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ON

INTERPOLATION BY RATIOAL FUCNTION

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Preface

In constructing numerical methods we often encounter the need for approximating functions of one or more variables. There are many ways of approximating functions, and the problem is to find a way that is suitable to a particular application. This usually means that the approximation has to be easy to work with that, that it gives good accuracy with a simple amount of work, and that there exists a body of knowledge about it that helps us analyze the results. The commonly used class of approximating functions includes polynomials, trigonometric, exponential and rational functions.

Usually, from the application point of view polynomial functions are used in approximating the given function in the given interval, however, approximating by rational function is preferable at the poles and near the poles as it leads to smaller maximum error than approximation by polynomial functions. It is also possible to use exponential and trigonometric functions.

This seminar report consists of six sections of which the first focuses on the general properties of interpolation by rational function which covers basic definitions, theorems with their proofs and examples. The second section deals with the construction of rational functions, which is Thiele's Continued fractions by using inverse and reciprocal differences. The third 0f this seminar is concerned deriving an algorithm of Neville type. Section four is about convergence of rational interpolants to analytic functions, and then the fifth one is deals with the comparison of rational and polynomial interpolation. Finally, this seminar lists advantages of interpolation by rational functions.

Habtamu Getachew

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INTRODUCTION

Interpolation is a basic tool for the approximation of a given function f. Consider a family of functions of a single variable x, $\Phi(x; a_0, a_1, \cdots, a_n)$ having n+1 parameters $a_0, a_1, \cdots, a_n$, whose values characterize the individual functions in this family. The interpolation problem for Φ consists of determining these parameters $a_0, a_1, \cdots, a_n$ so that for n+1 given real or complex pair of numbers $(x_i, f_i), i = 0,1,2, \cdots, n$ with $x_i \neq x_j$ for $i \neq j$ satisfying $\Phi(x_i; a_0, a_1, \cdots, a_n) = f_i \, , i = 0,1,2, \cdots, n$. We call the pair (x_i, f_i) support points, the locations of x_i is called the support abscissa and the values f_i support ordinates. There are two types of interpolation problem based on the dependence of Φ on the parameters. In the above if Φ depends linearly on the parameters, that is,

$$\Phi(x; a_0, a_1, \cdots, a_n) = a_0\Phi_0(x) + a_1\Phi_1(x) + \cdots + a_n\Phi_n(x),$$

the interpolation problem is a linear interpolation problem. This class of problems includes the classical one of polynomial interpolation,

$$\Phi(x; a_0, a_1, \cdots, a_n) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

as well as trigonometric interpolation,

$$\Phi(x; a_0, a_1, \cdots, a_n) = a_0 + a_1 e^{xi} + a_2 e^{2xi} + \cdots + a_n e^{nxi} \; (i^2 = -1).$$

If Φ does not linearly depends on the parameters, then the interpolation problem is non linear interpolation problem. One of the non linear interpolation problem scheme is rational interpolation,

$$\Phi(x; a_0, a_1, \cdots, a_n, b_0, b_1, \cdots, b_m) = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n}{b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m}.$$

Sometimes functions are not well interpolated by polynomials, but are well interpolated by rational functions because of it's ability to approximate with poles, that is zero of the denominator. Thus rational interpolation plays a great role in the process of approximating a given function by one which is readily evaluated on a digital computer.

1. General properties of rational interpolation

Consider a given set of support points $(x_i, f_i), i = 0,1,2,\cdots$ now we try to examine the use of rational functions

$$\Phi^{n,m}(x) = \frac{P^{n,m}(x)}{Q^{n,m}(x)} = \frac{a_0+a_1x+ a_2x^2+ \cdots+a_nx^n}{b_0+b_1x+b_2x^2+ \cdots+b_mx^m},$$

for interpolating these support points. Here the integers n and m denotes the maximum degree of the polynomial in the numerator and denominator respectively. We call the pair of integers (n, m) is the degree type of rational interpolation problem. The rational interpolation problem for $\Phi^{n,m}$ consists of determining the parameters $a_0, a_1, \cdots, a_n, b_0, b_1, \cdots, b_m$ satisfying

$$\Phi^{n,m}(x_i) = f_i \ , i = 0,1,\cdots,n+m.$$

Since the rational function $\Phi^{n,m}$ has coefficients $a_0, a_1, \cdots, a_n, b_0, b_1, \cdots, b_m$ we say that it is determined by it's $n+m+2$ coefficients. On the other hand, $\Phi^{n,m}$ determines these coefficients only up to a non zero common factor. This suggests that $\Phi^{n,m}$ is fully determined by
$n+m+1$ interpolation conditions,

$$\Phi^{n,m}(x_i) = f_i \ , i = 0,1,\cdots,n+m \qquad (1)$$

Let now denotes by $A^{n,m}$ the problem of calculating the rational function $\Phi^{n,m}$ from (1). It is clearly that the coefficients a_r, b_s, of $\Phi^{n,m}$ solve the homogeneous system of linear equations

$$P^{n,m}(x_i) - f_iQ^{n,m}(x_i) = 0 \ , i = 0,1,\cdots,n+m$$

Substituting

$$P^{n,m}(x_i) = a_0 + a_1x_i + \cdots + a_nx_i{}^n$$
$$Q^{n,m}(x_i) = b_0 + b_1x_i + \cdots + b_mx_i{}^m$$

We have

$$a_0 + a_1x_i + \cdots + a_nx_i{}^n - f_i(b_0 + b_1x_i + \cdots + b_mx_i{}^m) = 0,$$
$$i = 0,1,\cdots,n+m \ . \qquad (2)$$

We denote this linear system by $S^{n,m}$.

Definition 1: We say that a rational expression $\Phi^{n,m}$ is a solution of $S^{n,m}$ if its coefficients solve $S^{n,m}$.

Note that if $\Phi^{n,m}$ solves problem $A^{n,m}$ then it solves $S^{n,m}$, but the converse does not hold. This is why the reason that rational interpolation is complicated than polynomial interpolation.

Example 1: For the support points

x_i:	0	1	2
f_i:	1	2	2

and $n = m = 1$.

This show that $\Phi^{1,1}$ has the three interpolation conditions,

$$\Phi^{1,1}(x_i) = f_i, i = 0,1,2.$$

Since $\Phi^{1,1}$ uniquely determines the coefficients only up to a non zero common factor then the only coefficients are a_0, a_1, b_0, b_1. Thus, the system of linear equation $S^{1,1}$:

$$a_0 + a_1 x_i - f_i(b_0 + b_1 x_i) = 0, i = 0,1,2.$$

yields

$$a_0 - 1.b_0 = 0$$
$$a_0 + a_1 - 2(b_0 + b_1) = 0$$
$$a_0 + 2a_1 - 2(b_0 + 2b_1) = 0$$

From this system since the number of equation is less than the number of unknowns then we have infinitely many solutions. Now if $b_1 = 1$, then $a_1 = 2$ and $a_0 = b_0 = 0$ and the rational expression is $\Phi^{1,1}(x) = \frac{2x}{x}$. This shows that since the coefficient of $\Phi^{1,1}(x) = \frac{2x}{x}$ solves $S^{1,1}$. Then $\Phi^{1,1}$ is a solution of $S^{1,1}$. But it does not solve $A^{1,1}$ because of the following. If $x = 0$, the expression leads to an indeterminate of the form $\frac{0}{0}$. But, after canceling the common factor x from both numerator and denominator we arrive at the rational expression $\overline{\Phi}^{1,1}(x) = 2$. Both expressions $\Phi^{1,1}$ and $\overline{\Phi}^{1,1}$ represents the same rational function, namely the constant function of value 2. This function misses the first support point $(x_0, f_0) = (0,1)$. Therefore it does not solve $A^{1,1}$.

Definition 2: Suppose (x_i, f_i) , $i = 0,1,2,\cdots$ be support points and if the rational function $\Phi^{n,m}$ misses some of the support points (x_i, f_i) then such type of points are called inaccessible points.

Note that form the above example the point $(0,1)$ is inaccessible points and the other two points $(1,2)$ and $(2,2)$ are accessible points.

Remark: The interpolation problem $A^{n,m}$ is solvable if there are no inaccessible points.

Example 2: Given the support points

x_i:	0	1	3
f_i:	1	3	2

and $n = m = 1$.

Determine the rational function $\Phi^{1,1}$ for the above support points (x_i, f_i) , $i = 0,1,2$.
The homogenous linear system is

$$a_0 + a_1 x_i - f_i(b_0 + b_1 x_i) = 0, i = 0,1,2.$$

yield the equations

$$a_0 - 1.b_0 = 0$$
$$a_0 + a_1 - 3(b_0 + b_1) = 0$$
$$a_0 + 3a_1 - 2(b_0 + 3b_1) = 0$$

Solving this system we have the coefficients

$$a_0 = b_0 = -\frac{3}{5}, \qquad b_1 = 1,\ a_1 = \frac{9}{5}$$

Thus the rational expression

$$\Phi^{1,1}(x) = \frac{a_0 + a_1 x}{b_0 + b_1 x} = \frac{-3+9x}{-3+5x}$$

Therefore, since there is no inaccessible point the problem $A^{1,1}$ is solvable.
Note that $\Phi^{1,1}$ has a pole at $x = \frac{3}{5}$; this singularity is within the span of the data points and indicates that $\Phi^{1,1}$ may be a good approximation to f if f is known to have a pole somewhere in the interval [0,3]. However, if f is continuous in [0,3], then $\Phi^{1,1}$ may not be used to approximate f in this region.
Example1 shows that the rational interpolation problem $A^{n,m}$ need not be solvable. Indeed, if $S^{n,m}$ has a solution which leads to a rational function that does not solve $A^{n,m}$, as was the case in example, then the rational interpolation problem is not solvable. In order to examine this situation more closely, we have to distinguish between different representations of the same rational function $\Phi^{n,m}$ which arises from each other by cancelling or by introducing a common polynomial factor in numerator and denominator.

Definition 3: The two rational expression $\Phi_1(x) = \frac{P_1(x)}{Q_1(x)}$ and $\Phi_2(x) = \frac{P_2(x)}{Q_2(x)}$ are said to be equivalent if $P_1(x).\,Q_2(x) = P_2(x).\,Q_1(x)$ and denoted by $\Phi_1 \sim \Phi_2$.This precisely when the two rational expressions represent the same rational function.

Definition 4: A rational expression $\Phi^{n,m}(x) = \frac{P^{n,m}(x)}{Q^{n,m}(x)}$ is said to be relatively prime if its numerator and denominator are relatively prime. That is both numerator and denominator are not divisible by the same polynomial of positive degree (degree$\geq$ 1). If the rational expression is not relatively prime, then cancelling all common polynomial factors from both numerator and denominator leads to an equivalent rational expression which is relatively prime.

Theorem 1: The homogenous linear system of equations $S^{n,m}$ always has a non trivial solution. For such solutions,

$\Phi^{n,m}(x) = \frac{P^{n,m}(x)}{Q^{n,m}(x)}$, $Q^{n,m}(x) \neq 0$ holds.

That is all non trivial solution defines a rational functions.

Proof: The homogenous linear system $S^{n,m}$ has $n+m+1$ equations and $n+m+2$ unknowns by equation (2). As a homogenous linear system with more unknowns than equations has a non trivial solution then $S^{n,m}$ has a non trivial solution.

$$\text{i.e. } (a_0, a_1, \cdots, a_n, b_0, b_1, \cdots, b_m) \neq (0, \cdots, 0, 0, \cdots 0)$$

Now in order to show $Q^{n,m}(x) \neq 0$, assume in the contrary and we will arrive at a contradiction.

Suppose $Q^{n,m}(x_i) = b_0 + b_1x_i + \cdots + b_m{x_i}^m = 0$ then the homogenous linear system from equation (2)

$$a_0 + a_1x_i + \cdots + a_n{x_i}^n - f_i(b_0 + b_1x_i + \cdots + b_m{x_i}^m) = 0,$$

$i = 0,1,\cdots, n+m$ is reduced in to

$$a_0 + a_1x_i + \cdots + a_n{x_i}^n = 0 \qquad i = 0,1,\cdots, n+m,$$

This implies that the polynomial

$$P^{n,m}(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

has $n+m+1$ zeros

$$P^{n,m}(x_i) = 0, i = 0,1,\cdots, n+m.$$

Then it would follows that $P^{n,m}(x) = 0$, since the polynomial $P^{n,m}(x)$ has at most degree n, and it vanishes at $n+m+1 > n+1$.

i.e. $a_0 = a_1 = \cdots = a_n = 0$ which is a contradiction to

$(a_0, a_1, \cdots, a_n) \neq (0, \cdots, 0)$

Therefore $Q^{n,m}(x) \neq 0$ holds. Thus each non trivial solution of $S^{n,m}$ defines a rational expression.

The following theorem shows that the rational interpolation problem $A^{n,m}$ has unique solution if it has a solution at all.

Theorem 2: If Φ_1 and Φ_2 are both (non trivial) solution of the homogenous linear system $S^{n,m}$, then they are equivalent ($\Phi_1 \sim \Phi_2$), i.e. they determine the same rational function.

Proof: Let $\Phi_1(x) = \frac{P_1(x)}{Q_1(x)}$ and $\Phi_2(x) = \frac{P_2(x)}{Q_2(x)}$

If $\Phi_1(x) = \frac{P_1(x)}{Q_1(x)}$ and $\Phi_2(x) = \frac{P_2(x)}{Q_2(x)}$ are non trivial solutions of $S^{n,m}$, then their coefficients solves $S^{n,m}$.If both Φ_1 and Φ_2 solves $S^{n,m}$, then the polynomial defined by

$P(x) = P_1(x).\, Q_2(x) - P_2(x).\, Q_1(x)$ has $n + m + 1$ different roots

$$\begin{aligned} P(x_i) &= P_1(x_i).\, Q_2(x_i) - P_2(x_i).\, Q_1(x_i),\ i = 0,1,\cdots,n+m \\ &= f_i Q_1(x_i)\, Q_2(x_i) - f_i Q_2(x_i)\, Q_1(x),\ i = 0,1,\cdots,n+m \\ &= 0,\ i = 0,1,\cdots,n+m. \end{aligned}$$

Since the degree of the polynomial P does not exceed from $n + m$, it vanishes at $n + m + 1$ .i.e. $P(x) = 0$

$$\Rightarrow P_1(x).\, Q_2(x) - P_2(x).\, Q_1(x) = 0$$

$$\Rightarrow P_1(x).\, Q_2(x) = P_2(x).\, Q_1(x)$$

$$\Rightarrow \frac{P_1(x)}{Q_1(x)} = \frac{P_2(x)}{Q_2(x)}$$

Hence it follows that $\Phi_1(x) \sim \Phi_2(x)$.

Note that the converse of the above theorem theorem 2 does not hold. A rational expression Φ_1 may well solve $S^{n,m}$ where as some equivalent rational expressions Φ_2 does not. Example 1 furnishes this. From the given support points we found the two equivalent rational expressions

$\Phi^{1,1}(x) = \frac{2x}{x}$ and $\overline{\Phi}^{1,1}(x) = 2$.

Since the coefficient of $\Phi^{1,1}$ solves $S^{1,1}$, then $\Phi^{1,1}$ is a solution of $S^{1,1}$.But, since the coefficient of $\overline{\Phi}^{1,1}$ does not solve $S^{1,1}$ then $\overline{\Phi}^{1,1}$ is not a solution of $S^{1,1}$.

Combining the above two theorems we can find that for each rational interpolation problem $A^{n,m}$ their exists a unique rational function $\Phi^{n,m}$ up to equivalence that solves $S^{n,m}$. Either this rational function $\Phi^{n,m}(x)$ solves the problem by satisfying (1) or the problem $A^{n,m}$ is not solvable at all. In the latter case, their must be some inaccessible points (x_i, f_i) which is missed by $\Phi^{n,m}$. In summary, suppose, $\Phi^{n,m}(x) = \frac{P^{n,m}(x)}{Q^{n,m}(x)}$ is a solution of $S^{n,m}$ for each support point $(x_i, f_i), i = 0,1,\cdots,n+m$, the two cases occur.

1. $Q^{n,m}(x_i) \neq 0$ implies $\Phi^{n,m}(x_i) = f_i$.
2. $Q^{n,m}(x_i) = 0$ implies (x_i, f_i) may be inaccessible points.

In the first case, $\Phi^{n,m}(x_i) = f_i$. In the second case, $P^{n,m}(x_i) - f_i\, Q^{n,m}(x_i) = 0$ is reduced in to $P^{n,m}(x_i) = 0$. This implies that $P^{n,m}(x_i)$ and $Q^{n,m}(x_i)$ have common factor $(x - x_i)$.
$\Rightarrow$ $P^{n,m}(x)$ and $Q^{n,m}(x)$ are not relatively prime.
$\Rightarrow$ $\Phi^{n,m}(x)$ is not relatively prime.
Now by contra positive we have the following remark:
Remark: If $S^{n,m}$ has a solution $\Phi^{n,m}(x)$ which is relatively prime, then there is no inaccessible points. This implies that the problem $A^{n,m}$ is solvable.
Theorem 3: Given $\Phi^{n,m}(x) = \frac{P^{n,m}(x)}{Q^{n,m}(x)} = \frac{a_0+a_1x+a_2x^2+\cdots+a_nx^n}{b_0+b_1x+b_2x^2+\cdots+b_mx^m}$, Let $\overline{\Phi}^{n,m}(x)$ be an equivalent rational expression which is relatively prime. Suppose $\Phi^{n,m}$ solves $S^{n,m}$. Then $A^{n,m}$ is solvable and $\Phi^{n,m}$ is a solution if and only if $\overline{\Phi}^{n,m}$ solves $S^{n,m}$.

Proof: Suppose $\overline{\Phi}^{n,m}$ solves $S^{n,m}$. Then $\overline{\Phi}^{n,m}$ is a solution of $S^{n,m}$. Now since $\overline{\Phi}^{n,m}$ is relatively prime, then by the above remark there is no inaccessible points. This implies that $A^{n,m}$ is solvable. Also since it has a unique solution up to equivalence then $\Phi^{n,m}$ represents the solution.
Suppose $\overline{\Phi}^{n,m}$ does not solve $S^{n,m}$.
$\Rightarrow$ $\overline{\Phi}^{n,m}$ does not solve $S^{n,m}$.
$\Rightarrow$ $\Phi^{n,m}$ does not solve $S^{n,m}$.
$\Rightarrow$ $A^{n,m}$ is not solvable.
Thus, by contra positive if $A^{n,m}$ is solvable then $\overline{\Phi}^{n,m}$solves $S^{n,m}$.
Even if the linear system has full rank $n + m + 1$, the rational interpolation problem may not be solvable. However, since the solution of $S^{n,m}$ are, in this case, uniquely determined up to a non zero common factor. Example 1 confirms this. In this example $S^{1,1}$ has rank 3 but the problem $A^{1,1}$ is not solvable.
Theorem 4: If $S^{n,m}$ has a full rank, then $A^{n,m}$ is solvable if only if the solution $\Phi^{n,m}$ of $S^{n,m}$ is relatively prime.

Proof: Let $S^{n,m}$ has a full rank $n + m + 1$. Suppose that $A^{n,m}$ is solvable. Then there are no inaccessible points. Hence $S^{n,m}$ has a solution $\Phi^{n,m}$ by theorem (3). But since $S^{n,m}$ has full rank by hypothesis, the polynomials $P^{n,m}(x)$ and $Q^{n,m}(x)$ are relatively prime.
Thus, $\Phi^{n,m}(x) = \frac{P^{n,m}(x)}{Q^{n,m}(x)}$ is relatively prime.

Conversely, suppose $\Phi^{n,m}$ is a solution of $S^{n,m}$ which is relatively prime then by the above remark there is no inaccessible points. Hence $A^{n,m}$ is solvable.

Definition 5: The support points $(x_i, f_i), i = 0,1,2,\cdots,\delta$ are said to be in special position if they are interpolated by a rational expression of degree type (κ, λ) with $\kappa + \lambda < \delta$. In other words, the interpolation problem is solvable for a smaller combined degree of numerator and denominator than the suggested number of support points.

Theorem 5: The accessible support points of a non solvable interpolation problem $A^{n,m}$ are in special position.

Proof: Let $i_1, i_2, \cdots, i_\delta$ be the subscripts of the inaccessible points, and let $\Phi^{n,m}$ be a solution of $S^{n,m}$, then $\Phi^{n,m}$ is not relatively prime. The numerator and denominator have common factor $x - x_{i_1}, x - x_{i_2}, \cdots,$
$x - x_{i_\delta}$. Cancelling these common factors from both numerator and denominator leads to an equivalent rational expression $\Phi^{\kappa,\lambda}$ with $\kappa = n - \delta$ and $\lambda = m - \delta$. Now since $\Phi^{\kappa,\lambda}$ is relatively prime , then $\Phi^{\kappa,\lambda}$ solves $A^{\kappa,\lambda}$ which is consists of $n + m + 1 - \delta$ accessible support points. Since

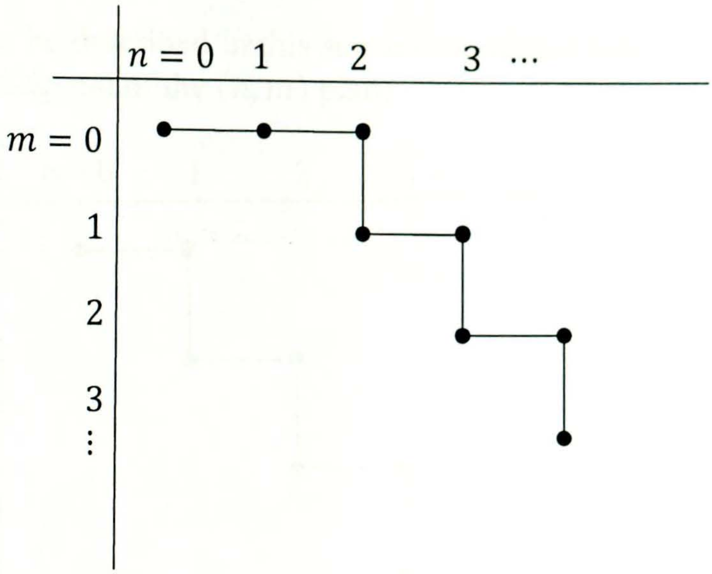
$$\kappa + \lambda + 1 = n - \delta + m - \delta + 1$$
$$= n + m + 1 - 2\delta$$
$$< n + m + 1 - \delta,$$

then the accessible points of $A^{n,m}$ are in special position.

The observation theorem (5) makes it clear that non solvability of the rational interpolation problem is a degeneracy phenomena: solvability can be achieved by arbitrary small perturbation of the support points. In what follows, we will therefore restrict our attention to fully non degenerate problems, that is, problems for which no subset of the support points is in special position. Not only $A^{n,m}$ is solvble in thiscase, but so are all problems $A^{\kappa,\lambda}$ for $\kappa + \lambda + 1$ of the original support points where $\kappa + \lambda \leq n + m$.

Most of the following discussion will be of recursive procedures for solving rational interpolation problems $A^{n,m}$. With each of such recursions there will be associated a rational expression $\prod^{n,m}$,of degree type (n, m) with $n \leq \kappa$ and $m \leq \lambda$, and either the numerator or the denominator of $A^{n,m}$ will be incre4ased by 1. Because of the availability of this choice, the recursion methods for rational interpolation are more varied than those for polynomial interpolation.

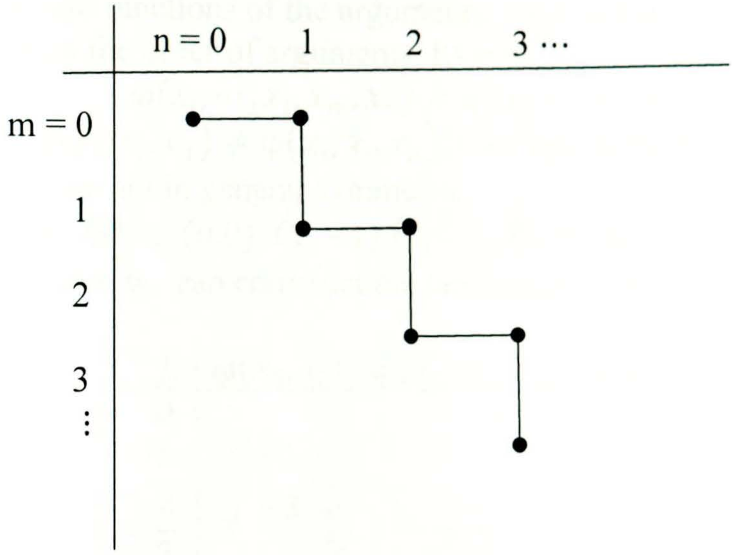
It is helpful to plot the sequence of degree type (n, m) which is encountered in a particular recursion as path in a diagram.



We will distinguish two kinds of algorithms. The first kind is analogous to Newton's method of interpolation. A tableau of quantities analogous to divided difference is generated from which coefficients are gathered for an interpolating rational expression. The second kind corresponds to the Neivel's-Aiteken approach of generating a tableau of values of intermediate rational function $\Phi^{n,m}$. These values relate to each other directly.

2. Inverse and Reciprocal Differences (Thiele's Continued Fraction)

The algorithm to be described in this section calculates rational expressions along the main diagonal of the (n, m) plane.



Starting from the support points $(x_i, f_i), i = 0,1,2,\cdots$, we construct the following tableau of inverse differences.

i	x_i	f_i				
0	x_0	f_0				
1	x_1	f_1	$\varphi(x_0, x_1)$			
2	x_2	f_2	$\varphi(x_0, x_2)$	$\varphi(x_0, x_1, x_2)$		
3	x_3	f_3	$\varphi(x_0, x_3)$	$\varphi(x_0, x_1, x_3)$	$\varphi(x_0, x_1, x_2, x_3)$	
4	x_4	f_4	$\varphi(x_0, x_4)$	$\varphi(x_0, x_1, x_4)$	$\varphi(x_0, x_1, x_2, x_4)$	$\varphi(x_0, x_1, x_2, x_3, x_4)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

The **inverse differences** are defined recursively as follows:

$$\varphi(x_i, x_j) = \frac{x_i - x_j}{f_i - f_j}$$

$$\varphi(x_i, x_j, x_k) = \frac{x_j - x_k}{\varphi(x_i, x_j) - \varphi(x_i, x_k)}$$

$$\varphi(x_i, \cdots, x_l, x_m, x_n) = \frac{x_m - x_n}{\varphi(x_i, \cdots, x_l, x_m) - \varphi(x_i, \cdots, x_l, x_n)}$$

On occasion, certain inverse differences becomes infinite because of the denominator in the above recursion vanishes.

Note that unlike divided differences, the inverse differences are, in general, not symmetric functions of the arguments. That is the inverse differences are dependent on the order of arguments. Even if,

$$\varphi(x_i,\cdots,x_l,x_m,x_n)=\varphi(x_i,\cdots,x_l,x_n,x_m)$$

but, since $\varphi(x_i,x_j,x_k)\neq\varphi(x_j,x_i,x_k)$, then this furnishes inverse differences are not in general symmetric.

Example 1: Given $(0,0)$, $(1,-1)$, $(2,-\frac{2}{3})$, $(3,9)$ be support points. By the above recursion we can construct the tableau of inverse differences.

i	x_i	f_i	$\varphi(x_0,x_i)$	$\varphi(x_0,x_1,x_i)$	$\varphi(x_0,x_1,x_2,x_3)$
0	0	0			
1	1	-1	-1		
2	2	$-\frac{2}{3}$	-3	$-\frac{1}{2}$	
3	3	9	$\frac{1}{3}$	$\frac{3}{2}$	$\frac{1}{2}$

Here each entry is obtained using entries in the previous column which are on the same row and on the main diagonal. Thus, for example,

$$\varphi(x_0,x_1,x_i)=\frac{x_1-x_i}{\varphi(x_0,x_1)-\varphi(x_0,x_i)},\mathrm{i}=2,3$$

$$\varphi(x_0,x_1,x_2)=\frac{x_1-x_2}{\varphi(x_0,x_1)-\varphi(x_0,x_2)}=\frac{1-2}{-1+3}=-\frac{1}{2}$$

$$\varphi(x_0,x_1,x_3)=\frac{x_1-x_3}{\varphi(x_0,x_1)-\varphi(x_0,x_3)}=\frac{1-3}{-1-\frac{1}{3}}=\frac{3}{2}$$

$$\varphi(x_0,x_1,x_2,x_3)=\frac{\mathrm{x}_2-\mathrm{x}_3}{\varphi(x_0,x_1,x_2)-\varphi(x_0,x_1,x_3)}=\frac{2-3}{-\frac{1}{2}-\frac{3}{2}}=\frac{1}{2}$$

Now we try to use the **inverse differences** to find the reciprocal differences. Let $P^{n,m}$ and $Q^{n,m}$ be polynomials whose degree is bounded by n and m respectively. Suppose $(x_i,f_i), i=0,1,2,\cdots,2k$, are given $(n=m=k)$.

Now we try to use inverse differences in order to find a rational expression

$$\Phi^{k,k}(x)=\frac{P^k(x)}{Q^k(x)} \text{ with } \Phi^{k,k}(x_i)=f_i\,, i=0,1,2,\cdots,2k.$$

For $i=0$, $\frac{P^k(x_0)}{Q^k(x_0)}=f_0$,

$$\Phi^{k,k}(x)=\frac{P^k(x)}{Q^k(x)}=f_0+\frac{P^k(x)}{Q^k(x)}-\frac{P^k(x_0)}{Q^k(x_0)}$$

Now since x_0 is the root of $\frac{P^k(x)}{Q^k(x)}-\frac{P^k(x_0)}{Q^k(x_0)}$, then $x-x_0$ is a factor of $\frac{P^k(x)}{Q^k(x)}-\frac{P^k(x_0)}{Q^k(x_0)}$.

This implies $\frac{P^k(x)}{Q^k(x)}-\frac{P^k(x_0)}{Q^k(x_0)}=(x-x_0)\frac{P^{k-1}(x)}{Q^k(x)}$

Thus, $\Phi^{k,k}(x)=\frac{P^k(x)}{Q^k(x)}=f_0+(x-x_0)\frac{P^{k-1}(x)}{Q^k(x)}$

$$=f_0+\frac{(x-x_0)}{\frac{Q^k(x)}{P^{k-1}(x)}} \tag{1}$$

From (1) we have, $\frac{P^k(x)}{Q^k(x)}=f_0+\frac{(x-x_0)}{\frac{Q^k(x)}{P^{k-1}(x)}}$

$$\Rightarrow\frac{P^k(x_i)}{Q^k(x_i)}=f_0+\frac{(x_i-x_0)}{\frac{Q^k(x_i)}{P^{k-1}(x_i)}} \text{ for } i=1,2,\cdots,2k.$$

$$\Rightarrow f_i-f_0=\frac{(x_i-x_0)}{\frac{Q^k(x_i)}{P^{k-1}(x_i)}}$$

$$\Rightarrow\frac{Q^k(x_i)}{P^{k-1}(x_i)}=\frac{(x_i-x_0)}{f_i-f_0}=\varphi(x_0,x_i), i=1,2,\cdots,2k. \tag{2}$$

For $i=1$,

$$\frac{Q^k(x_1)}{P^{k-1}(x_1)}=\frac{(x_1-x_0)}{f_1-f_0}=\varphi(x_0,x_1)$$

$$\frac{Q^k(x)}{P^{k-1}(x)}=\varphi(x_0,x_1)+\frac{Q^k(x)}{P^{k-1}(x)}-\frac{Q^k(x_1)}{P^{k-1}(x_1)}$$

$$=\varphi(x_0,x_1)+(x-x_1)\frac{Q^{k-1}(x)}{P^{k-1}(x)}$$

$$=\varphi(x_0,x_1)+\frac{(x-x_1)}{\frac{P^{k-1}(x)}{Q^{k-1}(x)}} \tag{3}$$

Substituting (3) in to (2)

$$\Phi^{k,k}(x)=\frac{P^k(x)}{Q^k(x)}=f_0+\frac{(x-x_0)}{\varphi(x_0,x_1)+\frac{(x-x_1)}{\frac{P^{k-1}(x)}{Q^{k-1}(x)}}} \tag{4}$$

From (3), we get

$$\frac{P^{k-1}(x_i)}{Q^{k-1}(x_i)} = \varphi(x_0, x_1) + \frac{(x_i - x_1)}{\frac{P^{k-1}(x_i)}{Q^{k-1}(x_i)}} \text{ for } i = 2, \cdots, 2k.$$

By equation (2)

$$\varphi(x_0, x_i) = \varphi(x_0, x_1) + \frac{(x_i - x_1)}{\frac{P^{k-1}(x_i)}{Q^{k-1}(x_i)}}$$

$$\Rightarrow \varphi(x_0, x_i) - \varphi(x_0, x_1) = \frac{(x_i - x_1)}{\frac{P^{k-1}(x_i)}{Q^{k-1}(x_i)}}, i = 2, \cdots, 2k.$$

$$\Rightarrow \frac{P^{k-1}(x_i)}{Q^{k-1}(x_i)} = \frac{(x_i - x_1)}{\varphi(x_0, x_i) - \varphi(x_0, x_1)} = \varphi(x_0, x_1, x_i), \mathrm{i} = 2,3, \cdots, 2\mathrm{k}.$$

For i = 2,

$$\frac{P^{k-1}(x_2)}{Q^{k-1}(x_2)} = \varphi(x_0, x_1, x_2)$$

$$\frac{P^{k-1}(x)}{Q^{k-1}(x)} = \varphi(x_0, x_1, x_2) + \frac{P^{k-1}(x)}{Q^{k-1}(x)} - \frac{P^{k-1}(x_2)}{Q^{k-1}(x_2)}$$

$$= \varphi(x_0, x_1, x_2) + (\mathrm{x} - \mathrm{x}_2) \frac{P^{k-2}(x)}{Q^{k-1}(x)}$$

$$= \varphi(x_0, x_1, x_2) + \frac{(\mathrm{x} - \mathrm{x}_2)}{\frac{Q^{k-1}(x)}{P^{k-2}(x)}} \qquad (5)$$

Substituting (5) in to (4)

$$\Phi^{k,k}(x) = \frac{P^k(x)}{Q^k(x)} = f_0 + \cfrac{(x - x_0)}{\varphi(x_0, x_1) + \cfrac{(x - x_1)}{\varphi(x_0, x_1, x_2) + \cfrac{(\mathrm{x} - \mathrm{x}_2)}{\cfrac{Q^{k-1}(x)}{P^{k-2}(x)}}}}$$

Continuing in this fashion, we arrive at the following expansions of $\Phi^{k,k}(x)$.

$$\Phi^{k,k}(x) = f_0 + \cfrac{x - x_0}{\varphi(x_0, x_1) + \cfrac{x - x_1}{\varphi(x_0, x_1, x_2) + \cfrac{\mathrm{x} - \mathrm{x}_2}{\varphi(x_0, x_1, x_2, x_3) + \cfrac{\mathrm{x} - \mathrm{x}_3}{\varphi(x_0, x_1, x_2, x_3, x_4) + \ddots + \cfrac{\mathrm{x} - \mathrm{x}_{2k-1}}{\varphi(x_0, x_1, \cdots, x_{2k})}}}}}$$

$(*)$

This formula is called **continued fraction** by using inverse differences.

$$\Phi^{0,0}(x) = f_0 = \varphi(x_0)$$

$$\Phi^{1,0}(x) = f_0 + \frac{(x-x_0)}{\varphi(x_0,x_1)}$$

$$\Phi^{1,1}(x) = f_0 + \frac{(x-x_0)}{\varphi(x_0,x_1)+\frac{(x-x_1)}{\varphi(x_0,x_1,x_2)}}$$

$$\Phi^{2,1}(x) = f_0 + \frac{(x-x_0)}{\varphi(x_0,x_1)+\frac{(x-x_1)}{\varphi(x_0,x_1,x_2)+\frac{(x-x_2)}{\varphi(x_0,x_1,x_2,x_3)}}} \quad \cdots$$

Example1: Given the support points

x_i:	0	1	3
f_i:	1	3	2

Find $\Phi^{1,1}$ such that $\Phi^{1,1}(x_i) = f_i$.

Solution: Now the inverse difference table is

i	x_i	f_i	$\varphi(x_0, x_i)$	$\varphi(x_0, x_1, x_i)$
0	0	1		
1	1	3	$\frac{1}{2}$	
2	3	2	3	$\frac{4}{5}$

By the above formula

$$\Phi^{1,1}(x) = f_0 + \frac{x-x_0}{\varphi(x_0,x_1)+\frac{x-x_1}{\varphi(x_0,x_1,x_2)}} = 1 + \frac{x-0}{\frac{1}{2}+\frac{x-1}{\frac{4}{5}}} = \frac{9x-3}{5x-3}$$

which is the solution obtained in example 2 of section 1.

Example 2: Find the rational expression $\Phi^{2,1}$ for the following support points

x_i	0	1	2	3
f_i	0	-1	$-\frac{2}{3}$	9

Solution: The table of inverse differences is

i	x_i	f_i	$\varphi(x_0,x_i)$	$\varphi(x_0,x_1,x_i)$	$\varphi(x_0,x_1,x_2,x_3)$
0	0	0			
1	1	-1	-1		
2	2	$-\frac{2}{3}$	-3	$-\frac{1}{2}$	
3	3	9	$\frac{1}{3}$	$\frac{3}{2}$	$\frac{1}{2}$

The continued fraction is

$$\Phi^{2,1}(x) = f_0 + \cfrac{x-x_0}{\varphi(x_0,x_1) + \cfrac{x-x_1}{\varphi(x_0,x_1,x_2) + \cfrac{x-x_2}{\varphi(x_0,x_1,x_2,x_3)}}}$$

$$= 0 + \cfrac{x-0}{-1 + \cfrac{x-1}{-\frac{1}{2} + \cfrac{x-2}{\frac{1}{2}}}}$$

$$= \cfrac{x}{-1+\cfrac{x-1}{-\frac{1}{2}+2x-4}} = \frac{4x^2-9x}{-2x+7}$$

Because of the **inverse differences** luck symmetry, the so called **reciprocal differences** $\rho(x_i, x_{i+1}, \cdots, x_{i+k})$ are often preferred. The **reciprocal differences** are defined by the following recursion:

$\rho(x_i) = f(x_i) = f_i$

$\rho(x_i, x_{i+1}) = \frac{x_i - x_{i+1}}{f_i - f_{i+1}}$

$\rho(x_i, x_{i+1}, x_{i+2}) = \frac{x_i - x_{i+2}}{\rho(x_i,x_{i+1}) - \rho(x_{i+1},x_{i+2})} + \rho(x_{i+1})$

$\vdots$

$$\rho(x_i, x_{i+1}, x_{i+2}, \cdots, x_{i+k-1}, x_{i+k}) = \frac{x_i - x_{i+k}}{\rho(x_i,x_{i+1},x_{i+2},\cdots,x_{i+k-1}) - \rho(x_{i+1},x_{i+2},\cdots,x_{i+k-1},x_{i+k})} + \rho(x_{i+1}, x_{i+2}, \cdots, x_{i+k-1})$$

Note that **reciprocal differences** are symmetrical. The reciprocal differences are closely related to inverse differences.

Theorem 2.1: Suppose $\rho(x_0, x_1, x_2, \cdots, x_{p-2}) = 0$ for $p = 1$. Then for $p = 1,2,3,\cdots,$ $\varphi(x_0, x_1, \cdots, x_p) = \rho(x_0, x_1, \cdots, x_p) - \rho(x_0, x_1, x_2, \cdots, x_{p-2})$

Proof: The proposition is correct for $p = 1$. Because

$$\rho(x_0, x_1) - \rho(x_0, x_{-1}) = \rho(x_0, x_1) - 0 = \rho(x_0, x_1) = \varphi(x_0, x_1)$$

Assume it is true for p, that is

$$\varphi(x_0, x_1, \cdots, x_p) = \rho(x_0, x_1, \cdots, x_p) - \rho(x_0, x_1, x_2, \cdots, x_{p-2})$$

We need to show that the proposition is true for $p + 1$.
By inverse differences formula

$$\varphi(x_0, x_1, \cdots, x_p, x_{p+1}) = \frac{x_p - x_{p+1}}{\varphi(x_0, x_1, \cdots, x_p) - \varphi(x_0, x_1, \cdots, x_{p-1}, x_{p+1})}$$

By the hypothesis

$$\varphi(x_0, \cdots, x_p, x_{p+1}) = \frac{x_p - x_{p+1}}{\rho(x_0, x_1, \cdots, x_p) - \rho(x_0, x_1, \cdots, x_{p-2}) - [\rho(x_0, x_1, \cdots, x_{p-1}, x_{p+1}) - \rho(x_0, x_1, \cdots, x_{p-2})]}$$

$$= \frac{x_p - x_{p+1}}{\rho(x_0, x_1, \cdots, x_p) - \rho(x_0, x_1, \cdots, x_{p-1}, x_{p+1})}$$

$$= \frac{x_p - x_{p+1}}{\rho(x_p, x_0, x_1, \cdots, x_{p-1}) - \rho(x_0, x_1, \cdots, x_{p-1}, x_{p+1})},$$

Since ρ is symmetric

$$= \rho(x_p, x_0, x_1, \cdots, x_{p-1}, x_{p+1}) - \rho(x_0, x_1, \cdots, x_{p-1})$$

(by the above recursive formula)

$$= \rho(x_0, x_1, \cdots, x_{p-1}, x_p, x_{p+1}) - \rho(x_0, x_1, \cdots, x_{p-1})$$

Since the proposition is true for $p + 1$, then by induction hypothesis

$$\varphi(x_0, x_1, \cdots, x_p) = \rho(x_0, x_1, \cdots, x_p) - \rho(x_0, x_1, x_2, \cdots, x_{p-2})$$

The **reciprocal differences** can be arranged in the tableau as follows:

i	x_i	f_i				
0	x_0	f_0				
			$\rho(x_0,x_1)$			
1	x_1	f_1		$\rho(x_0,x_1,x_2)$		
			$\rho(x_1,x_2)$		$\rho(x_0,x_1,x_2,x_3)$	
2	x_2	f_2		$\rho(x_1,x_2,x_3)$		$\rho(x_0,x_1,x_2,x_3,x_4)$
			$\rho(x_2,x_3)$			
					$\rho(x_1,x_2,x_3,x_4)$	
3	x_3	f_3		$\rho(x_2,x_3,x_4)$		
			$\rho(x_3,x_4)$			
4	x_4	f_4				
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Using theorem 2.1 to substituting reciprocal differences for inverse differences in the above equation $(*)$ yields **Thiele's continued fraction.**

$$\Phi^{k,k}(x) = f_0 + \cfrac{x-x_0}{\rho(x_0,x_1)+\cfrac{x-x_1}{\rho(x_0,x_1,x_2)-\rho(x_0)+\cfrac{x-x_2}{\rho(x_0,x_1,x_2,x_3)-\rho(x_0,x_1)+\cfrac{x-x_3}{\rho(x_0,x_1,x_2,x_3,x_4)-\rho(x_0,x_1,x_2)+ \ddots + \cfrac{x-x_{2k-1}}{\rho(x_0,x_1,\cdots,x_{2k})-\rho(x_0,x_1,\cdots,x_{2k-2})}}}}} \quad \cdots (**)$$

Example 1: Calculate the reciprocal differences for the support points

x_i	0	1	2	3
f_i	0	-1	$-\frac{2}{9}$	9

and by using Thiele's continued fraction determine the rational expression $\Phi^{2,1}$ for which $\Phi^{2,1}(x_i) = f_i$.

Solution: The table of reciprocal differences is as follows

i	x_i	f_i	$\rho(x_i,x_{i+1})$	$\rho(x_i,x_{i+1},x_{i+2})$	$\rho(x_0,x_1,x_2,x_3)$
0	0	0			
			-1		
1	1	-1			
			3	$-\frac{1}{2}$	
					$-\frac{1}{2}$
2	2	$-\frac{2}{9}$		$-\frac{57}{42}$	
			$\frac{3}{29}$		
3	3	9			

We compute the reciprocal differences of adjacent lines rather than use the first and the k^{th} ; we use only the top diagonal values. Thus, for example,

$$\rho(x_1,x_2,x_3) = \frac{x_1 - x_3}{\rho(x_1,x_2) - \rho(x_2,x_3)} + \rho(x_2)$$

$$= \frac{1-3}{3-\frac{3}{29}} - \frac{2}{9}$$

$$= -\frac{57}{42}$$

$$\rho(x_0,x_1,x_2,x_3) = \frac{x_0 - x_3}{\rho(x_0,x_1,x_2) - \rho(x_1,x_2,x_3)} + \rho(x_1,x_2)$$

$$= \frac{0-3}{-\frac{1}{2}+\frac{57}{42}} + 3$$

$$= -\frac{1}{2}$$

Hence by using Thiele's continued fraction, we have

$$\Phi^{2,1}(x) = f_0 + \cfrac{x - x_0}{\rho(x_0,x_1) + \cfrac{x - x_1}{\rho(x_0,x_1,x_2) - \rho(x_0) + \cfrac{x - x_2}{\rho(x_0,x_1,x_2,x_3) - \rho(x_0,x_1)}}}$$

$$= 0 + \cfrac{x-0}{-1 + \cfrac{x-1}{-\frac{1}{2} - 0 + \cfrac{x-2}{-\frac{1}{2} - (-1)}}}$$

$$= \cfrac{x}{-1 + \cfrac{x-1}{-\frac{1}{2} + 2x - 4}}$$

$$= \cfrac{x}{-1 + \frac{2x-2}{4x-9}}$$

$$= \frac{x\,(4x-9)}{-4x+9+2x-2}$$

$$= \frac{4x^2 - 9x}{-2x+7}$$

Which is a solution obtained in example 2 of section 2.

Example 2: Using the reciprocal differences for the support points

x_i	0	1	2	3	4	5	6
f_i	1	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{17}$	$\frac{1}{26}$	$\frac{1}{37}$

Determine the rational expression $\Phi^{2,2}(x)$ for which $\Phi^{2,2}(x_i) = f_i$.

Solution: The table of reciprocal differences is

i	x_i	f_i	$\rho(x_i, x_{i+1})$	$\rho(x_i, x_{i+1}, x_{i+2})$	$\rho(x_i, x_{i+1}, x_{i+2}, x_{i+3})$	$\rho(x_0, x_1, x_2, x_3)$
0	0	1				
			-2			
1	1	$\frac{1}{2}$		-1		
			$\frac{10}{3}$		0	
						0
2	2	$\frac{1}{5}$		$-\frac{1}{10}$		
			-10		40	
						0
3	3	$\frac{1}{10}$		$-\frac{1}{25}$		
			$-\frac{170}{7}$		140	
						0
4	4	$\frac{1}{17}$		$-\frac{1}{46}$		
			$-\frac{442}{9}$		324	
5	5	$\frac{1}{26}$		$-\frac{1}{73}$		
			$-\frac{962}{11}$			
6	6	$\frac{1}{37}$				

Here the column of zeros plays the same role as a column of zeros in the normal difference table; it indicates how far to go. Now we get from formula $(**)$ by direct substitution:

$\Phi^{2,2}(x) =$

$$f_0 + \cfrac{x - x_0}{\rho(x_0,x_1) + \cfrac{x - x_1}{\rho(x_0,x_1,x_2) - \rho(x_0) + \cfrac{x - x_2}{\rho(x_0,x_1,x_2,x_3) - \rho(x_0,x_1,x_2) + \cfrac{x - x_3}{\rho(x_0,x_1,x_2,x_3,x_4) - \rho(x_0,x_1,x_2,x_3)}}}}$$

$$= 1 + \cfrac{x-0}{-2+\cfrac{x-1}{-1-1+\cfrac{x-2}{0-(-2)+\cfrac{x-3}{0-(-2)}}}}$$

$$= 1 + \cfrac{x}{-2+\cfrac{x-1}{-2+\cfrac{x-2}{x-1}}}$$

$$= 1 + \cfrac{x}{-2+\cfrac{(x-1)^2}{-x}}$$

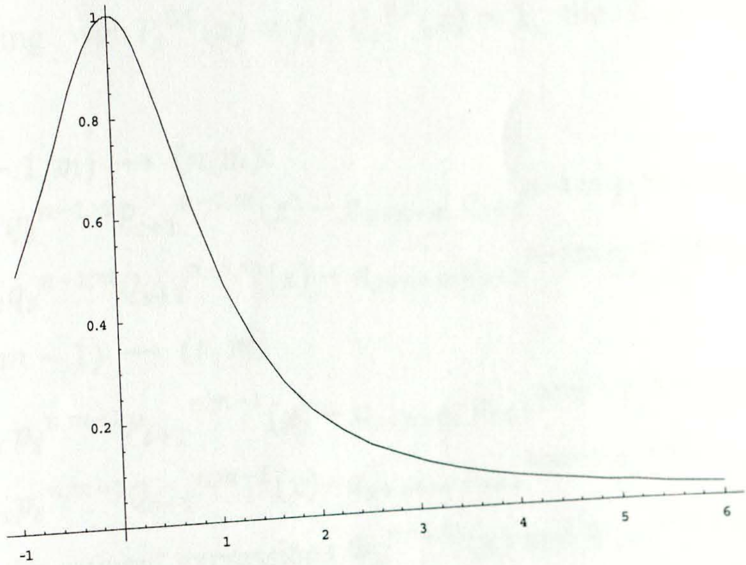
$$= 1 - \frac{x^2}{x^2+1}$$

$$= \frac{1}{x^2+1}$$

Note that applying **Lagrange's** and **Newton's divided difference** interpolation formula, we obtain the following polynomial:

$$P_6(x) = -\frac{73}{163540}x^6 + \frac{6257}{96356}x^5 - \frac{3391908}{144569360}x^4 + \frac{5271}{81770}x^3 + \frac{121}{8177}x^2 - \frac{86020272}{153604945}x + 1$$

This is a complicated expression. The graph look likes



3. Algorithms of the Neville Type

We proceed to derive an algorithm for rational interpolation which is analogous to Neville's algorithm for polynomial interpolation. A quick reminder that after discussing possible degeneracy effects in rational interpolation problems (section 1), we have assumed that such effects are absent in the problems whose solution we are discussing. Indeed such degeneracies are not likely to occur in Numerical problems.

$\Phi_s^{n,m}(x) = \frac{P_s^{n,m}(x)}{Q_s^{n,m}(x)}$ be the rational expressions with $\Phi_s^{n,m}(x_i) = f_i$ for $i = s, s+1, \cdots, s+n+m$. $P_s^{n,m}(x)$, $Q_s^{n,m}(x)$ being polynomials of degree not exceed n and m respectively. Let $p_s^{n,m}$ and $q_s^{n,m}$ be the leading coefficient of these polynomials:

$$P_s^{n,m}(x) = p_s^{n,m}\, x^n + \cdots, \text{ and } Q_s^{n,m}(x) = q_s^{n,m}\, x^m + \cdots$$

For brevity we put

$$a_i = x - x_i \text{ and } T_s^{n,m}(x,y) = P_s^{n,m}(x) - yQ_s^{n,m}(x)$$

Note that $T_s^{n,m}(x_i, f_i) = 0, i = s, s+1, \cdots, s+n+m.$

Theorem 3.1: Starting with $P_s^{0,0}(x) = f_s$, $Q_s^{0,0}(x) = 1$, the following recursion holds:

a) Transition $(n-1, m) \longrightarrow (n, m)$:

$$P_s^{n,m}(x) = a_s\, q_s^{n-1,m} P_{s+1}^{n-1,m}(x) - a_{s+n+m}\, q_{s+1}^{n-1,m}\, P_s^{n-1,m}(x)$$

$$Q_s^{n,m}(x) = a_s q_s^{n-1,m} Q_{s+1}^{n-1,m}(x) - a_{s+n+m} q_{s+1}^{n-1,m} Q_s^{n-1,m}(x)$$

b) Transition $(n, m-1) \longrightarrow (n, m)$

$$P_s^{n,m}(x) = a_s\, p_s^{n,m-1} P_{s+1}^{n,m-1}(x) - a_{s+n+m}\, p_{s+1}^{n,m-1}\, P_s^{n,m-1}(x)$$

$$Q_s^{n,m}(x) = a_s p_s^{n,m-1} Q_{s+1}^{n,m-1}(x) - a_{s+n+m}\, p_{s+1}^{n,m-1} Q_s^{n,m-1}(x)$$

Proof: (a) Suppose the rational expressions $\Phi_s^{n-1,m}(x)$ and $\Phi_{s+1}^{n-1,m}(x)$ meet the interpolation requirements

$$T_s^{n-1,m}(x_i, f_i) = 0, i = s, s+1, \cdots, s+n+m-1$$

$$T_{s+1}^{n-1,m}(x_i, f_i) = 0, i = s+1, s+2, \cdots, s+n+m \tag{3.2a}$$

If we define $P_s^{n,m}(x)$, $Q_s^{n,m}(x)$ by (a), then the degree of $P_s^{n,m}$ clearly does not exceed n. The polynomial expression for $Q_s^{n,m}(x)$ contains formally a term with x^{m+1}, whose coefficient, however, vanishes. The polynomial $Q_s^{n,m}(x)$ is therefore of degree at most m. Finally,

$$T_s^{n,m}(x,y) = a_s q_s^{n-1,m} T_{s+1}^{n-1,m}(x,y) - a_{s+n+m} q_{s+1}^{n-1,m} T_s^{n-1,m}(x,y)$$

From this and equation $(3.2a)$

$$\begin{aligned} T_s^{n,m}(x_i,f_i) &= a_s q_s^{n-1,m} T_{s+1}^{n-1,m}(x_i,f_i) - a_{s+n+m} q_{s+1}^{n-1,m} T_s^{n-1,m}(x_i,f_i) \\ &= a_s q_s^{n-1,m}(0) - a_{s+n+m} q_{s+1}^{n-1,m}(0) \\ &= 0 \end{aligned}$$

Hence $T_s^{n,m}(x_i,f_i) = 0$ for $i = s, s+1, \cdots, s+n+m$.

Under general hypothesis that no combination of (n,m,s) has in accessible points, the above result shows that (a) indeed defines the numerator and denominator of $\Phi_s^{n,m}(x)$. i.e. ,

$$\Phi_s^{n,m}(x) = \frac{a_s\, q_s^{n-1,m}\, P_{s+1}^{n-1,m}(x) - a_{s+n+m}\, q_{s+1}^{n-1,m}\, P_s^{n-1,m}(x)}{a_s\, q_s^{n-1,m}\, Q_{s+1}^{n-1,m}(x) - a_{s+n+m}\, q_{s+1}^{n-1,m}\, Q_s^{n-1,m}(x)}$$

(b) Assume that the rational expressions $\Phi_s^{n,m-1}(x)$ and $\Phi_{s+1}^{n,m-1}(x)$ satisfies

$$T_s^{n,m-1}(x_i,f_i) = 0, i = s, s+1, \cdots, s+n+m-1$$

$$T_{s+1}^{n,m-1}(x_i,f_i) = 0, i = s+1, s+2, \cdots, s+n+m \qquad (3.2b)$$

Now define $P_s^{n,m}(x)$ and $Q_s^{n,m}(x)$ by (b). Then the degree of $Q_s^{n,m}(x)$ does not exceed m. The polynomial expression for $P_s^{n,m}(x)$ contains a term with x^{n+1}, whose coefficient vanishes. The polynomial $P_s^{n,m}(x)$ is therefore of degree at most degree n. Hence,

$$T_s^{n,m}(x,y) = a_s\, p_s^{n,m-1} T_{s+1}^{n,m-1}(x,y) - a_{s+n+m}\, p_{s+1}^{n,m-1} T_s^{n,m-1}(x,y)$$

From this and equation $(3.2b)$

$$\begin{aligned} T_s^{n,m}(x_i,f_i) &= a_s\, p_s^{n,m-1} T_{s+1}^{n,m-1}(x_i,f_i) - a_{s+n+m}\, p_{s+1}^{n,m-1} T_s^{n,m-1}(x_i,f_i) \\ &= a_s p_s^{n-1,m}(0) - a_{s+n+m}\, p_{s+1}^{n-1,m}(0) \\ &= 0 \end{aligned}$$

Hence, $T_s^{n,m}(x_i, f_i) = 0$ for $i = s, s+1, \cdots, s+n+m$.

By the general hypothesis that no combination of (n, m, s) has inaccessible points, the above the result shows that

$$\Phi_s^{n,m}(x) = \frac{a_s\, p_s^{n,m-1} P_{s+1}^{n,m-1}(x) - a_{s+n+m}\, p_{s+1}^{n,m-1}\, P_s^{n,m-1}(x)}{a_s p_s^{n,m-1} Q_{s+1}^{n,m-1}(x) - a_{s+n+m}\, p_{s+1}^{n,m-1} Q_s^{n,m-1}(x)}$$

Unfortunately, the recursion in theorem (3.1) still contain the coefficients $p_s^{n,m-1}$ and $q_s^{n-1,m}$. The formulas are not yet suitable for the calculation of $\Phi_s^{\mu,\lambda}(x)$ for prescribed values of x. However, we can eliminate these coefficients on the basis of the following theorem.

Theorem 3.3:

(a) $\Phi_s^{n-1,m}(x) - \Phi_{s+1}^{n-1,m-1}(x) = C_1 \frac{(x-x_{s+1})\cdots(x-x_{s+n+m-1})}{Q_s^{n-1,m}(x)\, Q_{s+1}^{n-1,m-1}(x)}$

With $C_1 = -p_{s+1}^{n-1,m-1} q_s^{n-1,m}$

(b) $\Phi_{s+1}^{n-1,m}(x) - \Phi_{s+1}^{n-1,m-1}(x) = C_2 \frac{(x-x_{s+1})\cdots(x-x_{s+n+m-1})}{Q_{s+1}^{n-1,m}(x)\, Q_{s+1}^{n-1,m-1}(x)}$

With $C_2 = -p_{s+1}^{n-1,m-1} q_{s+1}^{n-1,m}$

Proof: (a) The numerator polynomial of the rational expression

$$\Phi_s^{n-1,m}(x) - \Phi_{s+1}^{n-1,m-1}(x) = \frac{P_s^{n-1,m}(x)}{Q_s^{n-1,m}(x)} - \frac{P_{s+1}^{n-1,m-1}(x)}{Q_{s+1}^{n-1,m-1}(x)}$$

$$= \frac{P_s^{n-1,m}(x)\, Q_{s+1}^{n-1,m-1}(x) - P_{s+1}^{n-1,m-1}(x)\, Q_s^{n-1,m}(x)}{Q_s^{n-1,m}(x)\, Q_{s+1}^{n-1,m-1}(x)}$$

of degree $(n-1) + m = n + m - 1$ and has $n + m - 1$ different zeros x_i, $i = s+1, s+2, \cdots, s+n+m-1$.

By definition of $\Phi_s^{n-1,m}(x)$ and $\Phi_{s+1}^{n-1,m-1}(x)$. It must therefore be of the form $C_1(x - x_{s+1})\cdots(x - x_{s+n+m-1})$ with $C_1 = -p_{s+1}^{n-1,m-1} q_s^{n-1,m}$

Hence, $\Phi_s^{n-1,m}(x) - \Phi_{s+1}^{n-1,m-1}(x) = C_1 \frac{(x-x_{s+1})\cdots(x-x_{s+n+m-1})}{Q_s^{n-1,m}(x)\, Q_{s+1}^{n-1,m-1}(x)}$ with $C_1 = -p_{s+1}^{n-1,m-1} q_s^{n-1,m}$.

(b) Since the numerator polynomial of the rational expression

$$\Phi_{s+1}{}^{n-1,m}(x) - \Phi_{s+1}{}^{n-1,m-1}(x) = \frac{P_{s+1}{}^{n-1,m}(x)}{Q_{s+1}{}^{n-1,m}(x)} - \frac{P_{s+1}{}^{n-1,m-1}(x)}{Q_{s+1}{}^{n-1,m-1}(x)}$$

$$= \frac{P_{s+1}{}^{n-1,m}(x)\, Q_{s+1}{}^{n-1,m-1}(x) - P_{s+1}{}^{n-1,m-1}(x)\, Q_{s+1}{}^{n-1,m}(x)}{Q_{s+1}{}^{n-1,m}(x)\, Q_{s+1}{}^{n-1,m-1}(x)}$$

is at most degree $(n-1)+m = n+m-1$ and has $n+m-1$ different zeros x_i for $i = s+1, s+2, \cdots, s+n+m-1$ by definition of $\Phi_{s+1}{}^{n-1,m}(x)$ and $\Phi_{s+1}{}^{n-1,m-1}(x)$. Therefore, it must be of the form $C_2(x-x_{s+1})\cdots(x-x_{s+n+m-1})$ with $C_2 = -\,p_{s+1}{}^{n-1,m-1} q_{s+1}{}^{n-1,m}$.
Hence,

$$\Phi_{s+1}{}^{n-1,m}(x) - \Phi_{s+1}{}^{n-1,m-1}(x) = C_2 \frac{(x-x_{s+1})\cdots(x-x_{s+n+m-1})}{Q_{s+1}{}^{n-1,m}(x) Q_{s+1}{}^{n-1,m-1}(x)} \text{ with}$$

$C_2 = -\,p_{s+1}{}^{n-1,m-1} q_{s+1}{}^{n-1,m}$.

Theorem 3.4: For $n \geq 1, m \geq 1$,

(a) $$\Phi_s{}^{n,m}(x) = \Phi_{s+1}{}^{n-1,m}(x) + \frac{\Phi_{s+1}{}^{n-1,m}(x) - \Phi_s{}^{n-1,m}(x)}{\frac{a_s}{a_{s+n+m}}\left[1 - \frac{\Phi_{s+1}{}^{n-1,m}(x) - \Phi_s{}^{n-1,m}(x)}{\Phi_{s+1}{}^{n-1,m}(x) - \Phi_{s+1}{}^{n-1,m-1}(x)}\right] - 1}$$

(b) $$\Phi_s{}^{n,m}(x) = \Phi_{s+1}{}^{n,m-1}(x) + \frac{\Phi_{s+1}{}^{n,m-1}(x) - \Phi_s{}^{n,m-1}(x)}{\frac{a_s}{a_{s+n+m}}\left[1 - \frac{\Phi_{s+1}{}^{n,m-1}(x) - \Phi_s{}^{n,m-1}(x)}{\Phi_{s+1}{}^{n,m-1}(x) - \Phi_{s+1}{}^{n-1,m-1}(x)}\right] - 1}$$

Proof: (a) by theorem (3.1)

$$\Phi_s{}^{n,m}(x) = \frac{a_s\, q_s{}^{n-1,m}\, P_{s+1}{}^{n-1,m}(x) - a_{s+n+m}\, q_{s+1}{}^{n-1,m}\, P_s{}^{n-1,m}(x)}{a_s\, q_s{}^{n-1,m}\, Q_{s+1}{}^{n-1,m}(x) - a_{s+n+m}\, q_{s+1}{}^{n-1,m}\, Q_s{}^{n-1,m}(x)}$$

We now assume that $p_{s+1}{}^{n-1,m-1} \neq 0$ and multiply both the numerator and denominator of the above fraction by

$$-\frac{p_{s+1}{}^{n-1,m-1}(x-x_{s+1})\cdots(x-x_{s+n+m-1})}{\Phi_{s+1}{}^{n-1,m}(x)\, \Phi_s{}^{n-1,m}(x)\, \Phi_{s+1}{}^{n-1,m-1}(x)}$$

and taking theorem (3.3) in account, we arrive at

$$\Phi_s{}^{n,m}(x) = \frac{a_s\, \Phi_{s+1}{}^{n-1,m}(x)\, \eta_1 - a_{s+n+m}\, \Phi_s{}^{n-1,m}(x)\, \eta_2}{a_s\, \eta_1 - a_{s+n+m}\, \eta_2}$$

Where,

$$\eta_1 = \Phi_s{}^{n-1,m}(x) - \Phi_{s+1}{}^{n-1,m-1}(x)$$
$$\eta_2 = \Phi_{s+1}{}^{n-1,m}(x) - \Phi_{s+1}{}^{n-1,m-1}(x)$$

Hence by a straight forward transformation (a) holds.

(b) By theorem (3.1), we have

$$\Phi_s^{\,n,m}(x) = \frac{a_s\, p_s^{\,n,m-1}\, P_{s+1}^{\,n,m-1}(x) - a_{s+n+m}\, p_{s+1}^{\,n,m-1}\, P_s^{\,n,m-1}(x)}{a_s\, p_s^{\,n,m-1}\, Q_{s+1}^{\,n,m-1}(x) - a_{s+n+m}\, p_{s+1}^{\,n,m-1}\, Q_s^{\,n,m-1}(x)}$$

Assume that $p_{s+1}^{\,n-1,m-1} \neq 0$ and multiply numerator and denominator of the above fraction by

$$-\frac{p_{s+1}^{\,n-1,m-1}\,(x-x_{s+1})\cdots(x-x_{s+n+m-1})}{\Phi_{s+1}^{\,n,m-1}(x)\,\Phi_s^{\,n,m-1}(x)\,\Phi_{s+1}^{\,n-1,m-1}(x)}$$

and taking theorem (3.3) in account, we arrive at

$$\Phi_s^{\,n,m}(x) = \frac{a_s\,\Phi_{s+1}^{\,n,m-1}(x)\,\delta_1 - a_{s+n+m}\,\Phi_s^{\,n,m-1}(x)\,\delta_2}{a_s\,\delta_1 - a_{s+n+m}\,\delta_2} \quad \text{where}$$

$$\delta_1 = \Phi_s^{\,n,m-1}(x) - \Phi_{s+1}^{\,n-1,m-1}(x)$$

$$\delta_2 = \Phi_{s+1}^{\,n,m-1}(x)\,\Phi_{s+1}^{\,n-1,m-1}(x)$$

Hence by a straight forward transformation (b) holds.

The formula in theorem (3.4) can now be used to calculate the value of rational expressions for prescribed x successively, alternatively increasing the degree of numerators and denominators. This corresponding to a zigzag path in the (n, m)- diagram.

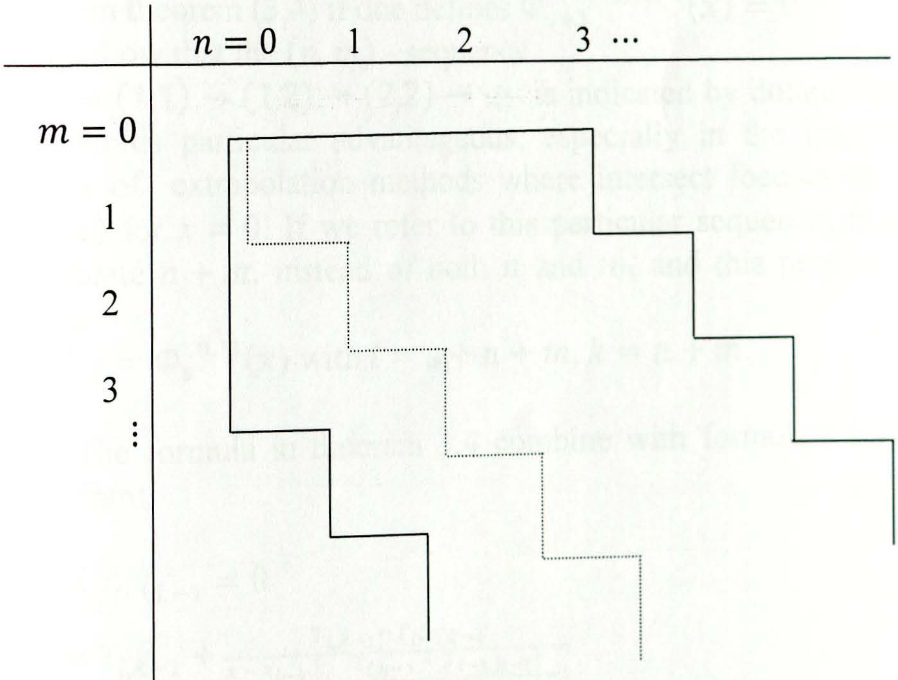


Diagram 3.1

Special recursive rules are still needed for initial straight portions: vertically and horizontally of such paths.
As long as $m = 0$ and only n is being increased one has a case of pure polynomial interpolation. One uses Neville's formulas:

$$\Phi_s^{0,0}(x) = f_s,$$

$$\Phi_s^{n,0}(x) = \frac{a_s \Phi_{s+1}^{n-1,0}(x) - a_{s+n+m} \Phi_s^{n-1,0}(x)}{a_s - a_{s+n}}, \quad n=1,2,3,\cdots$$

Actually these are specializations of theorem (3.4a) from $m = 0$, provided the convention $\Phi_{s+1}^{n-1,-1}(x) = \infty$ is adopted, which cases the quotient

$$\frac{\Phi_{s+1}^{n-1,m}(x) - \Phi_s^{n-1,m}(x)}{\Phi_{s+1}^{n-1,m}(x) - \Phi_{s+1}^{n-1,m-1}(x)}$$

in theorem (3.4a) vanishes.
If $n = 0$ and only m is being increased, then this case relates to polynomial interpolation with the support points $\left(x_i, \frac{1}{f_i}\right)$ and one can use the formulas

$$\Phi_s^{0,0}(x) = f_s$$

$$\Phi_s^{0,m}(x) = \frac{a_s - a_{s+m}}{\frac{a_s}{\Phi_{s+1}^{0,m-1}(x)} - \frac{a_{s+m}}{\Phi_s^{0,m-1}(x)}}, n = 1,2,3,\cdots \qquad (3.5)$$

which arises from theorem (3.4) if one defines $\Phi_{s+1}^{-1,m-1}(x) = 0$.
Experience has show that the (n, m) - sequence
$(0,0) \rightarrow (0,1) \rightarrow (1,1) \rightarrow (1,2) \rightarrow (2,2) \rightarrow \cdots$ is indicated by dotted line in diagram (3.1) holds particular advantageous, especially in the important application area of extrapolation methods where intersect focuses on the values $\Phi_s^{n,m}(x)$ for $x = 0$. If we refer to this particular sequence, then it suffices to indicate $n + m$, instead of both n and m, and this permits the shorter notation

$$T_{i,k} = \Phi_s^{n,m}(x) \text{ with } i = s + n + m, k = n + m.$$

Theorem 3.6: The formula in theorem 3.4 combine with formula (3.5) to yield the algorithm:

$$T_{i,0} = f_i, T_{i,-1} = 0$$

$$T_{i,k} = T_{i,k-1} + \frac{T_{i,k-1} - T_{i-1,k-1}}{\frac{x - x_{i-k}}{x - x_i}\left[1 - \frac{T_{i,k-1} - T_{i-1,k-1}}{T_{i,k-1} - T_{i-1,k-2}}\right] - 1}$$

$$1 \leq k \leq i, \; i = 0,1,2,\cdots \qquad (3.7)$$

Note that this recursion formula differs from the corresponding polynomial formula

$$T_{i,0} = f_i$$

$$T_{i,k} = \frac{(x-x_{i-k})\,T_{i,k-1} - (x-x_i)\,T_{i-1,k-1}}{x - x_{i-k}}$$

$$= T_{i,k-1} + \frac{T_{i,k-1} - T_{i-1,k-1}}{\frac{x - x_{i-k}}{x - x_i} - 1}$$

for $1 \le k \le i, i \ge 0$ only the expression in brackets, i.e.,

$$1 - \frac{T_{i,k-1} - T_{i-1,k-1}}{T_{i,k-1} - T_{i-1,k-2}}$$

which assumes that the value 1 in polynomial case if we display the values $T_{i,k} = T_{ik}$ in the table below, letting i count the ascending diagonal and k the columns, then each instances of the recursion formula (3.7) interrelates the four corners of a rhombus.

$(n,m) =$	$(0,0)$	$(0,1)$	$(1,1)$	$(1,2)$	$(2,2)$	$\cdots$
	$f_0 = T_{0,0}$					
$T_{0,-1} = 0$		$T_{1,1}$				
	$f_1 = T_{1,0}$		$T_{2,2}$			
$T_{1,-1} = 0$		$T_{2,1}$		$T_{3,3}$		
	$f_2 = T_{2,0}$		$T_{3,2}$			
					$T_{4,4}\cdots$	
$T_{2,-1} = 0$		$T_{3,1}$		$T_{4,3}$	$\vdots$	
	$f_3 = T_{3,0}$		$T_{4,2}$	$\vdots$		
$T_{3,-1} = 0$		$T_{4,1}$	$\vdots$			
$\vdots$	$f_4 = T_{4,0}$	$\vdots$				
	$\vdots$					

If one is interested in the rational function itself, i.e. its coefficients, then the method of section 2, involving inverse or reciprocal differences, are suitable. However, if one desires the values of the interpolating function for just one single argument, then algorithms of Neville type based on the formula of theorem 3.6 and formula (3.7) are preferred. The formula (3.7) is particularly useful in the context of extrapolation methods.

Example 1: Given the following support points

x_i	-1	0	1
f_i	1	0	1

Estimate the value of $f(0.5)$.

Solution: The tableau of values is the following:

(n,m)	$(0,0)$	$(0,1)$	$(1,1)$
	$f_0 = T_{0,0} = 1$		
		$T_{1,1} = 4$	
			$T_{2,2} = \frac{5}{2}$
$T_{0,-1} = 0$	$f_1 = T_{1,0} = 2$		
		$T_{2,1} = \frac{12}{5}$	
$T_{1,-1} = 0$	$f_2 = T_{2,0} = 3$		

Here the first column $(0,0)$ of the tableau contains the prescribed support ordinates f_i. Subsequent columns are filled by calculation of each entry recursively from its two neighbors $T_{i,k-1}$ and $T_{i-1,k-1}$ in the previous column before the previous column. The entries in our case, for instance, are given by formula

$$T_{i,0} = f_i \, , T_{i,-1} = 0$$

$$T_{i,k} = T_{i,k-1} + \frac{T_{i,k-1} - T_{i-1,k-1}}{\frac{x-x_{i-k}}{x-x_i}\left[1 - \frac{T_{i,k-1} - T_{i-1,k-1}}{T_{i,k-1} - T_{i-1,k-2}}\right] - 1};$$

For $1 \leq k \leq i, i = 0,1,2,\cdots$

$$T_{1,1} = T_{1,0} + \frac{T_{1,0} - T_{0,0}}{\frac{x-x_0}{x-x_1}\left[1 - \frac{T_{1,0} - T_{0,0}}{T_{1,0} - T_{0,-1}}\right] - 1}$$

$$T_{1,1}(0.5) = 2 + \frac{2-1}{\frac{1.5}{0.5}\left[1 - \frac{2-1}{2-0}\right] - 1} = 4$$

$$T_{2,1} = T_{2,0} + \frac{T_{2,0} - T_{1,0}}{\frac{x-x_0}{x-x_1}\left[1 - \frac{T_{2,0} - T_{1,0}}{T_{2,0} - T_{1,-1}}\right] - 1}$$

$$T_{2,1}(0.5) = 3 + \frac{3-2}{\frac{0.5}{-0.5}\left[1 - \frac{3-2}{3-0}\right] - 1} = \frac{12}{5}$$

$$T_{2,2} = T_{2,1} + \frac{T_{2,1}-T_{1,1}}{\frac{x-x_0}{x-x_2}\left[1-\frac{T_{2,1}-T_{1,1}}{T_{2,1}-T_{1,0}}\right]-1}$$

$$T_{2,2}(0.5) = \frac{12}{5} + \frac{\frac{12}{5}-4}{\frac{1.5}{-0.5}\left[1-\frac{\frac{12}{5}-4}{\frac{12}{5}-2}\right]-1} = \frac{5}{2}$$

Hence, $T_{2,2}(0.5) = \frac{5}{2}$

Example 2: Determine an approximate value for $f(2.5)$ in example 2 of section 2.

Solution: The rational interpolation with $(n,m) = (2,2)$ using the formula (3.8) gives:

(n,m)	$(0,0)$	$(0,1)$	$(0,1)$	$(1,2)$
	$T_{0,0} = 1$			
$T_{0,-1} = 0$		$T_{1,1} = \frac{2}{7}$		
	$T_{1,0} = \frac{1}{2}$		$T_{2,2} = \frac{1}{11}$	
$T_{1,-1} = 0$		$T_{2,1} = \frac{2}{13}$		$T_{3,3} = \frac{4}{29}$
	$T_{2,0} = \frac{1}{5}$		$T_{3,2} = \frac{7}{50}$	
$T_{2,-1} = 0$		$T_{3,1} = \frac{2}{15}$		$T_{4,3} = \frac{4}{29}$
	$T_{3,0} = \frac{1}{10}$		$T_{4,2} = \frac{13}{95}$	
$T_{3,-1} = 0$		$T_{4,1} = \frac{2}{13}$		$T_{5,3} = \frac{4}{29}$
	$T_{4,0} = \frac{1}{17}$		$T_{5,2} = \frac{19}{134}$	
$T_{4,-1} = 0$		$T_{5,1} = \frac{2}{7}$		$T_{6,3} = \frac{4}{29}$
	$T_{5,0} = \frac{1}{26}$		$T_{6,2} = \frac{5}{31}$	
$T_{5,-1} = 0$		$T_{6,1} = -\frac{2}{3}$		
	$T_{6,0} = \frac{1}{37}$			

Observe that the last column implies that $T_{i,k} = \frac{4}{29}$ for every i and k with $3 \le i \le 6$ and $3 \le k \le 6$.

Hence $f(2.5) = \frac{4}{29}$ which is the value obtained by using the rational interpolat $\Phi^{2,2}$ in example 2 of section 2.

4. Convergence of rational interplant

Let $x_0 < x_1 < \cdots < x_{n+m}$ be $n+m+1$ distinct points, and let $f_i = f(x_i)$, $i = 0,1,2,3,\cdots,n+m$ where f is an unknown function. They are changed in to a set of data $\{(x_0, f_0), (x_1, f_1), \cdots, (x_{n+m}, f_{n+m})\}$.

The classical algebraic problem $A^{n,m}$ of rational interpolation is to compute a pair of polynomials $P^{n,m}$ and $Q^{n,m}$ satisfying the relations:

I. $\deg P^{n,m} \leq n$, $\deg Q^{n,m} \leq m$,

II. $\Phi^{n,m}(x_i) = \frac{P^{n,m}(x_i)}{Q^{n,m}(x_i)} = f_i, i = 0,1,2,3,\cdots,n+m,$

for a given Pair of non negative integers n and m. The rational interpolation in (II) is computed by several methods. Example Thiele's interpolating continued fraction, linearized equations and so on. Here the linearized equations are expressed as follows.

$$P^{n,m}(x_i) - f_i\, Q^{n,m}(x_i) = 0\ , i = 0,1,2,3,\cdots,n+m.$$

If we put

$$P^{n,m}(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \text{ and}$$
$$Q^{n,m}(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^m,$$

then we can compute the values a_j and b_k for $j = 0,1,2,3,\cdots,n$ and $k = 0,1,2,3,\cdots,m$, by solving the equations. However, there are two difficulties in which a rational satisfying linearized equations would not be considered as a solution of problem above:

- There may arise the rational interpolation of the form $\frac{0}{0}$ at x_i , i.e. unattainable points.
- There may be poles between the points of interpolation.

In the later case, if a function f is continuous in $[x_0\,, x_{n+m}]$, an interpolated rational function $\Phi^{n,m}(x)$ of f_i becomes a poor approximation at the poles and near the poles. To avoid the difficulties, several rational interpolation has been proposed. One of them is called linear rational interpolation. This interpolation has no unattainable points and no poles in the interval of interpolation $[x_0\,, x_{n+m}]$. However, the degree of the approximation will become very high to obtain better approximation of f.

Let an infinite triangular matrix of interpolation points $c_{ij} \in \bar{\mathbb{C}}$ (called interpolation scheme) be given:

$$\Gamma = \begin{bmatrix} c_{00} & & & & & \\ c_{01} & c_{11} & & & & \\ \cdots & \cdots & \cdots & & & \\ c_{0n} & c_{1n} & \cdots & c_{nn} & & \\ \cdots & \cdots & & \cdots & \cdots & \cdots \end{bmatrix} \quad (4.1)$$

Each row $\psi_n = \{c_{0n}, c_{1n}, \cdots, c_{nn}\}$ of the matrix Γ defines an interpolation set with $n+1$ interpolation points. It not excluded that some or more points are identical. Hence in (4.2) we have in general a multi set with multiplicities of elements taken account of by representation. With each interpolation set ψ_n a polynomial

$$w_n(z) = \prod_{x \in \psi_n}(z - x) = \prod_{j=0}^{n}(z - c_{jn}) \quad (4.2)$$

is associated.

In the sequel it is assumed that the function f which will be interpolated is analytic at each point $z \in \psi_n, n = 1,2,3, \cdots$

Definition 4.1: A rational function $\Phi^{n,m}$ is called rational interpolant of degree $n; m$ to the function f at the $n+m+1$ interpolation points of the set ψ_{n+m} if the quotient $\frac{f - \Phi^{n,m}}{w_{n+m}}$ is bounded at each $x \in \psi_{n+m}$. (4.3)

Remarks:

1. Condition (4.3) implies that at each zero of the polynomial w_{n+m} the interpolation error $f - \Phi^{n,m}$ has a zero of at least the same order. Thus, $f - \Phi^{n,m}$ has a zero at each point $x \in \psi_{n+m}$ of at least the same order as the frequency of the point x in the set ψ_{n+m}, or in other words, the interpolant $\Phi^{n,m}$ and its derivatives $(\Phi^{n,m})^{(k)}$ coincides with the function f and its derivatives $f^{(k)}$ at the point x up to an order determined by the frequency of x in ψ_{n+m} .
2. The existence of a rational function $\Phi^{n,m}$ satisfying (4.3) is in general not guaranteed. If for positive integers n, m a rational function $\Phi^{n,m}$ exists that satisfies (4.3), then one says that the interpolation problem $A^{n,m}$ is solvable.
3. If the interpolation problem is solvable, then the solution is unique.

Example 1: Consider $n = m = 1, \psi_2 = \{-1,0,1\}$ and as function to be interpolated $f(z) = z^2$. Then the support points are

z_i	-1	0	1
f_i	1	0	1

and to find the rational function $\Phi^{1,1}(z) = \frac{a_0+a_1z}{b_0+b_1z}$ which takes the prescribed values, the system $S^{1,1}$ is

$$a_0 + a_1z_i - f_i(b_0 + b_1z_i) = 0, i = 0,1,2.$$

$$a_0 + a_1(-1) - 1(b_0 + b_1(-1)) = 0$$
$$a_0 + a_1(0) - 0(b_0 + b_1(0)) = 0$$
$$a_0 + a_1(1) - 1(b_0 + b_1(1)) = 0$$

which has the solution of the form $a_0 = b_0 = 0;\ a_1 = b_1 = r$ for which the corresponding rational function is given by $\Phi^{1,1}(z) = \frac{rz}{rz}$.

Any function $\Phi^{1,1}$ is either a Moebius transform or a constant. If $\Phi^{1,1}$ is a Moebius transform, then it is univalent in $\overline{\mathbb{C}}$ and therefore can not interpolate the values 1 at the two different points -1 and1. If $\Phi^{1,1}$ is a constant function, then it can not interpolate the two different values 0 and 1.Hence, already in this very simple situation a rational function $\Phi^{n,m}$ satisfying (4.3) does not exists.

The main reason for the non-existence in case of rational interpolants is caused by the non-linearity of the parameterization of the interpolants. In order to circumvent the difficulties one uses a linearized version of definition(4.1).

Definition 4.2: The rational function

$$\Phi^{n,m}(z) = \frac{P^{n,m}(z)}{Q^{n,m}(z)}, Q^{n,m}(z) \neq 0$$

is called linearized rational interpolant of degree $n; m$ to the function f at $n + m + 1$ points of the interpolation set ψ_{n+m} if the quotient

$$\frac{Q^{n,m}f - P^{n,m}}{w_{n+m}} \quad (4.4)$$

is bounded at each point $x \in \psi_{n+m}$.

Remarks:

1. In the definition (4.1) and (4.2) the same symbol $\Phi^{n,m}$ has been used on purpose, since if $\Phi^{n,m}$ satisfies (4.1) then it is automatically satisfies (4.4) with an approximate choice of the numerator and denominator polynomials. Note it may be necessary that the two polynomials $P^{n,m}$ and $Q^{n,m}$ contains common factors.
2. The linearized version of the rational interpolant $\Phi^{n,m}$ always exists. Indeed, relation (4.4) is equivalent to a system $S^{n,m}$ of $n+m+1$ linear homogenous equations for the $n+m+2$ unknown parameters in the polynomial $P^{n,m}$ and $Q^{n,m}$.Hence, a non-trivial solution always exists, and for such solution $Q^{n,m}(z)=0$ is impossible.
3. The rational function $\Phi^{n,m}$ is uniquely determined by (4.4). The is not true for the non-zero pair of polynomials $(P^{n,m}, Q^{n,m})$. In any case the polynomials $P^{n,m}$ and $Q^{n,m}$ can be multiplied by a common non-zero constant, but their may exist more essential non-uniqueness.

Definition: The **Pade approximation** is the form of rational approximation which is analogous to the Taylor polynomial approximation because it is based on the derivative of $f(x)$ at $x=0$. The rational approximation is writer in the form $f(x) \approx \Phi^{n,m}(x) = \frac{a_0+a_1x+a_2x^2+\cdots+a_nx^n}{b_0+b_1x+b_2x^2+\cdots+b_nx^m}$ is also called Pade approximation. Using the Maclaurin series expansion for $f(x)$ given by $f(x)=\sum_{j=0}^{\infty} c_jx^j$ in which $c_j's$ are known constants such that

$$c_j = \frac{f^j(0)}{j!}, \text{ where } j = 0,1,2,\cdots$$

$$f(x) - \Phi^{n,m}(x) = \sum_{j=0}^{\infty} c_jx^j - \frac{\sum_{j=0}^{n} a_jx^j}{\sum_{j=0}^{m} b_jx^j}$$

$$= \frac{\left(\sum_{j=0}^{\infty} c_jx^j\right)\left(\sum_{j=0}^{m} b_jx^j\right) - \sum_{j=0}^{n} a_jx^j}{\sum_{j=0}^{m} b_jx^j} \tag{1}$$

with out loss of generality we can take $b_0=1$. The $n+m+1$ constants $a_i, i=0,1,2,\cdots,n$ and $b_i=1,2,3,\cdots,m$ are determined by equating coefficients of $x^j, j=0,1,2,\cdots,n+m$ in the numerator of (1) to zero. The first non zero term in the numerator gives the order of approximation.

If $N=n+m$ and $m=0$, the Pade approximation, $\Phi^{n,0}(x)$ is the Taylor polynomial of degree $N=n$ expanded about 0 that is the Maclaurin polynomial of degree N,otherwise we get an approximation of order $N=n+m$.

Example: Obtain the rational approximation (Pade approximation) of the form

(i) $\frac{a_0+a_1x}{1+b_1x}$ (ii) $\frac{a_0+a_1x}{1+b_1x+b_2x^2}$ to $f(x) = e^x$.

Solution: (i) We have for $n = m = 1, N = 2$

$$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

We seek to choose a_0, a_1 and b_1 such that

$$e^x - \frac{a_0+a_1x}{1+b_1x} = 0, (1 + b_1x) \neq 0 \text{ Therefore,}$$

$$e^x - \frac{a_0+a_1x}{1+b_1x} = \frac{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)(1+b_1x) - (a_0+a_1x)}{1+b_1x}$$

Set $\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)(1 + b_1x) - (a_0 + a_1x) = \sum_{j=0}^{\infty} d_j x^j$

Put $d_j = 0, j = 0,1,2$. We have,

$$1 - a_0 = 0$$
$$1 + b_1 - a_1 = 0$$
$$\frac{1}{2} + b_1 = 0$$

which gives $a_0 = 1,\ a_1 = \frac{1}{2}, b_1 = -\frac{1}{2}$

Therefore, $e^x = \frac{1+x/2}{1-x/2} + o(x^3)$

(ii) In this Case, we have $n = 1, m = 2, N = 3$

Set $\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)(1 + b_1x) - (a_0 + a_1x) = \sum_{j=0}^{\infty} d_j x^j$

Putting $d_j = 0, j = 0,1,2,3$. We get

$$1 - a_0 = 0$$
$$1 + b_1 - a_1 = 0$$
$$\frac{1}{2} + b_1 + b_2 = 0$$
$$\frac{1}{6} + \frac{1}{2}b_1 + b_2 = 0$$

The solution of the system is

$$a_0 = 1,\ a_1 = \frac{1}{3}, b_1 = -\frac{2}{3}, b_2 = \frac{1}{6}$$

The rational approximation is given by

$$e^x = \frac{1+x/3}{1-(2x/3)+x^2/6} + o(x^4)$$

5. Comparison of Rational and Polynomial Interpolation

Interpolation, as mentioned before, is frequently used for approximating a given function $f(x)$. In many such instances, interpolation by polynomials is entirely satisfactory. The situation is different if the location x for which one desires an approximate value of $f(x)$ lies in the proximity of a pole or some other singularity of $f(x)$ like the value of $\tan x$ for x close to $\frac{\pi}{2}$. In such cases, polynomial interpolation does not give satisfactory results, where as rational interpolation does, because rational function them selves may have poles.

The rational interpolation can also be used for functions behaving as polynomials. The same or higher order of accuracy can be achieved by using lower order polynomials in $\Phi^{n,m}(x)$ than the direct polynomial interpolation. When the function f is such that it retains a finite value when $x \to \infty$, polynomial interpolation gives very poor results. In comparison, rational interpolations give such better results. We generally compute functions like $\operatorname{Sin} x$, $\operatorname{Cos} x$ etc for large arguments using rational interpolations.

Example: For the function $f(x) = cot\, x$ the values $cot\, 1^{\circ}, cot\, 2^{\circ}, cot\, 3^{\circ}$, ... have been tabulated. The problem is to determine an approximate value for $cot\, 2^{\circ}30'$.

Solution: Now first let us find the polynomial interpolation of order four, using the Neville's formula for polynomial interpolation:

x_i	f_i				
x_0	$f_0 = T_{0,0}$				
		$T_{1,1}$			
			$T_{2,2}$		
				$T_{3,3}$	
x_1	$f_1 = T_{1,0}$				$T_{4,4}$
		$T_{2,1}$			
			$T_{3,2}$		
				$T_{4,3}$	
x_2	$f_2 = T_{2,0}$				
		$T_{3,1}$			
			$T_{4,2}$		
x_3	$f_3 = T_{3,0}$				
		$T_{4,1}$			
x_4	$f_4 = T_{4,0}$				

Where

$$T_{i,0} = f_i \text{ , } i = 0,1,2,3,4$$

$$T_{i,k} = \frac{(x-x_{i-k})\,T_{i,k-1} - (x-x_i)\,T_{i-1,k-1}}{x - x_{i-k}}$$

$$= T_{i,k-1} + \frac{T_{i,k-1} - T_{i-1,k-1}}{\frac{x-x_{i-k}}{x-x_i} - 1} \quad \text{for } 1 \le k \le i$$

Using this we have the following table

x_i	f_i				
1°	57.28996163				
		14.30939911			
			21.47137102		
				22.36661762	
2°	28.63625328				
		23.85869499			
					22.**63519158**
			23.2618642		
3°	19.08113669				
				23.08281486	
		21.47137190			
			22.18756808		
4°	14.30066626				
		18.60658719			
5°	11.43005230				

The rational interpolation with $(n, m) = (2,2)$ using the formula (3.7) in contrast gives:

x_i	f_i				
1°	57.28996163				
		22.907660673			
			22.90341624		
				22.90369573	
2°	28.63625328				
		22.90201805			22.9037655**2**
			22.90411487		
				22.90384141	
3°	19.08113669				
		22.91041916			
			22.90201975		
4°	14.30066626				
		22.94418151			
5°	11.43005230				

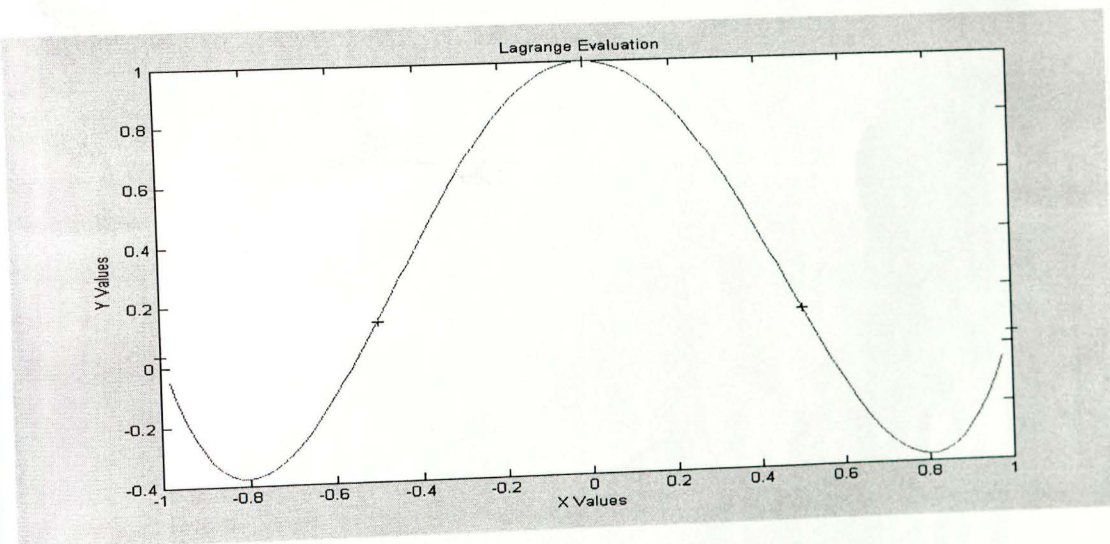
The exact value is $\cot 2^{\circ}30' = 22.9037655484 \cdots$; the incorrect digits are written in bold. From this example we can say that interpolation by rational function is more accurate than polynomial interpolation.

Example 2: The data for the rational function is written as:

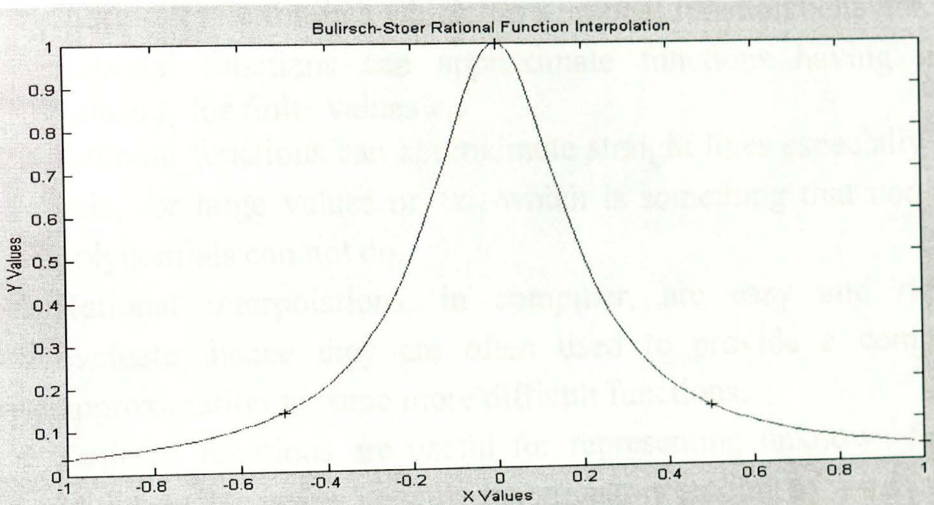
x	y
-1	0.0385
-0.5	0.1379
0	1
0.5	0.1379
1	0.0385

Solution: By using Lagrange for polynomial interpolation and rational interpolation we have the following:

The Lagrange fit of the data



The Rational Function interpolation fit of the data



6. Advantages of interpolation by rational function

- Rational interpolation generalizes polynomial interpolation
- More suitable for data which have rational function behavior.
- Rational functions can approximate functions having infinite values f_i for finite values x_i.
- Rational functions can approximate straight lines especially the x-axis, for large values of x_i, which is something that non trivial polynomials can not do.
- Rational interpolations, in computer, are easy and rapid to evaluate; hence they are often used to provide a computable approximation to some more difficult functions.
- Rational functions are useful for representing unknown function with possible poles, i.e with denominators tending to wards zero.
- Rational functions are a particular easy non linear interpolation schemes to fit.
- Rational functions are typically smoother and less oscillatory than polynomial interpolation.

CONCLUSION

In this seminar report, the use of rational functions to interpolate data has been discussed in detail. This is a more general class of functions as compared to the class of polynomial interpolation. The rational function interpolation deal with fractional polynomials depends on the entire set of data points. Since the rational functions have poles, then it has an ability to approximate such type of data points that polynomials can not do. This for the reason introducing rational function interpolation as a means of to approximate a function f is that of ordinary polynomial interpolation may not be applied. In addition, the graph rational function interpolation function is smoother and less oscillatory than the graph of the polynomial interpolating function.

Moreover, it has been showed that if one is interested the rational function itself, then using continued fraction or Thiele's continued fraction involving inverse and reciprocal differences respectively are suitable. However, if one desires the value of the interpolating function for a single argument, then algorithms of the Neville's type based on the formula of theorem 3.6 and formula (3.7) are preferred.

Finally, polynomial interpolation is a special case of rational interpolation at $m = 0$, where m is the maximum degree of the polynomial at the denominator, hence the theory of rational interpolation gives better results.

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