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MIXED INTEGER PROGRAMMING APPROACH FOR INVENTORY DISTRIBUTION SYSTEM

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Acronyms

VRP	**V**ehicle **R**outing **P**roblem
VMI	**V**ehicle **M**anaged **I**nventory
IRP	**I**nventory **R**outing **P**roblem
TSP	**T**ravelling **S**alesman **P**roblem
OR	**O**peration **R**esearch
B & B	**B**ranch and **B**ound
MIP	**M**ixed **I**nteger **P**rogramming
MILP	**M**ixed **I**nteger **L**inear **P**rogramming
IP	**I**nteger **P**rogramming
LP	**L**inear **P**rogramming
GFC	**G**omory **F**ractional **C**ut
JITP	**J**oint **I**nventory **T**ransportation **P**roblems

Abstract

Optimization is a mathematical problem that has many real-world applications. It is used to determine minimizers or maximizers of a multi-variable real function, under a restricted domain. The thesis presented here aims in determining an optimal joint inventory with transportation proposed by [10]. These problems are characterized by the presence of both transportation and inventory considerations, either as parameters or constraints. A supply chain class which helps to determine joint transport and inventory cost does have a wide variety of advantages for both company and customer. One of the optimization parts which is mixed integer programming is applied to find the optimal solution to Joint Transportation-and-Inventory Problems (JTIPs) which portioning of customers as well as the route and date of delivery. Two mixed-integer programming models will be discussed: time-discretized integer programming and the new approach with predefined quantities of delivery with the date of delivery. As is shown in this thesis, the approach allows us to use a simplex method and the technique for solving mixed-integer programming i.e., branch and bound Method, and cutting plane method. Solution procedures are clearly illustrated based on a hypothetical applications and modification of the model are present in this thesis.

Keywords: Inventory Routing Problem, Mixed Integer Programming Problem, Branch and Bound Method, Cutting Plane Method.

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Chapter 1

Introduction

Inventory is the term used to refer to a resource or usable items held or stored for future sale or use. Inventories have tangible and intangible costs.

- Item costs, carrying costs, ordering costs, and buying costs are all inventory costs that are called tangible costs.
- Inventories that cover manufacturing problems are intangible inventory costs; manufacturing problems cause more inventories and deteriorate the manufacturing system.

In the balance sheet of a company, inventory is listed as a current asset, and it acts as a bridge between output and order fulfillment. When an inventory item is sold, it moves the carrying expense to the cost of goods sold segment on the revenue statement. The management board of the company had to identify criteria for future deliveries. The inventory routing problem (IRP) is an interconnected problem of inventory and transport planning that collectively specifies the replenishment schedules for a given group of retailers and the routing decisions for a seller that wants to deliver a product to retailers over a limited planning horizon that usually consists of several periods.

The thesis presented here is about, where products are shipped directly to the customer from a processing facility and at the expense of the customer and a new solution suggests that a customer receives products from one of the new warehouses. However, to predict the parameters of such a distribution system, an effective implementation policy needs to be formulated.

This study is anextension of the work by [10]. In his article, he suggested that the joint inventory and transportation problem (JITP) optimization mechanism consists of using two iterative processes solved using the column generation method and system formulations presented. In this study, the optimization part applied to solve the model is the branch and bound ($B\&B$) method and cutting plane method. In the first stage of the first iteration, only the routes to one customer are considered. In the second step, the calculation in the first step, quantities of deliveries which are obtained in the first stage, are incorporated or integrated into the latter formulation and the problem is solved.

One year later, in 2012, Hanczar [11] published a second paper on the same topic using two optimization algorithms for solving the traveling sales-person problem the optimal routes for each partition are determined on fuel distribution. In the second paper, five different customers are added on the route of one customer to reduce the transportation cost. This thesis depends on the first paper and it is solved as mentioned above by using the branch and bound method and cutting plane method to avoid the complexity modification of the model is also done. Here, also we bring the model into the ground by solving for small variables. By using some techniques to solve mixed-integer programming problems the problem is solved and explained briefly.

The primary objective of this study attempted to find an optimal solution (minimum cost) and mathematical approach to the design of an inventory distribution system with less than truck-level deliveries to reduce the overall cost of distribution and inventory, assuming that none of the customer reports are unresolved. Depending on the solution obtained delivering the items on the selected route and at the period obtained for each customer. A comprehensive approach to allocating goods (delivery amounts and times, as well as the delivery routes for each vehicle) for the first model is a solution to a joint transport and inventory problem of [10]. Such a solution is known as a distribution and stock replenishment policy.

This thesis has sections devoted to background material to facilitate discussions and formulations, goals, literature review, conceptual approach, formulations, techniques, and a summary with conclusion.

1.1 Mixed Integer Linear Programming

Optimization is an activity to make the best use of a problem or resource effectively. To optimize means, in some cases, minimizing or maximizing any function relative to some set. Linear programming is simple optimization which is used to determine the best possible outcome or solution from a given set of a parameter in the form of linear. Linear programs have a linear objective function and linear constraints, which may include both equalities and inequalities [7]. There are two types of variables in mathematical models, continuous and discrete. Continuous variables can take any real number value, while discrete variables can take only integer values. There is a variable that can take only (0 or 1) these variables are also known as binary variables. Recall that the Linear Programming (LP) problem is written in standard form as:

$$LP: \quad \max\ c^T x \tag{1.1}$$

$$s.t\ Ax \leq b \tag{1.2}$$

$$x \geq 0. \tag{1.3}$$

where $c^T = (c_1, c_2, \ldots, c_n)$ represents coefficients in the objective function (1.1). The column vector $b = (b_1, b_2, \ldots, b_n)^T$ demonstrates right-hand side coefficients of constraints and A is a $m \times n$ matrix that contains coefficients of the unknown in constraints (1.2) and $x = (x_1, x_2, \ldots, x_n)^T$ symbolizes unknown variables which can take continuous or discrete values (1.3). The feasible set of the LP is a polyhedral $S = \{(x \epsilon R^n) : Ax \leq b, x \geq 0\}$. An optimal solution occurs at an extreme point of the polyhedral which corresponds to a basic feasible solution of the constraint $Ax \leq b$. We can use the simplex method to find an optimal solution of the LP [14].

We shall now consider, mixed-integer linear programming which has a linear objective function where some of the variables have integer values, but the rest are continuous variables. Mixed integer linear programming is constructed on the base of Dantzigs Simplex Method.

$$\begin{aligned} MILP: \quad \max\ & c^T x + h^T y \\ s.t\ & Ax + Gy \leq b \\ & x \geq 0 \\ & y \geq 0\,\&\ \text{integer} \end{aligned}$$

Where c, A and b are as in above of LP additionally, $h^T = (h_1, h_2, \ldots, h_n)$ are the coefficients in the objective function of MILP and G is $m \times n$ matrix of unknown variables and y is a variable which can take only integer value. If needed, converting a maximization problem to a minimization problem is quite simple. Given that z is an objective function for a maximization problem $max(z) = -min(-z)$. Recall also from the above Linear Programming (LP) problem:

$$\begin{aligned} LP: \quad & \max\, c^T x \\ & s.t\; Ax \leq b \\ & \quad x \geq 0. \end{aligned}$$

Where, A is $m \times n$ matrix of rank m, $c\epsilon R^n$ and $b\epsilon R^m$. Suppose $c^T x = c_B^T x_B + c_N^T x_N = z$ or $c^T x = c_B^T x_B + c_N^T x_N = z = 0$,where x_B and x_N are the vectors of basic and non-basic variables, respectively. Accordingly, partition A into $m \times m$ basic sub-matrix B and $m \times (n-m)$ non-basic sub-matrix N so that, A = (B, N), $c = (c_B, c_N)$ and $x = (x_B, x_N)$ [14]. Then, by writing this in the simplex tableau (where the upper row, called 0^{th} row, is for the objective function) and applying the usual pivot operations we get the following simplex tableau:

	x_B	x_N	
-z	0	$c_N - c_B B^{-1} N$	$-z_B$
x_B	1	$B^{-1}N$	$B^{-1}b$

Table 1.1: Simplex Tableau

Optimality: If $c_N - c_B B^{-1} N \leq 0$ then the current basic feasible solution $(x_B, x_N) = (B^{-1}b, 0)$ is optimal solution of LP with maximum objective values $\bar{z} = z_B$ [14].

1.1.1 Basic Definitions

Definition 1. *A solution that satisfies all constraints is called a feasible solution. If it is not feasible we call it infeasible.*

Definition 2. *Let $Z(t)$ be an objective function and S be a non empty subset of R^+ be the feasible set then $m\epsilon S$ is called optimal solution for minimization problem i.e.,* $\min Z(t)$ *s.t $t\epsilon S$, if $Z(m) \leq Z(t)\ \forall t\epsilon S$ then it is called global minimizer of constraints.*

Definition 3. *If $Z(m) \leq Z(t)$ then, there is $\varepsilon > 0$ such that $\forall t\, \epsilon N_\varepsilon(m) \cap S$, $t\epsilon S$ then it is called local minimizer of constraints.*

Definition 4. *If $m \epsilon S$ is an optimal solution then $Z(m)$ is an optimal(objective) value.*

Definition 5. *The incumbent solution is a known feasible solution with the best objective value at any stage.*

1.1.2 The Branch and Bound Method

The branch-and-bound (B&B) method first appeared in the literature for discrete programming in 1960 by contributions of Land and Doig [12] asserts that most MILP problems are solved by using the (B&B) method. Branch and bound is the method of divide and conquer technique [13]. The basic idea is to partition the set of feasible solutions S obtained using the simplex method for the linear programming relaxation (LPR) of the problem into disjoint smaller subsets $S_1, S_{2,} \ldots S_k$ such that $S = S_1 \cup S_2 \cup \ldots \cup S_k$ and find the maxima of the objective function on each of the smaller subsets. If Z_i is the maximum value of the objective function over S_i for each $i = 1, 2, \ldots, k$, then the optimal objective value (maximum of the given problem) is $z^* = max\{Z_1, Z_2, \ldots, Z_k\}$.

In this method also, the (LPR) of the problem is first solved using the simplex method. If the resulting solution has an integer value for all the y_j then it's the desired optimal solution. If the resulting solution is not an integer then, the optimal (maximum) value $\bar{z}$ of the LPR is an upper bound to the optimal (maximum) value z^* of the MILP i.e, $\bar{z} \geq z^*$ and on the other hand, if $\bar{y} \epsilon S$ is a feasible an integer solution then obviously $\bar{z} = c\bar{y}$. If it is not an integer, the feasible region is divided into two smaller parts: suppose y_j is a non-integer basic variable in the optimal solution of the LPR. That is, $y_j = b_j \notin Z$.

Then, there are only two possibilities for the integer value of y_j: either $y_j \leq \lfloor b_j \rfloor$ or $y_j \geq \lfloor b_j \rfloor + 1$.

These two conditions are mutually exclusive and when they are included separately into the LPR of the problem, two different subproblems (also called nodes) are formed with the upper bound U which is the optimal value of LPR. In particular, the feasible set S (as well as S_R) is divided into two. This can be shown by the following tree diagram. Such division is called branching; and y_j is called the branching variable.

For example, the left and right nodes that branched from the root node

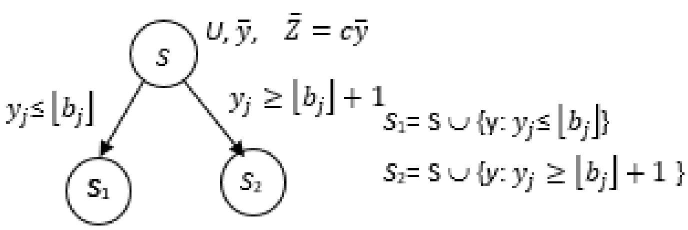

Figure 1.1: Branching Rule

in the above tree correspond to the subproblems P_1 and P_2, where s_1 and s_2 are included in the MILP respectively. Initially, let say L contains the original problem. Then, the original problem is taken out of the list, solved by simplex method, and then divided into two subproblems P_1 and P_2 so that L $= \{P_1, P_2\}$.

For example,

$$\begin{aligned} P_1 :\ \max c^T(x) + h^T(y) \\ s.t\ Ax + Gy \leq b \\ y_j \leq \lfloor b_j \rfloor \\ x \geq 0 \\ y \geq 0 \,\&\, \text{integer} \end{aligned} \qquad \begin{aligned} P_2 : \max c^T(x) + h^T(y) \\ s.t\ Ax + Gy \leq b \\ y_j \geq \lfloor b_j \rfloor + 1 \\ x \geq 0 \\ y \geq 0 \,\&\, \text{integer} \end{aligned}$$

We can now take P_1 to solve it using a simplex method and solve for P_2 separately if the solution obtained for y_j is aninteger we are done otherwise the process continues. This way we always make progress and only branch close to the fractional optimum, which is where the integer optimum is likely to be (if exists).

Example 1.

$$\begin{aligned} \max\ x_1 + x_2 \\ s.t \quad x_1 + 2x_2 \leq 4 \\ 3x_1 + 2x_2 \leq 5 \\ x_1, x_2 \geq 0 \\ x_2 \textit{ is integer.} \end{aligned}$$

Let S be the feasible set of the MILP where, $S = \{x = (x_1, x_2) : x_1+2x_2 \leq 4, 3x_1 + 2x_2 \leq 5, \;\; x_1, x_2 \geq 0 \; and \; x_2 \; is \; integer\}$. Initially, L = $\{MILP\}$, the given problem itself. Since the problem has only two variables, we can use a graphical method to solve the LPR of the problem for more variables we can use a simplex method.

Node 1: We obtained the optimal solution x =(0.5, 1.75) and optimal value z = 2.25 using graphical method. However, x_2 is not integer so, we should branch into two subproblems $P_1 \, and \, P_2$ with feasible sets $S_1 = S \cup \{(x_1, x_2) : x_2 \leq 1\} \, and \, S_2 = S \cup \{(x_1, x_2) : x_2 \geq 2\}$ respectively list L contain the two subproblems i.e, L = $\{P_1, P_2\}$.

Node 2: Now take P_1 from L and solve the subproblem P_1 (on the polyhedral S_1) using graphical method. This is the next active node, the optimal solution are x = (1, 1) and the optimal value is z = 2. L = $\{P_2\}$ and update the incumbent (i.e) $\bar{z} = z$ since x_2 is integer.

Node 3: The other active node from node 1 is node 3 ,Take P_2 from L and solve subproblem using graphical method, so L =$\{\}$. We got x = (2, 0.33) and z = 2.33 but, it is pruned by infeasibility. Therefore, we got the optimal solution from node 2.

The optimal solution for the given MILP are x=(1, 1) with optimal objective value is z = 2.

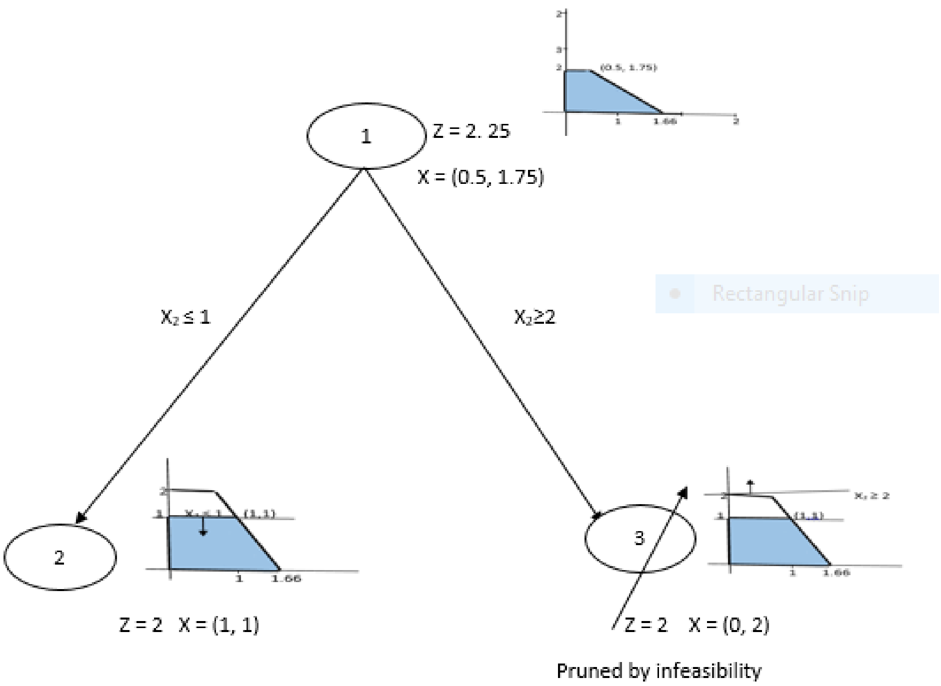

Figure 1.2: Branch and Bound Method

1.1.3 The Cutting Plane Method

The technique of cutting a plane is an alternative solution to the branch-and-bound method to solve MILP problems. The method of cutting plane was first developed in the late 1950s to solve MILP problems by Ralph E.Gomory [9] in the late 1950s. Cut means narrowing this plane with additional constraints that cut and reduce the size of the plane to draw near and detect the integer solution [14]. So for a typical MILP problem, any cut gets closer to the optimal solution. If the optimal solution is fractional, we will use rounding to obtain a cut (as explained below) in the example.

This cut would cut off the fractional solution and hence reduce the size of the feasible region. If the value in the final (optimal) simplex table of each specific feasible variable y is an integer, then the MIP problem was solved. Otherwise, the method selects a row of the final simplex tableau that contains a basic variable whose value is not integer as explained in the example below. This row is called the source row. The desired cut is then built in the source row from the fractional components of the numbers. For this function, it is known as the fractional cut.

Theorem 1. ***Gomory Fractional cut(GFC):*** *Let $\bar{y}$ be the optimal solution of LPR of MIP and let $\bar{y} \notin Z^n$. Suppose it is i^{th} component $b_i \notin Z$ and i^{th} constraints of the LPR at optimality is $y_{B_i} + \sum_{j \epsilon NB} a_{ij} y_j = b_i$. Then, $\sum_{j \epsilon NB} f_{ij} y_j \geq f_i$ where $f_{ij} = a_{ij} - \lfloor a_{ij} \rfloor$ and $f_i = b_i - \lfloor b_i \rfloor$ is a valid inequality and $\bar{y}$ from S.*

Briefly, follows the following for cutting plane method:

1. Solve for LP.
2. If the associated optimal basic solution y belongs to S is an integer,stop.
3. Otherwise using GFC solve the separation problem find a cutting plane $\alpha y_i + \gamma y_j \leq \beta$ separating $(y_i,\, y_j)$ from S,and repeat the recursive step.

Example 2.

$$\begin{aligned} \max \quad & x_1 + x_2 \\ s.t \quad x_1 + 2x_2 &\leq 4 \\ 3x_1 + 2x_2 &\leq 5 \\ x_1, x_2 &\geq 0 \\ x_2 \textit{ is integer.} \end{aligned}$$

	x_1	x_2	x_3	x_4	
-z	0	0	$\frac{-1}{4}$	$\frac{-1}{4}$	$\frac{-27}{12}$
x_2	0	1	$\frac{3}{4}$	$\frac{-1}{4}$	$\frac{7}{4}$
x_1	1	0	$\frac{-1}{2}$	$\frac{1}{2}$	$\frac{3}{6}$

Table 1.2: Simplex Tableau for Cutting Plane Method

Step 1: By using a simplex method the optimal solution and optimal value obtained are x = (0.5, 1.75) and z = 2.25 respectively. The needed variable x_2 is not integer so, using GFC new constraint is added to feasible set $S = \{x = (x_1,\, x_2) : x_1 + 2x_2 \leq 4,\ 3x_1 + 2x_2 \leq 5,\ \frac{-3}{4}x_3 + \frac{1}{4}x_4 \leq \frac{-3}{4},\ \ x_1, x_2 \geq 0 \textit{ and } x_2 \textit{ is integer}\}$ see table 1.3.

Step 2: Using the newly added constraint we solve for the MILP then, the value obtained for optimal solution and optimal value are x =(1, 1) and z = 2 respectively with x_2 has aninteger. The new incumbent solution found is that x = (1, 1) and z = 2 are the optimal solution and optimal values respectively for this step as shown in table 1.4.

	x_1	x_2	x_3	x_4	x_5	
-z	0	0	$\frac{-1}{4}$	$\frac{-1}{4}$	0	$\frac{-27}{12}$
x_2	0	1	$\frac{3}{4}$	$\frac{-1}{4}$	0	$\frac{7}{4}$
x_1	1	0	$\frac{-1}{2}$	$\frac{1}{2}$	0	$\frac{3}{6}$
x_5	0	0	$\frac{-3}{4}*$	$\frac{1}{4}$	1	$\frac{-3}{4}$

Table 1.3: Simplex Tableau for Cutting Plane Method for Step 1

	x_1	x_2	x_3	x_4	x_5	
-z	0	0	0	$\frac{-1}{3}$	$\frac{-1}{3}$	-2
x_2	0	1	0	0	1	1
x_1	1	0	0	$\frac{1}{3}$	$\frac{-1}{6}$	1
x_3	0	0	1	$\frac{-1}{3}$	$\frac{-4}{3}$	1

Table 1.4: Simplex Tableau for Cutting Plane Method for Step 2

Re-solving this tableau using the dual simplex method (with the new row as the first pivot row) we obtain the following optimal tableau. Therefore, the optimal solution of the given MILP is x = (1, 1) with the maximum objective value z = 2.

1.2 Background

Many problems of quantitative analysis of delivery routes have been solved by operations research, and dates are known as inventory routing problems (IRPs). This is applicable where, both the transportation and inventory cost for each customer with the delivery amount is taken into account. Inventory routing is important to firms handling the stock of their clients. The retailer takes replenishment decisions for goods based on clear inventory and supply chain policy in a vendor managed inventory (VMI) environment.

The most widely used parameters for classifying IRP problems include: the duration of the planning horizon, the integration/separation of storage costs, the demand of a customer (variable or constant in time, algorithmically or shared a similar expressed), and the form of solution looked for. The type of IRP issues in which both the cost of transportation and the cost of inventory is taken into account is known as combined or joint transport-and-inventory problems. The paper by Beltrami and Bodin [3] can be considered a seminal work on IRP problems. In the following studies by Fisher et al. and Bell et al [2] mixed-integer programming was used first to obtain a solution for the IRP instance.

The terminology used in this analysis has several common meanings. For this purpose, the following terminology is described to help explain the study:

- Customer: a general term, such as retail stores, distribution centers, and individual families.
- Stock: an investment that represents a share, or partial ownership, of a company.
- Goods: item to be sold.
- Vendor: seller that exchange goods or service for money.
- Demand: customer desire to purchase goods.
- Supply chain: a sequence of systems involved in the manufacture and distribution of goods or commodities.
- Planning horizon: the period of time before a strategic plan is planned when a company can look into the future.

The significance of the method of inventory distribution is to determine both the amount and the appropriate delivery path. The amount and route of travel were calculated within a defined period by the running discrete-time model. One of the standard methods of distribution planning is the vehicle routing problem. In this problem, a seller schedules delivery routes for his vehicle fleet, recognizing the amounts of orders issued by regionally scattered customers. A customer determines the appropriate arrival date after making an order.

A single period is used in each production schedule, and the preparation process is replicated over consecutive periods. The seller does not, of course, openly plan distribution paths. Stated standards of partnership conditions, such as minimum or maximum distribution quantity, minimum or maximum storage, or the maximum number of deliveries to be fulfilled by the seller, are also decided by the partners.

1.3 Problem Statement

Zakaria [16] emphasized the significant key role of (B&B) method for each supplier to satisfy customers demands with high service level requirements to meet the optimal total cost, which will ensure that the products are delivered at the given period, at the right location, in the right quantity.
Based on the time of the transition of ownership of products from the supplier to the buyer, these solutions are categorized as consignment inventory (where it is not an actual distribution that means a change of ownership, but only the selling or use of the products by the buyer in the manufacturing process) and the inventory handled by the vendor (VMI, where the customer claims at the time of delivery). In this case, the question:

- *When is the time to deliver goods?*
- *What amount of goods are to be delivered that it helps to minimize inventory cost.*

A second situation is also possible, where there is no change of ownership because both the vendor and the customer decide the number of goods to deliver. The main problem falls into place in this phase:

- *In which way we can choose the best delivery route so that both managers and company owners are satisfied then which optimal route is going to be taken?.*
- *How to minimize the inventory cost?*

By using OR methods, as indicated in this study, all these issues can be solved.

1.4 Study Goals

IRP has overcome many business issues nowadays. The problem of businesses and the work of suppliers can be strengthened by designing a successful IRP model. As mentioned in the introduction part, Hanczar [10] used column generation to solve the problem in his study. However, in this thesis, we use the branch and bound ($B\&B$) method and cutting plane method. The main goal of this analysis is to turn these mathematical formulations into iterative algorithms and solve the problem with a small number of variables, contributing to the cost of inventory being reduced. Which can help companies or vendors to decide the route, period, and delivery amount to visit their customer within a short period.

Mixed-integer linear programming plays a significant role in reducing costs, but this is seen in advance. Another purpose of the study is to improve the formulation of these existing mathematical models and to illustrate in this thesis how we can solve these models. The algorithm used to solve this issue can therefore be created.

1.5 Thesis Organization

In Chapter 2, literature regarding OR approaches to creating inventory distribution and IRP research parts are mentioned. Chapter 3 illustrates the mathematical model development procedure which includes the conceptual approach, variables, and parameters to create these guidelines with an OR based approach and strengthen the model. The consistently mentioned algorithm is depicted in Chapter 4. Afterward, an application with some variables is elaborately demonstrated. The final chapter infers the work done by the study and summarizes the whole study in Chapter 5.

Chapter 2

LITERATURE REVIEW

In this chapter, an OR point of view for inventory distribution systems, IRP parts and VRP are discussed briefly.

2.1 OR Perspective for Inventory Distribution

Inventory Routing helps a retailer to supply goods to a variety of clients to minimize inventory control, vehicle routing, and timing of distribution concurrently [8]. This is applicable where, subject to different restrictions, a supplier has to supply goods to a variety of clients in a region. In the calculation of delivery of products and periods, the use of OR techniques to help the supplier must decide simultaneously:

- When to service a customer in doubt.
- How much to deliver.
- How to merge car routes with customers.

The large distribution infrastructure for domestic and commercial users with liquid propane is studied by [6] uses the OR technique. The research by [2] also used various OR techniques to solve distribution problems, where the set partitioning formulation is incorporated into a problem of discrete-time inventory planning. It will be able to eradicate these problems in our methodology and allow them to apply the proposed approach effectively to handle the distribution problem.

The availability of high-potential microcomputers and methods of solving are now allowing operational researchers to solve extremely large-scale problems in incredibly short periods. In this research, taking advantage of technical advances and developing a single compact solution approach for the issue of inventory delivery was undertaken as a task.

2.1.1 Vehicle Routing Problem

Depending on selecting the shortest path, as we mention in the introduction part the vehicle routing problem (VRP) is to find the most effective route, given certain constraints, to send either passengers or goods from central sources to certain destinations. It first appeared in a paper by George Dantzig and John Ramser in 1959, in which the first algorithmic approach was written and applied to petrol deliveries [1]. Often, The context is that of supplying customers who have placed an order for such goods with goods located at a central depot is shown by [6].

Vehicle routing problem (VRP) is the way of compromising the design of a set routes with the lowest cost for each vehicle that is traveling from one step to another. The VRP concerns the service of delivery of goods from the company to their customer. How goods are shipped from one or more depots that have a certain number of residential vehicles that are operated by some drivers who can pass to a collection of customers on a given road network [3]. It provides for a series of routes (one route for each vehicle that must start and end at its depot) to be determined in such a manner that both customer needs and operating restrictions are fulfilled and the global cost of transport is minimized. Yun [15] studied the vehicles are located in one or more depots and operated by a set of drivers.

Besides, for any vehicle that goes from one-stop or plant to another or customer, the vehicle routing problem is one that requires the creation of a series of routes that have the lowest possible cost. In this thesis, if a vehicle can supply goods to several customers without the need to replenish the stock of goods transported, much of the fuel delivery in our country use this approach as a non-deterministic problem. Another factor is applied to the space of admissible solutions in the supply preparation for the technique mentioned above, where it is the seller that sets the release deadline: the delivery date.

Moreover, additional requirements, such as those which represent a particular task, need to be taken into account: the seller assumes responsibility for deficits, if any, or for exceeding the maximum quantity of products allowed.

In the supply chain, the major operational cost is the cost of the selected

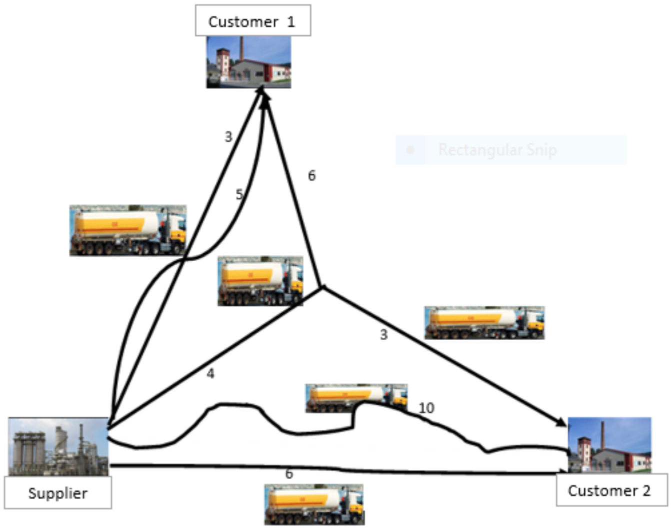

Figure 2.1: Flow Diagram of Single Supplier to two customer

route or transportation cost. Transport period, route costs and capacity vehicle or constraint play a role in decision making. In today's world, the particular life cycle and the many specialties of similar products have made the worldwide economy highly competitive. In the study of [16] to survive on the market, each company must be extremely competitive in terms of product quality, cost, and supply chains. For example, transporting in smaller amounts and at high frequency decreased the amount of inventory at the storage facility, but causes higher transport cost. Figure (2.1) shows an example of VRP with two different customers and it shows the transshipment of goods to different customers under certain restrictions.

Vehicles are situated in one or more terminals and are operated by a variety of drivers. They are going within an acceptable road network. In particular, the approach of the VRP calls for the development of a series of paths, each carried out by a single vehicle that begins and ends at its own depot, in such a manner that all customer needs are fulfilled, that all operating constraints are met and that global transport costs are reduced.

The delivery and inventory activities can operate independently if there is a reasonably large inventory reserve that completely decouples both of them. This would, however, reducing holding costs and longer lead times, since on one side the distribution planer, to minimize transportation cost, would prefer less than truck level and a minimum number of stops; and on the other side, the production planner would prefer less number of machine setups. Other thing we should consider are:

- Distance to travel: it is a common sense that the longer the truck travels, the more energy is consumed.
- Truck load: the more the vehicle carries, the more energy is needed.
- Truck speed: the faster the vehicle runs, the more energy is reduce. Depending on this three things we prefer less than truck level in this study. We focus on reducing inventory and cycle time in the supply chain forced many companies to investigate the possibility of closer coordination between manufacture and supply.

In this thesis, the delivery held in any route should be less than truck level and all the customers cannot be delivered in the same vehicle however if in the same vehicle the total demands of customers visited in one route should not be larger than the vehicle capacity. The presence of time windows can also have an influence on the energy consumption on the route. For example, in order to arrive in time for a customer, the vehicle might choose a fast route with more consumption than a slow one with less consumption.

2.1.2 IRP Research Parts

The Inventory Routing Problem (IRP) has been introduced since the 1980s. The inventory routing problem involves the integration and coordination of two components of the logistic value chain: inventory management and vehicle routing. The source (or supplier) is normally the producer in this model, but often it may be a distributor. He or she serves as a central decision-maker and physically or through electronic sensors and messaging tracks the inventory level of each buyer (customer or retailer). In this research, we divide it into three main parts.

In the first part, by taking the time horizon and inventory problems into account, the vehicle routing method is expanded. This field was initiated in research by [2], where the formulation of the set partitioning is integrated into a separate topic of time inventory planning. To decide the set of clients to be visited, this method is defined as the application of time-discretized integer programming models. In recent studies, this area includes research by [4] and [5], which covers the study of interrelationships between shipping costs and inventory costs for a single job of five consumers. Their other research analyzes the use of mixed-integer programming models that are similarly specified to determine suggested quantities of distribution over a long time horizon. The values thus calculated are then used in the second step to quantify the exact output amounts over a short period.

The second part, where the planning horizon is considered, is reduced by the estimation of the suggested replenishment amounts over a long time horizon and the corresponding assessment of a supply schedule and routes for the next few days, depending on the results received. Bard et al. (1998a, 1998b), performs with an IRP rolling horizon where for two weeks a short-term planning concern is established. According to the rotating horizon strategy, only choices for the first week are included.

The third part, based on their respective demands and other method-specific parameters, considers the division of a customer base into delivery groups. Then, with routes calculated using the classical VRP or TSP algorithms, each distribution is made to all customers in a given group. In these approaches, it has been assumed that each client belongs to one or more distribution classes in these methods. A specific fraction of its demand is allocated to each of the appropriate delivery groups if such a situation arises.

Mixed-integer programming models are used to determine with unsuggested delivery quantities by using simplex method. Subsequently, the values thus calculated are used, as described above, to quantify the exact delivery amounts over a short time horizon. The most promising formulation in this field was proposed by [1].

The company must manage its order involvement and production scheduling activities to ensure a practical distribution path for the distribution amounts listed. Regarding distribution volumes and timing, the vendor needs to make frequent resupply decisions. Briefly, stock control and vehicle routing and dispatching are built into the IRP. In general, there are a variety of suppliers and some customers, and each supplier has an inventory of a collection of items.

In industry, combined routing and inventory management problems are mainly found in road-based and maritime supply chains. This thesis focused on road-based. Nowadays many companies use the second part of IRPS of this thesis to visit their customer when the value of the delivery quantity is known. The route to the trucks is assigned according to Yun [15]. Types of supply chains according to the applications, different structures of supply chains are studied.

- One-to-one: where only one central provider and one particular customer are present. A balance between the cost of shipping and the cost of inventory.
- One-to-many: there are one vendor and a variety of clients. Among the range of clients, the driver takes a tour to distribute goods.

In this thesis, we use one to one-ways where there is one supplier and one customer are served. However, the modification is focused on the one to many where there is only one plant served many customers.

When we consider the planning horizon according to the application of the problem different type of planning horizon studied:

- The problem with a finite horizon is to find the lowest cost stabilization method in the long term. In this type of IRPs, in each time unit, the consumer requirements are typically provided by a rate (fixed or variable).
- The long-term problem is limited to a short period of time with a finite horizon and the acceptable costs are selected to represent long-run inventory costs.

Chapter 3

THE MATHEMATICAL MODEL OF IRP PROBLEM

In this chapter, we present two different mathematical formulations of the inventory routing problem, being mixed-integer linear programs (MILP). The first one, which is called model I, is only basically presented; finding minimum inventory cost with the undefined delivery amount is included for providing input for the second model (model II) which is useful to make decision.

3.1 Conceptual Approach

With its key sketch and simplifications, the mathematical models described in the previous sections are illustrated. Figure (3.1) shows a way to define and work on the model going to be illustrated. That begins with defining the variables, improve the constraints to construct the model, solving for small variables to show how the model decide for the variables and writing overall algorithms for both models.

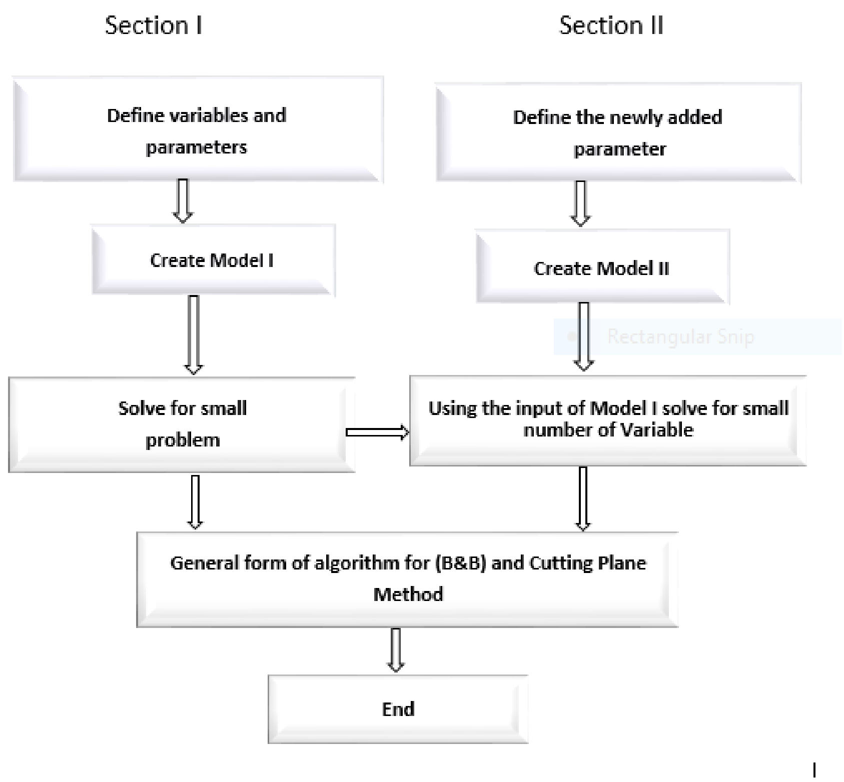

Figure 3.1: Guideline of Inventory Distribution

3.2 Model I

Considering this concept, a model I is borrowed from the article presented in [10] to design the optimal way of determining JTIPs. A set partitioning formulation for solving the vehicle routing problem using a discreet, cyclic supply planning model is the formulation of the inventory and transport model presented below. The nature of the problem and the region in which transport is carried out and the collection of possible routes to be carried out and the amounts to be transported is determined.

The first model of inventory and distribution (the preferred route) model was implemented and established where the partitioning formulation was set for solving vehicle routing. The set partitioning formulation was chosen for the possibility of deciding the price of the routes carried out. It uses the variable of the group of binary x_{jk} and the two groups of variables related to supplies where used y_{ik} and the decision variable z_{ijk} where x_{jk} have the value 1, when a route j is carried out in period k and attaining the value 0 otherwise. The variable y_{ik} is the stock for the customer i in period k and z_{ijk} delivery quantity for the customer i in route j at a period k. The notation (i,j) refers to a customer i in a route j. The main idea is to find a minimum cost for the model. When the optimal inventory and distribution cost is found the route to the customer, amount to be delivered and the period is known. The specificity of the problem and the area where transportation is carried out determines the set of routes possible to carry out. Then, z_{ijk} will be the input for the next model.

3.2.1 Definition of Variables and Parameters

Let n be the set of all customers. Since we are with scheduling a route of the customer to be visited with the amount to be delivered at the known period. Let m be the set of all routes to the customer and p be the set of periods to visit the customers.

Subscripts (indices)

i = 1,2,. . . ,n

j = 1,2,. . . ,m

k = 1,2,. . . ,p

The variables for this problem are defined as:

$$x_{jk} = \begin{cases} 1 & \text{if route } j \text{ is carried out in period } k. \\ 0 & \text{otherwise} \end{cases}$$

y_{ik} is stock of customer i serviced in period k.

The decision variables for this problem are defined as:

z_{ijk} is the amount of goods or items to be delivered to customer i, in route j at the period k.

There are also some parameters required to define an instance of the inventory routing problem. Let a_{ij} be the parameter whose value 1 if the customer i serviced in a route j and 0 otherwise, so it contains the possibility of carrying out the routes. However, while choosing those routes there is the case that we should consider the cost of carrying the route. So, c_j is introduced where it's the cost of carrying out route j but, after we choose the route the customer may want a different amount of item or goods.

Moreover, let b_i be the cost of inventory of goods per unit for the customer i

and which have to be less than the truck level. The parameter L denoting the capacity of the vehicle used in transportation in a selected route of the customer to be serviced. Another important parameter is the demand for each customer, which is denoted by d_i. Depending on the demand, sometimes if there is technical reasoning the parameter M is also used in the formulation, signifying a positive value greater than the sum of all receipt demands in the analyzed horizon. There is also the minimum inventory level g_i to be kept at a customer i.

Notation	Definition
a_{ij}	1, if customer i serviced route j and 0, otherwise.
c_j	cost of carrying route j.
b_i	cost of inventory of goods per unit for the customer i.
d_i	demand of customer i.
g_i	inventory level to be kept for the customer i.
L	capacity of vehicle holding.
M	sum of all receipts of demands for technical reasoning.

Table 3.1: Definition of Parameters of The Model

3.2.2 The Objective Function and constraints

It is not easy thing to describe the objective function of the inventory routing problem like this. The main purpose of this model is to reassign customers to an optimal inventory distribution system so that the total inventory costs are minimized, the normal candidate is the expense of the stock from the first day to the last day. In this case, this is not the only acceptable goal, since not every customer is available at the outset of the planning horizon. Another candidate is to minimize the total transportation cost i.e., the sum of the selected routes when the goods are delivered from the plant to the customer. Another factor to take into consideration is that shortage of goods is not desired and when solving the problem, some possibilities can cause choking. Goods are shipped only on schedule at their due date in an optimal setting.

The problem aims to determine a range of routes to be followed by considering a fleet of heterogeneous vehicles departing from one central depot or warehouse planned to meet the demand of different geographically scattered customer, reducing the overall cost of merchandise transport and the inventory cost of products in storage for the customer. The resulting mathematical model in mixed-integer linear programming is presented which can

solve the total transportation cost, this covers the cost of seeing the buyer, as well as the cost of storing the customer's products.
The objective function is expressed as:

$$\min \sum_{j=1,\ldots,m;k=1,\ldots,p} c_i x_{jk} + \sum_{i=1,\ldots,n;k=1,\ldots,p} b_i y_{ik}$$

The first constraints to ensure that deliveries will be carried out only in periods when a route with a given customer is executed. It can maintain the chosen route.

$$z_{ijk} - a_{ij} x_{jk} M \leq 0 \quad \forall i = 1, \ldots n, j = 1, \ldots, m, k = 1, \ldots, p.$$

We consider the inventory routing problem with less than the truck level, so while having delivery of goods to the executed customer at the given period of all the chosen route will be less than the capacity of vehicle holding. It prevents split deliveries.

$$\sum_{j=1,\ldots,m} z_{ijk} \leq L \quad \forall i = 1, \ldots n, j = 1, \ldots, m, k = 1, \ldots, p.$$

In high-level inventory, the cost of goods available minus cost of ending inventory at the end of the period equal to the cost of goods to be sold or in short the stock of consecutive period yields that:

$$y_{ik} + \sum_j z_{ijk} - d_i - y_{ik+1} = 0 \quad \forall i = 1, \ldots n, \\ j = 1, \ldots, m, k = 1, \ldots, p - 1.$$

The high level inventory policy and the transportation cycle yields that:

$$y_{ik} + \sum_j z_{ijk} - d_i - y_{i1} = 0 \quad \forall i = 1, \ldots n, j = 1, \ldots, m, k = 1, \ldots, p.$$

Depending on the characteristics of the variables defined in the model, binary (0,1) and continuous variables are established. This situation is established through restriction , given by the following expression:

$$x_{ik} \in \{0, 1\}, \quad y_{ik} \geq g_i \text{ and } \quad z_{ijk} \geq 0$$

The following is model I with its variables as presented in.

$$\min \sum_{j=1,\dots,m;k=1,\dots,p} c_j x_{jk} + \sum_{i=1,\dots,n;k=1,\dots,p} b_i y_{ik} \tag{3.1}$$

$$s.t \qquad z_{ijk} - a_{ij} x_{jk} M \leq 0 \tag{3.2}$$

$$\sum_{j=1,\dots,m} z_{ijk} \leq L \tag{3.3}$$

$$y_{ik} + \sum_j z_{ijk} - d_i - y_{ik+1} = 0 \tag{3.4}$$

$$y_{ik} + \sum_j z_{ijk} - d_i - y_{i1} = 0 \tag{3.5}$$

$$x_{ik} \in \{0, 1\} \tag{3.6}$$

$$y_{ik} \geq g_i \tag{3.7}$$

$$z_{ijk} \geq 0 \tag{3.8}$$

Briefly, the objective function (3.1) aims to minimize costs (inventory and transportation) cost. The first sum components are the route cost (transportation costs), while the second one is the inventory costs. The constraint (3.2) guarantees that deliveries will only take place in periods when a route involving a given customer will be carried out. The constraint (3.3) maintains the maximum truck level. The formulas (3.4) and (3.5) express the constraints guaranteeing the continuity of the high inventory policy. Regarding the constraint (3.5), as it is assumed that transportation is cyclic, combines states from the period p with those from period one. The constraints (3.6), (3.7), and (3.8) are setting bounds for the variables and decision variables like x, y, and z respectively.

The decision variable guarantees the optimum distribution quantity in the model presented. However, previous studies show a negative effect on the solution (leads to globally infeasible) due to a significant number of decision variables in this group (customers, routes, and product periods). Therefore, we should use the simplex method to improve this problem shown for the small set of routes for one customer to use this formulation for realistic instances and measured separately for each customer.
The solution to the first formulation suggests an appropriate delivery quantities for the second, while variable analysis of the second formulation is applied to the extension of the routes definition set in the first one. The optimization process starts with solving the former formulation with a restricted set of routes.

3.3 Model II

Observation of actual transport systems, in which one of the main considerations in the measurement of transport is the optimum use of vehicle capacity, indicates a method in which not only the presence of the customer on the road is indicated in the description of the road, but also the quantity of supply. According to this observation, the formulation is proposed where the quantity of deliveries is given.

Model II is developed by using the model proposed in the article presented in [10]. This model is the same as that of model I their difference is here the delivery quantity is represented as a parameter. In model I the delivery quantity is represented as a decision variable. The above definition under model I works for model II. However, there are some additional things:

Parameter

e_{ij} The value determine delivery quantity of customer i in the route j.

In stock out cost delivery quantities defined as parameters of the problem which could reduce the set of possible solutions in a radical manner, the sign of equality was replaced by the sign of minority.

$$y_{ik} + \sum_j e_{ij}x_{jk} - d_i - y_{i1} \geq 0.$$

The second model is presented below:

$$\min \sum_{j=1,\ldots,m;k=1,\ldots,p} c_j x_{jk} + \sum_{i=1,\ldots,n;k=1,\ldots,p} b_i y_{ik} \tag{3.9}$$

$$s.t \qquad y_{ik} + \sum_j e_{ij}x_{jk} - d_i - y_{ik+1} = 0 \tag{3.10}$$

$$y_{ik} + \sum_j e_{ij}x_{jk} - d_i - y_{i1} \geq 0 \tag{3.11}$$

$$x_{ik} \in \{0, 1\} \tag{3.12}$$

$$y_{ik} \geq g_i \tag{3.13}$$

The constraint (3.11) is modified from (3.5) from the previous model. In consideration of the number of deliveries specified as the parameters of the problem which could minimize the set of possible solutions in a radical way, the sign of equality was replaced by the sign of the greater or equal to sign. In this case, the mentioned constraint should make qualified transport policies in which, during the planning horizon, at least deliveries would be made to each customer under the customer's requirements.

Hanczar [10] modify this problem from model I. In order to reduce the number of variables in the model and accelerate the optimization process, a modification of the original model has been suggested. Moreover, the model have been simplified, we can apply the branch and bound method or cutting plane method to get the exact solution. The constraint (3.10) ensures the continuity of the inventory policy.

In the case of linear programming problems containing more than two variables or problems involving a large number of constraints, it is easier to use solutions that are adaptable to computers using the simplex method. As we consider in the introduction part, the simplex method is carried out by performing elementary row operations on a matrix that we call the simplex tableau. General way or steps to solve using the simplex method for any problem of the LP.

Step 1, set the initial simplex tableau. By adding slack and surplus variables as needed to convert to the simplex tableau.

	$x_1 \dots x_i \dots x_m \quad s_1 \dots\dots\dots s_k$	RHS
-z	$c_1 \dots c_i \dots c_m \quad 0 \dots\dots\dots 0$	
s_1	$a_{11}^* \dots a_{1i} \dots a_{1m} \quad 1 \dots\dots\dots 0$	b_1
$\vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots$
s_j	$a_{j1} \dots a_{ji} \dots a_{jm} \quad 0 \dots\dots\dots 0$	b_j
$\vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots$
s_k	$a_{k1} \dots a_{ki} \dots a_{km} \quad 0 \dots\dots\dots 1$	b_m

Table 3.2: Initial Simplex Method

Step 2, depending on the minimum ratio locate the pivot value for example, if it is on the first row first column which is stared $\frac{b_1}{a_{11}}$ and $c > 0$. If it is on the 1^{st} row dividing all the row by a_{11} where it is a positive element in that row and s_1 leaves x_1 enters. All the corresponding columns are zero with $a_{11} = 1$.

Step 3, Pivot to find a new tableau.

Step 4, repeat step 2 and step 3 until optimal solution is found. However, if $(\bar{c_1}, \ldots, \bar{c_k}) \leq 0$ and $(\bar{b_1}, \ldots, \bar{b_m}) \leq 0$ and all RHS is negative we solve it using dual simplex method.

The final two simplex tableau:

	$x_1 \ldots x_i \ldots x_m$	$s_1 \ldots\ldots\ldots s_k$	RHS
-z	$0 \ldots\ldots 0 \ldots 0$	$\bar{c_1} \ldots\ldots \bar{c_k}$	$-z_B$
x_1	$1 \ldots\ldots 0 \ldots 0$	$\bar{a_{11}} \ldots\ldots \bar{a_{1k}}$	$\bar{b_1}$
$\vdots$	$\vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots$
x_i	$0 \ldots\ldots 1 \ldots 0$	$\bar{a_{i1}} \ldots\ldots \bar{a_{ik}}$	$\bar{b_j}$
$\vdots$	$\vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots$
x_m	$0 \ldots\ldots 0 \ldots 1$	$\bar{a_{m1}} \ldots\ldots \bar{a_{mk}}$	$\bar{b_m}$

Table 3.3: Final Simplex table

Therefore we can solve any large number of constraints easily. We follow those steps to solve for the above models. The optimal solution of LPR is $\{x_1, \ldots, x_i, \ldots, x_m\} = \{\bar{b_1}, \ldots, \bar{b_i}, \ldots, \bar{b_m}\}$ and all non basic variable are zero. Alternatively, initialization is a serious problem caused by the constraints of other forms. Then, we apply the branch and bound method and cutting plane method to get the final minimum inventory cost. We will introduce the concept of artificial variable to find a starting basic feasible solution, and the two-phase method, that solves the expanded LP problem. We use the two phase method to solve for the above models.

Two-Phase Method as the name shows it is called, divides the process into two phases.
Phase 1: The goal is to find a basic feasible solution for the original LP. Indeed, we will ignore the original objective for a while, and instead, try to minimize the sum of all artificial variable. At the end of phase 1, a basic feasible solution is obtained if the minimum value of this LP is zero.
Phase 2 : Drop all the artificial variables, change the objective function back to the original one. Use just the regular above steps for the simplex algorithm, with the starting basic feasible solution obtained in Phase 1.

Chapter 4

SOME APPLICATION AND ALGORITHM

In this chapter, some applications carried out for a few variables, which carries out two mathematical models. The examples given below is used in the applications. In addition to the application procedures, a comprehensive qualitative analysis of the solution procedures, explanation of the solutions, algorithms and modification is included.

4.1 Sample Application Solution For Model I

In this approach, the supplier has the responsibility to determine the delivery amount and get the ability to group deliveries in the location by a short route at the appropriate time. Since the main purpose of this section is to minimize the cost. The $(B\&B)$ approach involves probing the tree nodes. To carry out these investigations for each node, the sub-problems are solved using the (two-phase) simplex method.

The analyzed distribution system consists of one customer and two depots. The planning horizon consists of 2 periods. The capacity of each vehicle used in the distribution process is 75 tons denoted as L in the model. The liquid in one tank must be completely delivered to one customer according to the demand of this customer. The goal is to determine the inventory distribution plan from the plant to the customer during two periods. Shortages of goods are not allowed in each plant. The objective of this model is minimizing the total length of routes performed in order to supply the appropriate amount of goods and finding the amount to be delivered.
Example used below are depending on two routes to the customer at a different period.

Example 3.

$$\begin{aligned}
\min \sum_{j=1,2;k=1,2} c_j x_{jk} + \sum_{i=1;k=1,2} b_i y_{ik}& \\
\text{s.t} \qquad z_{ijk} - 55x_{jk} &\leq 0 \\
\sum_{j=1}^{2} z_{ijk} &\leq 75 \\
y_{1k} + \sum_{j=1}^{2} z_{ijk} - d_1 - y_{1k+1} &= 0 \\
y_{1k} + \sum_{j=1}^{2} z_{ijk} - d_1 - y_{11} &= 0 \\
x_{1k} \in \{0,1\}, y_{1k} \geq g_i, z_{ijk} &\geq 0
\end{aligned}$$

Here in this example, we are assuming two different routes at different periods for one customer to minimize inventory cost corresponding to this customer only. We solve this problem by using the two-phase simplex method and the final solution of the problem found after eight iterations is not an integer so, we branch it into two subproblems or adding new constraint as mentioned under introduction the optimal solution and optimal value obtained by using $B\&B$ and cutting plane method for model I are:
$\bar{x} = \{\ x_{11} = 1,\ x_{12} = 0,\ x_{21} = 0,\ x_{22} = 0,\ y_{11} = 0,\ z_{111} = 55\}$ and $\bar{c} = 20$ respectively, with slack variable $s_2 = 20$. From the output we can assign the customer to the route one at period one since $x_{11} = 1$ i.e, route to service customer one is the route one and the delivery amount of good for this customer on the chosen route at period one is $z_{111} = 55$. Therefore, according to this output route one at period one with the amount to be delivered is 55tons as shown in the table (4.1) below.

	x_{11}	x_{12}	x_{21}	x_{22}	y_{11}	y_{12}	z_{111}	z_{112}	z_{121}	z_{122}	s_2	
z	0	0	0	0	0	0	0	0	0	0	0	-5
z_{111}	0	0	0	0	0	0	1	1	1	1	0	55
s_2	0	0	0	0	0	0	0	0	0	0	1	20
x_{21}	0	0	1	1	0	0	0	0	0	0	0	0
y_{11}	0	0	0	0	-1	1	0	0	0	0	0	0
x_{11}	1	1	0	0	0	0	0	0	0	0	0	1

Table 4.1: Simplex Table for Application Example for Model I

As we mentioned earlier, by using the first model we find the amount to be delivered to set the capacity of the vehicle and fulfill the demand of the customer. The targeted solution assumes that the business in question would use external warehouses owned by businesses that would take over the operation of the external warehouses and delivery of goods to customers. However, the fee for the operators services depends on the quantities of inventories stored in the warehouses. This requires that both the transportation cost and the inventory cost be included in the model used. Therefore, model two is used to find both inventory and transportation costs.

4.1.1 Algorithm of Branch & Bound Method for Model I

The branch and bound ($B\&B$) approach is a valuable technique for solving large non-deterministic problems. Let us take a minimization of optimization problem that is defined under chapter 3 for the model I. Assuming that minimizing problem with S denoting the feasible region of the problem, c denoting the objective function, lb and ub the lower and upper bound respectively in the region S, $\bar{c}$ the best objective encountered for the best current solution $\bar{x}$. In this algorithm, S denotes a list of feasible regions to be explored the same way Yun used in his study [15]. Global lower bound and global upper bound are Glb and Gub respectively. The c coefficients of variables and constraints are the parameters we use in the table (3.1).

When we say feasible region it is a set S containing: $S = \{z_{ijk} - a_{ij}x_{jk}M \leq 0, \sum_{j=1,\dots,m} z_{ijk} \leq L,\ y_{ik} + \sum_j z_{ijk} - d_i - y_{ik+1} = 0, y_{ik} + \sum_j z_{ijk} - d_i - y_{i1} = 0, x_{ik} \in \{0,1\}, y_{ik} \geq g_i$ and $z_{ijk} \geq 0\}$. The branching of the variables depends on the companies decision. The table below can use the general form according to the variables that companies need to branch, but not all variables needed to be an integer.

Algorithm 1

initialization Let $r_0 = (x_0, y_0, z_0) \in S$ be incumbent solution of $z = c(r_0)$ of LPR with their bounds $Glb = -\infty$, $Gub = +\infty$
while $S \neq \phi$ **do**
    Select the branching variable depend on the given, that is needed to be, an integer then feasible region which have fractional component
    if $x_1 \leq \lfloor \bar{a} \rfloor$ *or* $x_1 \geq \lfloor \bar{a} \rfloor + 1$ *where,* $\bar{a} \in (0, 1)$ **then**
        $x_0 \longleftarrow x_1$;
    else if $y_1 \leq \lfloor a \rfloor$ *and* $y_1 \geq \lfloor a \rfloor + 1$ **then**
        $y_0 \longleftarrow y_1 \, \forall a \in R$;
    else
        $z_1 \leq \lfloor b \rfloor$ and $z_1 \geq \lfloor b \rfloor + 1$ then $z_0 \longleftarrow z_1 \, \forall b \epsilon R$. If the needed variable is integer ;
    end
    $r = (x_1, y_1, z_1)$ is optimal solution and c(r) is optimal value. If it is not integer update the feasible region with $r = (x_1, y_1, z_1) \epsilon S$ and $lbS \leftarrow S \setminus \{r\}$
end
Bound objective function c in the region r :$lb(r) \leq c(r_0) \leq ub(r) \, \forall x \epsilon R$.
if $lb(r) \geq c(r_0)$ **then**
    Remove r ;
else
    Branch r by dividing it into k subsets $r_1, \ldots, r_k$ current section becomes this one $S \leftarrow S \cup \{r_1, \ldots, r_k\}$ return to while loop;
end
if $lb(r) \leq c(r_0)$ **then**
    the associated solution r is feasible ;
else
    $\bar{c} \leftarrow ub(r)$ and $\bar{x} \leftarrow r$;
end
Result: The branching variable,$\bar{x}$, $\bar{c}$ and ϵ for spatial branching

Table 4.2: Algorithm 1 Branch and Bound for Model I

The fundamental success of the $(B\&B)$ method depends on the quality of bounds used to guide the tree search and the branching strategies mentioned in chapter one. The optimal solution of the subproblem leads to the optimal solution of the original problem. The optimal percentages of inventory cost for their corresponding customer according to the results of both models, if the optimal solution is found, then we can set that as the best route to

deliver goods to the customer or to visit a customer with the amount of goods in need. During the adjustments of joint transportation and inventory cost assign customers to the most appropriate delivery route with the demand of customer and delivery quantities.
The output is: $\bar{c}$, $\bar{x}$ & , ϵ where, $\bar{x}$ represents the value of : x_{jk}, y_{ik} & z_{ijk} with optimal solution and the minimum objective value

$$\sum_{j=1,\ldots,m;k=1,\ldots,p} c_i x_{jk} + \sum_{i=1,\ldots,n;k=1,\ldots,p} b_i y_{ik} \text{ represented by } \bar{c}.$$

The algorithm uses the best optimal solution and optimal value in the feasible region S.

4.1.2 Algorithm on Cutting Plane Method For Model I

As implemented in the (B&B) method, the same experiment is carried out to explain the utility of the cutting-plane process for solving the problem for certain variables. Let $\bar{c}$ is the best objective encountered for the best current solution $\bar{x}$ and c is the biggest value of the objective function achieved by the feasible solutions of the original LPR. In this algorithm, S denotes a list of feasible regions to be explored coefficients of constraints and variables are parameters under the table (3.1).
Rounding and cutting of the fractional portion is also the secret to the performance of the cutting plane technique.

Algorithm 2

initialization Let S with a feasible region of LPR $r_0 = (x_0, y_0, z_0)$ be incumbent solution of $z = c(r_0)$ with their bounds
$Glb = -\infty$, $Gub = +\infty$
while $S \neq \phi$ **do**
  Select node $N = (r_0)$ have fractional component, cut by using GFC and add to constraints
  if x_0 *is a source row* **then**
    $a^t x_0 \leq b$ is added $\forall a \,\epsilon R^n, \forall b \epsilon R$;
  else if $a^t y_0 \leq b$ *is added* $\forall a \epsilon R^n$, $\forall b \,\epsilon R$ **then**
    when y_0 a source row;
  else
    $a^t z_0 \leq b$ is added $\forall a \epsilon R^n, \forall b \,\epsilon R$. If the needed variable is an integer ;
  end
  The feasible region shrinks when the fractional part is cut off and update $r = (x_1, y_1, z_1) \epsilon S$ and $S \leftarrow S \cup S(r)$
end
Bound objective function c in the region r :$lb(r) \leq c(r_0) \leq ub(r) \, \forall x \epsilon R$.
if $lb(r) \geq c$ **then**
  return to while loop ;
end
if $lb(r) \leq c$ **then**
  the associated solution r is feasible ;
else
  $\bar{c} \leftarrow ub(r)$ and $\bar{x} \leftarrow r$;
end
Result: $\bar{x}$, $\bar{c}$ and ϵ

Table 4.3: Algorithm 2 Cutting Plane Method for Model I

The output of the cutting plane method is the same as that of the branch and bound method when we solve the problem. The output yields : $\bar{x}$ and $\bar{c}$. The value of $\bar{x}$ represents the value of : x_{jk}, y_{ik} & z_{ijk} with optimal value and the minimum objective value

$$\bar{c} := \sum_{j=1,\ldots,m;k=1,\ldots,p} c_i x_{jk} + \sum_{i=1,\ldots,n;k=1,\ldots,p} b_i y_{ik}.$$

Branch and Cut: Whenever we use the branch and bound ($B\&B$) method and the cutting plane method together can help us to reach to optimal solution we can use the branch and cut method. The branch and cut method is that at each node in branch-and-bound tree we solve for the current MILP attempt to generate additional constraint by using Gomory Fractional Cut (GFC) that can cut of the fractional part. The process apply table (4.2) and table (4.3) starts from the while loop together.

4.2 Sample Application Solution For Model II

The sample taken corresponding to this model is presented below for the same number of the variable we take in model I. However, here the model included the amount to be delivered which is obtained by using model I as we mention in chapter 3.

Example 4.

$$\begin{aligned}
\min \sum_{j=1,2;k=1,2} c_j x_{jk} + \sum_{i=1;k=1,2} b_i y_{ik} \\
\text{s.t} \quad y_{1k} + \sum_{j=1}^{2} e_{ij} x_{jk} - d_1 - y_{1k+1} = 0 \\
y_{1k} + \sum_{j=1}^{2} e_{ij} x_{jk} - d_1 - y_{11} \geq 0 \\
x_{1k} \in \{0,1\}, y_{1k} \geq 0, z_{ijk} \geq 0
\end{aligned}$$

The second stage example of the study involved an attempt to calculate the minimum inventory distribution cost of the system service, assuming that the routes to a particular customer are bound by the predefined supply number in model I. We are assuming two different routes at two different periods for one customer to minimize inventory cost corresponding to this

	x_{11}	x_{12}	x_{21}	x_{22}	y_{11}	y_{12}	s_1	s_2	s_3	
z	0	0	0	0	0	0	0	0	0	-385
y_{11}	0	0	0	0	1	0	0	0	0	55
s_3	110	110	110	110	0	0	1	-1	1	110
y_{12}	-55	-55	-55	-55	0	1	-1	0	0	0

Table 4.4: Simplex Table for Application Example for Model II

customer only with the delivery amount we obtained using model I $z_{111} = 55$ which is represented by e_{11} in model II. The optimal solution and optimal value obtained by using two-phase simplex method for model two is $\bar{x} = \{y_{11} = 55 \text{ and } y_{12} = 0\}$ and $\bar{c} = 384$ respectively, where all the other basic

variables are zero.
Therefore, the stock of this customer is also known and minimum inventory and transportation cost of this companies with respect to this customer is 384 .
According to value we obtained from model I and model II the decision should made is that:

- route to visit customer one is route one.
- period to visit customer one is period one.
- the amount to be delivered is known which 55tons.

The optimal solution was calculated using the first formulation at the start of the optimization process. In the second step, the solution is determined using the second model. The delivery amount eij in this step was dependent on the solution obtained in the first formulation. The combination of these two models can easily reduce the intangible cost and tangible cost.

4.2.1 Algorithm of Branch and Bound Method for Model II

The general minimum optimization problem we use here is for model II. It is a minimizing problem with S denoting the feasible region of the problem, c denoting the objective function, lb and ub the lower and upper bound respectively in the region S, $\bar{c}$ is the objective value encountered for the best current solution $\bar{x}$. In this algorithm, S denotes a list of feasible regions to be explored. Global lower bound and global upper bound are Glb and Gub, respectively. The key to the success of the (B&B) method is often the quality of bounds used to guide the tree search. The coefficients of variables and constraints are the numerical form of the parameters defined under the table (3.1) with the delivery amount.

The feasible region here is a set S containing: $S = \{y_{ik} + \sum_j e_{ij}x_{jk} - d_i - y_{ik+1} = 0, y_{ik} + \sum_j e_{ij}x_{jk} - d_i - y_{i1} \geq 0,\ x_{ik} \in \{0, 1\}, y_{ik} \geq g_i.\}$

Algorithm 3

initialization Let S with a feasible region $r_0 = (x_0, y_0)$ be incumbent solution of $z = c(r_0)$ with their bounds $Glb = -\infty, Gub = +\infty$
while $S \neq \phi$ **do**
  Select branching variable depend on the given, that is needed to be, integer then feasible region which have fractional component
  if $x_1 \leq \lfloor \bar{a} \rfloor$ *or* $x_1 \geq \lfloor \bar{a} \rfloor + 1$ *where,* $\bar{a} \in (0, 1)$ **then**
    $x_0 \longleftarrow x_1$;
  else
    $y_1 \leq \lfloor a \rfloor$ and $y_1 \geq \lfloor a \rfloor + 1$ **then** $y_0 \longleftarrow y_1 \forall a \epsilon R$;
  end
  $r = (x_1, y_1)$ is optimal solution and c(r) is optimal value (i.e),if its not integer update the feasible region with $r = (x_1, y_1) \epsilon S$ and $lbS \leftarrow S \setminus \{r\}$
end
Bound objective function c in the region r :$lb(r) \leq c(r_0) \leq ub(r) \forall x \epsilon R$.
if $lb(r) \geq c(x_0)$ **then**
  Remove r ;
else
  Branch r by dividing it into k subsets $r_1, \ldots, r_k$ current section becomes this one $S \leftarrow S \cup \{r_1, \ldots, r_k\}$ return to while loop;
end
if $lb(r) \leq c(r_0)$ **then**
  the associated solution r is feasible ;
else
  $\bar{c} \leftarrow ub(r)$ and $\bar{x} \leftarrow r$;
end
Result: The branching variable,$\bar{x}$, $\bar{c}$ and ϵ for spatial branching

Table 4.5: Algorithm 3 Branch and Bound for Model II

Here are the output of the table below (4.3) branch and bound method when we solve the problem. The output yields that: $\bar{x}$ and $\bar{c}$. The value of $\bar{x}$ represents the values of : x_{jk} and y_{ik} with optimal value and the minimum objective value

$$\bar{c} := \sum_{j=1,\ldots,m;k=1,\ldots,p} c_i x_{jk} + \sum_{i=1,\ldots,n;k=1,\ldots,p} b_i y_{ik}.$$

We follow the same method for model II with defined delivery amount:

4.2.2 Algorithm of Cutting Plane Method for Model II

Here again a minimizing problem with S denoting the feasible region of the problem, c denoting the objective function, and $\bar{c}$ is the best objective value encountered for the best current solution $\bar{x}$. In this algorithm, S denotes a list of feasible regions to be explored. Global lower bound and global upper bound are Glb and Gub, respectively for model II. We use the table (3.1) and the delivery amount obtained in model I. A set S is the feasible region containing: $S = \{y_{ik} + \sum_j e_{ij}x_{jk} - d_i - y_{ik+1} = 0, y_{ik} + \sum_j e_{ij}x_{jk} - d_i - y_{i1} \geq 0,\ x_{ik} \in \{0, 1\}, y_{ik} \geq g_i.\}$. The output yields that: $\bar{x}$ and $\bar{c}$. The value of

Algorithm 4

initialization Let S with a feasible region of LPR $r_0 = (x_0, y_0)$ be the incumbent solution of $z = c(r_0)$ with their bounds
$Glb = -\infty, Gub = +\infty$
while $S \neq \phi$ **do**
    Select node $N = (r_0)$ have fractional component, cut by using GFC and add to constraints
    if x_0 *is selected* **then**
        $a^t x_0 \leq b$ is added $\forall a \epsilon R^n$, $\forall b \epsilon R$;
    else
        $a^t y_0 \leq b$ is added $\forall a \epsilon R^n$, $\forall b \epsilon R$ when y_0 is selected;
    end
    The feasible region shrinks when the fractional part is cut off and update $r = (x_1, y_1) \epsilon S$ and $S \leftarrow S \cup S(r)$
end
Bound objective function c in the region r :$lb(r) \leq c(r_0) \leq ub(r)\, \forall x \epsilon R$.
if $lb(r) \geq c$ **then**
    return to while loop ;
end
if $lb(r) \leq c$ **then**
    the associated solution r is feasible ;
else
    $\bar{c} \leftarrow ub(r)$ and $\bar{x} \leftarrow r$;
end
Result: $\bar{x}$, $\bar{c}$ and ϵ

Table 4.6: Algorithm 4 Cutting Plane Method for Model II

$\bar{x}$ represents the value of : x_{jk} and y_{ik} with optimal value and the minimum objective value

$$\bar{c} := \sum_{j=1,\ldots,m;k=1,\ldots,p} c_i x_{jk} + \sum_{i=1,\ldots,n;k=1,\ldots,p} b_i y_{ik}.$$

.

The model uses the branch and bound ($B\&B$) method and the cutting plane method together to approach to the optimal solution as illustrated in the above section.

Model I is globally infeasible, to improve this we need the modification of some constraints.

Modification As mentioned above i, j and k represents customers, routes and periods, respectively. For the definition of other variables please see chapter 3 section (3.2.1) and additional variables s_i and l_i are maximum stock level of customer i and amount to be delivered for customer i, respectively.

$$\min \sum_{j=1,\ldots m;k=1,\ldots,p} c_j x_{jk} + \sum_{i=1,\ldots,n;k=1,\ldots,p} b_i y_{ik} \tag{4.1}$$

$$s.t \qquad z_{ijk} - a_{ij} x_{jk} M \leq 0 \tag{4.2}$$

$$\sum_j z_{ijk} \leq L \tag{4.3}$$

$$y_{1k} + \sum_{j=1,\ldots,m} z_{ijk} - d_1 - y_{1k+1} = 0 \tag{4.4}$$

$$y_{1k} + \sum_{j=1,\ldots,m} z_{ijk} - d_1 - y_{11} \geq 0 \tag{4.5}$$

$$\sum_j x_{ij} \leq 1 \tag{4.6}$$

$$\sum_i y_{ik} \leq s_i \tag{4.7}$$

$$z_{ijk} \leq l_i \tag{4.8}$$

$$x_{jk} \in \{0, 1\}, y_{1k} \geq g_i, z_{ijk} \geq 0 \tag{4.9}$$

The goal function expressed by the formula (4.1) ensures minimizations of the inventory and transportation costs. The constraint (4.2) combines the visiting customer or delivery will be carried out only in period when a route with a given customer is executed. The constraint (4.3) is responsible for taking care of vehicle capacity not the delivery amount. The formulas (4.4) and (4.5) ensures that the stock of the customer in consecutive periods.
The constraints (4.6) maintain the route to visit the customer. Then constraints (4.7) are taken into consideration the beginning inventory levels for

each customer represent by s_i. The constraints(4.8) execute the delivery amount of the customer for the customer i represent by l_i. Finally the constraints(4.9) are bounds for the decision variables x, y and z.

Application In industry, the combined transportation and inventory management problems are mainly found in road-based and maritime supply chains. Campbell [5] according to his study road-based application is important for the distribution of industrial gases or petrol oil and blood or organic oil using trucks, where the VMI model is implemented. The main characteristics of this type of application are that the inventory monitoring are performed in graininess of during a very long period of time and that the stock-out is strictly forbidden or costs a lot. The problem could also be integrated with the scheduling of drivers and the assignment of driver to trucks. Furthermore, constraints on the customer side can also complicate the problem, such as customer time windows or consistency of visits.

Chapter 5

CONCLUSION AND FUTURE WORKS

In this chapter, conclusion of the above work on IRP and recommendation for the future work are explained.

5.1 Conclusion

Two formulations of mixed-integer programming used to solve the inventory routing problem are presented in this thesis. The use of the first formulation, considering a route to one customer which are solved using branch and bound ($B\&B$) and cutting plane method. We can solve the problem using the simplex method is guaranteed to global minimum moreover, uncertainty in the model complicated look like overkill. The second analyzed formulation determines the route and stock of customers under predefined quantities of delivery.

On the other hand, the research has strengthened the model that is combining deliveries may give measurable savings but the solving process is complex, time-consuming, and requires the use of advanced tools. The iterative algorithm to solve both model is also presented. The model solved using branch and bound ($B\&B$) method and cutting plane method for small variables analyzed for these models. Model one is globally infeasible therefore, modification is also done to improve the infeasibility which can lead to both minimum inventory and transportation cost.

It should be emphasized, though, that the limitation of the maximum number of customers visited along a single route had a material no effect on the results obtained in the tests presented here. In the case analyzed, the limitation directly resulted from the decision-making situation, giving rise to the thesis presented here is only to show how the model determines the route, delivery amount and period to visit the customer.

5.2 Future Works

Future work on this topic can be creating a simple mathematical model for many customers to have better inventory costs for companies and the transfer of goods between the buyers.
On the other hand, these models depend on only vehicle filling capacity but, not on the coherence of vehicle (driver) or scheduling of drivers and the assignment of driver to trucks. Future work on this topic can be creating simple mathematical code depending on the algorithm written which can solve the problem easily.

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