# Graduate Seminar Report

# On

# Generating Function Operator

By

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June 2004

Addis Ababa

# Acknowledgment

My special thanks goes to my advisor Dr. Yirgalem Tsegaye. With out the help of her I couldn't finish my learning. She helped me in producing relations, giving hints and over all equipped me to know much more about Generating Functions. I don't have words just Thank You.

## Out line

## Chapter One

### 1.1 Introduction

This seminar report is about generating functions, two operators of generating functions and some of their uses in solving recurrence relations. The subject is so vast that I am not attempted to give a diversified discussion. Rather I have tried to produce relations of Generating Functions which help us to solve recurrence relations.

In this report, we see two operators; one the generating function operator, denoted by $\varsigma$, and is applied to the general term $f_n$ of a sequence, to give the generating function f(t) of the sequence. The other is the "coefficient operator", denoted by $[t^n]$, and is applied to a generating function f(t) to extract the $n^{th}$ coefficient of the sequence whose generating function is f(t).

Generating functions are useful tools in solving combinatorial problems. For instance in finding closed forms for solutions to recurrence relations, in proving or verifying many known identities involving binomial coefficients, which are really important in combinatorics.

Here our approach is only algebraic. The operators are defined, a number of rules which can be considered as axioms are set, and are used to manipulate the operators so that a number of important generating functions are derived, and used to solve recurrence relations.

Generating functions are a bridge between discreet mathematics on the one hand and continuous analysis (particularly complex variable theory) on the other hand. It is possible to study them exclusively as tools for solving discreet mathematics. As such there is much that is powerful and magical in the way generating functions give unified methods for handling such problems.

In recent years there has been a vigorous trend in the direction of finding bijective proofs of combinatorial theorems. That is, if we want to prove that two sets have the same cardinality then we should be able to do it by exhibiting an explicit bijection between the sets. In many cases the fact that the two sets have the same cardinality was discovered in the first place by generating function arguments. Also, even though bijective arguments may be known, the generating function proofs may be shorter or more elegant.

## 1.2 Definitions

**Definition 1.2.1: Formal Power Series**

Let $f_0, f_1, f_2, f_3, \ldots$ be a sequence of real numbers, then a **formal power series** is an expression of the form

$$f_0 + f_1t + f_2t^2 + f_3t^3 \ldots = \sum_{n=0}^{\infty} f_n t^n$$

where t can be any variable or object which we can manipulate algebraic operations.

**Definition 1.2.2: Generating Function**

Given a sequence $f_0, f_1, f_2, f_3, \ldots$ of real numbers, the formal power series $\mathbf{f(t)} = \sum_{n=0}^{\infty} f_n t^n$ is the **ordinary generating function** of the sequence.

**Definition 1.2.3: The generating function operator**

Let $f_0, f_1, f_2, f_3, \ldots$ be a sequence of real numbers, then the generating function operator, denoted by $\varsigma$, is an operator, which acts on $f_n$ to give the generating function f(t). i.e

$$\varsigma\{f_n\} = f(t) = \sum_{n=0}^{\infty} f_n t^n$$

**Definition 1.2.4: The "Coefficient Operator"**

Let $f_0, f_1, f_2, f_3, \ldots$ be a sequence of real numbers, then the generating function operator, denoted by $[t^n]$, is an operator, which acts on f(t) to give the nth coefficient of the sequence $f_n$. i.e

$$[t^n]f(t) = f_n$$

**Definition 1.2.5: Equality**

$$f_n = g_n \quad (\forall n \in \mathbb{N}) \Rightarrow \varsigma\{f_n\} = \varsigma\{g_n\}$$

## 1.3 Operations

### Addition

Let $\varsigma\{ f_n \} = f(t) = \sum_{n=0}^{\infty} f_n t^n$ and $\varsigma\{ g_n \} = g(t) = \sum_{n=0}^{\infty} g_n t^n$ be power series, then

$$f(t) + g(t) = \varsigma\{ f_n \} + \varsigma\{ g_n \} = \varsigma\{ f_n + g_n \}$$

### Multiplication

$$f(t).g(t) = \sum_{n=0}^{\infty} f_n t^n . \sum_{n=0}^{\infty} g_n t^n = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} f_k g_{n-k} \right) t^n$$

### Derivatives

$$f'(t) = \sum_{n=1}^{\infty} n f_n t^{n-1}$$

### Integration

$$\int f(t)\, dt = \sum_{n=0}^{\infty} \int f_n t^n dt$$

$$= \sum_{n=0}^{\infty} f_n \frac{t^{n+1}}{n+1}$$

### Composition

$$\sum_{k=0}^{\infty} f_k (\varsigma\{g_n\})^k = \varsigma\{f_n\} \circ \varsigma\{g_n\}$$

### Convolution

$$\varsigma\left\{ \sum_{k=0}^{n} f_k g_{n-k} \right\} = \varsigma\{f_n\} \cdot \varsigma\{g_n\}$$

## 1.4 Rules

Let $\varsigma\{f_n\}$=f(t) and $\varsigma\{g_n\}$=g(t)

A1. $\varsigma\{\alpha f_n + \beta g_n\} = \alpha\varsigma\{f_n\} + \beta\varsigma\{g_n\}$ **Linearity**

$= \alpha f(t) + \beta g(t)$

A2. $\varsigma\{f_{n+1}\} = \frac{1}{t}(\varsigma\{f_n\} - f_0)$ **Shifting**

Proof: $\varsigma\{f_{n+1}\} = \sum_{n=0}^{\infty} f_{n+1}t^n = f_1 + f_2t + f_3t^2 + \ldots$

$$= \frac{(f_1t + f_2t^2 + f_3t^3 + \ldots)}{t}$$

$$= \frac{(f_0 + f_1t + f_2t^2 + \ldots) - f_0}{t} = \frac{1}{t}(\varsigma\{f_n\} - f_0)$$

A3. $\varsigma\{nf_n\} = tD\,\varsigma\{f_n\}$ **Differentiation**

Proof: $\varsigma\{nf_n\} = \sum_{n=0}^{\infty} nf_nt^n = 0 + f_1t + 2f_2t^2 + 3f_3t^3 + \ldots$

But we know that $f(t) = \sum_{n=0}^{\infty} f_nt^n$

$$\Rightarrow \quad f'(t) = \sum_{n=1}^{\infty} nf_nt^{n-1}$$

$$\Rightarrow \varsigma\{nf_n\} = \sum_{n=0}^{\infty} nf_nt^n = t.f'(t) = tD\,\varsigma\{f_n\}$$

**For the inverse operator**

| | |
|---|---|
| **$B_1$.** $[t^n](\alpha f(t) + \beta g(t)) = \alpha[t^n]f(t) + \beta[t^n]g(t)$ | **Linearity** |
| **$B_2$.** $[t^n](tf(t)) = [t^{n-1}]f(t)$ | **Shifting** |
| **$B_3$.** $[t^n]Df(t) = [t^n]\, f\,'(t) = (n+1)f_{n+1}$ | **Differentiation** |
| **$B_4$.** $[t^n]\,[f(t)\cdot g(t)] = \sum_{k=0}^{n}[t^k]f(t).[x^{n-k}]g(x)$ | **Convolution** |
| **$B_5$.** $[t^n]\,[f(t)\circ g(t)] = [t^n]\,\sum_{k=0}^{n}(g(t))^k[y^k]f(y)$ | **Composition** |

The above rules will be used to prove the following theorems, which serve as important tools to reach our scope.

# Chapter Two

## 2.1 Theorems and Relations

**Theorem 1:** If $\varsigma\ \{\mathbf{f_n}\}$ is the ordinary power series generating function of the sequence $\{f_n\}_{n=0}^{\infty}$, then

$$\varsigma\ \{\mathbf{f_{n+k}}\} = \frac{\varsigma\{f_n\} - f_0 - f_1 t - \ldots - f_{k-1}t^{k-1}}{t^k}$$

Proof: proof by inductive

From A2 we have, $\varsigma\ \{\mathbf{f_{n+1}}\} = \frac{1}{t}(\ \varsigma\{\mathbf{f_n}\} - \mathbf{f_0})$

$$\varsigma\ \{\mathbf{f_{n+2}}\} = \sum_{n=0}^{\infty} f_{n+2}t^n = f_2 + f_3t + f_4t^2 + \ldots$$

$$= \frac{(f_1 + f_2t + f_3t^2 + \ldots) - f_1}{t} = \frac{[\frac{(f_0 + f_1t + f_2t^2 + \ldots) - f_0}{t}] - f_1}{t}$$

$$= \frac{[\frac{(\varsigma\{f_n\} - f_0}{t}] - f_1}{t} = \frac{\varsigma\{f_n\} - f_0 - f_1t}{t^2}$$

Inductively we get

$$\varsigma\ \{\mathbf{f_{n+k}}\} = \frac{\varsigma\{n\} - f_0 - f_1 - \ldots - f_{k-1}t^{k-1}}{t^k}$$

**Theorem 2:** If $\varsigma\ \{\mathbf{f_n}\}$ is the ordinary power series generating function of the sequence $\{f_n\}_{n=0}^{\infty}$, then $\varsigma\ \{\mathbf{n^k f_n}\} = \mathbf{(tD)^k}\varsigma\{\mathbf{f_n}\}$

Proof: from A3 we have $\varsigma\ \{\mathbf{nf_n}\} = \mathbf{tD}\ \varsigma\{\mathbf{f_n}\}$

$$\varsigma\ \{\mathbf{n^2 f_n}\} = \sum_{n=k}^{\infty} n^2 f_n t^n = 0 + f_1t + 4f_2t^2 + 9f_3t^3 + \ldots$$

Obviously we re-apply the multiply-by-n operator tD, so the answer is $\mathbf{(tD)^2}\varsigma\{\mathbf{f_n}\}$

Inductively we get $\varsigma\ \{\mathbf{n^k f_n}\} = \mathbf{(tD)^k}\varsigma\{\mathbf{f_n}\}$

## Relations

R1. $\boldsymbol{\varsigma\{(n+1)f_{n+1}\} = D\varsigma\{f_n\}}$

Soln. $\varsigma\{(n+1)f_{n+1}\}$

$$= \varsigma\{nf_{n+1} + f_{n+1}\} = \varsigma\{nf_{n+1}\} + \varsigma\{f_{n+1}\}$$

$$= tD[\varsigma\{f_{n+1}\}] + \frac{\varsigma\{f_n\} - f_0}{t}$$

$$= tD[\frac{\varsigma\{f_n\} - f_0}{t}] + \frac{\varsigma\{f_n\} - f_0}{t}$$

$$= \frac{tD\varsigma\{f_n\} - \varsigma\{f_n\} + f_0}{t} + \frac{\varsigma\{f_n\} - f_0}{t} = \boldsymbol{D\varsigma\{f_n\}}$$

R2. $\boldsymbol{\varsigma\{\frac{1}{n}f_n\} =} \int \frac{\varsigma\{f_n\} - f_0}{t}dt$

Soln. let $g_n = \frac{1}{n} f_n \quad \Rightarrow \quad ng_n = f_n$, except for $n = 0$.

$\varsigma\{f_n\} = \varsigma\{ng_n\} = tD[\varsigma\{g_n\}]$ by A3

$$\Rightarrow \frac{\varsigma\{f_n\}}{t} = D[\varsigma\{g_n\}] + f_0$$

$$\Rightarrow \int \frac{\varsigma\{f_n\} - f_0}{t}dt = \varsigma\{g_n\}$$

R3. $\boldsymbol{\varsigma\{\frac{1}{n+1}f_n\} = t^{-1}[} \int \varsigma\{f_n\}dt + C \boldsymbol{]}$

Soln. Let $g_n = \frac{1}{n+1} f_n \quad \Rightarrow \quad (n+1)g_n = f_n$

$\varsigma\{f_n\} = \varsigma\{ng_n + g_n\} = \varsigma\{ng_n\} + \varsigma\{g_n\} = tD[\varsigma\{g_n\}] + \varsigma\{g_n\}$

This is a differential equation

Let $\varsigma\{f_n\} = f(t)$ and $\varsigma\{g_n\} = g(t)$

Then $\varsigma\{f_n\} = tD[\varsigma\{g_n\}] + \varsigma\{g_n\}$

$$\Rightarrow f(t) = t\,g'(t) + g(t)$$

$$\Rightarrow f(t) = \frac{d}{dt}[tg(t)]$$

$$\Rightarrow t\,g(t) = \int f(t)dt + C$$

$$\Rightarrow g(t) = \frac{1}{t}\int f(t)dt \quad + \quad C$$

$$\Rightarrow \varsigma\{\frac{1}{n+1}\mathbf{f_n}\} = \mathbf{t^{-1}}[\int \varsigma\{f_n\}dt \quad + C\,]$$

R4. If $\varsigma$ **{f$_n$} = f(t)**, then $\varsigma$ **{(-1)$^n$f$_n$} = f(-t)**

Let g(t) = -t

Then $\sum_{n=0}^{\infty} f_n(g(t))^n \;=\; f(g(t))$

$$\Rightarrow \sum_{n=0}^{\infty} f_n(-t)^n \;=\; f(-t)$$

$$\Rightarrow \sum_{n=0}^{\infty} (-1)^n f_n(t)^n \;=\; f(-t)$$

$$\Rightarrow \varsigma\{(-1)^n f_n\} \;=\; f(-t)$$

R5. $\varsigma\{\frac{f_n}{2n+1}\}$

Let $g_n = \frac{1}{2n+1} f_n \quad \Rightarrow \quad (2n+1)g_n = f_n$

$\varsigma\ \{f_n\} = \varsigma\ \{2ng_n + g_n\} = \varsigma\ \{2ng_n\} + \varsigma\ \{g_n\} = 2tD[\varsigma\ \{g_n\}] + \varsigma\ \{g_n\}$

This is a differential equation

Let $\varsigma$ {f$_n$} = f(t) and $\varsigma$\{g$_n$\} = g(t)

Then $\varsigma\ \{f_n\} = 2tD[\varsigma\{g_n\}] + \varsigma\{g_n\}$

$$\Rightarrow f(t) = 2t\, g'(t) + g(t)$$

$$\Rightarrow \frac{d}{dt}\left[\sqrt{t}\; g(t)\right] \;=\; \frac{1}{2\sqrt{t}} f(t)$$

$$\Rightarrow \sqrt{t}\; g(t) \;=\; \frac{1}{2}\int \frac{f(t)}{\sqrt{t}}dt \quad + \quad C$$

$$\Rightarrow g(t) \;=\; \frac{1}{2\sqrt{t}}\int \frac{f(t)}{\sqrt{t}}dt \quad + \quad \frac{C}{\sqrt{t}}$$

$$\Rightarrow \varsigma\{g_n\} = \varsigma\{\frac{1}{2n+1} f_n\} = \frac{1}{\sqrt{t}}\left[\frac{1}{2}\int \frac{f(t)}{\sqrt{t}}dt \quad + \; C\right]$$

R6. $\varsigma\{\sum_{k=0}^{n} f_k\} = \frac{1}{1-t}\varsigma\{\mathbf{f_n}\}$

Soln. Let $S_n = \sum_{k=0}^{n} f_k$

$$\Rightarrow \quad S_{n+1} = \sum_{k=0}^{n+1} f_k = f_{n+1} + S_n$$

$$\Rightarrow \quad S_{n+1} = f_{n+1} + S_n$$

$$\Rightarrow \quad f_{n+1} = S_{n+1} - S_n$$

$$\varsigma\{f_{n+1}\} = \varsigma\{S_{n+1}\} - \varsigma\{S_n\}$$

$$\frac{\varsigma\{f_n\} - f_0}{t} = \frac{\varsigma\{s_n\} - s_0}{t} - \varsigma\{S_n\} \qquad \text{but } S_0 = f_0$$

$$\frac{\varsigma\{f_n\} - f_0}{t} = \frac{\varsigma\{s_n\} - f_0}{t} - \varsigma\{S_n\}$$

$$\Rightarrow \quad \varsigma\{f_n\} = \varsigma\{S_n\} - t\,\varsigma\{S_n\}$$

$$\Rightarrow \quad \varsigma\{f_n\} = (1-t)\,\varsigma\{S_n\}$$

$$\Rightarrow \quad \varsigma\{S_n\} = \frac{1}{1-t}\varsigma\{\mathbf{f_n}\}$$

R7. If $\delta_{m,k} = \begin{cases} 1 & if\ k = m \\ 0 & otherwise \end{cases}$ then $\varsigma\{\delta_{m,k}\} = \mathbf{t^m}$

Soln. $\varsigma\{\delta_{j,m}\} = \sum_{j=0}^{\infty} \delta_{j,m} t^j = \delta_{0,m} + \delta_{1,m}\, t + \delta_{2,m}\, t^2 + \ldots + \delta_{m,m}\, t^m + \delta_{m+1,m}\, t^{m+1}$

$+ \ldots$

But here $\delta_{m,m} = 1$ and the other coefficients are 0.

Therefore, $\varsigma\{\delta_{j,m}\} = \mathbf{t^m}$

For instance, If $\delta_{0,k} = \begin{cases} 1 & if\ k = 0 \\ 0 & otherwise \end{cases}$ then $\varsigma\{\delta_{0,k}\} = ?$

Soln. From the above result we have $\varsigma\{\delta_{m,k}\} = \mathbf{t^m}$

Therefore, if m=0, we will get

$$\varsigma\{\delta_{0,k}\} = t^0 = 1$$

**Examples:**

1. $\varsigma\{1\} = \dfrac{1}{1-t}$

2. The constant sequence a, a, a, ... $a \in \mathcal{R}$

   i.e $f_n = a$

$$\varsigma\,\{\mathbf{f_n}\} = \sum_{n=0}^{\infty} f_n t^n = \sum_{n=0}^{\infty} a t^n = a \sum_{n=0}^{\infty} t^n = a.\,\varsigma\{1\} = \frac{a}{1-t}$$

3. For 1, 2, 3, ...

   i.e $f_n = n$

   $\varsigma\,\{\mathbf{f_n}\} = \varsigma\{n\}$

   From Theorem 3 and example 1 $\varsigma\,\{\mathbf{f_n}\} = tD\varsigma\,\{1\} = \dfrac{t}{(1-t)^2}$

4. For $1^2, 2^2, 3^2, \ldots$

   i.e $f_n = n^2$

   $\varsigma\,\{\mathbf{f_n}\} = \varsigma\{n^2\}$

   From Theorem 3 and example 1 $\varsigma\,\{\mathbf{f_n}\} = (tD)^2\varsigma\{1\} = \dfrac{t(1+t)}{(1-t)^3}$

5. For 1, $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

   From R2 we have $\varsigma\,\{\dfrac{1}{n}\mathbf{f_n}\} = \displaystyle\int \frac{\varsigma\{f_n\} - f_0}{t}\,dt$ in this case $f_n = 1$

   Therefore, $\varsigma\,\{\dfrac{1}{n}\mathbf{f_n}\} = \displaystyle\int \frac{\frac{1}{1-t} - 1}{t}\,dt = \int \frac{1}{1-t}\,dt = ln\frac{1}{1-t}$

6. For -a, a, -a, a, -a, ..., $(-1)^n a$, ... $a \in \mathcal{R}$

   $\varsigma\,\{(-1)^n a\}$, from example 1 and R4, we get

$$\varsigma\,\{(-1)^n a\} = \frac{a}{1+t}$$

7. For the sequence of partial sums of the harmonic series

$$\mathrm{H_n} = \sum_{k=1}^{n} f_k$$

From R6 we have $\varsigma\,\{\sum_{k=0}^{n} f_k\,\} = \dfrac{1}{1-t}\varsigma\,\{\mathbf{f_n}\}$ here $\mathrm{f_n} = \dfrac{1}{n}$

But from example 5, $\varsigma\left\{\dfrac{1}{n}\right\} = \ ln\dfrac{1}{1-t}$

Therefore, $\varsigma\,\{\sum_{k=0}^{n} f_k\,\} = \dfrac{1}{1-t}\varsigma\{\dfrac{1}{n}\}$

$$= \frac{1}{1-t}\,ln\frac{1}{1-t}$$

Hence, $\varsigma\,\{\sum_{k=0}^{n} f_k\,\} = \dfrac{1}{1-t}\,ln\dfrac{1}{1-t}$

8. $\varsigma\,\{\mathbf{nH_n}\} = ?$

From A3 and example 7, we have

$$\varsigma\,\{\mathbf{nH_n}\} = \mathrm{tD}[\varsigma\{\mathbf{H_n}\}]$$

$$= \mathrm{tD}[\frac{1}{1-t}\,ln\frac{1}{1-t}] = \frac{t}{(1-t)^2}(ln\frac{1}{1-t} + 1)$$

9. $\varsigma\,\{\dfrac{1}{n+1}\mathbf{H_n}\} = ?$

From R3 we have $\varsigma\,\{\dfrac{1}{n+1}\mathbf{f_n}\} = \mathbf{t^{-1}}[\ \int\varsigma\{f_n\}dt\ \ + C\,]$

Therefore, $\varsigma\,\{\dfrac{1}{n+1}\mathbf{H_n}\} = \mathbf{t^{-1}}[\ \int\varsigma\{H_n\}dt\ \ + C\,]$

$$= \mathbf{t^{-1}}[\ \int\frac{1}{1-t}ln\frac{1}{1-t}dt\ \ + C\,]$$

**More relations**

1. $\binom{p}{n} = \dfrac{p(p-1)\ldots(P-n+1)}{n!}$

$$\binom{p}{n+1} = \frac{p(p-1)\ldots(p-n+1)(p-n)}{(n+1)!}$$

$$\Rightarrow \binom{p}{n+1} = \frac{p-n}{n+1}\binom{p}{n}$$

If $f_n = \binom{p}{n}$, then

$$f_{n+1} = \binom{p}{n+1} = \frac{p-n}{n+1} f_n$$

$$\Rightarrow (n+1)f_{n+1} = (p-n)f_n = pf_n - nf_n$$

$$\Rightarrow \varsigma\{(n+1)f_{n+1}\} = \varsigma\{pf_n\} - \varsigma\{nf_n\}$$

$$= p\varsigma\{f_n\} - tD\varsigma\{f_n\}$$

$$= pf(t) - t\,f^{'}(t) \ldots\ldots\ldots\ldots\ldots (1)$$

Let $\quad nf_n = g_n$

$$\Rightarrow (n+1)f_{n+1} = g_{n+1}$$

$$\varsigma\{g_{n+1}\} = \frac{1}{t}[\varsigma\{g_n\} - g_0]$$

$$\varsigma\{(n+1)f_{n+1}\} = \frac{1}{t}[\varsigma\{nf_n\}] \qquad g_0 = 0$$

$$= \frac{1}{t}[tD\varsigma\{f_n\}]$$

$$= \frac{1}{t}[tf^{'}(t)]$$

$$= f^{'}(t) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

(1) & (2) gives

$$f^{'}(t) = pf(t) - tf^{'}(t)$$

$$(1+t)f'(t)=pf(t)$$

$$\Rightarrow\ f'(t)=\frac{p}{1+t}f(t)$$

$$\Rightarrow\ \int\frac{df}{f} = \int\frac{p}{1+t}dt$$

$$\Rightarrow\ \ln f = p\ln(1+t)+\ln c$$

$$\Rightarrow\ f(t) = c(1+t)^p$$

since $f(t) = \sum_{n=0}^{\infty} f_n t^n$

$$f(0)=f_0=\binom{p}{0}=1$$

$\Rightarrow$ c = 1

$$\Rightarrow \boxed{\varsigma\left\{\binom{p}{n}\right\} = (1+t)^p \qquad ,p\in\Re}$$

$$\Rightarrow\ [t^n](1+t)^p = \binom{p}{n}$$

2. $\varsigma\left\{\binom{p}{m+n}\right\} = ?$

$$\binom{p}{m+n} = [t^{m+n}](1+t)^p$$

$$=[t^m]\frac{(1+t)^p}{t^m}$$

$$\Rightarrow\ \varsigma\left\{\binom{p}{m+n}\right\} = \frac{(1+t)^p}{t^m}$$

3. $\varsigma\left\{\binom{p+n}{m}\right\}=?$

$$\binom{p+n}{m}=\binom{p+n}{p+n-m}=\binom{p+n-m-p-n-1}{p+n-m}(-1)^{p+n-m}$$

$$=\binom{-m-1}{p+n-m}(-1)^{p+n-m}$$

$$=\left[t^{p+n-m}\right](1-t)^{-m-1}$$

$$=\left[t^{p+n-m}\right]\frac{1}{(1-t)^{m+1}}$$

$$=\left[t^{n}\right]\frac{t^{m}}{t^{p}(1-t)^{m+1}}$$

$$=\left[t^{n}\right]\frac{t^{m-p}}{(1-t)^{m+1}}$$

Therefore, $\varsigma\left\{\binom{p+n}{m}\right\}=\dfrac{t^{m-p}}{(1-t)^{m+1}}$

4. $\varsigma\left\{\binom{p+n}{m+n}\right\}=?$

$$\binom{p+n}{m+n}=\binom{m+n-p-n-1}{m+n}(-1)^{m+n}$$

$$=\binom{m-p-1}{m+n}(-1)^{m+n}$$

$$=\left[t^{m+n}\right](1-t)^{m-p-1}$$

$$=\left[t^{m+n}\right]\frac{1}{(1-t)^{p+1-m}}$$

$$=\left[t^{n}\right]\frac{1}{t^{m}(1-t)^{p+1-m}}$$

Therefore, $\varsigma\left\{\binom{p+n}{m+n}\right\}=\dfrac{1}{t^{m}(1-t)^{p+1-m}}$

5. $\varsigma\left\{n\binom{p}{n}\right\} = ?$

Let $f_n = \binom{p}{n}$

$\varsigma\{nf_n\} = tD\varsigma\{f_n\}$

Therefore, $\varsigma\left\{n\binom{p}{n}\right\} = tD\varsigma\left\{\binom{p}{n}\right\}$

$$= t\left((1+t)^p\right)'$$

$$= tp(1+t)^{p-1}$$

6. $\varsigma\left\{n^2\binom{p}{n}\right\} = ?$

$$= (tD)^2\varsigma\{f_n\}$$

$$= t\left(tp(1+t)^{p-1}\right)'$$

$$= t\left[p(1+t)^{p-1} + tp(p-1)(1+t)^{p-2}\right]$$

$$= tp(1+t)^{p-1}\left[1 + t(p-1)(1+t)^{-1}\right]$$

$$= tp(1+t)^{p-1}\left[1 + \frac{t(p-1)}{1+t}\right]$$

$$= tp(1+t)^{p-2}(1+tp)$$

7. $\varsigma\left\{\frac{1}{n+1}\binom{p}{n}\right\} = ?$

We know that $\varsigma\left\{\frac{1}{n+1}f_n\right\} = \frac{1}{t}\left[\int\varsigma\{f_n\}dt + c\right]$

Therefore, $\varsigma\left\{\frac{1}{n+1}\binom{p}{n}\right\} = \frac{1}{t}\left[\int\varsigma\left\{\binom{p}{n}\right\}dt + c\right]$

$$= \frac{1}{t}\left[\int(1+t)^p\,dt + c\right]$$

$$= \frac{1}{t}\left[\frac{1}{p+1}(1+t)^{p+1} + c\right]$$

8. $\varsigma\left\{\binom{n}{m}\right\} = ?$

$$\binom{n}{m} = \binom{n}{n-m} = \binom{n-m-n-1}{n-m}(-1)^{n-m}$$

$$= \binom{-m-1}{n-m}(-1)^{n-m}$$

$$= \left[t^{n-m}\right](1-t)^{-m-1}$$

$$= \left[t^{n-m}\right]\frac{1}{(1-t)^{m+1}}$$

$$= \left[t^{n}\right]\frac{t^{m}}{(1-t)^{m+1}}$$

Therefore, $\varsigma\left\{\binom{n}{m}\right\} = \dfrac{t^{m}}{(1-t)^{m+1}}$

9. $\varsigma\left\{n\binom{n}{m}\right\} = ?$

$$\varsigma\left\{n\binom{n}{m}\right\} = tD\varsigma\left\{\binom{n}{m}\right\} = t\left(\frac{t^{m}}{(1-t)^{m+1}}\right)$$

$$= t\left[\frac{(t^{m})'(1-t)^{m+1} - t^{m}\left((1-t)^{m+1}\right)'}{(1-t)^{2m+2}}\right]$$

$$= \frac{t^{m}(m+t)}{(1-t)^{m+2}}$$

10. $\varsigma\left\{n^{2}\binom{n}{m}\right\} = (tD)^{2}\varsigma\left\{\binom{n}{m}\right\}$

$$= t\left[\frac{t^{m}(m+t)}{(1-t)^{m+2}}\right]'$$

$$= \frac{t^{m}(m^{2}+2tm+t^{2}+t)}{(1-t)^{m+3}}$$

11. $\varsigma\left\{\frac{1}{n}\binom{n}{m}\right\} = ?$

We know that $\varsigma\left\{\frac{1}{n}f_n\right\} = \int\frac{\varsigma\{f_n\}-f_0}{t}dt$

$$\varsigma\left\{\frac{1}{n}\binom{n}{m}\right\} = \int\frac{\varsigma\left\{\binom{n}{m}\right\}-f_0}{t}dt$$

$$= \int\frac{\frac{t^m}{(1-t)^{m+1}}-f_0}{t}dt$$

$$= \int\frac{t^{m-1}}{(1-t)^{m+1}}dt$$

Solving this we get the following result

$$\varsigma\left\{\frac{1}{n}\binom{n}{m}\right\} = \frac{(1-t)^{-m}t^m}{m}$$

12. $\varsigma\{p^n\} = ?$

We know that $p^{n+1} = pp^n$

Let $f_n = p^n$

$f_{n+1} = p^{n+1} = pp^n = pf_n$

$\varsigma\{f_{n+1}\} = \frac{1}{t}(\varsigma\{f_n\} - f_0)$ but $f_0 = p^0 = 1$

$= \frac{1}{t}(\varsigma\{f_n\} - 1)$ $f(t) = \varsigma\{f_n\}$

$\Rightarrow pf(t) = \frac{1}{t}\big(f(t) - 1\big)$

$\Rightarrow ptf(t) = \big(f(t) - 1\big)$

$\Rightarrow f(t) = \frac{1}{1-pt}$

Therefore, $\varsigma\{p^n\} = \dfrac{1}{1-pt}$

13. $\varsigma\{np^n\} = tD\varsigma\{p^n\} = t\left(\dfrac{1}{1-pt}\right)'$

$$= \frac{pt}{(1-pt)^2}$$

14. $\varsigma\left\{n^2p^n\right\} = (tD)^2\varsigma\{p^n\}$

$$= t\left(\frac{pt}{(1-pt)^2}\right)'$$

$$= \frac{pt(1+pt)}{(1-pt)^3}$$

15. $\varsigma\left\{\dfrac{1}{n}p^n\right\} = \displaystyle\int\frac{\varsigma\{p^n\}-1}{t}dt = \int\frac{\frac{1}{1-pt}-1}{t}dt$

$$= \int\frac{1-1+pt}{t(1-pt)}dt$$

$$= \int\frac{p}{(1-pt)}dt$$

let $\quad u = 1 - pt$

$du = -pdt$

$$\Rightarrow \varsigma\left\{\frac{1}{n}p^n\right\} = = ln\frac{1}{1-pt}$$

Therefore, $\varsigma\left\{\dfrac{1}{n}p^n\right\} = ln\dfrac{1}{1-pt}$

16. $\varsigma\left\{\displaystyle\sum_{k=0}^{n}p^k\right\} = ?$

We know that $\quad \varsigma\left\{\displaystyle\sum_{k=0}^{n}f_k\right\} = \dfrac{1}{1-t}\varsigma\{f_n\}$

$$\varsigma\left\{\sum_{k=0}^{n}p^k\right\} = \frac{1}{1-t}\varsigma\{p^n\}$$

$$= \frac{1}{(1-t)}\frac{1}{(1-pt)}$$

$$= \frac{1}{(1-t)(1-pt)}$$

17. $\varsigma\{f_n p^n\}$, Where $f_n = [t^n]f(t)$

$$\varsigma\{f_n p^n\} = \sum_{n=0}^{\infty} f_n p^n t^n$$

$$= \sum_{n=0}^{\infty} f_n (pt)^n$$

$$= f(pt)$$

18. $\varsigma\left\{\frac{1}{n!}\right\} = ?$

We know that $\frac{1}{(n+1)!} = \frac{1}{(n+1)}\frac{1}{n!}$

Let $f_n = \frac{1}{n!}$

$$\Rightarrow f_{n+1} = \frac{1}{(n+1)!} = \frac{1}{(n+1)}\frac{1}{n!} = \frac{1}{n+1}f_n$$

$$\Rightarrow (n+1)f_{n+1} = f_n$$

$$\varsigma\{f_n\} = \varsigma\{(n+1)f_{n+1}\}$$

$$= D\varsigma\{f_n\} = f'(t)$$

$$\Rightarrow f(t) = f'(t)$$

$\Rightarrow f(t) = e^t + c$ , take $c = 0$

$$\Rightarrow f(t) = e^t$$

Therefore, $\varsigma\left\{\frac{1}{n!}\right\} = e^t$

19. $\varsigma\left\{\frac{1}{n\,n!}\right\} = \varsigma\left\{\frac{1}{n}\frac{1}{n!}\right\} = \int \frac{\varsigma\{1/n!\} - f_0}{t}\,dt$

$$= \int \frac{e^t - 1}{t}\,dt$$

20. $\varsigma\left\{\dfrac{n}{(n+1)!}\right\} = ?$

$$\frac{n}{(n+1)!} = \frac{n}{(n+1)(n)(n-1)!} = \frac{1}{(n+1)(n-1)!}$$

First let us find $\varsigma\left\{\dfrac{1}{(n-1)!}\right\} =$

$$\varsigma\left\{\frac{1}{(n-1)!}\right\} = \varsigma\left\{\frac{n}{n!}\right\} = \varsigma\left\{n.\frac{1}{n!}\right\} = tD\varsigma\left\{\frac{1}{n!}\right\} = t(e^t)'$$

$$= te^t$$

Therefore, $\varsigma\left\{\dfrac{n}{(n+1)!}\right\} = \varsigma\left\{\dfrac{1}{(n+1)(n-1)!}\right\} = \dfrac{1}{t}\left[\displaystyle\int \varsigma\left\{\dfrac{1}{(n-1)!}\right\}dt + c\right]$

$$= \frac{1}{t}\left[\int t\,e^t dt + c\right]$$

$$= \frac{1}{t}\left[e^t(t-1) + c\right]$$

21. $\varsigma\left\{\displaystyle\sum_{k=0}^{n} \dfrac{k}{(k+1)!}\right\} = \dfrac{1}{1-t}\varsigma\left\{\dfrac{k}{(k+1)!}\right\}$

$$= \frac{1}{1-t}\frac{1}{t}\left[e^t(t-1)+1\right]$$

22. $\varsigma\left\{\dbinom{2n}{n}\right\} = ?$

$$\binom{2n}{n} = \frac{2(2n+1)}{n+1}\binom{2n}{n}$$

Let $f_n = \dbinom{2n}{n}$

$$\Rightarrow f_{n+1} = \frac{2(2n+1)}{n+1} f_n$$

$$\Rightarrow (n+1)f_{n+1} = 2(2n+1)f_n$$

$$\Rightarrow nf_{n+1} + f_{n+1} = 4nf_n + 2f_n$$

Then by using rules and relations derived before, we get

$$\Rightarrow f'(t) = \frac{2}{1-4t} f(t)$$

solving this differential equation gives us

$$f(t) = \frac{1}{\sqrt{1-4t}}$$

## Examples

## Solving recurrence relations

### i) Linear second order recurrence relation

$f_{n+1} = \alpha f_{n+1} + \beta f_n$ $\quad$ f₀, f₁ given initial condition

$\alpha, \beta \quad = \text{Constants}$

$$f_{n+2} = \alpha f_{n+1} + \beta f_n$$

$$\varsigma\{f_{n+2}\} = \varsigma\{\alpha f_{n+1}\} + \varsigma\{\beta f_n\}$$

$$\frac{\varsigma\{f_n\} - f_0 - f_1 t}{t^2} = \alpha\varsigma\{f_{n+1}\} + \beta\varsigma\{f_n\}$$

$$\frac{f(t) - f_0 - f_1 t}{t^2} = \alpha\left(\frac{f(t) - f_0}{t}\right) + \beta f(t)$$

$$f(t) - f_0 - f_1 t = \alpha t(f(t) - f_0) + \beta\varsigma\{f_n\}$$

$$\Rightarrow \boxed{f(t) = \frac{(1 - \alpha t)f_0 + f_1 t}{1 - \alpha t - \beta t^2}}$$

This is a generating function for linear second order recurrence relations.

For example

Fibonacci sequence

Example: Find the generating function of the following recurrence relation.

$$(n+2)f_{n+2} = 2(n+1)f_{n+1} + 3nf_n, \qquad f_0=0, f_1=1$$

Solution:

$\varsigma\{(n+2)f_{n+2}\} = 2\varsigma\{(n+1)f_{n+1}\} + 3\varsigma\{nf_n\}$

Now let us first find $\varsigma\{(n+2)f_{n+2}\}$ and $\varsigma\{(n+1)f_{n+1}\}$

$$\begin{aligned}
\varsigma\{(n+2)f_{n+2}\} &= \varsigma\{nf_{n+2} + 2f_{n+2}\} \\
&= \varsigma\{nf_{n+2}\} + 2\varsigma\{f_{n+2}\} \\
&= tD\varsigma\{f_{n+2}\} + 2\varsigma\{f_{n+2}\} \\
&= t\left(\frac{f(t) - f_0 - f_1 t}{t^2}\right)' + 2\left(\frac{f(t) - f_0 - f_1 t}{t^2}\right) \\
&= t\left(\frac{f(t) - t}{t^2}\right)' + 2\left(\frac{f(t) - t}{t^2}\right) \\
&= \frac{1}{t^2}\left(f'(t)t + t - 2f(t)\right) + \frac{2}{t^2}\left(f(t) - t\right) \\
&= \frac{1}{t^2}\left(f'(t)t + t - 2f(t) + 2f(t) - 2t\right) \\
&= \frac{1}{t^2}\left(tf'(t) - t\right) \\
&= \frac{1}{t}\left(f'(t) - 1\right)
\end{aligned}$$

and $$\begin{aligned}
\varsigma\{(n+1)f_{n+1}\} &= \varsigma\{nf_{n+1} + f_{n+1}\} \\
&= \varsigma\{nf_{n+1}\} + \varsigma\{f_{n+1}\} \\
&= tD\varsigma\{f_{n+1}\} + \varsigma\{f_{n+1}\} \\
&= t\left(\frac{f(t) - f_0}{t}\right)' + \frac{f(t) - f_0}{t} \\
&= t\left(\frac{f(t)}{t}\right)' + \frac{f(t)}{t} \\
&= \frac{f'(t).t - f(t)}{t} + \frac{f(t)}{t}
\end{aligned}$$

$$=\frac{1}{t}\left[f'(t)t - f(t) + f(t)\right]$$

$$= f'(t)$$

Therefore, $\varsigma\{(n+2)f_{n+2}\} = 2\varsigma\{(n+1)f_{n+1}\} + 3\varsigma\{nf_n\}$

$$\Rightarrow \frac{1}{t}\left(f'(t) - 1\right) = 2f'(t) + 3t.f'(t)$$

$$\Rightarrow \left(f'(t) - 1\right) = 2t.f'(t) + 3t^2.f'(t)$$

$$\Rightarrow (-3t^2 - 2t + 1)f'(t) = 1$$

$$\Rightarrow (1+t)(1-3t)f'(t) = 1$$

$$\Rightarrow f'(t) = \frac{1}{(1+t)(1-3t)}$$

$$\Rightarrow f(t) = \int \frac{1}{(1+t)(1-3t)}dt$$

But $\frac{1}{(1+t)(1-3t)} = \frac{A}{1+t} + \frac{B}{1-3t} = \frac{A - 3At + B + Bt}{(1+t)(1-3t)} = \frac{(B-3A)t + A + B}{(1+t)(1-3t)}$

$$\Rightarrow A = \tfrac{1}{4} \qquad , B = \tfrac{3}{4}$$

$$\Rightarrow f(t) = \frac{1}{4}\int \frac{1}{1+t}dt + \frac{3}{4}\int \frac{1}{1-3t}dt$$

$$= \frac{1}{4}\mathbf{ln}(1+t) - \frac{1}{4}\mathbf{ln}(1-3t)$$

$$= \mathbf{ln}\left(\frac{1+t}{1-3t}\right)^{\frac{1}{4}} + c$$

But $f(0) = f_0 = 0 = \ln 1 + c$

Therefore, $f(t) = \mathbf{ln}\left(\frac{1+t}{1-3t}\right)^{\frac{1}{4}}$

Example: Find the generating function of the following recurrence relation

$$f_n = g_n + \frac{2}{n}\sum_{k=0}^{n-1} f_k\,, \qquad \text{where } \varsigma\{g_n\} = g(t)$$

Solution:

$$\varsigma\{f_n\} = \varsigma\{g_n\} + \varsigma\left\{\frac{2}{n}\sum_{k=0}^{n-1} f_k\right\}$$

$$\Rightarrow\ f(t) = g(t) + 2\varsigma\left\{\frac{1}{n}\sum_{k=0}^{n-1} f_k\right\}$$

$$= g(t) + 2\int \frac{\varsigma\left\{\sum_{k=0}^{n-1} f_k\right\}}{t}\,dt$$

Now first let us find $\varsigma\left\{\sum_{k=0}^{n-1} f_k\right\}$

We know that $\varsigma\left\{\sum_{k=0}^{n} f_k\right\} = \frac{1}{1-t}\varsigma\{f_n\}$

Let $S_{n-1} = \sum_{k=0}^{n-1} f_k$

$$\Rightarrow S_n = \sum_{k=0}^{n} f_k = f_n + S_{n-1}$$

$$\Rightarrow f_n = S_n - S_{n-1}$$

$$\Rightarrow \varsigma\{f_n\} = \varsigma\{S_n\} - \varsigma\{S_{n-1}\}$$

$$\Rightarrow \varsigma\{S_{n-1}\} = \frac{1}{1-t}\varsigma\{f_n\} - \varsigma\{f_n\}$$

$$= \left(\frac{1}{1-t} - 1\right)\varsigma\{f_n\}$$

$$= \left(\frac{t}{1-t}\right)\varsigma\{f_n\}$$

Therefore, $\varsigma\left\{\sum_{k=0}^{n-1} f_k\right\} = \left(\frac{t}{1-t}\right)\varsigma\{f_n\}$

Hence, $f(t) = g(t) + 2\int \frac{\varsigma\left\{\sum_{k=0}^{n-1} f_k\right\}}{t}\,dt$

$$= g(t) + 2\int \frac{t/_{1-t}}{t} f(t)dt$$

$$= g(t) + 2\int \frac{1}{1-t} f(t)dt$$

$$\Rightarrow f'(t) = g'(t) + \frac{2}{1-t} f(t)$$

$$\Rightarrow f'(t) - \frac{2}{1-t} f(t) = g'(t)$$

here the integral factor of this differential equation is $e^{\int \frac{-2}{1-t}dt}$

but $\int \frac{-2}{1-t}dt = -2\int \frac{1}{1-t}dt$

$$= \mathbf{ln}(1-t)^2$$

Therefore, $e^{\int \frac{-2}{1-t}dt} = e^{\mathbf{ln}(1-t)^2} = (1-t)^2$

Therefore $\frac{d}{dt}\left[(1-t)^2 f\right] = (1-t)^2 \frac{df}{dt} - 2(1-t)f$

$$= (1-t)^2 \left[\frac{df}{dt} - \frac{2}{1-t} f\right]$$

Therefore, $\frac{d}{dt}\left[(1-t)^2 f\right] = g'(t)(1-t)^2$

$$\Rightarrow (1-t)^2 f = \int g'(t)(1-t)^2 dt + c$$

$$\Rightarrow f = \frac{1}{(1-t)^2}\left[\int g'(t)(1-t)^2 dt + c\right]$$

For example if $g_n = \frac{1}{n}$

$$g(t) = \varsigma\{g_n\} = \ln \frac{1}{1-t}$$

$$g'(t) = \frac{1}{\frac{1}{1-t}}\left(\frac{1}{1-t}\right)' = \frac{1-t}{1}\frac{1}{(1-t)^2} = \frac{1}{(1-t)}$$

Therefore, $f = \frac{1}{(1-t)^2}\left[\int \frac{1}{(1-t)}(1-t)^2\,dt \;+ c\right]$

$$f = \frac{1}{(1-t)^2}\left[\int (1-t)dt \;+ c\right]$$

$$f = \frac{1}{(1-t)^2}\left[t - \frac{t^2}{2} + C\right]$$

**Reference:**

Herebert S. Wilf. generatingfunctionology, Dept. of Math., University of Pensilinia (1990)