ADDIS ABABA UNIVERSITY
COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES
DEPARTMENT OF MATHEMATICS

INVERTIBILITY OF SINGLE LAYER POTENTIAL IN 2D

A Thesis Submitted to the Department of Mathematics of Addis Ababa University in partial Fulfillment of the Requirements of the Master of Science Degree in Mathematics

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The undersigned hereby certify that they have read and recommended to the Department of Mathematics for acceptance of this Thesis entitled **"Invertibility of single layer potential in 2D"** by **Getahun Melese** in partial fulfillment of the requirements for degree of Master of Science in Differential Equation.

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Abstract

In this work we investigate the invertibility of single layer potential operator in 2D. The single layer potential operator contains non trivial kernel. Consequently, we need to set conditions on the domain or the space to insure the boundedness and ellipticity of single layer potential operators and hence by Lax-Milgram Theorem the invertibility of single layer potential in 2D.

Notations

Ω**:** Bounded open set in $\mathbb{R}^2$.

$\partial\Omega$**:** The boundary of Ω.

$\bar{\Omega}$**:** The closure of Ω.

$B(x,r)$**:** Ball of radius r about x in $\mathbb{R}^2$.

$\partial B(x,r)$**:** Boundary of ball of radius r about x in $\mathbb{R}^2$.

$C^\infty(\Omega)$**:** The set of all infinitely differentiable function.

$C_0^\infty(\Omega)$**:** The space of functions in $c^\infty(\Omega)$ compactly supported in Ω.

Δ**:** The Laplace's operator.

∇**:** The gradient operator.

$V_\Delta g$**:** The single Layer potential operator with density function g.

$W_\Delta g$**:** The double Layer potential operator with density function g.

$C^k(\Omega)$**:** The space of functions which are bounded and k times continuously differentiable in Ω.

$L_p(\Omega)$**:** The space of all equivalence classes of measurable functions on Ω whose power of order p are integrable.

$L_2(\Omega)$**:** The space of all square integrable functions.

$L_\infty(\Omega)$**:** The space of functions which are measurable bounded almost everywhere.

$L_1^{loc}(\Omega)$**:** The space of locally integrable functions.

$\alpha(n)$**:** The volume of a unit ball in $\mathbb{R}^n$.

$n\alpha(n)$**:** The surface area of a unit ball in $\mathbb{R}^n$.

$ker(T)$**:** The kernel of the operator $T : X \to Y$, $Ker(T) = \{f \in X : Tf = 0\}$.

$div(V)$**:** The divergence of a vector V, $div(V) = \sum \frac{\partial V_i}{\partial x_i}$.

$\| T \|$**:** The operator norm of $T : X \to Y$, $\| T \| = \sup\{\| Tf \|_Y : \| f \|_X \leq 1\}$.

$\| . \|_{L^p(x)}$**:** The norm related to the Lebesgue space $L^p(x)$,$\| f \|_{L^p(x)} = (\int_x |f|^p dx)^{\frac{1}{p}}$.

$|.|$**:** The Euclidean norm, $|x| = \sqrt{x_1^2 + x_2^2 + ... + x_d^2}$.

$\langle ., . \rangle$**:** The Euclidean inner product,$\langle x, y \rangle = \sum x_i y_i$.

$\langle .|. \rangle$**:** The inner product related to $L^p(x)$.

Chapter 1

Introduction

This thesis consists of four chapters dealing with function spaces, Green's identities and Green's representation formula, properties of harmonic function, fundamental solutions of the Laplace's and Poisson's equation, potential theory, solutions of interior Dirichlet and interior Neumann problems using single layer and double layer potential operators, boundedness and ellipticity of single layer potential and finally invertibility of single layer potential in 2D.
In chapter 1, we introduce some selective points from the remaining 3 chapters as follows.

In chapter 2, we discuss some preliminaries which are important to study the Invertibility of Single Layer Potential Operators for constant coefficient in 2D.
Let $\Omega \subset \mathbb{R}^2$ be some open subset and assume $k \in \mathbb{N}_0$. $C^k(\Omega)$ is the space of functions which are bounded and k times continuously differentiable in Ω.
In particular for $u \in C^k(\Omega)$ the norm

$$\| u \| c^k(\Omega) := \sum_{|\alpha| \leq k} \sup_{x \in \Omega} |D^\alpha u(x)|$$

is finite.
Correspondingly, $C^\infty(\Omega)$ is the space of functions which are bounded and infinitely often continuously differentiable.
For $k \in N_0$ and $\kappa \in (0,1)$ we define $C^{k,\kappa}(\Omega)$ to be the space of Hölder

continuous equipped with the norm

$$\| u \|_{c^{k,\kappa}(\Omega)} := \| u \|_{c^k(\Omega)} + \sum_{|\alpha|=k} \sup_{x,y\in\Omega, x\neq y} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\kappa}.$$

The boundary of an open set $\Omega \in \mathbb{R}^2$ is defined as

$$\partial\Omega = \overline{\Omega} \cap (\mathbb{R}^2 \backslash \Omega).$$

By $L_p(\Omega)$ we mean the space of all equivalence classes of measurable functions on Ω whose powers of order p are integrable. For $p = 2$ we have $L_2(\Omega)$ to be the space of all square integrable functions, and Hölder's inequality turns out to be the Cauchy-Schwartz inequality.
In addition to the above points, we discuss about generalized derivatives, Sobolev spaces,and properties of Sobolev spaces.

In Chapter3, we discuss the detail part of the thesis by dividing the chapter in to four sections with their respective subsections.

In the first section, we study the fundamental solution of Laplace's equation in $\mathbb{R}^n$, $\Delta(u) = 0$ for$x \in \mathbb{R}^n$. Given the symmetric nature of Laplace's equation, we look for a radial solution. That is, we look for a harmonic function u on $\mathbb{R}^n$ such that $u(x) = v(|x|)$. In addition, to being a natural choice due to the symmetry of Laplace's equation,radial solutions are natural to look for because they reduce a PDE to an ODE, which is generally easier to solve.

In the second section of this chapter, we are interested in studying Green's identities, Green's function and representation formula. The primary use of Green's function in mathematics is to solve non-homogeneous boundary value problems.

In the third section, we prove a mean value property which all harmonic functions satisfy, and the converse to mean value property of harmonic functions. If u is a harmonic function on a bounded domain Ω in $\mathbb{R}^n$, then u attains its maximum value on the boundary of Ω. Moreover, we also prove that harmonic functions are C^∞. At last, we show that the only functions which are bounded and harmonic on $\mathbb{R}^n$ are constant functions.

In the last section of this chapter, we study properties of layer potentials for Laplace's equation. The reason why we are interested in layer potentials is that they are good candidates for being solutions to the Interior Dirichlet and Interior Neumann problems for Laplace's equation. In addition, they are also harmonic and they obey certain jump relations on the boundary.

In the last chapter, we discuss the main result of this thesis, which is the **Invertibility of Single Layer Potential Operators for constant coefficient in 2D.** Before we investigate the Invertibility of Single Layer Potential Operators, we discuss about the concept of logarithmic capacity, boundedness and ellipticity of single layer potentials. In this way we are attempting to demonstrate methods that can be used in determining the inverse of single layer potentials.
Finally, we investigate the Invertibility of Single Layer Potential Operators for constant coefficient in 2D.

Chapter 2

Preliminaries

2.1 Function Spaces

2.1.1 The Spaces $C^k(\Omega)$, $C^{k,\kappa}(\Omega)$ and $L_p(\Omega)$

If u is a sufficient smooth real valued function, then we can write partial derivatives as:

$$D^\alpha u(x) := (\frac{\partial}{\partial x_1})^{\alpha_1}, ...(\frac{\partial}{\partial x_d})^{\alpha_d} u(x_1, x_2..., x_d)$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d)$, $\alpha_i \in \mathbb{N}_0$, multi index with

$$|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_d.$$

Let $\Omega \in \mathbb{R}^2$ be some open subset and assume $k \in \mathbb{N}_0$. $C^k(\Omega)$ is the space of functions which are bounded and k times continuously differentiable in Ω.
In particular for $u \in C^k(\Omega)$ the norm

$$\| \; u \; \| \; c^k(\Omega) := \sum_{|\alpha| \leq k} \sup_{x \in \Omega} |D^\alpha u(x)|$$

is finite.
Correspondingly, $C^\infty(\Omega)$ is the space of functions which are bounded and infinitely often continuously differentiable.
For a function $u(x)$ defined for $x \in \Omega$ we denote

$$supp(u) \;\; = \;\; \{\overline{x \in \Omega : u(x) \neq 0}\}$$

to be the support of the function u.
Then,

$$C_0^{\infty}(\Omega) = \{u \in c^{\infty}(\Omega) : supp(u) \subset \Omega\}$$

is the space $C^{\infty}(\Omega)$ functions with compact support.
For $k \in \mathbb{N}_0$ and $\kappa \in (0,1)$, we define $C^{k,\kappa}(\Omega)$ to be the space of Hölder continuous equipped with the norm

$$\| u \|_{c^{k,\kappa}(\Omega)} := \| u \|_{c^k(\Omega)} + \sum_{|\alpha|=k} \sup_{x,y \in \Omega, x \neq y} \frac{|D^{\alpha}u(x) - D^{\alpha}u(y)|}{|x-y|^{\kappa}}.$$

In particular, for $\kappa = 1$ we have $C^{k,1}(\Omega)$ to be the space of functions $u \in c^k(\Omega)$ where the derivatives $D^{\alpha}u$ of order $|\alpha| = k$ are Lipschitz continuous.
The boundary of an open set $\Omega \in \mathbb{R}^2$ is defined as

$$\partial\Omega = \overline{\Omega} \cap (\mathbb{R}^2 \backslash \Omega).$$

By $L_p(\Omega)$ we mean the space of all equivalence classes of measurable functions on Ω whose powers of order p are integrable. The associated norm is:

$$\| u \|_{L_p(\Omega)} := \{\int_{\Omega} |u(x)|^p dx\}^{\frac{1}{p}}$$

for $1 \leq p \leq \infty$.
Two elements $u, v \in L_p(\Omega)$ are identified with each other if they are different only on a set K of zero measure i.e $\mu(K) = 0$. In addition, $L_{\infty}(\Omega)$ is the space of functions u which are measurable and bounded almost everywhere with the norm:

$$\| u \|_{L_{\infty}(\Omega)} := ess \sup_{x \in \Omega}\{|u(x)|\} := \inf_{x \subset \Omega, \mu(k)=0} \sup_{x \in \Omega \backslash k} |u(x)|.$$

The spaces$L_p(\Omega)$ are Banach spaces with respect to the norm $\| u \|_{L_p(\Omega)}$.
There holds the Minkowski inequality $U \in L_p(\Omega)$.

$$\boxed{\| u + v \|_{L_p(\Omega)} \leq \| u \|_{L_p(\Omega)} + \| v \|_{L_p(\Omega)}} \tag{2.1}$$

for all $u, v \in L_p(\Omega)$.
For $u \in L_p(\Omega)$ and $v \in L_q(\Omega)$ with adjoint parameters p and q,i.e $\frac{1}{p} + \frac{1}{q} = 1$, we further have Hölder's inequality

$$\boxed{\int_{\Omega} |u(x)v(x)| dx \leq \| u \|_{L_p(\Omega)} \| v \|_{L_q(\Omega)}}. \tag{2.2}$$

Defining the duality pairing

$$\langle u, v\rangle := \int_\Omega u(x)v(x)dx,$$

we obtain

$$\| v \|_{L_q(\Omega)} = \sup_{0 \neq u \in L_p(\Omega)} \frac{|\langle u, v\rangle_\Omega|}{\| u \|}_{L_p(\Omega)}$$

for $1 \leq p < \infty, \frac{1}{p} + \frac{1}{q} = 1$.
For $p = 2$ we have $L_2(\Omega)$ to be the space of all square integrable functions,and Hölder's inequality (2.2) turns out to be the Cauchy-Schwartz inequality:

$$\boxed{\int_\Omega |u(x)v(x)|dx \leq \| u \|_{L_2(\Omega)} \| v \|_{L_2(\Omega)}}. \tag{2.3}$$

Moreover, for $u, v \in L_2(\Omega)$ we can define the inner product

$$\langle u, v\rangle_{L_2(\Omega)} = \int_\Omega u(x)v(x)dx$$

and with

$$\langle u, u\rangle_{L_2(\Omega)} = \| u \|^2_{L_2(\Omega)}$$

for all $u \in L_2(\Omega)$ we conclude that $L_2(\Omega)$ is a Hilbert space.

2.2 Generalized Derivatives and Sobolev Spaces

By $L_1^{loc}(\Omega)$ we denote the space of locally integrable functions, i.e.$u \in L_1^{loc}(\Omega)$ is integrable with respect to any closed bounded subset $K \subset \Omega$.
For functions $\varphi, \psi \in C_0^\infty(\Omega)$ we may apply integration by parts,

$$\int_\Omega \frac{\partial}{\partial xi}\varphi(x)\psi(x_i)dx = -\int_\Omega \varphi(x)\frac{\partial}{\partial x_i}\psi(x)dx.$$

Note that all integrals may be defined for non-smooth functions. This motivates the following definition of a generalized derivative.

Definition 2.2.1. *A function $u \in L_1^{loc}(\Omega)$ has a generalized partial derivative with respect to x_i, if there exists a function $v \in L_1^{loc}(\Omega)$ satisfying*

$$\boxed{\int_\Omega v(x)\varphi(x)dx = -\int_\Omega u(x)\frac{\partial}{\partial x_i}\varphi(x)dx} \tag{2.4}$$

for all $\varphi \in C_0^\infty(\Omega)$.

Again we denote the generalized derivative by $\frac{\partial}{\partial x_i}u(x) := v(x)$. The recursive application of (2.4) enables us to define a generalized partial derivative $D^\alpha u(x) \in L_1^{loc}(\Omega)$ by

$$\boxed{\int_\Omega [D^\alpha u(x)]\varphi(x)dx = (-1)^{|\alpha|} \int_\Omega u(x) D^\alpha \varphi(x)dx} \tag{2.5}$$

for all $\varphi \in C_0^\infty(\Omega)$.
For $k \in \mathbb{N}_0$ we define norms

$$\boxed{\| u \|_{W_P^K(\Omega)} := \{\sum_{|\alpha|\leq k} \| D^\alpha u \|_{L_p(\Omega)}^p\}^{\frac{1}{p}}} \tag{2.6}$$

for $1 \leq p < \infty$.
By taking the closure of $C^\infty(\Omega)$ with respect to the norm $\| \, . \, \|_{W_P^K(\Omega)}$ we define the Sobolev space :

$$\boxed{W_P^K(\Omega) := \overline{C^\infty(\Omega)}_{\|\cdot\|_{W_P^K(\Omega)}}}. \tag{2.7}$$

In particular, for any $u \in W_P^K(\Omega)$ there exists a sequence $\{\varphi_j\}_{j\in N} \subset C^\infty(\Omega)$ such that

$$\lim_{j\to\infty} \| u - \varphi_j \|_{W_P^K(\Omega)} = 0.$$

Correspondingly, the closure of $C_0^\infty(\Omega)$ with respect to $\| \, . \, \|_{W_P^K(\Omega)}$ defines the Sobolev space

$$\boxed{\dot{W_P^K}(\Omega) := \overline{C_0^\infty(\Omega)}_{\|\cdot\|_{W_P^K(\Omega)}}}. \tag{2.8}$$

The definitions of Sobolev norms $\| \, . \, \|_{W_P^K(\Omega)}$ and therefore of the Sobolev spaces (2.7) and (2.8) can be extended for any arbitrary $s \in \mathbb{R}$.
We first consider $0 < s < \mathbb{R}$ with $s = k + \kappa$ and $k \in N_0, \kappa \in (0, 1)$. Then,

$$\| u \|_{W_P^s(\Omega)} := \{\| u \|_{W_P^k(\Omega)}^p + |u|_{W_P^s(\Omega)}^p\}^{\frac{1}{p}}$$

is the Sobolev-Slobodeckii norm, and

$$|u|_{W_P^s(\Omega)}^p = \sum_{|\alpha|\leq k} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{d+pk}} dxdy$$

is the associated semi-norm.
In particular for $p = 2$ we have $W_2^s(\Omega)$ to be a Hilbert space with inner product

$$\langle u, v\rangle_{W_2^k(\Omega)} := \sum_{|\alpha|\leq k} \int_\Omega D^\alpha u(x) D^\alpha v(x) dx$$

for $s = k \in N_0$ and

$$\boxed{\langle u, v\rangle_{W_2^s(\Omega)} := \langle u, v\rangle_{W_2^k(\Omega)} + \sum_{|\alpha|\leq k} \int_\Omega \int_\Omega \frac{(D^\alpha u(x) - D^\alpha u(y))(D^\alpha v(x) - D^\alpha v(y))}{|x-y|^{d+2k}} dxdy} \tag{2.9}$$

for $s = k + \kappa, k \in \mathbb{N}_0, \kappa \in (0, 1)$.

2.2.1 Properties of Sobolev spaces

Assuming a certain relation for the indices $s \in \mathbb{R}$ and $p \in \mathbb{N}$ a function $u \in W_P^s(\Omega)$ turns out to be bounded and continuous.

Theorem 2.2.1 (Imbedding Theorem of Sobolev)**.** *Let $\Omega \in \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let $d \leq s$ for $p = 1, \frac{d}{p} < s$ for $p > 1$. For $u \in W_P^s(\Omega)$ we obtain $u \in C(\Omega)$ satisfying*

$$\| u \|_{L_\infty(\Omega)} \leq c \| u \|_{W_P^s(\Omega)}$$

for all $u \in W_P^s(\Omega)$.

For the proof of these theorem see, for example $[Olaf - Steinbach, Theorem 1.4.6]$. The norm (2.6) of Sobolev space $W_2^1(\Omega)$ is

$$\| v \|_{W_2^1(\Omega)} = \{\| v \|_{L_2(\Omega)} + \| \nabla v \|^2_{L_2(\Omega)}\}^{\frac{1}{2}}$$

where

$$|v|_{W_2^1(\Omega)} = \| \nabla v \|_{L_2(\Omega)}$$

is a semi-norm. Applying the following theorem we may deduce equivalent norms in $W_2^1(\Omega)$.

Theorem 2.2.2 (Norm Equivalence Theorem of Sobolev)**.** *Let $f : W_2^1(\Omega) \to \mathbb{R}$ be a bounded linear functional satisfying $0 \leq |f(v)| \leq Cf \| v \|_{W_2^1(\Omega)}$ for all $v \in W_2^1(\Omega)$.*
If $f(constant) = 0$ is only satisfied for $constant = 0$, then

$$\boxed{\| v \|_{W_2^1(\Omega)}, f := \{|f(v)|^2 + \| \nabla v \|^2_{L_2(\Omega)}\}^{\frac{1}{2}}} \tag{2.10}$$

defines an equivalence norm in $W_2^1(\Omega)$.

2.3 Fundamental Solutions of Linear Differential Operators

The Fourier transform is applied to construct the fundamental solutions of linear differential operators having constant coefficients. Naturally, only fundamental solutions of slow growth can be obtained by this method.

Generalized Solutions of Linear Differential Operators

Let

$$\boxed{\sum_{|\alpha|=0}^{m} a_\alpha(x) D^\alpha u = f(x), f \in D'} \tag{2.11}$$

be a linear differential equation of order m with coefficients $a_\alpha(x) \in C^\infty(\mathbb{R}^n)$. Introducing the differential operator

$$L(x, D) = \sum_{|\alpha|=0}^{m} a_\alpha(x) D^\alpha, D = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n})$$

We shall rewrite this equation in the form

$$\boxed{L(x, D)u = f(x)} \tag{2.12}$$

Each generalized function $u \in D'$ which satisfies this equation in the region Ω in a generalized sense, that is for any $\varphi \in D$, $supp\varphi \subset \Omega$

$$\boxed{(L(x, D)u, \varphi) = (f, \varphi)} \tag{2.13}$$

is known as the generalized solution of Eq.(2.11) in the region Ω.
Equation (2.13)is equal in effect to the equation

$$\boxed{(u, L^*(x, D)\varphi) = (f, \varphi), \varphi \in D(\Omega)} \tag{2.14}$$

where

$$\boxed{L^*(x, D)\varphi = \sum_{|\alpha|=0}^{m} (-1)^{|\alpha|} D^\alpha (a_\alpha \varphi)} \tag{2.15}$$

In fact,

$$\begin{aligned}(L(x,D)u,\varphi) &= (\sum_{|\alpha|=0}^{m} a_\alpha D^\alpha u,\varphi)\\ &= \sum_{|\alpha|=0}^{m} (a_\alpha D^\alpha u,\varphi)\\ &= \sum_{|\alpha|=0}^{m} (D^\alpha u, a_\alpha\varphi)\\ &= \sum_{|\alpha|=0}^{m} (-1)^{|\alpha|}(u, D^\alpha(a_\alpha\varphi))\\ &= (u, \sum_{|\alpha|=0}^{m} (-1)^{|\alpha|} D^\alpha(a_\alpha\varphi))\\ &= (u, L^*(x,D)\varphi)\end{aligned}$$

It is clear that every classical solution is also a generalized solution. We shall formulate the converse result as the following lemma.

Lemma 2.3.1. *If the generalized solution $u(x)$ of Eq.(2.11) in the region Ω belongs to the class $C^m(\Omega)$ and $f \in C(\Omega)$, then it is also the classical solution of this equation in the region Ω.*

Proof. Since $u \in D' \cap C^m(\Omega)$, the classical and generalized derivatives of the function u up to and including the order m coincide in the region Ω. Since u is the generalized solution of Eq. (2.11) in the region Ω, then the function $L(X,D)u = f$ which is continuous in Ω in the sense of generalized functions. According to DU Bios Reymond's lemma, $L(X,D)u(x) - f(x) = 0$ at all points of the region Ω , so that u satisfies Eq.(2.11) in the region Ω in the classical sense. The lemma is proved. □

2.4 Fundamental Solution

Let L be an operator with constant coefficients $a_\alpha(x) = a_\alpha$:

$$\boxed{L(D) = \textstyle\sum_{|\alpha|=0} a_\alpha D_\alpha, L*(D) = L(-D).} \tag{2.16}$$

The generalized function $\Phi \in D'$ which satisfies equation

$$\boxed{L(D)\Phi = \delta(x)} \tag{2.17}$$

in $\mathbb{R}^n$ is said to be the fundamental solution (the function of influence) of the differential operator $L(D)$.
The fundamental solution $\Phi(x)$ of the operator $L(D)$, generally speaking, is not unique; it is defined accurately as far as the term Φ_0, which is an arbitrary solution of the homogeneous equation $L(D)\Phi_0 = 0$
In fact, the generalized function $\Phi(x) + \Phi_0(x)$ is also a fundamental solution of the operator $L(D)$,

$$L(D)(\Phi + \Phi_0) = L(D)\Phi + L(D)\Phi_0 = \delta(x).$$

Chapter 3

Potential Theory

3.1 Laplace's Equation

In mathematics, Laplace's equation is a second order partial differential equation named after Pierre-Simon Laplace who first studied its properties. This is often written as: $\Delta u = 0$ or $\nabla^2 u = 0$ where $\Delta = \nabla^2$ is the Laplace operator and u is a scalar function. Laplace's equation and poisson's equation are the simplest examples of elliptic partial differential equations.

The general theory of solutions to Laplace's equation is known as **potential theory**. The solutions of Laplace's equation are the harmonic functions which are important in many fields of science, notably the fields of electromagnetism, astronomy, and fluid dynamics because they can be used to accurately describe the behaviour of electric, gravitational, and fluid potentials.

3.1.1 The Fundamental Solution

Consider Laplace's equation in $\mathbb{R}^n$,

$$\Delta u = 0, x \in \mathbb{R}^n.$$

Given the symmetric nature of Laplace's equation, we look for a radial solution. That is, we look for a harmonic function u on $\mathbb{R}^n$ such that $u(x) = v(|x|)$. In addition, to being a natural choice due to the symmetry of Laplace's equation, radial solutions are natural to look for because

they reduce a PDE to an ODE, which is generally easier to solve. Therefore, we look for a radial solution.
If $u(x) = v(|x|)$, then

$$u_{x_i} = \frac{x_i}{|x|} v'(|x|), |x| \neq 0,$$

$$u_{x_i x_i} = \frac{1}{|x|} v'(|x|) - \frac{x_i^2}{|x|^3} v'(|x|) + \frac{x_i^2}{|x|^2} v''(|x|), |x| \neq 0,$$

Therefore,

$$\Delta u = \frac{n-1}{|x|} v'(|x|) + v''(|x|).$$

Letting $r = |x|$, we see that $u(x) = v(|x|)$ is a radial solution of Laplace's equation implies v satisfies

$$\frac{n-1}{r} v'(r) + v''(r) = 0.$$

Therefore,

$$v'' = \frac{1-n}{r} v'$$
$$\implies \frac{v''}{v'} = \frac{1-n}{r}$$
$$\implies \ln v' = (1-n) \ln r + c$$
$$\implies v'(r) = \frac{C}{r^{n-1}},$$

which implies

$$v(r) = \begin{cases} c_1 \ln r + c_2 & n = 2 \\ \frac{c_1}{(2-n) r^{n-2}} + c_2 & n \geq 3. \end{cases}$$

From the calculations, we see that for any constants c_1,c_2, the function

$$\boxed{u(x) \equiv \begin{cases} c_1 \ln(|x|) + c_2 & n = 2 \\ \frac{c_1}{(2-n)|x|^{n-2}} + c_2 & n \geq 3. \end{cases}} \tag{3.1}$$

For $x \in \mathbb{R}^n$, $|x| \neq 0$ is a solution of Laplace's equation in $\mathbb{R}^n - 0$. We notice that the function u defined in (3.1)satisfies $\Delta u(x) = 0$ for $x \neq 0$, but at $x = 0$, $\Delta u(0)$ is undefined. We claim that we can choose constants c_1 and c_2 appropriately so that

$$-\Delta_x u = \delta_0$$

in the sense of distributions. Recall that δ_0 is the distribution which is defined as follows. For all $\phi \in D$,

$$(\delta_0, \phi) = \phi(0).$$

Below, we will prove this claim. For now, though, assume we can prove this. That is, assume we can find constants c_1, c_2 such that u defined in (3.1)satisfies

$$\boxed{-\Delta_x u = \delta_0}. \tag{3.2}$$

Let Φ denote the solution of (3.2). Then, define

$$v(x) = \int_{\mathbb{R}^n} \Phi(x,y) f(y)dy.$$

Formally, we compute the Laplacian of v as follows,

$$\begin{aligned} -\Delta_x v &= -\int_{\mathbb{R}^n} \Delta_x \Phi(x,y) f(y)dy \\ &= -\int_{\mathbb{R}^n} \Delta_y \Phi(x,y) f(y)dy \\ &= \int_{\mathbb{R}^n} \delta_x f(y)dy = f(x). \end{aligned}$$

That is, v is a solution of Poisson's equation! Of course, this set of equalities above is entirely formal. We have not proven anything yet. However, we have motivated a solution formula for poisson's equation to(3.2). We now return to using the radial solution(3.1) to find a solution of (3.2).
Define the function Φ as follows. For $|x| \neq 0$, let

$$\boxed{\Phi(x) = \begin{cases} -\frac{1}{2\pi} \ln(|x|) & n = 2 \\ \frac{1}{(n(n-2)\alpha(n))} \frac{1}{|x|^{n-2}} & n \geq 3 \end{cases}} \tag{3.3}$$

where $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$. We see that Φ satisfies Laplace's equation on $\mathbb{R}^n - 0$. As we will show in the following claim, Φ satisfies $-\Delta_x \Phi = \delta_0$.
For this reason, we call Φ the **fundamental solution** of Laplace's equation.
Claim 1. For Φ defined in (3.3), Φ satisfies

$$-\Delta_x \Phi = \delta_0$$

in the sense of distributions. That is, for all $g \in D$,

$$-\int_{\mathbb{R}^n} \Phi(x)\Delta_x g(x)dx = g(0).$$

Proof. Let F_Φ be the distributions associated with the fundamental solution Φ. That is, let$F_\Phi : D \to \mathbb{R}$ be defined such that

$$(F_\Phi, g) = \int_{\mathbb{R}^n} \Phi(x)g(x)dx$$

for all $g \in D$. Recall that the derivative of a distribution F is defined as the distribution G such that

$$(G, g) = -(F, g')$$

for all $g \in D$. Therefore, the distributional Laplacian of Φ is defined as the distribution$F_\Delta\Phi$ such that

$$(F_\Delta\Phi, g) = (F_\Phi, \Delta g)$$

for all $g \in D$. We will show that

$$(F_\Delta\Phi, g) = -(\delta_0, g) = -g(0),$$

and, therefore,

$$(F_\Phi, g) = -g(0),$$

which means $-\Delta_x\Phi = \delta_0$ in the sense of distributions.
By definition,

$$(F_\Phi, \Delta g) = \int_{\mathbb{R}^n} \Phi(x)\Delta g(x)dx$$

Now we would like to apply the divergence theorem, but Φ has a singularity at $x = 0$. We get around this, by breaking up the integral into two pieces: one piece consisting of the ball of radius δ about the origin, $B(0, \delta)$ and the other piece consisting of the complement of this ball in $\mathbb{R}^n$. Therefore, we have

$$\begin{aligned}
(F_\Phi, \Delta g) &= \int_{\mathbb{R}^n} \Phi(x)\Delta g(x)dx \\
&= \int_{B(0,\delta)} \Phi(x)\Delta g(x)dx + \int_{\Re^n - B(0,\delta)} \Phi(x)\Delta g(x)dx \\
&= I + J.
\end{aligned}$$

We look first at term I. For $n = 2$, term I is bounded as follows,

$$\begin{aligned}
|-\int_{B(0,\delta)} \frac{1}{2\pi} \ln|x|\Delta g(x)dx| &\leq C|\Delta g|L^{\infty}| \int_{B(0,\delta)} \ln|x|dx| \\
&\leq C|\int_0^{2\pi}\int_0^{\delta} \ln|r|rdrd\theta| \\
&\leq C|\int_0^{\delta} \ln|r|rdr| \\
&\leq C\ln|\delta|\delta^2.
\end{aligned}$$

For $n \geq 3$, term I is bounded as follows,

$$\begin{aligned}
|\int_{B(0,\delta)} \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}}\Delta g(x)dx| &\leq C|\Delta g|L^{\infty} \int_{B(0,\delta)} \frac{1}{|x|^{n-2}}dx \\
&\leq C\int_0^{\delta}(\int_{\partial B(0,r)} \frac{1}{|y|^{n-2}}dS(y))dr \\
&= \int_0^{\delta} \frac{1}{r^{n-2}}(\int_{\partial B(0,r)} dS(y))dr \\
&= \int_0^{\delta} \frac{1}{r^{n-2}} n\alpha(n)r^{n-1}dr \\
&= n\alpha(n)\int_0^{\delta} rdr = \frac{n\alpha(n)}{2}\delta^2.
\end{aligned}$$

Therefore, as $\delta \to 0^+$,$|I| \to 0$.
Next, we look at term J. Applying the divergence theorem, we have

$$\begin{aligned}
\int_{\mathbb{R}^n - B(0,\delta)} \Phi(x)\Delta_x g(x)dx &= \int_{(\mathbb{R}^n - B(0,\delta))} \Phi(x)\Delta_x g(x)dx - \int_{\partial(\mathbb{R}^n - B(0,\delta))} \frac{\partial\Phi}{\partial n} g(x)dS(x) \\
&+ \int_{\partial(\mathbb{R}^n - B(0,\delta))} \frac{\partial\Phi}{\partial n} g(x)dS(x) \\
&= -\int_{\partial(\mathbb{R}^n - B(0,\delta))} \frac{\partial\Phi}{\partial n} g(x)dS(x) + \int_{\partial(\mathbb{R}^n - B(0,\delta))} \Phi(x)\frac{\partial\Phi}{\partial n} g(x)dS(x) \\
&= J_1 + J_2.
\end{aligned}$$

Using the fact that $\Delta_x\Phi(x) = 0$ for $\in \mathbb{R}^n - B(0,\delta)$.

We first look at term J_1. Now, by assumption, $g \in D$, and, therefore, g vanishes at ∞. Consequently, we only need to calculate the integral over $\partial B(0,\epsilon)$ where the normal derivative n is the outer normal to $\mathbb{R}^n - B(0,\delta)$. By a straightforward calculation, we see that

$$\nabla_x \Phi(x) = \frac{x}{n\alpha(n)|x|^n}.$$

The outer unit normal to $\mathbb{R}^n - B(0,\delta)$ on $B(0,\delta)$ is given by

$$n(x) = \frac{x}{|x|}.$$

Therefore, the normal derivative of Φ on $B(0,\delta)$ is given by

$$\frac{\partial \Phi}{\partial n} = (-\frac{x}{n\alpha(n)|x|^n}).(-\frac{x}{|x|}) = \frac{1}{n\alpha(n)|x|^{n-1}}.$$

Therefore, J_1 can be written as

$$-\int_{\partial B(0,\delta)} \frac{1}{n\alpha(n)|x|^{n-1}} g(x)dS(x) = -\frac{1}{n\alpha(n)\delta^{n-1}} \int_{\partial B(0,\delta)} g(x)dS(x) = -⨍_{\partial B(0,\delta)} g(x)dS(x).$$

Now if g is a continuous function, then

$$-⨍_{\partial B(0,\delta)} g(x)dS(x) \to -g(0) as \delta \to 0.$$

Lastly, we look at term J_2. Now using the fact that g vanishes as $|x| \to +\infty$, we only need to integrate over $\partial B(0,\delta)$ Using the fact that $g \in D$, and, therefore, infinitely differentiable, we have

$$\begin{aligned} |\int_{\partial B(0,\delta)} \Phi(x)\frac{\partial g}{\partial n}| &\leq |\frac{\partial g}{\partial n}|_{L^\infty \partial(B(0,\delta))} \int_{\partial B(0,\delta)} |\Phi(x)|dS(x) \\ &\leq C \int_{\partial B(0,\delta)} |\Phi(x)|dS(x). \end{aligned}$$

Now first, for $n = 2$,

$$\begin{aligned} \int_{\partial B(0,\delta)} |\Phi(x)|dS(x) &= C \int_{\partial B(0,\delta)} |\ln|x||dS(x) \\ &\leq C|\ln|\delta|| \int_{\partial B(0,\delta)} dS(x) \\ &= C|\ln|\delta||(2\pi\delta) \leq C_1 \delta |\ln|\delta||. \end{aligned}$$

Next, for $n \geq 3$,

$$\begin{aligned}\int_{\partial B(0,\delta)} |\Phi(x)| dS(x) &= C \int_{\partial B(0,\delta)} \frac{1}{|x|^{n-2}} dS(x) \\ &\leq \frac{C}{\delta^{n-2}} \int_{\partial B(0,\delta)} dS(x) \\ &= \frac{C}{\delta^{n-2}} n\alpha(n)\delta^{n-1} \leq C_2\delta.\end{aligned}$$

Therefore, we conclude that the term $J2$ is bounded in absolute value by

$$\begin{array}{cc} C\delta|\ln\delta| & n=2 \\ C\delta & n\geq 3 \end{array}.$$

Therefore, $J2 \to 0 as \delta \to 0^+$.
Combining these estimates, we see that

$$\int_{\mathbb{R}^n} \Phi(x)\Delta_x g(x)dx = \lim_{\delta\to 0^+} I + J1 + J2 = -g(0).$$

Therefore, our claim is proved. □

3.2 Green's Identities and Green's Function

In this section, we are interested in studying Green's identities and Green's function.

3.2.1 Green's Identities

The Gauss-Green Theorem

Theorem 3.2.1.

$$\boxed{\int_\Omega \frac{\partial(fg)}{\partial x} d\Omega = \int_\Omega \frac{\partial f}{\partial x} g d\Omega + \int_\Omega f\frac{\partial g}{\partial x} d\Omega = \int_{\partial\Omega} fgn_x dS.} \tag{3.4}$$

$$\boxed{\int_\Omega \frac{\partial(fg)}{\partial y} d\Omega = \int_\Omega \frac{\partial f}{\partial y} g d\Omega + \int_\Omega f\frac{\partial g}{\partial y} d\Omega = \int_{\partial\Omega} fgn_y dS.} \tag{3.5}$$

Equations(3.4) and (3.5) state the integration by parts in two dimensions and are known as the Gauss-Green Theorem.

The Divergence Theorem of Gauss

Theorem 3.2.2. *The divergence theorem results readily as an application of the Gauss-Green Theorem. Consider the vector field* $u = u_i + v_j$*,where* i *and* j *are the unit vectors along the* x *and the* y *axes, respectively. While* $u = u(x, y)$ *and* $v = v(x, y)$ *denotes its components.*

$$\boxed{\int_\Omega (\nabla .u) d\Omega = \int_{\partial\Omega} (u.n) dS.} \tag{3.6}$$

The quantity $\nabla .u$ *is referred to as the divergence of a vector* u *at a point inside the domain* Ω*, whereas the quantity* $u.n$ *is referred to as the flux of the vector field directed in the* n *direction at a single point on the boundary* Ω*. Equation (3.6) is known as the divergence theorem of Gauss, and relates the total divergence to the total flux of a vector field.*

Green's First Identity

Theorem 3.2.3. *Let* $u, v \in C^2(\overline{\Omega})$*. Then we have*

$$\int_\Omega u\Delta v dx = -\int_\Omega \nabla u.\nabla v dx + \int_{\partial\Omega} (u\frac{\partial v}{\partial n}) ds$$

Proof.

$$\begin{aligned} \nabla(u\nabla v) &= \nabla((uv_{x_1}, ...uv_{x_n})) \\ &= \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(u\frac{\partial v}{\partial x_i}) \\ &= \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}.\frac{\partial v}{\partial x_i} + \sum_{i=1}^{n} u\frac{\partial^2 v}{\partial x_i^2} \\ &= \nabla u \nabla v + u\Delta v \end{aligned}$$

Integrating with respect to dx on Ω,

$$\begin{aligned} \int_\Omega \nabla u.\nabla v dx + \int_\Omega u\Delta v dx &= \int_\Omega \nabla(u\nabla v) dx \\ &= \int_{\partial\Omega} u\nabla v.ndS \\ &= \int_{\partial\Omega} u\frac{\partial v}{\partial n} dS. \end{aligned}$$

Therefore,

$$\int_{\Omega} u\Delta v dx = -\int_{\Omega} \nabla u . \nabla v dx + \int_{\partial\Omega} u\frac{\partial v}{\partial n} dS$$

□

Green's Second Identity

Theorem 3.2.4. *Let $u, v \in C^2(\overline{\Omega})$. Then we have*

$$\int_{\Omega} (v\Delta u - u\Delta v) d\Omega = \int_{\partial\Omega} (v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}) dS.$$

Proof. Consider the functions $u = u(x, y)$ and $v = v(x, y)$ which are twice continuously differentiable in Ω and once in $\partial\Omega$. Applying (3.4) for $g = v$, $f = \frac{\partial u}{\partial x}$ and also (3.5) for $g = v$, $\frac{\partial u}{\partial y}$, and finally adding the resulting equations, results

$$\boxed{\int_{\Omega} v(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) d\Omega = -\int_{\Omega} (\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}) d\Omega + \int_{\partial\Omega} v(\frac{\partial u}{\partial x} n_x + \frac{\partial u}{\partial y} n_y) dS} \quad (3.7)$$

Similarly, applying (3.4)for $g = u$, $f = \frac{\partial v}{\partial x}$ and (3.5) for $g = u$ $f = \frac{\partial v}{\partial y}$, and finally adding the resulting equations

$$\boxed{\int_{\Omega} u(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) d\Omega = -\int_{\Omega} (\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}) d\Omega + \int_{\partial\Omega} u(\frac{\partial v}{\partial x} n_x + \frac{\partial v}{\partial y} n_y) dS.} \quad (3.8)$$

Subtracting(3.8)from (3.7)yields

$$\boxed{\int_{\Omega} (v\Delta u - u\Delta v) d\Omega = \int_{\partial\Omega} (v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}) dS} \quad (3.9)$$

where the Laplacian Δ is defined as

$$\Delta = \nabla^2 = \nabla . \nabla = (i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}).(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

while

$$\frac{\partial}{\partial n} = n.\nabla = (n_x i + n_y j).(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}) = n_x\frac{\partial}{\partial_y} + n_y\frac{\partial}{\partial y},$$

is the operator that produces the derivative of a scalar function in the direction of n. Equation (3.9)is known as Green's second identity for Δ. □

Representation Formula

Theorem 3.2.5. *If $u \in c^2(\Omega)$ is a solution of*

$$\begin{cases} -\Delta u = f & x \in \Omega \subset \mathbb{R}^n \\ u = g & \in \partial\Omega, \end{cases}$$

where f and g are continuous, then

$$\boxed{u(x) = -\int_{\partial\Omega} g(y)\frac{\partial G}{\partial n}(x,y)dS(y) + \int_{\Omega} f(y)G(x,y)dy} \qquad (3.10)$$

for $x \in \Omega$, where $G(x,y)$ is the Green's function for Ω.

corollary 3.2.1. *If u is harmonic in Ω and $u = g$ on $\partial\Omega$, then*

$$u(x) = -\int_{\partial\Omega} g(y)\frac{\partial G}{\partial n}(x,y)dS(y).$$

3.2.2 Green's Function

Motivation on Green's Function

Green's functions are named after the British mathematician George Green, who first developed the concept in the 1830's. In the modern study of linear partial differential equations, Green's functions are studied largely from the point of view of fundamental solutions instead.

Definition and Uses

The primary use of Green's functions in mathematics is to solve non-homogeneous boundary value problems.

Definition 3.2.1. *Let x_0 be an interior point of Ω. The Green's function $G(x, x_0)$ for the operator Δ and the domain Ω is a function defined for $x \in \Omega$ such that:*
(i) Let $K(x, x_0) = \frac{-1}{(4\pi|x-x_0|)}$. The function $H(x) = G(x, x_0) - K(x, x_0)$ has continuous second derivatives and is harmonic in Ω (including the point x_0).
(ii) $G(x, x_0) = 0$ for $x \in \partial\Omega$.

Theorem 3.2.6. *If $G(x, x_0) = 0$ is the Green's function, then the solution of the Dirichlet problem is given by the formula*

$$u(x_0) = \int_{\partial\Omega} u(x)\frac{\partial G(x, x_0)}{\partial n} dS.$$

Proof. Let us recall that the Green's representation formula is

$$u(x_0) = \int_{\partial\Omega} [u(x)\frac{\partial}{\partial n}K(x, x_0) - K(x, x_0)\frac{\partial}{\partial n}u(x)]dS.$$

The result of applying Green's second identity to the pair of harmonic functions U and H is

$$\int_{\partial\Omega} [u(x)\frac{\partial}{\partial n}H(x) - H(x)\frac{\partial}{\partial n}u(x)]dS = 0.$$

Adding the above two equations, yields

$$\int_{\partial\Omega} [u(x)\frac{\partial}{\partial n}G(x, x_0) - G(x, x_0)\frac{\partial}{\partial n}u(x)]dS = \int_{\partial\Omega} u(x)\frac{\partial G(x, x_0)}{\partial n} dS.$$

Therefore,

$$u(x_0) = \int_{\partial\Omega} u(x)\frac{\partial G(x, x_0)}{\partial n} dS.$$

□

3.3 Properties of Harmonic Functions

3.3.1 Mean value property

In this section, we prove a mean value property which all harmonic functions satisfy. First, we give some definitions. Let For a function u defined on $B(x, r)$, the average of u on $B(x, r)$ is given by

$$\fint_{\partial B(x,r)} u(y)ds(y) = \frac{1}{\alpha(n)r^n}\int_{B(x,r)} u(y)dy.$$

For a function u defined on $\partial B(x, r)$, the average of u on$\partial B(x, r)$ is given by

$$\fint_{\partial B(x,r)} u(y)ds(y) = \frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B(x,r)} u(y)ds(y).$$

Theorem 3.3.1 (Mean-Value Formulas)**.** *Let* $\Omega \subset \mathbb{R}^n$. *If* $u \in C^2(\Omega)$ *is harmonic, then*

$$u(x) = \fint_{\partial B(x,r)} u(y)dS(y) = \fint_{B(x,r)} u(y)dy$$

for every ball $B(x,r) \subset (\Omega)$.

Proof. Assume $u \in C^2(\Omega)$ is harmonic . For $r > 0$, define

$$\phi(r) = \fint_{\partial B(x,r)} u(y)dS(y).$$

For $r = 0$, define $\phi(r) = u(x)$. Notice that if u is a smooth function, then $\lim_{r\to 0^+} \phi(r) = u(x)$, and, therefore, ϕ is continuous function. Therefore, if we can show that $\phi'(r) = 0$, then we can conclude that ϕ is a constant function, and, therefore,

$$u(x) = \fint_{\partial B(x,r)} u(y)dS(y).$$

We proof $\phi(r) = 0$ as follows. First, making a change of variables, we have

$$\phi(r) = \fint_{\partial B(x,r)} u(y)dS(y) \quad = \quad \fint_{\partial B(0,1) u(x+rz)} dS(z).$$

Therefore,

$$\begin{aligned}
\phi'(r) &= \fint_{\partial B(0,1)} \nabla u(x+rz).zdS(z) \\
&= \fint_{\partial B(x,r)} \nabla u(y).\frac{y-x}{r} dS(y) \\
&= \fint_{\partial B(x,r)} \frac{\partial u}{\partial v}(y)dS(y) \\
&= \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} \frac{\partial u}{\partial v}(y)dS(y) \\
&= \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \nabla.(\nabla u)dy(by the Divergence Theorem) \\
&= \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u(y)dy = 0,
\end{aligned}$$

using the fact that u is harmonic. Therefore, we have proven the first part of the theorem. It remains to prove that

$$u(x) = \fint_{\beta(x,r)} u(y)dy.$$

We do so as follows, using the first result,

$$\begin{aligned}
\int_{B(x,r)} u(y)dy &= \int_0^r (\int_{\partial\beta(x,s)} u(y)dS(y))ds \\
&= \int_0^r (n\alpha(n)s^{n-1} \fint_{\partial\beta(x,s)} u(y)dS(y))ds \\
&= \int_0^r n\alpha(n)s^{n-1}u(x)ds \\
&= n\alpha(n)u(x)s^n|_{s=0}^{s=1} \\
&= \alpha(n)u(x)r^n.
\end{aligned}$$

Therefore,

$$\int_{B(x,r)} u(y)dy = \alpha(n)r^n u(x),$$

which implies

$$u(x) = \frac{1}{\alpha(n)r^n} \int_{B(x,r)} u(y)dy = \fint_{B(x,r)} u(y)dy,$$

as claimed. □

3.3.2 Converse to Mean Value Property

In this section, we prove that if a smooth function u satisfies the mean value property described above, then u must be harmonic.

Theorem 3.3.2. *If $u \in C^2(\Omega)$ satisfies*

$$u(x) = \fint_{\partial B(x,r)} u(y)dS(y)$$

for all $B(x,r) \subset \Omega$, then u is harmonic.

Proof. Let

$$\phi(r) = \fint_{\partial B(x,r)} u(y)dS(y).$$

If

$$u(x) = \fint_{\partial B(x,r)} u(y)dS(y)$$

for all $B(x,r) \subset \Omega$, then $\phi'(r) = 0$. As described in the previous theorem ,

$$\phi'(r) = \frac{r}{n} \fint_{B(x,r)} \Delta u(y)dy.$$

Suppose u is not harmonic. Then there exists some ball $B(x,r) \subset \Omega$ such that $\Delta u > 0$ or $\Delta u < 0$.
Without loss of generality, we assume there is some ball $B(x,r)$ such that $\Delta u > 0$. Therefore,

$$\phi'(r) = \frac{r}{n} \fint_{B(x,r)} \Delta u(y)dy > 0,$$

which contradicts the fact that $\phi'(r) = 0$. Therefore, u must be harmonic. □

3.3.3 Maximum Principle

In this section, we prove that if u is a harmonic function on a bounded domain Ω in $\mathbb{R}^n$, then u attains its maximum value on the boundary of Ω.

Theorem 3.3.3. *Suppose $\Omega \subset \mathbb{R}^n$ is open and bounded. Suppose $u \in C^2(\Omega) \cap C(\Omega)$ is harmonic. Then*

1. *(Maximum principle)*

$$\max u(x)_{\overline{\Omega}} = \max u(x)_{\partial(\Omega)}.$$

2. *(Strong maximum principle) If Ω is connected and there exists a point $x_0 \in \Omega$ such that*

$$u(x_0) = \max u(x)_{\overline{\Omega}}.$$

then u is constant within Ω.

Proof. We prove the second assertion. The first follows from the second. Suppose there exists a point x_0 in Ω such that

$$u(x_0) = M = \max u(x)_{\overline{\Omega}}.$$

Then for $0 < r < dist(x_0, \partial\Omega)$, the mean value property says

$$M = u(x_0) = \fint_{B(x_0,r)} u(y)dy \leq M.$$

But, therefore,

$$\fint_{B(x_0,r)} u(y)dy = M,$$

and $M \equiv \max_{\overline{\Omega}} u(x)$. Therefore, $u(y) \equiv M$ for $y \in B(x_0, r)$. To prove $u \equiv M$ throughout Ω, we continue with this argument, filling Ω with balls. □

Remark. *By replacing u by -u above, we can prove the Minimum principle.*

Next, we use the maximum principle to prove uniqueness of solutions to poisson's equation on bounded domains Ω in $\mathbb{R}^n$.

Theorem 3.3.4 (uniqueness)**.** *There exists at most one solution* $u \in C^2(\Omega) \cap C(\overline{\Omega})$ *of the boundary-value problem,*

$$\begin{cases} -\Delta u = f & x \in \Omega \\ u = g & x \in \partial\Omega. \end{cases}$$

Proof. Suppose there are two solutions u and v. Let $w = u - v$ and $\tilde{w} = v - u$.Then w and $\tilde{w}$ satisfy

$$\begin{cases} \Delta w = 0 & x \in \Omega \\ w = 0 & x \in \partial\Omega. \end{cases}$$

Therefore, using the maximum principal, we conclude

$$\max_{\overline{\Omega}} |u - v| = \max_{\partial\Omega} |u - v| = 0$$

which implies $u - v = 0$ and hence the uniqueness follows. □

3.3.4 Smoothness of Harmonic Functions

In this section, we prove that harmonic functions are C^∞.

Theorem 3.3.5. *Let Ω be an open, bounded subset of $\mathbb{R}^n$. If $u \in C(\Omega)$ and u satisfies the mean value property,*

$$u(x) = \fint_{\partial B(x,r)} u(y)dS(y)$$

for every ball $B(x,r) \subset \Omega$, then $u \in C^\infty(\Omega)$.

Remark.

1. *As proven earlier, if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and u is harmonic, then u satisfies the mean value property, and, therefore, $u \in C^\infty(\Omega)$.*
2. *In fact, if u satisfies the hypothesis of the above theorem, then u is analytic, but we will not prove that here. (See Evans.)*

Proof. First, we introduce the function η such that

$$\eta(x) \equiv \begin{cases} Ce^{\frac{1}{|x|^2-1}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

where the constant C is chosen such that $\int_{\mathbb{R}^n} \eta(x)dx = 1$. Notice that $\eta \epsilon C^\infty(\mathbb{R}^n)$ and η has compact support. Now define the function $\eta(x)$ such that

$$\eta(x) \equiv \frac{1}{\epsilon^n}\eta(\frac{x}{\epsilon}).$$

Therefore, $\eta_\epsilon C^\infty(\mathbb{R}^n)$ and$supp(\eta_\epsilon) \subset x : |x| < \epsilon$. Further,

$$\int_{\mathbb{R}^n} \eta_\epsilon(x)dx = 1.$$

Now chose ϵ such that $\epsilon < dist(x, \partial\Omega)$. Define

$$u_\epsilon(x) = \int_\Omega \eta_\epsilon(x-y)u(y)dy.$$

Now we claim

1. $u_\epsilon \epsilon C^\infty$

2. $u_\epsilon(x) = u(x)$.

First, for (1), $u_\epsilon \epsilon C^\infty$. We prove(2) as follows.Using the fact that $supp\eta_\epsilon(x-y) \subset \{y : |x-y| < \epsilon\}$. Therefore,

$$\begin{aligned}
u_\epsilon(x) &= \int_{B(x,\epsilon)} \eta_\epsilon(x-y)u(y)dy \\
&= \frac{1}{\epsilon^n}\int_{B(x,\epsilon)} \eta(\frac{|x-y|}{\epsilon})u(y)dy \\
&= \frac{1}{\epsilon^n}\int_0^\epsilon (\int_{\partial\Omega(x,r)} \eta(\frac{|x-y|}{\epsilon})u(y)dS(y))dr \\
&= \frac{1}{\epsilon^n}\int_0^\epsilon (\int_{\partial\beta(x,r)} \eta(\frac{r}{\epsilon})u(y)dS(y)dr \\
&= \frac{1}{\epsilon^n}\int_0^\epsilon \eta(\frac{r}{\epsilon}\int_{\partial B(x,r)} u(y)dS(y)dr \\
&= \frac{1}{\epsilon^n}\int_0^\epsilon \eta(\frac{r}{\epsilon})n\alpha(n)r^{n-1} ⨍_{\partial B(x,r)} u(y)dS(y)dr \\
&= \frac{1}{\epsilon^n}\int_0^\epsilon \eta(\frac{r}{\epsilon})n\alpha(n)r^{n-1}udr(x) \\
&= u(x)\frac{1}{\epsilon^n}\int_0^\epsilon \eta(\frac{r}{\epsilon})\int_{\partial B(x,r)} dS(y)dr \\
&= u(x)\frac{1}{\epsilon^n}\int_{B(0,\epsilon)} \eta(\frac{|y|}{\epsilon})dy \\
&= u(x)\int_{B(0,\epsilon)} \eta_\epsilon(y)dy \\
&= u(x).
\end{aligned}$$

□

3.3.5 Liouville's Theorem

In this section, we show that the only functions which are bounded and harmonic on $\mathbb{R}^n$ are constant functions.

Theorem 3.3.6. *Suppose $u : \mathbb{R}^n \to \mathbb{R}$ is harmonic and bounded. Then u is constant.*

Proof. Let $x_0 \in \mathbb{R}^n$. By the mean value property,

$$u(x_0) = \fint_{B(x_0,r)} u(y)dy$$

for all $B(x_0, r)$. Now by the previous theorem, we know that if $u \in c^2(\Omega) \cap C(\overline{\Omega})$ and u is harmonic, then u is C^∞. Therefore,

$$\Delta u = 0 \Longrightarrow \Delta u_{x_i} = 0$$

for $i = 1..., n$. Therefore, u_{x_i} is harmonic and satisfies the mean value property. Therefore,

$$\begin{aligned} u_{x_i}(x_0) &= \fint_{B(x_0,r} u_{x_i}(y)dy \\ &= \frac{1}{\alpha(\tilde{n})r^n} \int_{B(x_0,r)} u_{x_i}(y)dy \\ &= \frac{1}{\alpha(n)r^n} \int_{\partial B(x_0,r)} uv_i dS(y), \end{aligned}$$

by the divergence theorem, where $v = (v_1, ..., v_n)$ is the outward unit normal to $B(x_0, r)$. Therefore,

$$\begin{aligned} |u_{x_i}| \leq |\frac{1}{\alpha(n)r^n} \int_{\partial B(x_0,r)} uv_i dS(y)| &\leq |u|L^\infty(\partial B(x_0, r))|v_i|L^\infty|\frac{1}{\alpha(n)r^n} \int_{\partial B(x_0,r)} dS(y)| \\ &\leq |u|L^\infty(\mathbb{R}^n)|\frac{n\alpha(n)r^{n-1}}{\alpha(n)r^n}| \\ &\leq \frac{n}{r}|u|L^\infty(\mathbb{R}^n). \end{aligned}$$

Therefore,

$$\begin{aligned} &|u_{x_i(x_0}|\frac{n}{r}|u|L^\infty(\mathbb{R}^n) \\ &\leq C\frac{n}{r}, \end{aligned}$$

by the assumption that u is bounded. Now this is true for all r. Taking the limit as $r \to +\infty$, we see that $|u_{x_i}(x_0)| = 0$. Therefore, $u_{x_i}(x_0) = 0$. This is true for $i = 1, ..., n$ and for all $x_0 \in \mathbb{R}^n$.
Therefore, we conclude that $x_0 \equiv \mathit{constant}$. □

3.4 Potential Theory

3.4.1 Problems of Interest.

In what follows, we consider Ω an open, bounded subset of $\mathbb{R}^n$ with C^2 boundary. We are interested in studying the following two problems:

1. Interior Dirichlet problem.

$$\begin{cases} \Delta u = 0 & x \in \Omega \\ u = g & x \in \partial\Omega. \end{cases}$$

2. Interior Neumann problem.

$$\begin{cases} \Delta u = 0 & x \in \Omega \\ \frac{\partial u}{\partial v} = g & x \in \partial\Omega. \end{cases}$$

If u is a C^2 solution of the Interior Dirichlet problem, then by using Green's representation u is given by

$$u(x) = -\int_{\partial\Omega} g(y) \frac{\partial G}{\partial v_y}(x,y)\,dS(y),$$

where G(x,y) is the Green's function for Ω. However, in general, it is difficult to calculate an explicit formula for the Green's function. Here, we use a different approach to look for solutions to the Interior Dirichlet problem, as well as the Interior Neumann problem. Again, it is difficult to calculate explicit solutions, but we will discuss existence of solutions and give representations for them.

3.4.2 Definitions and Preliminary Theorems.

Let $\Phi(x)$ denote the fundamental solution of Laplace's equation. That is, let

$$\Phi(x) \equiv \begin{cases} -\frac{1}{2\pi} \ln|x| & n = 2 \\ \frac{1}{n(n-2)\alpha(n)} \cdot \frac{1}{|x|^{n-2}} & n \geq 3. \end{cases}$$

Let h be a continuous function on $\partial\Omega$. The **single layer potential with moment** h is defined as

$$\boxed{V_\Delta(x) = -\int_{\partial\Omega} h(y)\Phi(x-y)\,dS(y).} \tag{3.11}$$

The **double layer potential with moment** h is defined as

$$\boxed{W_{\Delta}(x) = -\int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial v_y}(x-y)dS(y).} \tag{3.12}$$

We plan to use these layer potentials to construct solutions of the problems listed above. Notice that Green's function gives us a solution to the Interior Dirichlet Problem which is similar to a double layer potential. We will see that for appropriate choice of h, we can write solutions of the Dirichlet problem(a) as double layer potentials and solutions of the Neumann problem(b) as single layer potentials.
First, we will prove that for a continuous function h, (3.11) and (3.12) are harmonic functions for all $x \notin \partial\Omega$.

Theorem 3.4.1. *For h a continuous function on*$\partial\Omega$,

1. V_{Δ} *and* W_{Δ} *are defined for all* $x \in \mathbb{R}^n$.
2. $\Delta V_{\Delta}(x)$=$\Delta W_{\Delta}(x) = 0$ *for all* $x \notin \partial\Omega$.

Proof. 1. We prove that W_{Δ} is defined for all $x \in \mathbb{R}^n$. A similar proof works for V_{Δ}. First, suppose $x \notin \partial\Omega$. Therefore, $\frac{\partial\Phi}{\partial v_y}(x-y)$ is defined for all $y \in \partial\Omega$. Consequently, for all $x \notin \partial\Omega$, we have

$$|W_{\Delta}(x)| \leq |h(y)|L^{\infty}(\partial\Omega)\int_{\partial\Omega}|\frac{\partial\Phi}{\partial v_y}(x-y)|dS(y) \leq C.$$

Next, consider the case when x is in $\partial\Omega$. In this case, the term $\frac{\partial\Phi}{\partial v_y}(x-y)$ in the integrand is undefined at $x = y$. We prove W_{Δ} is defined at this point x by showing that the integral in (3.12) still converges.
We need to look for a bound on

$$-\int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial v_y}(x-y)dS(y).$$

Recall

$$\Phi(x-y) = \begin{cases} \frac{-1}{2\pi}\ln|x-y| & n=2 \\ \frac{1}{n(n-2)\alpha(n)}\cdot\frac{1}{|x-y|^{n-2}} & n\geq 3 \end{cases}$$

Therefore,

$$\Phi_{y_i}(x-y) = \frac{x_i - y_i}{n\alpha(n)|y-x|^n},$$

and

$$\begin{aligned}\frac{\partial \Phi}{\partial v_y}(x-y) &= \nabla_y \Phi(x-y).v(y) \\ &= \frac{(x-y).(v(y)}{n\alpha(n)|y-x|^n},\end{aligned}$$

where $v(y)$ is the unit normal to $\partial\Omega$ at y.
Claim: Fix $x \in \partial\Omega$. for all $y \in \partial\Omega$, there exists a constant $C > 0$ such that

$$|(x-y).v(y)| \leq C|x-y|^2.$$

Proof of claim. By assumption, $\partial\Omega$ is C^2. This means at each point $x \in \partial\Omega$, there exists an $r > 0$ and a C^2 function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ such that upon relabelling and reorienting if necessary we have

$$\Omega \cap \beta(x,r) = \{z \in B(x,r) | z_n > f(z_1, ..., z_{n-1})\}.$$

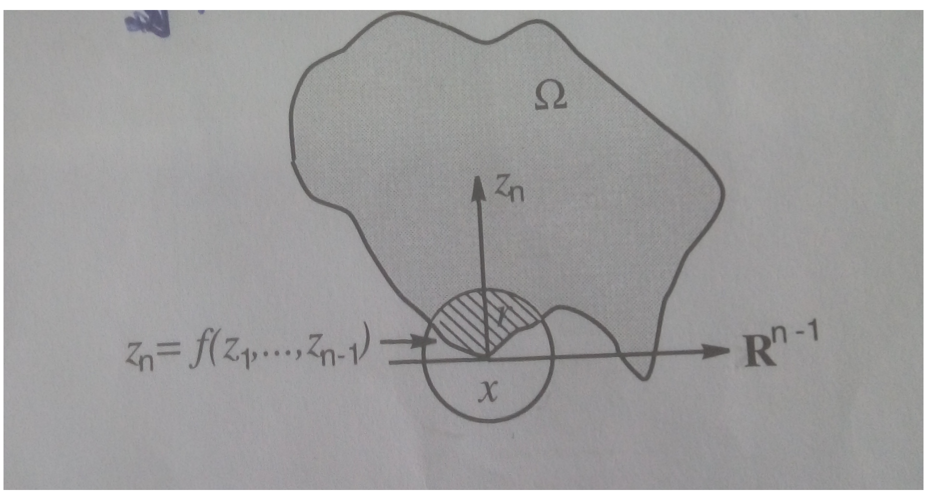

Figure 3.1:

Without loss of generality (by reorienting if necessary), we may assume $x = 0$ and $v(x) = (0, ..., 0, 1)$. Using the fact our boundary is C^2, we know there exists an $r > 0$ a C^2 function $f : B(0,r) \subset \mathbb{R}^{n-1} \to \mathbb{R}$ such

that $\partial\Omega$ is given by the graph of the function f near x.
First, consider $y \in \partial\Omega$ such that $|x-y| \geq r$. In this case

$$|(x-y).v(y)| \leq |x-y| \leq \frac{1}{r}|x-y|^2 = C(r)|x-y|^2.$$

Second, consider $y \in \partial\Omega$ such that $|x-y| \leq r$. In this case, we use the fact that

$$\begin{aligned}
|(x-y).v(y)| &= |(x-y).(v(x)+v(y)-v(x))| \\
&\leq |(x-y).v(x)| + |(x-y).(v(y)-v(x))| \\
&= |y_n| + |(x-y).(v(y)-v(x))|.
\end{aligned}$$

Now,

$$y_n = f(y_1, ..., y_{n-1})$$

where $f \in C^2$, $f(0) = 0$ and $\nabla f(0) = 0$. Therefore, by Taylor's Theorem, we have

$$\begin{aligned}
|y_n| &= |f(y_1, ..., y_{n-1})| \\
&\leq C|(y_1, ..., y_{n-1}|^2 \\
&\leq C|y|^2 \\
&= C|x-y|^2,
\end{aligned}$$

where the constant C depends only on the bound on the second partial derivatives of $f(y_1, ..., y_{n-1})$ for $|(y_1, ..., y_{n-1})| \leq r$, but this is bounded because by assumption $f \in C^2(B(0,r))$. Next, we look at $|(x-y).(v(y)-v(x))|$. By assumption, $\partial\Omega$ is C^2 and consequently, v is a C^1 function and therefore, there exists a constant $C > 0$ such that

$$|v(y) - v(x)| \leq C|y-x|.$$

Therefore,

$$|(x-y).(v(y)-v(x))| \leq C|y-x|^2.$$

Consequently, our claim is proven. We remark that the constant C will depend on r, but once x is chosen r is fixed.

Therefore, we conclude that for $x \in \partial\Omega$, all $y \in \partial\Omega$,

$$\begin{aligned} |\frac{\partial\Phi}{\partial v_y}(x-y)| &= |\frac{(x-y).v(y)}{n\alpha(n)|y-x|^n}| \\ &\leq C\frac{|x-y|^n}{|x-y|^n} \\ &= \frac{C}{|x-y|^{n-2}}. \end{aligned}$$

Therefore,

$$\begin{aligned} |-\int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial v_y}(x-y)dS(y)\,| &\leq |h(y)|L^{\infty}(\partial\Omega)\int_{\partial\Omega\Omega}|\frac{\partial\Phi}{\partial v_y}(x-y)|dS(y) \\ &\leq C\int_{\partial\Omega}\frac{1}{|x-y|^{n-2}}dS(Y) \leq C \end{aligned}$$

using the fact $\partial\Omega$ is of dimension $n-1$.
Therefore, we conclude that W_Δ is defined for all $x \in \partial\Omega$ and consequently for all $x \in \mathbb{R}^n$ as claimed.

2. Next, we will prove that $\Delta(W_\Delta(x) = 0)$ for all $x \in \Omega$. A similar proof works to prove that $\Delta(V_\Delta(x) = 0)$.
Fix $x \in \Omega$. We note that for all $y \in \partial\Omega$, $\frac{\partial\Phi}{\partial v_y}(x-y)$ is smooth function. Further, using the fact that $\Phi(x-y)$ is harmonic for all $x \neq y$, we conclude that $\Delta_x\frac{\partial\Phi}{\partial v_y}(x-y) = 0$ for all $y \in \partial\Omega$. Therefore, using the fact our integral is finite and $\frac{\partial\Phi}{\partial v_y}(x-y)$ is smooth, we conclude that

$$\begin{aligned} \Delta_x W_\Delta(X) &= -\Delta_X\int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial v_y}(x-y)dS(y) \\ &= -\int_{\partial\Omega} h(y)\Delta_x\frac{\partial\Phi}{\partial v_y}(x-y)dS(y) \\ &= 0. \end{aligned}$$

□

In the above theorem, we showed that as long as h is continuous function on $\partial\Omega$, then V_Δ and W_Δ in (3.11)and(3.12), respectively, are harmonic functions on Ω. Consequently, if we choose h appropriately so that our initial condition will be satisfied, then we can find a solution of our particular

problem (a) and (b).
We claim that by choosing h appropriately, W_Δ will give us a solution of our Interior Dirichlet problem. Similarly, we will show that by choosing h appropriately, V_Δ will give us a solution of our Interior Neumann problems. For a moment, consider the Interior Dirichlet problem (a). As proven above, for h a continuous function on $\partial\Omega$, V_Δ defined in (3.12) is harmonic. Now, if we can choose h appropriately, such that for all $x_0 \in \partial\Omega$,

$$\lim_{x\in\Omega\to x_0} W_\Delta(x) = g(x_0),$$

then we will have found a solution of the Interior Dirichlet problem. Consequently, we are interested in studying the limits of $W_\Delta$as we approach the boundary of Ω. In order to study this, we must first prove the following lemma.

Theorem 3.4.2 (Gauss' Lemma). *Consider the double layer potential,*

$$W_\Delta(x) = -\int_{\partial\Omega} \frac{\partial\Phi}{\partial v_y}(x-y)dS(y).$$

Then

$$W_\Delta(x) = \begin{cases} 0 & x \in \Omega^c \\ 1 & x \in \Omega \\ \frac{1}{2} & x \in \partial\Omega. \end{cases}$$

Proof. 1. First, for $x \in \Omega^c$,

$$\begin{aligned} W_\Delta(x) &= -\int_{\partial\Omega} \frac{\partial\Phi}{\partial v_y}(x-y)dS(y) \\ &= -\int_{\Omega} \Delta_y\Phi(x-y)dy \\ &= 0 \end{aligned}$$

using the Divergence Theorem and the fact that $\Phi(x-y)$ is smooth for $y \in \Omega, x \in \Omega^c$.
2. Now, for $x \in \Omega, \Phi(x-y)$ is not smooth for all $x \in \Omega$. In order to overcome this problem, we fix $\epsilon > 0$ sufficiently small such that $B(x,\epsilon)$ is contained within Ω. Then on the region $\Omega - B(x,\epsilon), \Phi(x-y)$ is smooth, and,

consequently, we can say

$$\begin{aligned} 0 &= \int_{\Omega - B(x,\epsilon)} \Delta_y \Phi(x-y)dy \\ &= \int \partial(\Omega - B(x,\epsilon)) \frac{\partial \Phi}{\partial v_y}(x-y)dS(y) \\ &= \int_{\partial\Omega} \frac{\partial \Phi}{\partial v_y}(x-y)dS(y) + \int_{\partial B(x,\epsilon)} \frac{\partial \Phi}{\partial v_y}(x-y)dS(y) \end{aligned}$$

where v is the outer unit normal to $\Omega - B(x,\epsilon)$. As mentioned above,

$$\Phi_{y_i}(x-y) = \frac{x_i - y_i}{n\alpha(n)|y-x|^n}.$$

For $y \in \partial B(x,\epsilon)$, the outer unit normal to $\Omega - B(x,\epsilon)$ is given by

$$v(y) = \frac{x-y}{|x-y|}.$$

Therefore, for$y\epsilon\partial B(x,\epsilon)$,

$$\begin{aligned} \frac{\partial \Phi}{\partial v_y}(x-y) &= \nabla_y \Phi(x-y).v(y) \\ &= \frac{x-y}{n\alpha(n)|x-y|^n}.\frac{x-y}{|x-y|} \\ &= \frac{|x-y|^2}{n\alpha(n)|x-y|^{n+1}} \\ &= \frac{1}{n\alpha(n)|x-y|^{n-1}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\partial B(x,\epsilon)} \frac{\partial \Phi}{\partial v_y}(x-y) &= \int_{\partial B(x,\epsilon)} \frac{1}{\alpha(n)|x-y|^{n-1}} dS(y) \\ &= \frac{1}{nalpha(n)\epsilon^{n-1} \int_{\partial B(x,\epsilon}} dS(y) \\ &= 1. \end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} 0 &= \int_{\partial\Omega} \frac{\partial \Phi}{\partial v_y} dS(y) + \int_{\partial B(x,\epsilon)} \frac{\partial \Phi}{\partial v_y}(x-y)dS(y) \\ &= \int_{\partial\Omega} \frac{\partial \Phi}{\partial v_y}(x-y)dS(y) + 1. \end{aligned}$$

Which implies

$$-\int_{\partial\Omega} \frac{\partial \Phi}{\partial v_y}(x-y)dS(y) = 1,$$

as desired.
3. Last, we consider the case $x \in \partial\Omega$. In this case, $\frac{\partial \Phi}{\partial v_y}(x-y)$ is not defined at$y = x$. Fix $x \in \partial\Omega$. Let $B(x, \epsilon)$ be the ball of radius ϵ about x. Let

$$\Omega_\epsilon \equiv \Omega - (\Omega \cap B(x, \epsilon)).$$

Let

$$C_\epsilon \equiv \{y \in \partial B(x, \epsilon) : v(x).y < 0\}.$$

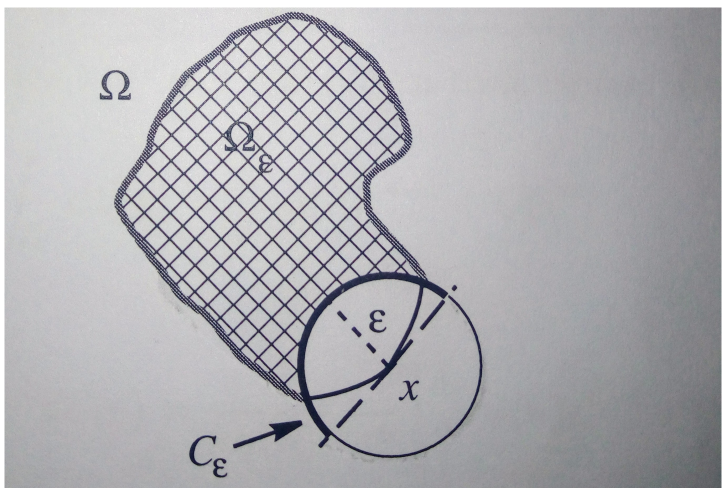

Figure 3.2:

Let

$$\tilde{C}_\epsilon \equiv \partial\Omega_\epsilon \cap C_\epsilon.$$

First, we note that

$$\begin{aligned} 0 &= \int_{\Omega_\epsilon} \Delta_y \Phi(x-y)dy \\ &= \int_{\partial\Omega_\epsilon} \frac{\partial \Phi}{\partial v_y}(x-y)dS(y) \\ &= \int_{\partial\Omega_{\epsilon-\tilde{C}_\epsilon}} \frac{\partial \Phi}{\partial v_y}(x-y)dS(y) + \int_{\tilde{C}_\epsilon} \frac{\partial \Phi}{\partial V_y}(x-y)dS(y), \end{aligned}$$

Therefore,

$$\boxed{0 = \int_{\partial\Omega_{\epsilon - \tilde{C}_\epsilon}} \frac{\partial\Phi}{\partial v_y}(x-y)dS(y) + \int_{\tilde{C}_\epsilon} \frac{\partial\Phi}{\partial V_y}(x-y)dS(y)} \qquad (3.13)$$

where v_y is the outer unit normal to Ω_ϵ.
Now, first, we recall that

$$\nabla_y \Phi(x-y) = \frac{x-y}{n\alpha(n)|y-x|^n}.$$

For all $y \in \tilde{C}_\epsilon$, the outer unit normal is given by

$$v(y) = \frac{x-y}{|x-y|}.$$

Therefore,

$$\begin{aligned} \int_{\tilde{C_\epsilon}} \frac{\partial\Phi}{\partial v_y}(x-y)dS(y) &= \int_{\tilde{C_\epsilon}} \frac{1}{n\alpha(n)|x-y|^{n-1}} dS(y) \\ &= \frac{1}{n\alpha(n)\epsilon^{n-1}} \int_{\tilde{C_\epsilon}} dS(y). \end{aligned}$$

Next, we use the fact that

$$\int_{\tilde{C_\epsilon}} dS(y) \approx \int_{C_\epsilon} dS(y).$$

In fact, as we will show below,

$$\boxed{\int_{\tilde{C}_\epsilon} dS(y) = \int_{C_\epsilon} dS(y) + O(\epsilon^n).} \qquad (3.14)$$

We omit the proof of (3.14) for now and will return to it below. Assuming this fact for now, we have

$$\int_{\tilde{C}_\epsilon} = \frac{1}{2} n\alpha(n)\epsilon^{n-1} + O(\epsilon^n).$$

Which implies

$$\begin{aligned} \int_{\tilde{C}_\epsilon} \frac{\partial\Phi}{\partial V_y}(x-y)dS(y) &= \frac{1}{n\alpha(n)\epsilon^{n-1}}[\frac{1}{2}n\alpha(n)\epsilon^{n-1} + O(\epsilon^n)] \\ &= \frac{1}{2} + \frac{1}{n\alpha(n)O(\epsilon)} \end{aligned}$$

Therefore,

$$\boxed{\int_{\tilde{C}_\epsilon} \frac{\partial \Phi}{\partial V_y}(x-y)dS(y) = \frac{1}{2} + \frac{1}{n\alpha(n)O(\epsilon)}.} \tag{3.15}$$

Combining (3.13) and (3.15),we have

$$0 = \int_{\partial\Omega_\epsilon - \tilde{C}_\epsilon} \frac{\partial \Phi}{\partial v_y}(x-y)dS(y) + \frac{1}{2} + \frac{1}{n\alpha(n)}O(\epsilon),$$

Which implies

$$\int_{\partial\Omega_\epsilon - \tilde{C}_\epsilon} \frac{\partial \Phi}{\partial v_y}(x-y)dS(y) = -\frac{1}{2} - \frac{1}{n\alpha(n)}O(\epsilon).$$

Taking the limit as $\epsilon \to 0^+$, we have

$$-\int_{\partial\Omega} \frac{\partial \Phi}{\partial v_y}(x-y)dS(y) = \frac{1}{2},$$

as claimed. □

Now we will prove(3.14).

Claim 3. For $\tilde{C_\epsilon}$ and C_ϵ as defined above, we have

$$\int_{\tilde{C}_\epsilon} dS(y) = \int_{C_\epsilon} dS(y) + O(\epsilon^n).$$

Proof. We just need to show that the surface area of $C_\epsilon - \tilde{C_\epsilon'}$ is $O(n)$. Then surface area is approximately the surface area of the base times the height. Now the surface area of the base is $O(\epsilon^{n-2})$. Therefore, we just need to show that the height is $O(\epsilon^2)$.

Without loss of generality, we let $x = 0$. Now, by assumption, $\partial\Omega$ is C^2. Therefore, $\partial\Omega$ can be written as the graph of a C^2 function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $f(0) = 0$ and $\nabla f(0) = 0$. Therefore, if $y \in C_\epsilon - \tilde{C_\epsilon}$, then

$$|y_n| \leq |f(y_1, ..., y_{n-1})| \leq C|(y_1, ..., y_{n-1})|^2 \leq C|y|^2 \leq C\epsilon^2,$$

using Taylor's Theorem. Therefore, the height is $O(\epsilon^2)$ and the claim follows. □

Now with Gauss' Lemma above, for V_Δ and $W_\triangle$ defined as in (3.11)and(3.12) for a continuous function h, we can find the limits of $V_\Delta(x)$ and $W_\triangle(x)$ as we approach $\partial\Omega$ from the interior or the exterior. We state these results in the following theoren.

Theorem 3.4.3. *Let h be a continuous function on $\partial\Omega$. Define single and double layer potentials as follows:*

$$V_\Delta(x) = -\int_{\partial\Omega} h(y)\Phi(x-y)dS(y)$$

and

$$W_\triangle(x) = -\int_{\partial\Omega} \frac{\partial\Phi}{\partial V_y}(x-y)dS(y).$$

Let $x_0 \in \partial\Omega$. Then

$$\boxed{\lim_{x\in\Omega\to x_0} V_\Delta(x) = V_\Delta(x_0)} \tag{3.16}$$

$$\lim_{x\in\Omega\to x_0} \frac{\partial V_\Delta(x)}{\partial n_x} = \frac{1}{2}h(x_0) + \frac{\partial V_{\Delta(x_0)}}{\partial n_x},$$

$$\boxed{\lim_{x\in\Omega^c\to x_0} \frac{\partial V_\Delta(x)}{\partial n_x} = \frac{-1}{2}h(x_0) + \frac{\partial V_{\Delta(x_0)}}{\partial n_x}} \tag{3.17}$$

$$\boxed{\lim_{x\in\Omega\to x_0} W_\Delta(x) = \frac{-1}{2}h(x_0 + W_\Delta(x_0), \lim_{x\in\Omega^c\to x_0} W_\Delta(x) = \frac{-1}{2}h(x_0) + W_\Delta(x_0).} \tag{3.18}$$

Proof. 1. Proof of equation (3.16). Let $x \in \Omega$, $x_0 \in \partial\Omega$. We have

$$V_\Delta(x) = -\int_{\partial\Omega} \Phi(x-y)h(y)dS(y)$$

and

$$V_\Delta(x_0) = -\int_{\partial\Omega} \Phi(x_0-y)h(Y)dS(y).$$

We need to show that

$$\lim_{x\in\Omega\to x_0} V_\Delta(x) = V_\Delta(x_0).$$

That is for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$| V_\Delta(x) - V_\Delta(x_0) | < \epsilon for \mid x - x_0 \mid < \delta.$$

Now,

$$V_\Delta(x) - V_\Delta(X_0) = -\int_{\partial\Omega} h(y)[\Phi(x-y) - \Phi(x_0 - y)]dS(y).$$

By assumption, h is continuous, and as we know $\Phi(x-y)$ is smooth for $x \neq y$. Therefore, to get a bound on $| V_\Delta(x) - V_\Delta(x_0) |$, we devide $\partial\Omega$ into two pieces:

1. $B(x_0, \gamma) \cap \partial\Omega$
2. $\partial\Omega - B(x_0, \gamma) \cap \partial\Omega.$

We look at these pieces below .
First for(1),

$$| V_\Delta(x) - V_\Delta(x_0) | \leq | h(y) |_{L^\infty(B(x_0,\gamma)\cap\partial\Omega)} \int_{B(x_0,\gamma)\cap\partial\Omega} | \Phi(x-y) - \Phi(x_0 - y) | dS(y).$$

By assumption, h is continuous. Therefore, for all $\tilde{\epsilon} > 0$ there exists a $\gamma > 0$ such that $|h(y) < \tilde{\epsilon}$ for $y < \gamma$. In addition that

$$\int_{B(x_0,\gamma)\cap\partial\Omega} |\Phi(x-y) - \Phi(x_0 - y)| dS(y) \leq C$$

using the fact that V_Δ is defined for all $x \epsilon \mathbb{R}$. Therefore, we conclude that for any $\tilde{\epsilon} > 0$,$|(1)| < C_1 \tilde{\epsilon}$ for γ chosen appropriately small. Next, for(2), we use the fact that $\Phi(x-y)$ is cotinuous in x for x away from y. Consequently, we have

$$| V_\Delta(x) - V_\Delta(x_0) | \leq$$

$$| h(y) | L^\infty(\partial\Omega - B(x_0,\gamma)\cap\partial\Omega) | \Phi(x-y) - \Phi(x_0 - y) | L^\infty(\partial\Omega - B(x_0,\gamma))\cap\partial\Omega) | \int_{\partial\Omega - (B(x_0,\gamma)\cap\partial\Omega)} dS(y$$

Now, first h is bounded on $\partial\Omega$. Therefore, $|h(y)| \leq C$.
Next, $| \int dS(y) | \leq C$. Lastly, using the fact $\Phi(x-y)$ is continuous in x uniformly for y, we conclude that there exists a δ such that

$$| \Phi(x-y) - \Phi(x_0 - y) | L^\infty(\partial\Omega - B(x_0,\gamma) \cap \partial\Omega) \leq \tilde{\epsilon}, for \mid x - x_0 \mid < \delta.$$

Therefore, $|(2)| \leq C_2\tilde{\epsilon}$ if $|x - x_0| < \gamma$ where δ is chosen appropriately small. Consequently, for $\epsilon > 0$ choose $\tilde{\epsilon} > 0$ such that $C_1\tilde{\epsilon} + C_2\tilde{\epsilon} < \epsilon$.
Then choosing $\gamma > 0$ sufficiently small such that $|(1)| \leq C_1\tilde{\epsilon}$ and $\delta > 0$ sufficiently small such that $|(2)| \leq C_2\tilde{\epsilon}$ when $|x - x_0| < \delta$, we conclude that

$$V_\Delta(x) - V_\Delta(x_0)| \leq C_1\tilde{\epsilon} + C_2\tilde{\epsilon} < \epsilon, for|x - x_0| < \delta.$$

2. Proof of equation (3.17) is similar to the following proof of equation (3.18). we will prove only the first case, when $x \in \Omega$. The second case works similarly. Let $x \in \Omega, x_0 \in \partial\Omega$. We have,

$$\begin{aligned}
W_\Delta(x) &= -\int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial n_y}(x - y)dS(y) \\
&= -\int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial n_y}(x-)dS(y) + h(x_0)\int_{\partial\Omega}\frac{\partial\Phi}{\partial n_y}(x - y)dS(y) - h(x_0)\int_{\partial\Omega}\frac{\partial\Phi}{\partial n_y}(x - y)dS(y) \\
&= -\int_{\partial\Omega}[h(y) - h(x_0)]\frac{\partial\Phi}{\partial n_y}(x - y)dS(y) + h(x_0) \\
&\equiv h(x_0) + I(x)
\end{aligned}$$

using the fact that

$$-\int_{\partial\Omega}\frac{\partial\Phi}{\partial n_y}(x - y)dS(y) = 1$$

, for $x \in \Omega$, proven in Gauss' Lemma. Similarly,

$$\begin{aligned}
W_\Delta(x_0) &= -\int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial n_y}(x_0 - y)dS(y) \\
&= -\int_{\partial\Omega}[h(y) - h(x_0)]\frac{\partial\Phi}{\partial n_y}(x_0 - y)dS(y) - h(x_0)\int_{\partial\Omega}\frac{\partial\Phi}{\partial n_y}(x_0 - y)dS(y) \\
&= -\int_{\partial\Omega}[h(y) - h(x_0)]\frac{\partial\Phi}{\partial n_y}(x_0 - y)dS(y) + \frac{1}{2}h(x_0) \\
&= I(x_0) + \frac{1}{2}h(x_0)
\end{aligned}$$

again using Gauss' Lemma.
Therefore,

$$W_\Delta(x) - W_\Delta(x_0) = I(x) + h(x_0) - I(x_0) - \frac{1}{2}h(x_0),$$

which implies

$$W_\Delta(x) = I(x) - I(x_0) + \frac{1}{2}h(x_0) + W_\Delta(x_0).$$

Therefore, to prove our theorem, we need only show that

$$\lim_{x\in\Omega\to x_0} [I(x) - I(x_0)] = 0,$$

where

$$I(x) \equiv - \int_{\partial\Omega} [h(y) - h(x_0)] \frac{\partial\Phi}{\partial n_y}(x-) dS(y).$$

Now,

$$I(x) - I(x_0) = - \int_{\partial\Omega} [h(y) - h(x_0)][\frac{\partial\Phi}{\partial n_y}(x - y) - \frac{\partial\Phi}{\partial n_y}(x_0 - y)]dS(y).$$

We need to show for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|I(x) - I(x_0)| < \epsilon$ for $|x - x_0)| < \delta$. By assumption, h is continuous, and as we know $\Phi(x - y)$ is smooth for $y \neq x$.
Therefore, to get a bound on $|I(x) - I(x_0)|$,we divide $\partial\Omega$ into two pieces:

1. $B(x_0, \gamma) \cap \partial\Omega$
2. $\partial\Omega - B(x_0, \gamma) \cap \partial\Omega.$

We look at these two pieces below. First for (1),

$$| - \int_{B(x_0,\gamma)\cap\partial\Omega} [h(y) - h(x_0)][\frac{\partial\Phi}{\partial n_y}(x - y) - \frac{\partial\Phi}{\partial n_y}(x_0 - y)]dS(y) |$$

$$\leq | h(y) - h(x_0) | L^\infty(B(x_0, \gamma) \cap \partial\Omega) \int_{B(x_0,\gamma)\cap\partial\Omega} | \frac{\partial\Phi}{\partial n_y}(x-y) - \frac{\partial\Phi}{\partial n_y}(x_0-y) | dS(y).$$

By assumption, h is continuous. Therefore, for all $\tilde{\epsilon} > 0$ there exists a $\gamma > 0$ such that $| h(y) - h(x_0) | < \tilde{\epsilon}$ if $| y - x_0 | < \gamma$. In addition,

$$\int_{B(x_0,\gamma)\cap\partial\Omega} | \frac{\partial\Phi}{\partial n_y}(x - y) - \frac{\partial\Phi}{\partial n_y}(x_0 - y) | dS(y) \leq C$$

using the fact that $W_\Delta(x)$ is defined for all $x \epsilon \mathbb{R}$. Therefore, we conclude that for any $\tilde{\epsilon} > 0$,

$$| (1) | \leq C_1 \tilde{\epsilon}$$

for γ chosen appropriately small.
Next, for (2), we use the fact that $\frac{\partial \Phi}{\partial n_y}(x-y)$ is continuous in x for x away from y.
Cosequently, we have

$$| - \int_{\partial\Omega - B(x_0,\gamma)\cap\partial\Omega} [h(y) - h(x_0)][\frac{\partial \Phi}{\partial n_y}(x-y) - \frac{\partial \Phi}{\partial n_y}(x_0 - y) dS(y) \mid$$

$$\leq \mid h(y) - h(x_0) \mid L^\infty \mid \frac{\partial \Phi}{\partial n_y}(x-y) - \frac{\partial \Phi}{\partial n_y}(x_0 - y) \mid L^\infty(\partial\Omega - B(x_0,\gamma)\cap\partial\Omega) \mid \int dS(y) \mid .$$

Now, first h is bounded on $\partial\Omega$. Therefore, $|h(y) - h(x_0)| < C$. Next, $|\int dS(y)| \leq C$. Lastly, using the fact that $\frac{\partial \Omega}{\partial n_y}(x-y)$ is continuous in x uniformly for y,we conclude that there exists a $\delta > 0$ such that

$$|\frac{\partial \Phi}{\partial n_y}(x-y) - \frac{\partial \Phi}{\partial n_y}(x_0 - y)|_{L^\infty(\partial\Omega - B(x_0,\gamma)\cap\partial\Omega)} \leq \tilde{\epsilon},$$

for $|x - x_0| < \delta$. Therefore,

$$|(2)| \leq C_2 \tilde{\epsilon}$$

if $|x - x_0| < \delta$ where δ is chosen appropriately small. Consequently, for $\epsilon > 0$ choose $\tilde{\epsilon} > 0$ such that

$$C_1\tilde{\epsilon} + C_2\tilde{\epsilon} < \epsilon.$$

Then choosing $\gamma > 0$ sufficiently small such that

$$|(2)| \leq C_2\tilde{\epsilon}.$$

When $|x - x_0| < \delta$, we conclude that

$$|I(x) - I(x_0)| \leq C_1\tilde{\epsilon} + C_2\tilde{\epsilon} \leq \epsilon,$$

for $|x - x_0| < \delta$, implies that

$$\lim_{x\in\Omega\to x_0} [I(x) - I(x_0)] = 0.$$

Consequently,

$$\lim_{x\in\Omega\to x_0} W_\Delta(x) = \lim_{x\in\Omega\to x_0} ([I(x) - I(x_0)] + \frac{1}{2}h(x_0) + W_\Delta(x_0))$$

$$= \frac{1}{2}h(x_0) + W_\Delta(x_0)$$

as claimed.

$\square$

In the next section, we use this theorem to construct solutions of the interior Dirichelet and interior Neumann problems.

3.4.3 Solutions of Laplace's Equation as a Double or Single Layer Potentials

We begin by considering the Interior Dirichelet Problem,

$$\begin{cases} \Delta u = 0 & x \in \Omega \\ u = g & x \in \partial\Omega \end{cases}$$

For a given function h, define the double-layer potential

$$W_\Delta(x) = -\int_{\partial\Omega} h(y)\frac{\partial \Phi}{\partial n_y}(x-y)\,dS(y).$$

In the previous section, we proved that W_Δ is a harmonic function in Ω. In addition, we proved that for $x_0 \in \partial\Omega$,

$$\lim_{x\in\Omega\to x_0} W_\Delta(x) = \frac{1}{2}h(x_0) + W_\Delta(x_0).$$

Therefore, if we can find a continuous function h such that for all $x_0 \in \partial\Omega$,

$$g(x_0) = \frac{1}{2}h(x_0) - \int_{\partial\Omega} h(y)\frac{\partial \Phi}{\partial n_y}(x_0-y)\,dS(y)$$

and we define

$$W_\Delta(x) = -\int_{\partial\Omega} h(y)\frac{\partial \Phi}{n_y}\,dS(y),$$

for that choice of h, then W_Δ will give us a solution of our interior Dirichelet problem.

Now, we consider the Neumann problems. We will find solutions below as single layer potentials. Consider the Neumann problem,

$$\begin{cases} \Delta u = 0 & x \in \Omega \\ \frac{\partial u}{\partial n_y} = g & x \in \partial\Omega \end{cases}$$

First, we note a compatibility condition on the boundary data in order for a solution to exist. By the Divergence Theorem, we know

$$\int_\Omega \Delta u = \int_{\partial\Omega} \frac{\partial u}{\partial n_y}\,dS(y).$$

Therefore, in order for a solution to exist, we need

$$\int_{\partial\Omega} g(y)dS(y) = 0.$$

For a continuous function h, define the single-layer potential

$$V_\Delta(x) = -\int_{\partial\Omega} h(y)\Phi(y-x)dy.$$

From the previous section, we know that V_Δ is harmonic in Ω. In order to choose h appropriately so that our boundary condition will be satisfied, we extend the notion of normal derivative to points not in $\partial\Omega$ as follows . Let $x_0 \in \partial\Omega$. Let $V(x_0)$ be the outer unit normal to Ω at x_0. For $t < 0$, such that $x_0 + tV(x_0)$ in Ω, we define

$$i^{x_0}(t) = \nabla V_\Delta(x_0 + tV(x_0)).V(x_0).$$

In a manner similar to the proof of Theorem 3.4.3 in the previous section, we can show that

$$\begin{aligned}\lim_{t\to 0^-} i^{x_0}(t) &= \frac{-1}{2}h(x_0) + \frac{\partial V_\Delta}{\partial V}(x_0) \\ &= \frac{-1}{2}h(x_0) - \int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial n_x}(x_0 - y)dS(y).\end{aligned}$$

Therefore, if we can find a continuous function h such that for all $x_0 \in \partial\Omega$,

$$g(x_0) = \frac{-1}{2}h(x_0) - \int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial n_x}(x_0 - y)dS(y),$$

then by defining the single-layer potential

$$V_\Delta(x) = -\int_{\partial\Omega} h(y)\Phi(x-y)dy,$$

for that choice of h, V_Δ will give us a solution of our interior Neumann problem.

Chapter 4

Invertiblity of single layer potential in 2D

4.1 Logarithmic Capacity

The domain on which no unique solution can be guaranteed are related to the so called logarithmic capacity. The logarithmic capacity is a real positive number being a function of the domain. This concept originates from the field of measure theory, but it is also appears in potential theory.
In the two dimensional case $n = 2$, the logarithmic capacity is defined by

$$cap_{\partial\Omega} := e^{-2\pi V_{\Delta}\psi_{eq}},$$

$\psi_{eq} \in H^{-\frac{1}{2}}(\partial\Omega)$ is natural density, so that

$$\frac{1}{2\pi} \ln \frac{1}{cap_{\partial\Omega}} = V_{\Delta}\psi_{eq}.$$

Note that $V_{\Delta}\psi_{eq} = 0$ if and only if $cap_{\partial\Omega} = 1$.

Proof. ($\Rightarrow$) Suppose $V_{\Delta}\psi_{eq} = 0$, then from the definition of logarithmic capacity for $n = 2$ we have $cap_{\partial\Omega} = e^{-2\pi(0)} = e^0 = 1$. Therefore, if $V_{\Delta}\psi_{eq} = 0$, then $cap_{\partial\Omega} = 1$.
($\Leftarrow$) Suppose $cap_{\partial\Omega} = 1$, then $e^{-2\pi V_{\Delta}\psi_{eq}} = 1$ which implies that $-2\pi V_{\Delta}\psi_{eq} = 0$ and therefore $V_{\Delta}\psi_{eq} = 0$. Hence, from ($\Rightarrow$) and ($\Leftarrow$) we can conclude that $V_{\Delta}\psi_{eq} = 0$ if and only if $cap_{\partial\Omega} = 1$.

□

There is a connection between the logarithmic capacity and the Euclidean diameter of Ω. In particular, $cap_{\partial\Omega} \leq diam(\Omega)$.
If the logarithmic capacity is strictly less than one, then we can conclude that $\lambda > 0$. Where $\lambda := \frac{1}{2\pi} \ln \frac{1}{cap_{\partial\Omega}}$. Therefore, to ensure $cap_{\partial\Omega} < 1$ a sufficient criteria is to assume $diam(\Omega) < 1$. This assumption can always granted when considering a suitable scaling of the domain $\Omega \subset \mathbb{R}^2$.

4.2 Boundedness And Ellipticity of Single Layer potential

Definition 4.2.1. *An operator $T : X \to Y$ is bounded if there is a constant $c > 0$ such that $\| T_f \|_Y \leq c\| f \|_X$ for every $f \in X$. The set of bounded operators from X to Y is denoted by $B(X, Y)$.*

Example 4.2.1. *The identity operator $I : X \to x, I(x) = x$ is bounded.*

Definition 4.2.2. *The operator $T : X \to Y$ is called $X-elliptic$ if*

$$\langle TV, V\rangle \geq C_1^T \| v \|_X^2$$

for all $v \in X$ is satisfied with some positive constant C_1^T.

Theorem 4.2.1 (Lax-Milgram Lemma)**.** *Let the operator $T : X \to Y$ be bounded and $X-elliptic$. For any $f \in Y$ there exists a unique solution $u \in X$ of the operator equation $Tu = f$ satistying the estimate $\| u \|_X \leq \frac{1}{C_1^T} \| f \|_Y$.*

Proof. For the proof of this theorem [see, e.g Olaf. S page (47)] □

Theorem 4.2.2. *The single layer potential operator $v_\triangle : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is bounnded with $\| v_\Delta\psi \|_{H^{\frac{1}{2}}(\partial\Omega)} \leq c\| \psi \|_{H^{-\frac{1}{2}}(\partial\Omega)}$ for all $\psi \in H^{-\frac{1}{2}}$ and $c > 0$.*

Theorem 4.2.3. *Let $dim(\Omega) < 1$ and therefore $\lambda > 0$ be satisfied. The single layer potential operator $v_\triangle : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$, is then $H^{-\frac{1}{2}}(\partial\Omega)$ elliptic, i.e, $\langle \nu_\triangle, \psi\rangle_{\partial\Omega} \geq C\| \psi \|^2_{H^{-\frac{1}{2}}(\partial\Omega)}$ for all $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$.*

Proof. For an arbitrary $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$ we consider the unique decomposition $\psi = \psi_0 + \alpha\psi_{eq}, \psi_0 \in H_*^{-\frac{1}{2}}(\partial\Omega), \alpha = \langle\psi, 1\rangle_{\partial\Omega}$ satisfying

$$\begin{aligned}
\| \psi \|^2_{H^{-\frac{1}{2}}(\partial\Omega)} &= \| \psi_0 + \alpha\psi_{eq} \|^2_{H^{-\frac{1}{2}}(\partial\Omega)} \\
&\leq [\| \psi_0 \|_{H^{-\frac{1}{2}}(\partial\Omega)} + \alpha\| \psi_{eq} \|_{H^{-\frac{1}{2}}(\partial\Omega)}]^2 \\
&\leq 2[\| \psi_0 \|^2_{H^{-\frac{1}{2}}(\partial\Omega)} + \alpha^2\| \psi_{eq} \|^2_{H^{-\frac{1}{2}}(\partial\Omega)}] \\
&\leq 2\max\{1, \| \psi_{eq} \|^2_{H^{-\frac{1}{2}}(\partial\Omega)}\}[\| \psi_0 \|^2_{H^{-\frac{1}{2}}(\partial\Omega)} + \alpha^2]
\end{aligned}$$

on the other hand by using $\langle v_\triangle, \psi_0\rangle_{\partial\Omega} = 0$, by the above decomposition and properties of inner products we have the following:

$$\begin{aligned}
\langle v_\triangle, \psi\rangle_{\partial\Omega} &= \langle v(\psi_0 + \alpha\psi_{eq}), \psi_0 + \alpha\psi_{eq}\rangle_{\partial\Omega} \\
&= \langle v_\triangle, \psi_0\rangle_{\partial\Omega} + 2\alpha\langle v\psi_{eq}, \psi\rangle_{(\partial\Omega)} + \alpha^2\langle \nu\psi_{eq}, \psi_{eq}\rangle_{(\partial\Omega)} \\
&\geq C\| \psi_0 \|^2_{H^{-\frac{1}{2}}(\partial\Omega)} + \alpha^2\lambda \\
&\geq \min\{C, \lambda\}[\| \psi_0 \|^2_{H^{-\frac{1}{2}}(\partial\Omega)} + \alpha^2],
\end{aligned}$$

and therefore the ellipticity estimate follows. □

4.3 Invertibility of single layer potential operator

The boundary integral operator $v_\triangle : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is Fredholm operator of index zero ($([1], theorem 7.6)$.For the case of $3D$,the following holds. For $\psi^* \in H^{-\frac{1}{2}}(\partial\Omega)$, if $\nu_\triangle\psi^*(y) = 0\ y \in \Omega$, then $\psi^* = 0$ which implies the invertibility of the single layer potential operator mapping from $H^{-\frac{1}{2}}(\partial\Omega)$ to $H^{\frac{1}{2}}(\partial\Omega)$. But this will not be true for two-dimensional case. This is well-known (see,eg[[4]Remark 1.42(ii)],[[3],proof of Theorem 6.22])that for some $2D$ domains the kernel of the operator $v_\triangle$ is non zero, i.e.$ker v_\triangle \neq \{0\}$ for some domains. The following eaxample illustrates this fact.

Example 4.3.1. *Take the density function $\phi \equiv 1$ and $\Omega = B_R(0)$ to be a disc of radius R centered at the origin and $\partial\Omega = S_R(0)$ be the circular boundary of the disc. We can show that;*

$$V_\triangle\phi(y) = \begin{cases} Rlog|y| & for|y| > R, \\ RlogR & for|y| \leq R. \end{cases}$$

Proof. Let $\phi \equiv 1$,then

$$(V_{\triangle}\phi)(y) = \frac{1}{2\pi}\int_{\partial\Omega} \log|x-y|ds_x.$$

For $|y| > R$,

$$(V_{\triangle}\phi)(y) = \frac{1}{2\pi}\int_{|x|=R} \log[|x-y| - log|y|]ds_x + \frac{1}{2\pi}\int_{|x|=R} \log|y|ds_x$$

For $y \to \infty$,the first integral tends to zero.Hence we have;

$$\begin{aligned}(V_{\triangle}\phi)(y) &= \frac{1}{2\pi}\int_{|x|=R} \log|y|ds_x \\ &= \frac{1}{2\pi}\log|y|\int_{|x|=R} ds_x \\ &= Rlog|y|.\end{aligned}$$

Therefore,for $|y| > R$ we have ;

$$\boxed{(V_{\triangle}\phi)(y) = Rlog|y|.} \tag{4.1}$$

For $|y| \le R$, in particular take $y = 0$,

$$\boxed{(V_{\triangle}\phi)(0) = \tfrac{1}{2\pi}\textstyle\int_{|x|=R} \log|x|ds_x = RlogR.} \tag{4.2}$$

The relation (4.1) implies that, the limit of the value of the single layer potential when y approach the boundary from exterior is given;

$$\boxed{\lim_{|y|\to R^+}(V_{\triangle}\phi)(y) = RlogR.} \tag{4.3}$$

Furthemore, since the single layer potential is continuous on $\mathbb{R}^2$ we have

$$(V_{\triangle}\phi)(y) = RlogR for|y| = R.$$

To dermine the value of the potential inside the disc we will use the maximum minimum principle .
Since the single layer potential is harmonic on Ω it has neither maximum and nor minimum value in the disc.Let

$$C_0 = (V_{\triangle}\phi)(y_0) for 0 < |y_0| < R.$$

If we assume $C_0 \neq R \ln R$, i.e., C_0 is different from the value of the potential on the boundary, we will arrive contradiction of the maximum principle. Thus $(V_{\triangle}\phi)(y)$ is constant on $\overline{\Omega}$.
Therefore, $(V_{\triangle}\phi)(y) = R \ln R$,for $|y| \leq R$. $\square$

Remark 1. *In the above example, if we take the value of $R = 1$, and since $a(y) \neq 0$,then $(V_a\phi)(y) = 0$ in $\overline{\Omega}$.*

Example 4.1 shows that, the kernel of the operator $\nu_a : H^{-\frac{1}{2}}(\partial\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega)$ contains non zero element for a unit ball. That is $ker\nu \neq 0$ for $\Omega = B(0, 1)$.
Thus ν is not one-to- one for this particular domain. The following question may arise; does the kernel of ν contains non-zero element on every bounded domain in $\mathbb{R}^2$? The answer is no.
In order to have have invertibility for the single layer potential operator in $2 - D$ we define the following subspace of the space $H^{-\frac{1}{2}}(\partial\Omega)$,

$$H_{**}^{-\frac{1}{2}}(\partial\Omega) := \{\phi \in H^{-\frac{1}{2}}(\partial\Omega) : \langle\phi, 1\rangle_{\partial\Omega} = 0\}$$

where the norm in $H_{**}^{-\frac{1}{2}}(\partial\Omega)$ is the induced norm in $H^{-\frac{1}{2}}(\partial\Omega)$.

Theorem 4.3.1. *Let $\psi \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$ satsfies $\nu_a\psi = 0$ on $\partial\Omega)$,then $\psi = 0$.*

Proof. The theorem holds for the operator $v_\Delta(see, [1], corollary 8.11(ii))$,

$$v_a\psi = 0$$

$$\Rightarrow \frac{1}{a(y)} v_\Delta\psi = 0$$

$$\Rightarrow \psi = 0, (since a(y) \neq 0, \Rightarrow v_\Delta \neq 0).$$

$\square$

Theorem 4.3.2. *Let $\Omega \subset \mathbb{R}^2$ have diameter $dim(\Omega) < 1$. Then the single layer potential $V_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega)$ is invertible.*

Proof. By theorem 4.2.3 for $dim(\Omega) < 1$ the operator $V_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega)$ is $H^{-\frac{1}{2}}(\partial\Omega)$-elliptic i.e

$$\|v_\Delta\psi\|_{H^{\frac{1}{2}}(\partial\Omega)} \geq C\|\psi\|_{H^{\frac{-1}{2}}(\partial\Omega)} for \psi \in H^{-\frac{1}{2}}(\partial\Omega)$$

and since it is also bounded by theorem 4.2.2 for $s = -\frac{1}{2}$, the Lax-Milgram theorem [i.e theorem 4.2.1] implies its invertibility.Then the invertibility of the operator $V_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ also follows. That is

$$v_\Delta^{-1} : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$$

is bounded and satisfying $\|v_\Delta^{-1}\psi\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq C\|\psi\|_{H^{\frac{1}{2}}(\partial\Omega)}$ for $\psi \in H^{\frac{1}{2}}(\partial\Omega)$. Hence, v_Δ is $H^{-\frac{1}{2}}(\partial\Omega)$ invertible. $\square$

Conclusion

From the study we observe that the single layer potential operator mapping from $H^{-\frac{1}{2}}(\partial\Omega)$ to $H^{\frac{1}{2}}(\partial\Omega)$ is invertible in the case of $3D$. But this is not true for two-dimentional case. For some $2D$ domains the kernel of the operator

$$v_{\Delta} : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$$

is non zero i.e $kerv_{\Delta} \neq \{0\}$. We have an example to illustrate this fact as follows. Take the density function $\phi \equiv 1$ and $\Omega = B_R(0)$ to be a disc of radius R centered at the origin and $\partial\Omega = S_R(0)$ be the circular boundary of the disc.We have shown that;

$$v_{\triangle}\phi(y) = \begin{cases} Rlog|y| & for|y| > R, \\ RlogR & for|y| \leq R. \end{cases}$$

This shows that the operator

$$v_{\Delta} : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$$

contains non zero element for a unit ball. Thus v_{Δ} is not one-to-one for this particular domain. In order to have invertibility for the single layer potential operator in $2D$ we define the following subspace of the space $H^{-\frac{1}{2}}(\partial\Omega)$,

$$H_{**}^{-\frac{1}{2}}(\partial\Omega) := \{\phi \in H^{-\frac{1}{2}}(\partial\Omega) : \langle \phi, 1 \rangle_{\partial\Omega} = 0\}$$

where the norm in $H_{**}^{-\frac{1}{2}}(\partial\Omega)$ is the induced norm in $H^{-\frac{1}{2}}(\partial\Omega)$ and by restricting the domain Ω such that $diam(\Omega) < 1$.

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