Graduate Seminar Report

on

THE CONVERGENCE OF FOURIER SERIES OF FUNCTIONS IN A HOMOGENEOUS BANACH SPACE

BY

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INTRODUCTION

In 1807, Joseph Fourier a mathematician and engineer discovered his series in connection with the theory of heat conduction. Fourier claimed that an arbitrary function, defined in a finite interval by an arbitrary capricious graph, can always be resolved into a sum of pure sine and cosine functions. But further investigations on Fourier's discovery were made by many mathematicians of the time and finally in 1829, L. Dirichlet circumscribed the function which allows expansion in Fourier series and set up the theorem on sufficient condition for the convergence of Fourier series.

The theory of representation of functions of a real variables by means of Fourier series is of highest importance not only on account of the fact that such mode of representation is at present an indispensable tool in the various branches of mathematical physics, but also because this theory has exercised the most far reaching influence upon the development of modern mathematical analysis. It is a significant fact that the theory of this mode of representation a function by a trigonometric series had its origin in the attempt to investigate the form of a stretching string in a state of vibration.

With regard to the convergence of the Fourier series the following problems are awaiting their solution:

1. Is there a continuous function with an everywhere divergent Fourier series?
2. What is the structure of the functions with absolutely convergent Fourier series?
3. What is the structure of the sets of uniqueness of Fourier series of functions?

In addition to the above problems, much remains to be done on the convergence or divergence almost everywhere of Fourier series of a function. Furthermore, the necessary and sufficient conditions for the convergence of the Fourier series at an individual point of the interval, or throughout any particular portion of the interval, have not been obtained.

Kolmogoroff constructed a function for which the Fourier series fails to converge at the points of a set of measure 2π; but this function is not continuous.

Lebesgue and Haar have investigated the general condition that, at a particular point of continuity of the function, the Fourier series should fail to converge. Haar also investigated the condition that, although the series converges at the point, the convergence should be non-uniform in any neighborhood of the point.

In this report we consider Fourier series of functions in a homogeneous Banach space on a circle group T. Part one mainly focus on the properties of Fourier coefficients and the relationship between the function and its Fourier coefficients. In part two we emphasis on the norm convergence and point wise convergence of Fourier series of functions in homogeneous Banach spaces.

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PART ONE
FOURIER SERIES ON THE UNIT CIRCLE T

The group **T** is defined as the quotient $\mathfrak{R}/2\pi Z$ where $2\pi Z$ is the integral multiples of 2π. Most often **T** is called the circle group. There is an obvious identification between functions on **T** and 2π-periodic functions on $\mathfrak{R}$. A function f on **T** is Lebesgue integrable if the corresponding 2π-periodic function, which we denote again by f, is Lebesgue integrable on $[0, 2\pi)$ and we set

$$\int_T f(t)dt = \int_0^{2\pi} f(x)dx.$$

We consider the interval $[0, 2\pi)$ as a model for **T** and the Lebesgue measure on **T** is the restriction of the Lebesgue measure of $\mathfrak{R}$ to $[0, 2\pi)$. We normalize the Lebesgue measure on **T** by writing the factor $\frac{1}{2\pi}$ in front of every integral. For each function f defined on **T** and $t_0 \in T$ we have

$$\int_T f(t-t_0)dt = \int_T f(t)dt.$$

1.1. FOURIER COEFFICIENTS

We denote by $\mathbf{L}^1(\mathbf{T})$ the space of all complex-valued Lebesgue integrable functions on **T**. Then $\mathbf{L}^1(\mathbf{T})$ is a Banach space with the norm defined by

$$\|f\|_{L^1} = \frac{1}{2\pi}\int_T |f(t)|dt.$$

DEFINITION. 1.1.1: A trigonometric polynomial on **T** is an expression of the form

$$P \sim \sum_{n=-N}^{N} a_n e^{int}.$$

The numbers n appearing in the expression are called the frequencies of P. The largest integer n such that $|a_n| + |a_{-n}| \neq 0$ is called the degree of P. Since each of the summands, $a_n e^{int}$, are functions and the sum is a finite sum, the expression is a function and we denote for each $t \in \mathbf{T}$

$$P(t) = \sum_{n=-N}^{N} a_n e^{int}.$$

Remarks:

1. For integers k, $\frac{1}{2\pi}\int_T e^{ikt}dt = \begin{cases} 1 \text{ if } k = 0 \\ 0 \text{ if } k \neq 0. \end{cases}$
2. Let P(t) = $\sum_{n=-N}^{N} a_n e^{\text{int}}$. Then $a_n = \frac{1}{2\pi}\int_T P(t)e^{-\text{int}}dt.$

DEFINITION.1.1.2: A trigonometric series on T is an expression of the form

$$S \sim \sum_{n=-\infty}^{\infty} a_n e^{\text{int}}.$$

The conjugate series $\widetilde{S}$ of the trigonometric series S is the series

$$\widetilde{S} \sim \sum_{j=-\infty}^{\infty} -iSgn(j)a_j e^{ijt}, \text{ where } Sgn(j) = \begin{cases} -1 \text{ if } j < 0 \\ 0 \text{ if } j = 0 \\ 1 \text{ if } j > 0 \end{cases}.$$

DEFINITION.1.1.3: Let $f \in$ **L**^{1}**(T)**. The n$^{\text{th}}$ Fourier coefficient of f, denoted by $\hat{f}(n)$, is defined as

$$\hat{f}(n) = \frac{1}{2\pi}\int_T f(t)e^{-\text{int}}dt.$$

DEFINITION.1.1.4: Let f$\in$ **L**^{1}**(T)**. The Fourier series S[f] of f is the trigonometric series

$$S[f] \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{\text{int}}.$$

The conjugate Fourier series of f, $\widetilde{S}[f]$, is the series conjugate to S[f]. We shall say that a trigonometric series is a Fourier series if and only if it is the Fourier series of some function in **L**^{1}**(T)**.

The following Theorem is the immediate consequence of the above definitions.

THEOREM 1.1.1: Let $f \in$**L**^{1}**(T)** and g$\in$**L**^{1}**(T)**. Then for $n = 0, \pm 1, \pm 2, ...$

i. $\widehat{(f+g)}(n) = \hat{f}(n) + \hat{g}(n)$.

ii. For any complex number α, $\left(\widehat{\alpha f}\right)(n) = \alpha\hat{f}(n)$.

iii. If $\overline{f}$ is the complex conjugate of f, i.e., $\overline{f}(t) = \overline{f(t)}$ for all t$\in$T, then

$$\hat{\overline{f}}(n) = \overline{\hat{f}(n)}.$$

iv. Denote $f_\tau(t) = f(t-\tau), \tau \in T$. Then $\hat{f}_\tau(n) = \hat{f}(n)e^{-in\tau}$.

v. $\left|\hat{f}(n)\right| \le \frac{1}{2\pi}\int_T |f(t)|dt = \|f\|_{L^1}$.

COROLLARY. 1.1.1: Assume $f_j \in L^1(T)$, j = 0,1,2… and $\lim_{j\to\infty}\|f_j - f_o\|_{L^1} = 0$. Then $\hat{f}_j(n) \xrightarrow{j\to\infty} \hat{f}_o(n)$ uniformly.

Proof: Let ε>0 be given. Then there exists N = N(ε) such that

$$\|f_j - f_o\|_{L^1} < \varepsilon$$

for all j ≥ N. Then

$$\left|\hat{f}_j(n) - \hat{f}_o(n)\right| \le \|f_j - f_o\|_{L^1} < \varepsilon$$

for all n∈Z and for all j≥N. Therefore, $\hat{f}_j(n) \xrightarrow{j\to\infty} \hat{f}_o(n)$ uniformly.//

THEOREM. 1.1.2: Let $f \in L^1(T)$. Assume $\hat{f}(0) = 0$ and define

$$F(t) = \int_0^t f(\tau)d\tau.$$

Then F is continuous 2π-periodic and $\hat{F}(n) = \frac{1}{in}\hat{f}(n)$ for all n≠0.

Proof: Since F is absolutely continuous, F is continuous. Now to show the periodicity of F,

$$F(t+2\pi) - F(t) = \int_0^{t+2\pi} f(\tau)d\tau - \int_0^t f(\tau)d\tau = \int_0^{2\pi} f(\tau)d\tau = \hat{f}(0) = 0.$$

Therefore, F(t+2π) = F(t) for all t∈T. Hence F is a 2π-periodic function. Finally for n≠0 $\hat{F}(n) = \frac{1}{2\pi}\int_T F(t)e^{-\text{int}}dt$. Since $F^{|}(t) = f(t)$, applying integration by parts we obtain

$$\hat{F}(n) = -\frac{1}{2\pi\, in}F(t)e^{-\text{int}}\Big|_o^{2\pi} + \frac{1}{2\pi\, in}\int_T F^{|}(t)e^{-\text{int}}dt$$

$$= -\frac{1}{2\pi\, in}[F(2\pi) - F(0)] + \frac{1}{2\pi\, in}\int_T f(t)e^{-\text{int}}dt$$

$$= \frac{1}{2\pi\, in}\hat{f}(n). \quad //$$

THEOREM. 1.1.3: Let $g, f \in L^1(T)$. For almost all t, the function $f(t-\tau)g(\tau)$ is integrable (as a function of τ on **T**). If we write

$$h(t) = \frac{1}{2\pi} \int_T f(t-\tau)g(\tau)d\tau$$

then $h \in L^1(T)$ and $\|h\|_{L^1} \le \|f\|_{L^1} \|g\|_{L^1}$. Moreover, $\hat{h}(n) = \hat{f}(n)\hat{g}(n)$ for all n.

Proof: The functions $f(t-\tau)$ and $g(\tau)$ are measurable functions of the two variables (t, τ). Hence the product function $F(t, \tau) = f(t-\tau)g(\tau)$ is also measurable. For almost all τ, $F(t, \tau)$ is just a constant multiple of f_τ hence integrable, and

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}\int_0^{2\pi}|F(t,\tau)|dt\right)d\tau = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}\int_0^{2\pi}|f(t-\tau)|\,|g(\tau)|dt\right)d\tau$$

$$= \frac{1}{2\pi}\int_0^{2\pi}|g(\tau)|\,\|f\|_{L^1}d\tau = \|f\|_{L^1}\|g\|_{L^1}$$

By the Theorem of Fubini, $F(t, \tau)$ is integrable over $[0, 2\pi]$ as a function of τ for almost all t. Now

$$\frac{1}{2\pi}\int_0^{2\pi}|h(t)|\,dt \le \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}|F(t,\tau)|\,dtd\tau = \|f\|_{L^1}\|g\|_{L^1}.$$

Thus h is integrable. For $n \in Z$,

$$\hat{h}(n) = \frac{1}{2\pi}\int_0^{2\pi} h(t)e^{-\text{int}}dt$$

$$= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(t-\tau)g(\tau)e^{-in(t-\tau)}e^{-in\tau}dtd\tau$$

$$= \hat{f}(n)\frac{1}{2\pi}\int_0^{2\pi} g(\tau)e^{-in\tau}d\tau = \hat{f}(n)\hat{g}(n). \quad //$$

DEFINITION. 1.1.5: The convolution $f*g$ of the $L^1(T)$ functions f and g is the function h defined in Theorem 1.1.3.

THEOREM. 1.1.4: The convolution operation in $L^1(T)$ is commutative, associative and distributive (with respect to the addition).

Proof: The change of variable $u = t - \tau$ gives

$$\frac{1}{2\pi}\int_T f(t-\tau)g(\tau)d\tau = \frac{1}{2\pi}\int_T g(t-u)f(u)du.$$

Hence we get $f*g = g*f$.

If $f_1, f_2, f_3 \in L^1(T)$ then

1. $$[(f_1*f_2)*f_3](t) = \frac{1}{2\pi}\int_T (f_1*f_2)(t-\tau)f_3(\tau)d\tau$$
$$= \frac{1}{2\pi}\int_T (\tfrac{1}{2\pi}\int_T f_1(t-\tau-u)f_2(u)du)f_3(\tau)d\tau$$
$$= \frac{1}{4\pi^2}\int_T\int_T f_1(t-\tau-u)f_2(u)f_3(\tau)dud\tau$$
$$= \frac{1}{4\pi^2}\int_T\int_T f_1(t-w)f_2(w-\tau)f_3(\tau)dwd\tau$$
$$= \frac{1}{2\pi}\int_T f_1(t-w)(\tfrac{1}{2\pi}\int_T f_2(w-\tau)f_3(\tau)d\tau\, dw$$
$$= \frac{1}{2\pi}\int_T f_1(t-w)(f_2*f_3)(w)dw$$
$$= [f_1*(f_2*f_3)](t).$$

$$[(f_1*(f_2+f_3)](t) = \frac{1}{2\pi}\int_T f_1(t-\tau)[f_2+f_3](\tau)d\tau$$
$$= \frac{1}{2\pi}\int_T f_1(t-\tau)f_2(\tau)d\tau + \frac{1}{2\pi}\int_T f_1(t-\tau)f_3(\tau)d\tau$$
$$= (f_1*f_2)(t) + (f_1*f_3)(t). \quad //$$

LEMMA. 1.1.1: Assume $f \in L^1(T)$ and let $\varphi(t) = e^{int}$ for some integer n. Then $(\varphi*f)(t) = \hat{f}(n)e^{int}$.

Proof:

$$(\varphi*f)(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{in(t-\tau)}f(\tau)d\tau$$
$$= (\frac{1}{2\pi}\int_0^{2\pi} e^{-in\tau}f(\tau)d\tau)e^{int}$$
$$= \hat{f}(n)e^{int}. \quad //$$

COROLLARY. 1.1.2: If $f \in L^1(T)$ and $k(t) = \sum_{n=-N}^{N} a_n e^{int}$ then

$$(k*f)(t) = \sum_{n=-N}^{N} a_n \hat{f}(n) e^{int}.$$

Proof:

$$(k*f)(t) = \frac{1}{2\pi} \int_T \sum_{n=-N}^{N} a_n e^{in(t-\tau)} f(\tau) d\tau$$

$$= \sum_{n=-N}^{N} a_n e^{int} \frac{1}{2\pi} \int_T e^{-in\tau} f(\tau) d\tau$$

$$= \sum_{n=-N}^{N} a_n \hat{f}(n) e^{int}. \ //$$

1.2. SUMMABILITY IN NORM AND HOMOGENEOUS BANACH SPACES ON T

THEOREM. 1.2.1: In the Banach space $L^1(T)$ the following properties hold.

(i) *The Translation Invariance.*

If $f \in L^1(T)$ and $\tau \in T$ then $f_\tau \in L^1(T)$ and $\|f\|_{L^1} = \|f_\tau\|_{L^1}$, where $f_\tau(t) = f(t-\tau)$.

(ii) *Continuity of theTranslation.*

The $L^1(T)$-valued function $\tau \to f_\tau$ is continuous on T, that is, for $f \in L^1(T)$ and $\tau_o \in T$ $\lim\limits_{\tau \to \tau_o} \|f_\tau - f_{\tau_o}\|_{L^1} = 0$.

Proof: Let $f \in L^1(T)$ and $\tau \in T$. Then

$$\frac{1}{2\pi} \int_T |f_\tau(t)| dt = \frac{1}{2\pi} \int_T |f(t-\tau)| dt = \frac{1}{2\pi} \int_T |f(t)| dt = \|f\|_{L^1} < \infty.$$

Therefore, $f_\tau \in L^1(T)$ and $\|f\|_{L^1} = \|f_\tau\|_{L^1}$.

Notice that a continuous function on T is uniformly continuous. Let f be a continuous function on T. Then given $\varepsilon > 0$ there exists $\delta > 0$ such that $|t-\tau| < \delta$ implies $|f(t) - f(\tau)| < \varepsilon$. Now

$$\left\|f_\tau - f_{\tau_o}\right\|_{L^1} = \frac{1}{2\pi}\int_T \left|f_\tau(t) - f_{\tau_o}(t)\right| dt$$

$$= \frac{1}{2\pi}\int_T \left|f(t-\tau) - f(t-\tau_o)\right| dt$$

$$< \frac{1}{2\pi}\int_0^{2\pi} \varepsilon dt = \varepsilon \quad provided\ that\ |\tau - \tau_o| < \delta.$$

Therefore, $\lim_{\tau \to \tau_o} \left\|f_\tau - f_{\tau_o}\right\|_{L^1} = 0.$

Let $f \in L^1(T)$. Since continuous functions are dense in $L^1(T)$, there exists a continuous function g such that $\|g - f\|_{L^1} < \frac{\varepsilon}{4}$. Since g is continuous on T there exists $\delta = \delta(\varepsilon) > 0$ such that $\tau, \tau_o \in T$ and $|\tau - \tau_o| < \delta$ implies $\left\|g_\tau - g_{\tau_o}\right\|_{L^1} < \frac{\varepsilon}{2}$. Now

$$\left\|f_\tau - f_{\tau_o}\right\|_{L^1} \le \left\|f_\tau - g_\tau\right\|_{L^1} + \left\|g_\tau - g_{\tau_o}\right\|_{L^1} + \left\|g_{\tau_o} - f_{\tau_o}\right\|_{L^1}$$

$$= \left\|(f-g)_\tau\right\|_{L^1} + \left\|g_\tau - g_{\tau_o}\right\|_{L^1} + \left\|(g-f)_{\tau_o}\right\|_{L^1}$$

$$= \|f - g\|_{L^1} + \left\|g_\tau - g_{\tau_o}\right\|_{L^1} + \|g - f\|_{L^1}$$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon \quad provided \quad that\ |\tau - \tau_o| < \delta. \quad //$$

DEFINITION. 1.2.1: A summability kernel is a sequence (k_n) of continuous 2π-periodic functions satisfying:

$$(S_1) \quad \frac{1}{2\pi}\int_T k_n(\tau) d\tau = 1.$$

$$(S_2) \quad \frac{1}{2\pi}\int_T |k_n(\tau)| d\tau \le const.$$

$$(S_3) \quad For\ all\ 0 < \delta < \pi,\ \lim_{n\to\infty} \int_\delta^{2\pi-\delta} |k_n(t)| dt = 0.$$

A positive summability kernel is one such that $k_n(t) \ge 0$ for all t and n.

EXAMIPLE. 1.2.1:
Fejer's kernel ($\mathbf{F_n}$) defined by

$$F_n(t) = \sum_{j=-n}^{n} (1 - \tfrac{|j|}{n+1})e^{ijt}$$

is a summability kernel.

EXAMPLE. 1.2.2:
The de la Vallee Poussin kernel defined by

$$\mathbf{V_n(t) = 2F_{2n+1}(t) - F_n(t)}$$

is a summability kernel.

LEMMA. 1.2.1: Let B be a Banach space, φ a continuous B-valued function on **T** and (k_n) a summability kernel. Then

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_T k_n(\tau)\varphi(\tau)d\tau = \varphi(0).$$

Proof: For $0 < \delta < \pi$, by (S_1) we have

$$\frac{1}{2\pi}\int_T k_n(\tau)\varphi(\tau)d\tau - \varphi(0) = \frac{1}{2\pi}\int_T k_n(\tau)[\varphi(\tau)-\varphi(0)]d\tau$$

$$= \frac{1}{2\pi}(\int_{-\delta}^{\delta} + \int_{\delta}^{2\pi})k_n(\tau)[\varphi(\tau)-\varphi(0)]d\tau$$

Now $\left\|\frac{1}{2\pi}\int_{-\delta}^{\delta} k_n(\tau)[\varphi(\tau)-\varphi(0)]d\tau\right\|_B \le \max_{|\tau|\le\delta}\|\varphi(\tau)-\varphi(0)\|_B \|k_n\|_{L^1}$, and

$$\left\|\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} k_n(\tau)[\varphi(\tau)-\varphi(0)]d\tau\right\|_B \le \max_{\tau}\|\varphi(\tau)-\varphi(0)\|_B \frac{1}{2\pi}\int_{\delta}^{2\pi-\delta}| k_n(\tau)| d\tau.$$

Let $\varepsilon>0$ be given. Then by the continuity of φ at 0 there exists $\delta=\delta(\varepsilon)$ such that $|\tau| < \delta$ implies

$$\|\varphi(\tau)-\varphi(0)\|_B < \varepsilon.$$

And by (S_3) there exists N such that

$$\int_{\delta}^{2\pi-\delta}| k_n(\tau)| d\tau < \varepsilon \quad \forall n > N.$$

Therefore,

$$\left\|\frac{1}{2\pi}\int_T k_n(\tau)\varphi(\tau)d\tau-\varphi(0)\right\|_B$$

$$\leq\left\|\frac{1}{2\pi}\int_{-\delta}^{\delta} k_n(\tau)[\varphi(\tau)-\varphi(0)]d\tau\right\|_B+\left\|\frac{1}{2\pi}\int_{\delta}^{2\pi} k_n(\tau)[\varphi(\tau)-\varphi(0)]d\tau\right\|_B$$

$$<\varepsilon[\|k_n\|_{L^1}+\max_\tau\|\varphi(\tau)-\varphi(0)\|_B] \quad \textit{provided that } n\geq N.$$

Hence $\lim_{n\to\infty}\frac{1}{2\pi}\int_T k_n(\tau)\varphi(\tau)d\tau=\varphi(0)$ in B-norm. //

THEOREM. 1.2.2: Let $f\in L^1(T)$ and (k_n) a summability kernel. Then

$$f=\lim_{n\to\infty}\frac{1}{2\pi}\int_T k_n(\tau)f_\tau d\tau$$

in the $L^1(T)$-norm.

Proof: From Theorem 1.2.1 the mapping $\tau\to f_\tau$ is continuous $L^1(T)$-valued function on T. Thus by Lemma 1.2.1 we get

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_T k_n(\tau)f_\tau d\tau=f_0=f. \text{ //}$$

LEMMA. 1.2.2: Let k be a continuous function on T and $f\in L^1(T)$. Then

$$\frac{1}{2\pi}\int_T k(\tau)f_\tau d\tau=k*f.$$

Proof: Assume first that f is continuous on T. We have

$$\frac{1}{2\pi}\int_T k(\tau)f_\tau d\tau=\frac{1}{2\pi}\lim_{\|P_j\|\to 0}\sum_j(\tau_{j+1}-\tau_j)k(\tau_j)f_{\tau_j}$$

the limit being taken in the $\mathbf{L^1(T)}$-norm as the subdivision $\{\tau_j\}$ of $[0, 2\pi]$ becomes finer and finer. On the other hand,

$$\frac{1}{2\pi}\lim_{\|P_j\|\to 0}\sum_j(\tau_{j+1}-\tau_j)k(\tau_j)f(t-\tau_j)=\frac{1}{2\pi}\int_T k(\tau)f(t-\tau)d\tau=(k*f)(t)$$

uniformly and the lemma is proved for continuous functions. For arbitrary $f\in L^1(T)$ and $\varepsilon>0$ there exists a continuous function g on **T** such that

$$\|f-g\|_{L^1}<\varepsilon.$$

Now since g is continuous we have

$$\frac{1}{2\pi}\int_T k(\tau)f_\tau d\tau - k*f = \frac{1}{2\pi}\int_T k(\tau)f_\tau d\tau - \frac{1}{2\pi}\int_T k(\tau)g_\tau d\tau + k*g - k*f$$
$$= \frac{1}{2\pi}\int_T k(\tau)(f_\tau - g_\tau)d\tau + k*(g-f)$$

Consequently,

$$\left\|\frac{1}{2\pi}\int_T k(\tau)f_\tau d\tau - k*f\right\|_{L^1} \le \left\|\frac{1}{2\pi}\int_T k(\tau)(f_\tau - g_\tau)d\tau\right\|_{L^1} + \|k*(g-f)\|_{L^1}$$
$$\le \|k\|_{L^1}\|g_\tau - f_\tau\|_{L^1} + \|k\|_{L^1}\|g-f\|_{L^1}$$
$$= 2\|k\|_{L^1}\|g-f\|_{L^1}$$
$$< 2\varepsilon\|k\|_{L^1}.$$

Since ε was arbitrary we have $\frac{1}{2\pi}\int_T k(\tau)f_\tau d\tau = k*f$. //

NOTATION:

1. For $f \in L^1(T)$ and for Fejer's kernel F_n we denote $F_n * f = \sigma_n(f)$.
2. $\sigma_n(f, t) = (F_n * f)(t)$.

REMARKS:

1. $\sigma_n(f,t) = \sum_{j=-n}^{n}(1-\frac{|j|}{n+1})\hat{f}(j)e^{ijt}$.
2. For each $f \in L^1(T)$ we have $\lim_{n\to\infty}\sigma_n(f) = f$ in the $L^1(T)$-norm.
3. Trigonometric polynomials are dense in $L^1(T)$.

THEOREM. 1.2.3: *The Uniqueness Theorem.* Let $f \in L^1(T)$. If $\hat{f}(n) = 0$ for all $n \in Z$ then $f = 0$.

Proof: By the above remark, $\sigma_n(f)=0$ for all n. Since $\sigma_n(f)\to f$ in $L^1(T)$ we have $0 = \lim_{n\to\infty}\sigma_n(f) = f$. //

COROLLARY. 1.2.1: Let $f, g \in L^1(T)$. If $\hat{f}(n) = \hat{g}(n)$ for all n then $f = g$.

THEOREM. 1.2.4: *The Riemann-Lebesgue Lemma.* If $f \in L^1(T)$ then

$$\lim_{|n|\to\infty}\hat{f}(n) = 0.$$

Proof: Let ε>0. Then there exists a trigonometric polynomial on T of degree N such that $\|f-P\|_{L^1}<\varepsilon$. Then $\hat{P}$(n) =0 for all |n| >N. Consequently,

$$\left|\hat{f}(n)\right|=\left|\hat{f}(n)-\hat{P}(n)\right|\leq\|f-P\|_{L^1}<\varepsilon \text{ for all } |n|>N.$$

Therefore, $\lim\limits_{|n|\to\infty}\hat{f}(n)=0$. //

THEOREM. 1.2.5: If K is a compact subset of $\mathbf{L}^1(\mathbf{T})$ and ε>0, there exist a finite number of trigonometric polynomials $P_1, P_2, \ldots, P_N$ such that for every $f\in$K there exists a j, 1≤j≤N, such that $\|f-P_j\|_{L^1}<\varepsilon$. If |n|> $\max\limits_{1\leq j\leq N} N_j$ where N_j is the degree of P_j, then $\left|\hat{f}(n)\right|<\varepsilon$ for all $f\in$K.

Proof: Since trigonometric polynomials are dense in L^1(T), there exists a collection Ω of ε-spheres about trigonometric polynomials that cover L^1(T). Then Ω also covers K. By the compactness of K there exist ε-spheres $S_1,\ldots, S_N$ such that $K\subseteq\bigcup\limits_{j=1}^{N}S_j$. By the definition of Ω we get trigonometric polynomials $P_1, P_2, \ldots, P_N$ such that $S_j = B(P_j, \varepsilon)$, j=1,2,…,N. Let $f\in$K. Then there exists j, 1≤j≤N, such that $f\in S_j$, that is, $\|f-P_j\|_{L^1}<\varepsilon$.

If |n| > $\max\limits_{1\leq j\leq N} N_j$ then $\hat{P}_j(n)=0$ for all j = 1,2,…, N, and

$$\left|\hat{f}(n)\right|\leq\|f-P_j\|_{L^1}<\varepsilon \text{ for some j, } 1\leq j\leq N. //$$

REMARK: The Riemann-Lebesgue Lemma holds uniformly on compact subsets of L^1(T).

NOTATION: For $f\in L^1$(T) we denote $S_n(f)$ the n^{th} partial sum of S[f], that is,

$$(S_n(f))(t)=S_n(f,t)=\sum_{j=-n}^{n}\hat{f}(j)e^{ijt}.$$

REMARKS:

1. $\sigma_n(f)=\dfrac{1}{n+1}\sum\limits_{j=0}^{n}S_j(f)$.

2. If S[f] converges in L^1(T) then the limit is necessarily f.

Proof: Assume $S_n(f) \xrightarrow{n\to\infty} g$ in L^1(T)-norm. Then

$$\hat{g}(j) = \underset{n\to\infty}{Lim}(\hat{S}_n(f))(j) = \underset{n\to\infty}{Lim}\,\hat{f}(j) = \hat{f}(j) \text{ for all } j\in Z.$$

By the Uniqueness Theorem, $f = g$. //

3. $S_n(f) = D_n * f$, where D_n is the Dirichlet kernel defined by

$$D_n(t) = \sum_{j=-n}^{n} e^{ijt} = \frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}.$$

Proof:

$$(D_n * f)(t) = \frac{1}{2\pi}\int_T D_n(\tau) f(t-\tau)d\tau$$

$$= \sum_{j=-n}^{n} \frac{1}{2\pi}\int_T f(t-\tau)e^{ij\tau}d\tau$$

$$= \sum_{j=-n}^{n} \hat{f}(j)e^{ijt} = S_n(f,t).//$$

DEFINITION. 1.2.2: A homogeneous Banach space on **T** is a linear subspace B of $\mathbf{L^1(T)}$ having a norm $\|.\|_B \geq \|.\|_{L^1}$ under which it is a Banach space, and having the following properties:

(H-1) *Translation invariance*
If $f\in B$ and $\tau\in T$ then $f_\tau\in B$ and $\|f\|_B = \|f_\tau\|_B$.

(H-2) *Continuity of the translation*
For all $f\in B$ and $\tau_o\in T$, $\underset{\tau\to\tau_o}{Lim}\|f_\tau - f_{\tau_o}\|_B = 0$.

REMARK: Let B be a homogeneous Banach space on T. If $f\in B$ and $\tau,\tau_o\in T$ then $\|f_\tau - f_{\tau_o}\|_B = \|f_{\tau-\tau_o} - f\|_B$.
By (H-1) we get

$$\|f_\tau - f_{\tau_o}\|_B = \|(f_\tau - f_{\tau_o})_{-\tau_o}\|_B = \|f_{\tau-\tau_o} - f_0\|_B = \|f_{\tau-\tau_o} - f\|_B.$$

LEMMA. 1.2.3: Let $B\subseteq L^1(T)$ be a Banach space satisfying (H-1). Denote by B_c the set of all $f\in B$ such that $\tau\to f_\tau$ is a continuous B-valued function. Then B_c is a closed subspace of B.

Proof: Let g be a limit point of B_c in B. Then given $\varepsilon>0$ there exists $f\in B_c$ such that $\|g-f\|_B < \frac{\varepsilon}{4}$. Then there exists also $\delta = \delta(\varepsilon) > 0$ such that $|\tau-\tau_o|<\delta$ implies $\|f_\tau - f_{\tau_o}\|_B < \frac{\varepsilon}{2}$. Now

$$\|g_\tau - g_{\tau_o}\|_B \le \|g_\tau - f_\tau\|_B + \|f_\tau - f_{\tau_o}\|_B + \|f_{\tau_o} - g_{\tau_o}\|_B$$
$$= 2\|g-f\|_B + \|f_\tau - f_{\tau_o}\|_B < \varepsilon \text{ provided that } |\tau-\tau_o|<\delta.$$

Thus $\tau\to g_\tau$ is a continuous B-valued function on T. Hence $g_\tau\in B_c$. Therefore, B_c is a closed subspace of B. //

EXAMPLES OF HOMOGENEOUS BANACH SPACES

1. By Theorem 1.2.1, $L^1(T)$ is a homogeneous Banach space.

2. C(T)- the space of all continuous 2π-periodic functions with the norm
$$\|f\|_\infty = \max_t |f(t)|.$$

   C(T) with the defined norm is a Banach space satisfying $\|.\|_B \ge \|.\|_{L^1}$. Let $f\in C(T)$ and $\tau\in T$. Since f is uniformly continuous, f_τ is continuous 2π-periodic function and thus $f_\tau\in C(T)$. By the uniform continuity of f for any given $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)$ such that $t_1, t_2\in T$ and $|t_1 - t_2| < \delta$ implies $|f(t_1)-f(t_2)| < \varepsilon$. Therefore,
$$\|f_\tau - f_{\tau_o}\|_\infty = \max_t |f_\tau(t) - f_{\tau_o}(t)| = \max_t |f(t-\tau) - f(t-\tau_o)| < \varepsilon$$
   provided that $|\tau-\tau_o|<\delta$. Thus C(T) is a homogeneous Banach space on T.

3. $C^{(n)}(T)$- the subspace of C(T) of all n-times continuously differentiable functions (n being a rational integer) with the norm
$$\|f\|_{C^n} = \sum_{j=0}^{n} \tfrac{1}{j!} \max_t |f^{(j)}(t)| = \sum_{j=0}^{n} \tfrac{1}{j!} \|f^{(j)}\|_\infty .$$

   With the defined norm $C^{(n)}(T)$ is a Banach space satisfying
$$\|f\|_{C^n} \ge \|f\|_\infty \ge \|f\|_{L^1}.$$
   For $0\le j\le n$ we have for all $t\in T$, $f_\tau^{(j)}(t) = f^{(j)}(t-\tau)$. Therefore,

$f_\tau \in C^n(T)$ and $\|f_\tau\|_{C^n} = \sum_{j=0}^{n} \frac{1}{j!}\|f_\tau^{(j)}\|_\infty = \sum_{j=0}^{n} \frac{1}{j!}\|f^{(j)}\|_\infty = \|f\|_{C^n}$.

Let $f \in C^n(T)$ and $\tau_0 \in T$. Then

$$\lim_{\tau\to\tau_o}\|f_\tau - f_{\tau_o}\|_{C^n} = \sum_{j=0}^{n} \frac{1}{j!}\lim_{\tau\to\tau_o}\|f_\tau^{(j)} - f_{\tau_o}^{(j)}\|_\infty = 0 \quad \text{since } f^{(j)} \in C(T).$$

Therefore, $C^n(T)$ is a homogeneous Banach space.

4. $L^p(T)$, $1 < p < \infty$ - the subspace of $L^1(T)$ consisting of all the functions f for which $\int_T |f(t)|^p\, dt < \infty$ with the norm

$$\|f\|_{L^p} = [\frac{1}{2\pi}\int_T |f(t)|^p\, dt]^{1/p}.$$

Clearly $L^p(T)$, $1 < p < \infty$, with the defined norm is a Banach space satisfying $\|f\|_{L^p} \geq \|f\|_{L^1}$. By the translation invariance of the Lebesgue measure we have $\int_T |f_\tau(t)|^p\, dt = \int_T |f(t-\tau)|^p\, dt = \int_T |f(t)|^p\, dt < \infty$. Hence $f_\tau \in L^p(T)$. Since continuous functions are dense in $L^p(T)$, given $\varepsilon>0$ for $f \in L^p(T)$ there exists a continuous function $g \in L^p(T)$ such that $\|f - g\|_{L^p} < \frac{\varepsilon}{4}$. For this $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that $\|g_\tau - g_{\tau_o}\|_{L^p} < \frac{\varepsilon}{2}$ whenever $|\tau-\tau_o| < \delta$. Now

$$\|f_\tau - f_{\tau_o}\|_{L^p} \leq \|f_\tau - g_\tau\|_{L^p} + \|g_\tau - g_{\tau_o}\|_{L^p} + \|g_{\tau_o} - f_{\tau_o}\|_{L^p}$$

$$< \frac{\varepsilon}{2} + \|g_\tau - g_{\tau_o}\|_{L^p} < \varepsilon \text{ provided that } |\tau - \tau_o| < \delta.$$

Therefore, $\lim_{\tau\to\tau_o}\|f_\tau - f_{\tau_o}\|_{L^p} = 0$.

Hence $L^p(T)$ is a homogeneous Banach space on T.

THEOREM. 1.2.6: Let B be a homogeneous Banach space on T, $f \in B$ and (k_n) a summability kernel. Then

$$\|k_n * f - f\|_B \xrightarrow{n\to\infty} 0.$$

Proof: Since $\|.\|_B \geq \|.\|_{L^1}$, the B-valued integral

$$\frac{1}{2\pi}\int_T k_n(\tau) f_\tau\, d\tau$$

is the same as the $L^1(T)$-valued integral which by Lemma 1.2.2 is equal to

$k_n * f$. By Lemma 1.2.1

$$\lim_{n\to\infty} k_n * f = f_0 = f \text{ in B-norm. //}$$

THEOREM. 1.2.7: Let B be a homogeneous Banach space on T. Then the trigonometric polynomials in B are everywhere dense.

Proof: For every $f \in B$ by Theorem 1.2.6,

$$\sigma_n(f) \xrightarrow{n\to\infty} f \text{ in B-norm. //}$$

COROLLARY. 1.2.2. *Weierstrass Approximation Theorem.* Every continuous 2π-periodic function can be approximated uniformly by trigonometric polynomials.

Proof: Let $f \in C(T)$. Then by Theorem 1.2.7 there exists a sequence (P_n) of trigonometric polynomials in C(T) such that

$$\lim_{n\to\infty} \|P_n - f\|_\infty = 0.$$

Since each P_n is uniformly continuous and f is also uniformly continuous on T, P_n converges to f uniformly. //

1.3. FOURIER SERIES OF SQUARE SUMMABLE FUNCTIONS

$L^2(T)$ with its inner product defined by

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}dt$$

is a Hilbert space. The exponentials $\{e^{int}\}$, n=0, ±2, ±2,..., form a complete orthonormal system in this Hilbert space.

THEOREM. 1.3.1: Let $f \in L^2(T)$. Then

$$a. \quad \sum_{n=-\infty}^{\infty} \left|\hat{f}(n)\right|^2 = \frac{1}{2\pi}\int_T |f(t)|^2 dt.$$

$$b. \quad f = \lim_{N\to\infty} \sum_{n=-N}^{N} \hat{f}(n)e^{int} \quad \textit{in the } L^2(T)-\textit{norm}.$$

Proof: Since $\{e^{int}\}$ is a complete orthonormal system in $L^2(T)$, for every $f \in L^2(T)$ we have

$$\|f\|_{L^2}^2 = \sum_{n=-\infty}^{\infty} \left|\left\langle f, e^{\text{int}} \right\rangle\right|^2 = \sum_{n=-\infty}^{\infty} \left|\hat{f}(n)\right|^2 \qquad (*)$$

Now to show (b) we have that the series in (*) converges. Therefore, for given ε>0 there exists N such that $\sum_{|n|=N+1}^{\infty} \left|\hat{f}(n)\right|^2 < \varepsilon$. Now

$$\left\| f - \sum_{n=-N}^{N} \hat{f}(n) e^{\text{int}} \right\|_{L^2}^2 = \left\langle f - \sum_{n=-N}^{N} \hat{f}(n) e^{\text{int}}, f - \sum_{n=-N}^{N} \hat{f}(n) e^{\text{int}} \right\rangle$$

$$= \langle f, f \rangle - \sum_{n=-N}^{N} \hat{f}(n) \left\langle e^{\text{int}}, f \right\rangle - \sum_{n=-N}^{N} \overline{\hat{f}(n)} \left\langle f, e^{\text{int}} \right\rangle$$

$$+ \left\langle \sum_{n=-N}^{N} \hat{f}(n) e^{\text{int}}, \sum_{n=-N}^{N} \hat{f}(n) e^{\text{int}} \right\rangle$$

$$= \|f\|_{L^2}^2 - 2 \sum_{n=-N}^{N} |\hat{f}(n)|^2 + \sum_{n=-N}^{N} |\hat{f}(n)|^2$$

$$= \|f\|_{L^2}^2 - \sum_{n=-N}^{N} |\hat{f}(n)|^2$$

$$= \sum_{|n|=N+1}^{\infty} |\hat{f}(n)|^2 < \varepsilon.$$

Therefore, $S_N(f) \xrightarrow{N \to \infty} f$ in L^2(T)- norm.//

THEOREM. 1.3.2: Let $(a_n)_{n=-\infty}^{\infty}$ be a sequence of complex numbers satisfying $\sum_{n=-\infty}^{\infty} |a_n|^2 < \infty$. Then there exists a unique $f \in L^2$(T) such that $a_n = \hat{f}(n)$.

Proof: Put $f(t) = \sum_{n=-\infty}^{\infty} a_n e^{\text{int}}$. Then clearly $a_n = \hat{f}(n)$. The uniqueness of *f* follows from the Uniqueness Theorem.//

THEOREM. 1.3.3: Let *f*, $g \in L^2$(T). Then

$$\langle f, g \rangle = \frac{1}{2\pi} \int_T f(t) \overline{g(t)} dt = \sum_{n=-\infty}^{\infty} \hat{f}(n) \hat{g}(n).$$

Proof: By Theorem 1.3.1 and Theorem 1.3.2 we have

$$f = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{\text{int}} \quad and\ g = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{\text{int}}.$$

Then

$$\langle f, g \rangle = \left\langle \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{\text{int}}, \sum_{n=-\infty}^{\infty} \hat{g}(n)e^{\text{int}} \right\rangle$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \hat{f}(n)\hat{g}(m) \underbrace{\left\langle e^{\text{int}}, e^{imt} \right\rangle}_{\delta_{nm}}$$

$$= \sum_{n=-\infty}^{\infty} \hat{f}(n)\hat{g}(n).//$$

REMARK: The space l^2 of sequences $a=(a_n)_{n=-\infty}^{\infty}$ of complex numbers such that $\sum_{n=-\infty}^{\infty} | a_n |^2 < \infty$ with point wise addition and scalar multiplication, and with the norm

$$\|a\| = (\sum_{n=-\infty}^{\infty} | a_n |^2)^{1/2}$$

or equivalently with the inner product

$$\langle a, b \rangle = \sum_{n=-\infty}^{\infty} a_n \overline{b_n}$$

where $a=(a_n)_{n=-\infty}^{\infty}$ and $b=(b_n)_{n=-\infty}^{\infty}$ is a Hilbert space. The correspondence $f \to (\hat{f}(n))_{n=-\infty}^{\infty}$ is an isometry between $L^2(T)$ and l^2.

1.4. POINTWISE CONVERGENCE OF $\sigma_n(f)$

THEOREM. 1.4.1: *Fejer's Theorem.* Let $f \in L^1(T)$.

(i). Assume that

$$\lim_{h \to 0}[f(t_o + h) + f(t_o - h)]$$

exists (we allow the values $-\infty$ and ∞). Then

$$\sigma_n(f, t_o) \xrightarrow{n \to \infty} \frac{1}{2} \lim_{h \to 0}[f(t_o + h) + f(t_o - h)].$$

In particular, if t_o is a point of continuity of f then $\sigma_n(f, t_0) \to f(t_o)$.

(ii). If every point of a closed interval I is a point of continuity of f then $\sigma_n(f, t)$ Converges to f(t) uniformly on I.

(iii). If for all t, $m \le f(t) \le M$ then $m \le \sigma_n(f, t) \le M$ for all n = 0, 1, ...

Proof: Recall that (F_n) is a positive summability kernel that has the following two properties:

(1) For $0 < \delta < \pi$, $\lim_{n\to\infty}[\sup_{\delta<t<2\pi-\delta} F_n(t)] = 0$,

(2) $F_n(t) = F_n(-t)$ for all t in T and n = 0, 1, ….

(i) Assume $\breve{f}(t_o) = \lim_{h\to 0}[f(t_o+h)+f(t_o-h)]$ is finite. Then

$$\sigma_n(f,t_o) - \breve{f}(t_o) = \frac{1}{2\pi}\int_T F_n(\tau)[f(t_o-\tau)-\breve{f}(t_o)]d\tau$$

$$= \frac{1}{2\pi}\int_{-\delta}^{\delta} F_n(\tau)[f(t_o-\tau)-\breve{f}(t_o)]d\tau$$

$$+ \frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} F_n(\tau)[f(t_o-\tau)-\breve{f}(t_o)]d\tau$$

$$= \frac{1}{\pi}\int_{0}^{\delta} F_n(\tau)[\frac{f(t_o+\tau)+f(t_o-\tau)}{2}-\breve{f}(t_o)]d\tau$$

$$+ \frac{1}{\pi}\int_{\delta}^{2\pi-\delta} F_n(\tau)[\frac{f(t_o+\tau)+f(t_o-\tau)}{2}-\breve{f}(t_o)]d\tau.$$

Given $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon)$ and $N = N(\varepsilon)$ such that

$\left|\frac{f(t_o+\tau)+f(t_o-\tau)}{2}-\breve{f}(t_o)\right| < \varepsilon$ whenever $|\tau| < \delta$ and

$\sup_{\delta<\tau<2\pi-\delta} F_n(\tau) < \varepsilon$ whenever $n \geq N$.

Therefore, $|\sigma_n(f,t_o) - \breve{f}(t_o)| < \varepsilon + \varepsilon\left\|f - \breve{f}(t_o)\right\|_{L^1}$ for all $n \geq N$.

Thus, $\lim_{n\to\infty}\sigma_n(f,t_o) = \breve{f}(t_o)$.

In the same procedure we can prove the case for $\breve{f}(t_o) = \pm\infty$.

(ii) Since the closed interval I is compact, f is uniformly continuous on I. Thus given $\varepsilon>0$ there exists $\delta>0$ such that $t, \tau \in T$ and

$$|t-\tau| < \delta \text{ implies } |f(t)-f(\tau)| < \varepsilon.$$

By (2) there exists n_o such that

$$\sup_{\delta<\tau<2\pi-\delta} F_n(\tau) < \varepsilon \text{ for } n>n_o.$$

Now

$$|\sigma_n(f,t)-f(t)| \le \tfrac{1}{\pi}\int_{-\delta}^{\delta} F_n(\tau)|f(t-\tau)-f(t)|d\tau$$
$$+\tfrac{1}{\pi}\int_{\delta}^{2\pi-\delta} F_n(\tau)|f(t-\tau)-f(t)|d\tau$$
$$<\varepsilon+2\varepsilon\|f\|_{L^1} \qquad \forall n>n_o \ and \ t\in T.$$

(iii) By (S_1) in the definition of summability kernel, we get

$$\sigma_n(f,t)-m=\tfrac{1}{2\pi}\int_T F_n(\tau)[\underbrace{f(t-\tau)-m}_{\ge 0}]d\tau \ge 0, \quad and$$

$$M-\sigma_n(f,t)=\tfrac{1}{2\pi}\int_T F_n(\tau)[\underbrace{M-f(t-\tau)}_{\ge 0}]d\tau \ge 0.$$

Therefore, $m \le \sigma_n(f,t) \le M$. //

COROLLARY. 1.4.1: If t_o is a point of continuity of f and if the Fourier series of f converges at t_o then its sum is $f(t_o)$.

Proof: Since the series $\sum_{n=-\infty}^{\infty}\hat{f}(n)e^{\text{int}_o}$ is convergent, the arithmetic means of the partial sums $S_n(f, t_o)$ are convergent. Let's assume that $S_n(f, t_o)$ converges to $g(t_o)$. Then it is easily shown that $\frac{1}{n+1}\sum_{j=0}^{n} S_j(f,t_o)\xrightarrow{n\to\infty} g(t_o)$. By Fejer's Theorem we have $f(t_o)= g(t_o)$.//

THEOREM. 1.4.2: *Lebesgue Theorem.* Let $f\in L^1(T)$. Then if

$$\lim_{h\to 0}\tfrac{1}{h}\int_0^h\left|\frac{f(t_o+\tau)+f(t_o-\tau)}{2}-\breve{f}(t_o)\right|d\tau=0 \qquad \textbf{(I)}$$

then $\sigma_n(f,t_o)\xrightarrow{n\to\infty}\breve{f}(t_o)$. In particular $\sigma_n(f,t)\xrightarrow{n\to\infty} f(t)$ almost everywhere.

Proof:

$$\sigma_n(f,t_o)-\breve{f}(t_o)=\tfrac{1}{\pi}\int_0^{\delta}F_n(\tau)[\frac{f(t_o+\tau)+f(t_o-\tau)}{2}-\breve{f}(t_o)]d\tau$$
$$+\tfrac{1}{\pi}\int_{\delta}^{\pi}F_n(\tau)[\frac{f(t_o+\tau)+f(t_o-\tau)}{2}-\breve{f}(t_o)]d\tau \qquad \textbf{(II)}$$

Since $F_n(t) = \frac{1}{n+1}\left(\frac{\sin(\frac{n+1}{2})t}{\sin\frac{t}{2}}\right)^2$ *and* $\sin\frac{t}{2} > \frac{t}{\pi}$, $0 < t < \pi$ we obtain

$$F_n(\tau) \le \min\{n+1, \frac{\pi^2}{(n+1)\tau^2}\}.$$

Therefore,

$$\tfrac{1}{\pi}\int_{\delta}^{\pi} F_n(\tau)\left|\frac{f(t_o+\tau)+f(t_o-\tau)}{2} - \breve{f}(t)\right| d\tau \le \frac{\pi}{(n+1)\delta^2}\left\|f - \breve{f}(t_o)\right\|_{L^1} \xrightarrow{n\to\infty} 0.$$

Now we turn to evaluate the first integral in (II). We pick $\delta = n^{-\frac{1}{4}}$ and denote

$$\Phi(h) = \int_0^h \left|\frac{f(t_o+\tau)+f(t_o-\tau)}{2} - \breve{f}(t_o)\right| d\tau.$$

Then

$$\left|\tfrac{1}{\pi}\int_0^{\delta} F_n(\tau)[\frac{f(t_o+\tau)+f(t_o-\tau)}{2} - \breve{f}(t_o)]d\tau\right|$$

$$\le \tfrac{1}{\pi}\left|\int_0^{1/n} F_n(\tau)[\frac{f(t_o+\tau)+f(t_o-\tau)}{2} - \breve{f}(t_o)]d\tau\right|$$

$$+ \tfrac{1}{\pi}\left|\int_{1/n}^{\delta} F_n(\tau)[\frac{f(t_o+\tau)+f(t_o-\tau)}{2} - \breve{f}(t_o)]d\tau\right|$$

$$\le \frac{n+1}{\pi}\Phi(\tfrac{1}{n}) + \frac{\pi}{n+1}\int_{1/n}^{\delta}\left|\frac{f(t_o+\tau)+f(t_o-\tau)}{2} - \breve{f}(t_o)\right|\frac{d\tau}{\tau^2}.$$

Since

$$\lim_{n\to\infty} n\int_0^{1/n}\left|\frac{f(t_o+\tau)+f(t_o-\tau)}{2} - \breve{f}(t_o)\right| d\tau = 0$$

we get

$$\lim_{n\to\infty}\frac{n+1}{\pi}\Phi(\tfrac{1}{n}) = 0,$$

and integrating by parts we obtain

$$\frac{\pi}{n+1}\int_{1/n}^{\delta}\left|\frac{f(t_o+\tau)+f(t_o-\tau)}{2} - \breve{f}(t_o)\right|\frac{d\tau}{\tau^2}$$

$$= \frac{\pi}{n+1}\left[\frac{\Phi(\tau)}{\tau^2}\right]_{\frac{1}{n}}^{\delta} + \frac{2\pi}{n+1}\int_{1/n}^{\delta}\frac{\Phi(\tau)}{\tau^3}d\tau$$

$$= \underbrace{\frac{\pi}{n+1}}_{\xrightarrow{n\to\infty} 0} \underbrace{\frac{\Phi(n^{-\frac{1}{4}})}{n^{-\frac{1}{2}}}}_{\xrightarrow{n\to\infty} 0} - \underbrace{\frac{\pi}{n+1}}_{\xrightarrow{n\to\infty} 0} \underbrace{\frac{\Phi(\frac{1}{n})}{n^{-2}}}_{\xrightarrow{n\to\infty} 0} + \frac{2\pi}{n+1}\int_{1/n}^{\delta}\frac{\Phi(\tau)}{\tau^3}d\tau \quad \text{(III)}$$

Now given ε>0 there exists n_o such that n > n_o and 0<τ<δ we have $\Phi(\tau) < \frac{\varepsilon}{3\pi}\tau$. Hence

$$\frac{2\pi}{n+1}\int_{1/n}^{\delta}\frac{\Phi(\tau)}{\tau^3}d\tau \le \frac{2\pi\varepsilon}{3\pi(n+1)}\int_{1/n}^{\delta}\frac{1}{\tau^2}d\tau \le \varepsilon.$$

Therefore, $\lim\limits_{n\to\infty}\frac{2\pi}{n+1}\int_{1/n}^{\delta}\frac{\Phi(\tau)}{\tau^3}d\tau = 0.$ (IV)

Then from (II), (III) and (IV) we get

$$\lim_{n\to\infty}\sigma_n(f,t_o) = \breve{f}(t_o).//$$

COROLLARY. 1.4.2: If the Fourier series of *f* in $L^1(T)$ converges on a set E of positive measure, its sum coincides with *f* almost everywhere on E. In particular if a Fourier series converges to zero almost everywhere, then all its Fourier coefficients must vanish.

Proof: By Fejer's Theorem,

$$S_n(f,t) \xrightarrow{n\to\infty} \breve{f}(t)$$

for all $t \in$ E and since $\breve{f}$ (t) = *f* (t) almost everywhere on E we obtain

$$S_n(f,t) \xrightarrow{n\to\infty} f(t)$$

almost everywhere on E. If the Fourier series of *f* converge to zero almost everywhere then *f* is zero almost everywhere. And thus $\hat{f}$ (n) =0, n = 0, ±1...//

THEOREM. 1.4.3: *Fatou.* If

$$\psi(h) = \int_0^h\left[\frac{f(t_o+\tau)+f(t_o-\tau)}{2} - \breve{f}(t_o)\right] = o(h)$$

then

$$\lim_{r\to 1}\sum_{-\infty}^{\infty}\hat{f}(j)r^{|j|}e^{ijt_o} = \breve{f}(t_o).$$

1.5 THE ORDER OF MAGNITUDE OF FOURIER COEFFICIENTS

THEOREM. 1.5.1: Let $(a_n)_{n=-\infty}^{\infty}$ be an even sequence of nonnegative numbers tending to zero at infinity. Assume that for n>0,

$$a_{n-1} + a_{n+1} - 2a_n \geq 0.$$

Then there exists a nonnegative function $f \in L^1(T)$ such that $\hat{f}\ (n) = a_n$.

Proof: From the hypothesis, $(a_n - a_{n+1})$ is monotonically decreasing with n,

$$\lim_{n\to\infty} n(a_n - a_{n+1}) = 0,$$

and consequently

$$\sum_{n=1}^{N} n(a_{n-1}a_{n+1} - 2a_n) = a_0 - a_N - N(a_N - a_{N+1})$$

converges to a_o as $N \to \infty$. Put

$$f(t) = \sum_{n=1}^{\infty} n(a_{n-1} + a_{n+1} - 2a_n)F_{n-1}(t),$$

F_n denoting as usual the Fejer's kernel of order n. Since $\|F_n\|_{L^1} = 1$, the series in the definition of *f* converges in L^1 (T) norm, and all its terms being nonnegative, its limit *f* is nonnegative. Now

$$\hat{f}(j) = \sum_{n=1}^{\infty} n(a_{n-1} + a_{n+1} - 2a_n)\hat{F}_{n-1}(j)$$

$$= \sum_{n=|j|+1}^{\infty} n(a_{n-1} + a_{n+1} - 2a_n)(1 - \tfrac{|j|}{n}) = a_{|j|}. //$$

DEFINITION. 1.5.1:

Let *f (t)* and *g(t)>0* be two functions defined for $t > t_0$. We say that

(i) *f(t)=o(g(t))* if and only if $\dfrac{f(t)}{g(t)} \to 0 \;\; as \;\; t \to \infty.$

(ii) *f(t)*=O*(g(t))*if and only if $\dfrac{f(t)}{g(t)}$ is bounded for all t sufficiently large.

(iii) *f* and g are asymptotically equal in the neighborhood of t_o if and only if

$$\frac{f(t)}{g(t)} \to 1 \text{ as } t \to t_0, \text{ and we write } f(t) \simeq g(t).$$

(iv). f and g are of the same order in the neighborhood of t_o if and only if there exists two constants A>0 and B >0 such that $A \le \frac{f(t)}{g(t)} \le B$ for all t Sufficiently near t_o, denoted $f(t) \sim g(t)$.

THEOREM 1.5.2: Let $f \in L^1(T)$. Then if f is absolutely continuous then

$$\hat{f}(n) = 0\left(\tfrac{1}{n}\right).$$

Proof: Since f is absolutely continuous, $f(t) = \int_0^t f'(t)\,dt$. Thus,

$$\hat{f}(n) = \tfrac{1}{in}\hat{f}'(n) \quad \text{for } n \neq 0$$

and by the Riemann-Lebesgue Lemma $\hat{f}'(n) \to 0$. Therefore,

$$n\hat{f}(n) = \tfrac{1}{i}\hat{f}'(n) \to 0 \text{ as } n \to \infty.$$

Hence $\hat{f}(n) = 0\left(\tfrac{1}{n}\right)$.//

REMARK: If f is k-times differentiable and $f^{(k)} \in L^1(T)$ then,

$$\hat{f}(n) = o\left(\frac{1}{n^k}\right) \text{ as } |n| \to \infty.$$

THEOREM 1.5.3: If f is k-times differentiable and $f^{(k)} \in L^1(T)$ then

$$\left|\hat{f}(n)\right| \le \min_{0 \le j \le k} \frac{\left\|f^{(j)}\right\|_{L^1}}{|n|^j}.$$

THEOREM 1.5.4: If f is of bounded variation on T, then

$$\left|\hat{f}(n)\right| \le \frac{\operatorname{var}(f)}{2\pi|n|} \qquad \text{for } \; n \neq 0.$$

Proof: We integrate by parts using Stieltjes integrals

$$\left|\hat{f}(n)\right| = \tfrac{1}{2\pi}\left|\int_T e^{-\mathrm{int}} f(t)dt\right| = \frac{1}{2\pi}\left|\int_T e^{-\mathrm{int}} df(t)\right| \leq \frac{\mathrm{var}(f)}{2\pi|n|}.//$$

DEFINITION. 1.5.2:

1. For $f \in C(T)$,

$$\omega(f,h) = \max_{|y|\leq h,t}\left|f(t+y) - f(t)\right|$$

is called the modulus of continuity of *f*.

2. For $f \in L^1(T)$, $\Omega(f,h) = \left\|f(t+h) - f(t)\right\|_{L^1}$ is called the integral modulus of continuity of *f*.

REMARK: From these definitions we have $\Omega(f,h) \leq \omega(f,h)$

THEOREM 1.5.5: For $n \neq 0$, $\left|\hat{f}(n)\right| \leq \tfrac{1}{2}\Omega(f,\frac{\pi}{|n|})$.

Proof: $\hat{f}(n) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(t)e^{-\mathrm{int}}dt = \dfrac{-1}{2\pi}\displaystyle\int_0^{2\pi} f(t)e^{-in(t+\frac{\pi}{n})t}dt$.

By a change of variable

$$\hat{f}(n) = \frac{-1}{4\pi}\int_0^{2\pi}[f(t+\tfrac{\pi}{n}) - f(t)]e^{-\mathrm{int}}dt.$$

Hence

$$\left|\hat{f}(n)\right| \leq \tfrac{1}{2}\Omega(f,\tfrac{\pi}{|n|}). \quad //$$

PART TWO

THE CONVERGENCE OF FOURIER SERIES

We saw in Part One that if $f \in L^1(T)$, then $\sigma_n(f)$ converges to f in the topology of any homogeneous Banach space that contains f. Moreover, if $\breve{f}(t_o) = \lim_{h\to 0} \frac{f(t_o+h)+f(t_o-h)}{2}$ exists (we allow $+\infty$ and $-\infty$) then $\sigma_n(f)$ converges to $\breve{f}(t_o)$. We are now going to see similar problems of convergence of partial sums $S_n(f)$ of Fourier series of functions in a homogeneous Banach spaces. In particular we have seen in section 1.3 that for every $f \in L^2(T)$, $S_n(f)$ converges to f in $L^2(T)$-norm. In this part we will see the following.

1. For every $f \in L^p(T)$, $1 < p < \infty$, $S_n(f)$ converges to f in $L^p(T)$-norm.
2. Existence of a continuous function whose Fourier series diverges at a point and two criterions for convergence at a point, and
3. For every set $E \subseteq T$ of measure zero there exists a continuous function on T such that $S_n(f)$ diverges for all $t \in E$.

2.1. CONVERGENCE IN NORM

DEFINITION 2.1.1: Let B be a homogeneous Banach space on T. Let $f \in B$ and $S_n(f) = S_n(f,t) = \sum_{j=-n}^{n} \hat{f}(j)e^{ijt}$. We say that B admits convergence in norm if and only if

$$\lim_{n\to\infty} \|S_n(f) - f\|_B = 0.$$

The operators S_n are well defined in every homogeneous Banach space. We denote their norms, as operators on B, by $\|S_n\|^B$.

THEOREM 2.1.1: A homogeneous Banach space B admits convergence in norm if and only if $\|S_n\|^B$ are bounded (as $n\to\infty$), that is, if there exists a constant K such that $\|S_n(f)\|_B \le K\|f\|_B$ for all $f \in B$ and for all $n \ge 0$.

Proof: 1. Assume B admits convergence in norm. Then for each $f \in B$ there exists $\gamma_f > 0$ such that $\|S_n(f)\|_B \le \gamma_f$, $n = 0,1,2,\ldots$. Thus the family ***(S_n)*** of continuous linear operators on B is point wise bounded on B. By the Uniform Bounded ness Theorem ($\|S_n\|^B$) is bounded.

2. Conversely, assume that there exists a constant K such that $\|S_n(f)\|_B \le K\|f\|_B$ for all $f \in B$ and for all $n \ge 0$. Let $f \in B$ and $\varepsilon > 0$ be given. Then there exists a trigonometric polynomial P of degree N such that $\|f - P\|_B < \dfrac{\varepsilon}{2K}$. For n>N we have $S_n(P) = P$. Now

$$
\begin{aligned}
\|S_n(f) - f\|_B &= \|S_n(f) - S_n(P) + P - f\|_B \ \forall n > N \\
&\le \|S_n(f - P)\|_B + \|P - f\|_B \ \forall n > N \\
&\le K\|f - P\|_B + \|P - f\|_B \ \forall n > N \\
&< K.\frac{\varepsilon}{2K} + \frac{\varepsilon}{2K} \quad \forall n > N \\
&\le \varepsilon \qquad \forall n > N.
\end{aligned}
$$

Therefore, $\underset{n\to\infty}{Lim}\|S_n(f) - f\|_B = 0 \ \forall f \in B,$ and thus B admits convergence in norm. //

Since $S_n(f) = D_n * f$, where D_n is the Dirichlet kernel,

$$D_n(t) = \sum_{j=-n}^{n} e^{ijt} = \frac{\sin(n + \frac{1}{2})t}{\sin\frac{t}{2}}$$

We get

$$\|S_n(f)\|_B = \|D_n * f\|_B \le \|D_n\|_{L^1} \cdot \|f\|_B \ \forall f \in B.$$

Therefore, we have a simple bound for each S_n,

$$\|S_n\|^B \le \|D_n\|_{L^1}.$$

DEFINITION. 2.1.2: The numbers $L_n = \|D_n\|_{L^1}$ are called the Lebesgue constants.

LEMMA. 2.1.1: The sequence *(L_n)* of the Lebesgue constants is not bounded. Specifically, $L_n > \frac{4}{\pi^2} Log n$ for all n∈N.

Proof: Since $t \geq 2\sin\frac{t}{2}$, we have $\frac{1}{t} \leq \frac{1}{2\sin\frac{t}{2}}$. Then we get

$$L_n = \frac{1}{2\pi}\int_0^{2\pi}|D_n(t)|dt = \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}\right|dt \geq \frac{2}{\pi}\int_0^{\pi}\frac{|\sin(n+\frac{1}{2})t|}{t}dt.$$

By the change of variable we get,

$$L_n \geq \frac{2}{\pi}\int_0^{(n+\frac{1}{2})\pi}\frac{|\sin u|}{u}du > \frac{2}{\pi}\sum_{k=1}^{n}\frac{1}{k\pi}\int_{(k-1)\pi}^{k\pi}|\sin u|du = \frac{4}{\pi^2}\sum_{k=1}^{n}\frac{1}{k} > \frac{4}{\pi^2}Log n.$$

Hence, $L_n \xrightarrow{n\to\infty} \infty$ .//

THEOREM. 2.1.2: ***L^1(T)*** does not admit convergence in norm.

Proof: We have seen that for a homogeneous Banach space B, $\|S_n\|^B \leq L_n$, n = 0, 1, 2, …. In particular, if B = L^1(T) then $\|S_n\|^{L^1(T)} \leq L_n$, $n = 0,1,2,\ldots$. It is known that $S_n(F_N) = \sigma_N(D_n)$. Hence, now we have

$$\|S_n\|^{L^1(T)} \geq \|S_n(F_N)\|_{L^1} = \|\sigma_N(D_n)\|_{L^1} \; and \; \|F_N\|_{L^1} = 1.$$

But since $\sigma_N(D_n) \xrightarrow{N\to\infty} D_n$ in L^1(T) norm, we have

$$\|S_n\|^{L^1(T)} \geq \lim_{N\to\infty}\|\sigma_N(D_n)\|_{L^1} = \|D_n\|_{L^1} = L_n, n = 0,1,2,\ldots.$$

Therefore, by Lemma 2.1.1and by Theorem 2.1.1, L^1(T) does not admit convergence in norm.//

THEOREM. 2.1.3: ***C(T)*** does not admit convergence in norm.

Proof: Let φ_n be continuous functions satisfying $\|\varphi_n\|_\infty = \sup_{t\in T}|\varphi_n(t)| \leq 1$ such that φ_n (t)=Sgn(D_n(t)) except in small intervals around the points of discontinuity of Sgn(D_n(t)). If the sum of the lengths of these intervals is less than $\frac{\varepsilon}{2n}$, we have

$$\|S_n\|^{C(T)} \geq \|S_n(\varphi_n)\|_\infty \geq |S_n(\varphi_n, 0)| > L_n - \varepsilon.$$

Since ε was arbitrary, $\|S_n\|^{C(T)} \geq L_n$. By Lemma 2.1.1 and by Theorem 2.1.1, ***C(T)*** does not admit convergence in norm.//

DEFINITION. 2.1.3: *The conjugate function.* If $\boldsymbol{f} \in \boldsymbol{L^1(T)}$ and if the series conjugate to the Fourier series of $\boldsymbol{f}$ is the Fourier series of some function $\boldsymbol{g} \in \boldsymbol{L^1(T)}$, then $\boldsymbol{g}$ is called the conjugate function of $\boldsymbol{f}$ and is denoted by $\tilde{f}$.

DEFINITION. 2.1.4: A space of functions $\boldsymbol{B} \subseteq \boldsymbol{L^1(T)}$ *admits conjugation* if and only if for every $\boldsymbol{f} \in \boldsymbol{B}$, $\tilde{f}$ is defined and belongs to ***B***.

LEMMA. 2.1.2: Let **B** be a homogeneous Banach space on **T**. If **B** admits conjugation then the mapping $f \to \tilde{f}$ is a bounded linear operator on **B**.

Proof: 1. Define $\Lambda : B \to B$ by Λ(f) = $\tilde{f}$. Let $\boldsymbol{f}, \boldsymbol{g} \in \boldsymbol{B}$ and α a complex number. Now if Λ(f+g) = h then for each integer n,

$$\hat{h}(n) = -iSgn(n)[\hat{f}(n) + \hat{g}(n)] = \hat{\tilde{f}}(n) + \hat{\tilde{g}}(n) = (\tilde{f} + \tilde{g})^{\wedge}(n).$$

By the Uniqueness Theorem, h = $\tilde{f} + \tilde{g}$. Therefore,

$$\Lambda\ (f+g) = \tilde{f} + \tilde{g} = \Lambda(f) + \Lambda(g).$$

Hence Λ is additive. Let $\Lambda(\alpha$ f)= k. Then

$$\hat{k}(n) = -iSgn(n)(\widehat{\alpha f})(n) = -iSgn(n).\alpha.\hat{f}(n) = \alpha\hat{\tilde{f}}(n) = (\widehat{\alpha\tilde{f}})(n)\ \forall n \in Z.$$

By the uniqueness Theorem, k=$\alpha\,\tilde{f}$, that is, $\Lambda(\alpha$ f)= $\alpha\ \Lambda(f)$. Therefore, Λ is ***homogeneous***. Therefore, Λ is ***linear***.

2. Let (f_n) be a sequence of functions in B such that

$$\lim_{n\to\infty} f_n = f \text{ and } \lim_{n\to\infty} \Lambda f_n = g$$

in B-norm. Now for each integer j,

$$\hat{g}(j) = \lim_{n\to\infty} \hat{\tilde{f}}_n(j) = \lim_{n\to\infty}(-iSgn(j)\hat{f}_n(j) = -iSgn(j)\lim_{n\to\infty} \hat{f}_n(j)$$
$$= -i\,\mathrm{sgn}(j)\hat{f}(j) = \hat{\tilde{f}}(j).$$

By the Uniqueness Theorem, g = $\tilde{f}$. Thus the operator Λ is closed. By the Closed Graph Theorem, Λ is ***continuous*** and hence ***bounded***. //

LEMMA. 2.1.3: Let **B** be a homogeneous Banach space on **T,** which admits conjugation. For each *f*∈***B***, define

$$f^b = \tfrac{1}{2}\hat{f}(0) + \tfrac{1}{2}(f + i\tilde{f}) \sim \sum_{j=0}^{\infty} \hat{f}(j)e^{ijt}.$$

Then the mapping $f \to f^b$ is a well-defined, bounded linear operator on **B**.

Proof: 1. Let *f, g*∈***B*** and α a complex number. Then

$$(f+g)^b = \tfrac{1}{2}(\widehat{f+g})(0) + \tfrac{1}{2}[(f+g) + i(\tilde{f}+\tilde{g})] = f^b + g^b, \text{ and}$$
$$(\alpha f)^b = \frac{1}{2}(\widehat{\alpha f})(0) + \frac{1}{2}[\alpha f + i(\widetilde{\alpha f})] = \alpha f^b.$$

Therefore, the mapping is linear.
2. Let (f_n) be a sequence of functions in B such that

$$f_n \xrightarrow{n\to\infty} f \quad \text{and} \quad f_n^b \xrightarrow{n\to\infty} g$$

in B-norm. For each integer j,

$$\hat{g}(j) = \lim_{n\to\infty} \hat{f}_n^b(j) = \begin{cases} \lim_{n\to\infty} \hat{f}_n(j) & \text{if } j \ge 0 \\ 0 & \text{if } j < 0 \end{cases}$$
$$= \begin{cases} \hat{f}(j) & \text{if } j \ge 0 \\ 0 & \text{if } j < 0 \end{cases} = \hat{f}^b(j).$$

By the Uniqueness Theorem, $g = f^b$. Hence the operator is closed. By the Closed Graph Theorem, the mapping is continuous and thus bounded. //

LEMMA. 2.1.4: Let **B** be a homogeneous Banach space on **T.** If the mapping $f \to f^b$ is well defined in **B** then **B** admits conjugation.

Proof: Let $f \in B$. Then by the hypothesis $f^b \in \mathbf{B}$. Then $\tilde{f} = -i\,[2f^b - f - \hat{f}(0)] \in \mathbf{B}$. Thus B admits conjugation.//

THEOREM. 2.1.4: Let **B** be a homogeneous Banach space on T and assume that for $f \in B$ and for every integer n, $e^{int}f \in$ **B** and $\left\|e^{\text{int}} f\right\|_B = \|f\|_B$. Then **B** admits conjugation if and only if B admits convergence in norm.

Proof: 1. Assume that B admits conjugation. Then the mapping $f \to f^b$ is a well-defined, bounded linear operator on **B**. Without loss of generality there exists $\gamma > 0$ such that $\left\|f^b\right\|_B \leq \gamma \|f\|_B$ for all $f \in B$. Define S_n^b: **B**→**B** by

$$S_n^b(f) = \sum_{j=0}^{2n} \hat{f}(j) e^{ijt} = e^{\text{int}} S_n(e^{-\text{int}} f).$$

Then $S_n^b(f) = f^b - e^{i(2n+1)t}(e^{-i(2n+1)t} f)^b$.

Now for each $f \in B$,

$$\begin{aligned}\left\|S_n^b(f)\right\|_B &\leq \left\|f^b\right\|_B + \left\|e^{i(2n+1)t}\left(e^{-i(2n+1)t} f\right)^b\right\|_B \\ &= \left\|f^b\right\|_B + \left\|(e^{-i(2n+1)t} f)^b\right\|_B \\ &\leq \gamma\|f\|_B + \gamma\left\|e^{-i(2n+1)t} f\right\|_B = 2\gamma\|f\|_B.\end{aligned}$$

By the uniform Bounded ness Theorem, the sequence $\left(\left\|S_n^b\right\|^B\right)$ is bounded.

$$\left\|S_n^b(f)\right\|_B = \left\|e^{\text{int}} S_n(e^{-\text{int}} f)\right\|_B \leq \|S_n\|^B . \|f\|_B,$$

for all $f \in B$. Therefore, $\left\|S_n^b\right\|^B \leq \|S_n\|^B$. Now since $S_n(e^{-\text{int}} f) = e^{-\text{int}} S_n^b(f)$,

$$\left\|S_n(e^{-\text{int}} f)\right\|_B = \left\|e^{-\text{int}} S_n^b(f)\right\|_B = \left\|S_n^b(f)\right\|_B \leq \left\|S_n^b\right\|^B . \|f\|_B$$

for all $f \in B$. This implies $\|S_n\|^B \leq \left\|S_n^b\right\|^B$. Therefore, $\left\|S_n^b\right\|^B = \|S_n\|^B$. Hence $\|S_n\|^B$ are bounded. By Theorem 2.1.1, B admits convergence in norm.

2. Conversely, assume that B admits convergence in norm. By Theorem 2.1.1, there exists K>0 such that $\|S_n\|^B \leq K$, n= 0,1,2,…. For the operators S_n^b defined in (1) above we have

$$\|S_n^b(f)\|_B = \|e^{\text{int}} S_n(e^{-\text{int}} f)\|_B \leq \|S_n\|^B . \|f\|_B \leq K\|f\|_B ,$$

for all $f \in B$. Thus we get $\|S_n^b\|^B \leq K \; for \; n = 0,1,2,\ldots$.

Let $f \in B$. Given $\varepsilon > 0$ there exists a trigonometric polynomial of degree N such that $\|f - P\|_B < \frac{\varepsilon}{2K}$. Then

$$\|S_n^b(f) - S_n^b(P)\|_B = \|S_n^b(f - P)\|_B \leq \|S_n^b\|^B \|f - P\|_B \leq \frac{\varepsilon}{2},$$

for all n>N.

If n>N and m>N then S_n^b (P) = S_m^b (P). Therefore, for n, m>N

$$\|S_n^b(f) - S_m^b(f)\|_B \leq \|S_n^b(f - P)\|_B + \|S_m^b(P - f)\|_B < \varepsilon.$$

Therefore, (S_n^b (*f*)) is a Cauchy sequence in **B**. Then there exists $g \in B$ such that (S_n^b (*f*)) converges to g in **B**-norm. By applying the Uniqueness Theorem we can show that g = f^b. Thus $f^b \in B$. Therefore the mapping $f \to f^b$ is well defined in **B**. By Lemma 2.1.4, **B** admits conjugation.//

In the remaining part of this section we are going to show that, for $1 < p < \infty$, $L^p(T)$ admits conjugation that is to prove the following theorem.

THEOREM 2.1.5: For $1 < p < \infty$, the Fourier series of every $f \in L^p(T)$ converges to *f* in the $L^p(T)$-norm.

Identify **T** with unit circumference {z: z = e^{it}} in the complex plane. The unit disc {z: |z|<1} is denoted by D and the closed unit disc, {z: |z|≤1} by $\overline{D}$. For $f \in L^1(T)$, we denote by $f(re^{it})$, **r<1**, the *Poisson integral* of *f*,

$$f(re^{it}) = (P(r,.) * f)(t) = \sum_{-\infty}^{\infty} r^{|n|} \hat{f}(n) e^{\text{int}} \qquad \textbf{(1)}$$

The harmonic conjugate of (1) is the function

$$\tilde{f}(re^{it}) = -i\sum_{-\infty}^{\infty} Sgn(j)r^{|j|}\hat{f}(j)e^{ijt} = (Q(r,.) * f)(t)$$

$$where\, Q(r,t) = -i\sum_{-\infty}^{\infty} Sgn(j)r^{|j|}e^{ijt} = \frac{2r\sin t}{1-2r\cos t + r^2}$$

is the harmonic conjugate of Poisson's kernel P(r, t).

LEMMA. 2.1.5: Every function harmonic and bounded in D is the Poisson integral of some bounded function on T.

Proof: Let F be harmonic and bounded function on D. Let $r_n \uparrow 1$ and write $f_n(e^{it}) = F(r_n e^{it})$. The sequence (f_n) is a bounded sequence in $\mathbf{L}^{\infty}(\mathbf{T})$. $\mathbf{L}^{\infty}(\mathbf{T})$ being the dual space of $\mathbf{L}^1(\mathbf{T})$, for some sequence $n_j \xrightarrow{j\to\infty} \infty$, f_{n_j} converges in the weak-star topology to some function $F(e^{it})$. Let $\rho e^{i\tau} \in D$. Then

$$\frac{1}{2\pi}\int_T P(\rho, t-\tau)F(e^{it})dt = \lim_{j\to\infty} \frac{1}{2\pi}\int_T P(\rho, t-\tau) f_{n_j}(e^{it})dt$$

$$= \lim_{j\to\infty} F(r_{n_j}\rho e^{i\tau}) = F(\rho e^{i\tau}).$$

Therefore, for $0<\rho<1$, $F(\rho e^{i\tau}) = \frac{1}{2\pi}\int_T P(\rho, t-\tau)F(e^{it})dt$.//

LEMMA. 2.1.6: Assume $f \in \mathbf{L}^1(\mathbf{T})$ and let $\tilde{f}(re^{it})$ be the harmonic conjugate to $f(re^{it})$. Then for almost all t, $\tilde{f}(re^{it})$ tends to a limit as $r \to 1$.

Proof: Since the mapping $f \to \tilde{f}(re^{it})$ is linear and any $\mathbf{f} \in \mathbf{L}^1(\mathbf{T})$ can be written as $f_1 - f_2 + if_3 - if_4$ with $f_j \geq 0$ in $\mathbf{L}^1(\mathbf{T})$, there is no loss of generality in assuming $f \geq 0$. The function $F(z) = e^{-f(z) - i\tilde{f}(z)}$ is holomorphic (hence harmonic) in D. Since the Poisson integral of a non negative function is $f(z) \geq 0$, and since $\tilde{f}$ is real-valued(being the harmonic conjugate of the real-valued function f), it follows that $|F(z)| \leq 1$ in D. By Theorem 1.4.3 and Lemma 2.1.5, F has a radial limit of modulus $e^{-f(e^{it})}$ almost everywhere. At every point where $F(e^{it})$ exists and is non-zero, $\tilde{f}(re^{it})$ has a finite radial limit.//

DEFINITION. 2.1.5: Let $f \in \mathbf{L}^1(\mathbf{T})$. The conjugate function of f is the function $\tilde{f}(e^{it}) = \lim_{r\to 1} \tilde{f}(re^{it})$.

REMARKS:

1. If the series conjugate to the Fourier series of f is the Fourier series of some function $g \in \mathbf{L}^1(\mathbf{T})$, then the Poisson integral of g is $\tilde{f}(re^{it})$.
2. If $\tilde{f} \in \mathbf{L}^1(\mathbf{T})$ then its Fourier series is $\widetilde{S}[f]$ so that if $\widetilde{S}[f]$ is not a Fourier series then $\tilde{f} \notin \mathbf{L}^1(\mathbf{T})$.

DEFINITION. 2.1.6. Denote the Lebesgue measure of a measurable set $E \subseteq T$ by $|E|$. The distribution function of a measurable, real-valued function f on T is the function

$$m(x) = m_f(x) = |\{t : f(t) \leq x\}|, \quad -\infty < x < \infty.$$

Property 2.1.1:

(a). Distribution functions are continuous to the right and monotone increasing from zero at $x = -\infty$ to 2π at $x = \infty$.

(b). For every continuous function F on $\mathfrak{R}$, $\int_T F(f(t))dt = \int F(x)dm_f(x)$.

DEFINITION. 2.1.7: A measurable function is of weak L^p type, $0<p<\infty$ if and only if there exists a constant C such that for all $\lambda>0$

$$m_{|f|}(\lambda) \geq 2\pi - C\lambda^{-p},$$

that is $|\{t: |f(t)| \geq \lambda\}| \leq C\lambda^{-p}$.

REMARK: If f is in $\mathbf{L}^p(\mathbf{T})$ then f is of weak $\mathbf{L}^p$ type.

Proof: Let $f \in \mathbf{L}^p(\mathbf{T})$. Define F: $\mathfrak{R} \rightarrow \mathfrak{R}$ by $F(x) = x^p$. Then for all $\lambda>0$,

$$\|f\|_{L^p}^p = \frac{1}{2\pi}\int_T |f(t)|^p dt = \frac{1}{2\pi}\int_T F(|f(t)|)dt = \frac{1}{2\pi}\int_0^\infty x^p dm_{|f|}(x)$$
$$\geq \frac{1}{2\pi}\int_\lambda^\infty x^p dm_{|f|}(x) \geq \frac{\lambda^p}{2\pi}\int_\lambda^\infty dm_{|f|}(x) = \frac{1}{2\pi}[2\pi - m_{|f|}(\lambda)]\lambda^p.$$

Taking $C = 2\pi\|f\|_{L^p}^p$, we have $m_{|f|}(\lambda) \geq 2\pi - C\lambda^{-p}$ and hence f is of weak $\mathbf{L}^p$ type. //

LEMMA. 2.1.7: If f is of weak $\mathbf{L}^p$ type then $f \in \mathbf{L}^r(\mathbf{T})$ for every $r<p$.

Proof: By the hypothesis there exists C such that $2\pi - m_{|f|}(\lambda) \le C\lambda^{-p}$ for all $\lambda>0$. Now

$$\int_T |f(t)|^r \, dt = \int_0^\infty x^r dm_{|f|}(x) \le m_{|f|}(1) + \int_1^\infty x^r dm_{|f|}(x)$$

$$= m_{|f|}(1) - [x^r(2\pi - m_{|f|}(x))]_1^\infty + \int_1^\infty [2\pi - m_{|f|}(x)] d(x^r)$$

$$\le m_{|f|}(1) + [2\pi - m_{|f|}(1)] + C\int_1^\infty x^{-p} d(x^r) = 2\pi + \frac{Cr}{r-p} < \infty$$

Therefore, $f \in \mathbf{L}^r(\mathbf{T})$. //

THEOREM. 2.1.6: If $f \in \mathbf{L}^1(\mathbf{T})$ then $\tilde{f}$ is of weak $\mathbf{L}^1$ type.

Proof: We assume first that $f \ge 0$. We normalize f by assuming $\|f\|_{L^1} = 1$. We want to evaluate the measure of the set of points where $|\tilde{f}| > \lambda$. The function

$$H_\lambda(z) = 1 + \frac{1}{\pi}\arg\frac{z - i\lambda}{z + i\lambda} = 1 + \frac{1}{\pi}\operatorname{Im}(Log[\frac{z - i\lambda}{z + i\lambda}])$$

is harmonic and nonnegative in the half-plane Re(z)>0, and its level lines are circular arcs passing through the points $i\lambda$ and $-i\lambda$. The level line $H_\lambda(z) = \frac{1}{2}$ is the half circle $z = \lambda e^{i\theta}$, $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$. Hence if $|z| \ge \lambda$, then $H_\lambda(z) \ge \frac{1}{2}$.

$$H_\lambda(1) = 1 + \frac{1}{\pi}\arg(\tfrac{1-i\lambda}{1+i\lambda}) = 1 + \tfrac{1}{\pi}\arg(e^{-2i\arctan\lambda}) = 1 + \frac{2}{\pi}\arctan\lambda$$

$$= \frac{2}{\pi}[\frac{\pi}{2} - \arctan\lambda] = \frac{2}{\pi}\arctan\tfrac{1}{\lambda} < \tfrac{2}{\pi}.\tfrac{1}{\lambda} = \frac{2}{\pi\lambda}.$$

Now $H_\lambda(f(z) + i\tilde{f}(z))$ is a well defined positive harmonic in D, and

$$\tfrac{1}{2\pi}\int_T H_\lambda(f(re^{it}) + i\tilde{f}(re^{it}))dt = H_\lambda(f(0)) = H_\lambda(1) < \frac{2}{\pi\lambda}, \text{ and}$$

$$H_\lambda(f + i\tilde{f}) \ge \tfrac{1}{2} \text{ if } |f + i\tilde{f}| \ge \lambda,$$

we obtain $|\{t: |\tilde{f}(re^{it})| > \lambda\}| \leq \frac{8}{\lambda}$. Since the mapping $f \rightarrow \tilde{f}$ is linear, it is clear that if we omit the normalization $\|f\|_{L^1} = 1$ we obtain, letting $r \rightarrow 1$, that for $f \geq 0$ in $\mathbf{L}^1(\mathbf{T})$,

$$|\{t: |\tilde{f}(re^{it})| > \lambda\}| \leq 8\|f\|_{L^1} \lambda^{-1}.$$

Every $f \in \mathbf{L}^1(\mathbf{T})$ can be written as $f = f_1 - f_2 + if_3 - if_4$ where $f_j \geq 0$ and $\|f_j\|_{L^1} \leq \|f\|_{L^1}$. Consequently

$$\{t: |\tilde{f}(re^{it})| > \lambda\} \subseteq \bigcup_{j=1}^{4} \{t: |\tilde{f}_j(e^{it})| > \tfrac{\lambda}{4}\}.$$

It follows that for C=128 and for every $f \in \mathbf{L}^1(\mathbf{T})$ we get

$$|\{t: |\tilde{f}(re^{it})| > \lambda\}| \leq 128\|f\|_{L^1} \lambda^{-1}.$$

Thus for $f \in \mathbf{L}^1(\mathbf{T})$ $\tilde{f}$ is of weak $\mathbf{L}^1$ type.//

COROLLARY. 2.1.1: If $f \in \mathbf{L}^1(\mathbf{T})$ then $\tilde{f} \in \mathbf{L}^{\alpha}(\mathbf{T})$ for all $\alpha < 1$.

Proof: Follows from Lemma 2.1.7 and Theorem 2.1.6.//

THEOREM 2.1.7: Let $f \in \mathbf{L}^1(\mathbf{T})$. Then if $f \,\mathrm{Log}^+|f| \in \mathbf{L}^1(\mathbf{T})$ then $\tilde{f} \in \mathbf{L}^1(\mathbf{T})$ where $\mathrm{Log}^+ x = \sup(\mathrm{Log}\, x, 0)$ for $x \geq 0$.

Proof: We shall use the fact that for $g \in \mathbf{L}^2(\mathbf{T})$ we have $\tilde{g} \in \mathbf{L}^2(\mathbf{T})$ and $\|\tilde{g}\|_{L^2} \leq \|g\|_{L^2}$. This implies that $m_{|g|}(\lambda) \geq 2\pi[1 - \|g\|_{L^2}^2 \lambda^{-2}],\ \lambda > 0$. We write $f = g + h$, where $g = f$ when $|f| \leq \lambda$ and $h = f$ when $|f| > \lambda$. We have $\tilde{f} = \tilde{g} + \tilde{h}$ and consequently

$$\{t: |\tilde{f}(e^{it})| > \lambda\} \subseteq \{t: |\tilde{g}(e^{it})| > \tfrac{\lambda}{2}\} \cup \{t: |\tilde{h}(e^{it})| > \tfrac{\lambda}{2}\}.$$

Therefore,

$$|\{t: |\tilde{g}(e^{it})| > \tfrac{\lambda}{2}\}| \leq 8\pi\lambda^{-2}\|g\|_{L^2}^2 = 8\pi\lambda^{-1}\int_0^{\lambda} x^2\, dm_{|f|} \quad and$$

$$|\{t: |\tilde{h}(e^{it})| > \tfrac{\lambda}{2}\}| \leq 2C\lambda^{-1}\|h\|_{L^1} = 2C\lambda^{-1}\int_{\lambda}^{\infty} x\, dm_{|f|}.$$

For $x \geq \lambda$, $[Log x]^{1/2} > [Log \lambda]^{1/2}$ and we obtain

$$|\{t : |\widetilde{h}(e^{it})| > \tfrac{\lambda}{2}\}| \leq \frac{2C}{\lambda\sqrt{Log\lambda}} \int_{\lambda}^{\infty} x(Logx)^{\frac{1}{2}} dm_{|f|}.$$

Therefore, $2\pi - m_{|\widetilde{f}|}(\lambda) \leq 8\pi\lambda^{-2} \int_0^{\lambda} x^2 dm_{|f|} + \dfrac{2C}{\lambda\sqrt{Log\lambda}} \int_{\lambda}^{\infty} x\sqrt{Logx}\, dm_{|f|}.$

Now from the hypothesis we have:

$$\int_1^{\infty} xLogx\, dm_{|f|}(x) \leq \int_T |f(t) Log|f(t)||dt < \infty,$$

and by Fubini's Theorem

$$\int_1^R \lambda^{-2}\left(\int_0^{\lambda} x^2 dm_{|f|}\right)d\lambda = \int_0^1 (1 - \tfrac{1}{R})x^2 dm_{|f|} + \int_1^R (\tfrac{1}{x} - \tfrac{1}{R})x^2 dm_{|f|}$$

$$\leq 2\pi + \int_1^R x\, dm_{|f|} = O(1),$$

$$\int_1^R \tfrac{1}{\lambda\sqrt{Log\lambda}}\left[\int_{\lambda}^{\infty} x\sqrt{Logx}\, dm_{|f|}\right]d\lambda = 2\int_1^R xLogx\, dm_{|f|} + 2\sqrt{LogR}\int_R^{\infty} x\sqrt{Logx}\, dm_{|f|}$$

$$= O(1).$$

THEOREM. 2.1.8: *M. Riesz*. For $1 < p < \infty$, the mapping $f \to \widetilde{f}$ is a bounded linear operator on $L^p(T)$.

Proof: If p and q are conjugate exponents, the mappings $f \to \widetilde{f}$ in $L^p(T)$ and $L^q(T)$ are except for a sign each other's adjoints and consequently if one is bounded so is the other and by the same bound. Thus it is enough to prove the theorem for $1 < p < 2$. Let $f \in L^p(T)$ be nonnegative. Denote by $f(re^{it})$ its Poisson integral, by $\widetilde{f}(re^{it})$ the harmonic conjugate and write

$$H(re^{it}) = f(re^{it}) + i\widetilde{f}(re^{it}).$$

Assume that f does not vanish identically. Since $f \geq 0$, $f(re^{it}) > 0$ and hence $H(re^{it}) \neq 0$ in D. Let $G(re^{it})$ be the branch of $[H(re^{it})]^p$ which is real at $r = 0$. Let γ be a real number satisfying $\gamma < \frac{\pi}{2}$, $p\gamma > \frac{\pi}{2}$. For $0 < r < 1$ we have

$$\frac{1}{2\pi}\int_T |G(re^{it})|\, dt = \frac{1}{2\pi}\int_I |G(re^{it})|\, dt + \tfrac{1}{2\pi}\int_{II} |G(re^{it})|\, dt,$$

where the first integral is taken over the set I satisfying $|\arg(H(z))| < \gamma$ and the second integral is taken over the set II satisfying $\gamma \leq |\arg(H(z))| < \frac{\pi}{2}$, where $z = re^{it}$. On the set I we have $|H(z)| < f(z)[\cos\gamma]^{-1}$, hence

$$\frac{1}{2\pi}\int_I |G(re^{it})|\, dt \leq \|f\|_{L^p}^p [\cos\gamma]^{-p},$$

and in particular,

$$\frac{1}{2\pi}\int_I \mathrm{Re}(G(re^{it}))dt \leq \|f\|_{L^p}^p [\cos\gamma]^{-p}.$$

On the other hand, on the set II $|G(z)| \leq \mathrm{Re}(G(z))(\cos p\gamma)^{-1}$ (both factors being nonnegative). Now since

$$\frac{1}{2\pi}\int_I \mathrm{Re}(G(re^{it}))dt = G(0) = (\hat{f}(0))^p,$$

we get

$$\frac{1}{2\pi}\int_I |\mathrm{Re}(G(re^{it}))|\, dt \leq (\hat{f}(0))^p + (\cos\gamma)^{-p}\|f\|_{L^p}^p.$$

Therefore,

$$\frac{1}{2\pi}\int_I |G(re^{it})|\, dt \leq \|f\|_{L^p}^p(\cos\gamma)^{-p} + (\hat{f}(0))^p + (\cos\gamma)^{-p}\|f\|_{L^p}^p$$

$$\leq \|f\|_{L^p}^p(\cos\gamma)^{-p} + \|f\|_{L^p}^p + \|f\|_{L^p}^p(\cos\gamma)^{-p} = c_p\|f\|_{L^p}^p$$

Where c_p is a constant depending only on p.
Since $|\tilde{f}(re^{it})|^p \leq |H(re^{it})|^p = |G(re^{it})|$, letting $r \to 1$ we get

$$\frac{1}{2\pi}\int_I |\tilde{f}(re^{it})|\, dt \leq c_p\|f\|_{L^p}^p, \text{and hence } \|\tilde{f}\|_{L^p}^p \leq c_p\|f\|_{L^p}^p.$$

Thus $\tilde{f} \in$ **L^p(T)**. Let $f \in$ **L^p(T)**. Then $f = f_1 - f_2 + if_3 - if_4$, $f_j \geq 0$, and

$$\|f_j\|_{L^p} \leq \|f\|_{L^p}, \text{ j=1,2,3,4.}$$

Then by the linearity of the mapping $f \to \tilde{f}$ we have $\tilde{f} = \tilde{f}_1 - \tilde{f}_2 + i\tilde{f}_3 - i\tilde{f}_4$. From the above case $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{f}_4 \in$ **L^p(T)**, and thus $\tilde{f} \in$ **L^p(T)**. Therefore, the mapping $f \to \tilde{f}$ is a bounded linear operator on **L^p(T)** for $1 < p < 2$. Since we have done for L^2(T) in section 1.3, for $1 < p \leq 2$, **L^p(T)** admits conjugation.//

2.2. CONVERGENCE AND DIVERGENCE AT A POINT

THEOREM. 2.2.1: There exists a continuous function whose Fourier series diverges at a point.

Proof: We give two proofs.

1. *Based on the Uniform boundedness Theorem.*

By Theorem 2.1.3, the mappings $f \to S_n(f,0)$ are continuous linear functionals on C(T) which are not uniformly bounded. By the Uniform Bounded ness Theorem, S_n are not point wise bounded. Therefore, there exists $f \in C(T)$ such that $(S_n(f,0))$ is not bounded, that is, the Fourier series of f diverges unboundedly at t =0.

2. *Construction of a Concrete Example.*

There exists a sequence of functions $\psi_n \in C(T)$ satisfying:

$$\|\psi_n\|_\infty \le 1$$

$$|S_n(\psi_n,0)| > \tfrac{1}{2}\|D_n\|_{L^1} \text{ or } |S_n(\psi_n,0)| > \tfrac{1}{10} Log n.$$

Put $\Phi_n(t) = \sigma_{n^2}(\psi_n, t)$ and notice that Φ_n is a trigonometric polynomial of degree n^2 satisfying: $\|\Phi_n\|_\infty \le 1$ *and* $|S_n(\Phi_n,t) - S_n(\psi_n,t)| < 2$. To show this:

$$\|\Phi_n\|_\infty = \|\sigma_{n^2}(\psi_n)\|_\infty \le \|F_{n^2}\|_{L^1} \cdot \|\psi_n\|_\infty = \|\psi_n\|_\infty \le 1, \quad \text{and}$$

$$|S_n(\Phi_n,t) - S_n(\psi_n,t)| = \left| \sum_{j=-n}^{n} \hat{\Phi}_n(j)e^{ijt} - \sum_{j=-n}^{n} \hat{\psi}_n(j)e^{ijt} \right|$$

$$= \left| \sum_{j=-n}^{n} (\hat{\Phi}_n(j) - \hat{\psi}_n(j))e^{ijt} \right|$$

$$= \left| - \sum_{j=-n}^{n} \frac{|j|}{n^2+1} \hat{\psi}_n(j)e^{ijt} \right|$$

$$\le \frac{1}{n^2+1} \sum_{j=-n}^{n} |j| = \frac{n(n+1)}{n^2+1} = 1 + \frac{n-1}{n^2+1} < 2.$$

Now in particular,

$$|S_n(\Phi_n,0)-S_n(\psi_n,0)|<2$$

and hence

$$-|S_n(\Phi_n,0)|+|S_n(\psi_n,0)|\le |S_n(\Phi_n,0)-S_n(\psi_n,0)|<2.$$

And thus

$$|S_n(\Phi_n,0)|>|S_n(\psi_n,0)|-2>\tfrac{1}{10}Log n-2.$$

With $\lambda_n = 2^{3^n}$ we define

$$f(t)=\sum_{n=1}^{\infty}\tfrac{1}{n^2}\Phi_{\lambda_n}(\lambda_n t).$$

<u>Claim</u>: f is a continuous function whose Fourier series diverges at t = 0.
To show the continuity of f consider the mappings

$$f_N(t)=\sum_{n=1}^{N}\tfrac{1}{n^2}\Phi_{\lambda_n}(\lambda_n t).$$

Then (f_N) is a sequence of continuous functions converging point wise to f. But it is easily shown that $f_N \xrightarrow{N\to\infty} f$ uniformly on T. Therefore, f is continuous on T. To show the divergence of the Fourier series of f at t = 0, we notice that

$$\begin{aligned}\Phi_{\lambda_j}(\lambda_j t)&=\sigma_{\lambda_j^2}(\psi_{\lambda_j},\lambda_j t)\\ &=\sum_{m=-\lambda_j^2}^{\lambda_j^2}(1-\tfrac{|j|}{\lambda_j^2+1})\hat{\psi}_{\lambda_j}(m)e^{i\lambda_j mt}\\ &=\sum_{m=-\lambda_j^2}^{\lambda_j^2}\hat{\Phi}_{\lambda_j}(m)e^{i\lambda_j mt}\end{aligned}$$

$$\sin ce\ \hat{\Phi}_{\lambda_j}(m)=\hat{F}_{\lambda_j^2}(m).\hat{\psi}_{\lambda_j}(m) = \begin{cases}(1-\tfrac{|m|}{\lambda_j^2+1})\hat{\psi}_{\lambda_j}(m) & if\ |m|\le\lambda_j^2\\ 0 & if\ |m|>\lambda_j^2.\end{cases}$$

Therefore,

$$\Phi_{\lambda_j}(\lambda_j t)=\sum_m \hat{\Phi}_{\lambda_j}(m)e^{i\lambda_j mt}.$$

Hence

$$\left|S_{\lambda_n^2}(f,0)\right| = \left|S_{\lambda_n^2}(\sum_1^n \tfrac{1}{j^2}\Phi_{\lambda_j}(\lambda_j t),0) + S_{\lambda_n^2}(\sum_{n+1}^\infty \tfrac{1}{j^2}\Phi_{\lambda_j}(\lambda_j t),0)\right|$$

$$= | S_{\lambda_n^2}(\sum_1^n \tfrac{1}{j^2}\sum_m \hat{\Phi}_{\lambda_j}(m)e^{i\lambda_j mt},0) + S_{\lambda_n^2}(\sum_{n+1}^\infty \tfrac{1}{j^2}\sum_m \hat{\Phi}_{\lambda_j}(m)e^{i\lambda_j mt},0) |$$

$$= \left|\sum_1^n \tfrac{1}{j^2}\sum_m \hat{\Phi}_{\lambda_j}(m)S_{\lambda_n^2}(e^{i\lambda_j mt},0) + \sum_{n+1}^\infty \tfrac{1}{j^2}\sum_m \hat{\Phi}_{\lambda_j}(m)S_{\lambda_n^2}(e^{i\lambda_j mt},0)\right|$$

$$= \left|\sum_1^{n-1} \tfrac{1}{j^2}\sum_m \hat{\Phi}_{\lambda_j}(m)S_{\lambda_n^2}(e^{i\lambda_j mt},0) + \sum_{n+1}^\infty \tfrac{1}{j^2}\hat{\Phi}_{\lambda_j}(0) + \frac{1}{n^2}\sum_m \hat{\Phi}_{\lambda_n}(m)S_{\lambda_n^2}(e^{i\lambda_n mt},0)\right|$$

$$= \left|\sum_1^{n-1} \tfrac{1}{j^2}\sum_m \hat{\Phi}_{\lambda_j}(m)S_{\lambda_n^2}(e^{i\lambda_j mt},0) + \sum_{n+1}^\infty \tfrac{1}{j^2}\hat{\Phi}_{\lambda_j}(0) + \frac{1}{n^2}S_{\lambda_n^2}(\Phi_{\lambda_n},0)\right|$$

$$= \left|\sum_1^{n-1} \tfrac{1}{j^2}\sum_{m=-\lambda_j^2}^{\lambda_j^2} \hat{\Phi}_{\lambda_j}(m) + \sum_{n+1}^\infty \tfrac{1}{j^2}\hat{\Phi}_{\lambda_j}(0) + \frac{1}{n^2}S_{\lambda_n^2}(\Phi_{\lambda_n},0)\right|$$

$$= \left|\sum_1^{n-1} \tfrac{1}{j^2}\Phi_{\lambda_j}(0) + \sum_{n+1}^\infty \tfrac{1}{j^2}\hat{\Phi}_{\lambda_j}(0) + \frac{1}{n^2}S_{\lambda_n^2}(\Phi_{\lambda_n},0)\right|$$

$$\geq \frac{1}{n^2}| S_{\lambda_n^2}(\Phi_{\lambda_n},0)| - \left(\left|\sum_1^{n-1} \tfrac{1}{j^2}\Phi_{\lambda_j}(0)\right| + \left|\sum_{n+1}^\infty \tfrac{1}{j^2}\hat{\Phi}_{\lambda_j}(0)\right|\right)$$

$$\geq \frac{1}{n^2}Log\lambda_n - 3$$

which tends to ∞. Therefore, $(S_n(f, 0))$ is not bounded and hence the Fourier series diverges unboundedly at t = 0.//

LEMMA. 2.2.1: Let $f \in L^1(T)$ and $\hat{f}(n) = O(\tfrac{1}{n})\ as\,|n| \to \infty$. Then for every $\varepsilon>0$ there exists $\lambda>1$ such that $\underset{n\to\infty}{Lim\ Sup} \sum_{n<|j|\leq \lambda n} |\hat{f}(j)| < \varepsilon$.

Proof: Since

$$\hat{f}(n) = O(\tfrac{1}{n}) \quad as \quad |n| \to \infty,$$

there exists M>0 such that $|n\hat{f}(n)| \leq M$, that is,

$$|\hat{f}(n)| \leq \tfrac{M}{|n|}\ as\,|n| \to \infty .$$

If n is sufficiently large and $\lambda = 1 + \frac{\varepsilon}{2M}$ then

$$\sum_{n<|j|\leq \lambda n} |\hat{f}(j)| \leq M \sum_{n<|j|\leq \lambda n} \tfrac{1}{|j|} < M.2(\lambda n - n)\tfrac{1}{n} = 2M(\lambda - 1) = \varepsilon .//$$

THEOREM. 2.2.2: *Tauberian Theorem due to Hardy.*

Let $f \in L^1(T)$ and $\hat{f}(n) = O(\frac{1}{n})\, as\, |n| \to \infty$. Then S_n(f, t) and σ_n(f, t) converge for the values of t and to the same limit. Also if σ_n(f, t) converges uniformly on some set, so does S_n(f, t).

Proof: Let ε>0 be given. By the above lemma there exists λ>1 such that

$$\underset{n\to\infty}{Lim\,Sup} \sum_{n<|j|\le\lambda n} |\hat{f}(j)| < \varepsilon .$$

$$S_n(f,t) = \frac{[\lambda n]+1}{[\lambda n]-n}\sigma_{[\lambda n]}(f,t) - \frac{n+1}{[\lambda n]-n}\sigma_n(f,t) - \frac{[\lambda n]+1}{[\lambda n]-n}\sum_{n<|j|\le\lambda n}(1-\tfrac{|j|}{[\lambda n]+1})\hat{f}(j)e^{ijt}$$

where [λn] denotes the integral part of λn. By Lemma 2.2.1, there exists an n_0 such that

$$n > n_0 \Rightarrow \left| \frac{[\lambda n]+1}{[\lambda n]-n}\sum_{n<|j|\le\lambda n}(1-\tfrac{|j|}{[\lambda n]+1})\hat{f}(j)e^{ijt} \right| < \frac{\varepsilon}{2}.$$

If σ_n (f, t_0) converge to a limit σ(f, t_o) then there exists n_1 sufficiently large such that for n > n_1 we have

$$\begin{aligned} &|S_n(f,t_o) - \sigma(f,t_o)| \\ &\le \frac{[\lambda n]+1}{[\lambda n]-n}|\sigma_{[\lambda n]}(f,t_o) - \sigma(f,t_o)| + \frac{n+1}{[\lambda n]-n}|\sigma(f,t_o) - \sigma_n(f,t_o)| \\ &\quad + \frac{[\lambda n]+1}{[\lambda n]-n}\left|\sum_{n<|j|\le\lambda n}(1-\tfrac{|j|}{[\lambda n]+1})\hat{f}(j)e^{ijt}\right| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, S_n(f, t_o) converges to σ(f, t_o). Thus if σ_n (f, t) converges uniformly on some set E, then also S_n(f, t) converges uniformly on E.//

COROLLARY. 2.2.1: Let f be a bounded variation on T. Then S_n(f, t) converge to $\frac{1}{2}[f(t+0) + f(t-0)]$ and in particular to f(t) at every point of continuity t. The convergence is uniform on closed intervals of continuity.

Proof: By Fejer's Theorem, σ_n(f, t)$\to \frac{1}{2}[f(t+0)+f(t-0)]$ as n$\to\infty$.
Since *f* is of bounded variation on T, we have $\hat{f}(n)=O(\frac{1}{n})\, as\, |n|\to\infty$.
Therefore, by Theorem 2.2.2,

$$S_n(f,t))\to \tfrac{1}{2}[f(t+0)+f(t-0)].$$

The remaining properties hold by Fejer's Theorem.//

LEMMA. 2.2.2: Let $f\in L^1(T)$ and assume $\int_{-1}^{1}\left|\frac{f(t)}{t}\right|dt<\infty$. Then $\lim_{n\to\infty} S_n(f,0)=0$.

Proof:

$$S_n(f,0)=\frac{1}{2\pi}\int_T \frac{f(t)}{\sin t/2}\sin(n+\tfrac{1}{2})tdt$$

$$=\frac{1}{2\pi}\int_T f(t)\cos nt dt+\frac{1}{2\pi}\int_T \frac{f(t)\cos\frac{t}{2}}{\sin\frac{t}{2}}\sin nt dt$$

By the hypothesis $\frac{f(t)\cos\frac{t}{2}}{\sin\frac{t}{2}}\in L^1(T)$. By the Riemann-Lebesgue Lemma,

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_T f(t)\cos nt dt=0,\ and\ \lim_{n\to\infty}\frac{1}{2\pi}\int_T \frac{f(t)\cos\frac{t}{2}}{\sin\frac{t}{2}}\sin nt dt=0.$$

Therefore, $\lim_{n\to\infty} S_n(f,0)=0$.//

THEOREM. 2.2.3: *Principle of Localization*. Let $f\in L^1(T)$ and assume that *f* vanishes in an open interval **I**. Then S_n(*f*, t) converges to zero for t$\in$**I**, and the convergence is uniform on closed subintervals of **I**.

Proof: Let $t_o\in$**I**. Then there exists $0<\delta<\pi$ such that $(t_o-\delta, t_o+\delta)\subseteq$**I**. Then

$$S_n(f,t_o)=\frac{1}{2\pi}\int_{-\delta}^{\delta} f(t+t_o)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}dt+\frac{1}{2\pi}\int_{\delta}^{\pi} f(t+t_o)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}dt$$

$$+\frac{1}{2\pi}\int_{-\pi}^{-\delta} f(t+t_o)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}dt$$

On the intervals $[\delta,\pi]$ and $[-\pi,-\delta]$, the function $\frac{1}{\sin\frac{t}{2}}$ is continuous (since $|t| \geq \delta$). Therefore, the function

$$\Phi(t) = \frac{f(t+t_o)}{\sin\frac{t}{2}}$$

is absolutely integrable, that is,

$$\frac{1}{2\pi}\int_{\delta}^{\pi}|\Phi(t)| = \frac{1}{2\pi}\int_{\delta}^{\pi}\left|\frac{f(t+t_o)}{\sin\frac{t}{2}}\right|dt \leq \frac{1}{2\pi}[\sin\tfrac{\delta}{2}]^{-1}\int_{\delta}^{\pi}|f(t+t_o)|dt < \infty.$$

By the Riemann-Lebesgue Lemma we have

$$(i)\quad \lim_{n\to\infty}\frac{1}{2\pi}\int_{\delta}^{\pi} f(t)\cos ntdt = 0,$$

$$(ii)\quad \lim_{n\to\infty}\frac{1}{2\pi}\int_{\delta}^{\pi}\frac{f(t)\cos\frac{t}{2}}{\sin\frac{t}{2}}\sin ntdt = 0,$$

$$(iii)\quad \lim_{n\to\infty}\frac{1}{2\pi}\int_{-\pi}^{-\delta} f(t+t_o)\cos ntdt = 0,$$

$$(iv)\quad \lim_{n\to\infty}\frac{1}{2\pi}\int_{-\pi}^{-\delta}\frac{f(t+t_o)\cos\frac{t}{2}}{\sin\frac{t}{2}}\sin ntdt = 0,$$

Therefore, $\lim_{n\to\infty} S_n(f,t_o) = 0$. Since t_o is arbitrary element of **I**, for all t in **I**

we have $S_n(f,t) \xrightarrow{n\to\infty} 0$ .//

REMARK: *Restatement of the principle of localization*

Let ***f***, ***g***$\in \mathbf{L}^1(\mathbf{T})$ and assume that ***f*** *(t)* = ***g****(t)* in some neighborhood of a point t_o. Then the Fourier series of ***f*** and ***g*** at t_o are either both convergent and to the same limit or both are divergent and in the same manner.

THEOREM. 2.2.4: *Dini's Test.*

Let $f \in L^1(T)$. If $\int_{-1}^{1}\left|\frac{f(t+t_o)-f(t_o)}{t}\right|dt < \infty$ then $\lim_{n\to\infty} S_n(f,t_o) = f(t_o)$.

Proof:

$$S_n(f, t_o) - f(t_o) = \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(t+t_o) - f(t_o)]D_n(t)dt$$

$$= \frac{1}{2\pi}\int_{0}^{\pi}[f(t+t_o) + f(t_o - t) - 2f(t_o)]D_n(t)dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(t+t_o) - f(t_o)]\cos ntdt$$

$$+ \frac{1}{2\pi}\int_{0}^{\pi}\frac{f(t+t_o) + f(t_o - t) - 2f(t_o)}{\tan\frac{t}{2}}\sin ntdt.$$

Since $\tan\frac{t}{2} \simeq \frac{t}{2}$ as $t \to 0$, from the hypothesis $\frac{f(t+t_o) - f(t_o)}{\tan\frac{t}{2}}$ is integrable.

Therefore, by the Riemann- Lebesgue Lemma

(i) $\lim\limits_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(t+t_o) - f(t_o)]\cos ntdt = 0$, and

(ii) $\lim\limits_{n\to\infty} \frac{1}{2\pi}\int_{0}^{\pi}\frac{f(t+t_o) + f(t_o - t) - 2f(t_o)}{\tan\frac{t}{2}}\sin ntdt = 0.$

Therefore, $\lim\limits_{n\to\infty}[S_n(f, t_o) - f(t_o)] = 0$,

that is,

$$\lim_{n\to\infty} S_n(f, t_o) = f(t_o). \quad //$$

2.3. SETS OF DIVERGENCE

DEFINITION. 2.3.1: Let **B** be a homogeneous Banach space on **T**. A set $\mathbf{E}\subseteq\mathbf{T}$ is a set of divergence for **B** if and only if there exists an $f\in B$ whose Fourier series diverges at every point of **E**.

NOTATION: Let $f\in L^1(T)$. We put

(i) $S_n^*(f, t) = \sup\limits_{m\le n}|S_m(f, t)|$

(ii) $S^*(f, t) = \sup\limits_{n}|S_n(f, t)|$

LEMMA. 2.3.1: Let **B** be a homogeneous Banach space on **T and** $g\in B$. Then there exist $f\in B$ and a positive even sequence (λ_k) such that $\lambda_k \to \infty$ monotonically with k and $\hat{f}(k) = \lambda_k \hat{g}(k)\ \forall k \in Z$.

Proof: For each n, let $\beta(n)$ be such that $\left\|\sigma_{\beta(n)}(g) - g\right\|_B < 2^{-n}$. We write

$$f = g + \sum_{n=1}^{\infty}(g - \sigma_{\beta(n)}(g)).$$

The series defining f converges in norm. Hence $f\in B$. Also $\hat{f}(k) = \lambda_k \hat{g}(k)$ where $\lambda_k = 1 + \sum_{n=1}^{\infty} \min(1, \frac{|k|}{\beta(n)+1})$. //

THEOREM. 2.3.1: Let **B** be a homogeneous Banach space on **T** and **E⊆T.** E is a set of divergence for **B** if and only if there exists an $f\in B$ such that $S^*(f,t) = \sup_n |S_n(f,t)| = \infty$ for $t\in E$.

Proof: 1. The condition $S^*(f,t) = \sup_n |S_n(f,t)| = \infty$ for $t\in E$ is clearly sufficient for the divergence of $\sum \hat{f}(j)e^{ijt}$ for all $t\in E$.

2. Assume that for some $g\in B$, $\sum \hat{g}(j)e^{ijt}$ diverges at every point of E. Let $f\in B$ and (λ_k) be the function and the sequence corresponding to g by Lemma 2.3.1. For n > m

$$S_n(g,t) - S_m(g,t) = \sum_{m+1}^{n}(S_j(f,t) - S_{j-1}(f,t))\lambda_j^{-1}$$

$$= \frac{S_n(f,t)}{\lambda_n} - \frac{S_m(f,t)}{\lambda_{m+1}} + \sum_{m+1}^{n-1}[\tfrac{1}{\lambda_j} - \tfrac{1}{\lambda_{j+1}}]S_j(f,t).$$

Hence $|S_n(g, t)-S_m(g, t)| \le 2S^*(f, t)\lambda_{m+1}^{-1}$. It follows that if $S^*(f, t)<\infty$, then the Fourier series of g converges and $t\notin E$.//

LEMMA. 2.3.2: Let **B** be a homogeneous Banach space on **T** such that if $f\in B$ and $n\in Z$ then $e^{int}f\in B$ and $\left\|e^{int}f\right\|_B = \|f\|_B$. Then E is a set of divergence for **B** if and only if there exists a sequence of trigonometric polynomials $P_j\in B$ such that $\sum \left\|P_j\right\|_B < \infty$ *and* $\sup_j S^*(P_j,t) = \infty$ on E.

Proof: 1. Assume that there exist trigonometric polynomials $P_j \in B$ such that

$$\sum \|P_j\|_B < \infty \quad and \quad \sup_j S^*(P_j, t) = \infty$$

on E. Denote by m_j the degree of P_j and let v_j be integers satisfying $v_j > v_{j-1} + m_{j-1} + m_j$. Put

$$f(t) = \sum e^{iv_j t} P_j(t).$$

For $n \le m_j$ we have

$$S_{v_j+n}(f,t) - S_{v_j-n-1}(f,t) = e^{iv_j t} S_n(P_j, t)$$

and thus $\sum \hat{f}(j) e^{ijt}$ diverges on E.

2. Conversely assume that E is a set of divergence for B. Then there exists a monotonic sequence $\omega_n \to \infty$ and a function $f \in B$ such that $|S_n(f, t)| > \omega_n$ infinitely often for every t in E. We now pick a sequence of integers (λ_k) such that

$$\|f - \sigma_{\lambda_k}(f)\|_B < 2^{-k}$$

and integers μ_j such that

$$\omega_{\mu_j} > 2 \sup_t S^*(\sigma_{\lambda_j}(f), t) \; and \; write \; P_j = V_{\mu_{j+1}} * (f - \sigma_{\lambda_j}(f))$$

where as usual V_μ denotes de la Vallee Poussin's kernel. Then

$$\sum \|P_j\|_B \le \|V_{\mu_{j+1}}\|_{L^1} \sum \|f - \sigma_{\lambda_j}(f)\|_B \le \|V_{\mu_{j+1}}\|_{L^1} \sum \frac{1}{2^j} < \infty.$$

If t is in E and n is an integer such that $|S_n(f, t)| > \omega_n$, then for some j, $\mu_j < n \le \mu_{j+1}$ and

$$S_n(P_j, t) = S_n(f - \sigma_{\lambda_j}(f), t) = S_n(f, t) - S_n(\sigma_{\lambda_j}(f), t).$$

Hence $|S_n(P_j, t)| > \frac{1}{2}\omega_n$. Therefore,

$$S^*(f, t) = \infty$$

for every t in E, and thus E is a set of divergence for B. //

THEOREM. 2.3.2: Let **B** be a homogeneous Banach space on **T** such that if $f\in$B and n$\in$Z then $e^{int}f\in$B and $\left\|e^{int} f\right\|_B = \|f\|_B$. If E_j, j=1,2... are sets of divergence for B, then E = $\bigcup_{j=1}^{\infty} E_j$ is a set of divergence for B.

Proof: Let (P_n^j) be the sequences of polynomials corresponding to E_j. Omitting a finite number of terms does not change $\underset{j}{Sup}\, S^*(P_j, t) = \infty$.

Therefore for each j there exists N_j such that $\sum_{n=N_j+1}^{\infty} \left\|P_n^j\right\|_B < 2^{-j}$. Consider the sequence of trigonometric polynomials $\{P_n^j\}_{n,j} = \bigcup_{j=1}^{\infty} \{P_n^j\}_{n=N_j+1}^{\infty}$. Then

$$\sum_{n,j}^{\infty} \left\|P_n^j\right\|_B = \sum_{j=1}^{\infty} \left(\sum_{n=N_j+1}^{\infty} \left\|P_n^j\right\|_B \right) \le \sum_{j=1}^{\infty} 2^{-j} < \infty,$$

and

$$\underset{j,n}{Sup}\, S^*(P_n^j, t) \ge \underset{n}{Sup}\, S^*(P_n^j, t) = \infty \;\; for\;\; t \in \bigcup_{j=1}^{\infty} E_j.$$

Therefore, E = $\bigcup_{j=1}^{\infty} E_j$ is a set of divergence for B. //

LEMMA. 2.3.3: Let E be a union of a finite number of intervals on T. Denote the measure of E by δ. There exists a trigonometric polynomial φ such that $S^*(\varphi, t) > \frac{1}{2\pi} Log(\frac{1}{3\delta})$ *on E and* $\|\varphi\|_\infty \le 1$.

Proof: It is convenient to identify T with the unit circumference {z: |z| = 1}. Let I be a small interval on T, I = $\{e^{it} : |t-t_o| \le \varepsilon\}$. The function

$$\psi_I = (1+\varepsilon - z e^{-it_o})^{-1}$$

has a positive real part throughout the unit disc, its real part is larger than $\frac{1}{3\varepsilon}$ on I, and its value at the origin (z=0) is $(1+\varepsilon)^{-1}$. We now write E$\subseteq \bigcup_{j=1}^{N} I_j$, the I_j being small intervals of equal length 2ε such that Nε < δ, and consider the function

$$\psi(z) = \frac{1+\varepsilon}{N} \sum \psi_{I_j}(z).$$

ψ has the following properties:

(i) Re $(\psi(z)) > 0$ for $|z| \leq 1$,

(ii) $\psi(0) = 1$, and

(iii) $|\psi(z)| \geq$ Re $(\psi(z)) > \frac{1}{3N\varepsilon} > \frac{1}{3\delta}$ on E.

Since the Taylor series of Logψ converges uniformly on T, we can take a particular sum

$$\Phi(z) = \sum_{n=1}^{M} a_n z^n$$

of that series such that $|\text{Im}(\text{Log}(\Phi(z)))| < \pi$ on T and

$$|\text{Log}(\Phi(z))| > Log \tfrac{1}{3\delta}$$

is valid for Φ in place of Logψ. We now put

$$\varphi(t) = \tfrac{1}{\pi} e^{-iMt} \,\text{Im}(\Phi(e^{it})) = \tfrac{1}{2\pi i} e^{-iMt} \left(\sum_{1}^{M} a_n e^{\text{int}} - \sum_{1}^{M} \overline{a}_n e^{-\text{int}}\right)$$

and notice that $|S_M(\varphi, t)| = \frac{1}{2\pi} |\Phi(e^{it})|$. //

THEOREM. 2.3.3: Every set of measure zero is a set of divergence for **C(T).**

Proof: If E is a set of measure zero, it can be covered by a union $\bigcup_{n=1}^{\infty} I_n$ the I_n being intervals of length $|I_n|$ such that $\sum_{n=1}^{\infty} | I_n | < 1$ and such that every $t \in E$ belongs to infinitely many I_n's. Grouping finite sets of intervals we can cover E infinitely often by $\bigcup_{n=1}^{\infty} E_n$ such that every E_n is a finite union of intervals and such that $|E_n| < e^{-2^n}$. Let φ_n be a trigonometric polynomial satisfying Lemma 2.3.3 for $E = E_n$ and put

$$P_n = n^{-2}\varphi_n.$$

We clearly have

$$\sum_{n=1}^{\infty} \|P_n\|_\infty < \infty \; and \; S^*(P_n, t) > \frac{2^{n-1}}{2\pi n^2}$$

on E_n. Since every $t \in E$ belongs to infinitely many E_n's,

$$\underset{j}{Sup}\, S^*(P_n,t)=\infty \quad as \quad \frac{2^{n-1}}{2\pi n^2}\xrightarrow{n\to\infty}\infty.$$

By Lemma 2.3.2, E is a set of divergence for B. //

THEOREM. 2.3.4: Let **B** be a homogeneous Banach space on **T** such that if $f \in B$ and $n \in Z$ then $e^{int}f \in B$ and $\left\| e^{int} f \right\|_B = \|f\|_B$. Assume $C(T) \subseteq B$. Then either T is a set of divergence for B or the sets of divergence for B are precisely the sets of measure zero.

Proof: By Theorem 2.3.3, it is clear that every set of measure zero is a set of divergence for B. Assume that E is a set of divergence for B of positive measure. For $\alpha \in T$ denote by E_α the translate of E by α. E_α is clearly a set of divergence for B. Let (α_n) be the sequence of all rational multiples of 2π and put $\widetilde{E} = \bigcup_{n=1}^{\infty} E_{\alpha_n}$. By Theorem 2.3.1, $\widetilde{E}$ is a set of divergence for B.

<u>Claim</u>: $T \setminus \widetilde{E}$ is a set of measure zero.

Consider the characteristic function $\chi_{\widetilde{E}}$. Then $\chi_{\widetilde{E}}(t-\alpha_n) = \chi_{\widetilde{E}}(t)$ for all t and for all α_n. This means

$$\sum_j \hat{\chi}_{\widetilde{E}}(j)e^{-i\alpha_n j}e^{ijt} = \sum_j \hat{\chi}_{\widetilde{E}}(j)e^{ijt} \; or \; \hat{\chi}_{\widetilde{E}}(j)e^{-i\alpha_n j} = \hat{\chi}_{\widetilde{E}}(j)$$

for all α_n.

If $j \neq 0$, this implies $\hat{\chi}_{\widetilde{E}}(j) = 0$. Hence $\chi_{\widetilde{E}}(t)$ = constant, almost everywhere and since $\chi_{\widetilde{E}}$ is a characteristic function this implies that the measure of $\widetilde{E}$ is either zero or 2π. Since $\widetilde{E} \supset E$, $\widetilde{E}$ is almost all of T. Since $T \setminus \widetilde{E}$ is a set of measure zero, it is a set of divergence for B. Hence T is a set of divergence for B. //

REMARK: For homogeneous Banach spaces satisfying the conditions of Theorem 2.3.4 and in particular for $\mathbf{B} = \mathbf{L^p(T)}$, $1 < p < \infty$, or $B = C(T)$, either there exists a function $f \in B$ whose Fourier series diverges everywhere, or the Fourier series of every $f \in B$ converges almost everywhere. The case of $B = L^2(T)$ was settled by only recently by L. Carleson, who proved the "Lusin Conjecture," namely that "The Fourier series of functions in $L^2(T)$ converge almost everywhere." This result was extended by Hunt to all $L^p(T)$ with $p > 1$.

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