid having different velocities. Thus angular momentum has been transported outward as a result of collisions, the effect of collisions in this case is to introduce a positive co-relation between azimuthal velocity fluctuations and radial velocity fluctuations, which in the language of turbulent fluids means that the Reynolds stress $\langle \rho v_r v_\varphi \rangle$ transports angular momentum outwards. Because the chaotic motion takes place in an equilibrium flow, the exchange of fluid elements cannot result in the net transfer of any matter between the two rings. Therefore, mass crosses the surface r =constant at equal rates in both directions, of the order $H\rho\dot{v}$ per unit arc length, where $\dot{v}$ is the typical velocity of the eddies. Since the two mass fluxes carry different angular momenta, there is a transport of angular momentum due to the chaotic process, i.e, a viscous torque exerted on the outer stream by the inner stream and an equal and opposite torque exerted by the outer stream on the inner.

The torque exerted on the outer ring by the inner ring is given by the net outward angular momentum flux. Since the mass flux due to chaotic motions is the same in both

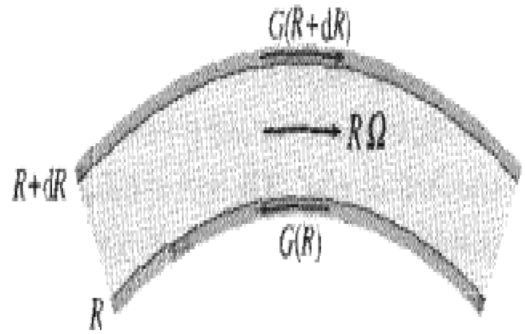

Figure 2.1: Differential viscous torque

directions, one obtains to first order in λ the torque per unit arc length as

$$-\rho \dot{v} H \lambda r^2 \Omega^{'} \tag{2.3.1}$$

where $\Omega^{'} = \frac{d\Omega}{dr}$ and λ is the spatial scale or characteristic wave length of the turbulence and we assume that the angular velocity changes slowly over the length scale of chaotic motions. The non-vanishing component of the stress in this case is the force in the φ direction per unit area, and is given by

$$f_{r\varphi} = -\eta r \Omega^{'} \sim -\rho \dot{v} \lambda r \Omega^{'} \tag{2.3.2}$$

yielding a positive kinematic viscosity $\nu \sim \lambda \dot{v}$. But generally in an accretion disc, the total torque is obtained simply by multiplying by the length $2\pi r$ of the circular boundary. Setting $\rho H = S$(the surface density), we can write the torque exerted by the outer ring on the inner (= -the torque of the inner on the outer) as

$$G(r) = 2\pi r \nu S r^2 \Omega^{'} \tag{2.3.3}$$

The sign of this equation tells us for a rotation law in which $\Omega(r)$ decreases outwards, hence $G(r)$ is negative, in this case the inner rings lose angular momentum to the outer ones and the gas slowly spirals in.

Now let us consider the net torque on a ring of gas between r and $r + dr$. As this has both an inner and an outer edge, it is subject to competing torques (Fig. 2.1); the net torque (trying to speed up) is

$$G(r + dr) - G(r) = \frac{\partial G}{\partial r} dr \tag{2.3.4}$$

Because this torque is acting in the sense of angular velocity $\Omega(r)$, there is a rate of working

$$\Omega \frac{\partial G}{\partial r} dr = \left[\frac{\partial}{\partial r}(G\Omega) - G\Omega^{'} \right] dr \tag{2.3.5}$$

by the torque. But the term

$$\frac{\partial}{dr}(G\Omega) \tag{2.3.6}$$

is just the rate of convection of rotational energy through the gas by the torques and the term $-G\Omega^{'} dr$ represents a local rate of mechanical energy to the gas. This lost energy must go in to internal (heat) energy.

The viscous torques therefore cause viscous dissipation with in the gas at a rate $G\Omega^{'} dr$ per ring width dr. Ultimately, this energy will be radiated over the upper and lower faces of the disc. We are therefore interested in the rate per unit plane surface area $Q(r)$.

2.4 Momentum Conservation

In real fluid the transport of momentum is expressed by the advective term in Euler's (momentum) equation. And the derivation of the Navier-Stockes equation begins with an application of Newtons second law, for a fluid of density ρ, flowing with a velocity v Euler's equation is given by

$$\rho\frac{dv}{dt} = -\nabla P - \rho\nabla\Phi \tag{2.4.1}$$

where P is the pressure and Φ is the gravitational potential. If dissipative terms representing the action of viscous forces are included on the right hand side of Eq.(2.4.1) it becomes the Navier-Stockes equation:

$$\rho\frac{dv}{dt} = -\nabla P + \rho\nabla\Phi + \nabla.f \tag{2.4.2}$$

Here f represents the stress tenser. As it is mentioned in the assumption part above, we neglect terms like $J \times B$, $\rho_e E$, which are all caused by the action of electromagnetic forces and eventhough the viscosity term is some what in general law, we include it to show the effect and contribution of the interactions of the particles of the fluid, and since detailed disc models can only be constructed if we know the magnitude of the viscosity. The source of the viscosity might be small scale turbulence and transfer of angular momentum by magnetic stresses, that is, random magnetic fields are amplified by the differential rotation and turbulence of the disc. At the interface between adjacent cells of the resulting chaotic field, the gradients become so strong that magnetic field line reconnection occurs. Clearly, we are not able to handle these complicated phenomena in quantitative manner. A reasonable parametrization has, however, been given by Shakura and Sunyaev.

Using

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v.\nabla,$$

we can write Eq.(2.4.2) as

$$\rho(\frac{\partial v}{\partial t} + v.\nabla v) = -\nabla P + \rho\nabla\Phi + \nabla.f$$

Therefore, the steady- state equation for conservation of momentum, the Navier-Stocke's equation can be written as

$$\rho(v.\nabla v) = -\nabla P + \rho\nabla\Phi + \nabla.f \tag{2.4.3}$$

Using the cylindrical coordinates(r, φ, z), and considering first each term separately then after a set of mathematical manipulations connecting the expansion terms according to their components, we have the following results as shown in different subsections.

2.4.1 The Azimuthal Component

The φ component of the momentum will be

$$\rho(v_r\partial_r v_\varphi + v_\varphi\frac{1}{r}\partial_\varphi v_\varphi + v_z\partial_z v_\varphi + \frac{v_\varphi v_r}{r}) = \frac{1}{r}\partial_\varphi P + \rho\frac{1}{r}v_\varphi\Phi + \frac{1}{r}\partial_r(rf_{\varphi r})\partial_z(f_{\varphi z}) + \frac{1}{r}f_{r\varphi}$$

Let

$$Q = v_r\partial_r + V_z\partial_z$$

and since by symmetry the fluid variables are independent of φ thus,

$$\rho(Qv_\varphi + \frac{v_\varphi v_r}{r}) = \frac{1}{r}\partial_r(rf_{\varphi r}) + \partial_z(f_{\varphi z}) + \frac{1}{r}f_{r\varphi}$$

Multiplying this equation with r then for φ independent functions we obtain

$$\rho Q(rv_\varphi) = \frac{1}{r}\partial_r(r^2 f_{r\varphi}) + \partial_z(rf_{r\varphi}) \tag{2.4.4}$$

Now we multiply the continuity equation

$$\frac{1}{r}\partial_r(r\rho v_r) + \partial_z(\rho v_z) = 0$$

with rv_φ and add the resulting equation to Eq.((2.4.4)) and obtain

$$\frac{1}{r}\partial_r(rv_r\rho rv_\varphi) + \partial_z(v_z v_\varphi r\rho) = \frac{1}{r}\partial_r(r^2 f_{r\varphi}) + \partial_z(rf_{z\varphi})$$

Integrating over z gives

$$\frac{1}{r}\partial_r \int (r^2 v_r \rho r v_\varphi)dz = \frac{1}{r}\partial_r(r^2 W_{r\varphi}) \tag{2.4.5}$$

where

$$W_{r\varphi} = \int f_{r\varphi} dz$$

for thin disc Eq.((2.4.5)) is approximately

$$\frac{1}{r}\partial_r(v_r S r^2 v_\varphi) = \frac{1}{r}\partial_r(r^2 W_{r\varphi})$$

since for steady state

$$\frac{1}{r}\partial_r(v_r S r) = 0,$$

thus we have

$$Sv_r\partial_r(rv_\varphi) = \frac{1}{r}\partial_r(r^2 W_{r\varphi})$$

where rv_φ is the specific angular momentum and the component $f_{r\varphi}$ has the form $f_{r\varphi} = \eta r\partial_r(\frac{v_\varphi}{r})$, where η is the dynamic vicosity.

2.4.2 The Radial Component

The radial component of the momentum equation is

$$\rho(v_r\partial_r v_r + \frac{v_\varphi}{r}\partial_\varphi v_r + v_z\partial_z v_r - \frac{v^2{}_\varphi}{r}) = -\partial_r P - \rho(\partial_r\Phi) + \frac{1}{r}\partial_r(rf_{rr}) + \partial_z f_{rz} - \frac{1}{r}f_{\varphi\varphi} \tag{2.4.6}$$

Similarly $\partial_\varphi v_r = 0$ (by symmetry). Except for $f_{r\varphi}$, all the viscous stresses can be neglected and ignoring the vertical variation (out flow), we have to sufficient accuracy

$$\rho(v_r\partial_r v_r = -\partial_r P - \rho(\frac{v^2{}_\varphi}{r} - \partial_r\Phi) \tag{2.4.7}$$

Since we use the fact that

$$\Phi = \frac{GM}{(r^2+z^2)^{\frac{1}{2}}}$$

thus,

$$\partial_r \Phi = -\frac{1}{2}\frac{2rGM}{(r^2+z^2)^{\frac{3}{2}}} = \frac{rGM}{(r^2+z^2)^{\frac{3}{2}}}$$

since $r >> z$

$$\Longrightarrow \partial_r \Phi \approx -\frac{GM}{r^2}$$

hence after integration over z Eq.(1.3.6) can be rewritten as

$$S v_r \partial_r v_r = S(\frac{v^2{}_\varphi}{r} - \frac{GM}{r^2}) - \partial_r W$$

where

$$W = \int P dz.$$

It can be also rewritten as

$$\rho \frac{v^2{}_\varphi}{r} = \partial_r \Phi + v_r \partial_r v_r + \partial_r P$$

Neglecting all the terms on the RHS of this equation except the first one gives

$$\frac{v^2{}_\varphi}{r} = \partial_r \Phi$$

This is due to the fact that for a thin accretion disc $v_\varphi >> c_s$ hence,

$$v^2{}_\varphi = \frac{GM}{r}$$

which shows that the disk rotates in the Keplerian fashion.

2.4.3 The Vertical Component

Since the z component of the momentum equation contains only the small components of the viscosity, then ignoring these and the φ derivative we have,

$$\rho(v_r \partial_r v_r + v_z \partial_z v_z) = -\partial_z P - \rho(\partial_z \Phi)$$

we assume that the motion in the z direction are subsonic. Thus, the LHS is a factor of $(\frac{z}{r})^2$. Now neglecting the vertical out flows, the equation reduces to the equation of hydrostatic equilibrium in the z- direction, i.e,

$$\partial_z P = \rho \frac{GM}{r^2} \frac{z}{r} \tag{2.4.8}$$

Using H as half thickness of the disc, the pressure at the mid plane of the disc is,

$$P = \int_0^H \rho \frac{GMz}{r^3} dz = \frac{1}{2} \rho H \frac{GM}{r^3} H = \frac{1}{2} HS \frac{GM}{r^3}$$

Making slight approximation over Eq.(1.3.7), i.e, letting

$$\frac{1}{\rho} \partial_z P \approx \frac{1}{\rho} \frac{P}{Z} = \frac{GMz}{r^3},$$

since the Keplerian velocity

$$v_k^2 = \frac{GM}{r}$$

and

$$c_s^2 = \frac{P}{\rho}$$

we can rewrite the hydrostatic equilibrium equation as

$$c_s^2 = v_k^2 (\frac{H}{r})^2$$

This implies

$$\frac{H}{r} = \frac{c_s}{v_k}$$

In this approximation we have used,

$$v_k = v_\varphi$$

The thin disc requirement thus says that the circular flow velocity is highly supersonic.

2.5 Conservation Of Energy

Energy balance requires that the energy deposited by viscous dissipation in each active layer is equal to the energy radiated away at each surface of the disc. The basic idea behind the accretion disc is that viscosity in the gas disc converts the free energy of differential rotation in to thermal energy, which is then radiated away. As the potential energy is released, the gas slowly spirals inward, completing many revolutions around the neutron star before significantly changing its distance from the central source. The amount of gravitational potential energy release by the gas in the disc increases as the gas draws closer to the central object. Since the dominant part of the scalar product of the stress tenser and the velocity is the radial component of magnitude $f_{r\varphi}v_{\varphi}$ and thus

$$\nabla.(f.v) \approx \frac{1}{r}\partial_r(rf_{r\varphi}v_{\varphi})$$

Ignoring again the z-component of the velocity and in addition the radial component of the energy flux vector, we have

$$\rho(v_r\partial_r)(\frac{1}{2}v_r^2 + \frac{1}{2}v_{\varphi}^2 + \Phi) = \frac{1}{r}\partial_r(rf_{r\varphi}v_{\varphi}) - \partial_z F$$

where F is the vertical energy flux density ($F = qz$). If we denote the energy flux per unit area emitted at the disc surface by $Q^- = 2F$ and the dissipation function $q^+ = f_{r\varphi}r\partial_r(\frac{v_{\varphi}}{r})$ then the energy produced per unit area Q^+ is given by

$$Q^+ = \int q^+ dz = W_{r\varphi}r\partial_r\frac{v_{\varphi}}{r}$$

Usually it is assumed that the energy dissipated in to heat is radiated on the spot in the vertical direction. Then we have $\partial_z F = q^+ = f_{r\varphi}r\partial_r\frac{v_{\varphi}}{r}$

Since we approximate the azimuthal component of velocity by the circular Keplerian velocity, i.e, $v_{\varphi} \approx \Omega r$, where $\Omega = (\frac{GM}{r^3})^{\frac{1}{2}}$

Then we obtain

$$\frac{\dot{M}\Omega r}{2} = -2\pi\partial_r(W_{r\varphi}r^2)$$

with

$$W_{r\varphi} = r\frac{d\Omega}{dr}\int \eta dz$$

and thus,

$$\dot{M}\Omega r^2 + 2\pi r^3\frac{d\Omega}{dr}\int \eta dz = I$$

where η is the turbulent viscosity and I is independent of r .

The Keplerian approximation follows if inertia and pressure gradient terms are neglected. Corrections are of order $(\frac{H}{r})^2$.

From the above relations we find

$$\partial_z F = \frac{9}{4}\eta\frac{GM}{r^3}$$

then the energy produced per unit area will be given by

$$Q^+ = \frac{9}{4}\frac{GM}{r^3}\int \eta dz$$

The angular momentum conservation implies

$$\dot{M}\Omega r^2 = -2\pi r^2 W_{r\varphi} + I$$

Here the constant I is the net inward flux of angular momentum, whose value is usually assumed to be of the order of $\dot{M}r_A^2\Omega(r_A)$, where r_A is the inner edge of the disc. Then we have for specific angular momentum $l = r^2\Omega$

$$\dot{M}[l(r) - l(r_A)] = -2\pi r^2 W_{r\varphi} \tag{2.5.1}$$

The torque $2\pi r^2 W_{r\varphi}$ is on the other hand determined from $W_{r\varphi} = -\frac{3}{2}\Omega\int \eta dz$

In the energy equation above we neglect derivatives of W, S, and v_r:

$\frac{d}{dr}[\dot{M}(\frac{1}{2}v_\varphi^2 - \frac{GM}{r}) + 2\pi r^2 W_{r\varphi}\Omega] = 2\pi r Q^-$. Finally we express the cooling rate Q^- independent of η as

$$Q^- = \frac{3}{4\pi}\dot{M}\frac{GM}{r^3}[1 - (\frac{r_A}{r})^{\frac{1}{2}}] \tag{2.5.2}$$

Chapter 3

LOCAL DISC VARIABLES

If the thin disc approximation hold, the task of computing the detailed disc structure is enormously simplified. Both the pressure and temperature gradients are essentially vertical, so that the vertical and radial structures are largely decoupled.
From the work of (Kippenhahn and weigert (1990)) we have the equations of hydrostatic equilibrium and energy transport to solve with the radial disc structure only entering the calculations in the fixing of the local energy generation rate.
The temperature T_c must it self be given by an energy equation relating the energy flux in the vertical direction to the rate of generation of energy by viscous dissipation. The vertical energy transport mechanism may be either radiative or convective, depending on whether or not the temperature gradient required for radiative transport is smaller or greater than the gradient given by the adiabatic assumption.

Then here we are assuming that the transport is radiative, this is indeed true in many important cases. Because of the thin disc approximation, the disc medium is essentially 'plane-parallel' at each radius, so that the temperature gradient is effectively in the z-direction, as we pointed out in chapter two. Under these circumstances, the flux of radiant energy though a surface z=constant is

$$F(z) = -(\frac{4acT^3}{3\chi\rho}\frac{dT}{dz})$$

whereχ is the Rosseland mean opacity.

It is implicitly assumed in writing the above equation that the disc is optically thick in the sense that

$$\tau = \rho H \chi = S\chi >> 1$$

So that the radiation field is locally very close to the black-body value, once the optical depth given above becomes less than(or equal) unity, the expression for radiative cooling given above breaks down as the radiation can escape directly.

The energy balance, as we have seen in chapter two above, is

$$\frac{\partial F}{\partial Z} = q^-$$

where q^- is the volume rate of energy emitted

$$F(H) - F(0) = \int_0^H q^-(z)dz = Q^-$$

This equation tells us the total dissipation rate through one-half of the vertical structure must give the dissipation rate per unit face area Q^-.

From $F(z) = -(\frac{4acT^3}{3\chi\rho}\frac{dT}{dz})$ we have

$$F(z) \sim (\frac{4\sigma}{3\tau})T^4$$

Using $\tau = \rho H\chi = S\chi >> 1$, so that provided the central temprature exceeds the surface temprature enough to make $T_c^4 >> T^4(H)$, $F(H) - F(0) = \int_0^H q^-(z)dz = Q^-$ becomes approximately

$$\frac{4\sigma}{3\tau}T_c^4 = Q^-$$

which is our required energy equation.

3.1 Central Density and Central Temperature With the Presence of Coulombic Pressure

From the hydrostatic equilibrium we have

$$\frac{dP}{dz} = -\rho \frac{GM}{r^2}\frac{z}{r} \tag{3.1.1}$$

and from the energy conservation equation we derived that

$$\frac{dF}{dz} = \frac{9}{4}\frac{GM}{r^3}\eta \tag{3.1.2}$$

We also have the equation of state (total pressure equation)

$$P = \frac{N_0 K_\beta \rho T}{\mu} - \frac{e^3}{3}(\frac{\pi}{k_\beta T})^{\frac{1}{2}}(\rho N_0 f)^{\frac{3}{2}} \tag{3.1.3}$$

We can treat the hydrodynamical equitions integrated exactly over the thickness of the disc. As for the viscous stress $f_{r\varphi}$ we have

$$f_{r\varphi} = \eta r \partial_r (\frac{v_\varphi}{r}) = \eta r \partial_r \Omega,$$

If we assume the turbulent viscosity to be a linear function of the turbulent velocity and the Keplerian angular velocity Ω the viscous stress is reduced to the familiar Shakura and Sunyaev α-model: $f_{r\varphi} = -\alpha P$.

The parameter $\alpha \sim \frac{v_t}{c_s}$ where c_s and v_t are the sound and turbulent velocities respectively. Thus the above expression for the viscous stress gives us the equation:

$$f_{r\varphi} = -\frac{3}{2}\eta(\frac{GM}{r^3})^{\frac{1}{2}} = -\alpha P$$

Comparing this with $f_{r\varphi} = \eta r \partial_r (\frac{v_\varphi}{r}) = \eta r \partial_r \Omega$ we get

$$\eta = \frac{2}{3}\alpha P (\frac{Gm}{r^3})^{-1/2}$$

Once a formula for the viscosity is given, we can determine the local structure of the disc. We assume a polytropic equation of state for fixed r:

$$P(z) = K\rho(z)^{1+\frac{1}{N}} \tag{3.1.4}$$

Eq.(3.1.1) can then immediately be solved with the result:

$K(1+N)\rho^{\frac{1}{N}} = \frac{1}{2}\frac{GM}{r}[(\frac{H}{r})^2 - (\frac{z}{r})^2]$

For the values in the central plane we obtain

$$\frac{P_c}{\rho_c} = \frac{1}{1+N}\frac{1}{2}\frac{GM}{r}(\frac{H}{r})^2 \tag{3.1.5}$$

Further more

$$W = 2P_C HI(N+1) \tag{3.1.6}$$

$$S = 2\rho_c HI(N) \tag{3.1.7}$$

where $I(N) = \frac{(2^N N!)^2}{(2N+1)!}$

Since we are dealing with the spirit of the modified α- prescription of Shakura and Sunyaev(1973) that we are modeling the disk as geometrically thin the energy transport equation is radiation transport dominated.

For optically thick parts of the disc, we use the expression

$$\frac{dT}{dz} = -\frac{3\chi\rho F}{4acT^3} \tag{3.1.8}$$

with $Q^- = 2F$ we have

$$Q^- = -2(\frac{4acT^3}{3\chi\rho}\frac{dT}{dz})_{surface} \tag{3.1.9}$$

If we let $A = \frac{N_0 K_\beta}{\mu}$ and $B = \frac{e^3}{3}(\frac{\pi}{k_\beta})^{\frac{1}{2}}(N_0 f)^{\frac{3}{2}}$

then Eq.(3.1.3) and (3.1.4) give us:

$$K\rho^{1+\frac{1}{N}} = A\rho T - B(\frac{\rho^3}{T})^{\frac{1}{2}}$$

$$K\rho^{\frac{1}{N}} = AT - B(\frac{\rho}{T})^{\frac{1}{2}}$$

with

$$\rho_c^{\frac{1}{N}} = \frac{1}{2K(N+1)} \frac{GM}{r} (\frac{H}{r})^2$$

Thus

$$K\rho^{\frac{1}{N}} T^{\frac{1}{2}} = AT^{\frac{1}{2}} - B\rho^{\frac{1}{2}}$$

$$T(z) = (B^2\rho)/(A - K\rho^{\frac{1}{N}})^2$$

$$T(z) = B^2\rho \frac{1}{\left[A - \frac{1}{2(N+1)}\frac{Gm}{r}\left(\left(\frac{H}{r}\right)^2 - \left(\frac{z}{r}\right)^2\right)\right]^2}$$

The temperature at the center of the disc plane is then

$$T_c = T(z=0) = B^2\rho_c \frac{1}{\left[A - \frac{1}{2(N+1)}\frac{Gm}{r}(\frac{H}{r})^2\right]^2} \tag{3.1.10}$$

$$\frac{T}{T_c} = \frac{\rho}{\rho_c} \frac{\left[A - \frac{1}{2(N+1)}\frac{Gm}{r}(\frac{H}{r})^2\right]^2}{\left[A - \frac{1}{2(N+1)}\frac{Gm}{r}\left(\left(\frac{H}{r}\right)^2 - \left(\frac{z}{r}\right)^2\right)\right]^2} \tag{3.1.11}$$

Let $Y = \frac{1}{2(N+1)}\frac{Gm}{r}(\frac{H}{r})^2$ then with

$$\frac{dT}{dz} = -T_c \frac{2z}{H^2} \frac{(A-Y)^2}{[A - Y(1-(\frac{z}{H})^2)]^3} (1 - (\frac{z}{H})^2)^{N-1} \left[N(A - Y(1 - (\frac{z}{H})^2) + 2Y(1 - (\frac{z}{H})^2)\right]$$

From Eq.(3.1.9) together with Eq.(3.1.11) we have

$$Q^- = \frac{16}{3}\frac{ac}{\chi\rho}\frac{z}{H^2}\frac{T_c^4 (A-Y)^8 (1-(\frac{z}{H})^2)^{4N-1}}{(A - Y(1-(\frac{z}{H})^2))^9}\left[N(A - Y(1 - (\frac{z}{H})^2)) + 2Y(1 - (\frac{z}{H})^2)\right]$$

Since the general form of the opacity is given by

$$\chi = \chi_o \rho^n T^{-s},$$

where

$n = 1, s = 3.5$ Kramer's law which is particularly good representation of the opacity when it is dominated by free-free absorption.

$n = .75, s = 3.5$ Schwarzschild's opacity which yields somewhat better results if bound-free opacity makes an important contribution, and

$n = 0, s = 0$ is used when electron scattering is dominated.

Note that for free-free opacity dominated $\chi_o = 5x10^{23}\frac{m^2}{kg}$ and for electron-scattering opacity dominated we have $\chi_o = 0.04\frac{m^2}{kg}$.

Thus,

$$\chi\rho = \chi_0\rho^{n+1}T^{-s} = \frac{T_c^{-s}\rho_c^{n+1}(A-Y)^{-2s}}{[A-Y(1-(\frac{z}{H})^2)]^{-2s}}[1-\left(\frac{z}{H}\right)^2]^{N(n+1)-sN}$$

$$Q^- = \frac{16}{3}\frac{acT_c^4}{\chi_0\rho_c^{n+1}T_c^{-s}}\frac{z}{H^2}\frac{(A-Y(1-(\frac{z}{H})^2))^{-2s-9}}{(A-Y)^{-8-2s}}(1-(\frac{z}{H})^2)^{4N-1-sN-N(n+1)}$$
$$*\left[N(A-Y(1-(\frac{z}{H})^2))+2Y(1-(\frac{z}{H})^2)\right]$$

The energy flux per unit area emitted at the disc surface is

$$(Q^-)_{surface} = \lim_{z\longrightarrow H} Q^-$$

The limit $z \longrightarrow H$ exists only if $4N - 1 + sN - N(n+1) = 0$ or $N = \frac{1}{s-n+3}$.

Then we have

$$Q^- = \left(\frac{16acT^4}{3\chi\rho}\right)_c \frac{NA}{H}\frac{A^{-2s-9}}{(A-Y)^{-2s-8}}$$

Or

$$Q^- = \left(\frac{16acT^4}{3\chi\rho}\right)_c \frac{N}{H}\frac{(A-Y)^{2s+8}}{A^{2s+8}} \tag{3.1.12}$$

Comparing these two Eqs. (2.4.2) and (3.1.12) we get

$$H = \left(\frac{64\pi acT^4}{9\chi\rho}\right)_c \frac{N}{\dot{M}}\frac{(A-Y)^{2s+8}}{A^{2s+8}}\frac{r^3}{Gm}(1-(\frac{r_A}{r})^{1/2})^{-1} \tag{3.1.13}$$

The vertical structure described in the previous paragraph depends on the parameters H, ρ_c which are all functions of the radius. The radial equations (chapter 2) together with the α-model allow us to determine these functions for a given $(m, \dot{M})$

From Eq.(2.5.1) and $W_{r\varphi} = -\alpha W$ we have

$$W(r) = \frac{\dot{M}}{2\pi\alpha r^2}[l(r) - l(r_0)] \tag{3.1.14}$$

with $l(r) = (Gmr)^{\frac{1}{2}}$

On the left hand side, we insert Eq.(3.1.6) to obtain

$$P_c(r) = \frac{1}{I(N+1)}\frac{\dot{M}}{4\pi r^2\alpha}\left(\frac{H}{r}\right)^{-1}\left(\frac{GM}{r}\right)^{\frac{1}{2}}[1-(\frac{r_0}{r})^{\frac{1}{2}}] \tag{3.1.15}$$

This equation contains still the parameter H. Using the relation (3.1.5) between P_c and ρ_c gives

$$\rho_c(r) = \frac{2(N+1)}{I(N+1)}\frac{\dot{M}}{4\pi r^2\alpha}\left(\frac{GM}{r}\right)^{-1/2}\left(\frac{H}{r}\right)^{-3}[1-(\frac{r_0}{r})^{1/2}] \tag{3.1.16}$$

$$H = \left(\frac{64\pi acT^{4+s}}{9\chi_0\rho^{n+1}}\right)_c\frac{N}{\dot{M}}\frac{(A-Y)^{2s+8}}{A^{2s+8}}\frac{r^3}{Gm}(1-(\frac{r_A}{r})^{1/2})^{-1} \tag{3.1.17}$$

3.1.1 Middle Region (Electron-scattering opacity dominated)

We said that the radiative cooling converges as $z \longrightarrow H$ only if $N = \frac{1}{s-n+3}$, then for $n = 0$, and $s = 0$ we get $N = \frac{1}{3}$

Eq.(3.1.12) can be rewritten as

$$Q^- = \left(\frac{16acT^4}{3\chi_o\rho}\right)_c\left(\frac{1}{3H}\right)\left(\frac{A-Y(N=\frac{1}{3})}{A}\right)^8 \tag{3.1.18}$$

and

$$H = \left(\frac{64\pi acT^4}{27\chi_0\rho}\right)_c\frac{1}{\dot{M}}\frac{(A-Y(N=\frac{1}{3}))^8}{A^8}\frac{r^3}{Gm}(1-(\frac{r_A}{r})^{1/2})^{-1}$$

where , $Y(N=\frac{1}{3}) = \frac{3}{8}\frac{Gm}{r}(\frac{H}{r})^2$ and

$$T_C^4 = \frac{B^8\rho_c^4}{\left[A - \frac{3}{8}\frac{Gm}{r}(\frac{H}{r})^2\right]^8} \tag{3.1.19}$$

$$\rho_c(r) = \frac{8}{3I(\frac{4}{3})}\frac{\dot{M}}{4\pi r^2\alpha}\left(\frac{GM}{r}\right)^{-1/2}\left(\frac{H}{r}\right)^{-3}[1-(\frac{r_A}{r})^{1/2}] \tag{3.1.20}$$

Then we have

$$H = \left(\frac{2}{3}\right)^{\frac{3}{5}}\left(\frac{8ac}{\pi^2\alpha^3\chi_o}\right)^{\frac{1}{10}}\left(\frac{B}{A}\right)^{\frac{4}{5}}I(\frac{4}{3})^{\frac{3}{10}}\frac{\dot{M}^5}{(Gm)^{\frac{1}{4}}}r^{\frac{9}{20}}\left[1-\left(\frac{r_A}{r}\right)^{\frac{1}{2}}\right]^{\frac{3}{10}}$$

For simplicity we approximate $I(\frac{4}{3}) \sim I(1) = \frac{2}{3}$, $I(1)^{\frac{3}{10}} = (\frac{2}{3})^{\frac{3}{10}}$.

Then we find the value for the half thickness as a function of the radial variable r as

$$H = \left(\frac{2^{12}ac}{3^9\pi^2\alpha^3\chi_o}\right)^{\frac{1}{10}} \left(\frac{B}{A}\right)^{\frac{4}{5}} \frac{\dot{M}^5}{(Gm)^{\frac{1}{4}}} r^{\frac{9}{20}} \left[1 - \left(\frac{r_A}{r}\right)^{\frac{1}{2}}\right]^{\frac{3}{10}} \tag{3.1.21}$$

3.1.2 Outer Region (free-free absorption dominated opacity)

Now $n = 1$ and $s = \frac{7}{2}$, from the condition for convergence of the limit of radiative cooling we set $N = \frac{2}{11}$

$$Q^- = \left(\frac{32acT^{\frac{15}{2}}}{33\chi_o\rho^2}\right)_c \left(\frac{1}{H}\right)\left(\frac{A - Y(N = \frac{2}{11})}{A}\right)^{15} \tag{3.1.22}$$

and

$$H = \left(\frac{128\pi acT^{\frac{15}{2}}}{99\chi_0\rho^2}\right)_c \frac{1}{\dot{M}} \left(\frac{A - \frac{11}{26}\frac{Gm}{r}(\frac{H}{r})^2}{A}\right)^{15} \frac{r^3}{Gm}(1 - (\frac{r_A}{r})^{1/2})^{-1}$$

Up on inserting

$$T_C^{\frac{15}{2}} = \frac{B^{15}\rho_c^{\frac{15}{2}}}{\left[A - \frac{11}{26}\frac{Gm}{r}(\frac{H}{r})^2\right]^{15}} \tag{3.1.23}$$

$$H = \left(\frac{128\pi acB^{15}\rho_{\frac{11}{2}}}{99\chi_0}\right)_c \frac{1}{\dot{M}}\frac{1}{A^{15}}\frac{r^3}{Gm}(1 - (\frac{r_A}{r})^{1/2})^{-1}$$

with

$$\rho_c^{\frac{11}{2}}(r) = \left(\frac{26\dot{M}}{44\pi\alpha I(\frac{13}{11})}\right)^{\frac{11}{2}} \frac{r^{\frac{33}{4}}}{(Gm)^{\frac{11}{4}} H^{\frac{33}{2}}} [1 - (\frac{r_A}{r})^{1/2}]^{\frac{11}{2}} \tag{3.1.24}$$

and approximating $I(\frac{13}{11}) \sim I(1) = \frac{2}{3}$, $I(1)^{\frac{3}{10}} = \frac{2}{3}^{\frac{3}{10}}$.

$$H = \left(\frac{128\pi ac}{99\chi_0}\right)^{\frac{2}{35}} \left(\frac{B}{A}\right)^{\frac{6}{7}} \left(\frac{39}{44\pi\alpha}\right)^{\frac{11}{35}} \frac{\dot{M}^{\frac{9}{35}}}{Gm^{\frac{3}{14}}} r^{\frac{9}{14}} \left(1 - (\frac{r_A}{r})^{1/2}\right)^{\frac{9}{35}} \tag{3.1.25}$$

3.2 Central Density and Central Temperature with out Coulombic Pressure

We shall now determine the disc variables in the absence of coulombic pressure in the region where the pressure is dominated by the gas pressure. This will allow us to check if the coulombic pressure in the thin disc approximation has a real contribution to the stability of the accretion disc by comparing those equations derived in Section 3.1 and here below.

Now the equation of state becomes

$$P = \frac{N_0 K_\beta \rho T}{\mu}$$
$$= A\rho T$$

Using still the polytropic relation $P(z) = K\rho(z)^{1+\frac{1}{N}}$ thus, $K\rho(z)^{\frac{1}{N}} = A\rho T$

Putting the expression for $\rho(z)^{\frac{1}{N}}$ given above we find

$$T(z) = \frac{1}{2A(N+1)}\frac{Gm}{r}\left[\left(\frac{H}{r}\right)^2 - \left(\frac{z}{r}\right)^2\right]$$

And the central temperature is

$$T(z=0) = T_c = \frac{1}{2A(N+1)}\frac{Gm}{r}\left(\frac{H}{r}\right)^2 \tag{3.2.1}$$

Thus we have

$$\frac{T}{T_c} = \left(1 - \left(\frac{z}{H}\right)^2\right) \tag{3.2.2}$$

Since for optically thick the energy transport equation is given by Eq.(3.1.9) with

$$T^3\frac{dT}{dz} = -2T_c^4\frac{z}{H}\left(1 - \left(\frac{z}{H}\right)^2\right)^3$$

$$\chi\rho = \chi_o\rho^{n+1}T^{-s}$$
$$= \chi_o\rho_c^{n+1}T_c^{-s}\left(1 - \left(\frac{z}{H}\right)^2\right)^{N(N+1)-S}$$

The equation

$$Q^- = -\frac{8acT^3}{3\chi_o\rho^{n+1}T^{-s}}\frac{dT}{dz}$$

becomes

$$Q^{-} = -\left(\frac{16acT^4}{3\chi\rho}\right)_c \left(\frac{z}{H}\right)\left(1-\left(\frac{z}{H}\right)^2\right)^{(3+s-N(n+1))}$$

Using the fact that $(Q^{-})_{surface} = \lim_{z\longrightarrow H} Q^{-}$, and the limit exists only if $3+s-N(N+1)=0$ or $N=\frac{3+s}{n+1}$ then

$$Q^{-} = -\left(\frac{16acT^4}{3\chi\rho}\right)_c \frac{1}{H} \tag{3.2.3}$$

Comparing this with Eq.(2.2.4) we get

$$H = \left(\frac{16acT^4}{3\chi\rho}\right)_c \frac{4\pi}{3}\frac{r^3}{Gm}\frac{1}{\dot{M}}\left(1-\left(\frac{r_A}{r}\right)^{\frac{1}{2}}\right)^{-1}$$

or

$$H = \left(\frac{16acT^{4+s}}{3\chi_o\rho^{n+1}}\right)_c \frac{4\pi}{3}\frac{r^3}{Gm}\frac{1}{\dot{M}}\left(1-\left(\frac{r_A}{r}\right)^{\frac{1}{2}}\right)^{-1} \tag{3.2.4}$$

Since T_c and ρ_c are still functions of H, using Eq.(3.1.16) for ρ_c

that is,

$$\rho_c^{n+1}(r) = \left(\frac{2(N+1)}{I(N+1)}\right)^{n+1}\left(\frac{\dot{M}}{4\pi r^2\alpha}\right)^{n+1}\left(\frac{GM}{r}\right)^{-\frac{(n+1)}{2}}\left(\frac{H}{r}\right)^{-3(n+1)}\left[1-\left(\frac{r_A}{r}\right)^{1/2}\right]^{n+1} \tag{3.2.5}$$

3.2.1 Middle Region (Electron-scattering opacity dominated)

For $n=0$, $s=0$ we have $N=3$, Eq.(3.2.5) becomes

$$\rho_c(r) = \frac{8}{I(4)}\left(\frac{\dot{M}}{4\pi r^2\alpha}\right)\left(\frac{GM}{r}\right)^{-\frac{1}{2}}\left(\frac{H}{r}\right)^{-3}\left[1-\left(\frac{r_A}{r}\right)^{1/2}\right]$$

And also for T_c^4 imply

$$T_c^4 = \left(\frac{1}{8A}\right)^4\left(\frac{Gm}{r}\right)^4\left(\frac{H}{r}\right)^8$$

$$H = \left(\frac{1152\chi_o A^4\dot{M}^2}{I(4)ac\alpha\pi^2(Gm)^{\frac{7}{2}}}\right)^{\frac{1}{10}} r^{\frac{21}{20}}\left(1-\left(\frac{r_A}{r}\right)^{\frac{1}{2}}\right)^{\frac{1}{5}}$$

since $I(4)=\frac{2^7}{315}$ then we have

$$H = \left(\frac{2.84x10^3\chi_o A^4\dot{M}^2}{ac\alpha\pi^2(Gm)^{\frac{7}{2}}}\right)^{\frac{1}{10}} r^{\frac{21}{20}}\left(1-\left(\frac{r_A}{r}\right)^{\frac{1}{2}}\right)^{\frac{1}{5}} \tag{3.2.6}$$

Up on inserting the constants (the used constants are given at the back of this material), we find the approximated value for the half thickness of the disc for this particular region:

$$H \simeq 1.41x10^{-3.05} r^{\frac{19}{20}} \left[r^{\frac{1}{2}} - 10^3 \right]^{\frac{1}{5}}$$

The the other disc variables will be expressed as purely in terms of the radial variable r.

$$\rho_c(r) = 2.624x10^{13.15} r^{-\frac{37}{20}} \left[r^{\frac{1}{2}} - 10^3 \right]^{\frac{2}{5}}$$

$$T_c = 3.34x10^{12.1} r^{-\frac{11}{10}} \left[r^{\frac{1}{2}} - 10^3 \right]^{\frac{2}{5}}$$

3.2.2 Outer Region (free-free absorption dominated opacity)

Now for $n = 1$, $s = 3.5$ we have $N = \frac{13}{4}$, Eq.(3.2.5) becomes

$$\rho_c^2(r) = \left(\frac{17}{2I(\frac{17}{4})} \right)^2 \left(\frac{\dot{M}}{4\pi r^2 \alpha} \right)^2 \left(\frac{GM}{r} \right)^{-1} \left(\frac{H}{r} \right)^{-6} \left[1 - \left(\frac{r_A}{r} \right)^{1/2} \right]^2$$

$I(\frac{17}{4})$ is approximated to $I(4) = \frac{2^7}{315}$, and also for $T_c^{\frac{15}{2}}$ imply

$$T_c^{\frac{15}{2}} = \left(\frac{2Gm}{17A} \right)^{\frac{15}{2}} H^{15} r^{-\frac{45}{2}}$$

Thus Eq.(3.2.4) gives

$$H = \left(\frac{7.5x10^5 \chi_o \dot{M}^3}{I(4) ac\alpha^2 \pi^3} \right)^{\frac{1}{20}} \left(\frac{A^{\frac{3}{2}}}{Gm} \right)^{\frac{1}{4}} r^{-1.92} \left(1 - \left(\frac{r_A}{r} \right)^{\frac{1}{2}} \right)^{\frac{1}{10}} \tag{3.2.7}$$

Chapter 4

STABILITY ANALYSIS OF THIN ACCRETION DISC

4.1 Thermal Instabilities

In the last chapters, we have dealt in some detail with the theory of steady thin accretion discs. We investigate now the work of Shakura and Sunyaev, the stability of the steady discs described in chapter two. Only stable models have a chance to be physically relevant. It will turn out that possible instabilities depend strongly on the assumed viscosity. There are several reasons for extending this study to time dependent discs. One is that we must check that the steady-state models are stable against small perturbations if not, we have probably made some assumption in the course of constructing these discs which is not compatible with the further assumption of steadiness. The time dependence of disc flow is controlled by the size of the viscosity then they offer one of the few sources of qualitative information.

We begin by identifying the typical timescales on which the disc structure may vary by encountering the viscous timescale

$$t_{visc} \sim \frac{r^2}{\nu} \sim \frac{r}{v_r}$$

which gives the timescale on which matter diffuses though the disc under the effect of the viscous torques. The shortest characteristic timescale of the disc is the dynamical timescale

$$t_\varphi \sim \frac{r}{v_\varphi} \sim \Omega_k^-$$

In equilibrium we must have $Q^+ = Q^-$. But if, when the central temperature T_c is increased by a small perturbation δT_c, Q^+ increases faster than Q^-, T_c will rise further because the cooling rate is inadequate. In other words, a steady state is impossible in a parameter regime where the instability would grow, despite the fact that formally an equilibrium solution can be found. If the energy balance is disturbed in the disc, any stability will grow on a timescale t_{th} given by the ratio of heat content per unit disc area to the dissipation rate per unit disc area. It is the timescale for readjustment to the thermal equilibrium, if, say, the dissipation rate is altered. Since the heat content per unit volume of a gas is

$\frac{\rho\ k_\beta T}{\mu m_p} \sim \rho c_s^2$

this gives $t_{th} \sim \frac{S c_s^2}{Q^+}$. The relation between Q^+ and νS means that t_{th} can be re-expressed in terms of the viscous timescale: for a keplerian disc, we have $(\Omega = \Omega_k)$

$t_{th} \sim \frac{r^3 c_s^2}{G m \nu} \sim \frac{c_s^2}{v_\varphi^2}\frac{r^2}{\nu} \sim \xi^{-2} t_{visc}$

Suppose now that a small perturbation is made to a putative equilibrium solution and that this perturbation continues to grow rather than being damped. The difference we have found in these timescales means we can distinguish different types of instabilities. If for example the energy balance is disturbed in the disc, any instability will grow on a timescale t_{th}, which is much less than t_{visc}.

If $\nu = \eta/\rho$ denotes the kinematic viscosity we have from Section 2.5

$Q^+ = \frac{9}{4}\nu S \Omega^2$

And for the α-model in Section 2.5 gives

$\nu = \alpha c_s H = \frac{2\alpha}{3}\frac{W}{S}\frac{1}{\Omega}$

Since t_{visc} is the time scale for significant changes in the surface density S to occur, we can assume that S is fixed during the growth time t_{th} we refer to this as a thermal instability and such instabilities arise when the local (volume) cooling rate, q^- with in the disc can no longer cope with the volume heating rate, q^+.

In general the customary way of writing the instability criterion is [1,2]

$$\frac{dQ^-}{dT_c} < \frac{dQ^+}{dT_c}$$

Or

$$\frac{d\ln Q^-}{d\ln T_c} < \frac{d\ln Q^+}{d\ln T_c}$$

Now in our case since we assume $H << r$ we can use the fact that $Y << A$ then Eq.(3.1.12) can be approximated to give volume cooling rate

$q^- = \frac{Q^-}{H} \sim \frac{\sigma T_c^4}{\chi\rho H^2}$

For free-free opacity it will be (omitting the constraints)

$q^- \sim \frac{{T_c}^{\frac{15}{2}}}{\rho^2 H^2} = \frac{{T_c}^{\frac{15}{2}}}{S^2} \sim {T_c}^{\frac{15}{2}}$

The volume heating rate q^+ is also given by

$q^+ \sim \frac{Q^+}{H} \sim \frac{\nu}{H} \sim \alpha c_s \sim \alpha T_c^{\frac{1}{2}}$

where we have used the α-parametrization $\nu = \alpha c_s H$

Comparing these two expression, we see that the thermal instability will grow if q^- increases less rapidly with T_c than does q^+

Similarly for the electron-scattering opacity we have

$q^- = \frac{Q^-}{H} \sim \frac{\sigma T_c^4}{\rho H^2}$

which can be furthermore approximated as $q^- \sim T_c^4$

Thus for gas pressure dominated region, unless α decreases more rapidly than (T_c^{-7} for Kramer's opacity and $T_c^{-\frac{7}{2}}$ in the case of electron-scattering), but obviously $\alpha < 1$, thus the optically thick parts of a disc will be thermally stable.

4.2 Viscous Instabilities

Let us now consider changes in the disc structure which takes place on the viscous timescale. This includes instabilities and the evolution of discs in response to changes in external conditions, such as the mass transfer rate. As $t_{visc} >> t_{th}$, we assume that the disc adjusts so rapidly that if always maintains both thermal and hydrostatic equilibrium. Hence, some equations describing steady discs apply also to time dependent discs.

Defining the mass transfer rate $\dot{M} = \dot{M}(r,t)$ at each radius with S and v_t be functions of r, i.e,

$$\frac{\partial \dot{M}}{\partial r} = 2\pi r \frac{\partial S}{\partial t}$$

Suppose that the the surface density in a steady disc is perturbed axsiymmetrically at each r, so that

$$S = S_o + \delta S$$

where S_o is the steady state distribution. In the Keplerian approximation for v_φ we get from Eq.(2.3.5) since v_φ is time independent

$$rSv_r = \frac{\partial_r(r^2 W_{r\varphi})}{\partial_r(rv_\varphi)}$$

using $\nu = \eta/\rho$ from Section 2.5

$$W_{r\varphi} = r\frac{d\Omega}{dr}\int \nu\rho dz = r\frac{d\Omega}{dr}\nu S \tag{4.2.1}$$

Hence, we obtain from Eq.(4.0.1)

$$rSv_r = -\frac{3}{r\Omega}\partial_r[\nu r^2 \Omega S] \tag{4.2.2}$$

Applying on this equation the operator $r^{-1}\partial_r$ and using the mass conservation equation gives

$$\partial_t S = \frac{3}{r}\partial_r\left(\frac{1}{r\Omega}\partial_r[\nu r^2\Omega S]\right) \tag{4.2.3}$$

Denoting $\mu = \nu S$ there will be a corresponding perturbation $\delta\mu$, since $\nu = \nu(r,S)$ thus $\mu = \mu(r,S)$ so that

$$\frac{\delta\mu}{\delta S} = \frac{\partial\mu}{\partial S}$$

$$\frac{\partial}{\partial t}(\delta S) = \frac{3}{r}\frac{\partial}{\partial r}\left[\frac{1}{r\Omega}\frac{\partial}{\partial r}(r^2\Omega\delta\mu)\right]$$

eliminating δS we obtain the equation governing the growth of the perturbation

$$\frac{\partial}{\partial t}(\delta\mu) = \frac{\partial\mu}{\partial S}\frac{3}{r}\frac{\partial}{\partial r}\left[\frac{1}{r\Omega}\frac{\partial}{\partial r}(r^2\Omega\delta\mu)\right]$$

$\delta\mu$ obeys a diffusion equation having the diffusion coefficient proportional to $\frac{\partial\mu}{\partial S}$.

If $\frac{\partial\mu}{\partial S}$ is positive the perturbation decays on a viscous timescale. However, if it is negative, more material will be fed into those regions of the disc that are denser than their surroundings and material will be removed from those regions that are less dense, so that the disc will tend to breakup in to rings. This breakup of the disc on a timescale t_{visc} constitutes the viscous instabilities, more precisely stated; steady disc flow is only possible provided [1]

$$\frac{\partial\mu}{\partial S} > 0 \tag{4.2.4}$$

4.3 The Stability Parameter β

Since the task of approximating we did above (viscous instability) may lead to wrong conclusion. Thus the right way of determining the instability is using the stability parameter, which is derived basically from the two dominant (radiation and gas) pressures then we apply the general condition to our case.

We consider only axially symmetric perturbations of wave length Λ, satisfying H << Λ << r, and which change little on the dynamical time scale Ω^{-1} (Ω^{-1} is also roughly the time it takes for a sound wave to cross the disc in the transverse direction). For a linear stability analysis, we to linearize the basic time dependent equations around the equilibrium solutions. For the type of perturbations,which we want to consider, we can still use in the vertical direction the hydrostatic equation (neglecting terms of order $(\frac{H}{r})^2$). Furthermore, v_φ is still Keplerian, up to terms of order $(\frac{H}{r})^2$,$\frac{H^2}{r\Lambda}$.

Using the continuity equation we define another form of the energy equation given in Chapter two

$$\partial_t(\epsilon\rho) + div[(\epsilon\rho + P)v] - \nabla_v P = q^+ - divq$$

Thus

$$\partial_t(\epsilon\rho) + \frac{1}{r}\partial_r[rv_r(\rho\epsilon + P)] + \partial_z[v_z(\rho\epsilon + P)] - v_r\partial_r P - v_z\partial_z P = q^+ - divq$$

Integrating over z gives

$$\partial_t \int \epsilon\rho dz + \frac{1}{r}\partial_r[\int rv_r(\rho\epsilon + P)dz] - \int v_r\partial_r P dz - \int v_z\partial_z P dz = Q^+ - Q^- \qquad (4.3.1)$$

Where ϵ is the internal specific energy given by

$$\epsilon = c_v T + \frac{aT^4}{\rho}$$

Let $\gamma = \frac{c_p}{c_v}$ and $P_g = \beta P$. Using $c_p = C_v + A$ we get

$$\epsilon = \frac{LP}{\rho}$$

In the thin disc approximation and using the hydrostatic equation for $\partial_z P$, we obtain

$$\partial_t(LW) + \frac{1}{r}\partial_r[rv_r(L+1)W] - v_r\partial_r W + \Omega^2 \int \rho V_z z dz = Q^+ - Q^- \qquad (4.3.2)$$

From now on, we choose for simplicity a constant density in the z-direction and assume that the perturbations in the z- direction preserve this property.

Then the hydrostatic equation in the z-direction gives

$$P(z) = P_c\left[1 - (\frac{z}{H})^2\right], P_c = \frac{1}{4}S\Omega^2 H$$

and thus the average pressure is

$$P = \frac{1}{6}S\Omega^2 H$$

we also have

$$W = \frac{4}{3}P_c H = \frac{1}{3}S\Omega^2 H^2$$

and thus for the α-models

$$\nu = \frac{2}{3}\alpha\Omega^2 H^2. \qquad (4.3.3)$$

Furthermore, since $v_z = \frac{z}{H}\partial_t H$

$$\int \rho V_z z dz = \frac{1}{3} S H \partial_t H.$$

Inserting these expressions into Eq.(4.3.2) gives

$$\frac{1}{3}\partial_t\Big[LS\Omega^2H^2\Big] + \frac{1}{3}\frac{1}{r}\partial_r\Big[rv_r(L+1)S\Omega^2H^2\Big] - \frac{1}{3}v_r\partial_r(S\Omega^2H^2) + \frac{1}{3}S\Omega^2H\partial_tH = Q^+ - Q^-$$

In the second term on the left we use Eq.(4.2.2) to get the second basic equation:

$$\frac{1}{3}\partial_t\Big[LS\Omega^2H^2\Big] - \frac{1}{r}\partial_r\Big[(L+1)\frac{\Omega H^2}{r}\partial_r(\nu\Omega r^2 S)\Big] - \frac{1}{3}v_r\partial_r(S\Omega^2H^2) + \frac{1}{3}S\Omega^2H\partial_tH = Q^+ - Q^- \tag{4.3.4}$$

Eqs.(4.2.3) and (4.3.4) have to be linearized now about the equilibrium.

Let us introduce the following notations for the changes of S and H from their equilibrium values

$$\frac{\delta S}{S} = u, \qquad \frac{\delta H}{H} = h, \qquad |u|, |h| << 1 \tag{4.3.5}$$

We set

$$\frac{\delta\nu}{\nu} = nu + mh + ... \tag{4.3.6}$$

If we use the α-law Eq.(4.3.3),then

$$n = 0, \qquad m = 2 \tag{4.3.7}$$

Inserting Eq.(4.3.5) and Eq.(4.3.6) gives for the linearizion of Eq.(4.2.3)

$$S\partial_t u = 3\nu S\partial_r^2[(n+1)u + mh]. \tag{4.3.8}$$

The linearizion of Eq.(4.3.4) is a bit more complicated.

Let

$$\frac{\delta Q^-}{Q^-} = lu + kh + ... \tag{4.3.9}$$

where the expansion coefficients depend on the opacity χ

One must also include variations of L. For definiteness we choose $\gamma = \frac{5}{3}$.

Then

$$L = \frac{3}{2}(1+\dot{\beta}), \qquad \dot{\beta} = 1 - \beta \tag{4.3.10}$$

First order changes of $\dot{\beta}$ are obtained from the equation of state

$$P = \dot{\beta}P + A\frac{S}{2H}\left(\frac{3\dot{\beta}P}{a}\right)^{\frac{1}{4}}$$

and from the expression $P = \frac{1}{6}S\Omega^2 H$.Computing the variations of these two equations gives

$$\frac{\delta\dot{\beta}}{\dot{\beta}} = \frac{1-\dot{\beta}}{1+3\dot{\beta}}(7h-u) \tag{4.3.11}$$

Now the linearizion of Eq.(4.3.4) is straightforward. Using the equilibrium conditions (in particular $Q^+ = Q^-$), one finds for the α-model, if only the determinant terms of order $(\frac{H}{\Lambda})^2$ are kept:

$$3(1+3\dot{\beta}+4\dot{\beta}^2)\partial_t u + (8+51\dot{\beta}-3\dot{\beta}^2)\partial_t h - 3(1+3\dot{\beta})\alpha\Omega[(n+1-l)u + (m-k)h]$$
$$= \frac{2}{3}\alpha\Omega H^2(5+18\dot{\beta}+9\dot{\beta}^2)\partial_r^2[(n+1)u+mh]. \tag{4.3.12}$$

For an ionized gas at T> 10^4K the major competitive opacity is electron scattering. Thus if the opacity is dominated by the electron scattering opacity($\chi \sim \frac{\sigma_T}{m_p} \sim 0.4 \quad cm^2/g$), using $P_c = \frac{1}{4}S\Omega^2 H$

$$Q^- = e_s\frac{8m_p c}{\sigma_T}\frac{\dot{\beta}P_c}{S} = 2e_s\frac{m_p c}{\sigma_T}\dot{\beta}H\Omega^2 \tag{4.3.13}$$

Thus

$$\frac{\delta Q^-}{Q^-} = \frac{\delta\dot{\beta}}{\dot{\beta}} + h \tag{4.3.14}$$

Using Eq.(4.3.11) this gives

$$k = \frac{8-4\dot{\beta}}{1+3\dot{\beta}}, \qquad l = \frac{\dot{\beta}-1}{1+3\dot{\beta}} \tag{4.3.15}$$

All perturbations are fully described by Eqs.(4.3.8) and (4.3.12) since all quantities of interest, in particular $\dot{M}$), can be expressed in terms of u and h. We consider harmonic perturbations

$$u(r,t) = u(r)e^{\omega t}, \qquad h(r,t) = h(r)e^{\omega t}$$

and write a single equation for the combination

$$\psi = (n+1)u + mh. \tag{4.3.16}$$

This amplitude describes the viscous perturbations, as can be seen from $Q^+ = \frac{9}{4}\nu S\Omega^2$ Eqs.(4.3.5), and (4.3.6).

From (4.3.8) we get for the α-law

$$u = \frac{2}{3}\alpha\frac{\Omega}{\omega}H^2\partial_r^2\psi. \tag{4.3.17}$$

Using Eqs.(4.3.16),(4.3.17) in Eq.(4.3.12) gives

$$\omega\frac{C\omega - 3(1+3\dot{\beta})\alpha\Omega(m-k)}{D\omega - 3(1+3\dot{\beta})\alpha\Omega[ml - k(n+1)]}\psi = \frac{2}{3}\alpha\Omega H^2\partial_r^2\psi \tag{4.3.18}$$

where $C(\dot{\beta}) = 8 + 51\dot{\beta} - 3\dot{\beta}^2$

and $D(\dot{\beta}) = (N+1)C(\dot{\beta}) + M(2 + 9\dot{\beta} - 3\dot{\beta}^2)$ For solutions proportional to $sin(r/\Lambda)$ we obtain the dissipation relation

$$C\left(\frac{\omega}{3\alpha\Omega}\right)^2 + \left[2D\left(\frac{H}{3\Lambda}\right)^2 - (1+3\dot{\beta})(m-k)\right]\frac{\omega}{3\alpha\Omega} - 2\left(\frac{H}{3\Lambda}\right)^2(1+3\dot{\beta})[ml-(n+1)k] = 0$$

$$-2\left(\frac{H}{3\Lambda}\right)^2(1+3\dot{\beta})[ml-(n+1)k] = 0 \tag{4.3.19}$$

for the special values of Eqs.(4.3.7) and (4.3.15) we obtain

$$\frac{\omega}{\alpha\Omega} = \frac{3}{C}\left[-\left(D\left(\frac{H}{3\Lambda}\right)^2 + (1+3\dot{\beta})\right)\right.$$

$$\left.\pm\left[\left(D\left(\frac{H}{3\Lambda}\right)^2 + (3-5\dot{\beta})\right)^2 - 4C(5-3\dot{\beta})\left(\frac{H}{3\Lambda}\right)^2\right]^{\frac{1}{2}}\right] \tag{4.3.20}$$

with

$$C(\dot{\beta}) = 8 + 51\dot{\beta} - 3\dot{\beta}^2 > 0, \qquad D(\dot{\beta}) = (N+1)C(\dot{\beta}) + M(2 + 9\dot{\beta} - 3\dot{\beta}^2) > 0$$

Obviously, $\mathrm{Re}\langle\omega\rangle < 0$,if $3 - 5\dot{\beta} > 0$. We thus have stability for $\dot{\beta} < \frac{3}{5}$ which is the case if the plasma pressure dominates.

In our case the total pressure is given by Eq.(1.0.6)

$$P = A\rho T - B\left[\frac{\rho^{\frac{3}{2}}}{T}\right]^{\frac{1}{2}}$$

Thus if the disc is stable in the region the condition

$$\beta = \frac{P_g}{P_g + P_{coul}} < 0.4$$

should fulfilled.

Using Eq.(3.1.15) the expression for the central pressure with H given by Eq.(3.1.17) in the case of total pressure and in the case of gas pressure only H is given by Eq.(3.2.4)

$$\frac{P_g}{P_g + P_{coul}} = \frac{\frac{1}{I(N+1)_{gas}} \frac{\dot{M}}{4\pi r^2 \alpha} \left(\frac{H_{gas}}{r}\right)^{-1} \left(\frac{GM}{r}\right)^{\frac{1}{2}} [1 - (\frac{r_0}{r})^{\frac{1}{2}}]}{\frac{1}{I(N+1)_{total}} \frac{\dot{M}}{4\pi r^2 \alpha} \left(\frac{H_{total}}{r}\right)^{-1} \left(\frac{GM}{r}\right)^{\frac{1}{2}} [1 - (\frac{r_0}{r})^{\frac{1}{2}}]}$$

$$\frac{P_g}{P_g + P_{coul}} = \frac{(I(N+1)H)_{total}}{(I(N+1)H)_{gas}}$$

$I(N+1)_{gas} = I(4) = \frac{2^7}{315}$ for outer region

$I(N+1)_{gas} \sim I(4) = \frac{2^7}{315}$for inner region

$I(N+1)_{total} \sim I(1) = \frac{2}{3}$ for both regions

Inserting all the required constants, we determine the approximate values of the half thickness in the two regions.

MIDDLE REGION

Eq.(3.1.21)

$$H_{total} \simeq 3.74 r^{\frac{3}{10}} \left[r^{\frac{1}{2}} - 10^3 \right]^{\frac{3}{10}}$$

and from Eq.(3.2.6) we find

$$H_{gas} \simeq 1.41 x 10^{-3.05} r^{\frac{19}{20}} \left[r^{\frac{1}{2}} - 10^3 \right]^{\frac{1}{5}}$$

Thus

$$\beta(r) \simeq 8.7 x 10^{3.05} r^{-\frac{13}{20}} \left[r^{\frac{1}{2}} - 10^3 \right]^{\frac{1}{10}} \tag{4.3.21}$$

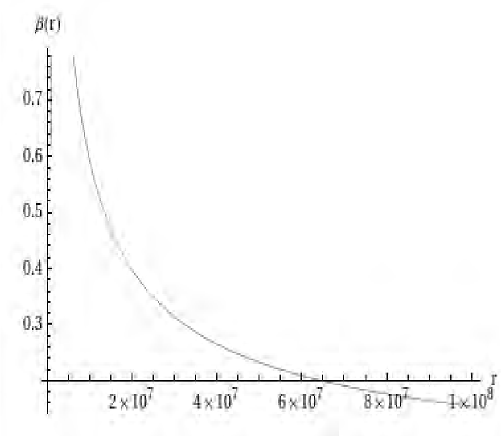

Figure 4.1: The stability parameter for middle region ($r \sim 10^6$ to 10^8)

OUTER REGION

From Eq.(3.1.25)

$$H_{total} \simeq 2.9x10^{-4} r^{0.51} \left[r^{\frac{1}{2}} - 10^3 \right]^{\frac{9}{35}}$$

and for H_{gas} Eq.(3.2.7) gives

$$H_{gas} \simeq 3.75x10^{-1.3} r^{-1.97} \left[r^{\frac{1}{2}} - 10^3 \right]^{\frac{1}{10}}$$

Then

$$\beta(r) \simeq 1.28x10^{-2.7} r^{-1.46} \left[r^{0.5} - 10^3 \right]^{0.36}$$

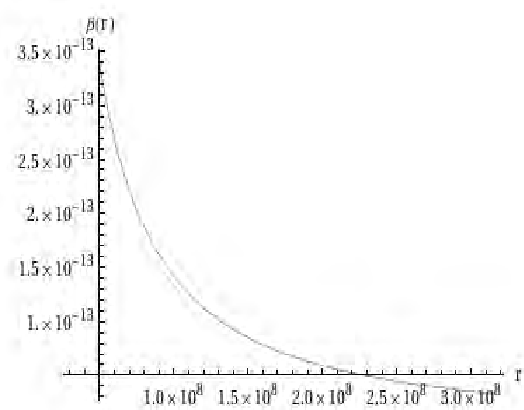

Figure 4.2: The stability parameter for outer region ($r \sim 10^{7.5}$ to $10^{8.5}$)

Chapter 5

CONCLUSION

The stability of gas pressure-dominated accretion disc with the addition of coulombic pressure has been studied with the consideration of only axially symmetric perturbations in the vertical direction. The model we considered is geometrically thin with electron-scattering opacity dominated (middle region) and free-free absorption opacity dominated (outer region) and the energy transport equation is radiation transport dominated. The disc is stable for the middle region for $r \sim 1.5x10^7$ to $r = 6.5x10^7$ and not stable for $r \sim 10^6$ up to 10^7.

Similarly for the case of outer region the disc is stable for $r < 2.4x10^8$. Since the disc is thermally stable as stated in Section 4.1 the instability shown in the graphical solution is mainly caused due to the viscosity. Thus with a set of such assumptions and with the result we obtain, it could be concluded that the disc is thermally stable but viscously unstable.

USED CONSTANTS

Speed of light	$c = 3x10^8$m/s
Constant of gravity	$G = 6.67X10^{-11}m^3/kgs$
Avogadro's number	$N_0 = 6.02x10^{23}part/mol$
Gas constant	$R = 8.31J/Kmol$
Boltzmann constant	$k_\beta = R/N_0 = 1.38X10^{-23}J/K$
Proton mass	$m_p = 1.67x10^{-27}kg$
Electron mass	$m_e = 9.11x10^{-31}kg$
Elementary charge	$e = 9.11x10^{-19}c$
Atomic weight for hydrogen	$A_z = 1.0071$
Mean molecular weight	$\mu = 0.6$
Stefan-Boltzmann constant	$\sigma = 5.67x10^{-5}erg/cm^2SK^4$
Radiation constant	$a = \frac{4\sigma}{c} = 7.57x10^{-16}Jm^{-3}K^{-4}$
Mass of the sun	$M_{\text{ffl}} = 1.99x10^{30}kg$
Mass of the Neutron star	$m = 1.4M_{\text{ffl}}$
Accretion rate	$\dot{M} = 10^{-9}M_{\text{ffl}}/year$
Alfven radius	$r_A = 10^6$
The free-viscous parameter	$\alpha = 0.1$
Electron-scattering Opacity	$\chi_o = 0.04\frac{m^2}{kg}$
Free-free absorption opacity	$\chi_o = 5x10^{23}\frac{m^2}{kg}$

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Declaration

This thesis is my original work, has not been presented for a degree in any other University and that all the sources of material used for the thesis have been dully acknowledged.

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This thesis has been submitted for examination with my approval as University advisor.

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